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On isometric immersions of a Riemannian space under weak regularity assumptions

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Abstract

We consider a Riemannian metric in an open subset of $\mathbb{R}^d$ and assume that its Riemann curvature tensor vanishes. If the metric is of class C^2, a classical theorem in differential geometry asserts that the Riemannian space is locally isometrically immersed in the d-dimensional Euclidean space. We establish that, if the metric belongs to the Sobolev space $W^{1,\infty}$ and its Riemann curvature tensor vanishes in the space of distributions, then the Riemannian space is still locally isometrically immersed in the d-dimensional Euclidean space. ***To cite this article: S. Mardare, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Résumé

Sur les immersions isométriques d'un espace de Riemann sous des hypothèses faibles de régularité. On considère une métrique Riemannienne dans un ouvert de $\mathbb{R}^d$ et on suppose que son tenseur de courbure de Riemann s'annule. Si la métrique est de classe C^2, un théorème classique en géométrie différentielle affirme que l'espace de Riemann peut être plongé localement dans l'espace euclidien d-dimensionnel par une immersion isométrique. On établit que, si la métrique est de classe $W^{1,\infty}$ et son tenseur de courbure de Riemann s'annule, alors l'espace de Riemann peut encore être plongé localement dans l'espace euclidien d-dimensionnel par une immersion isométrique. ***Pour citer cet article : S. Mardare, C. R. Acad. Sci. Paris, Ser. I 337 (2003).***

Version française abrégée

On considère une métrique Riemannienne définie dans un ouvert $\Omega \subset \mathbb{R}^d$ par un champ des matrices symétriques définies positives d'ordre d et l'on suppose que son tenseur de courbure de Riemann s'annule, i.e.,

$$\partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^l \Gamma_{jl}^p - \Gamma_{ij}^l \Gamma_{kl}^p = 0 \quad \text{dans } \Omega$$

pour tout $i, j, k, p \in \{1, 2, \dots, d\}$, où

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$$

sont les symboles de Christoffel associés à la métrique (g_{ij}).

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L'objet de cette Note est d'établir, sous des hypothèses faibles de régularité sur la métrique, que l'espace de Riemann $(\Omega; (g_{ij}))$ peut être plongé localement dans l'espace euclidien d-dimensionnel (identifié à $\mathbb{R}^d$) par une immersion isométrique, i.e., que pour tout point $x \in \Omega$, il existe un voisinage V de x et une application $\Theta : V \to \mathbb{R}^d$ telle que

$$g_{ij}(x) = \frac{\partial \Theta(x)}{\partial x_i} \cdot \frac{\partial \Theta(x)}{\partial x_j}$$

pour tout $x = (x_1, x_2, \ldots, x_d) \in V$. De plus, une telle immersion isométrique Θ est unique aux isométries de $\mathbb{R}^d$ près. Lorsque l'ouvert Ω est connexe et simplement connexe, il existe une immersion isométrique définie sur tout Ω (voir par exemple [1]), c'est-à-dire que $V = \Omega$ dans la définition ci-dessus. Comme le résultat local est évidemment une conséquence du résultat global, seulement ce dernier est présenté dans cette Note, sous la forme de deux théorèmes.

Dans le premier (Théorème 2.1), on établit l'existence d'une immersion isométrique $\Theta \in W^{2,\infty}(\Omega; \mathbb{R}^d)$ sous les hypothèses que la métrique (g_{ij}) appartient à l'espace $W^{1,\infty}(\Omega; \mathbb{S}^d_>)$ avec $(g_{ij})^{-1} \in L^\infty(\Omega; \mathbb{M}^d)$ et que l'ouvert Ω est borné, connexe, simplement connexe, et satisfait la propriété du cône.

Dans le second (Théorème 2.2), on établit l'existence d'une immersion isométrique $\Theta \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ sous les hypothèses que la métrique (g_{ij}) appartient à l'espace $W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^d_>)$ et que l'ouvert Ω est connexe et simplement connexe.

Les démonstrations complètes de ces deux théorèmes, esquissées dans la version anglaise, se trouvent dans [3].

1. Preliminaries

All functions and fields appearing in this paper are real-valued and the summation convention with respect to repeated indices and exponents is used.

The d-dimensional Euclidean space will be identified with $\mathbb{R}^d$ by fixing a Cartesian basis in it. Let $\boldsymbol{u} \cdot \boldsymbol{v}$ designate the Euclidean inner product of $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ and let $|\boldsymbol{u}|$ denote the Euclidean norm of $\boldsymbol{u} \in \mathbb{R}^d$. The distance between two subsets A and B of $\mathbb{R}^d$ is defined by

$$\mathrm{dist}(A, B) = \inf_{x \in A,\, y \in B} |x - y|.$$

Let $\mathbb{M}^d$ designate the space of all square matrices of order d and let $\mathbb{S}^d_>$ designate its subset consisting of all symmetric, positive definite matrices of order d. The notation (g_{ij}) stands for the matrix whose entries are the elements g_{ij}, where the first index is the row index and the second index is the column index.

Let $\mathcal{D}'(\Omega)$ denote the space of distributions defined over Ω, let $W^{m,p}(\Omega; \mathbb{R}^d)$ denote the usual Sobolev space, and let

$$W^{m,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^d) := \{\boldsymbol{v} \in \mathcal{D}'(\Omega; \mathbb{R}^d);\ \boldsymbol{v} \in W^{m,p}(U; \mathbb{R}^d) \text{ for all open set } U \Subset \Omega\},$$

where the notation $U \Subset \Omega$ means that the closure of U in $\mathbb{R}^d$ is a compact subset of Ω. For real-valued function spaces, we shall use the notation $W^{m,p}(\Omega)$ instead of $W^{m,p}(\Omega, \mathbb{R})$, $\mathcal{D}'(\Omega)$ instead of $\mathcal{D}'(\Omega, \mathbb{R})$, etc.

Let $x = (x_1, x_2, \ldots, x_d)$ and $x' = (x_1, x_2, \ldots, x_{d-1})$ respectively denote generic points in $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$ and let

$$\partial_i := \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}},$$

where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_d)$ is a multi-index and $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

In what follows, we make the following conventions: for any $\hat{f} \in L^\infty$, we will always choose a bounded representative f such that $\|\hat{f}\|_{L^\infty} = \sup_x |f(x)|$; for any $\hat{f} \in W^{1,\infty}$, we will always choose the continuous representative f of $\hat{f}$.

The following three lemmas are key ingredients in the proof of our main results (Theorems 2.1 and 2.2).

Lemma 1.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Then there exists a set $X_d \subset \mathbb{R}$ with zero measure such that*

$$f(\cdot, \bar{x}_d) \in L^1_{\mathrm{loc}}\big(\mathbb{R}^{d-1}\big) \quad \text{and} \quad \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\bar{x}_d-\varepsilon}^{\bar{x}_d+\varepsilon} \int_{\omega'} \big| f(x', x_d) - f(x', \bar{x}_d) \big| \,\mathrm{d}x = 0$$

for all bounded open sets $\omega' \subset \mathbb{R}^{d-1}$ and all $\bar{x}_d \in \mathbb{R} \setminus X_d$.

Lemma 1.2. *Let $(g_{ij}) \in \mathbb{S}^d_>$. Then there exist d vectors $\boldsymbol{g}_i \in \mathbb{R}^d$, $i \in \{1, 2, \ldots, d\}$, such that $\boldsymbol{g}_i \cdot \boldsymbol{g}_j = g_{ij}$.*

Lemma 1.3. *Let Ω be a bounded open subset of $\mathbb{R}^d$ that satisfies the cone property and let $1 \leqslant p \leqslant \infty$. Then*

$$W^{m,p}(\Omega) = \big\{ u \in \mathcal{D}'(\Omega);\ \partial^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| = m \big\},$$

where $W^{m,p}(\Omega) := \{u \in L^p(\Omega);\ \partial^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leqslant m\}$ denotes the usual Sobolev space.

The proof of Lemma 1.1 relies on an idea which has been already put to use in other contexts (see [2] for instance), the proof of Lemma 1.2 is straightforward, and a proof of Lemma 1.3 can be found in, e.g., [4].

2. Existence of an isometric immersion

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^d$ and let $(\Omega, (g_{ij}))$ be a Riemannian space whose metric is given by a matrix field $(g_{ij}) \in W^{1,\infty}_{\mathrm{loc}}(\Omega; \mathbb{S}^d_>)$. Define the Christoffel symbols

$$\Gamma^k_{ij}(x) = \frac{1}{2} g^{kl}(x) \big(\partial_i g_{jl}(x) + \partial_j g_{li}(x) - \partial_l g_{ij}(x) \big)$$

for almost all $x \in \Omega$, where $(g^{kl}(x))$ is the inverse of the matrix $(g_{ij}(x))$. We recall that, according to the conventions made in Section 1, the field (g_{ij}) is the continuous representative of the class, still denoted by, (g_{ij}). Therefore, $(g_{ij}(x)) \in \mathbb{S}^d_>$ for all $x \in \Omega$. This implies that the inverse matrix (g^{kl}) belongs to $C^0(\Omega; \mathbb{S}^d_>)$, then that the Christoffel symbols Γ^k_{ij} belong to the space $L^\infty_{\mathrm{loc}}(\Omega)$.

We assume that the corresponding Riemann curvature tensor vanishes, that is,

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik} \Gamma^p_{jl} - \Gamma^l_{ij} \Gamma^p_{kl} = 0 \quad \text{in } \mathcal{D}'(\Omega) \text{ for all } i, j, k, p \in \{1, 2, \ldots, d\}.$$

The aim of this section is to find a mapping $\Theta \in W^{2,\infty}_{\mathrm{loc}}(\Omega; \mathbb{R}^d)$ such that the restriction of the d-dimensional Euclidean metric to $\Theta(\Omega)$ is given by the matrix field (g_{ij}). The mapping Θ is thus an isometric immersion of the Riemannian space $(\Omega, (g_{ij}))$ in the d-dimensional Euclidean space. This result is established in Theorem 2.2.

If the set Ω is bounded and satisfies the cone property (the definition can be found in, e.g., [4]) and if the metric (g_{ij}) satisfies some additional assumptions, we show that the mapping Θ belongs to $W^{2,\infty}(\Omega; \mathbb{R}^d)$. This result is established in the theorem below:

Theorem 2.1. *Let Ω be a connected and simply-connected bounded open subset of $\mathbb{R}^d$ that satisfies the cone property. Let there be given a matrix field $(g_{ij}) \in W^{1,\infty}(\Omega; \mathbb{S}^d_>)$ such that $(g_{ij})^{-1} \in L^\infty(\Omega; \mathbb{M}^d)$ and assume that the Riemann curvature tensor associated with the metric (g_{ij}) vanishes, that is,*

$$\partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^l_{ik} \Gamma^p_{jl} - \Gamma^l_{ij} \Gamma^p_{kl} = 0 \quad \text{in } \mathcal{D}'(\Omega) \text{ for all } i, j, k, p \in \{1, 2, \ldots, d\}. \tag{1}$$

Then there exists a mapping $\Theta \in W^{2,\infty}(\Omega; \mathbb{R}^d)$, unique up to isometries in $\mathbb{R}^d$, such that

$$\partial_i \Theta \cdot \partial_j \Theta = g_{ij} \quad \text{in } \Omega.$$

Sketch of proof. Let the matrix field $\Gamma_i : \Omega \to \mathbb{M}^d$ be given by $\Gamma_i(x) = (\Gamma_{ij}^k(x)) \in \mathbb{M}^d$, where j is the column index and k is the row index of the matrix. Then the Riemann compatibility conditions (1) can be written as

$$\partial_i \Gamma_j - \partial_j \Gamma_i + \Gamma_i \Gamma_j - \Gamma_j \Gamma_i = 0 \quad \text{in } \mathcal{D}'\big(\Omega; \mathbb{M}^d\big) \text{ for all } i, j \in \{1, 2, \dots, d\}.$$

These relations make sense since $\Gamma_i \in L^\infty(\Omega; \mathbb{M}^d)$. Let also a point $x^0 \in \Omega$ and vectors $\Theta^0, \boldsymbol{g}_i^0 \in \mathbb{R}^d$ be given such that $\boldsymbol{g}_i^0 \cdot \boldsymbol{g}_j^0 = g_{ij}(x^0)$ (such vectors exist thanks to Lemma 1.2).

The outline of the proof, broken into five steps numbered (i) to (v), is as follows. We first prove the existence of a matrix field $F \in W^{1,\infty}(\Omega; \mathbb{M}^d)$ such that

$$\partial_i F = F \Gamma_i \quad \text{in } \Omega \quad \text{and} \quad F\big(x^0\big) = F^0, \tag{2}$$

where $F^0 \in \mathbb{M}^d$ is the matrix whose i-th column is $\boldsymbol{g}_i^0 \in \mathbb{R}^d$. This result is first proved locally (step (i)), then globally by glueing together the local solutions (step (iii)). This can be done thanks in particular to the local uniqueness result proved in step (ii). The columns $\boldsymbol{g}_i$ of the matrix $F(x)$ will turn out to be the derivatives of the sought mapping Θ, whose existence will then be given by a generalized form of Poincaré's theorem (step (iv)). The uniqueness up to isometries in $\mathbb{R}^d$ of such a mapping is proved in step (v).

(i) *Local existence of a solution to the system* (2). We first show that this system possesses a solution in any open subset of $\mathbb{R}^d$ of the form $\omega = \prod_{i=1}^d (\bar{x}_i - \varepsilon_i, \bar{x}_i + \varepsilon_i)$, where $\bar{x} \in \mathcal{A} \cap \Omega$ and $\varepsilon_i > 0$ are such that $\overline{\omega} \subset \Omega$. The set $\mathcal{A}$ is a dense subset of $\mathbb{R}^d$ defined in such a way that Lemma 1.1 can be applied for all functions $\widehat{\Gamma}_{ij}^k(\cdot, \bar{x}_{l+1}, \dots, \bar{x}_d)$, $2 \leqslant l \leqslant d$, where $\widehat{\Gamma}_{ij}^k$ are the extensions by zero in $\mathbb{R}^d$ of the functions Γ_{ij}^k.

To this end, let d vectors $\bar{\boldsymbol{g}}_i \in \mathbb{R}^d$ be given such that $\bar{\boldsymbol{g}}_i \cdot \bar{\boldsymbol{g}}_j = g_{ij}(\bar{x})$ (their existence is insured by Lemma 1.2) and let $\overline{F} \in \mathbb{M}^d$ denote the matrix whose i-th column is the vector $\bar{\boldsymbol{g}}_i$. We construct a solution to the system (2) defined over ω recursively. We define first a solution $F_1 \in W^{1,\infty}((\bar{x}_1 - \varepsilon_1, \bar{x}_1 + \varepsilon_1); \mathbb{M}^d)$ to the system

$$\partial_1 F_1 = F_1 \Gamma_1(\cdot, \bar{x}_2, \dots, \bar{x}_d) \quad \text{and} \quad F_1(\bar{x}_1) = \overline{F},$$

then a solution F_2 of class $W^{1,\infty}$ in $(\bar{x}_1 - \varepsilon_1, \bar{x}_1 + \varepsilon_1) \times (\bar{x}_2 - \varepsilon_2, \bar{x}_2 + \varepsilon_2)$ to the system

$$\partial_\alpha F_2 = F_2 \Gamma_\alpha(\cdot, \bar{x}_3, \dots, \bar{x}_d), \quad \alpha \in \{1, 2\}, \quad \text{and} \quad F_2(\bar{x}_1, \bar{x}_2) = \overline{F},$$

and finally, after d steps, a solution $F := F_d \in W^{1,\infty}(\omega; \mathbb{M}^d)$ to the system

$$\partial_i F_d(x) = F_d(x) \Gamma_i(x), \quad i \in \{1, 2, \dots, d\}, \quad \text{and} \quad F_d(\bar{x}) = \overline{F}. \tag{3}$$

Because of the lack of regularity of the coefficients Γ_i, we cannot integrate these equations directly, but only through a sequence of approximating solutions. For the last system above for instance, this sequence is defined by

$$F^0(x) = 0 \quad \text{and} \quad F^{n+1}(x) = F_{d-1}(x') + \int_{\bar{x}_d}^{x_d} \big(F^n \Gamma_d\big)(x', t_d)\, \mathrm{d}t_d,$$

for almost all $x \in \omega$. Then it is established by a recursion argument that, for all $n \geqslant 1$,

$$\begin{aligned}
&F^{n+1} \in W^{1,\infty}\big(\omega; \mathbb{M}^d\big),\\
&\partial_\alpha F^{n+1}(x) = \big(F^n \Gamma_\alpha\big)(x) + \int_{\bar{x}_d}^{x_d} \big(\big(\partial_\alpha F^n - F^{n-1}\Gamma_\alpha\big)\Gamma_d\big)(x', t_d)\, \mathrm{d}t_d\\
&\qquad + \int_{\bar{x}_d}^{x_d} \big(\big(F^n - F^{n-1}\big)(\Gamma_d \Gamma_\alpha - \Gamma_\alpha \Gamma_d)\big)(x', t_d)\, \mathrm{d}t_d \quad \text{for } \alpha \leqslant d-1,\\
&\partial_d F^{n+1}(x) = \big(F^n \Gamma_d\big)(x).
\end{aligned}$$

Letting n go to ∞ in these relations shows that the limit in $L^\infty(\omega;\mathbb{M}^d)$ of F^n satisfies the system (3). Besides, if $\boldsymbol{g}_i$ denotes the i-th column of F, then one can prove that

$$\boldsymbol{g}_i(x)\cdot\boldsymbol{g}_j(x)=g_{ij}(x)\quad\text{for all } x\in\omega.$$

This relation allows us to remove the assumption that $\bar{x}$ belongs to the subset $\Omega\cap\mathcal{A}$. Besides, since distances between parallelepipeds are not easy to estimate, the local solutions to the system (2) will finally be defined over open balls $B(y,r)$ such that $y\in\Omega$ and $0<r\sqrt{d}<\operatorname{dist}(y,\Omega^c)$, where $\Omega^c=\mathbb{R}^d\setminus\Omega$.

(ii) *Let U be a connected open subset of Ω and let $F,\widetilde{F}\in W^{1,\infty}_{\mathrm{loc}}(U;\mathbb{M}^d)$ be such that*

$$\partial_i F=F\Gamma_i\quad\textit{and}\quad\partial_i\widetilde{F}=\widetilde{F}\Gamma_i\quad\textit{a.e. in } U.$$

Assume that there exists a point $y\in U$ such that $F(y)=\widetilde{F}(y)$. Then $F(x)=\widetilde{F}(x)$ for all $x\in U$.

Define the set $S=\{x\in U;\ F(x)=\widetilde{F}(x)\}$ and let $x\in S$. Then we show, by means of an inequality of Poincaré type, that there exists an open ball $B(x,r)\Subset U$ such that $F=\widetilde{F}$ in this ball. Therefore the set S is open in U. But the set S is also closed in U, since the mapping $F-\widetilde{F}$ is continuous over U thanks to the Sobolev imbeddings. Then the connectedness of U implies that either $S=\emptyset$ or $S=U$. Since S contains at least the element y, we must have $S=U$.

(iii) *There exists a solution $F\in W^{1,\infty}(\Omega;\mathbb{M}^d)$ to the system*:

$$\begin{aligned}&\partial_i F=F\Gamma_i\quad\textit{a.e. in }\Omega,\\&F\big(x^0\big)=F^0,\end{aligned}\tag{4}$$

where F^0 is the matrix whose i-th column is $\boldsymbol{g}_i^0$. Moreover, $\boldsymbol{g}_i(x)\cdot\boldsymbol{g}_j(x)=g_{ij}(x)$ for all $x\in\Omega$, where $\boldsymbol{g}_i(x)$ denotes the i-th column of the matrix $F(x)$.

We define a solution F to the system above by glueing together some sequences of local solutions defined in step (i) along curves starting from the given point x^0. Let x be a fixed, but otherwise arbitrary, point of Ω. Let $\gamma\in C^0([0,1];\Omega)$ be a path joining x^0 to x ($\gamma(0)=x^0$ and $\gamma(1)=x$), let a number $R>0$ be such that $R\sqrt{d}<\operatorname{dist}(\gamma([0,1]),\Omega^c)$, and let a division $\Delta=\{t_0,t_1,t_2,\dots,t_N\}$ be such that $0=t_0<t_1<t_2<\cdots<t_N=1$ and such that, for all $i\in\{0,1,\dots,N\}$,

$$\gamma(t)\in B\big(x^i,R\big)\quad\text{for all } t\in[t_{i-1},t_{i+1}],$$

where $x^i:=\gamma(t_i)$, $t_{-1}:=0$ and $t_{N+1}:=1$. Let B_i denote the ball centered at x^i of radius R. For $i=0,1,2,\dots,N$, we successively define $F^i:=F^{[t_i]}\in W^{1,\infty}(B_i;\mathbb{M}^d)$ as the solutions to the systems

$$\begin{aligned}&\partial_j F^i=F^i\Gamma_j\quad\text{in }\mathcal{D}'\big(B_i;\mathbb{M}^d\big),\\&F^i\big(x^i\big)=F^{i-1}\big(x^i\big),\end{aligned}\tag{5}$$

with the convention that $F^{-1}(x^0):=F^0$. Since $R\sqrt{d}<\operatorname{dist}(x^i,\Omega^c)$, this system has a unique solution thanks to steps (i) and (ii). Then we define the solution to the system (4) by letting

$$F(x):=F^N(x).$$

In this way, we have associated a value for $F(x)$ for each γ, R, Δ defined as above. We prove that this definition is unambiguous, i.e., that it does not depend on the choice of γ, R, Δ, thanks to the uniqueness result proved in step (ii) and to the fact that the set Ω is simply-connected.

Next, we show that $F=F^N$ in the open ball $B(x,R)$. Since F^N belongs to the space $W^{1,\infty}(B(x,R);\mathbb{M}^d)$ and satisfies the system (5) with $i=N$, letting x vary in the set Ω shows that F belongs to the space $W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{M}^d)$ and satisfies the system (4). Besides, it can be shown as in step (i) that the columns $\boldsymbol{g}_i$ of F satisfy the relation $\boldsymbol{g}_i\cdot\boldsymbol{g}_j=g_{ij}$ in Ω.

Since $(g_{ij})\in L^\infty(\Omega;\mathbb{R}^d)$, this last relation implies that $\boldsymbol{g}_i\in L^\infty(\Omega;\mathbb{R}^d)$, hence that $F\in L^\infty(\Omega;\mathbb{M}^d)$. Since we know that $\partial_i F=F\Gamma_i\in L^\infty(\Omega;\mathbb{M}^d)$, the field F belongs to the space $W^{1,\infty}(\Omega;\mathbb{M}^d)$.

(iv) *Existence of an isometric immersion.* The columns $\boldsymbol{g}_i$ of the matrix F found in the previous step belong to the space $W^{1,\infty}(\Omega;\mathbb{R}^d)$ and satisfy the relations

$$\partial_j \boldsymbol{g}_i = \Gamma_{ji}^k \boldsymbol{g}_k = \Gamma_{ij}^k \boldsymbol{g}_k = \partial_i \boldsymbol{g}_j.$$

These relations, together with the assumption that Ω is simply-connected, imply the existence of a solution $\Theta \in W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$ to the system

$$\partial_i \Theta = \boldsymbol{g}_i \quad \text{in } \Omega,$$
$$\Theta(x^0) = \Theta^0.$$

To prove this generalized form of Poincaré's theorem, we follow the same ideas as those used in establishing the existence of a solution F to the system (2).

Since the set Ω is bounded and satisfies the cone property, Lemma 1.3 shows that the mapping Θ belongs to the space $W^{2,\infty}(\Omega;\mathbb{M}^d)$.

(v) The uniqueness up to isometries of $\mathbb{R}^d$ of the mapping Θ satisfying the conditions of the theorem can be established as a consequence of the uniqueness result of step (iii). This can be done as in the classical framework, where the metric is assumed to be of class C^2 over Ω. □

The assumptions that $(g_{ij}) \in W^{1,\infty}(\Omega;\mathbb{S}^d_>)$ and $(g_{ij})^{-1} \in L^\infty(\Omega;\mathbb{M}^d)$ were needed in Theorem 2.1 in order to prove that $\Gamma_{ij}^k \in L^\infty(\Omega)$. However, this regularity of the Christoffel symbols has been used only to prove that Θ belongs to $W^{2,\infty}(\Omega;\mathbb{R}^d)$ and to define the set $\mathcal{A}$ at the begining of the proof of Theorem 2.1. It is however not difficult to adapt this part of the proof under the assumption that Γ_{ij}^k belongs only to $L^\infty_{\mathrm{loc}}(\Omega)$, a condition which is satisfied if the matrix field (g_{ij}) belongs to $W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{S}^d_>)$. Therefore, we also have the following result:

Theorem 2.2. *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^d$ and let a metric be given in Ω by the means of a matrix field $(g_{ij}) \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{S}^d_>)$. Assume that the corresponding Riemann curvature tensor vanishes, that is,*

$$\partial_j \Gamma_{ik}^p - \partial_k \Gamma_{ij}^p + \Gamma_{ik}^l \Gamma_{jl}^p - \Gamma_{ij}^l \Gamma_{kl}^p = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Then there exists a mapping $\Theta \in W^{2,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^d)$, unique up to isometries in $\mathbb{R}^d$, such that

$$\partial_i \Theta \cdot \partial_j \Theta = g_{ij} \quad \text{in } \Omega.$$

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