On the Majorana condition for nonlinear Dirac systems ☆

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Abstract

For arbitrarily large initial data in an open set defined by an approximate Majorana condition, global existence and scattering results for solutions to the Dirac equation with Soler-type nonlinearity and the Dirac–Klein–Gordon system in critical spaces in spatial dimension three are established.
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1. Introduction

Let $m, M \geq 0$. Using the summation convention with respect to $\mu = 0, \ldots, 3$, the cubic Dirac equation (Soler model) for a spinor $\psi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ is given by

$$-i\gamma^\mu \partial_\mu \psi + M\psi = (\overline{\psi}\psi)\psi. \tag{1.1}$$

Here, $x^0 = t$, $\partial_0 = \partial_t$, and $\overline{\psi} = \psi^\dagger \gamma^0$ is the Dirac adjoint, where $\psi^\dagger$ denotes the complex conjugate transpose of the spinor ψ, and the matrices $\gamma^\mu \in \mathbb{C}^{4\times 4}$ are the standard Dirac matrices, see [9]. Writing $\Box = \partial_t^2 - \Delta$, the Dirac–Klein–Gordon system is

$$\begin{aligned}
-i\gamma^\mu \partial_\mu \psi + M\psi &= \phi\psi, \\
\Box\phi + m^2\phi &= \overline{\psi}\psi,
\end{aligned} \tag{1.2}$$

where $\phi : \mathbb{R}^{1+3} \to \mathbb{R}$ is a scalar field. These equations (1.1) and (1.2) arise as in relativistic quantum mechanics as toy models for interactions of elementary particles, see e.g. [6,16].

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In previous work, we have addressed the initial value problems for the above equations for small initial data of low regularity. Concerning the cubic Dirac equation, we have obtained small data global well-posedness and scattering in the massive case $M > 0$ [3,4] as well as the massless case $M = 0$ [7]. For the massive Dirac–Klein–Gordon system, we have obtained small data global well-posedness in the non-resonant regime for initial data of subcritical regularity [5] and both in the resonant and the non-resonant regime in the critical space with additional angular regularity [9]. Concerning a more complete account on earlier work on the low regularity well-posedness problem, we refer to the references therein. The purpose of the current article to gain insight into the asymptotic behaviour of an open set of large data solutions to (1.1) and (1.2).

In [10] Chadam and Glassey considered the equations (1.1) and (1.2) under the assumption that the initial data was of the form

$$\psi(0) = (f, g, -g^*, f^*)^t \tag{1.3}$$

where, given a complex scalar (or vector) $z \in \mathbb{C}^n$, we let z^* denote the complex conjugate, and $f, g : \mathbb{R}^3 \to \mathbb{C}$. This condition (1.3) is equivalent to

$$\psi(0) + z\gamma^2 \psi^*(0) = 0 \tag{1.4}$$

with $z = -i$, see [15]. A computation shows that the condition (1.3) is conserved under the evolution of (1.1) and (1.2), and moreover, that if ψ is of the form (1.3) then $\overline{\psi}\psi = 0$. Consequently, under the assumption (1.3), the cubic Dirac equation (1.1) and the Dirac–Klein–Gordon system (1.3) reduce to equations which are linear in ψ. In particular, the argument of Chadam–Glassey gives scattering and global well-posedness for (1.1) and (1.2) for a class of large data [10]. The structural condition (1.3) considered by Chadam and Glassey was introduced in the physics literature long before by Majorana [14] to describe fermions which are their own anti-particles, see [12] for an overview.

Our main Theorems 1.1 and 1.2 below pertain to solutions emanating from initial data which approximately satisfy the algebraic condition (1.4) with $|z| = 1$. For the results concerning the cubic Dirac equation (1.1), we rely on the estimates obtain in [3,4,7]. On the other hand, in the case of the Dirac–Klein–Gordon system (1.2), we require more refined estimates than those used in [9] to obtain the current sharpest small data global theory. The reason is that we have to deal with a large potential in the Dirac equation, which essentially is a free Klein–Gordon wave. Instead, we use refined estimates obtained in [8] which give a small power of a space–time $L_{t,x}^4$ norm on the right-hand side.

The main result regarding the cubic Dirac equation is the following.

Theorem 1.1. *Let $z \in \mathbb{C}$, $|z| = 1$, and $M \geqslant 0$. For any $A \geqslant 1$ there exists $\epsilon = \epsilon(A) > 0$ such that for all initial data satisfying*

$$\|\psi(0)\|_{H^1(\mathbb{R}^3)} \leqslant A \quad \text{and} \quad \|\psi(0) + z\gamma^2 \psi^*(0)\|_{H^1(\mathbb{R}^3)} \leqslant \epsilon,$$

the cubic Dirac equation (1.1) is globally well-posed and solutions scatter to free solutions as $t \to \pm\infty$.

To be more precise, we prove Theorem 3.1 on a reduced system instead, which is equivalent for smooth solutions. In Theorem 1.1 we are forced to take ϵ much smaller than A^{-1}. The regularity assumption in Theorem 1.1 is sharp, in the sense that $\dot{H}^1(\mathbb{R}^3)$ is the scale invariant space. In particular, the regularity assumptions match the optimal results known in the small data case [3,7]. The importance of Theorem 1.1 is that we can take A to be large, in particular, we obtain scattering for an open set of large data with essentially sharp regularity assumptions. Under stronger decay and regularity conditions, such results have been proven by Bachelot in [2]. Very recently, a similar result has been derived in [11] in the presence of a time independent potential and for initial data in $H^1(\mathbb{R}^3)$ with additional angular regularity.

We also have the corresponding version for the Dirac–Klein–Gordon system. Let $H^s_\sigma(\mathbb{R}^3) = (1 - \Delta_{\mathbb{S}^2})^{-\frac{\sigma}{2}} H^s(\mathbb{R}^3)$ be the subspace of the standard Sobolev space $H^s(\mathbb{R}^3)$ containing functions with σ angular derivatives in $H^s(\mathbb{R}^3)$, equipped with the norm

$$\|f\|_{H^s_\sigma} = \|(1 - \Delta_{\mathbb{S}^2})^{\frac{\sigma}{2}} f\|_{H^s}$$

see [9,8] for details. Note that $H^s(\mathbb{R}^3) = H^s_0(\mathbb{R}^3)$.

Theorem 1.2. *Let $z \in \mathbb{C}$, $|z| = 1$. Suppose that either $s > 0 = \sigma$ and $2M > m > 0$, or $\sigma > 0 = s$ and $M, m > 0$. For any $A \geqslant 1$, there exists $\epsilon = \epsilon(A) > 0$, such that if*

$$\|\psi(0)\|_{H^s_\sigma(\mathbb{R}^3)} \leqslant A, \quad \|\phi(0)\|_{H^{\frac{1}{2}+s}_\sigma(\mathbb{R}^3)} \leqslant A, \quad \|\partial_t \phi(0)\|_{H^{-\frac{1}{2}+s}_\sigma(\mathbb{R}^3)} \leqslant A,$$

and

$$\left\| \psi(0) + z\gamma^2 \psi^*(0) \right\|_{H^s_\sigma(\mathbb{R}^3)} \leqslant \epsilon,$$

then the system (1.2) is globally well-posed and solutions scatter to free solutions as $t \to \pm\infty$.

As for the cubic Dirac equation, we prove Theorem 4.1 on a reduced system instead, which is equivalent for smooth solutions.

We obtain an upper bound for ϵ which is the inverse exponential of a power of A, see Theorem 4.1 for more details. The Chadam–Glassey result in [10] corresponds to the case $z = i$ and $\epsilon = 0$ (with additional smoothness assumptions on the data). A result similar to Theorem 1.2 under strong decay and regularity conditions has been established in [1]. Notice that the small data results in [3,7,9] correspond to Theorems 1.1 and 1.2, respectively, in the case where A is very small, since it clearly implies the condition on $\psi(0) + z\gamma^2 \psi^*(0)$. Notice that $s = 0$ is the critical regularity for (1.2).

Organisation of the paper. In Section 2 we perform an initial reduction which decouples the small and the large parts of the spinors. In Section 3 we reformulate and prove the main results concerning the Soler model. In Section 4 we reformulate and prove the main results on the Dirac–Klein–Gordon system.

2. Initial reductions

Suppose we have data $\psi(0)$ satisfying the assumptions of Theorem 1.1. One way to proceed would be to linearise around the Chadam–Glassey type solutions. Thus decomposing

$$\psi(0) = \psi_N(0) + \psi_L(0)$$

where $\|\psi_N(0)\|_{H^1} \leqslant \epsilon$ and $\psi_L(0) + z\gamma^2 \psi_L^*(0) = 0$. Let ψ_L denote the solution to the linear Dirac equation with data $\psi_L(0)$. As mentioned in the introduction, for all times we have $\overline{\psi}_L \psi_L = 0$. Consequently, the remaining term $\psi_N = \psi - \psi_L$ satisfies the equation

$$-i\gamma^\mu \partial_\mu \psi_N + M\psi_N = \left(\overline{\psi}_L \psi_N + \overline{\psi}_N \psi_L \right)\psi + \overline{\psi}_N \psi_N \psi.$$

The last term is small since $\psi_N(0)$ is small. On the other hand, it is not at all clear that the first term $\left(\overline{\psi}_L \psi_N + \overline{\psi}_N \psi_L \right)\psi$ should be small, since it contains terms of the schematic form $\psi_L^2 \psi_N$, and ψ_L can be large. In particular, if we wanted to use the linearised equation to prove Theorem 1.1, we would be forced to absorb these terms into the left-hand side, which would significantly complicate the required multilinear estimates. It turns out that there is a better way to decompose ψ, which avoids this problem. In particular, we can exploit the multilinear estimates already contained in [3,7]. A similar comment applies to the proof for the Dirac–Klein–Gordon system, Theorem 1.2. However, a significant additional difficulty arises in the case where the data for ϕ is large.

We start with the following observation, see [14,13,10], we follow [15].

Lemma 2.1. *Assume that ψ is a classical solution of*

$$-i\gamma^\mu \partial_\mu \psi + M\psi = V\psi$$

for some real-valued, scalar, and locally integrable function $V : \mathbb{R}^{1+3} \to \mathbb{R}$. Then for any $z \in \mathbb{C}$ we have

$$\|\psi(t) + z\gamma^2 \psi^*(t)\|_{L^2_x} = \|\psi(0) + z\gamma^2 \psi^*(0)\|_{L^2_x}.$$

Proof. A computation shows that $\gamma^\mu \gamma^2 = -\gamma^2 (\gamma^\mu)^*$ which implies that

$$-i\gamma^\mu \partial_\mu (\psi + z\gamma^2 \psi^*) = -i\gamma^\mu \partial_\mu \psi + z\gamma^2 (-i\gamma^\mu \partial_\mu \psi)^* = -M(\psi + z\gamma^2 \psi^*) + V(\psi + z\gamma^2 \psi^*).$$

Result now follows by multiplying by $i(\psi + z\gamma^2 \psi^*)^\dagger \gamma^0$, taking the real part, and then integrating over $\mathbb{R}^3$. $\quad\square$

We can now rewrite the cubic Dirac equation (1.1). Let $\varphi, \chi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ be smooth enough and solve

$$\begin{aligned}
-i\gamma^\mu \partial_\mu \varphi + M\varphi &= (\overline{\varphi}\chi + \overline{\chi}\varphi)\varphi \\
-i\gamma^\mu \partial_\mu \chi + M\chi &= (\overline{\varphi}\chi + \overline{\chi}\varphi)\chi
\end{aligned} \tag{2.1}$$

with data

$$\varphi(0) = \frac{1}{2}(\psi(0) + z\gamma^2 \psi^*(0)), \qquad \chi(0) = \frac{1}{2}(\psi(0) - z\gamma^2 \psi^*(0)).$$

Then a computation using Lemma 2.1 implies that for all $t \in \mathbb{R}$ and $|z| = 1$ we have

$$\varphi(t) + z\gamma^2 \varphi^*(t) = 0, \qquad \chi(t) - z\gamma^2 \chi^*(t) = 0$$

and moreover that $\overline{\varphi}\varphi = \overline{\chi}\chi = 0$. Consequently, if we let $\psi = \varphi + \chi$, we obtain a solution to the cubic Dirac equation (1.1). Similarly, in the case of the Dirac–Klein–Gordon system (1.2), let $\varphi, \chi : \mathbb{R}^{1+3} \to \mathbb{C}^4$ and $\phi : \mathbb{R}^{1+3} \to \mathbb{R}$ be smooth enough and solve

$$\begin{aligned}
-i\gamma^\mu \partial_\mu \varphi + M\varphi &= \phi\varphi \\
-i\gamma^\mu \partial_\mu \chi + M\chi &= \phi\chi \\
\Box\phi + m^2\phi &= \overline{\varphi}\chi + \overline{\chi}\varphi
\end{aligned} \tag{2.2}$$

with data

$$\varphi(0) = \frac{1}{2}(\psi(0) + z\gamma^2 \psi^*(0)), \qquad \chi(0) = \frac{1}{2}(\psi(0) - z\gamma^2 \psi^*(0)).$$

As in the case of the cubic Dirac equation, an application of Lemma 2.1 implies that

$$\varphi(t) + z\gamma^2 \varphi^*(t) = 0, \qquad \chi(t) - z\gamma^2 \chi^*(t) = 0$$

and hence provided $|z| = 1$ we have $\overline{\varphi}\varphi = \overline{\chi}\chi = 0$. Consequently, letting $\psi = \varphi + \chi$ we get a solution to (1.2). For technical reasons, we prefer to work with a first order system. Defining $\phi_+ = \phi + i\langle\nabla\rangle^{-1}\partial_t\phi$, as ϕ is real-valued, we obtain

$$\begin{aligned}
-i\gamma^\mu \partial_\mu \varphi + M\varphi &= \mathrm{Re}(\phi_+)\varphi \\
-i\gamma^\mu \partial_\mu \chi + M\chi &= \mathrm{Re}(\phi_+)\chi \\
-i\partial_t\phi_+ + \langle\nabla\rangle_m \phi_+ &= \langle\nabla\rangle_m^{-1}(\overline{\varphi}\chi + \overline{\chi}\varphi)
\end{aligned} \tag{2.3}$$

with data

$$\varphi(0) = \frac{1}{2}(\psi(0) + z\gamma^2 \psi^*(0)), \quad \chi(0) = \frac{1}{2}(\psi(0) - z\gamma^2 \psi^*(0)),$$
$$\text{and } \phi_+(0) = \phi(0) + i\langle\nabla\rangle^{-1}\partial_t\phi(0).$$

Conversely, from ϕ_+ we can recover ϕ by taking the real part of ϕ_+.

3. Cubic Dirac equation

We begin by introducing some notation. Let $\Pi_\pm$ be the projection

$$\Pi_\pm = \tfrac{1}{2}\big(I \pm \langle\nabla\rangle_M^{-1}(-i\gamma^0\gamma^j\partial_j + M\gamma^0)\big),$$

let $\mathcal{U}_m^\pm(t) = e^{\mp it\langle\nabla\rangle_m}$ be the propagator for the homogeneous half-wave equation, let

$$\mathcal{U}_M(t) = \mathcal{U}_M^+(t)\Pi_+ + \mathcal{U}_M^-(t)\Pi_-$$

be the propagator for the homogeneous Dirac equation, and let

$$\mathcal{I}_{t_0}^{\pm,m}(F)(t) = i \int_{t_0}^{t} \mathcal{U}_m^{\pm}(t - t_0 - t') F(t') dt',$$

$$\mathcal{I}_{t_0}^{M}(G)(t) = i \int_{t_0}^{t} \mathcal{U}_M(t - t_0 - t') \gamma^0 G(t') dt'$$

be the corresponding Duhamel integrals.

The previous section implies that for smooth solutions (1.1) and (2.1) are equivalent, so that we focus on proving the following.

Theorem 3.1. *Let $z \in \mathbb{C}$, $|z| = 1$, and $M \geqslant 0$. There exists $c \in (0, 1)$, such that for any $A > 0$ and $\epsilon \leqslant cA^{-1}$, if the initial data satisfy*

$$\|\varphi(0)\|_{H^1} \leqslant \epsilon, \qquad \|\chi(0)\|_{H^1} \leqslant A,$$

then (2.1) is globally well-posed and the solutions scatter in $H^1(\mathbb{R}^3)$ to free solutions as $t \to \pm\infty$, i.e. there exist $\varphi_{\pm\infty} \in H^1(\mathbb{R}^3)$ and $\chi_{\pm\infty} \in H^1(\mathbb{R}^3)$, such that

$$\lim_{t \to \pm\infty} \|\varphi(t) - \mathcal{U}_M(t)\varphi_{\pm\infty}\|_{H^1} = 0 \text{ and } \lim_{t \to \pm\infty} \|\chi(t) - \mathcal{U}_M(t)\chi_{\pm\infty}\|_{H^1} = 0.$$

Proof. Let $X \subset C(\mathbb{R}, H^1(\mathbb{R}^3))$ be the Banach space constructed in [3] in the massive case ($M > 0$) and in [7] in the massless case ($M = 0$). Further, let $\|\cdot\|_X$ denote the norm obtained by multiplying by the norms from [3,7] by a small enough constant, such that for all solutions $\varphi \in X$ to the inhomogeneous Dirac equation

$$-i\gamma^\mu \partial_\mu \varphi + M\varphi = \left(\overline{\varphi^{(1)}} \varphi^{(2)}\right) \varphi^{(3)}$$

the bound

$$\|\varphi\|_X \leqslant \|\varphi(0)\|_{H^1(\mathbb{R}^3)} + C\|\varphi^{(1)}\|_X \|\varphi^{(2)}\|_X \|\varphi^{(3)}\|_X \tag{3.1}$$

holds. Consider the set

$$\mathcal{X} = \left\{ (\varphi, \chi) \in X \times X \mid \|\varphi\|_X \leqslant 2\|\varphi(0)\|_{H^1}, \|\chi\|_X \leqslant 2\|\chi(0)\|_{H^1} \right\}$$

and, for $A, \epsilon > 0$, the norm

$$\|(\varphi, \chi)\|_{\mathcal{X}} = \epsilon^{-1} \|\varphi\|_X + A^{-1} \|\chi\|_X.$$

$\mathcal{X}$ is a complete metric space. Let $\mathcal{T} = (T_1, T_2)$ denote the standard (inhomogeneous) solution map for (2.1) constructed from the Duhamel formula. The bound (3.1) together with the assumption on the initial data shows that if $(\varphi, \chi) \in \mathcal{X}$ then

$$\|T_1(\varphi, \chi)\|_X \leqslant \|\varphi(0)\|_{H^1} + 2C\|\varphi\|_X^2 \|\chi\|_X \leqslant \|\varphi(0)\|_{H^1} + 2^4 C\|\varphi(0)\|_{H^1}^2 \|\chi(0)\|_{H^1}$$
$$\leqslant (1 + 2^4 C A \epsilon) \|\varphi(0)\|_{H^1},$$

and similarly

$$\|T_2(\varphi, \chi)\|_X \leqslant \|\chi(0)\|_{H^1} + 2C\|\chi\|_X^2 \|\varphi\|_X \leqslant (1 + 2^4 C A \epsilon) \|\chi(0)\|_{H^1}.$$

Consequently, provided that $\epsilon \leq (2^4 CA)^{-1}$, we see that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$. Next, we verify that $\mathcal{T}$ is a contraction. For $(\varphi_1, \chi_1), (\varphi_2, \chi_2) \in \mathcal{X}$ another application of (3.1) gives

$$\|T_1(\varphi_1, \chi_1) - T_1(\varphi_2, \chi_2)\|_X \leqslant 2^4 C A \epsilon \|\varphi_1 - \varphi_2\|_X + 2^3 C\epsilon^2 \|\chi_1 - \chi_2\|_X,$$

and similarly

$$\|T_2(\varphi_1, \chi_1) - T_2(\varphi_2, \chi_2)\|_X \leqslant 2^4 C A \epsilon \|\chi_1 - \chi_2\|_X + 2^3 C A^2 \|\varphi_1 - \varphi_2\|_X.$$

This implies

$$\|\mathcal{T}(\varphi_1, \chi_1) - \mathcal{T}(\varphi_2, \chi_2)\|_{\mathcal{X}} \leqslant 2^6 C A \epsilon \|(\varphi_1, \chi_1) - (\varphi_2, \chi_2)\|_{\mathcal{X}}.$$

Therefore, choosing $\epsilon \leq (2^7 C A)^{-1}$, the map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a contraction with respect to $\| \cdot \|_{\mathcal{X}}$, hence it has a unique fixed point in $\mathcal{X}$, and standard arguments show the continuity of the flow map. The scattering claim follows from the finiteness of both $\|\varphi\|_X$ and $\|\chi\|_X$, because this implies that the pull-backs of φ and χ along the free evolution, as maps from $\mathbb{R}$ to $H^1(\mathbb{R}^3)$, have finite quadratic variation, see [3,7] for the details. $\quad\square$

4. The Dirac–Klein–Gordon system

Let P_λ be the standard Littlewood–Paley projections onto dyadic frequencies of size λ, and take H_N to be the projection onto angular frequencies of size N, see [8, Section 2] for precise definitions. If $s \geqslant 0$ and $\sigma = 0$, we define

$$\|f\|_{\mathbf{D}_0^s(I)} = \|\langle\nabla\rangle^s f\|_{L^4(I\times\mathbb{R}^3)}.$$

On the other hand, for $s \geqslant 0$ and $\sigma > 0$, we take

$$\|f\|_{\mathbf{D}_\sigma^s(I)} = \Big(\sum_{N\geqslant 1} N^{2\sigma} \|\langle\nabla\rangle^s H_N f\|^2_{L^4(I\times\mathbb{R}^3)} \Big)^{\frac{1}{2}}.$$

The results in Section 2 imply that for smooth solutions (1.2) and (2.3) are equivalent, so that we focus on proving the following.

Theorem 4.1. *Let $z \in \mathbb{C}$, $|z| = 1$. Suppose that either $s > 0 = \sigma$ and $2M > m > 0$, or $\sigma > 0 = s$ and $M, m > 0$. There exist $0 < c < 1$ and $\gamma > 1$, such that for any $A \geqslant 1$ and any $\epsilon \leqslant c\exp(-A^\gamma)$, if*

$$\|\varphi(0)\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant \epsilon, \quad \|\chi(0)\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant A, \quad \|\phi_+(0)\|_{H_\sigma^{\frac{1}{2}+s}(\mathbb{R}^3)} \leqslant A,$$

then the system (2.3) is globally well-posed and scatters to free solutions as $t \to \pm\infty$, i.e. there exist $\varphi_{\pm\infty} \in H_\sigma^s(\mathbb{R}^3)$, $\chi_{\pm\infty} \in H_\sigma^s(\mathbb{R}^3)$ and $\phi_{\pm\infty} \in H_\sigma^{\frac{1}{2}+s}(\mathbb{R}^3)$, such that

$$\lim_{t\to\pm\infty} \|\varphi(t) - \mathcal{U}_M(t)\varphi_{\pm\infty}\|_{H_\sigma^s} = 0, \quad \lim_{t\to\pm\infty} \|\chi(t) - \mathcal{U}_M(t)\chi_{\pm\infty}\|_{H_\sigma^s} = 0,$$

and $\lim_{t\to\pm\infty} \|\phi_+(t) - \mathcal{U}_m^+(t)\phi_{\pm\infty}\|_{H_\sigma^{s+\frac{1}{2}}} = 0.$

Before we turn to its proof, we summarise the results we require from [8].

Lemma 4.2. *Let $s, \sigma \in \mathbb{R}$, and I be any interval of the form $I = [t_1, t_2)$, $-\infty < t_1 < t_2 \leq \infty$. There exist Banach function spaces $\mathbf{F}_M^{s,\sigma}(I)$ and $\mathbf{V}_{+,m}^{s,\sigma}(I)$ and $C_0 \geqslant 1$ with the following properties:*

(i) *$C_0^\infty(I \times \mathbb{R}^3; \mathbb{C}^4) \subset \mathbf{F}_M^{s,\sigma}(I)$, $C_0^\infty(I \times \mathbb{R}^3; \mathbb{C}) \subset \mathbf{V}_{+,m}^{s,\sigma}(I)$, and*

$$\mathbf{F}_M^{s,\sigma}(I) \hookrightarrow C_b(I; H_\sigma^s(\mathbb{R}^3; \mathbb{C}^4)), \qquad \mathbf{V}_{+,m}^{s,\sigma}(I) \hookrightarrow C_b(I; H_\sigma^s(\mathbb{R}^3; \mathbb{C})).$$

(ii) *For $\psi \in \mathbf{F}_M^{s,\sigma}(I)$, $\phi_+ \in \mathbf{V}_{+,m}^{s,\sigma}(I)$, and for any $I' = [s_1, s_2) \subset I$, we have $\psi|_{I'} \in \mathbf{F}_M^{s,\sigma}(I')$, $\phi_+|_{I'} \in \mathbf{V}_{+,m}^{s,\sigma}(I')$, and*

$$\|\psi|_{I'}\|_{\mathbf{F}_M^{s,\sigma}(I')} \leqslant C_0 \|\psi\|_{\mathbf{F}_M^{s,\sigma}(I)}, \qquad \|\phi_+|_{I'}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I')} \leqslant C_0 \|\phi\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}.$$

(iii) *For $\psi_0 \in H_\sigma^s(\mathbb{R}^3; \mathbb{C}^4)$ and $\phi_0 \in H_\sigma^s(\mathbb{R}^3; \mathbb{C})$ we have $\mathcal{U}_M(t)\psi_0 \in \mathbf{F}_M^{s,\sigma}(I)$, $\mathcal{U}_m^+(t)\phi_0 \in \mathbf{V}_{+,m}^{s,\sigma}(I)$, and the bounds*

$$\|\mathcal{U}_M\psi_0\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \|\psi_0\|_{H_\sigma^s}, \qquad \|\mathcal{U}_m^+\phi_0\|_{\mathbf{V}_{+,m}^{s,\sigma}(I)} \leqslant \|\phi_0\|_{H_\sigma^s}. \tag{4.1}$$

(iv) *For $\psi \in \mathbf{F}_M^{s,\sigma}([t_1, t_2))$ and $\phi_+ \in \mathbf{V}_{+,m}^{s,\sigma}([t_1, t_2))$ the limits*

$$\lim_{t \to t_2} \mathcal{U}_M(-t)\psi(t) \in H^s(\mathbb{R}^3; \mathbb{C}^4) \text{ and } \lim_{t \to t_2} \mathcal{U}_m^+(-t)\phi_+(t) \in H^s(\mathbb{R}^3; \mathbb{C})$$

exist.

(v) *For $\phi_+ \in \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$ we have the Strichartz-type estimate*

$$\|\phi_+\|_{\mathbf{D}_\sigma^s(I)} \leqslant C_0 \|\phi_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}. \tag{4.2}$$

(vi) *Suppose that either $s > 0 = \sigma$ and $2M > m > 0$, or $\sigma > 0 = s$ and $M, m > 0$. There exists $\theta \in (0, 1)$, such that for any $t_0 \in I$ the Duhamel operators*

$$\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \ni (\phi_+, \varphi) \mapsto \mathcal{I}_{t_0}^M(\operatorname{Re}(\phi_+)\varphi) \in \mathbf{F}_M^{s,\sigma}(I),$$

$$\mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \ni (\chi, \varphi) \mapsto \mathcal{I}_{t_0}^{+,m}(\langle\nabla\rangle_m^{-1}(\overline{\chi}\varphi)) \in \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$$

are well-defined and the following estimates hold:

$$\|\mathcal{I}_{t_0}^M(\operatorname{Re}(\phi_+)\varphi)\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant C_0 \|\phi_+\|_{\mathbf{D}_\sigma^s(I)}^\theta \|\phi_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}^{1-\theta} \|\varphi\|_{\mathbf{F}_M^{s,\sigma}(I)}, \tag{4.3}$$

$$\|\mathcal{I}_{t_0}^{+,m}(\langle\nabla\rangle_m^{-1}(\overline{\chi}\varphi))\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)} \leqslant C_0 \|\chi\|_{\mathbf{F}_M^{s,\sigma}(I)} \|\varphi\|_{\mathbf{F}_M^{s,\sigma}(I)}. \tag{4.4}$$

Proof. For details see Section 2, Lemma 2.1, and Theorem 3.2 in [8]. $\quad\square$

The first step in the proof of Theorem 4.1, is to prove the following local result.

Theorem 4.3. *Suppose that either $s > 0 = \sigma$ and $2M > m > 0$, or $\sigma > 0 = s$ and $M, m > 0$. There exist $\theta, c \in (0, 1)$ and $C > 1$, such that for any $A, B \geqslant 1$ and any $0 < \alpha \leqslant cA^{-1}$ and $0 < \beta \leqslant cB^{\frac{\theta-1}{\theta}}$, and for any interval $I = [t_1, t_2) \subset \mathbb{R}$ and $t_0 \in I$, if we have*

$$\|\varphi_0\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant \alpha, \qquad \|\chi_0\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant A,$$

and

$$\|\mathcal{U}_m^+(\cdot - t_0)\phi_0\|_{\mathbf{D}_\sigma^s(I)} \leqslant \beta, \qquad \|\phi_0\|_{H_\sigma^{\frac{1}{2}+s}(\mathbb{R}^3)} \leqslant B,$$

then there exists a unique solution $(\varphi, \chi, \phi_+) \in \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$ of (2.3) on $I \times \mathbb{R}^3$ with initial condition $(\varphi, \chi, \phi_+)(t_0) = (\varphi_0, \chi_0, \phi_0)$. Moreover the solution depends continuously on the initial data and satisfies the bounds

$$\sup_{t \in I} \|\varphi(t)\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant 2\|\varphi_0\|_{H_\sigma^s(\mathbb{R}^3)}, \qquad \sup_{t \in I} \|\chi(t)\|_{H_\sigma^s(\mathbb{R}^3)} \leqslant 2\|\chi_0\|_{H_\sigma^s(\mathbb{R}^3)},$$

$$\sup_{t \in I} \|\phi_+(t) - \mathcal{U}_m^+(t - t_0)\phi_0(t_0)\|_{H_\sigma^{\frac{1}{2}+s}(\mathbb{R}^3)} \leqslant C\|\varphi_0\|_{H_\sigma^s(\mathbb{R}^3)} \|\chi_0\|_{H_\sigma^s(\mathbb{R}^3)}.$$

Proof. For convenience, let $\varphi_L(t) = \mathcal{U}_M(t - t_0)\varphi_0$, $\chi_L(t) = \mathcal{U}_M(t - t_0)\chi_0$, and $\phi_{+,L}(t) = \mathcal{U}_m^+(t - t_0)\phi_0$. Let $C_0 \geqslant 1$ and $\theta \in (0, 1)$ be as in Lemma 4.2. Define S as the set of all $(\varphi, \chi, \phi_+) \in \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$ satisfying

$$\|\varphi - \varphi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \|\varphi_0\|_{H_\sigma^s}, \qquad \|\chi - \chi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \|\chi_0\|_{H_\sigma^s},$$

$$\|\phi_+ - \phi_{+,L}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)} \leqslant 2^3 C_0 \|\varphi_0\|_{H_\sigma^s} \|\chi_0\|_{H_\sigma^s}.$$

It is a complete metric space with respect to the norm

$$\|(\varphi, \chi, \phi_+)\|_S := \alpha^{-1} \|\varphi\|_{\mathbf{F}_M^{s,\sigma}(I)} + A^{-1} \|\chi\|_{\mathbf{F}_M^{s,\sigma}(I)} + \eta^{-1} \|\phi_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)},$$

where $\eta > 0$ will be chosen later. Let

$$\mathcal{T} = (T_1, T_2, T_3) : \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I) \to \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$$

be defined as

$$\mathcal{T}(\varphi, \chi, \phi_+) = \begin{pmatrix} \mathcal{U}_M(\cdot - t_0)\varphi_0 + \mathcal{I}_{t_0}^M(\mathrm{Re}(\phi_+)\varphi) \\ \mathcal{U}_M(\cdot - t_0)\chi_0 + \mathcal{I}_{t_0}^M(\mathrm{Re}(\phi_+)\chi) \\ \mathcal{U}_m^+(\cdot - t_0)\phi_{+,0} + \mathcal{I}_{t_0}^{+,m}(\langle\nabla\rangle_m^{-1}(\overline{\varphi}\chi + \overline{\chi}\varphi)) \end{pmatrix},$$

see Lemma 4.2. Fixed points of $\mathcal{T}$ are solutions of (2.3) with the given data at time t_0. For $(\varphi, \chi, \phi_+) \in S$ we infer that

$$\|\varphi\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \|\varphi - \varphi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} + \|\varphi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant 2\|\varphi_0\|_{H_\sigma^s} \leqslant 2\alpha,$$

$$\|\chi\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \|\chi - \chi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} + \|\chi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant 2\|\chi_0\|_{H_\sigma^s} \leqslant 2A,$$

and similarly,

$$\|\phi_{+,L}\|_{\mathbf{D}_\sigma^s(I)}^\theta \|\phi_{+,L}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}^{1-\theta} \leqslant \beta^\theta B^{1-\theta},$$

$$\|\phi_+ - \phi_{+,L}\|_{\mathbf{D}_\sigma^s(I)}^\theta \|\phi_+ - \phi_{+,L}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}^{1-\theta} \leqslant 2^3 C_0^{1+\theta} \|\varphi_0\|_{H_\sigma^s} \|\chi_0\|_{H_\sigma^s} \leqslant 2^3 C_0^2 \alpha A.$$

If $\alpha \leqslant (2^5 C_0^3 A)^{-1}$ and $\beta \leqslant (4C_0 B^{1-\theta})^{-\frac{1}{\theta}}$, Lemma 4.2 implies

$$\|T_1(\varphi, \chi, \phi_+) - \varphi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \left(2C_0\beta^\theta B^{1-\theta} + 2^4 C_0^3 \alpha A\right)\|\varphi_0\|_{H_\sigma^s} \leqslant \|\varphi_0\|_{H_\sigma^s}, \tag{4.5}$$

and

$$\|T_2(\varphi, \chi, \phi_+) - \chi_L\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \left(2C_0\beta^\theta B^{1-\theta} + 2^4 C_0^3 \alpha A\right)\|\chi_0\|_{H_\sigma^s} \leqslant \|\chi_0\|_{H_\sigma^s}, \tag{4.6}$$

as well as

$$\|T_3(\varphi, \chi, \phi_+) - \phi_{+,L}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)} \leqslant 2^3 C_0 \|\varphi_0\|_{H_\sigma^s} \|\chi_0\|_{H_\sigma^s}. \tag{4.7}$$

We will now show that $\mathcal{T} : S \to S$ is a contraction, provided that α, β are chosen small enough. Let $(\varphi, \chi, \phi_+) \in S$ and $(\tilde{\varphi}, \tilde{\chi}, \tilde{\phi}_+) \in S$. Then, by Lemma 4.2,

$$\|T_1(\varphi, \chi, \phi_+) - T_1(\tilde{\varphi}, \tilde{\chi}, \tilde{\phi}_+)\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \left(C_0\beta^\theta B^{1-\theta} + 2^3 C_0^3 \alpha A\right)\|\varphi - \tilde{\varphi}\|_{\mathbf{F}_M^{s,\sigma}(I)}$$
$$+ 2C_0^2 \alpha \|\phi_+ - \tilde{\phi}_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)},$$

and

$$\|T_2(\varphi, \chi, \phi_+) - T_2(\tilde{\varphi}, \tilde{\chi}, \tilde{\phi}_+)\|_{\mathbf{F}_M^{s,\sigma}(I)} \leqslant \left(C_0\beta^\theta B^{1-\theta} + 2^3 C_0^3 \alpha A\right)\|\chi - \tilde{\chi}\|_{\mathbf{F}_M^{s,\sigma}(I)}$$
$$+ 2C_0^2 A \|\phi_+ - \tilde{\phi}_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)},$$

as well as

$$\|T_3(\varphi, \chi, \phi_+) - T_3(\tilde{\varphi}, \tilde{\chi}, \tilde{\phi}_+)\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)} \leqslant 2^2 C_0 \alpha \|\chi - \tilde{\chi}\|_{\mathbf{F}_M^{s,\sigma}(I)} + 2^2 C_0 A \|\varphi - \tilde{\varphi}\|_{\mathbf{F}_M^{s,\sigma}(I)}.$$

We obtain

$$\|\mathcal{T}(\varphi, \chi, \phi_+) - \mathcal{T}(\tilde{\varphi}, \tilde{\chi}, \tilde{\phi}_+)\|_S \leqslant 4C_0^2 \eta\eta^{-1} \|\phi_+ - \tilde{\phi}_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)}$$
$$+ \left(C_0\beta^\theta B^{1-\theta} + 2^3 C_0^3 \alpha A + 2^2 C_0 A \alpha\eta^{-1}\right)\alpha^{-1} \|\varphi - \tilde{\varphi}\|_{\mathbf{F}_M^{s,\sigma}(I)}$$
$$+ \left(C_0\beta^\theta B^{1-\theta} + 2^3 C_0^3 \alpha A + 2^2 C_0 A \alpha\eta^{-1}\right)A^{-1} \|\chi - \tilde{\chi}\|_{\mathbf{F}_M^{s,\sigma}(I)}.$$

By fixing $\eta = (2^4 C_0^2)^{-1}$, and choosing $\alpha \leqslant (2^{12} C_0^3 A)^{-1}$ and $\beta \leqslant (2^4 C_0 B^{1-\theta})^{-\frac{1}{\theta}}$, we have verified that $\mathcal{T} : S \to S$ is a contraction, hence it has a fixed point $(\varphi, \chi, \phi_+) \in S$ which is unique in S. For later purposes we note that we have chosen the thresholds for α and β small enough such that the same conclusion holds if α, A, and B are doubled. Similar estimates show that the fixed point depends continuously on the initial data. Due to (4.1), the claimed estimates on the Sobolev norms for $(\varphi(t), \chi(t), \phi_+(t))$ for $t \in I$ follow from (4.5), (4.6) and (4.7).

Finally, we prove uniqueness. Assume that $(\varphi', \chi', \phi'_+) \in \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{F}_M^{s,\sigma}(I) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)$ is another solution with the same data at t_0 such that

$$t' := \sup\{t \in I \mid (\varphi', \chi', \phi'_+)(t) = (\varphi, \chi, \phi_+)(t)\} < t_2.$$

Then,

$$\|\varphi'(t')\|_{H_\sigma^s} \leqslant 2\alpha, \qquad \|\chi'(t')\|_{H_\sigma^s} \leqslant 2A, \qquad \|\phi'_+(t')\|_{H_\sigma^{\frac{1}{2}+s}} \leqslant 2B.$$

Let $\|\phi'_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I)} \leq R$. By Lemma 4.2 we have

$$\|\phi'_+\|_{\mathbf{D}_\sigma^s(I')} \leqslant C_0 \|\phi'_+\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I')} \leqslant C_0^2 R$$

for any $I' \subseteq I$. For $\varepsilon \in (0, \beta)$ (which will be specified below), let $\delta > 0$ be small enough such that $I' := [t', t' + \delta) \subset I$ and $\|\phi'_+\|_{\mathbf{D}_\sigma^s(I')} \leqslant \varepsilon$. Let $\varphi'_L(t) := \mathcal{U}_M(t - t')\varphi(t')$, $\chi'_L(t) := \mathcal{U}_M(t - t')\chi(t')$, and $\phi'_{+,L}(t) := \mathcal{U}_m^+(t - t')\phi_+(t')$. Then,

$$\|\varphi' - \varphi'_L\|_{\mathbf{F}_M^{s,\sigma}(I')} \leqslant C_0 \varepsilon^\theta R^{1-\theta} \left(\|\varphi' - \varphi'_L\|_{\mathbf{F}_M^{s,\sigma}(I')} + \|\varphi'_L\|_{\mathbf{F}_M^{s,\sigma}(I')} \right),$$

so that if we fix some $\varepsilon \leqslant (2C_0 R^{1-\theta})^{-\frac{1}{\theta}}$, we obtain

$$\|\varphi' - \varphi'_L\|_{\mathbf{F}_M^{s,\sigma}(I')} \leqslant \|\varphi(t')\|_{H_\sigma^s}.$$

A similar estimate shows

$$\|\chi' - \chi'_L\|_{\mathbf{F}_M^{s,\sigma}(I')} \leqslant \|\chi(t')\|_{H_\sigma^s}.$$

Then,

$$\|\phi'_+ - \phi'_{+,L}\|_{\mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I')} \leqslant 2^3 C_0 \|\varphi(t')\|_{H_\sigma^s} \|\chi(t')\|_{H_\sigma^s}.$$

These estimates show that $(\varphi', \chi', \phi'_+)$ is contained in the set S defined as above, but with the modified initial condition at t' instead of t_0 and the interval I' instead of I. Also, the estimates with I replaced by I' in the first part of the proof imply that $(\varphi, \chi, \phi_+)|_{I'}$ is contained in this version of the set S. The uniqueness within S proven above implies that $(\varphi', \chi', \phi'_+) = (\varphi, \chi, \phi_+)$ in I', which contradicts the definition of t'. $\square$

We can now prove Theorem 4.1 as follows. By our hypothesis, the initial data at time 0 satisfy

$$\|\varphi_0\|_{H_\sigma^s} \leqslant \epsilon, \ \|\chi_0\|_{H_\sigma^s} \leqslant A, \ \|\phi_0\|_{H_\sigma^{\frac{1}{2}+s}} \leqslant A,$$

and $\epsilon > 0$ is chosen small enough, depending on A only (the precise threshold will be specified below). Let $\beta^*(B) = cB^{\frac{\theta-1}{\theta}}$ and $\alpha^*(A) = cA^{-1}$ be the thresholds as in Theorem 4.3. Then, by the Strichartz estimate from Lemma 4.2 (v), we have

$$\|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s(\mathbb{R}_+)} \leqslant C_0 A$$

with $C_0 \geqslant 1$. By monotone convergence, the function $T \mapsto \|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s([T_0, T))}$ is continuous in T and converges to zero as $T \searrow T_0$. Therefore, for $\beta := \beta^*(2A)$, we can choose $0 = s_0 < s_1 < \ldots < s_N$ such that

$$\|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s([s_{n-1}, s_n))} = \beta/4 \text{ and } \|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s([s_n, \infty))} \leqslant \beta/4.$$

With $s_{N+1} = \infty$, define the collection of intervals $I_n = [s_{n-1}, s_{n+1})$ for $n = 1, ..., N$. Then,

$$\beta/4 \leqslant \|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s(I_n)} \leqslant \beta/2$$

and, by Minkowski's inequality,

$$\sum_{n=1}^N \|\mathcal{U}_m^+(t)\phi_0\|_{\mathbf{D}_\sigma^s(I_n)}^4 \leqslant 2(C_0 A)^4,$$

therefore $N \leqslant N_0 := 2^6 (C_0 A)^4 \beta^{-4}$.

Now, fix $\epsilon \leqslant cC^{-1}C_0^{-1}2^{-2N_0}A^{-1}\beta$. We claim that for every $1 \leqslant n \leqslant N$, on I_n we have a unique solution $(\varphi^{(n)}, \chi^{(n)}, \phi_+^{(n)}) \in \mathbf{F}_M^{s,\sigma}(I_n) \times \mathbf{F}_M^{s,\sigma}(I_n) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}(I_n)$ with initial condition

$$(\varphi^{(n)}, \chi^{(n)}, \phi_+^{(n)})(s_{n-1}) = (\varphi^{(n-1)}, \chi^{(n-1)}, \phi_+^{(n-1)})(s_{n-1}) \quad (\text{if } 2 \leqslant n \leqslant N)$$
$$(\varphi^{(1)}, \chi^{(1)}, \phi_+^{(1)})(s_0) = (\varphi_0, \chi_0, \phi_0) \quad (\text{if } n = 1)$$

which satisfies the bounds

$$\|\mathcal{U}_m^+(\cdot - s_{n-1})\phi_+^{(n-1)}(s_{n-1})\|_{\mathbf{D}_\sigma^s(I_n)} \leqslant \beta,$$
$$\|\varphi^{(n)}(s_n)\|_{H_\sigma^s} \leqslant 2^n \epsilon, \qquad \|\chi^{(n)}(s_n)\|_{H_\sigma^s} \leqslant 2^n A, \tag{4.8}$$
$$\|\phi_+^{(n)}(s_n) - \mathcal{U}_m^+(s_n)\phi_0\|_{H_\sigma^{\frac{1}{2}+s}} \leqslant C 2^{2n} \epsilon A,$$

where C is the constant from Theorem 4.3. Indeed, for $n = 1$ the estimate in the first line follows by definition of I_1, and the estimates in the second and third line follow from an application of Theorem 4.3 (with $t_0 = 0$), where we use that $\epsilon \leqslant \alpha^*(A)$ and $\beta \leqslant \beta^*(A)$. As an induction hypothesis, let us suppose that holds (4.8) for some $1 \leqslant n \leqslant N - 1$.

By Lemma 4.2, the induction hypothesis, and the choice of ϵ we have

$$\begin{aligned}
\|\mathcal{U}_m^+(\cdot - s_n)\phi_+^{(n)}(s_n)\|_{\mathbf{D}_\sigma^s(I_{n+1})} &\leqslant \|\mathcal{U}_m^+\phi_0\|_{\mathbf{D}_\sigma^s(I_{n+1})} \\
&\quad + \|\mathcal{U}_m^+(\phi_0 - \mathcal{U}_m^+(-s_n)\phi_+^{(n)}(s_n))\|_{\mathbf{D}_\sigma^s(I_{n+1})} \\
&\leqslant \beta/2 + C_0 \|\phi_0 - \mathcal{U}_m^+(-s_n)\phi_+^{(n)}(s_n)\|_{H_\sigma^{\frac{1}{2}+s}} \\
&\leqslant \beta/2 + C C_0 2^{2n} \epsilon A \leqslant \beta.
\end{aligned}$$

From the estimate in the third line of the induction hypothesis and the smallness condition on ϵ we obtain

$$\|\phi_+^{(n)}(s_n)\|_{H_\sigma^{\frac{1}{2}+s}} \leqslant \|\mathcal{U}_m^+(s_n)\phi_0\|_{H_\sigma^{\frac{1}{2}+s}} + C 2^{2n} \epsilon A \leqslant A + C 2^{2n} \epsilon A \leqslant 2A.$$

Notice that due to our choices we have $\beta \leqslant \beta^*(2A)$ and $2^n \epsilon \leqslant \alpha^*(2^n A)$. Then, as $s_{n+1} \in I_{n+1}$, we obtain from Theorem 4.3 (with $t_0 = s_n$) that

$$\|\varphi^{(n+1)}(s_{n+1})\|_{H_\sigma^s} \leqslant 2\|\varphi^{(n)}(s_n)\|_{H_\sigma^s} \leqslant 2^{n+1} \epsilon,$$
$$\|\chi^{(n+1)}(s_{n+1})\|_{H_\sigma^s} \leqslant 2\|\chi^{(n)}(s_n)\|_{H_\sigma^s} \leqslant 2^{n+1} A,$$

and, using the induction hypothesis again,

$$\begin{aligned}
&\|\phi_+^{(n+1)}(s_{n+1}) - \mathcal{U}_m^+(s_{n+1})\phi_0\|_{H_\sigma^{\frac{1}{2}+s}} \\
&\leqslant \|\phi_+^{(n+1)}(s_{n+1}) - \mathcal{U}_m^+(s_{n+1} - s_n)\phi_+^{(n)}(s_n)\|_{H_\sigma^{\frac{1}{2}+s}} \\
&\quad + \|\mathcal{U}_m^+(s_{n+1} - s_n)\phi_+^{(n)}(s_n) - \mathcal{U}_m^+(s_{n+1})\phi_0\|_{H_\sigma^{\frac{1}{2}+s}} \\
&\leqslant C 2^{2n} \epsilon A + C 2^{2n} \epsilon A \leqslant C 2^{2(n+1)} \epsilon A.
\end{aligned}$$

The proof of the claim is complete.

By uniqueness, we have constructed a global solution

$$(\varphi, \chi, \phi_+) \in C_b(\mathbb{R}_+, H_\sigma^s) \times C_b(\mathbb{R}_+, H_\sigma^s) \times C_b(\mathbb{R}_+, H_\sigma^{\frac{1}{2}+s}),$$

and due to $(\varphi, \chi, \phi_+)|_{[s_N,\infty)} \in \mathbf{F}_M^{s,\sigma}([s_N,\infty)) \times \mathbf{F}_M^{s,\sigma}([s_N,\infty)) \times \mathbf{V}_{+,m}^{s+\frac{1}{2},\sigma}([s_N,\infty))$ it scatters as $t \to \infty$, see Lemma 4.2 Part (iv). The claim for $t \to -\infty$ follows by time reversibility. Continuous dependence also follows from the local result, we omit the details. This completes the proof of Theorem 4.1.

Conflict of interest statement

There is no conflict of interests.

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