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Differential Geometry

Homogeneous quaternionic Kähler structures of linear type

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Abstract

A classification of homogeneous quaternionic Kähler structures by real tensors is given and related to Fino's representation theoretic decomposition. A relationship between the modules whose dimension grows linearly and quaternionic hyperbolic space is found. ***To cite this article: M. Castrillón López et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Résumé

Structures quaternion-kählériennes homogènes de type linéaire. Nous donnons une classification des structures kählériennes quaternioniennes homogènes en termes de tenseurs réels, ainsi qu'une rélation avec la décomposition donnée par Fino en utilisant la théorie des représentations. Nous donnons aussi une rélation entre les modules ayant dimension à croissance linéaire et l'espace hyperbolique quaternionien. ***Pour citer cet article : M. Castrillón López et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).***

Version française abrégée

Classification des structures kählériennes quaternioniennes homogènes en termes de tenseurs réels

Soit (M, g, υ) une variété presque-hermitienne quaternionienne, c'est-à-dire [9] telle que le groupe structural du fibré tangent admet une réduction au groupe $\mathsf{Sp}(n)\,\mathsf{Sp}(1)$. Une base standard $J_1,\ J_2,\ J_3$ quelconque définit les 2-formes $\omega_a(X, Y) = g(X, J_aY)$ et la 4-forme globale $\Omega = \sum_{a=1}^{3} \omega_a \wedge \omega_a$. On dit que la variété est quaternion-kählérienne si et seulement si $\nabla\Omega = 0$, où ∇ est la connexion de Levi-Civita. Une telle variété est dite une variété quaternion-kählérienne homogène si elle admet un groupe transitif d'isométries préservant le sous-fibré υ. Le théorème de Kiričenko [10] donne le corollaire suivant (cf. [5]) : *Une variété quaternion-kählérienne, connexe, simplement connexe et complète* (M, g, υ)*, est homogène si et seulement s'il existe un champ de tenseurs* S *de type* $(1, 2)$ *sur* M *satisfaisant les equations d'Ambrose–Singer* [2] : $\widetilde{\nabla} g = 0$, $\widetilde{\nabla} R = 0$, $\widetilde{\nabla} S = 0$, *avec l'équation*

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$\widetilde{\nabla}\Omega = 0$, *où* $\widetilde{\nabla} = \nabla - S$. Alors, S est dite *structure quaternion-kählérienne homogène* sur M. Puisque $\widetilde{\nabla}g = 0$, l'équation $\widetilde{\nabla}\Omega = 0$ est équivalente à l'existence de certaines 1-formes différentielles locales $\tilde{p}_a$ telles que $\widetilde{\nabla}_X J_1 = \tilde{p}_3(X)J_2 - \tilde{p}_2(X)J_3$, etc., et on montre que $S_{XYZ} = g(S_X Y, Z)$ satisfait les équations $S_{XYZ} = -S_{XZY}$ et

$$S_{X,J_1Y,J_1Z} - S_{XYZ} = \pi_3(X)g(J_2Y, J_1Z) - \pi_2(X)g(J_3Y, J_1Z), \quad \text{etc.}, \tag{1}$$

pour d'autres 1-formes locales π_a. Donc S_X agit comme un élément de l'algèbre de Lie $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ sur T_pM. On a le théorème suivant [5, Lemma 5.1] :

Théorème 0.1 (Fino). *L'espace* $\mathcal{T}(V)_+$ *de structures kählériennes quaternioniennes homogènes se décompose comme* $\mathcal{T}(V)_+ = [EH] \otimes (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \cong [EH] \oplus [ES^3H] \oplus [EH] \oplus [S^3EH] \oplus [KH]$.

Ici, $EH = E \otimes H$, où E et H dénotent les représentations standard de $\mathsf{Sp}(n)$ et de $\mathsf{Sp}(1)$ sur $\mathbb{C}^{2n}$ et $\mathbb{C}^2$ respectivement, et K la représentation de plus haut poids $(2, 1, 0, \ldots, 0)$. Les crochets indiquent la représentation réelle induite. Soit $V = T_pM$ l'espace tangent sur un point $p \in M$ quelconque, muni de la métrique $\langle\,,\,\rangle = g_p$ et d'une base standard fixe sur p, $J_{1|p}, J_{2|p}, J_{3|p}$. On considére les espaces $\mathcal{T}(V) = \{S \in \otimes^3 V^*\colon\ S_{XYZ} = -S_{XZY}\}$ et $\mathcal{V} = \{S \in \mathcal{T}(V)\colon\ S$ satisfait (1) pour quelques $\pi_a \in V^*\}$. Alors $\mathcal{V} = \mathcal{T}(V)_+$ dans la notation de Fino. Pour étudier la décomposition de $\mathcal{V}$, d'abord nous écrivons $\mathcal{V}$ comme la somme directe $\mathcal{V} = \widehat{\mathcal{V}} \oplus \check{\mathcal{V}}$, où $\widehat{\mathcal{V}} = \{T \in \mathcal{T}(V)\colon\ T_{X,J_1Y,J_1Z} = T_{XYZ}, \text{etc.}\}$ et $\check{\mathcal{V}} = \{\Theta \in \otimes^3 V^*\colon\ 2\Theta_{XYZ} = \sum_{a=1}^3 \pi_a(X)\langle J_aY, Z\rangle,\ \pi_a \in V^*\}$. Plus précisément, pour $S \in V$ nous écrivons $S = T^S + \Theta^S$, où $\Theta^S \in \check{\mathcal{V}}$ est determiné par les π_a de S, et $4T^S_{XYZ} = S_{XYZ} + \sum_{a=1}^3 S_{X,J_aY,J_aZ}$. Cette décomposition de $\mathcal{V}$ est orthogonale par rapport au produit interieur induit par $\langle\,,\,\rangle$ sur $\otimes^3 V^*$ et invariant sous l'action de $\mathsf{Sp}(n)\,\mathsf{Sp}(1) \subset \mathsf{SO}(4n)$ donnée par $(A(S))_{XYZ} = S_{A^{-1}X,A^{-1}Y,A^{-1}Z}$. L'espace $\check{\mathcal{V}}$ se décompose en plus comme $\check{\mathcal{V}} = \mathcal{QK}_1 \oplus \mathcal{QK}_2$, où $\mathcal{QK}_1, \mathcal{QK}_2 \subset \otimes^3 V^*$ sont, respectivement, $\{\Theta_{XYZ} = \sum_{a=1}^3 \theta(J_aX)\langle J_aY, Z\rangle,\ \theta \in V^*\}$ et $\{\Theta_{XYZ} = \sum_{a=1}^3 \theta_a(X)\langle J_aY, Z\rangle,\ \theta_a \in V^*,\ \sum_{a=1}^3 \theta_a \circ J_a = 0\}$. Pour décomposer $\widehat{\mathcal{V}}$, nous considérons l'application $L\colon\widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$ définie par $L(T)_{XYZ} = T_{ZXY} + T_{YZX} + \sum_{a=1}^3 (T_{J_aZ,X,J_aY} + T_{J_aY,J_aZ,X})$. Il est facile de voir que L est bien définie, linéaire, auto-adjointe et qu'elle satisfait $L(A(T)) = A(L(T))$ et $L \circ L = 8\,\mathrm{Id} - 2L$. Donc L est diagonalisable avec deux espaces propres $\widehat{\mathcal{V}}^2$ et $\mathcal{QK}_5$ avec valeurs propres respectives 2 et -4. Des calculs ultérieurs montrent que $\mathcal{QK}_5 = \{T \in \widehat{\mathcal{V}}\colon \mathfrak{S}_{XYZ}T_{XYZ} = 0\}$ et que $\widehat{\mathcal{V}}^2$ est l'ensemble des $T \in \otimes^3 V^*$ tels que $6T_{XYZ} = \mathfrak{S}_{XYZ}T_{XYZ} + \sum_{a=1}^3 \mathfrak{S}_{X,J_aY,J_aZ}T_{X,J_aY,J_aZ}$. On définit maintenant $(c_{12}T)(X) = \sum_{i=1}^{4n} T_{e_i,e_i,X}$, pour une base orthonormale $\{e_i\}_{i=1,\ldots,4n}$ de V quelconque. Si l'on écrit $\mathcal{QK}_4 = \ker c_{12} \cap \widehat{\mathcal{V}}^2$, $\mathcal{QK}_3 = \{T^\theta : \theta \in V^*\}$ et $T^\theta_{XYZ} = \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y) + \sum_{a=1}^3 (\langle X, J_aY\rangle\theta(J_aZ) - \langle X, J_aZ\rangle\theta(J_aY))$, on a que $c_{12}T^\theta = 4(n+1)\theta$ et $\widehat{\mathcal{V}}^2 = \mathcal{QK}_3 \oplus \mathcal{QK}_4$. De plus, cette décomposition est orthogonale et invariante par rapport a l'action de $\mathsf{Sp}(n)\,\mathsf{Sp}(1)$. Nous avons le théorème suivant.

Théorème 0.2. *Pour* $n \geqslant 2$, *la décomposition* $\mathcal{T}(V)_+ = \mathcal{V} = \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3 \oplus \mathcal{QK}_4 \oplus \mathcal{QK}_5$ *est une décomposition en représentations irréductibles orthogonales de* $\mathsf{Sp}(n)\,\mathsf{Sp}(1)$.

Démonstration. Il suffit de montrer l'irréductibilité des cinq sous-espaces. Pour ceci, il est suffisant d'identifier les modules $\mathcal{QK}_i$ avec ceux de la décomposition de Fino et appliquer le Lemme de Schur. □

$\mathsf{Sp}(n)\,\mathsf{Sp}(1)$*-modules à croissance linéaire et l'espace hyperbolique quaternionien*

Nous trouvons que $\dim[EH] = 4n$, $\dim[ES^3H] = 8n$, $\dim[S^3EH] = 2n(2n+1)(2n+2)/3$, $\dim[KH] = 16n(n^2-1)/3$. Donc $\mathcal{QK}_1$, $\mathcal{QK}_2$ et $\mathcal{QK}_3$ sont les modules dans $\mathcal{T}(V)_+$ ayant des dimensions croissant de façon linéaire avec $\dim M$.

Théorème 0.3. *Une variété quaternion-kählérienne connexe, simplement connexe et complète de dimension $4n \geqslant 8$, qui admet une structure quaternion-kählérienne homogène $S \in \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$ avec projection non-zero sur $\mathcal{QK}_3$ est isométrique à l'espace hyperbolique quaternionien.*

Démonstration. Une variété quaternion-kählérienne de dimension $4n \geqslant 8$ est d'Einstein et on a une formule bien connue pour $R_{XYJ_1ZW} + R_{XYZJ_1W}$ à travers du tenseur de Ricci $\mathbf{r}$. Puisque g est d'Einstein nous avons $\mathbf{r} = \nu(n+2)g$ pour une certaine constante ν, qui est la courbure scalaire réduite. Si l'on écrit un élément générique $S \in \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$ comme $S_X Y = g(X,Y)\xi - g(\xi,Y)X + \sum_{a=1}^{3}(g(\xi, J_a Y)J_a X - g(X, J_a Y)J_a\xi + g(X,\zeta_a)J_a Y)$, où $\xi = \theta^\sharp$, en utilisant la 2ème et la 3ème équations d'Ambrose–Singer et les deux identités de Bianchi, on a la formule $0 = \theta \wedge \{R_{WU} - \frac{1}{4}\nu[W^\flat \wedge U^\flat + \sum_a (J_a W)^\flat \wedge (J_a U)^\flat + 2\sum_a \omega_a(W,U)\omega_a]\}$, de laquelle on déduit facilement que (M, g, υ) est un espace de courbure quaternionienne constante ν. Pour déterminer la valeur de ν, nous calculons $R_{XY}\xi$ directement d'après sa définition, d'abord avec $X, Y \in (\mathbb{H}\xi)^\perp$ puis avec $X = \xi$, $Y \in (\mathbb{H}\xi)^\perp$. Ces expressions donnent finalement que $\nu = -4g(\xi,\xi)$. □

1. Introduction and preliminaries

Representation theory has been applied with success to the classification of several geometric structures on differentiable manifolds, beginning with the almost-Hermitian structures (Gray and Hervella [8]). In [5], Fino gave an abstract representation theoretic decomposition of the space of tensors determining homogeneous quaternionic Kähler structures (see Theorem 1.2 below). The first purpose of this note is to give a concrete description of the modules in Fino's classification. It is seen that three of these modules ($\mathcal{QK}_1$, $\mathcal{QK}_2$, $\mathcal{QK}_3$) have dimensions that are linear functions of the dimension of the manifold. The second aim of this note is to show that if a quaternionic Kähler manifold has a homogeneous structure determined by a generic element in $\mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$, then that manifold is quaternionic hyperbolic space. We conclude with some remarks about the expected detailed behaviour of these structures.

Let M be a C^∞ manifold. An almost quaternionic structure on M is a rank 3 subbundle υ of the bundle of $(1,1)$ tensors, which locally admits a *standard* basis J_1, J_2, J_3, i.e., $J_1^2 = J_2^2 = J_3^2 = -I$ and $J_1 J_2 = -J_2 J_1 = J_3$, etc. [Here and in the sequel 'etc.' denotes the equations obtained by cyclically permuting the indices.] Such an (M, υ) is called an almost quaternionic manifold, and has dimension $4n$ ($n \geqslant 1$). If (M, υ) has a Riemannian metric g such that $g(\sigma X, Y) + g(X, \sigma Y) = 0$, for each section σ of υ, then (M, g, υ) is called an almost quaternion-Hermitian manifold. A manifold M admits such a structure if and only if the structure group of TM is reducible to $\mathsf{Sp}(n)\,\mathsf{Sp}(1) \subset \mathsf{SO}(4n)$. Here a choice of standard basis gives $T_p M$ the structure of an $\mathbb{H}$-module and $(B, q) \in \mathsf{Sp}(n) \times \mathsf{Sp}(1)$ acts on $v \in \mathbb{H}^n$ via $(B,q)v = Bvq^*$. Different choices of standard basis give isomorphic representations. We write $TM = [EH]$, where E is the standard representation of $\mathsf{Sp}(n)$ on $\mathbb{C}^{2n}$, H is the standard representation of $\mathsf{Sp}(1) = \mathsf{SU}(2)$ on $\mathbb{C}^2$, and $[EH]$ indicates the vector bundle associated to the real module underlying the complex representation $E \otimes H$. For (M, g, υ) almost quaternion-Hermitian, a standard basis gives two-forms $\omega_a(X,Y) = g(X, J_a Y)$, locally, but the four-form $\Omega = \sum_{a=1}^{3} \omega_a \wedge \omega_a$ is globally defined. The manifold is said to be quaternionic Kähler if and only if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection. Ishihara [9] showed that this is equivalent to the conditions $\nabla_X J_1 = p_3(X)J_2 - p_2(X)J_3$, etc., for some local differential 1-forms p_a. A quaternionic Kähler manifold is said to be a homogeneous quaternionic Kähler manifold if it admits a transitive group of isometries preserving the subbundle υ. Kiričenko's theorem [10] has the following corollary (cf. [5]):

Theorem 1.1. *A connected, simply connected and complete quaternionic Kähler manifold (M, g, υ) is homogeneous if and only if there exists a tensor field S of type $(1,2)$ on M satisfying the Ambrose–Singer equations* [2]

$$\widetilde{\nabla} g = 0, \qquad \widetilde{\nabla} R = 0, \qquad \widetilde{\nabla} S = 0, \tag{2}$$

together with the equation $\widetilde{\nabla}\Omega = 0$, where $\widetilde{\nabla} = \nabla - S$.

The tensor field S is called a *homogeneous quaternionic Kähler structure* on M. Since $\widetilde{\nabla} g = 0$, the equation $\widetilde{\nabla}\Omega = 0$ is equivalent to the existence of local differential 1-forms $\tilde{p}_a$ such that $\widetilde{\nabla}_X J_1 = \tilde{p}_3(X)J_2 - \tilde{p}_2(X)J_3$, etc., and one may show that $S_{XYZ} = g(S_X Y, Z)$ satisfies both $S_{XYZ} = -S_{XZY}$ and

$$S_{X,J_1Y,J_1Z} - S_{XYZ} = \pi_3(X)g(J_2Y, J_1Z) - \pi_2(X)g(J_3Y, J_1Z), \quad \text{etc.,} \tag{3}$$

for some local one-forms π_a. This implies that S_X acts an element of the Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ on T_pM, which leads to [5, Lemma 5.1]:

Theorem 1.2 (*Fino*). *The space $\mathcal{T}(V)_+$ of quaternionic Kähler homogeneous structures decomposes as $\mathcal{T}(V)_+ = [EH] \otimes (\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)) \cong [EH] \oplus [ES^3H] \oplus [EH] \oplus [S^3EH] \oplus [KH]$.*

Here S^3E and S^3H are the third symmetric powers of E and H, and K denotes the irreducible representation of $\mathsf{Sp}(n)$ of highest weight $(2, 1, 0, \ldots, 0)$ (zero for $n = 1$).

2. A classification in terms of real tensors

We wish to identify the tensors lying in each component of Fino's decomposition. Let $V = T_pM$, write $\langle\,,\,\rangle = g_p$ and fix a standard basis $J_1,\ J_2,\ J_3$ at p. Put

$$\mathcal{T}(V) = \left\{S \in \otimes^3 V^*\colon\ S_{XYZ} = -S_{XZY}\right\}, \qquad \mathcal{V} = \left\{S \in \mathcal{T}(V)\colon\ S \text{ satisfies (3) for some } \pi_a \in V^*\right\}.$$

Thus $\mathcal{V} = \mathcal{T}(V)_+$ in Fino's notation. We begin by splitting $\mathcal{V}$ as a direct sum $\mathcal{V} = \widehat{\mathcal{V}} \oplus \check{\mathcal{V}}$, where

$$\widehat{\mathcal{V}} = \left\{T \in \mathcal{T}(V)\colon\ T_{X,J_1Y,J_1Z} = T_{XYZ},\ \text{etc.}\right\}, \quad \check{\mathcal{V}} = \left\{\Theta \in \otimes^3 V^*\colon\ 2\Theta_{XYZ} = \sum_a \pi_a(X)\langle J_aY, Z\rangle, \pi_a \in V^*\right\}.$$

For $S \in V$ we have $S = T^S + \Theta^S$, where $\Theta^S \in \check{\mathcal{V}}$ is the determined by the π_a for S, and $4T^S_{XYZ} = S_{XYZ} + \sum_{a=1}^3 S_{X,J_aY,J_aZ}$. This decomposition of $\mathcal{V}$ is orthogonal with respect to the inner product induced by $\langle\,,\,\rangle$ on $\otimes^3 V^*$ and is invariant for the action of $\mathsf{Sp}(n)\mathsf{Sp}(1) \subset \mathsf{SO}(4n)$ given by $(A(S))_{XYZ} = S_{A^{-1}X, A^{-1}Y, A^{-1}Z}$. The space $\check{\mathcal{V}}$ decomposes further as $\check{\mathcal{V}} = \mathcal{QK}_1 \oplus \mathcal{QK}_2$, where

$$\mathcal{QK}_1 = \left\{\Theta \in \otimes^3 V^*\colon\ \Theta_{XYZ} = \sum_{a=1}^3 \theta(J_aX)\langle J_aY, Z\rangle,\ \theta \in V^*\right\},$$

$$\mathcal{QK}_2 = \left\{\Theta \in \otimes^3 V^*\colon\ \Theta_{XYZ} = \sum_{a=1}^3 \theta_a(X)\langle J_aY, Z\rangle,\ \theta_a \in V^*,\ \sum_{a=1}^3 \theta_a \circ J_a = 0\right\}.$$

To decompose $\widehat{\mathcal{V}}$, consider the map $L : \widehat{\mathcal{V}} \to \widehat{\mathcal{V}}$ defined by $L(T)_{XYZ} = T_{ZXY} + T_{YZX} + \sum_{a=1}^3 (T_{J_aZ,X,J_aY} + T_{J_aY,J_aZ,X})$. It is easily proved that L is well defined, linear, self-adjoint and satisfies $L(A(T)) = A(L(T))$ and $L \circ L = 8\,\mathrm{Id} - 2L$. Thus L is diagonalizable with two eigenspaces $\widehat{\mathcal{V}}^2$ and $\mathcal{QK}_5$ with respective eigenvalues 2 and -4. One sees $\mathcal{QK}_5 = \{T \in \widehat{\mathcal{V}}\colon\ \mathfrak{S}_{XYZ} T_{XYZ} = 0\}$, and further calculations show that $\widehat{\mathcal{V}}^2$ is the set of $T \in \otimes^3 V^*$ such that

$$6T_{XYZ} = \mathop{\mathfrak{S}}_{XYZ} T_{XYZ} + \sum_{a=1}^3 \mathop{\mathfrak{S}}_{X,J_aY,J_aZ} T_{X,J_aY,J_aZ}.$$

Define $(c_{12}T)(X) = \sum_{i=1}^{4n} T_{e_i,e_i,X}$, for $\{e_i\}_{i=1,\ldots,4n}$ an orthonormal basis of V. Put

$$\mathcal{QK}_4 = \ker c_{12} \cap \widehat{\mathcal{V}}^2, \quad \mathcal{QK}_3 = \{T^\theta \colon \theta \in V^*\},$$

$$T^\theta_{XYZ} = \langle X, Y\rangle\theta(Z) - \langle X, Z\rangle\theta(Y) + \sum_{a=1}^{3}\bigl(\langle X, J_aY\rangle\theta(J_aZ) - \langle X, J_aZ\rangle\theta(J_aY)\bigr).$$

Then $c_{12}T^\theta = 4(n+1)\theta$ and $\widehat{\mathcal{V}}^2 = \mathcal{QK}_3 \oplus \mathcal{QK}_4$ is an invariant orthogonal decomposition.

Theorem 2.1. *For $n \geqslant 2$, the decomposition $\mathcal{T}(V)_+ = \mathcal{V} = \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3 \oplus \mathcal{QK}_4 \oplus \mathcal{QK}_5$ is a decomposition into orthogonal irreducible representations of* $\mathsf{Sp}(n)\,\mathsf{Sp}(1)$.

Proof. It only remains to prove the irreducibility of the five subspaces. For this, it suffices to identify the modules $\mathcal{QK}_i$ with those in Fino's decomposition. It is clear that $\mathcal{QK}_1 \cong [EH] \cong \mathcal{QK}_3$, with $\mathcal{QK}_1$ being the copy in $[EH] \otimes [S^2H]$, and $\mathcal{QK}_3$ the copy in $[EH] \otimes [S^2E]$. The definition of $\mathcal{QK}_2$ shows that it is the complement of $[EH] \cong \mathcal{QK}_1$ in $[EH] \otimes [S^2H]$, so $\mathcal{QK}_2 \cong [ES^3H]$. Now we have $\mathcal{QK}_3 \oplus \mathcal{QK}_4 \oplus \mathcal{QK}_5 \cong [EH] \otimes [S^2E] \cong [EH] \oplus [S^3EH] \oplus [KH]$, and know that $\mathcal{QK}_3 \cong [EH]$. Using Schur's Lemma one may now show that $\mathcal{QK}_5 \cong [KH]$ and hence $\mathcal{QK}_4 \cong [S^3EH]$. □

3. Modules of linear growth and quaternionic hyperbolic space

Computing dimensions of modules, e.g., via Young diagrams, one finds $\dim[EH] = 4n$, $\dim[ES^3H] = 8n$, $\dim[S^3EH] = 2n(2n+1)(2n+2)/3$, $\dim[KH] = 16n(n^2-1)/3$. Thus $\mathcal{QK}_1$, $\mathcal{QK}_2$ and $\mathcal{QK}_3$ are the modules in $\mathcal{T}(V)_+$ whose dimensions grow linearly with $\dim M$. For Riemannian homogeneous spaces, Tricerri and Vanhecke [11, p. 39] showed that a homogeneous structure in the module with linear growth occurs only on manifolds of constant negative sectional curvature. Similar results have been proved for pseudo Riemannian manifolds [6, p. 19] and for Kähler manifolds [7], where the distinguished class has constant negative holomorphic sectional curvature.

Theorem 3.1. *A connected, simply connected and complete quaternionic Kähler manifold of dimension $4n \geqslant 8$ admitting a homogeneous quaternionic Kähler structure $S \in \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$ with non-zero projection to $\mathcal{QK}_3$, is isometric to quaternionic hyperbolic space.*

Proof. Let $R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_YZ + \nabla_Y\nabla_XZ$ and $R_{XYZW} = g(R_{XY}Z, W)$ be the curvature tensors of g. A quaternionic Kähler manifold with dimension $4n \geqslant 8$ is Einstein [1,3], and one has

$$R_{XYJ_1ZW} + R_{XYZJ_1W} = \frac{1}{n+2}\bigl(\mathbf{r}(J_2X, Y)g(J_3Z, W) - \mathbf{r}(J_3X, Y)g(J_2Z, W)\bigr), \quad \text{etc.,} \tag{4}$$

where $\mathbf{r}$ is the Ricci tensor (see, for instance, [9, formula 2.13], with a different curvature convention, or note the minor misprint in [4, p. 404]). As g is Einstein we have $\mathbf{r} = \nu(n+2)g$ for some constant ν. A generic element $S \in \mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$ is given by

$$S_XY = g(X, Y)\xi - g(\xi, Y)X + \sum_{a=1}^{3}\bigl(g(\xi, J_aY)J_aX - g(X, J_aY)J_a\xi + g(X, \zeta_a)J_aY\bigr), \tag{5}$$

where ξ, ζ_a are vector fields on M. The ζ_a are the metric duals of the one-forms π_a, from $\check{\mathcal{V}} = \mathcal{QK}_1 \oplus \mathcal{QK}_2$, and ξ is dual to θ in the definition of $\mathcal{QK}_3$. As this latter component is non-zero, we have $\xi \neq 0$. First we compute $\widetilde{\nabla}\xi$ and $\widetilde{\nabla}\zeta_a$ using the third Ambrose–Singer equation (2): $\widetilde{\nabla}_Z(S_XY) = S_{\widetilde{\nabla}_ZX}Y + S_X\widetilde{\nabla}_ZY$. Taking the covariant derivative of (5) with respect to Z and recalling $\widetilde{\nabla}_XJ_1 = \tilde{p}_3(X)J_2 - \tilde{p}_2(X)J_3$, we get $0 = g(X, Y)\widetilde{\nabla}_Z\xi - g(\widetilde{\nabla}_Z\xi, Y)X + \sum_a(g(\widetilde{\nabla}_Z\xi, J_aY)J_aX + g(X, J_aY)J_a\widetilde{\nabla}_Z\xi) + \mathfrak{S}_{1,2,3}g(X, \widetilde{\nabla}_Z\zeta_1 - \tilde{p}_3(Z)\zeta_2 + \tilde{p}_2(Z)\zeta_3)J_1Y$, for all X, Y, Z. This is the original expression for an element of $\mathcal{QK}_1 \oplus \mathcal{QK}_2 \oplus \mathcal{QK}_3$, with ξ replaced by $\widetilde{\nabla}_Z\xi$ and ζ_1 replaced by $\widetilde{\nabla}_Z\zeta_1 - \tilde{p}_3(Z)\zeta_2 + \tilde{p}_2(Z)\zeta_3$, etc. So $\widetilde{\nabla}_Z\xi = 0$ and $\widetilde{\nabla}_Z\zeta_1 = \tilde{p}_3(Z)\zeta_2 - \tilde{p}_2(Z)\zeta_3$, etc. The first relation gives $\nabla_Z\xi = S_Z\xi$ and that $g(\xi, \xi)$ is a constant function. The second Ambrose–Singer equation can be written as

$$(\nabla_X R)_{YZWU} = -R_{S_X YZWU} - R_{YS_X ZWU} - R_{YZS_X WU} - R_{YZWS_X U}. \tag{6}$$

Substituting (5) in (6), one sees from (4) that for each a, the terms containing ζ_a sum to zero. The differential Bianchi identity shows that the cyclic sum with respect to X, Y, Z of (6) is zero. Writing this out, applying the two Bianchi identities and the relation (4), we get

$$R_{XY}\xi = \frac{\nu}{4}\left\{g(X,\xi)Y - g(Y,\xi)X + \sum_a \bigl(g(J_aX,\xi)J_aY - g(J_aY,\xi)J_aX + 2g(J_aX,Y)J_a\xi\bigr)\right\}, \tag{7}$$

which is the expression of the curvature tensor $R_{XY}Z$, for $Z = \xi$, of a space of constant quaternionic curvature ν (see [1,9]). We only need to prove that the expression similar to (7) is true for arbitrary Z. For this, we successively apply the second Bianchi identity to the second Ambrose–Singer equation $\widetilde{\nabla} R = 0$ and expand the terms S_X that appear. Applying formulas (5), (4) and (7) one gets that $0 = \theta \wedge \{R_{WU} - \frac{1}{4}\nu[W^\flat \wedge U^\flat + \sum_a (J_aW)^\flat \wedge (J_aU)^\flat + 2\sum_a \omega_a(W,U)\omega_a]\}$. Contracting with ξ, by virtue of formula (4) we obtain that (M, g, υ) is a space of constant quaternionic curvature ν. To determine the specific value of ν, we compute $R_{XY}\xi$ directly from its definition, obtaining that

$$\begin{aligned} R_{XY}\xi = {} & -g(Y,\nabla_X\xi)\xi - g(Y,\xi)\nabla_X\xi + g(Y,\nabla_X J_a\xi)J_a\xi + g(Y,J_a\xi)\nabla_X J_a\xi \\ & - g(Y,\nabla_X\zeta_a)J_a\xi - g(Y,\zeta_a)\nabla_X J_a\xi + g(X,\nabla_Y\xi)\xi - g(X,\nabla_Y J_a\xi)J_a\xi \\ & - g(X,J_a\xi)\nabla_Y J_a\xi + g(X,\xi)\nabla_Y\xi + g(X,\nabla_Y\zeta_a)J_a\xi + g(X,\zeta_a)\nabla_Y J_a\xi. \end{aligned} \tag{8}$$

Then, we first take $X, Y \in (\mathbb{H}\xi)^\perp$ in (8); after some calculations we deduce that $\zeta_a \in \mathbb{H}\xi$, for any $a = 1,2,3$. Taking then $X = \xi$ and $Y \in (\mathbb{H}\xi)^\perp$, from (7), (8), and since $g(\xi,\xi)$ has constant modulus, one obtains after some computations that the quaternionic curvature is equal to $-4g(\xi,\xi)$, thus concluding. □

Homogeneous structures of type $\mathcal{QK}_3$ may be constructed on quaternionic hyperbolic space, either by using a generalisation of the Cayley transform technique in [7], or by computing directly the homogeneous structure, where one sees that $\widetilde{\nabla}$ has holonomy $\mathsf{Sp}(1)$. In the case that $\xi = 0$, i.e., $S \in \mathcal{QK}_1 \oplus \mathcal{QK}_2$, one may show that M is a quaternionic symmetric space (of non-compact type). From preliminary calculations we conjecture that in this case S is zero and no non-trivial homogeneous structures of type $\mathcal{QK}_1 \oplus \mathcal{QK}_2$ exist. This and other questions will be pursued in future work.

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