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C. R. Acad. Sci. Paris, Ser. I 338 (2004) 139–144



Partial Differential Equations/Complex Analysis

\mathcal{D} -modules associated to 3×3 matrices

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Received 29 August 2003; accepted 4 November 2003

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Abstract

We classify regular holonomic \mathcal{D} -modules whose characteristic variety is contained in the union of conormal bundles to the orbits of the group of invertible matrices. The main result is an equivalence between the category of such \mathcal{D} -modules and the one of graded modules of finite type over a Weyl algebra. *To cite this article: P. Nang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).* © 2003 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

\mathcal{D} -modules associés aux matrices 3×3 . On classe les \mathcal{D} -modules holonomes réguliers dont la variété caractéristique est contenu dans la réunion des fibrés conormaux aux orbites du groupe des matrices inversibles. Le résultat principal est une équivalence entre la catégorie de tels \mathcal{D} -modules et celle des modules gradués de type fini sur une algèbre de Weyl. *Pour citer cet article : P. Nang, C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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Version française abrégée

Soient X l'espace vectoriel complexe des matrices carrées d'ordre 3, \mathcal{D}_X le faisceau des opérateurs différentiels sur X . Le groupe $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$ opère sur X par : $((g, h), A) \mapsto gAh^{-1}$. Nous noterons G le quotient de $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$ par le noyau de cette action et $\tilde{G} := SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times \mathbb{C}$ son revêtement universel. Le groupe G a quatre orbites X_i : les ensembles de matrices de rang $i = 0, 1, 2, 3$. On se propose de classer les \mathcal{D}_X -modules holonomes réguliers dont la variété caractéristique est contenue dans la réunion Λ des fibrés conormaux aux orbites. Nous noterons $\text{Mod}_\Lambda^{\text{rh}}(\mathcal{D}_X)$ cette catégorie. On prouve le théorème suivant : soient θ le champ d'Euler sur X et \mathcal{B} l'algèbre de Weyl des opérateurs différentiels \tilde{G} -invariants opérant sur les fonctions \tilde{G} -invariantes. Notons $\text{Mod}^{\text{gr}}(\mathcal{B})$ la catégorie des \mathcal{B} -modules gradués de type fini T tels que $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty \forall u \in T$.

Théorème 0.1. *Les catégories $\text{Mod}_\Lambda^{\text{rh}}(\mathcal{D}_X)$ et $\text{Mod}^{\text{gr}}(\mathcal{B})$ sont équivalentes.*

De tels objets peuvent être interprétés en termes de diagrammes finis d'applications linéaires. Bien avant notre étude, des résultats similaires ont été obtenus par plusieurs auteurs. Boutet de Monvel [1] a classifié les $\mathcal{D}_{\mathbb{C}}$ -modules

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holonomes réguliers en utilisant des paires de \mathbb{C} -espaces vectoriels liés par certaines applications linéaires. Galligo, Granger et Maisonobe [2] ont obtenu, en utilisant la correspondance de Riemann–Hilbert, une classification des $\mathcal{D}_{\mathbb{C}^n}$ -modules à singularités régulières le long de $x_1 \dots x_n$ par des 2^n -tuples de \mathbb{C} -espaces vectoriels munit d'un ensemble d'applications linéaires. MacPherson et Vilonen [9] ont traité le cas à singularités le long de la courbe $y^n = x^m$. Enfin, l'auteur [11] a donné une présentation explicite dans le cas du cône quadratique dans \mathbb{C}^n (cf. aussi [3]) etc.

1. Introduction

Let X be the complex vector space of square matrices of order 3, \mathcal{D}_X the sheaf of differential operators on it. The group $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$ acts on X by: $((g, h), A) \mapsto gAh^{-1}$. Denote by G the quotient of $GL_3(\mathbb{C}) \times GL_3(\mathbb{C})$ by the kernel of this action and $\tilde{G} := SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times \mathbb{C}$ its universal covering. The group G has four orbits X_i : the sets of matrices of rank $i = 0, 1, 2, 3$. Our aim is to classify regular holonomic \mathcal{D}_X -modules whose characteristic variety is contained in the union Λ of conormal bundles to these orbits. Denote by $\text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X)$ this category. We prove the following theorem: let θ be the Euler vector field on X and \mathcal{B} the Weyl algebra of \tilde{G} -invariant differential operators acting on \tilde{G} -invariant functions. Denote by $\text{Mod}^{\text{gr}}(\mathcal{B})$ the category of graded \mathcal{B} -modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty \forall u \in T$.

Theorem 1.1. *The categories $\text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X)$ and $\text{Mod}^{\text{gr}}(\mathcal{B})$ are equivalent.*

Let us mention that the objects in the category $\text{Mod}^{\text{gr}}(\mathcal{B})$ can be understood in terms of finite diagrams of linear maps. Even before our study, results along these lines have been obtained by several authors. Boutet de Monvel [1] classified regular holonomic $\mathcal{D}_{\mathbb{C}}$ -modules by using pairs of finite dimensional \mathbb{C} -vector spaces related by certain linear maps. Galligo, Granger and Maisonobe [2] obtained, using the Riemann–Hilbert correspondance, a classification of holonomic $\mathcal{D}_{\mathbb{C}^n}$ -modules with singularities along $x_1 \dots x_n$ by using 2^n -tuples of \mathbb{C} -vector spaces with a set of linear maps. MacPherson and Vilonen [9] treated the case with singularities along the curve $y^n = x^m$. Finally, the author [11] constructed an explicit presentation in the case of the quadratic cone in \mathbb{C}^n (see also [3]) etc.

2. Preliminary notions and results

2.1. Homogeneous \mathcal{D} -modules

We will use the theory of analytic \mathcal{D} -modules developed in [4–8].

Definition 2.1. Let \mathcal{M} be a \mathcal{D}_X -module. We say that \mathcal{M} is homogeneous if there is a good filtration stable under the action of the Euler vector field θ on X . We say that a section s of \mathcal{M} is homogeneous if $\dim_{\mathbb{C}} \mathbb{C}[\theta]s < \infty$. The section s is said to be homogeneous of degree $\lambda \in \mathbb{C}$ if there exists $j \in \mathbb{N}$ such that $(\theta - \lambda)^j s = 0$.

Theorem 2.2 [11, Theorem 1.3]. *Let \mathcal{M} be a coherent homogeneous \mathcal{D}_X -module with a good filtration $(F_k \mathcal{M})_{k \in \mathbb{Z}}$ stable by θ . Then*

- (i) \mathcal{M} is generated by a finite number of global sections $(s_i)_{i=1, \dots, p}$ in \mathcal{M} such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]s_i < \infty$,
- (ii) the vector space $\Gamma(X, F_k \mathcal{M}) \cap [\bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p]$ of homogeneous global sections of $F_k \mathcal{M}$ of degree λ is finite dimensional $\forall k \in \mathbb{Z}, \forall \lambda \in \mathbb{C}$.

2.2. Algebra of invariant operators

Now denote by \mathcal{W} the Weyl algebra on X . Let us describe the algebra $\bar{\mathcal{B}} := \Gamma(X, \mathcal{D}_X)^{\tilde{G}} \subset \mathcal{W}$ of \tilde{G} -invariant differential operators. Let $x_1 = (x_{ij})$, $d_1 = {}^t(\partial/\partial x_{ij})$ be matrices with entries in \mathcal{D}_X . The group \tilde{G} acts on

these matrices by: $g \cdot (x_1, d_1) = (ax_1b^{-1}, bd_1a^{-1}) \forall g = (a, b) \in \tilde{G}$. Let $x_2 := \det(x_1)x_1^{-1}$ (resp. d_2) be the adjoint matrix of x_1 (resp. d_1). Denote by Tr the trace map. We set $\delta := \frac{1}{3} \text{Tr} x_1 x_2 = \det(x_{ij})$, $\Delta := \frac{1}{3} \text{Tr} d_1 d_2 = \det(\partial/\partial x_{ij})$, $\theta := \text{Tr} x_1 d_1$, $q := \text{Tr} x_2 d_2$. We have the following proposition.

Proposition 2.3. *The algebra $\bar{\mathcal{B}}$ is generated over \mathbb{C} by $\delta, \Delta, \theta, q$ such that*

$$\begin{aligned} (r_1) \quad & [\theta, \delta] = +3\delta, \quad [\theta, \Delta] = -3\Delta, \\ (r_2) \quad & [q, \delta] = 2\theta\delta, \quad [q, \Delta] = -2\Delta\theta, \\ (r_3) \quad & [\Delta, \delta] = \frac{q}{3} + 2\left(\frac{\theta}{3} + 1\right)\left(\frac{\theta}{3} + 3\right), \quad [\theta, q] = 0. \end{aligned}$$

Denote by $\mathcal{I} \subset \mathcal{W}$ the left ideal generated by infinitesimal generators of G . If we put $\mathcal{J} := \bar{\mathcal{B}} \cap \mathcal{I}$, then \mathcal{J} is the two sided ideal generated by $\frac{q}{3} - \frac{\theta}{3}(\frac{\theta}{3} + 1)$ and $\delta\Delta - \frac{\theta}{3}(\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)$ and $\mathcal{B} := \bar{\mathcal{B}}/\mathcal{J}$ is the algebra of \tilde{G} -invariant differential operators acting on \tilde{G} -invariant functions.

3. Modules with support on matrices of rank ≤ 2

Let us denote by $\bar{X}_i := \{x \in X \mid \text{rank}(x) \leq i\}$ ($i = 0, 1, 2$). We still denote by δ , the determinant map $\delta: X \rightarrow \mathbb{C}$, $x = (x_{ij}) \mapsto \det(x)$. Then we have $\bar{X}_2 := \{x \in X, \delta(x) = 0\}$. In this section, we study the \mathcal{D}_X -modules in $\text{Mod}_\Lambda^{\text{rh}}(\mathcal{D}_X)$ with support on \bar{X}_i . This study will be effectively used in the next section to prove that any \mathcal{D}_X -module \mathcal{M} in $\text{Mod}_\Lambda^{\text{rh}}(\mathcal{D}_X)$ is generated by its \tilde{G} -invariant homogeneous global sections.

3.1. Study of the inverse image

For a $\mathcal{D}_{\mathbb{C}}$ -module \mathcal{N} , we denote by $\delta^+\mathcal{N}$ its inverse image by δ . Let t be a coordinate of \mathbb{C} and $\partial_t := \frac{\partial}{\partial t}$, I_3 the identity matrix in X . The transfer module $\mathcal{D}_{X \xrightarrow{\delta} \mathbb{C}}$ is generated by an element K subject to the following relations

$$\delta K = Kt, \quad d_1 K = x_2 K \partial_t. \tag{1}$$

These relations imply the following

$$x_1 d_1 K = I_3 K t \partial_t, \quad \theta K = 3 K t \partial_t, \tag{2}$$

$$d_2 K = x_1 K \partial_t (t \partial_t + 1), \quad x_2 d_2 K = I_3 K t \partial_t (t \partial_t + 1), \tag{3}$$

$$q K = 3 K t \partial_t (t \partial_t + 1), \quad \Delta K = K \partial_t (t \partial_t + 1) (t \partial_t + 2). \tag{4}$$

The transfer module $\mathcal{D}_{X \xrightarrow{\delta} \mathbb{C}}$ is flat over $\delta^{-1}(\mathcal{D}_{\mathbb{C}})$ so the inverse image functor δ^+ is exact. If \mathcal{N} is a regular holonomic $\mathcal{D}_{\mathbb{C}}$ -module with singularity at $t = 0$, then its inverse image $\delta^+\mathcal{N}$ decomposes as \mathcal{N} . If the operator of multiplication by t is invertible on the $\mathcal{D}_{\mathbb{C}}$ -module \mathcal{N} , then δ is invertible on $\delta^+\mathcal{N}$. In particular, in this case any meromorphic section defined on $X \setminus \bar{X}_2$ extends to the whole X .

3.2. Characterization of $\delta^+(\mathcal{O}_{\mathbb{C}}(\frac{1}{t}))$

Now let us describe the submodules of $P := \delta^+(\mathcal{O}_{\mathbb{C}}(\frac{1}{t})) = \mathcal{O}_X(\frac{1}{\delta})$. The \mathcal{D}_X -module P is generated by its \tilde{G} -invariant homogeneous sections $e_k = K t^k = \delta^k K$, $k \in \mathbb{Z}_{\leq 0}$. We have

$$\delta e_k = e_{k+1}, \quad d_1 e_k = k x_2 e_{k-1}, \tag{5}$$

$$d_2 e_k = k(k+1) x_1 e_{k-1}, \quad \Delta e_k = k(k+1)(k+2) e_{k-1}. \tag{6}$$

The \mathcal{D}_X -module P has 4 submodules P_j , generated by e_j ($j = 0, -1, -2, -3$). Denote by P^j the quotient module associated to P_j : $P^0 = P_0$; $P^j = P_j/P_{j+1}$ if $j = -1, -2, -3$. The quotient P^j is an irreducible holonomic \mathcal{D}_X -module of multiplicity 1, whose microsupport is $\Lambda_{3+j} = \overline{T_{X_{3+j}}^* X}$. Indeed we have $P^0 = \mathcal{O}_X$ and the following description

Irreducible \mathcal{D} -modules	Associated generators and relations
P^{-1} with $\text{char}(P^{-1}) = \overline{T_{X_2}^* X}$	generator \tilde{e}_{-1} , $\begin{cases} \delta \tilde{e}_{-1} = 0, \\ d_2 \tilde{e}_{-1} = 0, \\ x_1 d_1 \tilde{e}_{-1} = -I_3 \tilde{e}_{-1}, \\ \theta \tilde{e}_{-1} = -3 \tilde{e}_{-1}, \end{cases} \tag{7}$

P^{-2} with $\text{char}(P^{-2}) = \overline{T_{X_1}^* X}$	generator \tilde{e}_{-2} , $\begin{cases} x_2 \tilde{e}_{-2} = 0, \\ \Delta \tilde{e}_{-2} = 0, \\ x_1 d_1 \tilde{e}_{-2} = -2I_3 \tilde{e}_{-2}, \\ \theta \tilde{e}_{-2} = -6 \tilde{e}_{-2}, \end{cases} \tag{8}$
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P^{-3} with $\text{char}(P^{-3}) = \overline{T_{X_0}^* X}$	generator \tilde{e}_{-3} , $\begin{cases} x \tilde{e}_{-3} = 0, \\ \theta \tilde{e}_{-3} = -9 \tilde{e}_{-3}. \end{cases} \tag{9}$
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Therefore using the relations (5), (6) and the fact that the P^j are irreducible modules, we see that if \mathcal{N} is a submodule of P which is not contained in P_j then \mathcal{N} contains P_{j+1} . Thus we have proved the following lemma.

Lemma 3.1. $P_0, P_{-1}, P_{-2}, P_{-3}$ are the only submodules of P .

Next we have the following proposition.

Proposition 3.2. Any section $s \in \Gamma(X \setminus \overline{X_1}, P_{-2})$ (resp. $\Gamma(X \setminus X_0, P_{-1})$) of the \mathcal{D}_X -module P_{-2} (resp. P_{-1}) defined on the complementary of $\overline{X_1}$ (resp. $X_0 = \{0\}$) extends to the whole X .

Proof. The \mathcal{D}_X -module P_j is the union of modules $\mathcal{O}_X \tilde{e}_k$ such that the associated graded module $\text{gr}(P_j)$ is the sum of modules $\mathcal{O}_{X_{3+j}} \tilde{e}_k$ ($j \leq k \leq 0, j = -1, -2$). In this case the property of “extension” here is true for functions because the hypersurface $\overline{X_2}$ is normal (along $\overline{X_1}$) and $\overline{X_1}$ is normal. \square

4. Invariant sections

Let \mathcal{M} be an object in $\text{Mod}_\Lambda^{\text{rh}}(\mathcal{D}_X)$. In this section, we show that \mathcal{M} is generated by \tilde{G} -invariant homogeneous global sections. This fact is at the heart of the proof of our main theorem. In an attempt to do it, first we see that there exists a $\mathcal{D}_\mathbb{C}$ -module \mathcal{N} such that \mathcal{M} is isomorphic to $\delta^+ \mathcal{N}$ in $X \setminus \overline{X_2}$. Let $i: \mathbb{C} \rightarrow X, t \mapsto \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be a section of the determinant map $\delta: X \rightarrow \mathbb{C}, x \mapsto \det(x)$, i.e., $\delta \circ i(\mathbb{C}) = \mathbb{C}$. Denote by $D := i(\mathbb{C})$ its image. The line D is non characteristic for \mathcal{M} , i.e., $\overline{T_D^* X} \cap \text{char}(\mathcal{M}) \subset T_X^* X$. Then \mathcal{M} is canonically isomorphic to $\delta^+ i^+(\mathcal{M})$ in the neighborhood of D . We know from [4] that $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta^+ i^+(\mathcal{M}))$ is a constructible sheaf. Also $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta^+ i^+(\mathcal{M}))$ is locally constant on the fibers $\delta^{-1}(t), t \in \mathbb{C}$. The sheaf $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \delta^+ i^+(\mathcal{M}))$ has a canonical section u defined in the neighborhood of the line D (corresponding with the isomorphism $\mathcal{M} \xrightarrow{\sim} \delta^+ i^+(\mathcal{M})$ which induces the identity on D). Since the fibers $\delta^{-1}(t), t \neq 0 \in \mathbb{C}$ are simply connected (see Theorem 5.2 of [10]), we have the following proposition.

Proposition 4.1. *The canonical isomorphism $u : \mathcal{M} \xrightarrow{\sim} \delta^+ i^+ (\mathcal{M})$ defined in the neighborhood of D such that $i^+ \cdot u = \text{Id}_D$, extends to $X \setminus \overline{X_2}$.*

Now we can prove the following theorem.

Theorem 4.2. *The \mathcal{D}_X -module \mathcal{M} is generated by its \tilde{G} -invariant homogeneous global sections.*

Proof. Recall $P := \delta^+ (\mathcal{O}_{\mathbb{C}}(\frac{1}{3})) = \mathcal{O}_X(\frac{1}{3})$. It has 4 submodules denoted by P_j , generated by the invariant homogeneous section e_j ($j = 0, -1, -2, -3$) (see Section 3.2). Let $\mathcal{M}^G \subset \mathcal{M}$ be the submodule generated over \mathcal{D}_X by \tilde{G} -invariant homogeneous global sections. We are going to show successively that the quotient $\mathcal{M}/\mathcal{M}^G$ is a module with support on $\overline{X_i}$ ($i = 0, 1, 2$).

First $\mathcal{M}/\mathcal{M}^G$ is with support on $\overline{X_2}$: indeed, we know from Proposition 4.1 that \mathcal{M} is isomorphic in $X \setminus \overline{X_2}$ to a module $\delta^+ \mathcal{N}$. We may assume that the operator of multiplication by t is invertible on \mathcal{N} such that there is a morphism $v : \mathcal{M} \rightarrow \delta^+ \mathcal{N}$ which is an isomorphism out of $\overline{X_2}$ (see Section 3.1). The image $v(\mathcal{M})$ is a submodule of $\delta^+ \mathcal{N}$ thus it is generated by its invariant global sections, and any invariant section s of a quotient of \mathcal{M} lifts to an invariant section \tilde{s} of \mathcal{M} (i.e. $\tilde{s} \in \Gamma(X, \mathcal{M})^G$). Therefore $\mathcal{M}/\mathcal{M}^G$ is with support on $\overline{X_2}$.

If \mathcal{M} is with support on $\overline{X_2}$, it is isomorphic out of $\overline{X_1}$ to a direct sum of copies of P_{-3}/P_0 , then there is a morphism $\mathcal{M} \rightarrow (P_{-3}/P_0)^N$ whose sections extend such that $\mathcal{M}/\mathcal{M}^G$ is with support on $\overline{X_1}$ because the submodules of P_{-3}/P_0 are also generated by their invariant sections. In the same way, if \mathcal{M} is with support on $\overline{X_1}$ then there is a morphism $\mathcal{M} \rightarrow (P_{-3}/P_{-1})^N$ which is an isomorphism out of $X_0 = \{0\}$, such that $\mathcal{M}/\mathcal{M}^G$ is with support on $\{0\}$ because the submodules of P_{-3}/P_{-1} are also generated by their invariant sections. Finally, if \mathcal{M} is with support on $X_0 = \{0\}$ the result is obvious. \square

5. Main result

Recall that \mathcal{W} indicates the Weyl algebra on X and $\overline{\mathcal{B}} \subset \mathcal{W}$ is the algebra of \tilde{G} -invariant differential operators generated by $\delta, \Delta, \theta, q$ satisfying the relations $(r_i)_{i=1,2,3}$ of Proposition 2.3. We denoted by \mathcal{B} the quotient of $\overline{\mathcal{B}}$ by the two sided ideal \mathcal{J} generated by $\delta\Delta - \frac{\theta}{3}(\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)$ and $q - \theta(\frac{\theta}{3} + 1)$. As in the introduction $\text{Mod}^{\text{gr}}(\mathcal{B})$ stands for the category of graded \mathcal{B} -modules T of finite type such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty \forall u \in T$.

If \mathcal{M} is an object in $\text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X)$, denote by $\Psi(\mathcal{M})$ the submodule of $\Gamma(X, \mathcal{M})$ consisting of \tilde{G} -invariant global sections u of \mathcal{M} such that $\dim_{\mathbb{C}} \mathbb{C}[\theta]u < \infty$. Recall $\Psi(\mathcal{M})_{\lambda} := [\Psi(\mathcal{M})] \cap [\bigcup_{p \in \mathbb{N}} \ker(\theta - \lambda)^p]$ is the \mathbb{C} -vector space of homogeneous global sections of degree λ of $\Psi(\mathcal{M})$ and that $\Psi(\mathcal{M}) = \bigoplus_{\lambda \in \mathbb{C}} \Psi(\mathcal{M})_{\lambda}$ (see Theorem 2.2). Then $\Psi(\mathcal{M})$ is an object in $\text{Mod}^{\text{gr}}(\mathcal{B})$. Conversely, if T is an object in $\text{Mod}^{\text{gr}}(\mathcal{B})$, one associates to it the \mathcal{D}_X -module $\Phi(T) := \mathcal{M}_0 \otimes_{\mathcal{B}} T$ where $\mathcal{M}_0 := \mathcal{W}/\mathcal{I}$ with \mathcal{I} the left ideal generated by infinitesimal generators of G . Then $\Phi(T)$ is an object in $\text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X)$. Thus, we have defined two functors $\Psi : \text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X) \rightarrow \text{Mod}^{\text{gr}}(\mathcal{B})$ and $\Phi : \text{Mod}^{\text{gr}}(\mathcal{B}) \rightarrow \text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X)$.

Theorem 5.1. *The functors Ψ and Φ induce an equivalence of categories*

$$\text{Mod}_{\Lambda}^{\text{rh}}(\mathcal{D}_X) \xrightarrow{\sim} \text{Mod}^{\text{gr}}(\mathcal{B}). \tag{10}$$

Proof. First, denote by ε the canonical generator of $\mathcal{M}_0 := \mathcal{W}/\mathcal{I}$. If $h \in \mathcal{W}$, denote by $\tilde{h} \in \overline{\mathcal{B}}$ its average over $SU_3(\mathbb{C}) \times SU_3(\mathbb{C})$. The average operator $\mathcal{W} \rightarrow \overline{\mathcal{B}}, h \mapsto \tilde{h}$ induces a surjective morphism of \mathcal{B} -modules $v : \mathcal{M}_0 \rightarrow \mathcal{B}$. More generally, for any T in $\text{Mod}^{\text{gr}}(\mathcal{B})$ the morphism $v \otimes 1_T$ is a surjective map $v_T : \mathcal{M}_0 \otimes_{\mathcal{B}} T \rightarrow \mathcal{B} \otimes_{\mathcal{B}} T = T$ which is the left inverse of $u_T : T \rightarrow \mathcal{M}_0 \otimes_{\mathcal{B}} T, t \mapsto \varepsilon \otimes t$. Thus u_T is injective. Next, the image of u_T is exactly the set of invariant sections of $\mathcal{M}_0 \otimes_{\mathcal{B}} T := \Phi(T)$ that is $\text{Im } u_T = \Psi(\Phi(T))$. Therefore u_T is bijective from T to $\Psi(\Phi(T))$. Now consider the canonical morphism $w : \Phi(\Psi(\mathcal{M})) \rightarrow \mathcal{M}$. From Theorem 4.2 \mathcal{M} is generated by its invariant sections $\Psi(\mathcal{M})$ thus w is surjective. The kernel $\mathcal{Q} := \ker w$ is also generated by $\Psi(\mathcal{Q})$. Then

$\Psi(\mathcal{Q}) \subset \Psi[\Phi(\Psi(\mathcal{M}))] = \Psi(\mathcal{M})$. Since $\Psi(\mathcal{M}) \rightarrow \mathcal{M}$ is injective we obtain $\Psi(\mathcal{Q}) = 0$. Thus $Q = \mathcal{D}_X \Psi(\mathcal{Q}) = 0$ (see Theorem 4.2). Therefore w is bijective. \square

5.1. *Diagram associated to a D-module*

A graded \mathcal{B} -module T in $\text{Mod}^{\text{gr}}(\mathcal{B})$ defines an infinite diagram consisting of finite dimensional vector spaces T_λ (with $(\theta - \lambda)$ being nilpotent on each T_λ , $\lambda \in \mathbb{C}$) and linear maps between them deduced from the multiplication by δ , Δ , $q: \dots \xrightarrow{\delta} T_\lambda \xrightarrow{\Delta} T_{\lambda+3} \xrightarrow{\delta} \dots$ satisfying the relations $(r_i)_{i=1,2,3}$ of Proposition 2.3 and $\delta\Delta = \frac{\theta}{3}(\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)$, $\Delta\delta = (\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)(\frac{\theta}{3} + 3)$, $q = \theta(\frac{\theta}{3} + 1)$. Such a diagram is completely determined by a finite subset of objects and arrows. Indeed

(a) For $\sigma \in \mathbb{C}/3\mathbb{Z}$, denote by $T^\sigma \subset T$ the submodule $T^\sigma = \bigoplus_{\lambda=\sigma \bmod 3\mathbb{Z}} T_\lambda$. Then T is generated by the finite direct sum of T^σ 's

$$T = \bigoplus_{\sigma \in \mathbb{C}/3\mathbb{Z}} T^\sigma = \bigoplus_{\sigma \in \mathbb{C}/3\mathbb{Z}} \left(\bigoplus_{\lambda=\sigma \bmod 3\mathbb{Z}} T_\lambda \right).$$

(b) If $\sigma \neq 0 \bmod 3\mathbb{Z}$ ($\lambda = \sigma \bmod 3\mathbb{Z}$), then the linear maps δ and Δ are bijective. Therefore T^σ is completely determined by one element T_λ .

(c) If $\sigma = 0 \bmod 3\mathbb{Z}$ ($\lambda = \sigma \bmod 3\mathbb{Z}$), then T^σ is completely determined by a diagram of four elements

$$T_{-9} \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\Delta} \end{array} T_{-6} \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\Delta} \end{array} T_{-3} \begin{array}{c} \xrightarrow{\delta} \\ \xleftarrow{\Delta} \end{array} T_0.$$

In the other degrees δ or Δ are bijective thanks to the relations $\Delta\delta = (\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)(\frac{\theta}{3} + 3)$, $\delta\Delta = \frac{\theta}{3}(\frac{\theta}{3} + 1)(\frac{\theta}{3} + 2)$.

Acknowledgements

We are deeply grateful to Professor L. Boutet de Monvel for valuable suggestions and stimulating discussions, to Professor M. Kashiwara for bringing this problem to our attention and for his kind help. Also, we wish to thank Professor K. Takeuchi for his comments.

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