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THE DUAL BRAID MONOID

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ABSTRACT. – We study a new monoid structure for Artin groups associated with finite Coxeter systems. Like the classical positive braid monoid, the new monoid is a Garside monoid. We give several equivalent constructions: algebraically, the new monoid arises when studying Coxeter systems in a "dual" way, replacing the pair (W,S) by (W,T), with T the set of all reflections; geometrically, it arises when looking at the reflection arrangement from a certain basepoint. In the type A case, we recover the monoid constructed by Birman, Ko and Lee.

RÉSUMÉ. – Nous étudions une nouvelle structure de monoïde pour les groupes d'Artin associés aux systèmes de Coxeter finis. Ce nouveau monoïde est, tout comme le classique monoïde des tresses positives, un monoïde de Garside. Nous en donnons différentes constructions : algébriquement, le nouveau monoïde apparaît quand on étudie les systèmes de Coxeter avec un point de vue "dual", qui consiste à remplacer la paire (W,S) par (W,T), où T est l'ensemble de toutes les réflexions ; géométriquement, il apparaît quand on observe l'arrangement de réflexions depuis un point-base particulier. Pour les systèmes de type A, nous retrouvons le monoïde construit par Birman, Ko et Lee.

Introduction

Combinatorics of Coxeter systems provide very powerful tools to understand finite real reflection groups, their geometry and their braid groups. The goal of this article is to describe an alternate approach to finite real reflection groups. This new approach can be seen as natural "twin" or "dual" of the classical theory of Coxeter groups and Artin groups. Our main object of study is a monoid, the *dual braid monoid*, which we construct in three different but equivalent ways. The first two constructions are of algebraic/combinatorial nature, the third is more geometric; each of them mirrors a standard construction of the classical positive braid monoid.

Let W be a finite real reflection group. Choosing S to be the set of reflections with respect to the walls of a chamber, we have a Coxeter presentation for W:

$$W \simeq \left\langle S \mid \forall s \in S,\ s^2 = 1;\ \forall s,t \in S,\ \underbrace{sts\ldots}_{m_{s,t}} = \underbrace{tst\ldots}_{m_{s,t}} \right\rangle_{\text{group}},$$

where $(m_{s,t})_{s,t\in S}$ is the Coxeter matrix of the Coxeter system (W,S). Let $\mathbf{B}(W,S)$ be the corresponding Artin group. To have simple yet precise notations, it is convenient to introduce a formal copy $\mathbf{S} \simeq S$. For each $\mathbf{s} \in \mathbf{S}$, we write s the corresponding element of S. With this convention, $\mathbf{B}(W,S)$ is defined as the abstract group

$$\mathbf{B}(W,S) := \langle \mathbf{S} \mid \forall \mathbf{s},\mathbf{t} \in \mathbf{S},\ \underbrace{\mathbf{sts}\ldots}_{m_{s,t}} = \underbrace{\mathbf{tst}\ldots}_{m_{s,t}} \rangle_{\text{group}}.$$

The map $\mathbf{s} \mapsto s$ extends to a surjective morphism $p : \mathbf{B}(W,S) \to W$.

Since the defining relations are between positive words, the presentation of $\mathbf{B}(W,S)$ can also be seen as a monoid presentation. We set

$$\mathbf{B}_{+}(W,S) := \langle \mathbf{S} \mid \forall \mathbf{s},\mathbf{t} \in \mathbf{S},\ \underbrace{\mathbf{sts}\ldots}_{m_{s,t}} = \underbrace{\mathbf{tst}\ldots}_{m_{s,t}} \rangle_{\text{monoid}}.$$

This monoid is often called the *positive braid monoid*. We prefer here the term of *classical braid monoid* (short for Artin–Brieskorn–Deligne–Garside–Saito–Tits monoid).

The structure of $\mathbf{B}(W,S)$ and $\mathbf{B}_{+}(W,S)$ has been studied in great detail by Deligne and Brieskorn–Saito [18,10]. One of the main results is that $\mathbf{B}_{+}(W,S)$ satisfies the *embedding property*, i.e., the morphism $\mathbf{B}_{+}(W,S) \to \mathbf{B}(W,S)$ is injective. In other words, $\mathbf{B}_{+}(W,S)$ is isomorphic to the submonoid of $\mathbf{B}(W,S)$ generated by $\mathbf{S}$. This explains why we did not bother to introduce another formal copy of $\mathbf{S}$ when defining the classical braid monoid. Another important result is the existence of a nice normal form in $\mathbf{B}(W,S)$, which, for example, gives practical solutions to the word and conjugacy problems.

The notion of *Garside monoid* has been introduced by Dehornoy and Paris [17,16], as a formal setting in which the strategies and results of [18] and [10] still hold. For example, the embedding property, the nice normal form and solutions to the word and conjugacy problems are general properties of Garside monoids, and some results of [18] and [10] implicitly express that the classical braid monoid is a Garside monoid.

Our dual braid monoid is also a Garside monoid, and its group of fractions is isomorphic to $\mathbf{B}(W,S)$ – but the dual braid monoid is not isomorphic to the classical braid monoid. In other words, we obtain a new presentation for $\mathbf{B}(W,S)$, a new normal form, a new solution to the word and conjugacy problems (and actually much more: a new coherence rule for action on categories, a new simplicial $K(\pi,1),\ldots$), which are analog but not identical to the classical ones. In the type A case, the dual braid monoid coincides with the new monoid introduced by Birman, Ko and Lee in their 1998 paper [3].

Let us now summarize the algebraic/combinatorial approach, which occupies the first half of our work. Define on W a relation $\prec_S$ by

$$w \prec_S w' \quad \Leftrightarrow \quad l_S(w) + l_S\big(w^{-1}w'\big) = l_S(w')$$

(where l_S is the usual length function on the Coxeter group (W,S)). A crucial property in [18] and [10] is that $(W,\prec_S)$ is a lattice; as we explain in Section 0, the main structural properties of $\mathbf{B}_{+}(W,S)$ follow from this lattice property. The starting point of the dual approach is quite naive: replace S by the set T of **all** reflections in W. We have a new length function l_T, from which we may define a relation $\prec_T$. Unfortunately, $(W,\prec_T)$ is generally not a lattice. Let $c \in W$ be a Coxeter element; let $P_c := \{w \in W \mid w \prec_T c\}$. One of our main results is that $(P_c,\prec_T)$ is a lattice (Fact 2.3.1). From this lattice, the tools introduced in Section 0 allow to construct a Garside monoid $\mathbf{M}(P_c)$ (the dual braid monoid). Another important result (Theorem 2.2.5) is that the group of fractions of $\mathbf{M}(P_c)$ is isomorphic to $\mathbf{B}(W,S)$.

These results suggest that the pair (W,T) should have some "Coxeter-like" features. We give in Section 1 the rudiments of what should be a "dual Coxeter theory".

A second definition of the dual braid monoid is by means of generators and relations. In Section 2, we define *dual braid relations* on the generating set T. Whereas classical braid relations involve only two generators but have arbitrary length (mostly two or three), dual braid relations may involve two or three generators but always have length two. The monoid defined by these relations is isomorphic to $\mathbf{M}(P_c)$ (Theorem 2.1.4); viewing the new presentation as a group presentation, we obtain $\mathbf{B}(W,S)$.

The geometric approach involves looking at reflection arrangements from a new *viewpoint*. Let $V_{\mathbb{R}}$ be the real vector space on which our reflection group W acts. By complexifying the representation, we may view W as a complex reflection group acting on $V_{\mathbb{C}} := V_{\mathbb{R}} \otimes \mathbb{C}$. Let $\mathcal{A}$ be the set of all (complex) reflecting hyperplanes. The *braid group* of W is the fundamental group of the regular orbit space $W \backslash (V - \bigcup_{H\in\mathcal{A}} H)$. This definition involves choosing a basepoint, an operation which will prove to be crucial.

Note that we make a distinction between the Artin group and the braid group. The Artin group is a group defined by a presentation, the braid group is a fundamental group. Of course, they are isomorphic (this was proved by Brieskorn, see [9] or Theorem 3.3.2 below), but the isomorphism is not canonical. When looking carefully at the standard isomorphism, one may observe that it essentially assumes that the basepoint has a trivial imaginary part. Hence the Artin presentation encodes properties of the real structure on V.

In a previous article [1], we studied a certain class of presentations for complex reflection groups. We proved that one may expect to have several competing presentations, obtained by choosing different *regular elements* (in the sense of [30]).

In Section 3, for each choice of a basepoint $v \in V - \bigcup_{H\in\mathcal{A}} H$, we define a *local braid monoid* M_v, which is given as a submonoid of $\pi_1(W\backslash(V - \bigcup_{H\in\mathcal{A}} H), p(v))$ (where $p(v)$ is the image of v in the quotient). The structure of M_v varies according to the position of v with respect to the *visibility stratification* – this indeed is a matter of how $\mathcal{A}$ appears when looked at from v.

The classical braid monoid is an example of local braid monoid. It is obtained when the basepoint is in a real chamber (Proposition 3.4.3).

When v is an eigenvector for an element $w \in W$ (this implies that w is a regular element), the monoid M_v admits certain automorphisms. This occurs, for example, when v is an eigenvector for a Coxeter element of W (it is well-known that Coxeter elements are regular). We prove that the local braid monoid corresponding to this particular example is isomorphic to the dual braid monoid (Theorem 3.6.1).

Just like the algebraic approach suggests that there is a “dual Coxeter theory”, the geometric approach calls for a “dual Coxeter geometry”, a new way to look at reflection arrangements, where real structure, walls and chambers would be replaced by new objects (which remain to be defined). In Section 4, we sketch some aspects of the dual geometries of types A, B and D.

The last two sections contain complements and applications. We compile numerological evidences of some unexplained “duality” between the classical and dual braid monoids – this motivates our terminology. We also include a formula involving generalized Catalan numbers (Section 5.2).

Note. After the first version of the present paper was circulated, the author was informed by T. Brady and C. Watt that they were working on the same problem. They have independently obtained some of our results, namely the lattice structure of P_c when W is of type B or D (see Theorem 2.3.2 below) as well as explicit embeddings of the monoids in the corresponding Artin groups [7]; the types I_2 and H_3 had also been studied independently by Brady.

0. Garsiditude

This preliminary section is an introduction to the theory of *Garside groups*, *Garside monoids* and *Garside pre-monoids*. These tools will be constantly used throughout this article. Some results in this section are quoted without proofs – the main references for the material are [17, 16,2].

According to the MathSciNet database, F.A. Garside published only one mathematical paper ([21], 1969). It contains a solution to the word and conjugacy problems in type A Artin groups. In 1972 appeared simultaneously two articles, by Brieskorn and Saito and by Deligne, generalizing Garside's techniques and results to the context of an arbitrary finite type Artin group [10,18].

These generalizations were however not ultimate, in the sense that most of the crucial proofs actually work for a larger class of groups. The notions of *Garside group* and *Garside monoid* were introduced by Dehornoy and Paris ([17], 1999). A Garside group is a group which satisfies a certain number of axioms, sufficient to apply the techniques of Garside, Brieskorn and Saito and Deligne. A slightly different axiomatic was introduced independently by Corran [15].

Garside groups and monoids share remarkable algebraic and algorithmic properties (see Subsection 0.7 below), and identifying a group as a Garside group yields answers to many questions about this group. But there can be more than one way of seeing a given group as a Garside group. The whole point of the present article is to explain that there are (at least) two ways to see braid groups as Garside groups.

Inspired by an earlier work of Michel ([24] – which itself reformulates results of Charney), we proposed with Digne and Michel a variant approach to Garside monoids (and "locally Garside monoids"), via what we called in [2] *pre-Garside structures*; the properties of the monoid are derived from properties of a subset of the monoid, a *pre-monoid*, on which the product is only partially defined.

The latter approach is the one retained in the present work. The basic example, which served as a model for the theory, is the classical braid monoid (the corresponding *pre-monoid* is described in Subsection 0.6 below). The dual braid monoid can also be constructed from a pre-monoid (see Sections 1 and 2).

The purpose of this section is to give a survey of this approach, with a new language. The terminology is probably more abstract than required, not that we have any pretention to universality or exhaustivity, but rather that we feel that it simplifies the exposition.

0.1. Pre-monoids

A pre-monoid can be thought of as a "fragment of monoid" or, more metaphorically, as a "seed" containing all the information to build a monoid.

DEFINITION 0.1.1. – A *pre-monoid* is a triple (P, D, m), where P is a set, D is a subset of $P \times P$ and m is a map $D \to P$, satisfying:

(assoc) For all $a, b, c \in P$, the condition "$(a,b) \in D$ and $(m(a,b),c) \in D$" is equivalent to "$(b,c) \in D$ and $(a, m(b,c)) \in D$", and, when they are satisfied, one has $m(m(a,b),c) = m(a, m(b,c))$.

A pre-monoid P is *unitary* when it satisfies in addition:

(unit) There exists an element $1 \in P$, such that, for all $a \in P$, $(a,1) \in D$ and $(1,a) \in D$, and $m(a,1) = m(1,a) = a$.

The map m should be seen as a "partial product", with domain D. Practically, it is convenient to omit to explicitly refer to m and D: we write "ab" for "$m(a,b)$", and "ab is defined" instead of "$(a,b) \in D$". A trivial lemma on binary trees shows that, thanks to the (assoc) axiom, for any sequence $a_1, a_2, \ldots, a_n$ of elements of P, the fact that the product $a_1 a_2 \ldots a_n$ is defined, and its value, do not depend on how one chooses to put brackets.

Let P be a pre-monoid. Let $p, q \in P$. We say that p is left (resp. right) divisor of q, or equivalently that q is a right (resp. left) multiple of p, and we write $p \prec q$ (resp. $q \succ p$), if there exists $r \in P$ such that $pr = q$ (resp. $rq = p$) in P.

0.2. The functor $\mathbf{M}$

Pre-monoids form a category $\mathbf{preMon}$, where a morphism $\varphi : P \to P'$ between two pre-monoids is defined to be a set-theoretical map such that, for all $a, b \in P$ such that ab is defined, the product $\varphi(a)\varphi(b)$ is defined in P', and equal to $\varphi(ab)$.

The category $\mathbf{Mon}$ of monoids can be defined as the full subcategory of $\mathbf{preMon}$ with objects being those pre-monoids for which the product is everywhere defined. The embedding functor $\mathbf{Mon} \to \mathbf{preMon}$ has a left adjoint $\mathbf{M}$, defined as follows:

– Let P be a pre-monoid. Let P^* be the free monoid on P, i.e., the set of finite sequences of elements of P, for the concatenation product. Let $\sim$ be the smallest equivalence relation on P^* compatible with concatenation and satisfying $(a, b) \sim (ab)$ whenever ab is defined in P. We set $\mathbf{M}(P) := P^*/\sim$.
  Note that one has a natural pre-monoid morphism $P \to \mathbf{M}(P),\ p \mapsto (p)$.
– If $\varphi : P \to Q$ is a pre-monoid morphism, we take $\mathbf{M}(\varphi)$ to be the (unique) monoid morphism which makes the following diagram commute:

$$\begin{array}{ccc} P & \longrightarrow & \mathbf{M}(P) \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\mathbf{M}(\varphi)} \\ Q & \longrightarrow & \mathbf{M}(Q) \end{array}$$

The empty sequence provides the unit of $\mathbf{M}(P)$, even when P is not unitary. Note that $\mathbf{M}$ is essentially surjective: for any monoid M, one has $\mathbf{M}(M) \simeq M$.

For any pre-monoid P, the monoid $\mathbf{M}(P)$ can be described by the monoid presentation with P as set of generators, and a relation $pq = r$ for all $p, q, r \in P$ such that $pq = r$ in P.

Formally, an element $m \in \mathbf{M}(P)$ is an equivalence class of sequences of elements of P, called *P-decompositions of m*. A P-decomposition is *strict* if it contains no occurrence of the unit of P (if P is not unitary, the condition is empty).

DEFINITION 0.2.1. – Let P be a pre-monoid. Let $m \in \mathbf{M}(P)$. We denote by $E(m)$ the set of strict P-decompositions of m.

We denote by $\leqslant$ the smallest partial order relation on $E(m)$ such that, for all $(a_1, \ldots, a_n) \in E(m)$ and for all i such that $a_i a_{i+1}$ is defined in P, we have, if $a_1 a_{i+1} \neq 1$,

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_n) \leqslant (a_1, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n),$$

or, if $a_i a_{i+1} = 1$,

$$(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, a_{i+2}, \ldots, a_n) \leqslant (a_1, \ldots, a_{i-1}, a_{i+2}, \ldots, a_n).$$

There is a classical notion of dimension for posets. Let $(E, \leqslant)$ be a poset, let $e_0 < \cdots < e_n$ be a chain in E; the length of the chain is, by definition, the integer n. The *dimension* of $(E, \leqslant)$ is set to be the supremum of the set of lengths of all chains in E. This dimension is an element of $\mathbb{Z}_{\geqslant 0} \cup \{\infty\}$. It coincides with usual notion of dimension for the simplicial realization of E.

DEFINITION 0.2.2. – Let P be a pre-monoid. We say that P is *atomic* if and only if for all $p \in P$, $E((p))$ is finite dimensional.

For monoids, this definition coincides with the usual one.

0.3. The functor G

In a similar way, the embedding functor $\mathbf{Grp} \to \mathbf{preMon}$ has a left adjoint $\mathbf{G}$, acting on objects as follows: for any pre-monoid P, the group $\mathbf{G}(P)$ can be described by the group presentation with P as set of generators, and a relation $pq = r$ for all $p, q, r \in P$ such that $pq = r$ in P.

We have $\mathbf{GM} \simeq \mathbf{G}$. If M is a monoid satisfying Ore's condition, $\mathbf{G}(M)$ is isomorphic to the group of fractions of M.

0.4. Generated groups

A general way of constructing a pre-monoid is from a pair (G, A) where G is a group and $A \subset G$ generates G as a monoid (we call such a pair a *generated group*). Let (G, A) be a generated group. An A-*decomposition* of $g \in G$ is a sequence $(a_1, \ldots, a_n) \in A^*$ such that $g = a_1 \ldots a_n$. An A-decomposition of g of minimal length is said to be *reduced*. We denote by $\mathrm{Red}_A(g)$ the set of reduced decompositions of g. We denote by $l_A(g)$ the common length of the elements of $\mathrm{Red}_A(g)$. The function l_A is sub-additive: for all $g, h \in G$, we have

$$l_A(gh) \leqslant l_A(g) + l_A(h).$$

We write $g \prec_A h$ if $l_A(g) + l_A(g^{-1}h) = l_A(h)$, and $g \succ_A h$ if $l_A(gh^{-1}) + l_A(h) = l_A(g)$.

DEFINITION 0.4.1. – Let (G, A) be a generated group. Let $g \in G$. We say that g is A-*balanced* (or simply *balanced*) if $\forall h \in G, h \prec_A g \Leftrightarrow g \succ_A h$.

Let g be a balanced element of G. The set

$$\{h \in G \mid h \prec_A g\} = \{h \in G \mid g \succ_A h\}$$

is denoted by $P_{G,A,g}$ (or simply by P_g). Let

$$D_g := \big\{(h, h') \in P_g \times P_g \mid hh' \in P_g,\ l_A(hh') = l_A(h) + l_A(h')\big\}$$

and let m_g be the restriction of the group product to D_g. The triple (P_g, D_g, m_g) is a unitary pre-monoid (to prove the associativity axiom, use the fact that g is balanced).

DEFINITION 0.4.2. – The pre-monoid (P_g, D_g, m_g) (or simply P_g) is called *pre-monoid of divisors of* g in (G, A).

Note that the restriction to P_g of the relation $\prec_A$ (resp. $\succ_A$) is really the left (resp. right) divisibility relation for the pre-monoid structure.

DEFINITION 0.4.3. – A pre-monoid P is said to be $\mathbf{M}$-*cancellative* if

$$\forall m \in \mathbf{M}(P),\ \forall p, q \in P, \quad \big((pm = qm) \text{ or } (mp = mq)\big) \quad \Rightarrow \quad p = q.$$

Note that this is formally weaker than the cancellativity of $\mathbf{M}(P)$. A first property of divisors pre-monoids is:

LEMMA 0.4.4. – *Let (G,A) be a generated group. Let g be a balanced element of G. The pre-monoid P_g is $\mathbf{M}$-cancellative.*

Proof. – Since the defining relations are valid in G, the monoid $\mathbf{M}(P_g)$ comes equipped with a natural morphism $\pi : \mathbf{M}(P_g) \to G$. If for example $pm = qm$, then $\pi(p)\pi(m) = \pi(q)\pi(m)$ in G. Since G is cancellative, $\pi(p) = \pi(q)$. To obtain the first claim, observe that the composition of the natural pre-monoid morphism $P_g \to \mathbf{M}(P_g)$ with π is the restriction of identity map of G. □

0.5. Garside monoids

The terminology has been fluctuating in the recent years, between several non-equivalent but similar sets of axioms. The following version seems to emerge as "consensual" [16].

DEFINITION 0.5.1. – A monoid M is a *Garside monoid* if:
– the monoid M is atomic;
– the monoid M is left and right cancellative;
– the posets $(M, \prec)$ and $(M, \succ)$ are lattices;
– there exists an element $\Delta \in M$ such that

$$\forall m \in M, \quad (m \prec \Delta) \quad \Leftrightarrow \quad (\Delta \succ m),$$

and $\{m \in M \mid m \prec \Delta\}$ is finite and generates M. (An element Δ satisfying this property is called a *Garside element*.)

Saying that $(M, \prec)$ and $(M, \succ)$ are lattices can be rephrased, in arithmetical terms, as the existence of left and right lcm's and gcd's.

Let M be a Garside monoid, with Garside element Δ. Let $P := \{m \in M \mid m \prec \Delta\}$. View P as a pre-monoid, the product of $p, q \in P$ being defined as the product pq in M (when $pq \in P$; otherwise, the product is not defined). We call a pre-monoid P obtained this way a *Garside pre-monoid*. The monoid M can be recovered from P: we have $M \simeq \mathbf{M}(P)$.

In [2] is given an axiomatic characterization of Garside pre-monoids (axioms (i)–(vi) $+$ existence of a common multiple). As J. Michel pointed out to us, in the context of generated groups, most of them are straightforward:

THEOREM 0.5.2. – *Let (G,A) be a finite generated group. Let g be a balanced element in G. Assume that $A \subset P_g$, and that all pairs $a, b \in A$ have a left lcm and a right lcm in P_g. Then P_g is a Garside pre-monoid. As a consequence, $\mathbf{M}(P_g)$ is a Garside monoid.*

The existence of left/right lcm's for pairs of elements of A follows, for example, if $(P_g, \prec_A)$ and $(P_g, \succ_A)$ are lattices. Conversely, a consequence of the theorem is that if pairs of elements of A have left/right lcm's, then $(P_g, \prec_A)$ and $(P_g, \succ_A)$ are lattices.

This theorem is a convenient tool, hiding most of the technical machinery (the long list of easy axioms). But the whole issue remains to check that $(P_g, \prec_A)$ and $(P_g, \succ_A)$ are lattices.

Proof. – The pre-monoid P_g is unitary (axioms (i) and (ii) of [2]); the length function l_A satisfies axiom (iii). With the assumption $A \subset P_g$, A is the set of atoms of P_g. The existence of left and right lcm's for elements of A are exactly axioms (iv) and (iv′). Axiom (v): let $h \in P_g$, $a, b \in A$, such that $h \prec_A g$, $ha \prec_A g$ and $hb \prec_A g$; then $a \prec_A h^{-1}g$, $b \prec_A h^{-1}g$, so $\mathrm{lcm}(a,b) \prec_A h^{-1}g$ and $h\mathrm{lcm}(a,b) \prec_A g$. Axiom (vi) is $\mathbf{M}$-cancellativity, which we have proved in Lemma 0.4.4. The element g is a common multiple of all elements of P_g. We conclude using Theorem 2.24 in [2]. □

It would be interesting to characterize Garside monoids arising from triples (G, A, g).

0.6. A basic example: the classical braid monoid

Let (W,S) be a finite Coxeter system; we view it as a generated group. Some crucial results from [18] and [10] show that the longest element w_0 is S-balanced (actually, $P_{w_0} = W$ as sets), and that the posets $(P_{w_0}, \prec_S)$ and $(P_{w_0}, \succ_S)$ are lattices. We have $\mathbf{B}_+(W,S) \simeq \mathbf{M}(P_{w_0})$. Our first construction of the dual monoid will be very similar, S being replaced by T and w_0 by a Coxeter element c.

0.7. Properties of Garside monoids

Let us conclude this section by compiling some remarkable properties of Garside monoids. Any Garside monoid M satisfies the *embedding property*, i.e., the canonical map $M \to \mathbf{G}(M)$ is injective. This implies that M is cancellative. Any finite subset of M admits a right lcm, a left lcm, a left gcd and a right gcd. In particular, M satisfies Ore's conditions on the left and on the right. In all examples considered here, the lcm of the atoms is a Garside element. Let Δ be a Garside element, with set of divisors P. Any element $m \in M$ has a unique decomposition as a product $m = p_1 \ldots p_k$ of elements of P such that, for all $i \in \{1, \ldots, k\}$, p_i is the left gcd of Δ and $p_i \ldots p_k$. The sequence $(p_1, \ldots, p_k)$ is called the *normal form* of m. One has a similar notion in $\mathbf{G}(P)$. This gives rise to solutions of the word problem. A sequence $(p_1, \ldots, p_k)$ is the normal form of $p_1 \ldots p_k$ if and only if, for all $i \in \{1, \ldots, k-1\}$, p_i is the left gcd of Δ and $p_i p_{i+1}$. In other words, the normality can be checked locally, by looking at consecutive terms. This has important algorithmic consequences ($\mathbf{G}(P)$ is biautomatic). The conjugation action by Δ on $\mathbf{G}(M)$ restricts to an automorphism of P. In particular, it is of finite order d. We call it the *diagram automorphism*, by analogy with the case of the classical braid monoid. It is easy to describe the submonoid of fixed points under a given power of the diagram automorphism. The element Δ^d is central in $\mathbf{G}(M)$. Some other properties are given in Section 6 of this paper, including a coherence rule for actions on categories.

1. Reduced T-decompositions

This section contains the first steps of what could be a "dual Coxeter theory".

1.1. Reflection groups

We call *abstract* (*finite real*) *reflection group* a pair (W,T) where W is a finite group, T a generating subset of W and there exists a faithful representation $\rho : W \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$, with $V_{\mathbb{R}}$ a finite dimensional $\mathbb{R}$-vector space, satisfying

$$\forall w \in W, \quad \operatorname{codim}\bigl(\ker(\rho(w) - \mathrm{Id})\bigr) = 1 \quad \Leftrightarrow \quad w \in T.$$

The group $\rho(W)$ is a *geometric* (*finite real*) *reflection group*, with set of reflections $\rho(T)$. We say that ρ is a *realization* of W.

Unless otherwise specified, all reflection groups considered in this paper are finite and real, and all Coxeter systems are finite ("spherical type").

Since geometric reflection groups are classified by (finite) Coxeter systems, all abstract reflection groups can be obtained as follows: let (W,S) be a (finite) Coxeter system; let T be the closure of S under conjugation; then (W,T) is an abstract reflection group. Conversely, if (W,T) is an abstract reflection group, one may always choose $S \subset T$ such that (W,S) is a Coxeter system. The type of (W,S) does not depend on the choice of $S \subset T$. The *rank* of (W,T) is the rank $|S|$ of (W,S).

Question 1.1.1. – Is there a nice combinatorial description of abstract reflection groups, similar to Coxeter systems, allowing for example a direct classification (not using the classification of Coxeter systems)?

We do not have an answer to this question, but we do obtain here some strong combinatorial properties of (W,T).

1.2. The reflection length l_T

An (abstract) reflection group (W,T) is a particular example of *generated group*, as this notion is defined in Section 0.4. We have a notion of reduced T-decomposition, a length function l_T and two partial orders $\prec_T$ and $\succ_T$ on W (see 0.4). The function l_T is called *reflection length*. Since T is invariant by conjugation, it is clear that $\prec_T$ and $\succ_T$ coincide.

Carter gave a geometric interpretation of the function l_T:

LEMMA 1.2.1. – *Let ρ be a realization of a reflection group (W,T).*

(i) *Let $w \in W$ and $t \in T$. We have*

$$t \prec_T w \quad \Leftrightarrow \quad \ker\bigl(\rho(t) - \mathrm{Id}\bigr) \supset \ker\bigl(\rho(w) - \mathrm{Id}\bigr).$$

(ii) *For all $w \in W$, $l(w) = \mathrm{codim}(\ker(\rho(w) - \mathrm{Id}))$.*

Proof. – See [13], Lemma 2.8. (Carter actually works with Weyl groups, but his argument can be used with an arbitrary finite geometric reflection group.) □

1.3. Chromatic pairs and Coxeter elements

DEFINITION 1.3.1. – A *chromatic pair* for an (abstract) reflection group (W,T) is an ordered pair (L,R) of subsets of T, such that:

- the intersection $L \cap R$ is empty;
- the subgroups $\langle L\rangle$ and $\langle R\rangle$ are abelian;
- the pair $(W, L \cup R)$ is a Coxeter system.

When unambiguous, we will sometimes write the pair $L \cup R$ instead of (L,R). The term "chromatic" comes from the fact that the Coxeter graph of $(W, L \cup R)$ comes equipped with a 2-colouring: elements of L are said to be "left" (let us pretend this is a colour), elements of R are "right". If (W,S) is an irreducible Coxeter system, there are exactly two 2-colourings of the Coxeter graph of (W,S).

If $L \cup R$ is a chromatic pair, we set

$$s_L := \prod_{s \in L} s, \qquad s_R := \prod_{s \in R} s, \qquad c_{L,R} := s_L s_R.$$

DEFINITION 1.3.2. – The *Coxeter elements* of (W,T) are the elements of the form $c_{L,R}$, where (L,R) is a chromatic pair. A *dual Coxeter system* is a triple (W,T,c) where (W,T) is a reflection group, and c is a Coxeter element in (W,T).

Note that our definition does not coincide with the one from [4]. It is not specific to a choice of S, not even to the choice of a realization. For example, in dihedral groups, not all Coxeter elements are conjugate. As they are defined by Bourbaki ([4], Ch. 5, §6), "Coxeter transformations" are relative to the choice of a geometric realization and chamber. Our Coxeter elements are those elements which, for a certain choice of a realization and chamber, become

Coxeter transformations in the sense of Bourbaki. All Coxeter elements have the same order, the *Coxeter number*, denoted by h.

In the "dual" approach, choosing a Coxeter element c plays a similar role as choosing a Coxeter generating set S (or, in geometric terms, a chamber) in the classical approach.

LEMMA 1.3.3. – *Let (W,T,c) be a dual Coxeter system of rank n. We have $l_T(c)=n$ and $\forall t\in T, t\prec_T c$.*

Proof. – Let ρ be an essential realization of W for which c is a Coxeter transformation (in the sense of Bourbaki). We have $\ker(c-\mathrm{Id})=\{1\}$ (this is a consequence of [4], Ch. V, §6, Th. 1, p. 119). The result then follows from Lemma 1.2.1. □

The last statement of the above lemma will be refined in 1.4.2.

LEMMA 1.3.4. – *Let (W,T) be an irreducible reflection group, with Coxeter number h. Let (L,R) be a chromatic pair, let $S:=L\cup R$. Then T is the closure of S under the conjugacy action of $c_{L,R}$. Moreover, if $\Omega\subset T$ is an orbit for the conjugacy action of $c_{L,R}$, then either*
(i) *Ω has cardinal h and $\Omega\cap S$ has cardinal 2; or*
(ii) *Ω has cardinal $h/2$ and $\Omega\cap S$ has cardinal 1.*

Proof. – Write $L=\{s_1,\dots,s_k\}$, $R=\{s_{k+1},\dots,s_n\}$, and $c:=c_{L,R}=s_1\dots s_n$.
Let $s_i,s_j\in S$. Assume $s_ic^m=c^ms_j$, for some integer $m>0$. Then we have $m\geqslant\lfloor h/2\rfloor$.
Indeed, assume that $m<\lfloor h/2\rfloor$; we will find a contradiction. According to [4], Ch. V, §6, Ex. 2 (p. 140), $(s_1\dots s_n)^m$ is a reduced S-decomposition of c^m, and

$$(\star)\qquad s_{k+1}\dots s_n(s_1\dots s_n)^m s_1\dots s_k$$

is a reduced S-decomposition of $c_{R,L}^{m+1}$.
Assume that $s_i\in L$ ($i\leqslant k$); by comparing the S-lengths of s_ic^m and of c^ms_j, we see that this implies $s_j\in R$. Then

$$(\dagger)\qquad s_1\dots\hat{s}_i\dots s_n(s_1\dots s_n)^{m-1}$$

and

$$(\ddagger)\qquad (s_1\dots s_n)^{m-1}s_1\dots\hat{s}_j\dots s_n$$

are two reduced S-decompositions of $s_ic^m=c^ms_j$. But while left-multiplying by s_i increases the length of (†), it decreases the length of (‡). We have a contradiction.
Now assume that $s_i\in R$. Then $s_i(s_1\dots s_n)^m$ is S-reduced. Since

$$s_i(s_1\dots s_n)^m=(s_1\dots s_n)^m s_j,$$

the word $(s_1\dots s_n)^m s_j$ is also S-reduced, and $s_j\in L$. The word $s_i(s_1\dots s_n)^m s_j$ must also be S-reduced (view it as a subword of ($\star$)). Since $s_i(s_1\dots s_n)^m s_j=(s_1\dots s_n)^m$, we have a contradiction.
This implies that for each orbit Ω, we have $|\Omega|\geqslant h/2|\Omega\cap S|$. Since c has order h, we also have $|\Omega|\leqslant h$. Using the well-known relation $|T|=h/2|S|$, we obtain the claimed results. □

1.4. Parabolic Coxeter elements

DEFINITION 1.4.1. – Let (W,T) be a reflection group. Let $S\subset T$ be such that (W,S) is a Coxeter system. Let $I\subset S$. Let $W_I:=\langle I\rangle$ and $T_I:=T\cap W_I$. The reflection group (W_I,T_I) is a *parabolic subgroup* of (W,T).

An element $w \in W$ is a *parabolic Coxeter element* if it is a Coxeter element in some parabolic subgroup of (W,T).

Let (W_I, T_I) be a parabolic reflection group of (W,T). Let $w \in W_I$. An easy consequence of Lemma 1.2.1(i) is that reduced T-decompositions of w consist only of elements of T_I. In particular, we have:

$$\mathrm{Red}_T(w) = \mathrm{Red}_{T_I}(w).$$

LEMMA 1.4.2. – *Let (W,T,c) be a dual Coxeter system of rank n. Let $t \in T$. There exists a chromatic pair (L,R) such that $t \in L$ and $c = c_{L,R}$. In particular, there exists $(t_1,\dots,t_n) \in \mathrm{Red}_T(c)$ such that $t_1 = t$ and $(W,\{t_1,\dots,t_n\})$ is a Coxeter system.*

Proof. – Let (L,R) be a chromatic pair such that $c = c_{L,R}$. Let $t \in T$. By Lemma 1.3.4, t is of the form $c^k s c^{-k}$, with $s \in L \cup R$.

- Assume $s \in L$. Then $(L',R') := (c^k L c^{-k}, c^k R c^{-k})$ is as required.
- If $s \in R$, we note that $c = c_{L,R} = c_{R, s_R^{-1} L s_R}$, so, by modifying the chromatic pair, we are back to the case already discussed. □

This allows the following characterization of parabolic Coxeter elements.

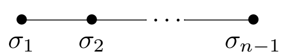
LEMMA 1.4.3. – *Let (W,T) be a reflection group. Let $w \in W$. The following assertions are equivalent*:

(i) *There exists a Coxeter element $c \in W$, such that $w \prec_T c$.*
(ii) *The element w is a parabolic Coxeter element.*

Proof. – Let c be a Coxeter element in W. Let $t \in T$. By the previous lemma, we can find $(t_1,\dots,t_n) \in \mathrm{Red}_T(c)$ such that $t_1 = t$ and $(W,\{t_1,\dots,t_n\})$ is a Coxeter system. Thus $tc = t_2 \dots t_n$ is a Coxeter element in the parabolic subgroup generated by $\{t_2,\dots,t_n\}$. By induction, this proves (i) ⇒ (ii). The converse is easy. □

1.5. Generating sets closed under conjugation

Let (G,A) be a generated group and assume that A is invariant by conjugation. Let n be a positive integer. Consider the Artin group B_n of type A_{n-1}:

B_n :


It is clear that the assignment

$$\sigma_i(a_1,\dots,a_{i-1},a_i,a_{i+1},a_{i+2},\dots,a_n) := (a_1,\dots,a_{i-1},{}^{a_i}a_{i+1},a_i,a_{i+2},\dots,a_n)$$

(we write ${}^a b$ for the conjugate aba^{-1}) extends to an action of B_n on A^n. The product map

$$\begin{aligned} A^n &\longrightarrow G \\ (a_1,\dots,a_n) &\longmapsto \prod_{i=1}^n a_i \end{aligned}$$

is invariant with respect to this action. In particular, for any $g \in G$, one has an action of $B_{l_A(g)}$ on $\mathrm{Red}_A(g)$.

1.6. A dual Matsumoto property

In classical Coxeter theory, the Matsumoto[1] property expresses that two reduced S-decompositions of the same $w \in W$ can be transformed one into the other by successive uses of braid relations. The "dual braid relations" will be defined in the next section – the next proposition will then appear as a dual analog of the Matsumoto property.

PROPOSITION 1.6.1. – *Let (W,T) be a reflection group. Let $w \in W$. If w is a parabolic Coxeter element of (W,T), then the action of $B_{l_T(w)}$ on $\mathrm{Red}_T(w)$ is transitive.*

This proposition indicates why Coxeter elements play a special part in the dual approach. Indeed, the $B_{l_T(w)}$-action on $\mathrm{Red}_T(w)$ may not be transitive for an arbitrary w. Take for example the Coxeter system of type B_2: let $W = \langle s,t \mid stst = tsts,\ s^2 = t^2 = 1\rangle$. The set T consists of s, t, $u := tst$ and $v := sts$, and we have $\mathrm{Red}_T(stst) = \{(s,u),(u,s),(t,v),(v,t)\}$. Since s commutes with u and t commutes with v, the action of B_2 has two orbits.

Proof. – Thanks to Lemma 1.4.3, it is enough to deal with the case of Coxeter elements: indeed, if w is a parabolic Coxeter in (W,T), it is a Coxeter element in some (W_I,T_I), and $\mathrm{Red}_T(w) = \mathrm{Red}_{T_I}(w)$.

We prove the proposition, for Coxeter elements, by induction on the rank n of (W,T). It is obvious when n is 0 or 1.

Assume $n > 1$, and assume the proposition is known for Coxeter elements in parabolic subgroups of rank $n-1$. Let $c \in W$ be a Coxeter element in W. Our goal is to prove that $\mathrm{Red}_T(c)$ forms a single orbit under the action of B_n. Denote by $\bullet$ the concatenation of finite sequences; we write $(t) \bullet \mathrm{Red}_T(tc)$ for $\{(t) \bullet u \mid u \in \mathrm{Red}_T(tc)\}$. We have

$$\mathrm{Red}_T(c) = \bigcup_{t \in T} (t) \bullet \mathrm{Red}_T(tc).$$

For all $t \in T$, tc is a parabolic Coxeter element. The induction assumption ensures that the action of B_{n-1} is transitive on $\mathrm{Red}_T(tc)$. In particular, since the B_{n-1}-action on the last $n-1$ terms is a restriction of the B_n-action, for any $u \in \mathrm{Red}_T(tc)$, the B_n-orbit of $(t) \bullet u$ contains the whole $(t) \bullet \mathrm{Red}_T(tc)$. To conclude, it is enough to exhibit a particular element of $\mathrm{Red}_T(c)$ such that its orbit under the action of B_n contains at least one element in each of the $(t) \bullet \mathrm{Red}_T(tc)$. This may be done as follows:

Let (L,R) be a chromatic pair such that $c = c_{L,R}$. Write $L = \{s_1,\ldots,s_k\}$, $R = \{s_{k+1},\ldots,s_n\}$. Clearly, $(s_1,\ldots,s_n) \in \mathrm{Red}_T(c)$. A direct computation shows that, for all $i \in \{1,\ldots,n\}$, the word

$$\sigma_1^{-1}\sigma_2^{-1}\ldots\sigma_{i-1}^{-1}(s_1,\ldots,s_n)$$

starts with s_i. Another straightforward computation yields the following:

$$\forall (t_1,\ldots,t_n) \in \mathrm{Red}_T(c), \quad (\sigma_{n-1}\ldots\sigma_1)^n(t_1,\ldots,t_n) = \big(ct_1c^{-1},\ldots,ct_nc^{-1}\big).$$

As a consequence, for all $i \in \{1,\ldots,n\}$ and for all positive integer k,

$$\big((\sigma_{n-1}\ldots\sigma_1)^{nk}\sigma_1^{-1}\sigma_2^{-1}\ldots\sigma_{i-1}^{-1}\big)(s_1,\ldots,s_n)$$

[1] As it was pointed out to us by the referee, according to Brown (K.S. Brown, *Buildings*, Springer-Verlag, 1989), Matsumoto property is actually due to Tits.

is an element in the B_n-orbit of $(s_1, \ldots, s_n)$ starting by $c^k s_i c^{-k}$. Lemma 1.3.4 ensures that all elements of T are of the form $c^k s_i c^{-k}$. □

We have the following immediate consequence.

COROLLARY 1.6.2. – *Let (W, T) be a reflection group. Let $w \in W$ be a parabolic Coxeter element. Let $(t_1, \ldots, t_k) \in \mathrm{Red}_T(w)$. The subgroup $\langle t_1, \ldots, t_k \rangle \subset W$ does not depend on the choice of $(t_1, \ldots, t_k)$ in $\mathrm{Red}_T(w)$.*

In the context of the corollary, we denote the subgroup $\langle t_1, \ldots, t_k \rangle$ by W_w. Let $T_w := W_w \cap T$. The reflection group (W_w, T_w) is a parabolic subgroup of (W, T). All parabolic subgroups may be obtained this way.

DEFINITION 1.6.3. – Let (W, T, c) be a dual Coxeter system. A *standard parabolic subgroup* (with respect to c) is a parabolic subgroup of the form (W_w, T_w), where $w \preccurlyeq_T c$.

Let ρ be a realization of (W, T) in $\mathrm{GL}(V_{\mathbb{R}})$. For each $w \in W$, we set $K_w := \ker(\rho(w) - \mathrm{Id})$. The next proposition summarizes the main results from [6]; it is a refinement of Lemma 1.2.1.

PROPOSITION 1.6.4. – *Let $w \in W$.*
(1) *For all $w' \in W$, we have $w' \preccurlyeq_T w \Leftrightarrow K_{w'} \supset K_w$.*
(2) *Let $w', w'' \in W$. Assume that both $w' \preccurlyeq_T w$ and $w'' \preccurlyeq_T w$, and that $K_{w'} = K_{w''}$. Then $w' = w''$.*

In particular, the map $w \mapsto K_w$ is a poset isomorphism from

$$\big(\{w \in W \mid w \preccurlyeq_T c\}, \preccurlyeq_T\big)$$

(the underlying set is denoted by P_c in 0.4) to its image in the subspaces poset of $V_{\mathbb{R}}$. To each subspace of $V_{\mathbb{R}}$ corresponds a parabolic subgroup of (W, T). Subspaces in the image of $w \mapsto K_w$ correspond to standard parabolic subgroups.

2. The dual braid monoid

Throughout this section, we work with a fixed reflection group (W, T). We often use "light" notations, not explicitly refering to (W, T), though of course all constructions are depending on (W, T).

2.1. The dual braid relations

DEFINITION 2.1.1. – Let c be a Coxeter element. We say that two reflections $s, t \in T$ are *non-crossing* (with respect to (W, T, c)), and we write $s \parallel_c t$, if $st \preccurlyeq_T c$ or $ts \preccurlyeq_T c$.

For any $s, t \in T$, the property $s \parallel_c t$ is equivalent to the existence of an element of $\mathrm{Red}_T(c)$ in which both s and t appear (use the Artin group action from the previous section). This relation is symmetric but in general not transitive. Note that the notion really depends on c.

Throughout this section, if A is an alphabet, we call *relation* an unordered pair of words in A^*. We write it $u \cong v$, or equivalently $v \cong u$, where u and v are the two words. E.g., in the next definitions, the dual braid relations are identities between length 2 words on the alphabet T.

DEFINITION 2.1.2. – Let c be a Coxeter element. The *dual braid relations* (with respect to (W, T, c)) are the relations of the form $st \cong {}^s t s$, where $s, t \in T$ are such that $st \preccurlyeq_T c$.

A consequence of Proposition 1.6.4(2) is that if we have both $st \prec_T c$ and $ts \prec_T c$, then $st = ts$ (use that $K_{st} = K_{ts}$ in any realization). The dual braid relations associated with st and ts are then both equal to $st \cong ts$. Therefore dual braid relations are indexed by unordered pairs of non-crossing reflections.

We use the terminology from Section 0.4. The pair (W, T) is a generated group. As noted before, since l_T is invariant by conjugacy, we have $\forall w, w' \in W, w \prec_T w' \Leftrightarrow w' \succ_T w$, and all elements of W, and in particular Coxeter elements, are T-balanced.

DEFINITION 2.1.3. – Let c be a Coxeter element. Let P_c be the pre-monoid of divisors of c in the generated group (W, T). The corresponding monoid $\mathbf{M}(P_c)$ is called the *dual braid monoid* (with respect to (W, T, c)).

The object of this section is the study of the combinatorics of $\mathbf{M}(P_c)$.

We start by deducing from the "dual Matsumoto property" that the dual braid monoid is presented by the dual braid relations. In other words, only a small fragment of the pre-monoid structure of P_c is needed to recover all relations in $\mathbf{M}(P_c)$:

THEOREM 2.1.4. – *Let c be a Coxeter element. The embedding $T \hookrightarrow P_c$ between generating sets induces an isomorphism*

$$\langle T \mid \textit{dual braid relations}\rangle_{\text{monoid}} \simeq \mathbf{M}(P_c).$$

Proof. – The monoid $\mathbf{M}(P_c)$ is generated by its atoms, which are exactly the elements of T. A presentation for $\mathbf{M}(P_c)$, with respect to this generating set, is obtained by taking all relations of the form $u \cong v$, where u and v are reduced T-decompositions of the same $w \in W$, with $w \prec_T c$. Let us prove that such a relation $u \cong v$ is a consequence of the dual braid relations with respect to (W, T, c). By Lemma 1.4.3, the corresponding w is a Coxeter element in a parabolic subgroup (W_I, T_I). Of course, u and v are reduced T_I-decompositions of w. By Proposition 1.6.1, any two reduced T_I-decompositions of w are in the same orbit for the type A Artin group action. By the very definition of this Artin group action, this ensures that the relation $u \cong v$ is a consequence of the dual braid relations with respect to W_I and w. Since $w \prec_T c$, these "parabolic" dual braid relations constitute a subset of the set of dual braid relations with respect to W and c. $\square$

2.2. Dual relations and classical relations

The goal of this subsection is to prove that the group of fractions of the dual braid monoid is isomorphic to the Artin group associated to W. In terms of presentations, this means that the dual braid relations are, as group-defining relations, "equivalent" to the classical braid relations.

LEMMA 2.2.1. – *Let c be a Coxeter element. Let $s, t \in T$, with $s \neq t$. We denote by $m_{s,t}$ the order of st in W. The following assertions are equivalent*:

(i) $s \parallel_c t$.

(ii) *The classical braid relation*

$$\underbrace{sts\dots}_{m_{s,t}\text{ terms}} = \underbrace{tst\dots}_{m_{s,t}\text{ terms}}$$

is satisfied in $\mathbf{M}(P_c)$.

Proof. – The implication (ii) $\Rightarrow$ (i) is obvious, since if s and t are crossing, no dual braid relation can be applied to $\underbrace{sts\dots}_{m_{s,t}\text{ terms}}$.

Let us now prove (i) $\Rightarrow$ (ii). We set $m := m_{s,t}$. Without loss of generality, we may assume $st \prec_T c$. Let $s_1 := t$, $s_2 := s$ and, for $n > 2$, $s_{n+1} := {}^{s_n}s_{n-1}$. We have, for all n, $s_{n+m} = s_n$, and

$$s_2s_1 = s_3s_2 = s_4s_3 = \cdots = s_{m-1}s_{m-2} = s_ms_{m-1} = s_1s_m$$

is a sequence of dual braid relations.

Assume m is even. By multiple use of the above relations, we have

$$\underbrace{s_2s_1s_2s_1\ldots s_2s_1s_2s_1}_{m \text{ terms}} = \underbrace{(s_1s_m)(s_{m-1}s_{m-2})\ldots(s_5s_4)(s_3s_2)}_{m \text{ terms}}$$
$$= \underbrace{s_1(s_2s_1)(s_2\ldots s_1)(s_2s_1)s_2}_{m \text{ terms}}.$$

Assume m is odd. We have

$$\underbrace{s_2s_1s_2s_1\ldots s_2s_1s_2s_1s_2}_{m \text{ terms}} = \underbrace{(s_1s_m)(s_{m-1}s_{m-2})\ldots(s_6s_5)(s_4s_3)s_2}_{m \text{ terms}}$$
$$= \underbrace{s_1(s_2s_1)(s_2\ldots s_1)(s_2s_1)(s_2s_1)}_{m \text{ terms}}. \qquad \square$$

From now on, we fix a chromatic pair (L,R). Let $S := L \cup R$, let $c := c_{L,R}$. We write $L = \{s_1,\ldots,s_k\}$, $R = \{s_{k+1},\ldots,s_n\}$, such that $c = s_1\ldots s_n$. In $\mathbf{B}(W,S)$, we consider the corresponding $\mathbf{S}$, $\mathbf{s}_i$ and $\mathbf{c}$. We set

$$\mathbf{T} := \{\mathbf{c}^k\mathbf{s}\mathbf{c}^{-k} \mid k \in \mathbb{Z},\ \mathbf{s} \in \mathbf{S}\}.$$

The next lemma is a “braid version” of Lemma 1.3.4.

LEMMA 2.2.2. – *Let $\mathbf{t},\mathbf{u} \in \mathbf{T}$, and let t,u be the corresponding elements of W. For all $m \in \mathbb{Z}$, we have*

$$\mathbf{u} = \mathbf{c}^m\mathbf{t}\mathbf{c}^{-m} \text{ in } \mathbf{B}(W,S) \quad \Leftrightarrow \quad u = c^mtc^{-m} \text{ in } W.$$

The canonical morphism $\mathbf{B}(W,S) \twoheadrightarrow W$ restricts to a bijection

$$\mathbf{T} \xrightarrow{\sim} T.$$

Proof. – It is enough to prove the result when W is irreducible.

The implication $\mathbf{u} = \mathbf{c}^m\mathbf{t}\mathbf{c}^{-m}$ in $\mathbf{B}(W,S) \Rightarrow u = c^mtc^{-m}$ in W is obvious.

Let $s,s' \in S$, let $m,l \in \mathbb{Z}_{\geqslant 0}$ such that $c^msc^{-m} = c^ls'c^{-l}$ in W. To obtain the converse implication, we have to prove that $\mathbf{c}^m\mathbf{s}\mathbf{c}^{-m} = \mathbf{c}^l\mathbf{s}'\mathbf{c}^{-l}$ in $\mathbf{B}(W,S)$. It suffices to deal with the case $l = 0$. From now on, we assume $c^msc^{-m} = s'$.

By Lemma 1.3.4, this implies that m is a multiple of $h/2$ (of course, if h is odd, then m must be a multiple of h). According to [4], Ch. V, § 6, Ex. 2, p. 140, we have $\mathbf{c}^{h/2} = \mathbf{w}_0$ when h is even, and $\mathbf{c}^h = \mathbf{w}_0^2$ with no restriction on h. In any case, we have $\mathbf{c}^m = \mathbf{w}_0^{2m/h}$. The conjugation by $\mathbf{w}_0$ is a diagram automorphism of the classical diagram for $\mathbf{B}(W,S)$; the relation $\mathbf{c}^m\mathbf{s}\mathbf{c}^{-m} = \mathbf{s}'$ follows immediately.

Since the natural map $\mathbf{T} \to T$ is $\mathbb{Z}/h\mathbb{Z}$-equivariant (for the respective conjugacy actions by powers of $\mathbf{c}$ and c), the description of T given in Lemma 1.3.4 and the definition of $\mathbf{T}$ prove that $\mathbf{T} \to T$ is a bijection. $\square$

LEMMA 2.2.3. – *Let $t, u \in T$. For all $m \in \mathbb{Z}$, we have*

$$uc^m = c^m t \text{ in } \mathbf{M}(P_c) \quad \Leftrightarrow \quad u = c^m t c^{-m} \text{ in } W.$$

Proof. – The $\Rightarrow$ implication is obvious. Let us prove $\Leftarrow$. An obvious induction reduces the lemma to the case $m = 1$. Let $t \in T$. Let $u := ctc^{-1}$. Let $(t_1, \ldots, t_n) \in \mathrm{Red}_T(c)$ such that $t_1 = u$ (Lemma 1.4.2). The relation $ut_2 \ldots t_n = t_2 \ldots t_n t$ is a consequence of the dual braid relations. Thus, in $\mathbf{M}(P_c)$, we have $uc = uut_2 \ldots t_n = ut_2 \ldots t_n t = t_2 \ldots t_n tt = ct$. □

We call **c**-*conjugacy relations* the relations of the form $\mathbf{t}(\mathbf{s}_1 \ldots \mathbf{s}_n)^m = (\mathbf{s}_1 \ldots \mathbf{s}_n)^m \mathbf{s}$ (with $\mathbf{t} \in \mathbf{T}$, $\mathbf{s} \in \mathbf{S}$ and m a positive integer) which are true in $\mathbf{B}(W, S)$.

The group $\mathbf{B}(W, S)$ has the presentation

$$\mathbf{B}(W, S) = \langle \mathbf{S} \mid \text{classical braid relations} \rangle_{\text{group}}.$$

Since the **c**-conjugacy relations allow the elements of $\mathbf{T}$ to be expressed as conjugates of the elements of $\mathbf{S}$, a successive use of Schreier transformations introducing the redundant generators $\mathbf{T} - \mathbf{S}$ leads to the following presentation:

$$\mathbf{B}(W, S) \simeq \left\langle \mathbf{T} \;\middle|\; \begin{array}{c} \text{classical braid relations on } \mathbf{S} \\ + \, \mathbf{c}\text{-conjugacy relations} \end{array} \right\rangle_{\text{group}}.$$

The group $\mathbf{G}(P_c)$ has the presentation

$$\mathbf{G}(P_c) = \langle T \mid \text{dual braid relations} \rangle_{\text{group}}.$$

According to the Lemma 2.2.3, the "c-conjugacy relations" are consequences of the dual braid relations. If $s, s' \in S$, we have $s \parallel_c s'$; by Lemma 2.2.1, the classical braid relation involving s and s' is true in $\mathbf{G}(P_c)$. Adding these two sets of redundant relations, we obtain

$$\mathbf{G}(P_c) = \left\langle T \;\middle|\; \begin{array}{c} \text{dual braid relations} \\ + \, c\text{-conjugacy relations} \\ + \text{ classical braid relations on } S \end{array} \right\rangle_{\text{group}}.$$

This proves that the bijection $\mathbf{T} \xrightarrow{\sim} T$ from Lemma 2.2.2 extends to a group morphism

$$\mathbf{B}(W, S) \to \mathbf{G}(P_c).$$

The morphism is invertible, thanks to the following fact:

FACT 2.2.4. – *Let $\mathbf{t}, \mathbf{u} \in \mathbf{T}$, and let t, u be the corresponding elements of W. Assume that $tu \prec_T c$. Let $tu \cong uv$, with $v \in T$, be the corresponding dual braid relation. Then $\mathbf{tu} = \mathbf{uv}$ in $\mathbf{B}(W, S)$ (where $\mathbf{v} \in \mathbf{T}$ corresponds to v).*

Proof. – We only have a case-by-case proof. It is enough to deal with the irreducible case. The exceptional types are dealt with by computer, using the package CHEVIE of GAP. The dihedral case is obvious. For the infinite families A, B and D, see Section 4.

Note however that the geometric interpretation of the next section allows a reformulation of this fact which, we hope, could lead to a general proof. □

This completes the proof of:

THEOREM 2.2.5. – *The bijection* $T \xrightarrow{\sim} \mathbf{T}$ *extends to a group isomorphism*

$$\mathbf{G}(P_c) \xrightarrow{\sim} \mathbf{B}(W,S).$$

We will later see that $\mathbf{M}(P_c)$ embeds in $\mathbf{G}(P_c)$, and therefore that $\mathbf{M}(P_c)$ is isomorphic to the submonoid of $\mathbf{B}(W,S)$ generated by $\mathbf{T}$.

For all $s,t \in T$ such that $st \prec_T c$, let us denote by $t \xrightarrow{s} {}^s t$ the dual braid relation relation $st \cong {}^s ts$. Viewing each relation $t \xrightarrow{s} {}^s t$ as a labelled oriented edge connecting t and ${}^s t$, and putting together all dual braid relations with respect to c, we obtain a *labelled oriented graph* with T as set of vertices (the edges are themselves labelled by vertices). Together, Theorems 2.1.4 and 2.2.5 show that $\mathbf{B}(W,S)$ admits what is called a *labelled oriented graph presentation* or *LOG presentation*. These presentations have been studied by various authors (see for example [22]), and are related to topological properties. A typical example of such a presentation is the Wirtinger presentation for the fundamental group of a link complement. The author thanks Ruth Corran for pointing out this interpretation.

Since the elements of $\mathbf{T}$ are conjugates of elements in $\mathbf{S}$, any presentation of $\mathbf{B}(W,S)$ with $\mathbf{T}$ as set of generators yields, by addition of quadratic relations, a presentation for W. As a corollary of the above theorem, we obtain a "dual Coxeter presentation" for W.

COROLLARY 2.2.6. – *The group* W *has the following presentation*:

$$W \simeq \left\langle T \mid \textit{dual braid relations relative to } c + \forall t \in T,\ t^2 = 1 \right\rangle_{\text{group}}.$$

Instead of deducing this corollary from Theorem 2.2.5, we could have given a direct proof without case-by-case, using the trivial analog of Fact 2.2.4 where the conclusion "$\mathbf{tu} = \mathbf{uv}$ in $\mathbf{B}(W,S)$" is replaced by "$tu = uv$ in W".

There is a well-known example of a presentation for W involving all the reflections: the Steinberg presentation of W, where, in addition to quadratic relations, all true relations of the form $st = tu$ are taken (not just those corresponding to non-crossing reflections). But, when removing the quadratic relations, the Steinberg presentation does not give a presentation of the braid group.

2.3. The dual monoid is a Garside monoid

Since T consists of involutions, word reversing provides a bijection between $\mathrm{Red}_T(c)$ and $\mathrm{Red}_T(c^{-1})$, and the posets $(P_c, \prec_T)$ and $(P_{c^{-1}}, \succ_T)$ are isomorphic. Since T is invariant by conjugacy and the Coxeter elements c and c^{-1} are conjugate, the posets $(P_{c^{-1}}, \succ_T)$ and $(P_c, \succ_T)$ are isomorphic. Hence $(P_c, \prec_T) \simeq (P_c, \succ_T)$ (but, in general, the identity $P_c \to P_c$ is not an isomorphism).

FACT 2.3.1. – *The poset* $(P_c, \prec_T)$ *is a lattice.*

Proof. – Here again, we only have a case-by-case proof (the reduction to the irreducible case is obvious). We hope that the geometric approach of the next section will eventually provide a general proof.

The exceptional types are dealt with by computer, using GAP. The type $I_2(e)$ is trivial (the poset has height 2, with only one maximal element).

For type A, B and D, see Section 4. Note that, using Theorem 0.5.2, we only have to check that pairs of reflections have a right lcm. □

Applying Theorem 0.5.2, we obtain the following:

THEOREM 2.3.2. – *The dual braid monoid* $\mathbf{M}(P_c)$ *is a Garside monoid.*

As explained in the preliminary section, being Garside is a very strong property for a monoid (see also Section 6 for more applications). This justifies the study of the dual braid monoid.

COROLLARY 2.3.3. – *The dual braid monoid* $\mathbf{M}(P_c)$ *is isomorphic to the submonoid of* $\mathbf{B}(W,S)$ *generated by* $\mathbf{T}$.

Proof. – Garside monoids satisfy the embedding property: the natural monoid morphism $\mathbf{M}(P_c) \to \mathbf{G}(P_c)$ is injective. We conclude using Theorem 2.2.5. □

2.4. Automorphisms of the dual braid monoid

DEFINITION 2.4.1. – We say that a monoid M is *symmetric* if it admits a generating set A such that the identity map $A \to A$ extends to an anti-automorphism of M.

Clearly, this is equivalent to the existence of a presentation such that whenever $u = v$ is a relation, $\overline{u} = \overline{v}$ is also a relation (where $\overline{u}$ and $\overline{v}$ are the reversed words).

For example, the classical braid monoids are symmetric monoids.

Dual braid monoids are (in general) not symmetric. Consider for example the reflection group of type A_2. The set T consists of three elements s,t,u, such that $st = tu = us$. Choose $c = st$ (the other choice is $c = ts$). Since T is the set of atoms of $\mathbf{M}(P_c)$, any generating set for $\mathbf{M}(P_c)$ must contain T. But the reversed defining relation $ts = ut$ is not true in $\mathbf{M}(P_c)$ (if it were true, then ut would be a minimal common right multiple of t and u, thus equal to their right lcm tu – but $ut \neq tu$ in W, which is a quotient of $\mathbf{M}(P_c)$).

Nonetheless, $\mathbf{M}(P_c)$ admits anti-automorphisms:

Let c be Coxeter element. Then c^{-1} is a Coxeter element and the identity map $T \to T$ induces an anti-isomorphism

$$\psi_c : \mathbf{M}(P_c) \underset{\text{op}}{\xrightarrow{\sim}} \mathbf{M}(P_{c^{-1}}).$$

For any $w \in W$, the conjugate ${}^{w}c$ is a Coxeter element. The bijection $T \to T, t \mapsto {}^{w}t$ extends to an isomorphism

$$\phi_{c,w} : \mathbf{M}(P_c) \xrightarrow{\sim} \mathbf{M}(P_{{}^{w}c}).$$

Let (L,R) be a chromatic pair such that $c = c_{L,R}$. Then $c^{-1} = c_{R,L}$. We have $s_R c s_R^{-1} = c^{-1}$. If (W,T) is irreducible, the centralizer of c in W is the cyclic subgroup generated by c, so any $w \in W$ such that $wcw^{-1} = c^{-1}$ is of the form $s_R c^k$.

We set

$$\Theta := \psi_c^{-1} \circ \phi_{c,s_R}$$

and

$$\Theta' := \psi_c^{-1} \circ \phi_{c,s_R c}.$$

By looking at the conjugacy action of Θ and Θ' on T, we obtain the following result:

PROPOSITION 2.4.2. – *Assume* (W,T) *is irreducible with Coxeter number* h. *The maps* Θ *and* Θ' *are involutive anti-automorphisms of* $\mathbf{M}(P_c)$. *They satisfy a classical braid relation of length* h. *This defines an action of the dihedral group* $I_2(h)$ *on* $\mathbf{M}(P_c)$, *such that reflections act by anti-automorphisms and rotations by automorphisms.*

If the center ZW *is trivial, this representation of* $I_2(h)$ *is faithful. Otherwise,* ZW *has order* 2, h *is even, and the kernel of the representation is the center of* $I_2(h)$.

It should not be too difficult to answer the following:

Question 2.4.3. – Let C be a conjugacy class of Coxeter elements in W. Is there a natural transitive system of isomorphisms between the $(\mathbf{M}(P_c))_{c\in C}$?

3. Local braid monoids

We give in this section a geometric description of the dual monoid. The classical monoid has an interpretation in terms of walls and chambers or, in other words, in terms of the convex geometry of the hyperplane arrangement, seen from a real basepoint. We prove that the dual monoid has an analogous interpretation, except that one has to look to the complexified hyperplane arrangement from a h-regular eigenvector. Hence the dual monoid is indeed a new *point of view* on braid groups ...

The structure of the section is as follows: 3.1 and 3.2 only contain generalities; the material in 3.3 is probably more or less standard, we include it to justify certain computations; in 3.4, we construct for each basepoint a "local" set of generators and a "local" submonoid of the braid group; when the basepoint is a regular eigenvector, the monoid has certain symmetries, as we will see in 3.5. A real basepoint yields the classical monoid. The main results of this section are in 3.6, where we interpret the dual monoid as a certain local monoid.

3.1. Conventions

Let γ and γ' be two paths in a topological space X. Our convention for composing paths is that the path $\gamma\gamma'$ is defined when the ending point of γ coincides with the starting point of γ'.

Let G be a group together with a left-action on X, such that $X \overset{p}{\twoheadrightarrow} G\backslash X$ is a regular covering. Let $x \in X$. The fibration exact sequence is

$$1 \longrightarrow \pi_1(X,x) \longrightarrow \pi_1\big(G\backslash X, p(x)\big) \xrightarrow{\alpha} G \longrightarrow 1,$$

where the morphism α is defined as follows: let γ be a loop in $(G\backslash X, p(x))$ representing an element $b \in \pi_1(G\backslash X, p(x))$; let $\tilde{\gamma}$ be the only path in X lifting γ and such that $\tilde{\gamma}(0) = x$; there is a unique $g \in G$ such that $\tilde{\gamma}(1) = gx$; we set $\alpha(b) = g$. Note that this indeed defines a morphism and not an anti-morphism (even though bb' means "b then b'" while gg' means "g' then g").

Our convention is opposite to the one used in certain papers about braid groups (e.g., [12]). One reason why we have to be very careful here is that the dual braid monoid is not symmetric (see 2.4), while many geometric statements about the classical monoid remain correct independently of the convention, due to the symmetry of the classical braid relations.

3.2. Braid groups

Let $W \hookrightarrow \mathrm{GL}(V_{\mathbb{R}})$ be a finite real reflection group. For simplicity, we assume throughout this section that this representation is irreducible. Let T be the set of (all) reflections in W (thus (W,T) is an abstract reflection group), and $\mathcal{A}_{\mathbb{R}}$ be the set of reflecting hyperplanes. Let V be the complexified representation $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. As a complex representation, V is irreducible; as a real representation, $V = (V_{\mathbb{R}} \otimes 1) \oplus (V_{\mathbb{R}} \otimes i)$, with $V_{\mathbb{R}} \otimes 1 \simeq V_{\mathbb{R}} \otimes i \simeq V_{\mathbb{R}}$; we denote by $\Re$ and $\Im$ the two corresponding W-equivariant projections $V \to V_{\mathbb{R}}$.

We denote by $\mathcal{A}$ the set of (complex) reflecting hyperplanes in V. More generally, we often use curly letters for subsets of $\mathcal{A}$ and plain letters for the corresponding subsets of T (we preferred not to change the standard notation $\mathcal{A}$ into $\mathcal{T}$). We also use curly letters for chambers: a (*real*) *chamber* $\mathcal{C}$ is a connected component of $V_{\mathbb{R}} - \bigcup_{H_{\mathbb{R}} \in \mathcal{A}_{\mathbb{R}}} H_{\mathbb{R}}$. To a chamber $\mathcal{C}$, we associate the set $\mathcal{S} \subset \mathcal{A}$ of walls of $\mathcal{C}$. The corresponding $S \subset T$ is such that (W, S) is a Coxeter system.

We set

$$V^{\mathrm{reg}} := V - \bigcup_{H \in \mathcal{A}} H.$$

The covering $V^{\mathrm{reg}} \twoheadrightarrow W \backslash V^{\mathrm{reg}}$ is unramified and regular.

The *braid group* of W is the fundamental group of $W \backslash V^{\mathrm{reg}}$. Of course, this is well-defined only up to the choice of a basepoint – this choice will appear to be crucial here. When consistent basepoints are chosen in V^{reg} and $W \backslash V^{\mathrm{reg}}$, the fibration exact sequence gives an epimorphism from the braid group of W to W.

3.3. The Brieskorn basepoint

We describe here a couple of tricks for computing in braid groups, inspired by [9]. In type A, given a loop representing a braid, one may write down a word by looking at where and how the strings cross in the real projection. This can be generalized to all types.

Let $\mathcal{C}$ be a real chamber, with set of walls $\mathcal{S}$. For each $H \in \mathcal{A}$, we fix a linear form $l_H : V_{\mathbb{R}} \to \mathbb{R}$ with kernel $H_{\mathbb{R}}$, and such that $\forall x \in \mathcal{C}$, $l_H(x) > 0$. By extension of scalars, we view l_H as a linear form $V \to \mathbb{C}$ with kernel H. It is uniquely defined, up to multiplication by an element of $\mathbb{R}_+^*$ (we could normalize l_H using the invariant scalar product on $V_{\mathbb{R}}$, but this is not crucial here). The following conditions are equivalent, for a given $v \in V_{\mathbb{R}}$:

(i) The vector v is in $\mathcal{C}$.
(ii) For all $H \in \mathcal{A}$, we have $l_H(v) > 0$.
(iii) For all $H \in \mathcal{S}$, we have $l_H(v) > 0$.

Since $(l_H)_{H \in \mathcal{S}}$ is a basis of the dual of $V_{\mathbb{R}}$, for any $H' \in \mathcal{A}$, we have $l'_H = \sum_{H \in \mathcal{S}} \alpha_{H',H} l_H$. A consequence of $(\mathrm{ii}) \Rightarrow (\mathrm{iii})$ is that all coefficients are in $\mathbb{R}_{\geqslant 0}$.

The space $\Re^{-1}(\mathcal{C}) = \mathcal{C} \otimes 1 + V_{\mathbb{R}} \otimes i \subset V^{\mathrm{reg}}$ is contractible. Thus we may choose it as a basepoint for V^{reg}. More precisely, for any $v \in \Re^{-1}(\mathcal{C})$, the homotopy exact sequence of the triple $\{v\} \subset \Re^{-1}(\mathcal{C}) \subset V^{\mathrm{reg}}$ yields a canonical isomorphism

$$\pi_1\big(V^{\mathrm{reg}}, v\big) \simeq \pi_1\big(V^{\mathrm{reg}}, \Re^{-1}(\mathcal{C})\big).$$

Denote by p the quotient map $V^{\mathrm{reg}} \to W \backslash V^{\mathrm{reg}}$. The space $B_0 := p(\Re^{-1}(\mathcal{C}))$ is contractible and can be used as a "basepoint" for $W \backslash V^{\mathrm{reg}}$. We call B_0 the *Brieskorn basepoint* of $W \backslash V^{\mathrm{reg}}$. For any $w, w' \in W$ and any path γ in V^{reg} such that $\Re(\gamma(0)) \in w\mathcal{C}$ and $\Re(\gamma(1)) \in w'\mathcal{C}$ unambiguously defines an element of $\pi_1(W \backslash V^{\mathrm{reg}}, B_0)$, the latter group being, for any $x_0 \in B_0$, canonically isomorphic to $\pi_1(W \backslash V^{\mathrm{reg}}, x_0)$.

DEFINITION 3.3.1. – Let γ be a differentiable path in $[a,b] \to V^{\mathrm{reg}}$. We say that $t \in [a,b]$ is a *critical time* for γ if $p(\gamma(t)) \notin B_0$. Let C_γ be the set of critical times. We say that γ is *non-singular* if all three conditions hold:

(a) We have $a \notin C_\gamma$ and $b \notin C_\gamma$.
(b) The set C_γ is finite.
(c) For each $t \in C_\gamma$, there is a unique $H_t \in \mathcal{A}$ such that $l_{H_t}(\Re(\gamma(t))) = 0$, and the tangent line to $\Re \circ \gamma$ at t is not included in $\Re(H_t)$.

Condition (b) actually follows from (c), which could be rephrased as "$\Re \circ \gamma$ is transverse to each stratum of the real hyperplane arrangement".

This notion allows a practical reformulation of the main results in [9]. Though not explicitly stated by Brieskorn, this reformulation follows easily from his construction. We leave the details to the reader.

THEOREM 3.3.2 (after Brieskorn). – *There exists a* (*unique, generating*) *subset* $(\mathbf{s}_H)_{H\in\mathcal{S}}$ *of* $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)$ *such that, for any non-singular differentiable path* $\gamma:[0,1]\to V^{\mathrm{reg}}$ *such that*

- $\gamma(0)\in\Re^{-1}(\mathcal{C})$,
- γ *has a unique critical time* t_0,

if we denote by H_0 *the hyperplane such that* $l_{H_0}(\Re(\gamma(t_0)))=0$ (H_0 *is always a wall of* $\mathcal{C}$), *we have*:

- *if* $\mathrm{im}(l_H(\gamma(t_0)))>0$, *then* γ *represents* $\mathbf{s}_{H_0}$,
- *if* $\mathrm{im}(l_H(\gamma(t_0)))<0$, *then* γ *represents* $\mathbf{s}_{H_0}^{-1}$.

These generators realize an explicit isomorphism $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)\simeq\mathbf{B}(W,S)$.

Remark. – There are two natural morphisms $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)\to W$: one comes from the fibration exact sequence, the other is the composition of the Brieskorn isomorphism $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)\simeq\mathbf{B}(W,S)$ with the canonical quotient morphism $\mathbf{B}(W,S)\twoheadrightarrow W$. We leave to the reader to check that these two morphisms coincide (thanks to the convention from 3.1).

We deduce from the theorem a recipe for translating non-singular paths into elements of the Artin group. Let $\gamma:[a,b]\to V^{\mathrm{reg}}$ be a non-singular differentiable path. Start by ordering

$$t_1<t_2<\cdots<t_k$$

the critical times. Let $a_0=a<a_1<\cdots<a_{k-1}<a_k=b$ be such that

$$a_0<t_1<a_1<t_2<\cdots<a_{k-1}<t_k<a_k.$$

For $i=1,\dots,k$, we denote by γ_i the restriction of γ to $[a_{i-1},a_i]$. The problem reduces to determining the image in $\mathbf{B}(W,S)$ of a given γ_i. Note that

$$\gamma_i(a_{i-1})\in p^{-1}(B_0)=\bigcup_{w\in W}\Re^{-1}(w\mathcal{C})$$

and that there is a unique $w_i\in W$ such that $\gamma_i(a_{i-1})\in\Re^{-1}(w_i\mathcal{C})$. The path $w_i^{-1}\gamma_i$ represents the same element of $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)$ as γ_i, and this element (of the form $\mathbf{s}_{j_i}^{\varepsilon_i}$) can be determined according to Theorem 3.3.2. The image of γ in $\mathbf{B}(W,S)$ is $\mathbf{s}_{j_1}^{\varepsilon_1}\mathbf{s}_{j_2}^{\varepsilon_2}\dots\mathbf{s}_{j_k}^{\varepsilon_k}$.

Note that these constructions do rely on the choice of a chamber.

What can we do with a singular path $\gamma:[0,1]\to V^{\mathrm{reg}}$? If the endpoints of γ are not in B_0, then the real projection is **really** ambiguous, since γ is not a relative loop in the "pointed" space $(W\backslash V^{\mathrm{reg}}, B_0)$. If the endpoints are in B_0, then we may always find γ' non-singular in the homotopy class of γ. Being non-singular is actually a "generic" property, and one may desingularize γ by arbitrary small perturbations. (Alternatively, desingularization could be avoided by replacing the above rudimentary recipe by a more sophisticated one, able to handle certain paths crossing more than one real hyperplane at a time.)

3.4. Local monoids

For $v,v'\in V$, we denote by $[v,v']$ the affine segment between v and v' (in other words, the convex hull of $\{v,v'\}$).

DEFINITION 3.4.1. – Let $v \in V^{\mathrm{reg}}$. We say that an hyperplane $H \in \mathcal{A}$ is *visible from* v if and only

$$\forall H' \in \mathcal{A}, \quad [v, s_H v] \cap H' \neq \emptyset \quad \Rightarrow \quad H = H'.$$

We set $\mathcal{V}_v := \{H \in \mathcal{A} \mid H \text{ is visible from } v\}$.

Assume we are given, for each $H \in \mathcal{A}$, a linear form l_H with kernel H. Clearly: "H is visible from v" $\Leftrightarrow$ "$\forall H' \in \mathcal{A} - \{H\}, 0 \notin [l_{H'}(v), l_{H'}(s_H v)]$".

Let $v \in V^{\mathrm{reg}}$, with image x in $W \backslash V^{\mathrm{reg}}$. Let $H \in \mathcal{A}$. Assume that H is visible from v. Then the path

$$\gamma : [0,1] \to V^{\mathrm{reg}}, \quad t \mapsto (1-t)v + t\frac{v + s_H(v)}{2}$$

is a path from v to H in V^{reg} (in the sense of [1], Section 2.1). The composition $\overline{\gamma}$ of γ with the quotient map $V^{\mathrm{reg}} \twoheadrightarrow W \backslash V^{\mathrm{reg}}$ is a path from $\overline{v}$ to the discriminant. As explained in [1], Section 2.1, this path defines a generator-of-the-monodromy in $\pi_1(W \backslash V^{\mathrm{reg}}, x)$. Let us denote by $\mathbf{s}_{v,H}$ this generator-of-the-monodromy.

DEFINITION 3.4.2. – Let $v \in V^{\mathrm{reg}}$. The elements of $\{\mathbf{s}_{v,H} \mid H \in \mathcal{V}_v\}$ are called *local generators* at v. The *local* (*braid*) *monoid at* v, denoted by M_v, is the submonoid of $\pi_1(W \backslash V^{\mathrm{reg}}, p(v))$ generated by all the local generators.

For any $v \in V^{\mathrm{reg}}$ and any $w \in W$, we clearly have $\mathcal{V}_{wv} = w\mathcal{V}_v$, and $\forall H \in \mathcal{V}_v, \mathbf{s}_{v,H} = \mathbf{s}_{wv,wH}$. Thus $M_v = M_{wv}$. Let $x := p(v)$. We set $M_x := M_v$ call it the *local monoid at* x. It does not depend on the choice of v in $p^{-1}(x)$.

Remark. – The different visibility conditions define a certain stratification (in the weak sense, i.e., without border condition) of V^{reg}, the *visibility stratification*. The structure of the local monoid only depends on the position of v with respect to this stratification. Some basic properties of this stratification are:

- The maximal strata, with real dimension $\dim_{\mathbb{R}}(V)$, are those from which all hyperplanes are visible. Generically, a point $v \in V^{\mathrm{reg}}$ lies in a maximal stratum, all hyperplanes are visible from v and the structure of the local monoid is stable by small modification of v. When some hyperplanes are not visible from v, the local monoid is not stable.
- The structure of the local monoid does not only depend on which hyperplanes are visible: "how" they are visible is important. For example, the dual monoid will appear later to be a particular example of local monoid, corresponding a certain maximum stratum, but there are maximal strata such that the corresponding local monoid is not the dual monoid.
- The visibility stratification is compatible with the action of W, and we may define a *quotient visibility stratification* on $W \backslash V^{\mathrm{reg}}$. For $W = \mathfrak{S}_4$, the quotient visibility stratification has four strata, two of which are maximal.

Rephrased in terms of local monoids, Brieskorn's theorem implies that when the basepoint is chosen in a real chamber, the local monoid is the classical positive braid monoid:

PROPOSITION 3.4.3. – *Let $v \in V^{\mathrm{reg}}$. Let $\mathcal{C}$ be a real chamber, with set of walls $\mathcal{S}$. Assume that $\Re(v) \in \mathcal{C}$; using Brieskorn's basepoint, we identify $\pi_1(W \backslash V^{\mathrm{reg}}, p(v))$ with $\mathbf{B}(W,S)$.*

(1) *Any $H \in \mathcal{S}$ is visible from v, and $\mathbf{s}_{v,H} = \mathbf{s}_H$.*

(2) *Assume that $\Im(v) = 0$. Then $\mathcal{V}_v = \mathcal{S}$, and the isomorphism $\pi_1(W \backslash V^{\mathrm{reg}}, p(v)) \simeq \mathbf{B}(W,S)$ restricts to an isomorphism $M_v \simeq \mathbf{B}_+(W,S)$.*

Proof. – (1) Let $H \in \mathcal{S}$. Since all hyperplanes have real equations, we have for all $H' \in \mathcal{A}$

$$[v, s_H v] \cap H' \neq \emptyset \quad \Rightarrow \quad \big[\Re(v), \Re(s_H v)\big] \cap H'_{\mathbb{R}} \neq \emptyset.$$

The chamber $s_H\mathcal{C}$ is separated from $\mathcal{C}$ by only one wall, H. Thus the segment $[\Re(v),\Re(s_Hv)]$ intersects only one real hyperplane, $H_{\mathbb{R}}$. This proves that $H\in\mathcal{V}_v$. The identity $\mathbf{s}_{v,H}=\mathbf{s}_H$ is easy: choose a path representing $\mathbf{s}_{v,H}$ and use Theorem 3.3.2.

(2) If $\Im(v)=0$, then for any $H,H'\in\mathcal{A}$, we have

$$[v,s_Hv]\cap H'=\emptyset \quad\Leftrightarrow\quad \big[\Re(v),\Re(s_Hv)\big]\cap H'_{\mathbb{R}}=\emptyset.$$

If $H\notin\mathcal{S}$, then the path $[\Re(v),\Re(s_Hv)]$, which exits the chamber $\mathcal{C}$, must cross at least a wall of $\mathcal{C}$, and $H\notin\mathcal{V}_v$. The second part of the statement follows immediately. □

LEMMA 3.4.4. – *Let L be a complex line* (*through the origin*) *in V. Let $L^* := L-\{0\}$. Assume $L^*\subset V^{\mathrm{reg}}$.*

(1) *Let $v,v'\in L^*$. Let γ be a path in L^* starting at v and ending at v'. The corresponding isomorphism*

$$\phi_\gamma:\pi_1\big(W\backslash V^{\mathrm{reg}},p(v)\big)\xrightarrow{\sim}\pi_1\big(W\backslash V^{\mathrm{reg}},p(v')\big)$$

does not depend on the choice of γ. Let us denote it by $\phi_{v,v'}$. The family $(\phi_{v,v'})_{v,v'\in L^*}$ is a transitive system of isomorphisms between the $(\pi_1(W\backslash V^{\mathrm{reg}},p(v)))_{v\in L^*}$.

(2) *Let $v,v'\in L^*$. Let $\phi_{v,v'}$ be the corresponding isomorphism, as in* (1). *We have $\mathcal{V}_{v'}=\mathcal{V}_v$, and*

$$\forall H\in\mathcal{V}_v,\quad \phi_{v,v'}(\mathbf{s}_{v,H})=\mathbf{s}_{v',H}.$$

The family $(\phi_{v,v'})_{v,v'\in L^*}$ induces by restriction a transitive system of isomorphisms between the $(M_v)_{v\in L^*}$.

The concrete meaning of the lemma is that it makes sense to use the notations $\mathbf{s}_{L^*,H}$ and M_{L^*}.

Proof. – (1) *A priori*, the isomorphism ϕ_γ only depends on the homotopy class of γ. To prove that it does not depend on γ, it is enough to check it when $v=v'$, i.e., to prove that the conjugacy action of $\pi_1(L^*,v)$ on $\pi_1(W\backslash V^{\mathrm{reg}},p(v))$ is trivial. But $\pi_1(L^*,v)$ is cyclic, generated by an element which is well-known to be central in $\pi_1(W\backslash V^{\mathrm{reg}},p(v))$ (see for example [12], Lemma 2.4). The transitivity of the system of isomorphisms follows from the independence of the choice of γ.

(2) The visibility condition is invariant by scalar multiplication. The rest is an easy computation. □

PROPOSITION 3.4.5. – *Let $v\in V^{\mathrm{reg}}$. The group $\pi_1(W\backslash V^{\mathrm{reg}},p(v))$ is generated* (*as a group*) *by the local generators at v.*

Proof. – By Lemma 3.4.4, if $\lambda\in\mathbb{C}^*$, the statement "$\pi_1(W\backslash V^{\mathrm{reg}},p(v))$ is generated (as a group) by the local generators at v" is equivalent to "$\pi_1(W\backslash V^{\mathrm{reg}},p(\lambda v))$ is generated (as a group) by the local generators at λv".

Since $\mathcal{A}$ is finite, it is always possible to find $\lambda\in\mathbb{C}^*$ such that $\forall H\in\mathcal{A},\mathrm{re}(l_H(\lambda v))\neq 0$ or, in other words, $\Re(\lambda v)$ is in a chamber $\mathcal{C}$.

By Proposition 3.4.3(1), the set of local generators at λv contains a classical Artin-type generating subset for $\pi_1(W\backslash V^{\mathrm{reg}},B_0)$. □

3.5. Local monoids and regular elements

A regular element in W is an element which has an eigenvector in V^{reg}. The connection between regular elements and finite order automorphisms of braid groups was first noticed in [11].

PROPOSITION 3.5.1. – *Let w be a regular element of W, of order d. If the center of W is non-trivial and d is even, set $d' := d/2$; otherwise set $d' := d$. Let v be a regular eigenvector for w.*

The set $\mathcal{V}_v$ is stable by the action of w, and the local monoid M_v admits an automorphism ϕ of order d', such that

$$\forall H \in \mathcal{V}_v, \quad \phi(\mathbf{s}_{v,H}) = \mathbf{s}_{v,wH}.$$

Proof. – By assumption, we have $wv = \zeta v$, where ζ is a primitive dth root of unity. Write $\zeta = e^{2i\pi k/d}$. Applying Lemma 3.4.4 to the path $\gamma : [0,1] \to e^{-2i\pi tk/d}$, we obtain an isomorphism

$$\phi : \pi_1\big(W\backslash V^{\mathrm{reg}}, p(v)\big) \xrightarrow{\sim} \pi_1\big(W\backslash V^{\mathrm{reg}}, p\big(\zeta^{-1}v\big)\big) = \pi_1\big(W\backslash V^{\mathrm{reg}}, p(v)\big)$$

such that, whenever $\forall H \in \mathcal{V}_v$, $\mathbf{s}_{v,H} \mapsto \mathbf{s}_{\zeta^{-1}v,H} = \mathbf{s}_{w^{-1}v,H} = \mathbf{s}_{v,wH}$. In particular, ϕ restricts to an automorphism of M_v.

The order of ϕ is the same as the order of the action w on $\mathcal{V}_v$; this action is isomorphic to the conjugation action of w on $S_v := \{s_H \mid H \in \mathcal{V}_v\}$. By Proposition 3.4.5, the set S_v generates W. Thus the order of ϕ is the smallest $k > 1$ such that w^k is central in W. If $ZW = 1$, then $k = d$. Otherwise, the only non-trivial central element is the (unique) regular element of order 2. The conclusion follows. □

3.6. The dual monoid as a local monoid

This subsection is devoted to the proof of the following theorem, which is an analog of Proposition 3.4.3(2) for the dual monoid.

THEOREM 3.6.1. – *Let $\mathcal{C}$ be a chamber of the real arrangement with set of walls $\mathcal{S}$. Decompose the corresponding S in a chromatic pair $L \cup R$; we have the corresponding partition $\mathcal{S} = \mathcal{L} \cup \mathcal{R}$. Let v be a non-zero $e^{2i\pi/h}$-eigenvector for $c := c_{L,R}$. Then all hyperplanes are visible from v, and the assignment*

$$\forall H \in \mathcal{A}, \quad \mathbf{s}_{v,H} \longmapsto s_H$$

extends to a unique monoid isomorphism

$$M_v \xrightarrow{\sim} \mathbf{M}(P_c).$$

Remark. – The space $\ker(c - e^{2i\pi/h}\mathrm{Id})$ is a complex line (since $a(h) = 1$, in the notations of [30] 3.4 (i)). The different spaces $\ker(c - e^{2i\pi/h}\mathrm{Id})$ corresponding to different choices of c, are transitively permuted by the action of W (see [30], 3.4 (iii)). These observations immediately imply that the structure of M_v does not depend on the choice of c and

$$v \in \ker\big(c - e^{2i\pi/h}\mathrm{Id}\big) \cap V^{\mathrm{reg}}.$$

We fix $\mathcal{C}$, $\mathcal{S}$, $\mathcal{L}$, $\mathcal{R}$, S, L and R as in the theorem.

The next proposition is a refinement, for Coxeter elements, of a general remark by Springer ([30], bottom of p. 173). We use the notation $\arg$ for the standard retraction from $\mathbb{C}^*$ to the unit circle S^1.

PROPOSITION 3.6.2. – *Let* $v \in \ker(c_{L,R} - e^{2i\pi/h}) \cap V^{\mathrm{reg}}$. *Consider the map*

$$\begin{aligned}\theta : \mathcal{A} &\longrightarrow S^1\\ H &\longmapsto \arg\big(l_H(v)\big).\end{aligned}$$

(1) *The partition* $\mathcal{S} = \mathcal{L} \cup \mathcal{R}$ *can be recovered from* θ*, in the following way*: *when* $H' \in \mathcal{L}$ *and* $H'' \in \mathcal{R}$*, we have*

$$\theta(H')/\theta(H'') = e^{i\frac{h-1}{h}\pi}.$$

In particular, $\theta(\mathcal{S})$ *consists of exactly two points, at angle* $\frac{h-1}{h}\pi$.

(2) *The image* $\theta(\mathcal{A})$ *consists of* h *consecutive points on a regular* $2h$*-gon.*

COROLLARY 3.6.3. – *The intersection* $\ker(c_{L,R} - e^{2i\pi/h}) \cap \Re^{-1}(\mathcal{C})$ *is non-empty.*

Proof. – Since $\ker(c_{L,R} - e^{2i\pi/h})$ has complex dimension 1, and since the claimed properties are invariant under multiplication of v by a non-zero complex number, we only have to prove the proposition for a particular v. It is easy to build one from the information provided by Bourbaki.

Let us summarize various results from pp. 118–120 in [4], Ch. V, §6. According to Bourbaki, it is possible to find $z', z'' \in V_{\mathbb{R}}$ such that:

- For any $H' \in \mathcal{L}$ and any $H'' \in \mathcal{R}$, we have

$$l_{H'}(z') = 0, \quad l_{H''}(z') > 0, \quad l_{H'}(z'') > 0 \quad \text{and} \quad l_{H''}(z'') = 0.$$

- The $\mathbb{R}$-plane P generated by z' and z'' is stable by s_L and s_R.
- The element s_L (resp. s_R) acts on P as a reflection with hyperplane $\mathbb{R}z'$ (resp. $\mathbb{R}z''$). Note that there is a unique (up to scalar multiplication) scalar product on P invariant by s_L and s_R and therefore there is a well-defined notion of angle in P. We have $(\widehat{z'', z'}) = \pi/h$.

Since the conditions specifying z' and z'' are stable by multiplication by an element of $\mathbb{R}_+^*$, we may assume that both of their norms are 1. The vector

$$n := \frac{z' - z'' \cos\pi/h}{\sin\pi/h}$$

is such that (z'', n) is an orthonormal basis. A direct computation shows that the element $v \in V$ defined by

$$v := z'' \otimes 1 - n \otimes i$$

is an $e^{2i\pi/h}$-eigenvector for $c_{L,R} = s_L s_R$.

Assume that $H' \in \mathcal{L}$ and $H'' \in \mathcal{R}$. We have

$$\begin{aligned}\frac{\arg(l_{H'}(v))}{\arg(l_{H''}(v))} = \arg\left(\frac{l_{H'}(v)}{l_{H''}(v)}\right) &= \arg\left(\frac{l_{H'}(z'') - il_{H'}(n)}{l_{H''}(z'') - il_{H''}(n)}\right)\\ &= \arg\left(\frac{l_{H'}(z'') + il_{H'}(z'')\cot\pi/h}{-il_{H''}(z')/\sin\pi/n}\right)\\ &= \arg\left(\frac{l_{H'}(z'')}{l_{H''}(z')}\,\frac{\sin\pi/h + i\cos\pi/h}{-i}\right)\\ &= \arg\left(\frac{\sin\pi/h + i\cos\pi/h}{-i}\right)\\ &= -\cos\pi/h + i\sin\pi/h\\ &= e^{i\pi\frac{h-1}{h}}.\end{aligned}$$

This proves claim (1).

(2) Let $H \in \mathcal{A}$. For any $w \in W$, the linear form $wl_H : x \mapsto l_H(w^{-1}x)$ has kernel wH, thus $wl_H = \lambda l_{wH}$, with $\lambda \in \mathbb{R}$. In particular, for any $k \in \mathbb{Z}$, we have

$$\begin{aligned}\theta\big(c_{L,R}^k H\big) = \arg\big(l_{c_{L,R}^k H}(v)\big) &= \pm \arg\big(c_{L,R}^k l_H(v)\big)\\ &= \pm \arg\big(l_H\big(c_{L,R}^{-k} v\big)\big)\\ &= \pm \arg\big(l_H\big(e^{-2i\pi \frac{k}{h}} v\big)\big)\\ &= \pm e^{-2i\pi\frac{k}{h}}\theta(H).\end{aligned}$$

By Lemma 1.3.4, T is the closure of S for the conjugacy action of $c_{L,R}$; rephrased in terms of hyperplanes, this says that $\mathcal{A}$ is the closure of $\mathcal{S}$ for the multiplication action of $c_{L,R}$. Using (1), we see that $\theta(\mathcal{A}) \cup (-\theta(\mathcal{A}))$ is the regular $2h$-gon containing $\theta(\mathcal{S})$. Since all l_H are linear combinations of the $(l_{H'})_{H' \in \mathcal{S}}$ with real positive coefficients, $\theta(\mathcal{A})$ must consist of the h consecutive points from $\theta(\mathcal{R})$ to $\theta(\mathcal{L})$. □

To simplify notations, we now work with an eigenvector

$$v \in \ker\big(c_{L,R} - e^{2i\pi/h}\mathrm{Id}\big) \cap \Re^{-1}(\mathcal{C})$$

such that, when $H' \in \mathcal{L}$ and $H'' \in \mathcal{R}$, one has

$$\arg\big(l_{H'}(v)\big) = e^{i\pi\frac{h-1}{2h}} \quad \text{and} \quad \arg\big(l_{H''}(v)\big) = e^{-i\pi\frac{h-1}{2h}}$$

(the existence of such a v is a consequence of the previous proposition).

We identify $\pi_1(W\backslash V^{\mathrm{reg}}, B_0)$ with the Artin group $\mathbf{B}(W,S)$ via Brieskorn theorem. By Proposition 3.4.3(1), $\mathcal{S} \in \mathcal{V}_v$, and for all $H \in \mathcal{S}$, $\mathbf{s}_{v,H} = \mathbf{s}_H$.

LEMMA 3.6.4. – (i) *The element* $\mathbf{c} = \mathbf{c}_{L,R} = \mathbf{s}_L\mathbf{s}_R \in \pi_1(W\backslash V^{\mathrm{reg}}, B_0)$ *is represented by the path* $\gamma : [0,1] \to V^{\mathrm{reg}}, t \mapsto ve^{2i\pi t/h}$.

(ii) *For all* $H \in \mathcal{A}$ *and all* $k \in \mathbb{Z}$, *we have*

$$\mathbf{c}^k \mathbf{s}_{v,H} \mathbf{c}^{-k} = \mathbf{s}_{v,c^k H}.$$

Proof. – (i) follows from Proposition 3.6.2 by an easy computation, done with the technique described in Subsection 3.3; we leave the details to the reader (the path is singular, but easy to handle, since the hyperplanes crossed simultaneously in the real projection correspond to commuting reflections).

From (i) and the proof of Proposition 3.5.1, it follows that the automorphism ϕ from Proposition 3.5.1 is the conjugation by $\mathbf{c}$. Assertion (ii) follows. □

An immediate consequence of the lemma is that we have a geometric interpretation of the set $\mathbf{T}$ defined in the previous section:

PROPOSITION 3.6.5. – *Via the identification*

$$\pi_1\big(W\backslash V^{\mathrm{reg}}, p(v)\big) \simeq \mathbf{B}(W,S),$$

the set of local generators at v *coincides with* $\mathbf{T}$.

Note that we did not use Fact 2.2.4, nor any case-by-case argument, to prove the last proposition. The proposition provides a geometric setting to check Fact 2.2.4. In the next section, we indicate how to do it for types A, B and D; the dual braid relations between the elements of $\mathbf{T}$ will appear to be particular Sergiescu relations (or, for type D, some analogs of Sergiescu relations).

Theorem 3.6.1 follows from the last proposition and Corollary 2.3.3.

3.7. Are there other Garside local monoids?

Ko and Han [23] have studied a certain class of submonoids of the type A braid groups. As this class contains all local monoids, their main theorem has the following consequence (X_n denotes the set of subsets of $\mathbb{C}$ of cardinal n, which is canonically homeomorphic to the space $W\backslash V^{\mathrm{reg}}$, where W is the Coxeter group if type A_{n-1}):

THEOREM 3.7.1 (after Ko-Han). – *Let $n \in \mathbb{Z}_{\geqslant 1}$ and $x \in X_n$. If the local monoid M_x is a Garside monoid, then x is included in an affine line or is the set of vertices of a strictly convex polygon.*

In other words, in the type A case, the classical monoid and the dual monoid are the only local monoids which are Garside monoids.

4. The dual geometries of types A, B and D

The previous section provides a geometric framework to study the dual monoid. When W is of type A, B and D, this framework can be used to prove Facts 2.2.4 and 2.3.1.

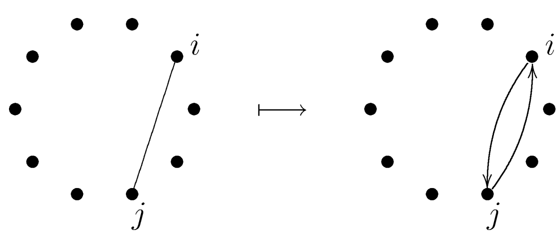
4.1. Type A

The type A dual monoid coincides with the Birman–Ko–Lee monoid [3]. In [2], we gave a geometric interpretation of this monoid, via non-crossing partitions (a similar approach is used independently in [5]); this interpretation can be seen as a particular case of the general one given in Section 3.

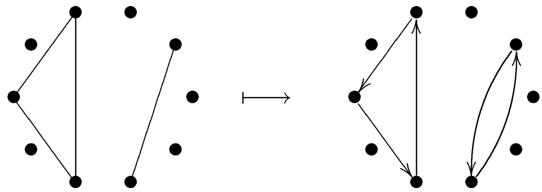
Instead of just quoting [2] for the lattice property (Fact 2.3.1), we give a survey of the main results, since they provide an intuitive illustration of some results from the previous sections. Formal definitions and complete proofs can be found in [2].

Let W be the symmetric group $\mathfrak{S}_n$, let $T \subset W$ be the subset of all transpositions. The group (W, T) is an abstract reflection group of type A_{n-1}. The Coxeter elements are the n-cycles. We choose the standard monomial realization. The space $W\backslash V^{\mathrm{reg}}$ (see Section 3) can be identified with the space X_n of subsets of $\mathbb{C}$ of cardinal n. The fiber of V^{reg} above $x \in X$ is indexed by the total orderings of x: a n-tuple $(x_1, \ldots, x_n) \in \mathbb{C}^n$ is above x if and only if $\{x_1, \ldots, x_n\} = x$.

Let $\mu_n \in X_n$ be the set of complex nth roots of unity. To fix notations, we choose $v = (e^{2i\pi\frac{1}{n}}, e^{2i\pi\frac{2}{n}}, \ldots, e^{2i\pi\frac{n}{n}})$. It lies in the fiber over μ_n. The vector v is a regular $e^{2i\pi/n}$-eigenvector for the n-cycle $c := (1\,2\,\ldots\,n)$. To a transposition $(i\,j)$, we associate the braid $\mathbf{s}_{i,j}$ represented as follows, by a path where only the ith and jth strings are moving:

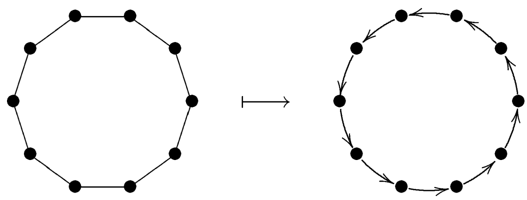


(All pictures here are with $n = 10$.) One easily checks that the reflecting hyperplane $H_{i,j}$ of $(i\ j)$ is visible from v, and that $\mathbf{s}_{i,j}$ is the corresponding local generator. More generally, to any non-crossing partition of μ_n (cf. [2] or [26]), we associate an element of $\pi_1(X_n, \mu_n)$ in the following manner:

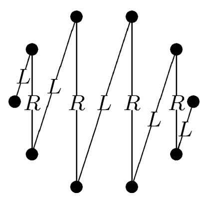


The planar oriented graph above may be interpreted, upon need, in three distinct but consistent ways: as an actual path (see Section 4 in [2]), as an element of $\pi_1(X_n, \mu_n)$, or as the graph of the corresponding permutation, via the morphism $\pi_1(X_n, \mu_n) \to \mathfrak{S}_n$. The elements of $\mathfrak{S}_n$ obtained this way are exactly the elements of P_c. This correspondence is a poset isomorphism between the poset of non-crossing partitions (for the “is finer than” order) and $(P_c, \prec_T)$. Since non-crossing partitions form a lattice, this proves Fact 2.3.1 in this case.

The Coxeter element c corresponds to the partition with only one part:



The Coxeter element c is the element $c_{L,R}$, where (L, R) is the following chromatic pair (or any other chromatic pair obtained by rotating the picture):



These pictures provide good illustrations of many of our results. For example, conjugating by c is the same as “rotating pictures by one nth of a turn”. The isomorphism $\mathbf{T} \simeq T$ from Lemma 2.2.2 is explained by the fact that the above graph picturing the chromatic pair (L, R) generates, by rotation, the complete graph on μ_n. Proposition 3.6.2 is also easy to figure out: for any $\zeta, \zeta' \in \mu_n$, we have $\frac{\zeta - \zeta'}{|\zeta - \zeta'|} \in \mu_{2n}$. The type A case of Theorem 3.6.1 is also clear.

For $s, t \in T$, we have $s \parallel_c t$ if and only if the edges corresponding to s and t have a common endpoint or no common point. All relations claimed in Fact 2.2.4 are particular Sergiescu relations [28].

4.2. Type B

Let (W, T) be the reflection group of type B_n, in its usual monomial realization. The Coxeter number is $2n$. It is well known that the orbit space $W \backslash V^{\mathrm{reg}}$ can be identified with the space of subsets of $\mathbb{C}^*$ of cardinal n or, equivalently, with the fixed subspace $X_{2n}^{\mu_2}$ for the action of μ_2 on X_{2n}. A particular case of Proposition 5.1 in [2] identifies $\pi_1(X_{2n}^{\mu_2}, \mu_{2n})$ with $\pi_1(X_{2n}, \mu_{2n})^{\mu_2}$.

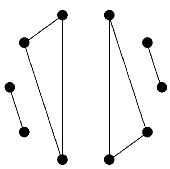
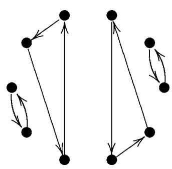
Let $x \in X_{2n}^{\mu_2}$. The identification $W \backslash V^{\mathrm{reg}}$ is such that the fiber in V^{reg} above x is the set of n-tuples $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $x = \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}$. In particular, the vector

$$v := \left(e^{2i\pi \frac{1}{2n}}, e^{2i\pi \frac{2}{2n}}, \ldots, e^{2i\pi \frac{n}{2n}}\right)$$

lies above μ_{2n}. It is a regular $e^{2i\frac{\pi}{2n}}$-eigenvector for the Coxeter element

$$c := \begin{pmatrix} 0 & 1 & 0 \ldots\ldots & 0 \\ 0 & 0 & 1 \quad 0 \ldots & 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ 0 & \ldots\ldots\ldots & 0 & 1 \\ -1 & 0 \ldots\ldots\ldots & & 0 \end{pmatrix}.$$

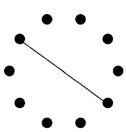
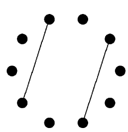
We say that a partition of μ_{2n} is μ_2-symmetric if each part is stable by multiplication by -1. Let λ be a μ_2-symmetric non-crossing partition of μ_{2n}. Let σ_λ be the corresponding element of $\mathfrak{S}_{2n}$ (identified, as in the type A discussion above, with $\mathfrak{S}_{\mu_{2n}}$). For any $k \in \{1, \ldots, n\}$, there is a unique pair $(l_k, \varepsilon_k) \in \{1, \ldots, n\} \times \{\pm 1\}$ such that $\sigma_\lambda(e^{2i\pi \frac{k}{2n}}) = \varepsilon_k e^{2i\pi \frac{l_k}{2n}}$. To λ, we associate the monomial matrix $w_\lambda := (\varepsilon_k \delta_{l_k, l})_{l,k}$ in W. An example with $n = 5$ is illustrated below:

λ :  σ_λ : 

$$w_\lambda : \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

One can easily deduce from the type A case that this construction identifies the poset of μ_2-symmetric non-crossing partitions of μ_{2n} with P_c. The type B case of Fact 2.3.1 follows, since μ_2-symmetric partitions form a lattice (this lattice is studied in [26]).

The reflections in W correspond to minimal symmetric non-crossing partitions. There are two types of them, corresponding to the two conjugacy classes of reflections in W: partitions with one symmetric part $\{\zeta, -\zeta\}$ (and all other parts being points), and partitions with two non-symmetric parts $\{\zeta, \zeta'\}$ and $\{-\zeta, -\zeta'\}$ (with $\zeta \neq \pm\zeta'$), as illustrated below. We call the first type “long” and the second “short”.

long:  short: 

Here again, the corresponding braids are the local generators at v, two reflections are non-crossing if and only if the corresponding edges have no common point (except possibly endpoints; the two reflections pictured above are crossing) and Fact 2.2.4 follows from the usual type A Sergiescu relations.

4.3. Type D

Let (W, T) be the reflection group of type D_n, with $n \geqslant 3$, seen in its usual monomial realization. The degrees of D_n are $2, 4, 6, \ldots, 2(n-1), n$. The Coxeter number is $2(n-1)$.

A Coxeter element is

$$c := \begin{pmatrix} 0 & 1 & 0 \ldots\ldots & & 0 & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \vdots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \vdots \\ 0 & \ldots\ldots\ldots & & 0 & 1 & \vdots \\ -1 & 0 \ldots\ldots\ldots & & & 0 & 0 \\ 0 & \ldots\ldots\ldots\ldots & & & 0 & -1 \end{pmatrix}$$

(the matrix has two diagonal blocks: a $(n-1)\times(n-1)$ block corresponding to a type B_{n-1} Coxeter element, and -1 as last diagonal coefficient). As this matrix suggests, the dual geometry of type D_n is related to the dual geometry of type B_{n-1}.

A regular $e^{2i\frac{\pi}{2(n-1)}}$-eigenvector for c is

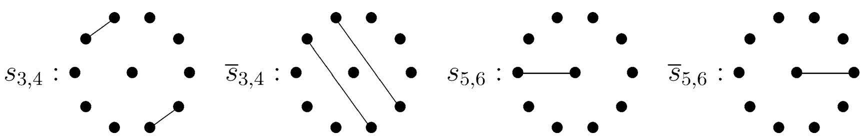
$$v := \left(e^{2i\pi\frac{1}{2(n-1)}}, e^{2i\pi\frac{2}{2(n-1)}}, \ldots, e^{2i\pi\frac{n-1}{2(n-1)}}, 0\right).$$

Consider the map

$$p : V^{\mathrm{reg}} \to \mathcal{P}(\mathbb{C})$$
$$(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}.$$

For $1 \leqslant i, j \leqslant n$, we denote $H_{i,j}$ (resp. $\overline{H}_{i,j}$) the reflecting hyperplane with equation $X_i = X_j$ (resp. $X_i = -X_j$). We denote by $s_{i,j}$ and $\overline{s}_{i,j}$ the corresponding reflections. Contrary to the type B case, the hyperplanes with equation $X_i = 0$ are not reflecting hyperplanes. The image of p is in $X_{2n} \cup X_{2n-1}$. We have $p(v) = \mu_{2(n-1)} \cup \{0\} \in X_{2n-1}$.

For $i \in \{1, \ldots, n-1\}$, we set $\zeta_i := e^{2i\pi\frac{i}{2(n-1)}}$. If $1 \leqslant i < j \leqslant n-1$, we represent the reflection $s_{i,j}$ (resp. $\overline{s}_{i,j}$) by the planar graph on $p(v)$ with edges $[\zeta_i, \zeta_j]$ and $[\overline{\zeta}_i, \overline{\zeta}_j]$ (resp. $[\zeta_i, \overline{\zeta}_j]$ and $[\overline{\zeta}_i, \zeta_j]$). If $1 \leqslant i \leqslant n-1$, we represent the reflection $s_{i,n}$ (resp. $\overline{s}_{i,n}$) by the planar graph with only edge $[\zeta_i, 0]$ (resp. $[\overline{\zeta}_i, 0]$). Here are some examples with $n = 6$:

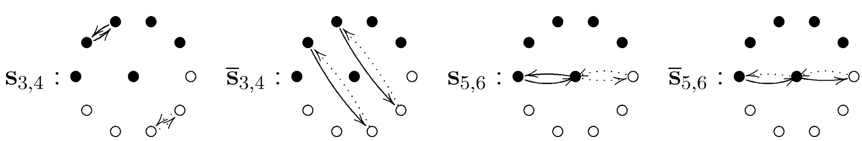


We say that $s_{i,j}$ (resp. $\overline{s}_{i,j}$) is B-like if both $i < n$ and $j < n$. This notion is of course specific to our choice of c.

We leave to the reader the following lemma:

LEMMA 4.3.1. – *Two reflections in T are non-crossing* (*with respect to c*) *if and only if the associated graphs are non-crossing* (*i.e, their edges have no common points except possibly endpoints*).

The corresponding local generators at v are easy to compute. We may represent each of them by a path in V^{reg} starting at v and ending at sv, according to the pictures below:



These pictures should be interpreted as follows: the black dots indicate the starting values of the coordinates (the coordinates of v); these coordinates vary continuously according to the plain arrows. The white dots and the dotted arrows complete the picture by symmetry. Together, the plain and dotted arrows represent the image of the path by p.

Remark. – The “folding” of the D_n Dynkin diagram onto the B_{n-1} diagram has a dual analog: the type B_{n-1} dual monoid is isomorphic to the submonoid of the type D_n dual monoid generated by the (short) B-like reflections and the (long) products $s_{k,n}\overline{s}_{k,n}(=\overline{s}_{k,n}s_{k,n})$, for $k=1,\ldots,n-1$.

The dual relations needed for Fact 2.2.4 are easy variations on Sergiescu relations, left to the reader. Using Allcock’s “orbifold” pictures (which are quotient modulo ± 1 of the pictures used here), Picantin explicitly described a presentation of the type D braid monoid [25]; Fact 2.2.4 can also be checked in the presentation in [25].

A detailed combinatorial proof of Fact 2.3.1 in type D can be found in [7]. Let us sketch a more geometric proof. As noted after Fact 2.3.1, the lattice property would follow if we prove that any pair of crossing reflections has a right lcm. The case of two B-like reflections follows easily from the type B combinatorics. Since any pair of crossing reflections always contain a B-like reflection, the only case left is when a B-like reflection $s_{i,j}$ (or $\overline{s}_{i,j}$) is crossing with a reflection of the form $s_{k,n}$. The lcm may be computed explicitly, using convex hulls.

In types A and B, the lattice P_c is isomorphic to the corresponding lattice of non-crossing partitions (as defined in Reiner, [26]). For type A, this isomorphism was observed in [5] and [2]; for type B, in [7] (see also [25]). The local geometry at a h-regular eigenvector provides a natural explanation. In type D, it ought to be possible to encode elements of P_c by planar graphs (extending what is done here for reflections). This should give a natural definition for planar partitions of type D. However, as it is observed in [7] (Section 4.2), Reiner’s type D planar partitions lattice is not isomorphic to P_c (the author thanks Frédéric Chapoton for bringing this to his attention), though they have the same cardinal (see further discussion in 5.2).

5. Numerology

5.1. The duality

This subsection is an attempt to convince the reader that there is some sort of “duality” between the classical braid monoid and the dual braid monoid. Unfortunately, at the present time, we are not able to formalize the nature of this duality.

Let $(W,S=L\cup R)$ be an irreducible finite Coxeter system of rank n, with set of reflections T. The notation $N:=|T|$ is standard. The set of atoms of $\mathbf{B}_+(W,S)$ is $\mathbf{S}$. Let $\mathbf{c}:=\mathbf{c}_{L,R}$. We denote by p the morphism $\mathbf{B}(W,S)\to W, \mathbf{s}\mapsto s$. Let $c:=p(\mathbf{c})=c_{L,R}$. We use Corollary 2.3.3 to identify $\mathbf{M}(P_c)$ with the submonoid of $\mathbf{B}(W,S)$ generated by $\mathbf{T}=\bigcup_{k\in\mathbb{Z}}\mathbf{c}^k\mathbf{S}\mathbf{c}^{-k}$.

The lcm (in $\mathbf{B}_+(W,S)$) of the atoms of $\mathbf{B}_+(W,S)$ is $\mathbf{w}_0$; it has length N for the natural length function on $\mathbf{B}(W,S)$; its image w_0 in W has order 2. The lcm (in $\mathbf{M}(P_c)$) of the atoms of $\mathbf{M}(P_c)$ is $\mathbf{c}$; it has length n; the order of c is the Coxeter number h.

Write $L=\{s_1,\ldots,s_k\}$, $R=\{s_{k+1},\ldots,s_n\}$. We have $\mathbf{c}=\mathbf{s}_1\ldots\mathbf{s}_n$. In other words, $\mathbf{c}$ is the product of the atoms of $\mathbf{B}_+(W,S)$, taken in a suitable order. Similarly, the next lemma proves that the product of atoms of $\mathbf{M}(P_c)$, taken in a suitable order, is $\mathbf{w}_0$.

We extend the notation $\mathbf{s}_m$ to all positive integers m, in such a way that $\mathbf{s}_m$ only depends on $m \bmod n$.

LEMMA 5.1.1. – *For any positive integer* m, *set*

$$\mathbf{t}_m := \left(\prod_{i=1}^{m} \mathbf{s}_i\right)\left(\prod_{i=1}^{m-1} \mathbf{s}_i\right)^{-1}.$$

We have $\mathbf{T} = \{\mathbf{t}_1, \ldots, \mathbf{t}_N\}$, *and*

$$\mathbf{w}_0 = \prod_{m=1}^{N} \mathbf{t}_{N-m+1}.$$

Proof. – Set $t_m := p(\mathbf{t}_m)$. By [4], Ch. V, §6 Ex. 2, pp. 139–140, we have $T = \{t_1, \ldots, t_N\}$. Using the commutation relations within L, we see that, when $1 \leqslant m \leqslant k$, $\mathbf{t}_m = \mathbf{s}_m$. When $k+1 \leqslant m \leqslant n$, the commutation relations within R yield

$$\mathbf{t}_m = \mathbf{s}_1 \ldots \mathbf{s}_{m-1}\mathbf{s}_m\mathbf{s}_{m-1}^{-1} \ldots \mathbf{s}_1^{-1} = \mathbf{s}_1 \ldots \mathbf{s}_n\mathbf{s}_m\mathbf{s}_n^{-1} \ldots \mathbf{s}_1^{-1} = \mathbf{c}\mathbf{s}_m\mathbf{c}^{-1}.$$

We have proved

$$\{\mathbf{s}_1, \ldots, \mathbf{s}_k, \mathbf{c}\mathbf{s}_{k+1}\mathbf{c}^{-1}, \ldots, \mathbf{c}\mathbf{s}_n\mathbf{c}^{-1}\} = \{\mathbf{t}_1, \ldots, \mathbf{t}_n\}.$$

For all m, we have $\mathbf{t}_{m+n} = \mathbf{c}\mathbf{t}_m\mathbf{c}^{-1}$. From this and the above description of $\{\mathbf{t}_1, \ldots, \mathbf{t}_n\}$, we deduce $\mathbf{T} = \{\mathbf{t}_1, \ldots, \mathbf{t}_N\}$.

From the Bourbaki exercice quoted above, we also get $w_0 = \prod_{m=1}^{N} s_m$. Since

$$(s_1, \ldots, s_N) \in \operatorname{Red}_S(w_0),$$

we have $\mathbf{w}_0 = \prod_{m=1}^{N} \mathbf{s}_m = \prod_{m=1}^{N} \mathbf{t}_{N-m+1}$. □

These facts are summarized in Table 1.

The final line has the following explanation: in [1], a certain class of presentations of braid groups is constructed. Each of these presentations corresponds to a regular degree d. The product of the generators, raised to the power d (which is the order of the image of this product in the reflection group), is always central.

For an irreducible Coxeter group, 2 and h are the respectively smallest and largest degrees; they are always regular; it is possible to choose intermediate regular degrees but they do not seem to yield Garside monoids.

Table 1

	Classical monoid	Dual monoid
Set of atoms	$\mathbf{S}$	$\mathbf{T}$
Number of atoms	n	N
Δ	$\mathbf{w}_0$	$\mathbf{c}$
Length of Δ	N	n
Order of $p(\Delta)$	2	h
Product of the atoms	$\mathbf{c}$	$\mathbf{w}_0$
Regular degree	h	2

5.2. Catalan numbers

Reiner [26] suggests a definition for what should be the "Catalan number" attached to a finite Coxeter group. Though he has no definition for exceptional types non-crossing partitions, the Catalan number should be the number of non-crossing partitions. The usual Catalan numbers

$$c_n := \frac{1}{n+1}\binom{2n}{n}$$

correspond to type A.

As explained in the last paragraph of Section 4, Reiner's lattice coincides with ours for types A and B, but does not for type D.

For all types, our lattice has the expected cardinality (the generalized Catalan number):

PROPOSITION 5.2.1. – *Let W be an irreducible Coxeter group, with degrees $d_1, \ldots, d_n = h$. Let c be a Coxeter element. The number of simple elements in the dual monoid is given by the formula*

$$|P_c| = \prod_{i=1}^{n} \frac{d_i + h}{d_i}.$$

Proof. – Case-by-case, using for example the list given in [25]. □

We may now answer some of the questions raised in [26], Remark 2 (at the cost of modifying Reiner's definition for the type D – our description having the advantage of being uniform and intrinsic). Fix a Coxeter element c. An element $w \in W$ should be called *non-crossing* if it is in P_c. A subspace in the intersection lattice generated by the reflecting hyperplanes should be called *non-crossing* if it is of the form K_w, with w non-crossing. According to Proposition 1.6.4, this defines a one-to-one correspondence between non-crossing elements and non-crossing subspaces (and standard parabolic subgroups). A more detailed study of the local geometry at a h-regular eigenvector is likely to provide an explanation.

Question 5.2.2. – The function l_T gives a natural grading on P_c. What should be the formula for the Poincaré polynomial of P_c?

Example. – For the reflection group of type E_8, this Poincaré polynomial is

$$1 + 120q + 1540q^2 + 6120q^3 + 9518q^4 + 6120q^5 + 1540q^6 + 120q^7 + q^8$$

(the palindromicity of this polynomial is a general fact, easy to prove: consider the bijection $P_c \to P_c$, $w \mapsto w^{-1}c$). The value of this polynomial at $q = 1$ is the cardinal of P_c (the corresponding Catalan number). Here, this value is 25080. Note that the order of $W(E_8)$ is 696729600; while the enumeration of the elements of $W(E_8)$ is presently beyond reach, the poset P_c is small enough to be enumerated by computer; checking the lattice property does not require much computing power. Using Lemma 1.2.1 and a formula due to Solomon (main result from [29]), we see that the Poincaré polynomial of $W(E_8)$ for the length function l_T is

$$\begin{aligned}(1+q)&(1+7q)(1+11q)(1+13q)(1+17q)(1+19q)(1+23q)(1+29q)\\ &= 1 + 120q + 6020q^2 + 163800q^3 + 2616558q^4 + 24693480q^5 + 130085780q^6\\ &\quad + 323507400q^7 + 215656441q^8.\end{aligned}$$

A final remark about the arithmetic of P_c. In type A_n, the lcm of two crossing reflections has length 3. In type E_8, there are pairs of crossing reflections whose lcm is c, of length 8.

6. Applications and problems

6.1. The dual normal form

As mentioned in the preliminary section, Garside monoids admit natural normal forms. Therefore, the dual monoid yields a new solution to the word problem. In the type A, the complexity of this solution has been studied by Birman, Ko and Lee [3] and has been proved to be better than the one deriving from the classical monoid. The general case has yet to be studied. A possible advantage that can already be observed is that the Catalan number $|P_c|$ is much smaller than $|W|$ (in the E_8 example above, $|P_c|$ is not far from being the square root of $|W|$).

Another specificity of the dual normal form is that it is compatible with the conjugacy action of a Coxeter element. In [2] is mentioned a conjecture about centralizers in generalized braid groups of certain dth roots of central elements, and the Birman–Ko–Lee monoid is used to prove the conjecture for the type A case. A specificity of the Birman–Ko–Lee monoid, used in Section 4 of [2], is that it is possible to associate to each element P_c a "geometric normal form" (a particular loop which is the shortest loop in its homotopy class, for a suitable metric). We suspect the same can be done with the dual braid monoid. New cases for the centralizer conjecture would follow (the case of W being a Coxeter group, and d dividing the Coxeter number h).

6.2. Braid groups actions on categories

To illustrate how the dual monoid can be used as a replacement for the classical braid monoid, we discuss the problem of braid groups actions on categories. This problem has been studied by Deligne and has applications in representation theory; the present discussion is nothing more than a straightforward reformulation of [19] in the more general context of Garside monoids.

An action of a pre-monoid P on a category $\mathcal{C}$ is a collection of endofunctors $(T(f))_{f\in P}$ and of natural isomorphisms $c_{f,g}: T(f)\circ T(g)\to T(fg)$ (one for each pair (f,g) in the domain of the partial product) with the following compatibility condition: whenever the product fgh is defined in P, the diagram

$$\begin{array}{ccc} T(f)\circ T(g)\circ T(h) & \longrightarrow & T(fg)\circ T(h) \\ \downarrow & & \downarrow \\ T(f)\circ T(gh) & \longrightarrow & T(fgh) \end{array}$$

is commutative.

For monoids, one recovers the notion of action on a category defined in [19]. An action of $\mathbf{M}(P)$ on $\mathcal{C}$ gives, by restriction, an action of P on $\mathcal{C}$. The actions of $\mathbf{M}(P)$ (resp. P) on $\mathcal{C}$ form a category and the restriction is functorial. The analog for the dual braid monoid of the main result (Theorem 1.5) in [19] is a special case of:

THEOREM 6.2.1 (after Deligne). – *Let P be a Garside pre-monoid. Let $\mathcal{C}$ be a category. The restriction functor from the category of actions of $\mathbf{M}(P)$ on $\mathcal{C}$ to the category of actions of P on $\mathcal{C}$ is an equivalence of categories.*

(An unital action by auto-equivalences of $\mathbf{M}(P)$ extends to an action of the braid group $\mathbf{G}(P)$; see [19], Prop. 1.9.)

The construction of the quasi-inverse is virtually identical to the one in [19], and the proofs can be reproduced with only minor adaptations. Let $m\in\mathbf{M}(P)$. In the preliminary section on Garsiditude, we defined a poset $(E(m),\leqslant)$ (our definition actually mimics the one from *loc.*

cit.). Generalizing Théorème 2.4 in *loc. cit.*, one can prove that the geometric realization $|E(m)|$ is contractible.

Denote by U the set of atoms of P which are left-divisors of m. For $u \in U$, denote by $E_u(m)$ the subset of $E(m)$ consisting of those sequences $(p_1, \ldots, p_k)$ such that $u \prec p_1$. For any non-empty subset $V \subset U$, let $E_V(m) := \bigcap_{u \in V} E_u(m)$. Deligne's proof can be easily adapted to establish the contractibility of $|E(m)|$. For the convenience of the reader, we precise how Lemme 2.5 and its proof should be modified to get rid of galleries and chambers:

LEMMA 6.2.2. – *Let $m \in \mathbf{M}(P)$. Let U be as above. Let V be a non-empty subset of U. The geometric realization $|E_V(m)|$ is contractible.*

Proof. – Let δ_V be the right lcm of the elements of V. Since $\forall u \in V, u \prec m$, one has $\delta_V \prec m$. More precisely, for all $(p_1, \ldots, p_k) \in E_V(m)$, one has $\delta_V \prec p_1$. Let $n \in M(P)$ be the element uniquely defined by $\delta_V n = m$. As V is non-empty, one has $l(n) < l(m)$, and Deligne's proof's induction hypothesis implies that $|E(n)|$ is contractible.

The map

$$\begin{aligned} f : E(n) &\longrightarrow E_V(m) \\ (p_1, \ldots, p_k) &\longmapsto (\delta_V, p_1, \ldots, p_k) \end{aligned}$$

is increasing and induces an isomorphism between $E(n)$ and an initial segment of $E_V(m)$.

The map

$$\begin{aligned} f^* : E_V(m) &\longrightarrow E(n) \\ (p_1, \ldots, p_k) &\longmapsto \begin{cases} (\delta_V^{-1} p_1, \ldots, p_k) & \text{if } \delta_V \neq p_1, \\ (p_2, \ldots, p_k) & \text{if } \delta_V = p_1 \end{cases} \end{aligned}$$

is increasing and one has, for all $x \in E(n)$ and all $y \in E_V(m)$,

$$f(x) \leqslant y \quad \Leftrightarrow \quad x \leqslant f^*(y),$$

and one concludes as in Deligne's proof. □

6.3. New $K(\pi, 1)$'s for braid groups

A motivation for Brady's work on the Birman–Ko–Lee monoid was to construct new finite simplicial complexes which are $K(\pi, 1)$'s for braid groups [5,7]. His techniques are modelled on a construction of Bestvina. Following the same approach, Charney, Meier and Whittlesey have extended Bestvina's construction to the context of an arbitrary Garside monoid [14]. They note that the $K(\pi, 1)$ constructed from the dual monoid has the minimal possible dimension.

For a general Garside monoid, the $K(\pi, 1)$ constructed in [14] is related to the complexes $E(m)$ from the previous subsection (more specifically, to $E(\Delta)$, where Δ is the Garside element).

6.4. Problems

We conclude with a list of problems.

(1) Formalize and complete the "dual Coxeter theory".

(2) What can be done with infinite Coxeter groups?

(3) Provide proofs of Facts 2.2.4 and 2.3.1, and of Proposition 5.2.1, not relying on the classification of finite Coxeter systems.

(4) Classify all local monoids which are Garside.

(5) Study the relations between the three natural orders on W: $\prec_S$, $\prec_T$, and the Bruhat order. Does the order $\prec_T$ have a geometric interpretation similar to the ones known for the Bruhat order?

(6) Study Hecke algebras with the dual point of view. Elements T_w are easy to define when $w \in P_c$. The work of Bremke and Malle is a possible source of inspiration on how to define T_w when $w \notin P_c$ (see [8], Prop. 2.4). More generally, study objects classified by Weyl groups (Lie groups, algebraic groups, ...) with the dual point of view.

(7) Explain and formalize the "duality" between the classical and the dual monoid.

(8) For crystallographic types, there should be a bijection between P_c and the number of regions inside the fundamental chamber in the double Shi hyperplane arrangement (described p. 219 in [26]). Give a general construction of such a bijection.

(9) The cardinal of P_c coincides with the number of clusters (in the sense of Fomin and Zelevinsky – see Prop. 3.8 in [20]). Give a bijective proof.

(10) (Related to (4) and (7)) Let M be a Garside monoid. Is it a frequent phenomenon to have another Garside monoid N such that $\mathbf{G}(M) \simeq \mathbf{G}(N)$? The pair classical monoid/dual monoid is an example. Here is another one: the fundamental group $T_{m,n}$ of the complement of the torus link $L_{m,n}$ (obtained by closing on itself the type A_{n-1} braid $(\sigma_1 \ldots \sigma_{n-1})^m$) has the presentation with m generators $s_1, \ldots, s_m$ and relations

$$\underbrace{s_1 s_2 s_3 \ldots}_{n \text{ terms}} = \underbrace{s_2 s_3 s_4 \ldots}_{n \text{ terms}} = \cdots = \underbrace{s_m s_1 s_2 \ldots}_{n \text{ terms}}$$

(if $n > m$, the s_i are cyclically repeated). As noted in [17] (Section 5, Example 5), the monoid $M_{m,n}$ defined by this positive presentation is a Garside monoid. But the links $L_{m,n}$ and $L_{n,m}$ are isotopic. So $\mathbf{G}(M_{m,n}) \simeq \mathbf{G}(M_{n,m})$. Some of these groups appear as braid groups attached to certain complex reflection groups: according to the tables of [12], $\mathbf{B}(G_{12}) \simeq T_{3,3}$, $\mathbf{B}(G_{13}) \simeq T_{3,4}$ and $\mathbf{B}(G_{22}) \simeq T_{3,5}$; hence we may define "dual monoids" for these braid groups.

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