Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 337 (2003) 809–813
COMPTES RENDUS MATHEMATIQUE

Mathematical Problems in Mechanics

Modelling of a fluid: homogenization and mixed scales

Isabelle Gruais

Université de Rennes 1, I.R.M.A.R., campus de Beaulieu, 35042 Rennes cedex, France

Received 27 February 2003; accepted after revision 10 October 2003

Presented by Philippe G. Ciarlet

Abstract

The limit behavior of a mixture made of drops of some fluid immersed in another fluid is analysed in the framework of periodic homogenization when the size of the drops is critical compared with the period of the underlying network. More precisely, a strange term is obtained in the limit thus leading to a Brinkmann law. ***To cite this article: I. Gruais, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Modélisation d'un fluide par homogénéisation et échelles multiples. On étudie le comportement limite d'un mélange composé de gouttes d'un fluide immergées dans un deuxième fluide lorsque la répartition – théorique – des gouttes permet de se placer dans les conditions d'application de l'homogénéisation périodique. On suppose que la taille relative de ces gouttes par rapport à la petite période du problème a la valeur critique qui permet de faire apparaître un terme étrange caractérisant la loi de Brinkmann des fluides. ***Pour citer cet article : I. Gruais, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

Présentation du problème. Soit Ω un ouvert de $\mathbb{R}^3$ que l'on suppose immergé dans un réseau périodique de petite période tendant vers 0. On construit une partition (1)–(3) de Ω décrivant la répartition des deux fluides : des petites gouttes de volume ε^9 réparties dans un fluide ambiant avec la périodicité ε. Le problème consiste alors en le couplage (4)–(9) de deux équations de Stokes avec les conditions classiques d'interface (10)–(12).

Le problème limite. Pour tenir compte de la structure particulière du mélange, et en particulier de la taille des gouttes, on doit utiliser une variante de la méthode double-échelle (Définition 3.1). A la limite, on obtient, outre la vitesse macroscopique du mélange, deux vitesses microscopiques que l'on peut décrire en termes de profiles résolvant des problèmes de type Stokes (Théorèmes 4.1, 4.2). Elles apparaissent dans le problème homogénéisé sous la forme d'un terme étrange caractéristique de la loi de Brinkmann des fluides.

E-mail address: gruais@univ-rennes1.fr (I. Gruais).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2003.10.016

1. Motivation

Our aim is to modelize a fluid containing drops of another fluid differing by its low viscosity. The anisotropy of the mixture is assumed to be well described by a periodic network allowing arguments from [4,1,5,6]. Until now, the two-scale convergence method was mainly used in problems with little holes; in our case however, the holes are filled with a so-called mixed fluid while the ambiant fluid denotes the exterior of the holes and we are left with a transmission problem. We quote [7] for analoguous questions about a bi-porous media, and [8] for the modelisation of a visco-elastic fluid. We made another choice based on [9] that introduced a variant of the two-scale convergence that allows to consider the critical size of holes in the sense of [2,3]. In accordance with the latter reference, we prove that such a mixture leads to a Brinkmann law which is characterized by the creation of a "strange term" as in [10,11].

2. Description of the problem

Let $\Omega \subset \mathbf{R}^3$ be an open set in $\mathbf{R}^3$ and let $Q_m \subset \mathbf{R}^3$ be an open set in the cube $Q = (-\frac{1}{2}, +\frac{1}{2})^3$, $Q_m \Subset Q$. We denote by $\mathcal{A}_\varepsilon$ the periodic network:

$$\mathcal{A}_\varepsilon = \big\{k \in \mathbf{Z}^3;\ (\varepsilon k + \varepsilon Q) \cap \Omega \neq \emptyset\big\}. \tag{1}$$

We define Ω^ε as the countable union of translated cells $\varepsilon k + \varepsilon Q$ when k varies in $\mathbf{Z}^3$, namely:

$$\Omega^\varepsilon = \operatorname{Int}\left(\bigcup_{k \in \mathcal{A}^\varepsilon} \varepsilon k + \varepsilon \overline{Q}\right). \tag{2}$$

Then Ω^ε may be decomposed as the disjoint union of the set

$$\Omega_m^\varepsilon = \bigcup_{k \in \mathcal{A}^\varepsilon} \varepsilon k + \varepsilon^3 Q_m \tag{3}$$

which is the mixed fluid and its complement $\Omega^\varepsilon \setminus \overline{\Omega_m^\varepsilon}$ which stands for the ambiant fluid. The bifluid under consideration is described by the following coupled boundary value problem in $J \times \Omega^\varepsilon$ where $J = [0, T]$ denotes some time interval, namely: for some given $f \in L^2(J \times \Omega)^3$, find

$$(u^\varepsilon, p^\varepsilon) \in L^\infty\big(J; H^1\big(\Omega_f^\varepsilon\big)\big)^3 \times L^\infty\big(J; L^2\big(\Omega_f^\varepsilon\big)\big), \tag{4}$$

$$(v^\varepsilon, q^\varepsilon) \in L^\infty\big(J; H^1\big(\Omega_m^\varepsilon\big)\big)^3 \times L^\infty\big(J; L^2\big(\Omega_m^\varepsilon\big)\big), \tag{5}$$

solution of

$$u_t^\varepsilon - \operatorname{div}\big(M(x)\nabla u^\varepsilon\big) + \nabla p^\varepsilon = f, \quad J \times \Omega_f^\varepsilon, \tag{6}$$

$$\operatorname{div}(u^\varepsilon) = 0, \quad \Omega_f^\varepsilon, \tag{7}$$

$$v_t^\varepsilon - \operatorname{div}\big(\mu^\varepsilon(x)\nabla v^\varepsilon\big) + \nabla q^\varepsilon = \varepsilon^3 f, \quad J \times \Omega_m^\varepsilon, \tag{8}$$

$$\operatorname{div}(v^\varepsilon) = 0, \quad \Omega_m^\varepsilon. \tag{9}$$

We assume that the viscosity μ^ε is defined in Ω_m^ε and behaves like ε^6, that is:

$$\varepsilon^6 C_0 \leqslant \mu^\varepsilon(x) \leqslant \varepsilon^6 C_1, \quad \text{a.e. } x \in \Omega_m^\varepsilon,$$

and that it is periodic of period εQ, namely:

$$\mu^\varepsilon(x + \varepsilon k) = \mu^\varepsilon(x), \quad \forall x \in \Omega_m^\varepsilon,\ \forall k \in \mathcal{A}^\varepsilon.$$

The viscosity M is defined in Ω_f^ε and is assumed to be a continuous function of the variable $x \in \Omega_f^\varepsilon$: this assumption will be necessary to get asymptotic formulas. Moreover, it satisfies a coercivity condition: $M \geqslant \alpha > 0$. Notice that the mixed fluid Ω_m^ε is actually the union of little inclusions of volume $\varepsilon^9|Q_m|$.

The boundary conditions include transmission across the interface $\partial\Omega_m^\varepsilon$ and Dirichlet conditions on the exterior boundary $\partial\Omega^\varepsilon$, which read, if n^ε denotes the unit outward normal to $\partial\Omega_m^\varepsilon$:

$$u^\varepsilon = v^\varepsilon, \quad \partial\Omega_m^\varepsilon, \tag{10}$$

$$-Mn^\varepsilon \cdot \nabla u^\varepsilon + p^\varepsilon n^\varepsilon = \varepsilon^{-6}(-\mu^\varepsilon n^\varepsilon \cdot \nabla v^\varepsilon + q^\varepsilon n^\varepsilon), \quad \partial\Omega_m^\varepsilon, \tag{11}$$

$$u^\varepsilon = 0, \quad \partial\Omega^\varepsilon. \tag{12}$$

Finally, initial conditions read:

$$u^\varepsilon(0) = u^{0\varepsilon} \in H^1\big(\Omega_f^\varepsilon\big)^3, \qquad v^\varepsilon(0) = v^{0\varepsilon} \in H^1\big(\Omega_m^\varepsilon\big)^3 \tag{13}$$

with

$$\chi_{\Omega_f^\varepsilon} u^{0\varepsilon} + \chi_{\Omega_m^\varepsilon} v^{0\varepsilon} \in H_0^1\big(\Omega^\varepsilon\big)^3.$$

Weak convergence of the velocities and the pressures towards macroscopic quantities can be established, namely:

Proposition 2.1. *There exist functions $u \in L^2(J, H^1(\Omega))^3$ and $p \in L^2(J; L^2_{\mathrm{loc}}(\Omega)/\mathbf{R})$ such that*:

$$\chi_{\Omega_f^\varepsilon}(u^\varepsilon, p^\varepsilon) \rightharpoonup (u, p) \quad \textit{in } L^2\big(J, L^2(\Omega)\big)^3 \times L^2\big(J; L^2_{\mathrm{loc}}(\Omega)/\mathbf{R}\big), \tag{14}$$

$$\chi_{\Omega_f^\varepsilon} \nabla u^\varepsilon \rightharpoonup \nabla u \quad \textit{in } L^2\big(J, L^2(\Omega)\big)^{3\times 3}, \tag{15}$$

$$\chi_{\Omega_m^\varepsilon} v^\varepsilon \to 0 \quad \textit{in } L^2\big(J, H^1(\Omega)\big)^3, \tag{16}$$

$$\chi_{\Omega_m^\varepsilon} q^\varepsilon \to 0 \quad \textit{in } L^2\big(J; L^2_{\mathrm{loc}}(\Omega)/\mathbf{R}\big). \tag{17}$$

3. The change of variable

More information about the microscopic structure of the limit fluid can be obtained after a change of variable taking into account the very structure of the mixture.

Definition 3.1. For every $k \in \mathbf{Z^3}$, let C_k^ε be the translated cube of $\mathbf{R^3}$ of size $\varepsilon^3|Q|$ and center εk, namely:

$$C_k^\varepsilon = \varepsilon k + \varepsilon^3 Q.$$

Then, from any sequence of measurable functions $u^\varepsilon : \Omega \to \mathbf{R^3}$, we derive a sequence $\tilde{u}^\varepsilon$ of functions after the following change of variable: for almost every $x \in \Omega$, if C_k^ε is the cube that contains x, we set

$$\tilde{u}^\varepsilon(x, y) = u^\varepsilon(\varepsilon k + \varepsilon^3 y).$$

Remark 1. In each cube C_k^ε, $k \in \mathbf{Z^3}$, $\tilde{u}^\varepsilon$ depends on x through the constant k, so it is piecewise constant as a function of x. Besides, as a function of y, it is derived from u^ε after the unfolding:

$$\tilde{u}^\varepsilon\big(x, y^\varepsilon(x)\big) = u^\varepsilon(x), \qquad y^\varepsilon(x) = \frac{1}{\varepsilon^3}(x - \varepsilon k).$$

Remark 2. The function $x \in \mathbf{R}^3 \mapsto k^\varepsilon(x) \in \mathbf{Z^3}$ may be defined as follows. For almost every $x \in \mathbf{R^3}$, $k^\varepsilon(x) = k(\frac{x}{\varepsilon})$ is such that x lies in the translated unit cube $\varepsilon k + \varepsilon Q$ and satisfies

$$\left\|x - \varepsilon k^\varepsilon(x)\right\|_\infty = \sup\left\{|x_i - \varepsilon k_i|,\ i = 1, 2, 3\right\} \leqslant \varepsilon.$$

Then, $y \mapsto k(y)$ being well defined outside $\bigcup_{k' \in \mathbf{Z^3}} k' + \partial Q$ which is a subset of $\mathbf{R^3}$ of measure zero, it is almost everywhere defined and is a measurable function of x.

The periodicity of μ^ε may be expressed in terms of this change of variable. More precisely, it amounts to assume that μ^ε reads:

$$\mu^\varepsilon(x) = \varepsilon^6 \mu\big(y^\varepsilon(x)\big),$$

where $\mu : \mathbf{R}^3 \to \mathbf{R}_+^*$ is Q-periodic.

Remark 3. With our notations:

$$\Omega \subset \Omega^\varepsilon$$

and Ω^ε is a finite union of ε- cells in $\mathbf{R^3}$.

4. The main result

Now we are in a position to state our main result.

Theorem 4.1. *There exists $u \in L^2(J; H_0^1(\Omega))^3$ such that*

$$\chi_{\Omega_f^\varepsilon} u^\varepsilon \rightharpoonup u \quad \text{in } L^2\big(J; H^1(\Omega)\big)^3 \cap H^1\big(J; L^2(\Omega)\big)^3, \tag{18}$$

$$\chi_{\Omega_m^\varepsilon} v^\varepsilon \to 0 \quad \text{in } L^2(J \times \Omega)^3. \tag{19}$$

Moreover, there exist functions $(u^0, v^0) : \Omega \times J \times \mathbf{R^3} \to \mathbf{R^3} \times \mathbf{R^3}$ such that:

$$\phi(\cdot)\big(\chi_{\mathbf{R}^3 \setminus Q_m} u^0 + \chi_{Q_m} v^0\big) \in L^2\big(\Omega \times J; H^1(\mathbf{R}^3)\big)^3, \quad \forall \phi \in \mathcal{D}(\mathbf{R}^3),\ \textit{for a.e. } x \in \Omega, \tag{20}$$

$$u^0 - u \in L^2\big(J \times \Omega; D^{1,2}\big(\mathbf{R}^3 \setminus Q_m\big)\big)^3, \tag{21}$$

$$\chi_{\Omega \times K^\varepsilon \setminus Q_m} \tilde{u}^\varepsilon \rightharpoonup u^0, \quad L^2\big(\Omega \times J; L^6(B_R \setminus Q_m)\big)^3, \quad \forall R > 0; \tag{22}$$

$$\chi_{\Omega \times K^\varepsilon \setminus Q_m} \nabla_y \tilde{u}^\varepsilon \rightharpoonup \nabla_y u^0, \quad L^2\big(\Omega \times J; L^2(\mathbf{R}^3 \setminus Q_m)\big)^{3\times 3}, \tag{23}$$

$$\chi_{\Omega \times Q_m} \tilde{v}^\varepsilon \rightharpoonup v^0, \quad L^2\big(J; H^1(Q_m)\big)^3, \tag{24}$$

$$\frac{1}{|K^\varepsilon|} \int\limits_{K^\varepsilon \setminus Q_m} \tilde{u}^\varepsilon(x, \zeta)\, \mathrm{d}\zeta = \varepsilon^{-3} \int\limits_{C_f^\varepsilon(x)} u^\varepsilon(\rho)\, \mathrm{d}\rho \rightharpoonup u \quad \text{in } L^2\big(\Omega; L^2(B_R \setminus Q_m \times J)\big)^3\ \forall R > 0, \tag{25}$$

where we set:

$$C_f^\varepsilon(x) = \varepsilon k^\varepsilon(x) + \varepsilon Q \setminus \varepsilon^3 Q_m,$$

and where we recall that

$$\mathcal{D}^{1,2} = \big\{u \in L^6;\ \nabla u \in L^2\big\}.$$

There exist functions $p^0 \in L^2(J; L^2_{\mathrm{loc}}(\Omega \times \mathbf{R}^3 \setminus Q_m)/\mathbf{R})$ and $q^0 \in L^2(J; L^2_{\mathrm{loc}}(\Omega \times Q_m)/\mathbf{R})$ such that

$$\varepsilon^3\tilde{p}^\varepsilon \rightharpoonup p^0, \quad L^2\big(J; L^2_{\mathrm{loc}}(\Omega\times\mathbf{R}^3\setminus Q_m)/\mathbf{R}\big), \tag{26}$$

$$\varepsilon^3\tilde{q}^\varepsilon \rightharpoonup q^0, \quad L^2\big(J; L^2_{\mathrm{loc}}(\Omega\times Q_m)/\mathbf{R}\big). \tag{27}$$

The limit problem is described in the following theorem.

Theorem 4.2. *Besides the hypotheses of previous sections, assume that the whole sequence of initial data converge, namely*:

$$\chi_{\Omega_f^\varepsilon}u^{0\varepsilon} \rightharpoonup u^{00} \quad \text{in } H^1(\Omega)^3, \tag{28}$$

$$\varepsilon^{-3}\chi_{\Omega_m^\varepsilon}v^{0\varepsilon} \rightharpoonup v^{00} \quad \text{in } H^1(\Omega)^3. \tag{29}$$

Then, there exists

$$\big(p^0,q^0\big)\in L^2\big(J; L^2_{\mathrm{loc}}(\Omega\times\mathbf{R}^3\setminus Q_m)/\mathbf{R}\big)\times L^2\big(J; L^2_{\mathrm{loc}}(\Omega\times Q_m)/\mathbf{R}\big)$$

such that (u,p,u^0,v^0,p^0,q^0), *solves the following boundary value problem*:

$$-M(x)\Delta_y u^0+\nabla_y p^0=0, \quad \mathbf{R}^3\setminus Q_m, \tag{30}$$

$$\mathrm{div}_y\, u^0=0, \quad \mathbf{R}^3\setminus Q_m, \tag{31}$$

$$v^0_t-\mathrm{div}_y\big(\mu(y)\nabla_y v^0\big)+\nabla_y q^0=0, \quad \Omega\times Q_m\times J, \tag{32}$$

$$\mathrm{div}_y\big(v^0\big)=0, \quad \Omega\times Q_m\times J, \tag{33}$$

$$v^0=u^0, \quad \partial Q_m, \tag{34}$$

$$-M\frac{\partial}{\partial n}u^0+p^0n=-\mu(y)\frac{\partial}{\partial n}v^0+q^0n, \quad \partial Q_m, \tag{35}$$

$$v^0(0)=v^{00}, \quad \Omega\times Q_m, \tag{36}$$

$$u_t-\mathrm{div}(M\nabla u)+M^\sharp u+\nabla p=f, \quad J\times\Omega, \tag{37}$$

$$\mathrm{div}(u)=0, \quad \Omega, \tag{38}$$

where $M^\sharp(x)$ *denotes a* 3×3 *matrix whose generic term depends on profiles.*

References

[1] G. Allaire, Continuity of the darcy's law in the low volume fraction limit, Ann. Scuola Norm. Sup. Pisa 4 (18) (1991) 475–499.
[2] G. Allaire, Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. i: Abstract framework, a volume distribution of holes, Arch. Rational Mech. Anal. 113 (3) (1991) 209–260.
[3] G. Allaire, Homogenization of the Navier–Stokes equations in open sets perforated with tiny holes. ii: Non-critical size of the holes for a volume distribution and a surface distribution of holes, Arch. Rational Mech. Anal. 113 (3) (1991) 261–298.
[4] G. Allaire, Homogenization of the Navier–Stokes equations with a slip boundary condition, Comm. Pure Appl. Math. 44 (1991) 605–641.
[5] G. Allaire, Homogenization of the unsteady stokes equations in porous media, in: C. Brandle, et al. (Eds.), Progress in Partial Differential Equations: Calculus of Variations, Applications, in: Pitman Res. Notes in Math., Vol. 267, Longman Higher Education, New York, 1992.
[6] G. Allaire, Sur quelques méthodes en homogénéisation et leurs applications, Habilitation à diriger des recherches, Université Pierre et Marie Curie, Paris, 1993.
[7] T. Arbogast, J. Falk, U. Hornung, Derivation of the double porosity model of single phase flow via homogenization theory, SIAM J. Math. Anal. 21 (4) (1990) 823–836.
[8] F. Bentalha, Communication personnelle, Thèse de doctorat, Université de Metz, 2002.
[9] J. Casado-Diaz, Two-scale convergence for nonlinear Dirichlet problems in perforated domains, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 249–276.
[10] D. Cioranescu, F. Murat, Un terme étrange venu d'ailleurs, in: Nonlinear Partial Differential Equations and their Applications, College de France Seminar, Vol. II (Paris, 1979/1980), in: Pitman Res. Notes in Math., Vol. 60, Pitman, Boston, 1982, pp. 98–138.
[11] D. Cioranescu, F. Murat, Un terme étrange venu d'ailleurs ii, in: Nonlinear Partial Differential Equations and their Applications, College de France Seminar, Vol. III (Paris, 1980/1981), in: Pitman Res. Notes in Math., Vol. 70, Pitman, Boston, 1982, pp. 154–178.