

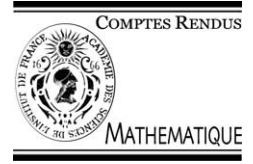


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Partial Differential Equations

# Asymptotic behavior of diffusion problems in a domain made of two cylinders of different diameters and lengths

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## Abstract

We study the asymptotic behavior of the solution of a diffusion problem posed in the union of a cylinder of small diameter and fixed length with another cylinder with much smaller diameter and length. The Dirichlet condition is assumed to hold at both extremities of this domain. Depending on the relative size of the parameters, we show that the boundary condition of the one-dimensional limit problem is a Dirichlet, Fourier or Neumann condition. We also prove a corrector result for every case. **To cite this article:** *J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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## Résumé

**Comportement asymptotique de problèmes de diffusion posés dans un domaine formé de deux cylindres de diamètres et longueurs différentes.** Nous étudions le comportement asymptotique de la solution d'un problème de diffusion posé sur l'union d'un cylindre de petit diamètre et de longueur fixe et d'un autre cylindre de longueur et de diamètre beaucoup plus petits. La condition de Dirichlet est imposée aux deux extrémités. Nous démontrons que selon les valeurs relatives des paramètres, la condition au bord du problème unidimensionnel limite est une condition de Dirichlet, de Fourier ou de Neumann. Nous démontrons aussi dans chaque cas un résultat de correcteur. **Pour citer cet article :** *J. Casado-Díaz et al., C. R. Acad. Sci. Paris, Ser. I 338 (2004).*

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## Version française abrégée

Dans cette Note, nous étudions le comportement asymptotique de la solution d'un problème de diffusion posé sur un domaine formé de deux cylindres dont l'un est un beaucoup plus petit que l'autre, à la fois en diamètre et en longueur (voir Fig. 1). Mathématiquement, le problème se formule de la façon suivante.

Pour  $\varepsilon > 0$ , on considère  $r_\varepsilon > 0$  et  $t_\varepsilon \geq 0$  deux paramètres qui convergent vers zéro avec  $\varepsilon$ . Soient  $S^-$  et  $S^+$  deux domaines réguliers bornés de  $\mathbb{R}^2$ , avec  $0 \in S^+$ . On définit  $\Omega_\varepsilon = \Omega_\varepsilon^- \cup \Omega_\varepsilon^0 \cup \Omega_\varepsilon^+$ , avec  $\Omega_\varepsilon^- = (-t_\varepsilon, 0) \times \varepsilon r_\varepsilon S^-$ ,  $\Omega_\varepsilon^0 = \{0\} \times \varepsilon r_\varepsilon S^-$ ,  $\Omega_\varepsilon^+ = (0, 1) \times \varepsilon S^+$ , et on note  $\Gamma_\varepsilon^- = \{-t_\varepsilon\} \times \varepsilon r_\varepsilon S^-$ ,  $\Gamma_\varepsilon^+ = \{1\} \times \varepsilon S^+$ ,  $\Gamma_\varepsilon = \Gamma_\varepsilon^- \cup \Gamma_\varepsilon^+$  (voir Fig. 1 pour le cas où  $S^-$  et  $S^+$  coïncident avec le disque unité de  $\mathbb{R}^2$ ).

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Pour  $A \in C^0([-1, 1] \times \overline{S^+}; M_{33})$  coercive, on définit  $A_\varepsilon \in C^0(\overline{\Omega_\varepsilon}; M_{33})$  par  $A_\varepsilon(x) = A(x_1, \frac{x'}{\varepsilon})$ ,  $\forall x \in \Omega_\varepsilon$ . De façon analogue, pour  $F \in L^2((-1, 1) \times S)$  on définit  $F_\varepsilon \in L^2(\Omega_\varepsilon)$  par  $F_\varepsilon(x) = F(x_1, \frac{x'}{\varepsilon})$ , p.p.  $x \in \Omega_\varepsilon$ .

Dans le domaine  $\Omega_\varepsilon$  on considère le problème de diffusion

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla U_\varepsilon) = F_\varepsilon & \text{dans } \Omega_\varepsilon, \\ A_\varepsilon \nabla U_\varepsilon n_\varepsilon = 0 & \text{sur } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \quad U_\varepsilon = 0 & \text{sur } \Gamma_\varepsilon. \end{cases}$$

Le problème que nous étudions dans cette Note est le comportement asymptotique de  $U_\varepsilon$  quand  $\varepsilon$  tend vers zéro. Le résultat est donné dans le Théorème 2.1 et les Propositions 2.1–2.4 de la version anglaise ci-dessous. Nous montrons que le comportement limite de  $U_\varepsilon$  dépend de la valeur des limites (qui existent toujours pour une sous-suite)

$$\mu = \lim_{\varepsilon \rightarrow 0} \frac{t_\varepsilon}{r_\varepsilon^2} \in [0, +\infty], \quad \nu = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{r_\varepsilon} \in [0, +\infty].$$

Si  $\mu = +\infty$  ou  $\nu = +\infty$ , le petit cylindre  $\Omega_\varepsilon^-$  est trop petit et  $U_\varepsilon$  se comporte comme si ce cylindre n’existait pas ; dans ce cas, on obtient une condition de Neumann en  $x_1 = 0$  pour la limite unidimensionnelle  $u$  de  $U_\varepsilon$ . Si  $\mu$  et  $\nu$  sont finis et si au moins l’un des deux est différent de zéro, il y a une certaine transmission du petit cylindre  $\Omega_\varepsilon^-$  au grand cylindre  $\Omega_\varepsilon^+$ , transmission qui se traduit par une condition de Fourier pour  $u$  en  $x_1 = 0$  ; cette condition de transmission est liée à un problème de type capacité défini (quand  $\nu > 0$ ) par (7) et (8) de la version anglaise ci-dessous. Quand  $\mu = \nu = 0$ , le petit cylindre  $\Omega_\varepsilon^-$  est si gros que la condition  $U_\varepsilon = 0$  sur  $\Gamma_\varepsilon^-$  implique  $u(0) = 0$ .

**1. Position of the problem and notation**

In the present paper we study the asymptotic behavior of the solution of a diffusion problem posed in a thin domain which is made of two cylinders, one of them being of length 1 and of diameter  $\varepsilon$ , and the second one being of length  $t_\varepsilon$  tending to zero and of diameter  $\varepsilon r_\varepsilon$  much smaller than  $\varepsilon$  (see Fig. 1). Mathematically the problem can be formulated as follows.

For  $\varepsilon > 0$ , we consider  $r_\varepsilon > 0$  and  $t_\varepsilon \geq 0$  two parameters which tend to zero with  $\varepsilon$ . Let  $S^-$  and  $S^+$  be two bounded smooth domains in  $\mathbb{R}^2$ , with  $0 \in S^+$ . We define the domain  $\Omega_\varepsilon = \Omega_\varepsilon^- \cup \Omega_\varepsilon^0 \cup \Omega_\varepsilon^+$ , where  $\Omega_\varepsilon^- = (-t_\varepsilon, 0) \times \varepsilon r_\varepsilon S^-$ ,  $\Omega_\varepsilon^0 = \{0\} \times \varepsilon r_\varepsilon S^-$ ,  $\Omega_\varepsilon^+ = (0, 1) \times \varepsilon S^+$ , and  $\Gamma_\varepsilon = \Gamma_\varepsilon^- \cup \Gamma_\varepsilon^+$ , where  $\Gamma_\varepsilon^- = \{-t_\varepsilon\} \times \varepsilon r_\varepsilon S^-$ ,  $\Gamma_\varepsilon^+ = \{1\} \times \varepsilon S^+$ . The domain  $\Omega_\varepsilon$  is represented by Fig. 1 in the case where  $S^-$  and  $S^+$  are both equal to the unitary ball of  $\mathbb{R}^2$ .

The elements of  $\mathbb{R}^3$  are decomposed as  $x = (x_1, x')$ ,  $x' = (x_2, x_3) \in \mathbb{R}^2$ , and we denote by  $M_{mn}$  the space of  $m \times n$  matrices.

We consider  $A \in C^0([-1, 1] \times \overline{S^+}; M_{33})$  such that there exists  $\alpha > 0$  with

$$A(x_1, x') \xi \xi \geq \alpha |\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \quad \forall (x_1, x') \in (-1, 1) \times S^+, \tag{1}$$

and we define  $A_\varepsilon \in C^0(\Omega_\varepsilon; M_{33})$  by

$$A_\varepsilon(x) = A\left(x_1, \frac{x'}{\varepsilon}\right), \quad \forall x \in \Omega_\varepsilon.$$

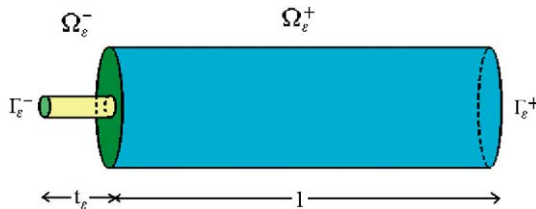


Fig. 1. The domain with  $\Omega_\varepsilon$ .

Fig. 1. Le domaine  $\Omega_\varepsilon$ .

We also consider  $F \in L^2((-1, 1) \times S^+)$  and we define  $F_\varepsilon \in L^2(\Omega_\varepsilon)$  by

$$F_\varepsilon(x) = F\left(x_1, \frac{x'}{\varepsilon}\right), \quad \text{a.e. } x \in \Omega_\varepsilon.$$

In the thin domain  $\Omega_\varepsilon$  we consider the diffusion problem

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla U_\varepsilon) = F_\varepsilon & \text{in } \Omega_\varepsilon, \\ A_\varepsilon \nabla U_\varepsilon n_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \quad U_\varepsilon = 0 & \text{on } \Gamma_\varepsilon. \end{cases} \tag{2}$$

The aim of this Note is to describe the asymptotic behavior of the solutions  $U_\varepsilon$ , as  $\varepsilon$  tends to zero. We show that it depends on the relative size of the parameters  $\varepsilon$ ,  $r_\varepsilon$  and  $t_\varepsilon$ , and more exactly on the values of the following limits

$$\mu = \lim_{\varepsilon \rightarrow 0} \frac{t_\varepsilon}{\varepsilon r_\varepsilon^2} \in [0, +\infty], \quad \nu = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{r_\varepsilon} \in [0, +\infty]. \tag{3}$$

Through this paper we assume that these limits exist (this always holds for a subsequence of  $\varepsilon$ ).

**Remark 1.** We assume the dimension of  $\Omega_\varepsilon$  to be  $N = 3$ . The case where  $N > 3$  is an easy extension of it, while the case  $N = 2$  requires more attention, due to the special properties of the  $H^1$ -capacity in the two dimensional case. More general conditions on  $A_\varepsilon$  and  $F_\varepsilon$  can be supposed (see the hypotheses in the paper [1], which is closely related to the present paper). Moreover, the results below can be easily generalized to monotone nonlinear problems.

To express the result and to make the proofs, we will use two changes of variables. The first change is the usual one used to study thin beams (see, e.g., [4–6]), which is given by  $y_1 = x_1$ ,  $y' = \frac{x'}{\varepsilon} \forall x \in \Omega_\varepsilon$ . This first change of variables does not provide the information we need about the behavior of  $U_\varepsilon$  in the thin cylinder  $\Omega_\varepsilon^-$  and in the part of  $\Omega_\varepsilon^+$  close to it. Thus we introduce a second change of variables defined by

$$z = \frac{x}{\varepsilon r_\varepsilon} \quad \forall x \in \Omega_\varepsilon^+ \cup \Omega_\varepsilon^0; \quad z_1 = \begin{cases} \frac{x_1}{r_\varepsilon^2} & \text{if } \mu = 0, \\ \frac{\mu x_1}{t_\varepsilon} & \text{if } \mu \in (0, +\infty), \end{cases} \quad z' = \frac{x'}{\varepsilon r_\varepsilon} \quad \forall x \in \Omega_\varepsilon^-,$$

which gives a suitable rescaling near  $x_1 = 0$ . In the case where  $\mu = +\infty$  and/or  $\nu = +\infty$ , we do not need to introduce this second change of variables.

We denote by  $Z^-$  the “limit” of  $z(\Omega_\varepsilon^-)$ , i.e.,  $Z^- = (-\mu, 0) \times S^-$ , by  $Z^0$  the “limit” of  $z(\Omega_\varepsilon^0)$ , i.e.,  $Z^0 = \{0\} \times S^-$ , by  $Z^+$  the “limit” of  $z(\Omega_\varepsilon^+)$ , i.e.,  $Z^+ = (0, +\infty) \times \mathbb{R}^2$ , and by  $Z = Z^- \cup Z^0 \cup Z^+$ .

We write the matrix  $A$  as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & A' \end{pmatrix},$$

where  $a_{11}$ ,  $a_{12}$ ,  $a_{21}$  and  $A'$  take their values respectively in  $M_{11}$ ,  $M_{12}$ ,  $M_{21}$  and  $M_{22}$ .

We introduce  $\eta \in L^\infty(0, 1; H^1(S^+))$  as the unique solution of

$$\begin{cases} -\operatorname{div}_{y'}(A' \nabla_{y'} \eta + a_{21}) = 0 & \text{in } S^+, \\ (A' \nabla_{y'} \eta + a_{21}) n' = 0 & \text{on } \partial S^+, \\ \int_{S^+} \eta(x_1, y') \, dy' = 0, & \text{a.e. } x_1 \in (0, 1). \end{cases} \tag{4}$$

We also define  $a \in L^\infty(0, 1)$ ,  $f \in L^2(0, 1)$  and (for  $0 < r < +\infty$ )  $\mathcal{A} \in L^\infty(Z; M_{33})$  by

$$\begin{aligned}
 a(x_1) &= \frac{1}{|S^+|} \int (a_{11}(x_1, y') + a_{12}(x_1, y') \nabla_{y'} \eta(x_1, y')) dy', \\
 f(x_1) &= \frac{1}{|S^+|} \int_{S^+} F(x_1, y') dy', \quad \text{a.e. } x_1 \in (0, 1), \\
 \mathcal{A}(z) &= \begin{pmatrix} a_{11}(0, 0) & v^{-1} a_{12}(0, 0) \\ v^{-1} a_{21}(0, 0) & v^{-2} A'(0, 0) \end{pmatrix} \chi_{Z^-}(z) + v^{-1} A(0, 0) \chi_{Z^+}(z), \quad \forall z \in Z.
 \end{aligned} \tag{5}$$

**Remark 2.** Taking into account the fact that  $\eta$  satisfies (4) and using (1), we get

$$a(x_1) = \frac{1}{|S^+|} \int_{S^+} A \begin{pmatrix} 1 \\ \nabla_{y'} \eta \end{pmatrix} \begin{pmatrix} 1 \\ \nabla_{y'} \eta \end{pmatrix} dy' \geq \frac{\alpha}{|S^+|} \int_{S^+} (1 + |\nabla_{y'} \eta|^2) dy' \geq \alpha, \quad \text{a.e. } x_1 \in (0, 1).$$

## 2. The result

The asymptotic behavior of  $U_\varepsilon$  depends on the values of  $\mu$  and  $\nu$ . Several situations appear which will be described separately.

**Theorem 2.1.** *The solution  $U_\varepsilon$  of (2) satisfies*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} (|U_\varepsilon(x) - u(x) \chi_{\Omega_\varepsilon^+}(x)|^2 + |\nabla U_\varepsilon(x) - P_\varepsilon(x)|^2) dx = 0,$$

where, depending on the values of  $\mu$  and  $\nu$ , the functions  $u$  and  $P_\varepsilon = (P_\varepsilon^1, P_\varepsilon')$  are defined by Propositions 2.1–2.4 below.

**Proposition 2.1.** *If  $\mu = +\infty$  and/or  $\nu = +\infty$ , then  $u \in H^1(0, 1)$  is the unique solution of*

$$\begin{cases} -\frac{d}{dx_1} \left( a(x_1) \frac{du}{dx_1} \right) = f(x_1) & \text{in } (0, 1), \\ -a(0) \frac{du}{dx_1}(0) = 0, \quad u(1) = 0, \end{cases}$$

and  $P_\varepsilon \in L^2(\Omega_\varepsilon)^3$  is given by

$$P_\varepsilon^1(x) = \frac{du}{dx_1}(x_1) \chi_{\Omega_\varepsilon^+}(x), \quad P_\varepsilon'(x) = \frac{du}{dx_1}(x_1) \nabla_{y'} \eta \left( x_1, \frac{x'}{\varepsilon} \right) \chi_{\Omega_\varepsilon^+}(x), \quad \text{a.e. } x \in \Omega_\varepsilon. \tag{6}$$

**Proposition 2.2.** *If  $\mu \in [0, +\infty)$  and  $\nu \in (0, +\infty)$ , then  $u \in H^1(0, 1)$  is the unique solution of*

$$\begin{cases} -\frac{d}{dx_1} \left( a(x_1) \frac{du}{dx_1} \right) = f(x_1) & \text{in } (0, 1), \\ -a(0) \frac{du}{dx_1}(0) + \lambda u(0) = 0, \quad u(1) = 0, \end{cases}$$

where

$$\lambda = \int_Z \mathcal{A} \nabla \zeta \nabla \zeta dz, \tag{7}$$

with  $\zeta \in H^1_{\text{loc}}(Z)$  the unique solution of

$$\begin{cases} -\operatorname{div}(\mathcal{A}\nabla\zeta) = 0 & \text{in } Z, \\ \zeta = 1 & \text{on } \{-\mu\} \times S^-, \quad \mathcal{A}\nabla\zeta n = 0 & \text{on } \partial Z(\{-\mu\} \times S^-), \end{cases} \quad \zeta \in L^6(Z), \quad \nabla\zeta \in L^2(Z)^3, \tag{8}$$

and  $P_\varepsilon \in L^2(\Omega_\varepsilon)^3$  is given by

$$\begin{cases} P_\varepsilon^1(x) = \frac{du}{dx_1}(x_1)\chi_{\Omega_\varepsilon^+}(x) - u(0)\left[\frac{1}{r_\varepsilon^2} \frac{\partial\zeta}{\partial z_1}\left(\frac{\mu x_1}{t_\varepsilon}, \frac{x'}{\varepsilon r_\varepsilon}\right)\chi_{\Omega_\varepsilon^-}(x)H(\mu) + \frac{1}{\varepsilon r_\varepsilon} \frac{\partial\zeta}{\partial z_1}\left(\frac{x}{\varepsilon r_\varepsilon}\right)\chi_{\Omega_\varepsilon^+}(x)\right], \\ P'_\varepsilon(x) = \frac{du}{dx_1}(x_1)\nabla_{y'}\eta\left(x_1, \frac{x'}{\varepsilon}\right)\chi_{\Omega_\varepsilon^+}(x) - \frac{u(0)}{\varepsilon r_\varepsilon}\left[\nabla_{z'}\zeta\left(\frac{\mu x_1}{t_\varepsilon}, \frac{x'}{\varepsilon r_\varepsilon}\right)\chi_{\Omega_\varepsilon^-}(x)H(\mu) + \nabla_{z'}\zeta\left(\frac{x}{\varepsilon r_\varepsilon}\right)\chi_{\Omega_\varepsilon^+}(x)\right], \end{cases} \tag{9}$$

a.e.  $x \in \Omega_\varepsilon$ , with  $H(\mu) = 0$  if  $\mu = 0$  and  $H(\mu) = 1$  if  $\mu > 0$ .

**Remark 3.** The number  $\lambda$  defined by (7) is, in some sense, as a generalization of the capacity (and the function  $\zeta$  is a generalization of the capacitary potential) of the set  $\{-\mu\} \times S^-$  in the set  $Z$  with respect to the matrix  $\mathcal{A}$ . In the particular case where  $\mathcal{A}$  is symmetric,  $\lambda$  is the minimum of the integral  $\int_Z \mathcal{A}\nabla\varphi\nabla\varphi \, dz$  among the functions  $\varphi$  such that  $\varphi = 1$  on  $\{-\mu\} \times S^-$  and  $\varphi = 0$  on  $|z| = +\infty$ . In the very particular case where  $\mu = 0$  and where the matrix  $A$  is the identity, it is easy to see by defining  $\zeta(-x_1, x') = \zeta(x_1, x')$  for  $x_1 > 0$  that  $\lambda$  is half of the (classical) capacity of the set  $\{0\} \times S^-$  in  $\mathbb{R}^3$ . Although we do not indicate this explicitly by using indices  $\mu$  and  $\nu$ , let us observe that the set  $Z$  depends on  $\mu$ , and that the matrix  $\mathcal{A}$  depends on  $\nu$ ; therefore  $\zeta$  and  $\lambda$  depend on  $\mu$  and  $\nu$ .

**Remark 4.** If we take a sequence  $\delta_\varepsilon > 0$  such that  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = 0$ ,  $\lim_{\varepsilon \rightarrow 0} \frac{\delta_\varepsilon}{\varepsilon r_\varepsilon} = +\infty$ , and if we define  $G_\varepsilon = \Omega_\varepsilon^+ \cap B(0; \delta_\varepsilon)$ , then in (9) we can replace  $\chi_{\Omega_\varepsilon^+}$  by  $\chi_{G_\varepsilon}$ . This gives the size of the boundary layer for  $x_1 > 0$  around  $x_1 = 0$ .

**Proposition 2.3.** If  $\mu \in (0, +\infty)$  and  $\nu = 0$ , then  $u \in H^1(0, 1)$  is the unique solution of

$$\begin{cases} -\frac{d}{dx_1}\left(a(x_1)\frac{du}{dx_1}\right) = f(x_1) & \text{in } (0, 1), \\ -a(0)\frac{du}{dx_1}(0) + \gamma u(0) = 0, \quad u(1) = 0, \end{cases}$$

where

$$\gamma = \frac{|S^-| \det(A(0, 0))}{\mu \det(A'(0, 0))}, \tag{10}$$

and  $P_\varepsilon \in L^2(\Omega_\varepsilon)^3$  is given by

$$\begin{cases} P_\varepsilon^1(x) = \frac{du}{dx_1}(x_1)\chi_{\Omega_\varepsilon^+}(x) + \frac{u(0)}{r_\varepsilon^2\mu}\chi_{\Omega_\varepsilon^-}(x), \\ P'_\varepsilon(x) = \frac{du}{dx_1}(x_1)\nabla_{y'}\eta\left(x_1, \frac{x'}{\varepsilon}\right)\chi_{\Omega_\varepsilon^+}(x) - \frac{u(0)}{r_\varepsilon^2\mu}A'(0, 0)^{-1}a_{21}(0, 0)\chi_{\Omega_\varepsilon^-}(x), \end{cases} \quad \text{a.e. } x \in \Omega_\varepsilon. \tag{11}$$

**Proposition 2.4.** If  $\mu = 0$  and  $\nu = 0$ , then  $u \in H^1(0, 1)$  is the unique solution of

$$\begin{cases} -\frac{d}{dx_1}\left(a(x_1)\frac{du}{dx_1}\right) = f(x_1) & \text{in } (0, 1), \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

and  $P_\varepsilon \in L^2(\Omega_\varepsilon)^3$  is given by (6).

**Remark 5.** The meaning of Propositions 2.1–2.4 is the following. If  $\mu = +\infty$  or  $\nu = +\infty$  (Proposition 2.1), then the small cylinder  $\Omega_\varepsilon^-$  is too small and the homogeneous Dirichlet condition on  $\Gamma_\varepsilon^-$  does not contribute to the

asymptotic behavior of  $u_\varepsilon$ , so that, passing to the the limit we obtain an homogeneous Neumann condition at  $x_1 = 0$ . If  $\mu$  and  $\nu$  are both finite and if at least one of them is different of zero (Propositions 2.2 and 2.3), the Dirichlet condition on  $\Gamma_\varepsilon^-$  gives at the limit a Fourier condition at  $x_1 = 0$ , i.e., there is some transmission process from  $\Omega_\varepsilon^-$  to  $\Omega_\varepsilon^+$ ; this Fourier condition is defined through a capacitary problem defined (when  $\nu > 0$ ) by (7) and (8). Finally, if  $\mu = 0$  and  $\nu = 0$  (Proposition 2.4), the small cylinder  $\Omega_\varepsilon^-$  is so thick that in this case  $U_\varepsilon$  behaves as if it vanishes on  $(\{0\} \times \varepsilon S^+)$ , so that passing to the limit we obtain an homogeneous Dirichlet condition at  $x_1 = 0$ .

**Remark 6.** Proposition 2.2 is in some sense the general case, and Propositions 2.1, 2.3 and 2.4 appear as limit cases of it. Indeed, when  $\mu$  tends to  $+\infty$  or  $\nu$  tends to  $+\infty$  in Proposition 2.2, then  $\nabla\zeta$  tends to zero and  $\lambda$  tends to zero, giving the result of Proposition 2.1.

When  $0 < \mu < +\infty$  and  $\nu$  tends to zero in Proposition 2.2, we consider for the sake of simplicity the case where the matrix  $A$  is symmetric. Then

$$\lambda = \inf_{\varphi} \int_{Z^-} A(0, 0) \begin{pmatrix} \frac{\partial\varphi}{\partial z_1} \\ \frac{1}{\nu} \nabla_{z'}\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial\varphi}{\partial z_1} \\ \frac{1}{\nu} \nabla_{z'}\varphi \end{pmatrix} dz + \frac{1}{\nu} \int_{Z^+} A(0, 0) \nabla\varphi \nabla\varphi dz,$$

where  $\varphi$  satisfies  $\varphi = 1$  on  $\{-\mu\} \times S^-$  and  $\varphi = 0$  on  $|z| = +\infty$ . One can prove (see, e.g., [5,1]) that  $\lambda$  converges to

$$\gamma = \inf_{\varphi_0, \varphi_1} \int_{Z^-} A(0, 0) \begin{pmatrix} \frac{d\varphi_0}{dz_1} \\ \nabla_{z'}\varphi_1 \end{pmatrix} \begin{pmatrix} \frac{d\varphi_0}{dz_1} \\ \nabla_{z'}\varphi_1 \end{pmatrix} dz, \quad (12)$$

where  $\varphi_0 \in H^1(-\mu, 0)$  with  $\varphi_0(-\mu) = 1$ ,  $\varphi_0(0) = 0$ , and where  $\varphi_1 \in L^2(-\mu, 0; H^1(S^-))$ , and that  $(\frac{\partial\zeta}{\partial z_1}, \frac{1}{\nu} \nabla_{z'}\zeta)$  converges to the minimizer  $(\frac{d\varphi_0}{dz_1}, \nabla_{z'}\varphi_1)$  of (12) in  $Z^-$ , while  $\frac{1}{\sqrt{\nu}} \nabla\zeta$  converges to zero in  $Z^+$ . An explicit computation then gives  $\frac{d\varphi_0}{dz_1} = -\frac{1}{\mu}$ ,  $\nabla_{z'}\varphi_1 = \frac{1}{\mu} A'(0, 0)^{-1} a_{21}(0, 0)$ , therefore  $\gamma = \frac{|S^-| \det(A(0, 0))}{\mu \det(A'(0, 0))}$ , i.e., the result of Proposition 2.3. Finally, when  $\mu$  tends to zero in Proposition 2.3, then  $\gamma$  tends to  $+\infty$ , implying that  $u(0)$  should vanish to the limit; this gives the result of Proposition 2.4.

The proofs of the above results are inspired by the proofs given in [1], and will be detailed in [2]. The case of elasticity problems will be considered in [3].

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