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Analysis of the adiabatic piston problem via methods of continuum mechanics

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Abstract

We consider a system modelling the motion of a piston in a cylinder filled by a viscous heat conducting gas. The piston is moving longitudinally without friction under the influence of the forces exerted by the gas. In addition, the piston is supposed to be thermally insulating (adiabatic piston). This fact raises several challenges which received a considerable attention, essentially in the statistical physics literature. We study the problem via the methods of continuum mechanics, specifically, the motion of the gas is described by means of the Navier–Stokes–Fourier system in one space dimension, coupled with Newton's second law governing the motion of the piston. We establish global in time existence of strong solutions and show that the system stabilizes to an equilibrium state for $t \to \infty$.

Keywords: Piston problem; Navier–Stokes–Fourier system; Free boundary problem

1. Introduction and statement of the main results

The adiabatic piston problem received a considerable attention, namely in the statistical physics oriented literature, during the last two decades (see, for instance, Lieb [15], Gruber et al. [10,9,11], Neishtadt and Sinai [19], Wright [25–27] and the references therein). This problem consists in studying the dynamics of a system composed of a gas in a cylindrical container with a piston which can move freely in the longitudinal direction. The piston and the exterior walls are supposed to be thermally insulating. Most of the above results are obtained using discrete models for the gas, meaning, the gas is supposed to be formed by a finite number of particles. Gruber [9] uses the kinetic approach, where the gas is modelled by a Boltzmann type equation. A fundamental question raised and discussed in these references is the large time behavior of the trajectories of the associated dynamical system. Making various assumptions, namely that the number of gas particles on each side of the piston is finite, that these particles make

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purely elastic collisions on the boundaries of the cylinder and on the piston, and that the mass of the piston is much larger than the mass of the gas, the results in [11], [25] or [26] assert that the system evolves in at least two stages. First, the system relaxes to a state where the forces exerted by the gas on each side of the piston are equal, but with possibly different gas temperatures on each side. In the second, much longer stage, the system moves to a state where the temperatures of the gas on the two sides of the piston equilibrate. Thus, in a certain sense, the piston is no longer adiabatic.

To the best of our knowledge, the problem has never been analyzed in the framework of continuum mechanics. The latter describes the state of the system by means of the macroscopic observable variables: The mass density ρ, the absolute temperature θ, the fluid velocity u, and the piston position h. Given the geometry with dominating longitudinal coordinate, the restriction to only one space variable seems appropriate. Accordingly, the trajectory of the piston being described by the curve

$$\gamma_h = \{(t, h(t)) \in [0, T] \times (-1, 1) \ | \ t \in [0, T]\}, \tag{1.1}$$

the time evolution of the state variables is governed by the Navier–Stokes–Fourier system of equations

$$\partial_t \rho + \partial_x (\rho u) = 0 \tag{1.2}$$

$$\partial_t (\rho u) + \partial_x (\rho u^2) + \partial_x p = \mu \partial_{x,x} u \tag{1.3}$$

$$c_v \left(\partial_t (\rho\theta) + \partial_x (\rho\theta u)\right) - \partial_x \left(\kappa \partial_x \theta\right) = \mu |\partial_x u|^2 - p \partial_x u. \tag{1.4}$$

satisfied in the open set

$$\begin{aligned} Q_T \subset (0, T) \times (-1, 1),\ \ Q_T &= Q_T^L \cup Q_T^R, \\ Q_L &= \left\{(t, x) \ \middle| \ t \in (0, T),\ x \in (-1, h(t))\right\}, \\ Q_R &= \left\{(t, x) \ \middle| \ t \in (0, T),\ x \in (h(t), 1)\right\}. \end{aligned}$$

Here, $\mu > 0$, $c_v > 0$, and $\kappa > 0$ stand for the gas viscosity, the specific heat at constant volume, and the heat conductivity, respectively; the pressure p is given by the ideal gas law.

$$p = p(\rho, \theta) = \rho\theta. \tag{1.5}$$

The dynamics of the piston is coupled to the gas motion via two equations. The first one asserts that the velocity of the piston coincides with the gas velocities on both sides of the piston and it can be written

$$\frac{\mathrm{d}h}{\mathrm{d}t}(t) = u(t, h(t)^-) = u(t, h(t)^+). \tag{1.6}$$

The second condition comes from Newton's second law and the assumption that the only forces acting on the piston are those exerted by the gas. The corresponding equation is

$$M \frac{\mathrm{d}^2 h}{\mathrm{d}t^2}(t) = [\ \mu \partial_x u - p\ ]_{x \to h(t)-}^{x \to h(t)+}, \tag{1.7}$$

where $M > 0$ is the mass of the piston.

We close the system by prescribing the remaining boundary conditions at the extremal points $x = \pm 1$, specifically,

$$\partial_x \theta(t, -1) = \partial_x \theta(t, h(t)^-) = \partial_x \theta(t, h(t)^+) = \partial_x \theta(t, 1) = 0, \tag{1.8}$$

and

$$u(t, -1) = u(t, 1) = 0, \tag{1.9}$$

meaning the piston is thermally insulating, and the whole system is both mechanically and thermally closed.

Note that the *complementary* problem involving a perfectly thermally conducting piston has been tackled in Shelukhin [21]. The main results in [21] assert that the system admits global in time strong solutions and that the state trajectories converge towards an equilibrium where the densities and the temperatures have constant values which are the same on both sides of the piston. Strangely enough, this result coincides with the scenario expected by statistical mechanics for the *thermally insulating* piston. In contrast with the prognosis of statistical mechanics, the dynamics of the continuous model with a thermally insulating piston features rather complex behavior that may depend sensitively on the initial state. As we show below, problem (1.2)–(1.9) admits a continuum of equilibria including those for which the ultimate temperature attained can be different on the two sides of the piston. Although each individual trajectory approaches an equilibrium state in the asymptotic limit $t \to \infty$, the latter is basically unpredictable in terms of the initial state.

Note also that simplified versions of these models, where the compressible Navier–Stokes equations were replaced by the one-dimensional viscous Burgers equation has been studied in Vázquez and Zuazua [23] and Cîndea et al. [4]. To complete the above list of quotations, we mention that the free piston problem, without considering thermal effects, has been studied in Shelukhin [20] for viscous gases and Liu [17] in the non viscous case.

We end this introduction by briefly describing the strategy adopted to prove our main results.

- To begin, we establish existence and uniqueness of strong solutions defined globally in time. In contrast with the situation treated by Shelukhin [21], we consider a larger class of initial data giving rise to the strong rather than classical solutions obtained in [21]. Because of technical difficulties connected with the no-heat flux condition imposed on the piston, we introduce an additional hypothesis that the heat conductivity coefficient κ is a coercive function of the temperature, specifically,
$$0 < \underline{\kappa}(1+\theta^\alpha) \leqslant \kappa(\theta) \leqslant \overline{\kappa}(1+\theta^\alpha), \tag{1.10}$$
where $\underline{\kappa}$, $\overline{\kappa}$ are strictly positive constants and $\alpha \geqslant 2$. As already mentioned, the above condition on α is essentially of technical nature. Its use is physically justified in certain high temperature regimes, see, for instance, Zel'dovich and Raizer [28, Ch. 3].
- In accordance with the Second law of thermodynamics, the equilibrium solutions minimizing the entropy production rate represent the terminal states for $t \to \infty$. We show that they are uniquely determined by the total energy of the system, the mass of the gas in the domain left and right to the piston, and by the limit entropy attained for $t \to \infty$.
- Finally, we use the relative energy functional introduced in [7] to show that each individual trajectory approaches an equilibrium solution for $t \to \infty$.

The paper is organized as follows. In Section 2, we introduce the preliminary material and rigorously state our main results. Section 3 addresses the problem of existence and uniqueness of global-in-time solutions. The convergence to equilibrium solutions is shown in Section 4.

2. Main results

In this section, we introduce the necessary definitions and state our main results on the existence of global-in-time solutions and their asymptotic behavior for large times.

2.1. Existence of global-in-time strong solutions

We start by a definition of (strong) solutions to problem (1.2)–(1.9) supplemented with the initial data
$$\rho(0,\cdot) = \rho_0,\ \theta(0,\cdot) = \theta_0,\ u(0,\cdot) = u_0,\ h(0) = h_0. \tag{2.1}$$

Definition 2.1. We say that ρ, θ, u, h is a (strong) solution to problem (1.2)–(1.9) on the time interval $(0, T)$ with the initial data (2.1) if:

- $h : [0, T] \to (-1, 1)$ is a Lipschitz continuous function;
- the functions ρ, θ, u, along with their distributional derivatives $\partial_t \rho, \partial_t \theta, \partial_t u, \partial_x \rho, \partial_x \theta, \partial_x u$, and $\partial_x^2 \theta, \partial_x^2 u$ belong to $L^2(Q_T^L) \cap L^2(Q_T^R)$;
- $\rho > 0, \theta > 0$ in $\overline{Q_T^L} \cup \overline{Q_T^R}$;
- equations (1.2)–(1.4) are satisfied[1] a.e. in Q_T^L and Q_T^R;
- the boundary conditions (1.6)–(1.9) hold for a.e. $t \in (0, T)$;
- the initial conditions (2.1) are satisfied.

In view of the embedding theorems for anisotropic Sobolev spaces, see Besov, Iljin, Nikolskij [2, Chapter III, Theorem 10.4], the functions θ, u are Hölder continuous on the sets $\overline{Q_T^L}$ and $\overline{Q_T^R}$, in particular, their boundary and initial traces are well defined. Similarly, the density ρ, being determined by the transport equation (1.2), satisfies

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\partial_x \rho\|_{L^2(-1,h(t))} + \operatorname*{ess\ sup}_{t\in(0,T)} \|\partial_x \rho\|_{L^2(h(t),1)} < \infty,$$

therefore, by [12, Lemma 2.2], ρ is Hölder continuous on the sets $\overline{Q_T^L}$, $\overline{Q_T^R}$. Finally, we have

$$\partial_x u(t, h(t)^-),\ \partial_x u(t, h(t)^+),\ \partial_x \theta(t, h(t)^-),\ \partial_x \theta(t, h(t)^+) \in L^2(0, T);$$

whence the free boundary conditions (1.7), (1.8) are well defined a.e. in $(0, T)$.

We claim the following existence result.

Theorem 2.2. *Let the initial data*

$$h_0 \in (-1, 1),$$
$$\rho_0 \in W^{1,2}(-1, h_0) \cup W^{1,2}(h_0, 1),\ \ \rho_0 > 0,$$
$$\theta_0 \in W^{1,2}(-1, h_0) \cup W^{1,2}(h_0, 1),\ \ \theta_0 > 0,$$
$$u_0 \in W_0^{1,2}(-1, 1)$$

be given. Let $\kappa \in C^2[0, \infty)$ satisfy hypothesis (1.10) with $\alpha \geqslant 2$.

Then, for every $T > 0$, the problem (1.2)–(1.9) admits a strong solution ρ, θ, u, h in $(0, T)$, unique in the class specified in Definition 2.1.

Theorem 2.2 is proved in Section 3 using the description of the problem in Lagrangian mass coordinates.

2.2. *Entropy and equilibrium solutions*

Dividing the thermal energy balance by θ we obtain the entropy balance equation

$$\partial_t(\rho s(\rho, \theta)) + \partial_x(\rho s(\rho, \theta)u) - \partial_x \left(\frac{\kappa(\theta)}{\theta} \partial_x \theta \right) = \frac{1}{\theta} \left(\mu |\partial_x u|^2 + \frac{\kappa(\theta)|\partial_x \theta|^2}{\theta} \right) \tag{2.2}$$

satisfied in Q_T^L and Q_T^R, where the specific entropy is given as

$$s(\rho, \theta) = \log \left(\frac{\theta^{c_v}}{\rho} \right) \qquad\qquad (\rho,\ \theta > 0). \tag{2.3}$$

Equilibrium solutions $[\rho_\infty, \theta_\infty, u_\infty, h_\infty]$ are those with vanishing entropy production rate. Accordingly, we get

$\partial_x u_\infty = 0$ yielding, in view of (1.9), $u_\infty \equiv 0$,

[1] Hereinafter, ‘a.e.’ stands for ‘almost everywhere’ or ‘almost every’.

and $h_\infty = \text{const}$. By the same token

$$\partial_x \theta_\infty = 0 \text{ in } (-1, h_\infty) \cup (h_\infty, 1);$$

whence

$$\theta_\infty = \begin{cases} \theta_L = \theta_L(t) \text{ for } x \in (-1, h_\infty), \\ \theta_L = \theta_R(t) \text{ for } x \in (h_\infty, 1) \end{cases}$$

Moreover, plugging $u = u_\infty = 0$ in the field equations (1.2), (1.3) we get

$$\partial_t \rho_\infty = 0, \partial_x p = \partial_x(\rho_\infty \theta_\infty) = \theta_\infty \partial_x \rho_\infty = 0$$

yielding

$$\rho_\infty = \begin{cases} \rho_L \text{ for } x \in (-1, h_\infty), \\ \rho_R \text{ for } x \in (h_\infty, 1) \end{cases},$$

where ρ_L, ρ_R are positive constants. Similarly, we deduce from (1.4) that $\partial_t \theta_\infty = 0$ concluding that θ_L, θ_R are positive constants independent of time.

Finally, we observe that the free boundary constraint (1.7) enforces continuity of the pressure therefore

$$\rho_L \theta_L = \rho_R \theta_R. \tag{2.4}$$

Obviously, any choice of (positive) ρ_L, ρ_R, θ_L, θ_R, h_∞ satisfying the compatibility condition (2.4) give rise to a strong (equilibrium) solution of problem (1.2)–(1.9). In particular, the states $[\rho_L, \theta_L]$, $[\rho_R, \theta_R]$ may be different.

2.3. Long-time behavior

The long time behavior of solutions to problem (1.2)–(1.9) is determined by several flow invariants. We start by recalling a version of transport theorem

$$\begin{aligned} &\int_{Q_\tau^L} [\partial_t(\rho D) + \partial_x(\rho D u)] \,\mathrm{d}x \,\mathrm{d}t = \int_{-1}^{h(\tau)} \rho D(\tau, \cdot) \,\mathrm{d}x - \int_{-1}^{h(0)} \rho D(0, \cdot) \,\mathrm{d}x \\ &\int_{Q_\tau^R} [\partial_t(\rho D) + \partial_x(\rho D u)] \,\mathrm{d}x \,\mathrm{d}t = \int_{h(\tau)}^{1} \rho D(\tau, \cdot) \,\mathrm{d}x - \int_{h(0)}^{1} \rho D(0, \cdot) \,\mathrm{d}x, \ 0 \le \tau \le T, \end{aligned} \tag{2.5}$$

which can be easily verified by means of the Sobolev version of Gauss–Green theorem (see e.g. Chen, Torres, Ziemer [3]) for any solution ρ, u of problem (1.2)–(1.9) belonging to the regularity class specified in Definition 2.1 and any

$$D \in W^{1,2}(Q_T^L) \cup W^{1,2}(Q_T^R).$$

2.3.1. Mass conservation

Applying (2.5) to $D = 1$ we get

$$m_L = \int_{-1}^{h(\tau)} \rho \,\mathrm{d}x = \int_{-1}^{h_0} \rho_0 \,\mathrm{d}x, \ m_R = \int_{h(\tau)}^{1} \rho \,\mathrm{d}x = \int_{h_0}^{1} \rho_0 \,\mathrm{d}x \text{ for any } \tau \ge 0, \tag{2.6}$$

which can be seen as a mathematical formulation of the conservation of mass.

2.3.2. Energy conservation

Multiplying momentum equation (1.3) by u and taking the sum with (1.4) we deduce the energy balance

$$\partial_t \left(\rho \left[\frac{1}{2} u^2 + c_v \theta \right] \right) + \partial_x \left(\rho \left[\frac{1}{2} u^2 + c_v \theta \right] u \right) = -\partial_x (\rho\theta u) + \mu \partial_x (\partial_x u u) + \partial_x (\kappa(\theta)\theta_x) \tag{2.7}$$

satisfied in each set Q_T^L, Q_T^R. Consequently, integrating (2.7) over Q_τ^L, Q_τ^R, applying (2.5) with

$$D = \left[\frac{1}{2} u^2 + c_v \theta \right],$$

and using (1.6), (1.7), (1.8), we deduce that

$$\int_{-1}^{1} \left[\frac{1}{2} \rho u^2 + c_v \rho \theta \right] (\tau, \cdot) \, \mathrm{d}x + \frac{M}{2} \left| \frac{\mathrm{d}}{\mathrm{d}t} h(\tau) \right|^2 = \int_{-1}^{1} \left[\frac{1}{2} \rho_0 u_0^2 + c_v \rho_0 \theta_0 \right] \mathrm{d}x + \frac{M}{2} |u_0(h_0)|^2 \equiv E_0 \tag{2.8}$$

meaning the total energy of the system is a constant of motion. Note that (2.8) holds for any solution belonging to the regularity class in Definition 2.1.

2.4. Entropy production

Applying (2.5) with $D = s(\rho, \theta)$ to (2.2) and using (1.8) we finally observe that the functions

$$\tau \mapsto \int_{-1}^{h(\tau)} \rho s(\rho, \theta) \, \mathrm{d}x, \ \tau \mapsto \int_{h(\tau)}^{1} \rho s(\rho, \theta) \, \mathrm{d}x$$

are non-decreasing for $\tau \geqslant 0$. Further, as $-\rho \log \rho \le e^{-1}$, $\rho \log \theta \le \rho\theta$ and the total energy of the system remains bounded by (2.8), it is easy to check that these functions admit an upper bound and consequently

$$\int_{-1}^{h(\tau)} \rho s(\rho, \theta) \, \mathrm{d}x \to S_L, \ \int_{h(\tau)}^{1} \rho s(\rho, \theta) \, \mathrm{d}x \to S_R \text{ as } \tau \to \infty. \tag{2.9}$$

2.5. Stabilization to equilibria

We show that the long time behavior of solutions to problem (1.2)–(1.9) is completely determined by the constants m_L, m_R, S_L, S_R identified in (2.6), (2.9) and the total energy of the system E_0. Note that, unlike m_L, m_R, E_0 that are fixed by the choice of the initial data, the asymptotic values of the entropy S_L, S_R are *a priori* unknown parameters.

The rather obvious idea, justified by rigorous arguments in Section 4, asserts that:

- Solution trajectories stabilize to equilibrium solutions.
- The limit equilibrium is uniquely identified by m_L, m_R, S_L, S_R and E_0.

Theorem 2.3. *Under the hypotheses of Theorem 2.2, let ρ, θ, u, h be a global-in-time (defined on $[0, T]$ for any $T > 0$) solution to problem (1.2)–(1.8) emanating from the initial data (2.1).*

Then

$$h(t) \to h_\infty \in (-1,1) \tag{2.10}$$

$$\rho|u|^2(t,\cdot) \to 0 \text{ in } L^1(-1,1) \tag{2.11}$$

$$\rho(t,\cdot) \to \rho_\infty \text{ in } L^q(-1,1) \text{ for any } 1 \le q < \infty \tag{2.12}$$

$$\theta(t,\cdot) \to \theta_\infty \text{ in } L^1(-1,1) \tag{2.13}$$

as $t \to \infty$, where

$$\rho_\infty = \begin{cases} \rho_L \text{ for } x \in (-1, h_\infty) \\ \rho_R \text{ for } x \in (h_\infty, 1), \end{cases}$$

$$\theta_\infty = \begin{cases} \theta_L \text{ for } x \in (-1, h_\infty) \\ \theta_R \text{ for } x \in (h_\infty, 1), \end{cases}$$

ρ_L, ρ_R, θ_L, θ_R positive constant satisfying

$$\begin{aligned} \rho_L\theta_L = \rho_R\theta_R &= \frac{1}{2c_v}E_0, \\ (1+h_\infty)\rho_L = m_L,\ (1-h_\infty)\rho_R &= m_R, \\ (1+h_\infty)\rho_L s(\rho_L,\theta_L) &= S_L, \\ (1-h_\infty)\rho_R s(\rho_R,\theta_R) &= S_R, \end{aligned} \tag{2.14}$$

with the quantities m_L, m_R, E_0, and S_L, S_R identified through (2.6), (2.8), and (2.9), respectively.

Theorem 2.3 will be proved in Section 4. It is easy to see that the equilibrium solution h_∞, ρ_∞, θ_∞ is uniquely determined by the constants m_L, m_R, E_0, S_L, S_R. Indeed we deduce that

$$\rho_L s(\rho_L,\theta_L) = \frac{S_L}{h_\infty + 1},\ \ \rho_R s(\rho_R,\theta_R) = \frac{S_R}{1-h_\infty}, \tag{2.15}$$

and, furthermore,

$$\begin{aligned} \log\left(c_v\frac{\theta_L^{c_v+1}}{E_0}\right) = \log\left(\frac{\theta_L^{c_v}}{\rho_L}\right) = s(\rho_L,\theta_L) = \frac{S_L}{m_L}, \\ \log\left(c_v\frac{\theta_R^{c_v+1}}{E_0}\right) = \log\left(\frac{\theta_R^{c_v}}{\rho_R}\right) = s(\rho_R,\theta_R) = \frac{S_R}{m_R}. \end{aligned} \tag{2.16}$$

Finally, it is also easy to check that the equilibrium state with $\rho_L = \rho_R$, $\theta_L = \theta_R$ actually *maximizes* the limit entropy of the system

$$\int_{-1}^{1} \rho_\infty s(\rho_\infty,\theta_\infty)\ \mathrm{d}x$$

under the given constraints m_L, m_R and E_0. This can be seen as a kind of "stability" of this state reflected by the predictions made by the statistical approach.

3. Global-in-time existence

3.1. Result statement using Lagrangian mass coordinates

Our goal in this section is to prove Theorem 2.2. Following the standard approach (see e.g. Kazhikhov and Shelukhin [14]) we consider the problem in Lagrangian mass coordinates, choosing $h(t)$, supposed to lie in $(-1,1)$ (see (1.1)), as origin for the space variable. More precisely, we set

$$y=\Psi(t,x), \qquad \Psi(t,x)=\int_{h(t)}^{x}\rho(t,\xi)\,\mathrm{d}\xi \qquad (t\geqslant 0,\ x\in[-1,1]). \tag{3.1}$$

Accordingly,

$$\Psi(t,-1)=-r_1, \qquad \Psi(t,1)=r_2, \qquad (t\geqslant 0), \tag{3.2}$$

where

$$r_1=m_L=\int_{-1}^{h_0}\rho_0(x)\,\mathrm{d}x, \qquad r_2=m_R=\int_{h_0}^{1}\rho_0(x)\,\mathrm{d}x.$$

More precisely, for every $t\geqslant 0$, $\Psi(t,\cdot)$ is one to one from $[-1,h(t)]$ onto $[-r_1,0]$ and from $[h(t),1]$ onto $[0,r_2]$. It follows that the problem in Lagrangian mass coordinates is considered on a fixed domain. For each $t\geqslant 0$ we denote by $\Psi^{-1}(t,\cdot)$ the inverse map of $\Psi(t,\cdot)$.

The *specific volume in Lagrangian mass coordinates* is defined by

$$\begin{cases}\tilde v(t,y)=\frac{1}{\rho(t,\Psi^{-1}(t,y))},\\ \rho(t,x)=\frac{1}{\tilde v(t,\Psi((t,x))}, \end{cases} \qquad (t\geqslant 0,\ y\in[-r_1,r_2]\setminus\{0\},\ x\in[-1,1]\setminus\{h(t)\}). \tag{3.3}$$

Similarly, the *velocity field in Lagrangian mass coordinates* writes

$$\tilde u(t,y)=u(t,\Psi^{-1}(t,y)), \qquad u(t,x)=\tilde u(t,\Psi(t,x)) \quad (t\geqslant 0,\ y\in[-r_1,r_2],\ x\in[-1,1]). \tag{3.4}$$

Finally, the *temperature field in Lagrangian mass coordinates* writes

$$\begin{cases}\tilde\theta(t,y)=\theta(t,\Psi^{-1}(t,y)),\\ \theta(t,x)=\tilde\theta(t,\Psi((t,x)), \end{cases} \qquad (t\geqslant 0,\ y\in[-r_1,r_2]\setminus\{0\},\ x\in[-1,1]\setminus\{h(t)\}). \tag{3.5}$$

With the above notation, the system formed by (1.2)–(1.9), can be written in the form:

$$\partial_t\tilde v-\partial_y\tilde u=0 \qquad (t\geqslant 0,\ y\in(-r_1,r_2)\setminus\{0\}), \tag{3.6}$$

$$\partial_t\tilde u+\partial_y\left(\frac{\tilde\theta}{\tilde v}\right)=\mu\partial_y\left(\frac{1}{\tilde v}\partial_y\tilde u\right) \qquad (t\geqslant 0,\ y\in(-r_1,r_2)\setminus\{0\}), \tag{3.7}$$

$$c_v\partial_t\tilde\theta-\partial_y\left(\frac{\kappa(\tilde\theta)}{\tilde v}\partial_y\tilde\theta\right)=\mu\frac{1}{\tilde v}|\partial_y\tilde u|^2-\frac{\tilde\theta}{\tilde v}\partial_y\tilde u \qquad (t\geqslant 0,\ y\in(-r_1,r_2)\setminus\{0\}), \tag{3.8}$$

$$\tilde u(t,-r_1)=\tilde u(t,r_2)=0 \qquad (t\geqslant 0), \tag{3.9}$$

$$\tilde u(t,0^{\pm})=g(t),\ M\frac{\mathrm{d}g}{\mathrm{d}t}(t)=\left[\frac{\mu}{\tilde v}\partial_y\tilde u-\frac{1}{\tilde v}\tilde\theta\right]_{y\to 0^-}^{y\to 0^+} \qquad (t\geqslant 0), \tag{3.10}$$

$$\partial_y\tilde\theta(t,0^{\pm})=\partial_y\tilde\theta(t,-r_1)=\partial_y\tilde\theta(t,r_2)=0 \qquad (t\geqslant 0), \tag{3.11}$$

$$g(0)=g_0, \tag{3.12}$$

$$\tilde v(0,y)=\tilde v_0(y), \quad \tilde u(0,y)=\tilde u_0(y) \qquad (y\in(-r_1,r_2)\setminus\{0\}), \tag{3.13}$$

where $g(t):=\frac{\mathrm{d}h}{\mathrm{d}t}(t)$ for every $t\geqslant 0$ and

$$\tilde{v}_0(y) = \frac{1}{\rho_0(\Psi^{-1}(0,y))}, \quad \tilde{u}_0(y) = u_0(\Psi^{-1}(0,y)), \quad \tilde{\theta}_0(y) = \theta_0(\Psi^{-1}(0,y)) \qquad (y \in (-r_1, r_2) \setminus \{0\}).$$

As is well known, see Wagner [24], formulation (3.6)–(3.13) is equivalent to the original problem (1.2)–(1.9), (2.1) even at the level of weak solutions as long as the specific volume v satisfies,

$$0 < \underline{v} \leqslant v \leqslant \overline{v}.$$

We claim the following result on global in time existence of strong solutions of (3.6)–(3.13):

Theorem 3.1. *Assume that $\kappa \in C^2[0,\infty)$ satisfies (1.10) for some $\alpha \geqslant 2$. Let the initial data g_0, $\tilde{v}_0$, $\tilde{u}_0$, $\tilde{\theta}_0$ be given in the class*

$$\tilde{v}_0 \in W^{1,2}(-r_1, 0) \cup W^{1,2}(0, r_2),\ \tilde{v}_0 > 0 \tag{3.14}$$
$$\tilde{u}_0 \in W_0^{1,2}(-r_1, r_2),\ \tilde{u}_0(0) = g_0 \tag{3.15}$$
$$\tilde{\theta}_0 \in W^{1,2}(-r_1, 0) \cup W^{1,2}(0, r_2),\ \tilde{\theta}_0 > 0. \tag{3.16}$$

Then for any $\tau > 0$, the system (3.6)–(3.13) admits a solution $[g, \tilde{v}, \tilde{u}, \tilde{\theta}]$ on the time interval $[0, \tau]$, unique in the class

$$g \in W^{1,2}(0,\tau), \tag{3.17}$$
$$\tilde{v} \in C([0,\tau]; W^{1,2}(-r_1, 0)) \cap C([0,\tau]; W^{1,2}(0, r_2)), \tag{3.18}$$
$$\tilde{v}(t,\cdot) \geqslant \underline{v}(t) > 0 \qquad (t \in [0,\tau]), \tag{3.19}$$
$$\tilde{u} \in C([0,\tau]; W_0^{1,2}(-r_1, r_2)), \tag{3.20}$$
$$\partial_t \tilde{u},\ \partial_{y,y}\tilde{u} \in L^2([0,\tau]; L^2[-r_1, 0]) \cup L^2([0,\tau]; L^2[0, r_2]), \tag{3.21}$$
$$\tilde{\theta} \in C([0,\tau]; W^{1,2}(-r_1, 0)) \cup C([0,\tau]; W^{1,2}(0, r_2)), \tag{3.22}$$
$$\partial_t \tilde{\theta},\ \partial_{y,y}\tilde{\theta} \in L^2([0,\tau]; L^2[-r_1, 0]) \cup L^2([0,\tau]; L^2[0, r_2]), \tag{3.23}$$
$$\tilde{\theta}(t,\cdot) \geqslant \underline{\theta}(t) > 0 \qquad (t \in [0,\tau]). \tag{3.24}$$

Remark 3.2. Besides the fact that we consider an insulating piston and a temperature dependent heat conductivity coefficient, the above theorem requires definitely less regularity of the data compared to Shelukhin's corresponding result [21] based on classical solution framework.

Moreover, the obtained $\tilde{v}$ is bounded from above and bounded away from zero and therefore, going back to the eulerian density (see (3.3)),

$$0 < m_\tau \leqslant \rho(t,x) \leqslant M_\tau < \infty \qquad (t \in [0,\tau],\ x \neq h(t)),$$
$$\int_{-1}^{h(t)} \rho(t,x)\,\mathrm{d}x = r_1, \quad \int_{h(t)}^{1} \rho(t,x)\,\mathrm{d}x = r_2 \qquad (t \in [0,\tau]),$$

where $h(t) = h_0 + \int_0^\tau g(\sigma)\,\mathrm{d}\sigma$, with $h_0 \in (-1,1)$. The above facts imply the expected property

$$h(t) \in (-1,1) \qquad (t \in [0,\tau]).$$

Passing to the original Eulerian variables we easily check that Theorem 3.1 yields strong solutions in the regularity class specified in Definition 2.1; whence justifying the existence statement claimed in Theorem 2.2. On the other hand, any solution enjoying the level of smoothness imposed by Definition 2.1 has the regularity claimed in Theorem 3.1 when expressed in the Lagrangian coordinates. Indeed relations (3.21), (3.23) in fact imply continuity claimed in (3.20), (3.22), respectively, while the continuity of the specific volume $\tilde{v}$ obtained in (3.18) follows from the fact that the density $\rho > 0$ satisfies the transport equation (1.2) in the Eulerian coordinates and therefore can be computed on the streamlines by the method of characteristics. In other words, the statement of Theorem 3.1 implies Theorem 2.2. In the remaining part of this section, we therefore focus on the proof of Theorem 3.1.

3.2. Local existence and uniqueness of solutions

In this subsection we give a local in time existence and uniqueness result, whose precise statement requires some notation. More precisely, for every $R > 0$ we denote by B_R the closed ball of radius R centered at the origin in X, where

$$X = \left\{ \begin{bmatrix} \tilde{v}_0 \\ \tilde{u}_0 \\ g_0 \\ \tilde{\theta}_0 \end{bmatrix} \;\middle|\; \begin{matrix} \tilde{v}_0 \in W^{1,2}(-r_1, 0) \cap W^{1,2}(0, r_2) \\ \tilde{u}_0 \in W_0^{1,2}(-r_1, r_2) \\ g_0 \in \mathbb{R} \\ \tilde{\theta}_0 \in W^{1,2}(-r_1, 0) \cap W^{1,2}(0, r_2) \end{matrix} \quad \text{and} \;\; \tilde{u}_0(0) = g_0 \right\}. \tag{3.25}$$

Moreover, given $R,\ m_0, m_1 > 0$, with $m_0 < m_1$, we denote by B_{R,m_0,m_1} the subset of B_R formed by the states satisfying

$$m_1 \geqslant \tilde{v}_0(y) \geqslant m_0 \qquad (y \in (-r_1, 0) \cup (0, r_2)), \tag{3.26}$$

$$m_1 \geqslant \tilde{\theta}_0(y) \geqslant m_0 \qquad (y \in (-r_1, 0) \cup (0, r_2)), \tag{3.27}$$

$$\|\tilde{v}_0\|_{W^{1,2}(-r_1,0)} + \|\tilde{v}_0\|_{W^{1,2}(0,r_2)} + \|\tilde{u}_0\|_{W^{1,2}(-r_1,r_2)} + \|\tilde{\theta}_0\|_{W^{1,2}(-r_1,0)} + \|\tilde{\theta}_0\|_{W^{1,2}(0,r_2)} \leqslant R.$$

Our local in time existence and uniqueness states as follows:

Theorem 3.3. *Let $R > 0$ and $m_1 > m_0 > 0$. Let $\begin{bmatrix} \tilde{v}_0 \\ \tilde{u}_0 \\ g_0 \\ \tilde{\theta}_0 \end{bmatrix} \in B_{R,m_0,m_1}$. Then there exists $T > 0$, depending only on m_0, m_1 and R such the system (3.6)–(3.13) admits an unique solution (in the sense of Theorem 3.1) on* $[0, T]$.

An important point is that the proof of the above theorem is based on a “monolithic” linearization of the system (3.6)–(3.13), followed by an application of the Banach fixed point theorem. The term *monolithic linearization* designs the fact that the linearized system is still a coupled one, in which both the velocities of the gas and of the piston are supposed to be the unknowns. Using this type of linearization is important in order to obtain the local existence and uniqueness in spaces which are less regular than those used in the existing literature.

With the exception of the above mentioned point, the proof of Theorem 3.3 is based on standard maximal regularity theory for linear parabolic equations and on standard Sobolev embedding theorems. We refer to the proof of Theorem 3.1 in Debayan et al. [18] for a detailed presentation of the method in a situation which is close to the one encountered in Theorem 3.3.

In the remaining part of this section we deduce the main estimates which are necessary to show that the local in time solution constructed in Theorem 3.3 can be extended to a solution defined on $[0, \tau)$ for every $\tau > 0$. Our approach has numerous common points with the methodology used by Shelukhin [21], Antontsev et al. [1] and Kazhikhov [13] but it requires new estimates in order to tackle the fact that the piston is assumed thermally insulating, which gives the homogeneous Neumann boundary condition for the temperature on the piston.

3.3. Energy type estimates and first density and temperature bounds

In this subsection and for the remaining part of this section $0 < \tau < \infty$ is fixed and $\begin{bmatrix} \tilde{v} \\ \tilde{u} \\ g \\ \tilde{\theta} \end{bmatrix}$ is a solution of (3.6)–(3.11) on $[0, T] \subset [0, \tau)$ having the properties stated in Theorem 3.3. All the constants appearing in the estimates below may depend on τ and on the constants $m_0,\ m_1,\ R$ in Theorem 3.3, but are independent of T.

In this subsection we derive, following closely ideas from [21], some energy bounds and we provide the first estimates on the density and temperature fields.

Denote, recalling that $g(t) := \frac{\mathrm{d}h}{\mathrm{d}t}(t)$ and that the constructed local solution satisfies $\tilde{\theta} > 0$ and $\tilde{v} > 0$ on the existence domain,

$$E_1(t) = \left\| c_v\tilde{\theta}(t,\cdot) + \frac{1}{2}\tilde{u}^2(t,\cdot) \right\|_{L^1[-r_1,r_2]} + M\,\frac{g^2(t)}{2} \qquad (t \in [0,T]), \tag{3.28}$$

$$E_2(t) = \left\| c_v\phi\left(\frac{\tilde{\theta}(t,\cdot)}{a}\right) + \phi\left(\frac{\tilde{v}(t,\cdot)}{b}\right) + \frac{\tilde{u}^2(t,\cdot)}{2a} \right\|_{L^1[-r_1,r_2]} + \frac{M}{2a}\,g^2(t) \qquad (t \in [0,T]),$$

$$E_3(t) = \left\| \left(\frac{\kappa(\tilde{\theta})}{\tilde{\theta}^2\,\tilde{v}} \left|\partial_x\tilde{\theta}\right|^2\right)(t,\cdot) + \left(\frac{\mu}{\tilde{v}\,\tilde{\theta}}\,|\partial_x\tilde{u}|^2\right)(t,\cdot) \right\|_{L^1[-r_1,r_2]} \qquad (t \in [0,T]), \tag{3.29}$$

where $a,\ b$ are arbitrary positive constants and

$$\phi(\theta) = \theta - \ln\theta - 1 \qquad (\theta > 0). \tag{3.30}$$

Lemma 3.4. *We have*

$$E_1(t) = E_1(0) \qquad (t \in [0,T]), \tag{3.31}$$

$$\|\tilde{v}(t,\cdot)\|_{L^1[-r_1,r_2]} = \|\tilde{v}_0\|_{L^1[-r_1,r_2]} := \beta \qquad (t \in [0,T]), \tag{3.32}$$

$$E_2(t) + \int_0^t E_3(s)\,\mathrm{d}s = E_2(0) \qquad (t \in [0,T]). \tag{3.33}$$

Proof. Taking the product of both sides of (3.7) by $\tilde{u}$, integrating on $[-r_1,0]$, on $[0,r_2]$, using (3.9) and summing up the obtained results it follows that, for every $t \geqslant 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2} \tilde{u}^2(t,y)\,\mathrm{d}y + g(t)\left[\frac{\mu}{\tilde{v}}\partial_y\tilde{u} - \frac{\tilde{\theta}}{\tilde{v}}\right]_{y\to 0^-}^{y\to 0^+}(t) + \int_{-r_1}^{r_2} \partial_y\tilde{u}(t,y)\left(\frac{\mu}{\tilde{v}}\partial_y\tilde{u} - \frac{\tilde{\theta}}{\tilde{v}}\right)(t,y)\,\mathrm{d}y = 0.$$

By combining the above formula and (3.10) it follows that, for every $t \geqslant 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2} \tilde{u}^2(t,y)\,\mathrm{d}y + \frac{M}{2}\frac{\mathrm{d}}{\mathrm{d}t}g^2(t) + \int_{-r_1}^{r_2} (\partial_y\tilde{u})(t,y)\left(\frac{\mu}{\tilde{v}}\,\partial_y\tilde{u} - \frac{\tilde{\theta}}{\tilde{v}}\right)(t,y)\,\mathrm{d}y = 0. \tag{3.34}$$

On the other hand, integrating (3.8) on $[-r_1,0]$, on $[0,r_2]$, using (3.11) and summing up the obtained results it follows that, for every $t \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2} c_v\tilde{\theta}(t,y)\,\mathrm{d}y - \int_{-r_1}^{r_2} \partial_y\tilde{u}\left[\left(\frac{\mu}{\tilde{v}}\partial_y\tilde{u} - \frac{\tilde{\theta}}{\tilde{v}}\right)\right](t,y)\,\mathrm{d}y = 0.$$

Summing up the above formula and (3.34) we obtain (3.31).

In order to prove (3.32), it suffices to integrate (3.6) on $[-r_1,0]$, on $[0,r_2]$, to sum up the obtained results and to use the first condition in (3.10).

In order to prove (3.33) we multiply (3.8) by $\frac{1}{a}\phi'\left(\frac{\tilde{\theta}}{a}\right) = \frac{1}{a} - \frac{1}{\tilde{\theta}}$, we integrate on $[-r_1,0]$ and on $[0,r_2]$ and finally we sum up. For the first term we obtain

$$\frac{1}{a}\int_{-r_1}^{0} c_v\left(\partial_t\tilde{\theta}(t,y)\right)\phi'\left(\frac{\tilde{\theta}(t,y)}{a}\right)\mathrm{d}y + \frac{1}{a}\int_0^{r_2} c_v\partial_t\tilde{\theta}(t,y)\phi'\left(\frac{\tilde{\theta}(t,y)}{a}\right)\mathrm{d}y$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2} c_v\phi\left(\frac{\tilde{\theta}(t,y)}{a}\right)\mathrm{d}y \qquad (t \in [0,T]). \tag{3.35}$$

The same operations and integration by parts applied to the second term in (3.8) give

$$-\frac{1}{a}\int_{-r_1}^{0}\partial_y\left(\frac{\kappa(\tilde\theta)}{\tilde v}\partial_y\tilde\theta\right)\phi'\left(\frac{\tilde\theta}{a}\right)\mathrm{d}y-\frac{1}{a}\int_{0}^{r_2}\partial_y\left(\frac{\kappa(\tilde\theta)}{\tilde v}\partial_y\tilde\theta\right)\phi'\left(\frac{\tilde\theta}{a}\right)\mathrm{d}y$$
$$=\int_{-r_1}^{r_2}\left(\frac{\kappa(\tilde\theta)}{\tilde v}\left|\partial_y\tilde\theta\right|^2\frac{1}{\tilde\theta^2}\right)\mathrm{d}y\qquad(t\in[0,T]).\tag{3.36}$$

To evaluate the contribution of the third term in (3.8), we note that

$$-\frac{1}{a}\int_{-r_1}^{r_2}\left\{\partial_y\tilde u\left(\frac{\mu}{\tilde v}\partial_y\tilde u-\frac{\tilde\theta}{\tilde v}\right)\right\}\phi'\left(\frac{\tilde\theta}{a}\right)\mathrm{d}y=-\frac{1}{a}\int_{-r_1}^{r_2}\left\{\partial_y\tilde u\left(\frac{\mu}{\tilde v}\partial_y\tilde u-\frac{\tilde\theta}{\tilde v}\right)\right\}\mathrm{d}y$$
$$+\int_{-r_1}^{r_2}\left(\frac{\mu}{\tilde v\tilde\theta}\left|\partial_y\tilde u\right|^2\right)\mathrm{d}y-\int_{-r_1}^{r_2}\left(\frac{1}{\tilde v}\partial_y\tilde u\right)\mathrm{d}y\qquad(t\in[0,T]).\tag{3.37}$$

Summing up (3.35)–(3.37) and using (3.6) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2}\left\{c_v\phi\left(\frac{\tilde\theta}{a}\right)-\ln\left(\frac{\tilde v}{b}\right)\right\}\mathrm{d}y=-\int_{-r_1}^{r_2}\left(\frac{\kappa(\tilde\theta)}{\tilde v}\left|\partial_y\tilde\theta\right|^2\frac{1}{\tilde\theta^2}\right)\mathrm{d}y$$
$$-\int_{-r_1}^{r_2}\left(\frac{\mu}{\tilde v\tilde\theta}\left|\partial_y\tilde u\right|^2\right)\mathrm{d}y+\frac{1}{a}\int_{-r_1}^{r_2}\left\{\partial_y\tilde u\left(\frac{\mu}{\tilde v}\partial_y\tilde u-\frac{\tilde\theta}{\tilde v}\right)\right\}\mathrm{d}y\qquad(t\in[0,T]).\tag{3.38}$$

On the other hand, multiplying (3.7) by $\frac{1}{a}\tilde u$ and integrating on $[-r_1,0]$ and on $[0,r_2]$ we obtain

$$\frac{1}{2a}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2}\tilde u^2\,\mathrm{d}y+\frac{1}{a}\int_{-r_1}^{r_2}\left\{\partial_y\tilde u\left(\frac{\mu}{\tilde v}\partial_y\tilde u-\frac{\tilde\theta}{\tilde v}\right)\right\}\mathrm{d}y+\frac{M}{2a}\frac{\mathrm{d}}{\mathrm{d}t}|g(t)|^2=0\qquad(t\in[0,T]).\tag{3.39}$$

Taking the sum of (3.38), (3.39) and using (3.32) we obtain the conclusion (3.33). □

Lemma 3.5. *Denote*

$$\sigma(t,y)=\frac{\mu}{\tilde v(t,x)}\partial_y\tilde u(t,y)-\frac{\tilde\theta(t,y)}{\tilde v(t,y)}\qquad(t\in[0,T],\ y\in[-r_1,r_2]\setminus\{0\}).\tag{3.40}$$

Then, for every $t\in[0,T]$ and every $y\in[-r_1,r_2]\setminus\{0\}$ we have

$$\tilde v(t,y)=\left(\tilde v_0(y)+\int_0^t\frac{1}{\mu}\tilde\theta(\xi,y)\exp\left\{-\int_0^\xi\frac{1}{\mu}\sigma(s,y)\,\mathrm{d}s\right\}\mathrm{d}\xi\right)\exp\left(\int_0^t\frac{1}{\mu}\sigma(s,y)\,\mathrm{d}s\right),\tag{3.41}$$

$$\int_0^t\sigma(s,y)\,\mathrm{d}s=-\frac{1}{\beta}\int_0^t\|\tilde u^2(s,\cdot)+\tilde\theta(s,\cdot)\|_{L^1[-r_1,r_2]}\mathrm{d}s+A(t,y)+B(t,y),\tag{3.42}$$

where β is the constant in (3.32) and

$$\beta A(t,y) = \int_{-r_1}^{r_2} \tilde{v}(t,\xi) \left\{ \int_{\xi}^{y} \tilde{u}(t,\eta)\,\mathrm{d}\eta - \int_{0}^{y} \tilde{u}_0(\eta)\,\mathrm{d}\eta \right\} \mathrm{d}\xi$$

$$+ \int_{-r_1}^{r_2} \tilde{v}_0(\xi) \int_{0}^{\xi} \tilde{u}_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi \qquad (t \in [0,T],\quad y \in [-r_1, r_2]),$$

$$\beta B(t,y) = -M \int_0^t g^2(s)\,\mathrm{d}s - (\|\tilde{v}(t,\cdot)\|_{L^1[0,r_2]} M\, g(t) - \|\tilde{v}_0\|_{L^1[0,r_2]} M\, g_0) \qquad (t \in [0,T],\quad y \in [-r_1, 0)),$$

$$\beta B(t,y) = -M \int_0^t g^2(s)\,\mathrm{d}s + (\|\tilde{v}(t,\cdot)\|_{L^1[-r_1,0]} M\, g(t) - \|\tilde{v}_0\|_{L^1[-r_1,0]} M\, g_0) \qquad (t \in [0,T],\quad y \in (0, r_2]).$$

Proof. By combining (3.6) and (3.40) it follows that

$$\partial_t \tilde{v}(t,y) = \frac{1}{\mu}(\sigma(t,y)\tilde{v}(t,y) + \tilde{\theta}(t,y)) \qquad (t \in [0,T],\ y \in [-r_1, r_2] \setminus \{0\}), \tag{3.43}$$

so that (3.41) is nothing else but the variation of constants formula applied to (3.43), seen as a linear ODE of unknown $\tilde{v}(\cdot, y)$ and depending on the parameter $y \in [-r_1, r_2] \setminus \{0\}$.

In order to prove (3.42) we set

$$\gamma(t,y) = \int_0^t \sigma(s,y)\,\mathrm{d}s + \int_0^y \tilde{u}_0(\eta)\,\mathrm{d}\eta \qquad (t \in [0,T],\quad y \in (0, r_2)). \tag{3.44}$$

Using the fact that equation (3.8) rewrites $\partial_t \tilde{u} = \partial_y \sigma$, it follows that

$$\partial_t \gamma(t,y) = \sigma(t,y), \qquad \partial_y \gamma(t,y) = \tilde{u}(t,y) \qquad (t \in [0,T],\ y \in [-r_1, r_2] \setminus \{0\}). \tag{3.45}$$

Consequently, for $t \in [0,T]$ and $y \in [-r_1, r_2] \setminus \{0\}$ we have

$$\partial_t(\tilde{v}\gamma) = (\partial_y \tilde{u})\gamma + \partial_y \tilde{u} - \tilde{\theta} = (\partial_{y,y}\gamma)\gamma + \partial_{y,y}\gamma - \tilde{\theta} = \partial_{y,y}\gamma + \partial_y\left[(\partial_y\gamma)\gamma\right] - (\partial_y\gamma)^2 - \tilde{\theta}.$$

Integrating the above formula on $[0,t] \times [-r_1, 0]$, on $[0,t] \times [0, r_2]$ and summing up we obtain:

$$\int_{-r_1}^{r_2} \tilde{v}(t,y)\gamma(t,y)\,\mathrm{d}y - \int_{-r_1}^{r_2} \tilde{v}_0(y) \int_0^y \tilde{u}_0(\xi)\,\mathrm{d}\xi\,\mathrm{d}x = \int_0^t g(s)(\gamma(s,0^-) - \gamma(s,0^+))\,\mathrm{d}s$$

$$- \int_0^t \|\tilde{u}^2(s,\cdot) + \tilde{\theta}(s,\cdot)\|_{L^1[-r_1,r_2]}\,\mathrm{d}s \qquad (t \in [0,T]). \tag{3.46}$$

Using the facts that

$$\gamma(s,0^-) = \int_0^s \partial_t \gamma(\eta, 0^-)\,\mathrm{d}\eta + \gamma(0,0^-) = \int_0^s \sigma(\eta, 0^-)\,\mathrm{d}\eta,$$

$$\gamma(s,0^+) = \int_0^s \partial_t \gamma(\eta, 0^+)\,\mathrm{d}\eta + \gamma(0,0^+) = \int_0^s \sigma(\eta, 0^+)\,\mathrm{d}\eta,$$

it follows that

$$\gamma(s,0^-)-\gamma(s,0^+)=-\int_0^s [\sigma(\eta,\cdot)]_{y\to 0^-}^{y\to 0^+}\,\mathrm{d}\eta=-M\int_0^s \dot{g}(\tau)\,\mathrm{d}\tau=-M(g(s)-g_0).$$

Inserting the above formula in (3.46) yields

$$\int_{-r_1}^{r_2}\tilde{v}(t,y)\gamma(t,y)\,\mathrm{d}y=-M\int_0^t g(s)(g(s)-g_0)\,\mathrm{d}s-\int_0^t \|\tilde{u}^2(s,\cdot)+\tilde{\theta}(s,\cdot)\|_{L^1[-r_1,r_2]}\,\mathrm{d}s$$
$$+\int_{-r_1}^{r_2}\tilde{v}_0(y)\int_0^y \tilde{u}_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}y \qquad (t\in[0,T]).$$

Since

$$\int_0^t g(s)\,\mathrm{d}s=\int_0^t \tilde{u}(s,0)\,\mathrm{d}s=\int_0^t\int_{-r_1}^0 \partial_y\tilde{u}(s,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$=\int_{-r_1}^0\int_0^t \partial_s\tilde{v}(s,y)\,\mathrm{d}s\,\mathrm{d}y=\|\tilde{v}(t,\cdot)\|_{L^1[-r_1,0]}-\|\tilde{v}_0\|_{L^1[-r_1,0]} \qquad (t\in[0,T)),$$

we deduce that

$$\int_{-r_1}^{r_2}\tilde{v}(t,y)\gamma(t,y)\,\mathrm{d}y=-M\int_0^t g^2(s)\,\mathrm{d}s+Mg_0\left(\|\tilde{v}(t,\cdot)\|_{L^1[-r_1,0]}-\|\tilde{v}_0\|_{L^1[-r_1,0]}\right)$$
$$-\int_0^t \|\tilde{u}^2(s,\cdot)+\tilde{\theta}(s,\cdot)\|_{L^1[-r_1,r_2]}\,\mathrm{d}s+\int_{-r_1}^{r_2}\tilde{v}_0(y)\int_0^y \tilde{u}_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}y \qquad (t\in[0,T]). \tag{3.47}$$

On the other hand, by combining (3.44) and the second formula in (3.45) it follows that

$$\int_0^t \sigma(s,y)\,\mathrm{d}s=\gamma(t,\xi)+\int_\xi^y \tilde{u}(t,\eta)\,\mathrm{d}\eta-\int_0^y \tilde{u}_0(\eta)\,\mathrm{d}\eta+\zeta(t,\xi), \tag{3.48}$$

for every $t\in[0,T]$, $y\in(0,r_2)$ and $\xi\in(-r_1,r_2)$, where, for every $t\in[0,T]$

$$\zeta(t,\xi)=\begin{cases}0 & \text{if } \xi\in(0,r_2]\\ \int_0^t[\sigma(s,\cdot)]_0=M(g(t)-g_0) & \text{if } \xi\in[-r_1,0).\end{cases}$$

Multiplying (3.48) by $\tilde{v}(t,\xi)$ and integrating with respect to ξ it follows that

$$\beta\int_0^t \sigma(s,y)\,\mathrm{d}y=\int_{-r_1}^{r_2}\tilde{v}(t,\xi)\gamma(t,\xi)\,\mathrm{d}\xi+\int_{-r_1}^{r_2}\tilde{v}(t,\xi)\left(\int_\xi^y \tilde{u}(t,\eta)\,\mathrm{d}\eta-\int_0^y \tilde{u}_0(\eta)\,\mathrm{d}\eta\right)\mathrm{d}\xi$$
$$+\|\tilde{v}(t,\cdot)\|_{L^1[-r_1,0]}M(g(t)-g_0) \qquad (t\in[0,T]).$$

Combining the above formula and (3.47) we obtain (3.42) for $y\in(0,r_2]$. The proof of (3.42) for $y\in[-r_1,0)$ is completely similar so it can be omitted. □

By combining (3.42) with (3.32) and (3.33) we obtain:

Corollary 3.6. *With the notation in Lemma 3.5, there exists a constant $c > 0$ such that*

$$\left| \int_0^t \sigma(s, y)\, \mathrm{d}s \right| \leqslant c \qquad (t \in [0, T],\ y \in [-r_1, r_2] \setminus \{0\}). \tag{3.49}$$

Given a function $f : [0, T] \times [-r_1, 0) \cup (0, r_2] \to (0, \infty)$, we denote, for the remaining part of this work

$$m_f(t) = \inf_{y \in [-r_1,0) \cup (0,r_2]} f(t, y), \qquad M_f(t) = \sup_{y \in [-r_1,0) \cup (0,r_2]} f(t, y) \qquad (t \in [0, T]).$$

Lemma 3.7. *With ϕ defined in (3.30), we have the inequalities:*

$$M_{\tilde{\theta}}(t) \leqslant 4a \left\{ 1 + K_0 \left\| \phi\left(\frac{\tilde{\theta}(t, \cdot)}{a} \right) \right\|_{L^1[-r_1,r_2]} M_{\tilde{v}}(t)\, J(t) \right\} \qquad (t \in [0, T]), \tag{3.50}$$

where K_0 is a universal constant, $a = \max\{r_1^{-1} E_1(0), r_2^{-1} E_1(0)\}$, where E_1 has been defined in (3.28) and

$$J(t) = \left\| \left(\frac{\left| \partial_y \tilde{\theta} \right|^2}{\tilde{v}\, \tilde{\theta}^2} \right)(t, \cdot) \right\|_{L^1[-r_1,r_2]} \qquad (t \in [0, T]). \tag{3.51}$$

Proof. In order to prove (3.50) we fix $t \in [0, T]$ and we show that $\tilde{\theta}(t, y)$ is smaller than the right hand side of (3.50) for every $y \in [-r_1, 0]$. The proof of the similar estimate for $y \in [0, r_2]$ can be done following line by line the same steps.

To achieve this goal, we first note that from the energy estimate (3.31) it follows that

$$\min_{y \in [-r_1,0]} \tilde{\theta}(t, y) \leqslant \frac{E_1(0)}{c_v r_1}.$$

Then we distinguish two cases. Firstly, if we assume that

$$\max_{y \in [-r_1,0]} \tilde{\theta}(t, y) \leqslant \frac{2E_1(0)}{c_v r_1}, \tag{3.52}$$

then (3.50) is obviously verified.

If (3.52) is false then, using the continuity of the map $y \mapsto \tilde{\theta}(t, y)$ on $[-r_1, 0]$ it follows that there exists $y_1 \in [-r_1, 0]$ (depending on t) such that

$$\tilde{\theta}(t, y_1) = \frac{E_1(0)}{c_v r_1}.$$

In this case, if y is such that $\tilde{\theta}(t, y) \leqslant 4\frac{E_1(0)}{c_v r_1}$ then there is nothing to prove. We can thus assume, without loss of generality, that $\tilde{\theta}(t, y) > 4\frac{E_1(0)}{c_v r_1}$. In this case we define

$$\Gamma_1(t, y) = \int_{\tilde{\theta}(t,y_1)}^{\tilde{\theta}(t,y)} \frac{1}{s} \sqrt{\phi\left(\frac{s}{a_1} \right)}\, \mathrm{d}s, \tag{3.53}$$

where $a_1 := \frac{E_1(0)}{c_v r_1}$ and ϕ has been defined in (3.30). We have

$$|\Gamma_1(t,y)| = \left|\int_{y_1}^{y} \partial_\eta \Gamma_1(t,\eta)\,\mathrm{d}\eta\right| \leqslant \int_{-r_1}^{0} \left|\partial_\eta \Gamma_1(t,\eta)\right| \mathrm{d}\eta = \int_{-r_1}^{0} \left|\frac{1}{\tilde\theta(t,\eta)}\sqrt{\phi\left(\frac{\tilde\theta(t,\eta)}{a_1}\right)}\,\frac{\partial\tilde\theta}{\partial\eta}(t,\eta)\right| \mathrm{d}\eta$$

$$\leqslant \left(\int_{-r_1}^{0} \phi\left(\frac{\tilde\theta(t,\eta)}{a_1}\right)\mathrm{d}\eta\right)^{\frac12}\left(\int_{-r_1}^{0} \frac{1}{\tilde\theta^2(t,\eta)}\left|\frac{\partial\tilde\theta}{\partial\eta}(t,\eta)\right|^2 \mathrm{d}\eta\right)^{\frac12} \leqslant \left(\left\|\phi\left(\frac{\tilde\theta(t,\cdot)}{a_1}\right)\right\|_{L^1[-r_1,r_2]} M_{\tilde v}(t)J(t)\right)^{\frac12}. \tag{3.54}$$

On the other hand, (3.53) and elementary inequalities imply that there exists an universal constant $K_0 > 0$ such that for every $y \in [-r_1, 0]$ with $\tilde\theta(t,y) > 4a_1$ we have

$$\Gamma_1(t,y) = \int_1^{\frac{\tilde\theta(t,y)}{a_1}} \frac{\sqrt{\phi(s)}}{s}\,\mathrm{d}s \geqslant K_0^{-1/2}\int_1^{\frac{\tilde\theta(t,y)}{a_1}} \frac{1}{\sqrt{s}}\,\mathrm{d}s = 2K_0^{-1/2}\left(\sqrt{\frac{\tilde\theta(t,y)}{a_1}} - 1\right) \geqslant K_0^{-1/2}\sqrt{\frac{\tilde\theta(t,y)}{a_1}}\ .$$

The last inequality and (3.54) imply the announced estimate. □

3.4. Pointwise density and temperature bounds

Lemma 3.8. *There exists $c > 0$ such that*

$$c^{-1} \leqslant \tilde v(t,y) \leqslant c \qquad (t \in [0,T],\ y \in (-r_1, r_2)). \tag{3.55}$$

Proof. Using (3.49) in (3.41) and the positivity of $\tilde\theta$ we obtain the first inequality in (3.55).

To prove the second one, we use again (3.49) and (3.41) to obtain that there exists a constant $c > 0$ such that

$$M_{\tilde v}(t) \leqslant c + c\int_0^t M_{\tilde\theta}(s)\,\mathrm{d}s \qquad (t \in [0,T]).$$

Inserting (3.50) into the last inequality we obtain that there exists a constant $c > 0$ such that

$$M_{\tilde v}(t) \leqslant c + c\int_0^t \left\{1 + \left\|\phi\left(\frac{\tilde\theta}{a}\right)(s,\cdot)\right\|_{L^1[-r_1,r_2]} M_{\tilde v}(s)J(s)\,\mathrm{d}s\right\}, \tag{3.56}$$

where J has been defined in (3.51). The above inequality, combined with Gronwall's lemma and with (3.33), implies the second inequality (the upper bound) in (3.55). □

By combining estimate (3.33) with Lemma 3.7 and Lemma 3.8 we obtain the following upper bound for the temperature

Corollary 3.9. *There exists $c > 0$ such that*

$$\int_0^t M_{\tilde\theta}(s)\,\mathrm{d}s \leqslant c \qquad (t \in [0,T]).$$

We also have the following upper bound for the velocity $\tilde u$.

Corollary 3.10. *There exists $c > 0$ such that*

$$\int_0^t M_{\tilde u^2}(s)\,\mathrm{d}s \leqslant c \qquad (t \in [0,T]). \tag{3.57}$$

Proof. Note that, using a simple Sobolev embedding and the Cauchy–Schwarz inequality,

$$M_{|\tilde{u}|}(s) \leqslant \|\partial_x \tilde{u}(s,\cdot)\|_{L^1[-r_1,r_2]} \leqslant \left\| \frac{1}{\sqrt{\tilde{v}\,\tilde{\theta}}}\,\partial_y \tilde{u}(s,\cdot)\right\|^{\frac{1}{2}}_{L^2[-r_1,r_2]} \|(\sqrt{\tilde{v}\,\tilde{\theta}})(s,\cdot)\|^{\frac{1}{2}}_{L^2[-r_1,r_2]}.$$

The above inequality together with (3.31) and (3.55) yield

$$M_{\tilde{u}^2}(s) \leqslant \frac{c}{\mu} E_1(0)E_3(s) \qquad (s\in[0,t],\ t\in[0,T]),$$

where E_1 and E_3 have been defined in (3.28) and (3.29), respectively. Integrating the above inequality with respect to time and using (3.33) we obtain the conclusion (3.57). □

Let us now show that the temperature $\tilde{\theta}$ admits a strictly positive lower bound on $[0,T]\times[-r_1,r_2]$.

Proposition 3.11. *There exists $c>0$ such that*

$$m_{\tilde{\theta}}(t) \geqslant c \qquad (t\in[0,T]).$$

Proof. We define $w=\frac{1}{\tilde{\theta}}$. By equation (3.8)

$$c_v\partial_t w - \partial_y\left(\frac{\kappa(\tilde{\theta})}{\tilde{v}}\partial_y w\right) = \frac{1}{4\mu\tilde{v}} - \frac{1}{\tilde{v}\tilde{\theta}^2}\left(\sqrt{\mu}\partial_y\tilde{u} - \frac{\tilde{\theta}}{2\sqrt{\mu}}\right)^2 - \frac{2\kappa(\tilde{\theta})}{\tilde{\theta}\tilde{v}}\left(\frac{1}{\tilde{\theta}}\partial_y\tilde{\theta}\right)^2. \tag{3.58}$$

Multiplying the above formula by w^{p-1}, with $p>1$, and integrating over $(-r_1,r_2)$ we obtain

$$c_v\|w(t,\cdot)\|_p^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|_p \leqslant \left\|\frac{1}{4\mu\tilde{v}(t,\cdot)}\right\|_p \|w(t,\cdot)\|_p^{p-1}. \tag{3.59}$$

This yields

$$c_v\frac{\mathrm{d}}{\mathrm{d}t}\|w(t,\cdot)\|_p \leqslant \frac{1}{4\mu}\left\|\frac{1}{\tilde{v}(t,\cdot)}\right\|_p \tag{3.60}$$

and, after integration with respect to time, we obtain

$$\|w(t,\cdot)\|_p \leqslant \|w(0)\|_p + \frac{1}{4c_v\mu}\int_0^t \left\|\frac{1}{\tilde{v}(s,\cdot)}\right\|_p \mathrm{d}s. \tag{3.61}$$

Tending with p to infinity in the above estimate and using Lemma 3.8 it follows that

$$\sup_{t\in[0,T]}\left\|\frac{1}{\tilde{\theta}(t,\cdot)}\right\|_\infty = \sup_{t\in[0,T]}\|w(t,\cdot)\| \leqslant c. \quad \Box \tag{3.62}$$

Proposition 3.12. *There exists $c>0$ such that*

$$\|\partial_y\tilde{v}(t,\cdot)\|^2_{L^2[-r_1,r_2]} + \int_0^t \|\partial_y\tilde{v}(s,\cdot)\|^2_{L^2[-r_1,r_2]}\,\mathrm{d}s \leqslant c \qquad (t\in[0,T]). \tag{3.63}$$

Proof. Let $q=\tilde{u}-\mu\partial_y(\ln\tilde{v}) = \tilde{u}-\frac{\mu}{\tilde{v}}\partial_y\tilde{v}$. Using (3.7) it follows that

$$\partial_t q(t,y) = -\partial_y\left(\frac{\tilde{\theta}}{\tilde{v}}\right)(t,y) \qquad (t\in[0,T),\ y\in[-r_1,r_2]\setminus\{0\}).$$

Multiplying the above equation by q and integrating on $[-r_1,r_2]$ we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q(t,\cdot)\|^2_{L^2[-r_1,r_2]}+\left\|\mu\frac{\tilde\theta}{\tilde v^3}\left|\partial_y\tilde v\right|^2(t,\cdot)\right\|_{L^1[-r_1,r_2]}=F_3(t)\qquad (t\in[0,T)),\tag{3.64}$$

where

$$F_3(t)=\int_{-r_1}^{r_2}\left(\tilde u\,\frac{\tilde\theta}{\tilde v^2}\,(\partial_y\tilde v)-\frac{1}{\tilde v}\,(\partial_y\tilde\theta)\,\tilde u+\frac{\mu}{\tilde v^2}(\partial_y\tilde v)(\partial_y\tilde\theta)\right)\mathrm{d}y\qquad (t\in[0,T]).$$

Using elementary inequalities it is easily checked that

$$F_3(t)\leqslant\frac{1}{2}\left\|\mu\frac{\tilde\theta}{\tilde v^3}\left|\partial_y\tilde v\right|^2(t,\cdot)\right\|_{L^1[-r_1,r_2]}+c(\mu)\int_{-r_1}^{r_2}\frac{\tilde\theta}{\tilde v}\tilde u^2\,\mathrm{d}y+c(\mu)\int_{-r_1}^{r_2}\frac{\mu}{\tilde v\,\tilde\theta}\left|\partial_y\tilde\theta\right|^2\mathrm{d}y,$$

where J has been defined in (3.51). Using (3.55) and Proposition 3.11 it follows that there exists $c>0$ such that

$$F_3(t)\leqslant\frac{1}{2}\left\|\mu\frac{\tilde\theta}{v^3}\left|\partial_y\tilde v\right|^2(t,\cdot)\right\|_{L^1[-r_1,r_2]}+c(\mu)M_{\tilde u^2}(t)\int_{-r_1}^{r_2}\tilde\theta\,\mathrm{d}y+c(\mu)\int_{-r_1}^{r_2}\left|\partial_y\tilde\theta\right|^2\mathrm{d}y.$$

The above inequality and (3.64) yield

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q(t,\cdot)\|^2_{L^2[-r_1,r_2]}+\frac{1}{2}\left\|\mu\frac{\tilde\theta}{\tilde v^3}\left|\partial_y\tilde v\right|^2(t,\cdot)\right\|_{L^1[-r_1,r_2]}&\leqslant c(\mu)M_{\tilde u^2}(t)\int_{-r_1}^{r_2}\tilde\theta\,\mathrm{d}y\\&\quad+c(\mu)\int_{-r_1}^{r_2}\left|\partial_y\tilde\theta\right|^2\mathrm{d}y\qquad (t\in[0,T]).\end{aligned}\tag{3.65}$$

On the other hand, we note that from (3.31) it follows that

$$\int_{-r_1}^{r_2}\tilde\theta(s,y)\,\mathrm{d}y\leqslant\frac{E_1(0)}{c_v}\qquad (s\in[0,T]),$$

whereas (3.33), combined with Lemma 3.8 and Corollary 3.10, implies that there exist $c>0$ such that

$$\int_0^t\left(M_{\tilde u^2}(s)+\int_{-r_1}^{r_2}\left|\partial_y\tilde\theta(s,y)\right|^2\right)\mathrm{d}y\,\mathrm{d}s<cE_2(0)+c\qquad (t\in[0,T]).$$

Therefore, integrating (3.65) with respect to time, using again the lower bound for $\tilde\theta$ and the upper bound for $\tilde v$, together with some elementary inequalities, we obtain the conclusion (3.63). □

3.5. *Proof of Theorem 3.1*

In this paragraph we derive the last estimates necessary to prove our main existence result and we give the formal proof of Theorem 3.1.

Proposition 3.13. *There exists a constant $c>0$ such that*

$$\left\|\partial_y\tilde u(t,\cdot)\right\|^2_{L^2[-r_1,r_2]}+\int_0^t\left\|\partial_{y,y}\tilde u\right\|^2_{L^2[-r_1,r_2]}\mathrm{d}t\leqslant c\qquad (t\in[0,T]).\tag{3.66}$$

Proof. We multiply (3.7) by $-\partial_{y,y}\tilde{u}$ and we integrate on $[-r_1,0]$ and on $[0,r_2]$. We get

$$-\int_{-r_1}^{r_2}(\partial_t\tilde{u})\,(\partial_{y,y}\tilde{u})\,\mathrm{d}y+\int_{-r_1}^{r_2}\frac{\mu}{\tilde{v}}\left|\partial_{y,y}\tilde{u}\right|^2\,\mathrm{d}y=\int_{-r_1}^{r_2}\frac{1}{\tilde{v}^2}(\partial_y\tilde{v})(\partial_y\tilde{u})(\partial_{y,y}\tilde{u})\,\mathrm{d}y$$
$$+\int_{-r_1}^{r_2}\frac{1}{\tilde{v}}\,(\partial_y\tilde{\theta})\,(\partial_{y,y}\tilde{u})\,\mathrm{d}y-\int_{-r_1}^{r_2}\frac{1}{\tilde{v}^2}\,(\partial_y\tilde{v})\,\tilde{\theta}\,(\partial_{y,y}\tilde{u})\,\mathrm{d}y\qquad(t\in[0,T]).\tag{3.67}$$

Integrating by parts the first term in the left hand side of the above formula and using (1.6), we see that

$$-\int_{-r_1}^{r_2}(\partial_t\tilde{u})\,(\partial_{y,y}\tilde{u})\,\mathrm{d}y=-\int_{-r_1}^{0}(\partial_t\tilde{u})(\partial_{y,y}\tilde{u})\,\mathrm{d}y-\int_{0}^{r_2}(\partial_t\tilde{u})\,(\partial_{y,y}\tilde{u})\,\mathrm{d}y$$
$$=\int_{-r_1}^{0}(\partial_{t,y}\tilde{u})\,(\partial_y\tilde{u})\,\mathrm{d}y-\frac{\mathrm{d}g}{\mathrm{d}t}(t)\,(\partial_y\tilde{u})(t,0^-)+\int_{0}^{r_2}(\partial_{t,y}\tilde{u})(\partial_y u)\,\mathrm{d}y+\frac{\mathrm{d}g}{\mathrm{d}t}(t)\,(\partial_y\tilde{u})(t,0^+)$$
$$=\frac{1}{2}\int_{-r_1}^{r_2}\partial_t\left|\partial_y\tilde{u}\right|^2\,\mathrm{d}y+\frac{\mathrm{d}g}{\mathrm{d}t}(t)\left[\partial_y\tilde{u}(t,\cdot)\right]_{y\to0^-}^{y\to0^+}\qquad(t\in[0,T]).$$

For the last term in the right hand side of the above formula we have that

$$\frac{\mathrm{d}g}{\mathrm{d}t}(t)\left[\partial_y\tilde{u}(t,\cdot)\right]_{y\to0^-}^{y\to0^+}\leqslant c\,\frac{\mathrm{d}g}{\mathrm{d}t}(t)\left(\left\|\partial_y\tilde{u}(t,\cdot)\right\|_{C[-r_1,0]}+\|\partial_y\tilde{u}(t,\cdot)\|_{C[0,r_2]}\right)$$
$$=c\,\frac{1}{M}\left[\left(\frac{\mu}{\tilde{v}}\partial_y\tilde{u}-\frac{1}{\tilde{v}}\tilde{\theta}\right)(t,\cdot)\right]_{y\to0^-}^{y\to0^+}\left(\left\|\partial_y\tilde{u}(t,\cdot)\right\|_{C[-r_1,0]}+\|\partial_y\tilde{u}(t,\cdot)\|_{C[0,r_2]}\right)$$
$$\leqslant c\left(\left\|\partial_y\tilde{u}\right\|^2_{L^\infty[-r_1,r_2]}+\|\tilde{\theta}\|^2_{W^{1,2}(-r_1,0)}+\|\tilde{\theta}\|^2_{W^{1,2}(0,r_2)}\right)\qquad(t\in[0,T]),$$

where we have used (3.10) and (3.55). On the other hand, Gagliardo–Nirenberg inequality together with Young's inequality yield

$$\left\|\partial_y\tilde{u}\right\|_{L^\infty[-r_1,r_2]}\leqslant c\|\tilde{u}\|^{\frac{3}{4}}_{W^{2,2}(-r_1,0)\cap W^{2,2}(0,r_2)}\|\tilde{u}\|^{\frac{1}{4}}_{L^2[-r_1,r_2]}$$
$$\leqslant\varepsilon\left\|\partial_{y,y}\tilde{u}\right\|_{L^2[-r_1,r_2]}+\varepsilon\left\|\partial_y\tilde{u}\right\|_{L^2[-r_1,r_2]}+c(\varepsilon)\|\tilde{u}\|_{L^2[-r_1,r_2]},$$

where $\varepsilon>0$ is arbitrary and $c(\varepsilon)>0$ depends only on ε. Inserting the last two estimates into (3.67) and using again Young's inequality we get

$$\frac{1}{2}\int_{-r_1}^{r_2}\partial_t\left|\partial_y\tilde{u}\right|^2\,\mathrm{d}y+\int_{-r_1}^{r_2}\left|\partial_{y,y}\tilde{u}\right|^2\,\mathrm{d}y\leqslant\varepsilon\left\|\partial_{y,y}\tilde{u}\right\|^2_{L^2[-r_1,r_2]}+c(\varepsilon)\left(\left\|\partial_y\tilde{u}\right\|^2_{L^2[-r_1,r_2]}+\|\tilde{u}\|^2_{L^2[-r_1,r_2]}\right.$$
$$\left.+\int_{-r_1}^{r_2}\left|\partial_y\tilde{v}\right|^2\left|\partial_y\tilde{u}\right|^2\,\mathrm{d}y+\left\|\partial_y\tilde{\theta}\right\|^2_{L^2[-r_1,r_2]}+\left\|\partial_y\tilde{v}\right\|^2_{L^2[-r_1,r_2]}\right).\tag{3.68}$$

For ε small enough we may absorb first term on the right hand side. Since $\alpha\geqslant2$ in (1.10), Lemma 3.4 yields $\int_0^T\left\|\partial_y\tilde{\theta}\right\|^2_{L^2[-r_1,r_2]}\leqslant c$. Consequently, it suffices to integrate (3.68) with respect to time, to use Proposition 3.12 and Gronwall's inequality in order to obtain the desired estimate. □

Proposition 3.14. *There exists a constant* $c > 0$ *such that*

$$\left\| \partial_y \tilde{\theta}(t,\cdot) \right\|^2_{L^2[-r_1,r_2]} + \int_0^t \left\| \partial_t \tilde{\theta} \right\|^2_{L^2[-r_1,r_2]} \mathrm{d}t \leqslant c \qquad (t \in [0,T]). \tag{3.69}$$

Proof. We multiply (3.8) by $\kappa(\tilde{\theta})\partial_t\tilde{\theta}$ and we integrate over interval $(-r_1, r_2)$. Using the fact that $\partial_y\left(\kappa(\tilde{\theta})\partial_t\tilde{\theta}\right) = \partial_t\left(\kappa(\tilde{\theta})\partial_y\tilde{\theta}\right)$, we get

$$\int_{-r_1}^{r_2} c_v \kappa(\tilde{\theta}) \left|\partial_t\tilde{\theta}\right|^2 \mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t}\int_{-r_1}^{r_2} \frac{1}{\tilde{v}} \left|\kappa(\tilde{\theta})\partial_y\tilde{\theta}\right|^2 \mathrm{d}y \leqslant$$

$$\int_{-r_1}^{r_2} \frac{1}{2\tilde{v}^2} |\partial_t\tilde{v}| \left|\kappa(\tilde{\theta})\partial_y\tilde{\theta}\right|^2 \mathrm{d}y + \int_{-r_1}^{r_2} \partial_y\tilde{u}\left(\frac{1}{\tilde{v}}\partial_y\tilde{u} - \frac{\tilde{\theta}}{\tilde{v}}\right)\kappa(\tilde{\theta})\partial_t\tilde{\theta}\,\mathrm{d}y. \tag{3.70}$$

Due to (3.6) and Proposition 3.13 we have $\partial_t\tilde{v} \in L^2([0,T], L^\infty[-r_1,r_2])$. We use Young inequality in order to estimate the last term and we may absorb part of it to the left hand side. By Gronwall inequality we get the claim. $\square$

We are now in a position to prove our main existence and uniqueness result.

Proof of Theorem 3.1. Let $R,\ m_0,\ m_1 > 0$ be such that $\begin{bmatrix} \tilde{v}_0 \\ \tilde{u}_0 \\ \tilde{g}_0 \\ \tilde{\theta}_0 \end{bmatrix} \in B_{R,m_0,m_1}$ (recall that B_{R,m_0,m_1} is defined at the beginning of Subsection 3.2). Due to Theorem 3.3 there exists a strong solution defined on $[0,T]$, with $T > 0$ depending only on $R,\ m_0,\ m_1$. This solution can obviously be extended to a maximal one defined on $[0,\tau)$, with $\tau \in [0,\infty]$. Assume, by contradiction, that $\tau < \infty$. Then, combining Lemma 3.4, Lemma 3.8, Propositions 3.11, 3.12, 3.13 and 3.14 it follows that $\begin{bmatrix} \tilde{v}(t,\cdot) \\ \tilde{u}(t,\cdot) \\ g(t) \\ \tilde{\theta}(t,\cdot) \end{bmatrix} \in B_{\widetilde{R},\widetilde{m}_0,\widetilde{m}_1}$, with $\widetilde{R}, \widetilde{m}_0, \widetilde{m}_1$ depending only on $R,\ m_0,\ m_1$ and τ. Applying again Theorem 3.3 it follows that there exists $T > 0$ such that our strong solution can be defined on $[t, t+T]$ for every $t \in [0,\tau)$, which contradicts the supposed maximality of this solution. $\square$

4. Long time behavior

Our ultimate goal is to show Theorem 2.3. To this end, we first derive bounds on global-in-time solutions that are uniform in time. We denote by $c(\text{data})$ a constant depending only on the initial data, in particular, independent of time.

4.1. Mass conservation, energy and entropy estimates

We recall the following estimates proved in Section 2.3.

Proposition 4.1. *Denote* $m_L = \int_{-1}^{h_0} \rho_0(x)\,\mathrm{d}x,\ m_R = \int_{h_0}^1 \rho_0(x)\,\mathrm{d}x$.
Then the mass of gas on each side of the piston is conserved, specifically,

$$\int_{-1}^{h(\tau)} \rho(\tau,x)\,\mathrm{d}x = m_L, \qquad \int_{h(\tau)}^{1} \rho(\tau,x)\,\mathrm{d}x = m_R \qquad (\tau \geqslant 0). \tag{4.1}$$

In addition, the total energy is conserved, i.e.,

$$\int_{-1}^{1} \left(\frac{1}{2}(\rho u^2)(\tau, x) + c_v(\rho\theta)(\tau, x) \right) \, \mathrm{d}x + \frac{M}{2}|u(\tau, h(\tau))|^2 = E_0 \qquad (\tau \geqslant 0), \tag{4.2}$$

where

$$E_0 = \int_{-1}^{1} \left(\frac{1}{2}\rho_0(x)u_0^2(x) + c_v\rho_0(x)\theta_0(x) \right) \, \mathrm{d}x + \frac{M}{2}\, |u(0, h_0)|^2 . \tag{4.3}$$

Finally, the entropy is being produced in the course of motion,

$$\begin{aligned} &\int_0^\tau \left[\int_{-1}^{h(t)} \frac{1}{\theta} \left(\mu|\partial_x u|^2 + \frac{\kappa(\theta)|\partial_x\theta|^2}{\theta} \right) \, \mathrm{d}x + \int_{h(t)}^{1} \frac{1}{\theta} \left(\mu|\partial_x u|^2 + \frac{\kappa(\theta)|\partial_x\theta|^2}{\theta} \right) \, \mathrm{d}x \right] \, \mathrm{d}t \\ &= \left[\int_{-1}^{h(t)} \rho s(\rho, \theta) \, \mathrm{d}x \right]_{t=0}^{t=\tau} + \left[\int_{h(t)}^{1} \rho s(\rho, \theta) \, \mathrm{d}x \right]_{t=0}^{t=\tau} \qquad (\tau \geqslant 0). \end{aligned} \tag{4.4}$$

4.2. *Uniform bounds for the temperature and density fields*

As a consequence of Proposition 4.1 we get the following uniform bounds for the temperature.

Proposition 4.2. *Under the hypotheses of Theorem 2.3, let ρ, θ, u, h be a strong solution in $[0, \infty)$ of problem (1.2)–(1.9), (2.1). Then there exists a positive function $\Phi \in L^2(0, \infty)$, with $\|\Phi\|_{L^2(0,\infty)} \leqslant c(\mathrm{data})$ such that*

$$\max_{x\in[-1,1]} |\log(\theta)(\tau, x)| \leqslant c(\mathrm{data})\Big[1 + \Phi(\tau)\Big] \qquad (\tau \geqslant 0), \tag{4.5}$$

$$\max_{x\in[-1,1]} \theta(\tau, x) \leqslant c(\mathrm{data})\Big[1 + \Phi(\tau)\Big] \qquad (\tau \geqslant 0). \tag{4.6}$$

Proof. We first note that the terms on the right-hand side of (4.4) evaluated at $t = 0$ are controlled by the initial data. Consequently, using (4.1), (4.2) and hypothesis (1.10) we may infer that for every $\tau \geqslant 0$ we have

$$\int_0^\tau \int_{-1}^{1} \frac{1}{\theta}|\partial_x u|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant c(\mathrm{data}), \tag{4.7}$$

$$\int_0^\tau \left[\int_{-1}^{h(t)} \Big(|\partial_x\theta|^2 + |\partial_x \log(\theta)|^2 \Big) \, \mathrm{d}x + \int_{h(t)}^{1} \Big(|\partial_x\theta|^2 + |\partial_x \log(\theta)|^2 \Big) \, \mathrm{d}x \right] \mathrm{d}t \leqslant c(\mathrm{data}), \tag{4.8}$$

and

$$\int_{-1}^{1} \rho|\log(\theta)|(\tau, x) \, \mathrm{d}x + \int_{-1}^{1} \rho|\log(\rho)|(\tau, x) \, \mathrm{d}x \leqslant c(\mathrm{data}). \tag{4.9}$$

To proceed we need the following observation proved in [8, Lemma 3.1]:

$$\max_{x\in I} |w(x)| \leqslant \int_I |\partial_x w| \, \mathrm{d}x + \frac{1}{m}\int_I \rho|w| \, \mathrm{d}x \tag{4.10}$$

for any interval $I \subset (-1,1)$, any w absolutely continuous in I, and any non-negative measurable ρ,

$$\int_I \rho \, \mathrm{d}x = m > 0.$$

Applying (4.10) to $w = \log(\theta)$ we obtain, in accordance with (4.1), (4.9)

$$\max_{x\in[-1,h(t)]} |\log(\theta)(t,x)| \leqslant \int_{-1}^{h(t)} |\partial_x \log(\theta)|(t,\cdot)\, \mathrm{d}x + \frac{1}{r_1}\int_{-1}^{h(t)} \rho|\log(\theta)|(t,\cdot)\, \mathrm{d}x$$
$$\leqslant c(\text{data})\left[1 + \left(\int_{-1}^{h(t)} |\partial_x \log(\theta)|^2 \, \mathrm{d}x\right)^{1/2}\right],$$

where, by virtue of (4.8), the integral on the right-hand side belongs to $L^2(0,\infty)$. Repeating the same arguments on the interval $(h(t),1)$ we obtain the conclusion (4.5).

The method of the proof of (4.6) is similar. It is sufficient to use (4.2) instead of (4.9), so we skip it. □

Remark 4.3. A simple but important observation is that the piston always maintains a positive distance from the fixed boundary of both chambers, namely

$$-1 < \underline{h}(\text{data}) \leqslant h(t) \leqslant \overline{h}(\text{data}) < 1 \qquad (t \geqslant 0). \tag{4.11}$$

Indeed it follows from (4.1), (4.9) that

$$0 < m_L = \int_{-1}^{h(t)} \rho(t,x)\, \mathrm{d}x = \int_{x\in(-1,h(t)),\rho(t,x)\leqslant K} \rho(t,x)\, \mathrm{d}x + \int_{x\in(-1,h(t)),\rho(t,x)>K} \rho(t,x)\, \mathrm{d}x$$
$$\leqslant K(h(t)+1) + \frac{1}{\log(K)}\int_{-1}^{1} \rho|\log(\rho)|(t,\cdot)\, \mathrm{d}x \leqslant K(h(t)+1) + \frac{c(\text{data})}{\log(K)}$$

for any $K > 1$; whence $h(t)$ admits the lower bound $\underline{h} > -1$. The existence of the upper bound can be shown in a similar manner.

We are ready to show a crucial estimate for the density ρ.

Proposition 4.4. *Under the hypotheses of Theorem 2.3, let ρ, θ, u, h be a strong solution in $[0,\infty)$ of problem (1.2)–(1.9), (2.1).*

Then

$$0 < \rho(\tau,x) \leqslant \overline{\rho}(\text{data}) \qquad (\tau \geqslant 0,\ x \in [-1,1]). \tag{4.12}$$

Proof. Following Straškraba [22] we note that

$$\partial_t \log(\rho) + u\partial_x \log(\rho) = -\partial_x u \qquad (t \geqslant 0, \quad -1 < x < h(t)). \tag{4.13}$$

By the momentum equation we get

$$\partial_x u = \frac{1}{\mu}\left(p + \rho u^2 + \partial_t \int_{-1}^{x} \rho u \mathrm{d}z\right) + c$$

for some constant c depending on t which can be determined by a condition

$$\frac{\mathrm{d}}{\mathrm{d}t} h(t) = u(t,h(t)) - u(t,-1) = \int_{-1}^{h(t)} \partial_x u.$$

Consequently, the derivative $\partial_x u$ can be expressed as

$$\partial_x u = \frac{1}{\mu}\left[p - \frac{1}{1+h(t)}\int_{-1}^{h(t)} p \, \mathrm{d}x + \rho u^2 - \frac{1}{1+h(t)}\int_{-1}^{h(t)} \rho u^2 \, \mathrm{d}x \right] + \frac{1}{\mu}\partial_t \int_{-1}^{x} \rho u \, \mathrm{d}z - \frac{1}{\mu}\frac{1}{1+h(t)}\int_{-1}^{h(t)} \left(\partial_t \int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\log(1+h(t)).$$

Furthermore, in accordance with (1.6),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{1+h(t)}\int_{-1}^{h(t)} \left(\int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x \right] = -\frac{u(t,h(t))}{(1+h(t))^2}\int_{-1}^{h(t)} \left(\int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x + \frac{u(t,h(t))}{1+h(t)}\int_{-1}^{h(t)} \rho u \, \mathrm{d}x + \frac{1}{1+h(t)}\int_{-1}^{h(t)} \left(\partial_t \int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x.$$

Going back to (4.13) we obtain

$$\begin{aligned} &\partial_t \left(\log(\rho) + \chi\right) + u\partial_x \left(\log(\rho) + \chi\right) \\ &\quad = \frac{1}{\mu}\left[\frac{1}{1+h(t)}\int_{-1}^{h(t)} p \, \mathrm{d}x - p + \frac{1}{1+h(t)}\int_{-1}^{h(t)} \rho u^2 \, \mathrm{d}x \right] \\ &\quad + \frac{u(t,h(t))}{\mu(1+h(t))^2}\int_{-1}^{h(t)} \left(\int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x - \frac{u(t,h(t))}{\mu(1+h(t))}\int_{-1}^{h(t)} \rho u \, \mathrm{d}x, \end{aligned} \tag{4.14}$$

where we have set

$$\chi = \log(1+h(t)) + \frac{1}{\mu}\left[\int_{-1}^{x} \rho u \, \mathrm{d}z - \frac{1}{1+h(t)}\int_{-1}^{h(t)} \left(\int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x \right].$$

It is worth pointing out that χ is bounded due to Proposition 4.1 and Remark 4.3.

We observe that the scalar functions

$$\begin{aligned} &t \mapsto \frac{1}{1+h(t)}\int_{-1}^{h(t)} \rho u^2 \, \mathrm{d}x \\ &t \mapsto \frac{u(t,h(t))}{1+h(t)}\int_{-1}^{h(t)} \left(\int_{-1}^{x} \rho u \, \mathrm{d}z \right) \mathrm{d}x \\ &t \mapsto \frac{u(t,h(t))}{1+h(t)}\int_{-1}^{h(t)} \rho u \, \mathrm{d}x \end{aligned} \tag{4.15}$$

are integrable over the half-line $(0,\infty)$ with the $L^1(0,\infty)$-norm bounded in terms of the initial data. To see this first observe that thanks to (4.5), (4.6), there exist positive constants $\underline{\theta}(\text{data})$, $\overline{\theta}(\text{data})$ and a set $\mathcal{S} \subset [0,\infty)$ of finite Lebesgue measure such that

$$0 < \underline{\theta} \leqslant \theta(t,\cdot) \leqslant \overline{\theta} \qquad (t \in (0,\infty) \setminus \mathcal{S}). \tag{4.16}$$

Consequently, in view of the energy bounds (4.2), (4.7), and Poincaré's inequality,

$$\begin{aligned}\int_0^\tau \int_{-1}^1 \rho u^2 \,\mathrm{d}x &\leqslant \int_{\mathcal{S}} \int_{-1}^1 \rho u^2 \,\mathrm{d}x \,\mathrm{d}t + (r_1 + r_2) \int_{(0,\tau)\setminus\mathcal{S}} \|u(t,\cdot)\|^2_{L^\infty(-1,1)} \,\mathrm{d}t \\ &\leqslant E_0|\mathcal{S}| + c(r_1+r_2) \int_{(0,\tau)\setminus\mathcal{S}} \|\partial_x u(t,\cdot)\|^2_{L^2(-1,1)} \,\mathrm{d}t \\ &\leqslant E_0|\mathcal{S}| + (r_1+r_2)\,\overline{\theta}\, c(\mathrm{data}),\end{aligned}$$

uniformly for $\tau \to \infty$.

Seeing that the remaining two functions in (4.15) can be handled in a similar fashion we may rewrite (4.14) as

$$\partial_t \left(\log(\rho) + \chi\right) + u\partial_x \left(\log(\rho) + \chi\right) = \frac{1}{\mu}\left[\frac{1}{1+h(t)}\int_{-1}^{h(t)} p \,\mathrm{d}x - p\right] + k(t), \tag{4.17}$$

with

$$\|k\|_{L^1\cap L^\infty(0,\infty)} \leqslant c(\mathrm{data}).$$

For the new quantity

$$\eta(t,x) = \exp\left(\chi(t,x) + \int_t^\infty k(s)\,\mathrm{d}s\right),$$

relation (4.17) reads

$$\partial_t\left(\frac{1}{\rho\eta}\right) + u\partial_x\left(\frac{1}{\rho\eta}\right) = \frac{\theta}{\mu\eta} - \left(\frac{1}{\rho\eta}\right)\frac{1}{\mu(1+h(t))}\int_{-1}^{h(t)} p \,\mathrm{d}x. \tag{4.18}$$

By virtue of the energy estimates (4.2) and (4.11) we conclude that

$$\partial_t\left(\frac{1}{\rho\eta}\right) + u\partial_x\left(\frac{1}{\rho\eta}\right) \geqslant K_1\theta - K_2\left(\frac{1}{\rho\eta}\right) \tag{4.19}$$

for certain $K_1 > 0$, $K_2 > 0$ depending solely on the initial data. Thus a simple comparison argument yields

$$\frac{1}{\rho\eta}(\tau,x) \geqslant Y(\tau) \text{ for } -1 < x < h(\tau), \tag{4.20}$$

where Y solves

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = K_1\tilde{\theta}(t) - K_2 Y(t),\ Y(0) = \min_{0<x<h_0} \frac{1}{\rho_0(x)\eta(0,x)} > 0, \tag{4.21}$$

with

$$\tilde{\theta}(t) = \inf_{x\in(-1,1)} \theta(t,x).$$

Consequently, in accordance with (4.16),

$$
\begin{aligned}
Y(\tau) &= \exp(-K_2\tau)Y(0) + K_1 \exp(-K_2\tau)\int_0^\tau \exp(K_2 t)\tilde{\theta}(t)\,\mathrm{d}t \\
&\geqslant \exp(-K_2\tau)Y(0) + K_1 \exp(-K_2\tau)\int_{(0,\tau)\setminus\mathcal{S}} \exp(K_2 t)\tilde{\theta}(t)\,\mathrm{d}t \\
&\geqslant \exp(-K_2\tau)Y(0) + \underline{\theta}K_1\int_0^\tau \exp(K_2(t-\tau))\,\mathrm{d}t \\
&\quad - \underline{\theta}K_1\exp(-K_2\tau)\int_{\mathcal{S}\cap(0,\tau)} \exp(K_2 t)\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\underline{\theta}K_1\exp(-K_2\tau)\int_{\mathcal{S}\cap(0,\tau)} \exp(K_2 t)\,\mathrm{d}t \to 0 \text{ for } \tau\to\infty.
$$

As η is uniformly bounded below and above in $(0,\infty)$ we conclude that

$$
0 < \rho(\tau,x) \leqslant \overline{\rho}(\text{data}) \text{ for all } \tau \geqslant 0,\ -1 < x < h(\tau),
$$

and, repeating step by step the previous arguments on the set $h(\tau) < x < 1$ we obtain (4.12). □

4.3. Convergence to equilibria

Having established all necessary uniform bounds we are ready to complete the proof of Theorem 2.3. To this end, we make use of the relative energy method developed in [7]. Specifically, we introduce the ballistic free energy function

$$
H_\Theta(\rho,\theta) = \rho e(\rho,\theta) - \rho\Theta s(\rho,\theta) = c_v\rho\theta - \rho\Theta\log\left(\frac{\theta^{c_v}}{\rho}\right),
$$

and the associated *relative energy functional*

$$
\begin{aligned}
&\mathcal{E}\left(\rho,\theta,u,h \,\middle|\, r,\Theta,U\right) = \\
&\int_{-1}^{1}\left[\frac{1}{2}\rho|u-U|^2 + H_\Theta(\rho,\theta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\rho}(\rho-r) - H_\Theta(r,\Theta)\right]\mathrm{d}x + \frac{M}{2}|u(h)|^2,
\end{aligned}
\tag{4.22}
$$

with r,Θ,U suitable "test" functions. As shown in [6, Chapter 5, Lemma 5.1], the functional $\mathcal{E}$ represents a "distance" between the quantities $[\rho,\theta,u]$ and $[r,\Theta,U]$. More precisely, for any compact set $K\subset(0,\infty)^2$, there exists a positive constant $c(K)$, depending solely on the structural properties of the thermodynamic functions e and s such that

$$
\begin{aligned}
&H_\Theta(\rho,\theta) - \frac{\partial H_\Theta(r,\Theta)}{\partial\rho}(\rho-r) - H_\Theta(r,\Theta) \\
&\quad\geq c(K)\begin{cases} |\rho-r|^2 + |\theta-\Theta|^2 \text{ if } [\rho,\theta]\in K,\ [r,\Theta]\in K \\ 1+\rho e(\rho,\theta)+\rho|s(\rho,\theta)| \text{ if } [\rho,\theta]\in(0,\infty)^2\setminus K,\ [r,\Theta]\in K. \end{cases}
\end{aligned}
\tag{4.23}
$$

Our strategy is based on the following steps:

- We take

$$
U=0,\ r(t,x)=\begin{cases}\rho_L \text{ for } x\in(-1,h(t)),\\ \rho_R \text{ for } x\in(h(t),1),\end{cases}\quad \Theta(t,x)=\begin{cases}\theta_L \text{ for } x\in(-1,h(t)),\\ \theta_R \text{ for } x\in(h(t),1),\end{cases}
\tag{4.24}
$$

  where $\rho_L,\rho_R,\theta_L,\theta_R$ are the constants determined through (2.14) as test functions in (4.22).

- We observe that
$$\mathcal{E}(t) \to \mathcal{E}_\infty$$
as $t \to \infty$.
- We show that $\mathcal{E}_\infty = 0$, which, in view of (4.23), yields the convergence claimed in Theorem 2.3.

To carry out the above delineated programme, we first observe that, with the ansatz (4.24),

$$\begin{aligned}
\mathcal{E}\left(\rho,\theta,u,h\,\middle|\,r,\Theta,U\right)(t) &= \int_{-1}^{1}\left[\frac{1}{2}\rho u^2 + c_v\rho\theta\right](t,\cdot)\,\mathrm{d}x + \frac{M}{2}|u(t,h(t))|^2 \\
&- \theta_L\int_{-1}^{h(t)}\rho s(\rho,\theta)\ \mathrm{d}x - \theta_R\int_{h(t)}^{1}\rho s(\rho,\theta)\ \mathrm{d}x \\
&- c_v\theta_L\int_{-1}^{h(t)}\rho\ \mathrm{d}x - c_v\theta_R\int_{h(t)}^{1}\rho\ \mathrm{d}x \\
&+ \theta_L\left[\log(\theta_L^{c_v}) - \log(\rho_L) - 1\right]\int_{-1}^{h(t)}\rho\ \mathrm{d}x + \theta_R\left[\log(\theta_R^{c_v}) - \log(\rho_R) - 1\right]\int_{h(t)}^{1}\rho\ \mathrm{d}x \\
&+ \frac{1}{c_v}E_0,
\end{aligned}$$

where we have used the identity

$$\frac{\partial H_\Theta(r,\Theta)}{\partial\rho}r - H_\Theta(r,\theta) = p(r,\Theta),$$

together with (2.14), specifically,

$$\rho_L\theta_L = \rho_R\theta_R = \rho_\infty\theta_\infty = \frac{1}{2c_v}E_0.$$

Consequently, in accordance with (2.9) and Proposition 4.1, the function

$$t \mapsto \mathcal{E}\left(\rho,\theta,u,h\,\middle|\,r,\Theta,U\right)(t),\ \text{with the test functions (4.24)},$$

is non-increasing in t, more specifically,

$$\begin{aligned}
\mathcal{E}\left(\rho,\theta,u,h\,\middle|\,r,\Theta,U\right)(t) \searrow\ & E_0 - \theta_L S_L - \theta_R S_R - c_v\theta_L m_L - c_v\theta_R m_R \\
&+ \theta_L s(\rho_L,\theta_L)m_L - \theta_L m_L + \theta_R s(\rho_R,\theta_R)m_R - \theta_R m_R + \frac{1}{c_v}E_0
\end{aligned} \tag{4.25}$$

as $t \to \infty$. Thus our task reduces to showing that the right-hand side of (4.25) is actually equal to zero. Moreover, as $\mathcal{E}$ is a non-increasing function of time, it is enough to show that any sequence of times $T_n \to \infty$ contains a subsequence (not relabeled) such that

$$\int_{T_n}^{T_n+1}\mathcal{E}\left(\rho,\theta,u,h\,\middle|\,r,\Theta,U\right)\ \mathrm{d}t \to 0 \text{ as } T_n \to \infty. \tag{4.26}$$

4.3.1. Vanishing velocity time averages

We first show that the velocity field tends to 0 when $t \to \infty$ and provide some information on the large time behavior of the piston position h.

Proposition 4.5. *The velocity field u satisfies*

$$\lim_{T\to\infty} \int_T^{T+1} \left(\|u(t,\cdot)\|_{L^\infty(-1,1)} + \|\partial_x u(t,\cdot)\|_{L^1(-1,1)} \right) \mathrm{d}t = 0. \tag{4.27}$$

Moreover, every sequence $T_n \to \infty$ contains a subsequence (not relabeled) such that

$$\sup_{t\in[T_n,T_n+1]} |h(t) - \hat{h}_\infty| \to 0 \textit{ for } n \to \infty, \tag{4.28}$$

for some $\hat{h}_\infty \in (-1,1)$.

Proof. Using Poincaré's inequality, we have

$$\begin{aligned}
\max_{x\in[-1,1]} |u(t,x)| &\leqslant c \left[\int_{-1}^{h(t)} |\partial_x u|(t,x)\ \mathrm{d}x + \int_{h(t)}^{1} |\partial_x u|(t,x)\ \mathrm{d}x \right] = \\
&= c \left[\int_{-1}^{h(t)} \frac{\sqrt{\theta}}{\sqrt{\theta}} |\partial_x u|(t,x)\ \mathrm{d}x + \int_{h(t)}^{1} \frac{\sqrt{\theta}}{\sqrt{\theta}} |\partial_x u|(t,x)\ \mathrm{d}x \right] \\
&\leqslant \varepsilon \left[\int_{-1}^{h(t)} \theta\ \mathrm{d}x + \int_{h(t)}^{1} \theta\ \mathrm{d}x \right] \\
&\quad + c(\varepsilon) \left[\int_{-1}^{h(t)} \frac{1}{\theta} |\partial_x u|^2(t,x)\ \mathrm{d}x + \int_{h(t)}^{1} \frac{1}{\theta} |\partial_x u|^2(t,x)\ \mathrm{d}x \right] \qquad (t \geqslant 0)
\end{aligned} \tag{4.29}$$

for any $\varepsilon > 0$. Furthermore, by virtue of (4.10) (applied to $w = \sqrt{\theta}$),

$$\int_{-1}^{h(t)} \theta \mathrm{d}x \le 2\|\sqrt{\theta}\|^2_{L^\infty(-1,h(t))} \le c \left(\int_{-1}^{h(t)} |\partial_x \sqrt{\theta}| \mathrm{d}x \right)^2 + 2 \left(\frac{1}{m_L} \int_{-1}^{h(t)} \rho \sqrt{\theta} \mathrm{d}x \right)^2,$$

where, by Hölder inequality,

$$\frac{1}{m_L} \int_{-1}^{h(t)} \sqrt{\rho}\sqrt{\rho\theta} \mathrm{d}x \le \frac{1}{r_1} \left(\int_{-1}^{h(t)} \rho \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{-1}^{h(t)} \rho\theta \mathrm{d}x \right)^{\frac{1}{2}} = \frac{1}{\sqrt{m_L}} \left(\int_{-1}^{h(t)} \rho\theta \mathrm{d}x \right)^{\frac{1}{2}}.$$

The same considerations could be done for $\int_{h(t)}^1 \theta \mathrm{d}x$ therefore

$$\int_{-1}^{h(t)} \theta\ \mathrm{d}x \leqslant c \int_{-1}^{h(t)} |\partial_x \sqrt{\theta}|^2\ \mathrm{d}x + \frac{2}{m_L} \int_{-1}^{h(t)} \rho\theta\ \mathrm{d}x \qquad (t \geqslant 0),$$

$$\int_{h(t)}^{1} \theta\ \mathrm{d}x \leqslant c \int_{h(t)}^{1} |\partial_x \sqrt{\theta}|^2\ \mathrm{d}x + \frac{2}{m_R} \int_{h(t)}^{1} \rho\theta\, \mathrm{d}x \qquad (t \geqslant 0).$$

Next, we integrate (4.29) over a time interval $(T, T+1)$ and use (4.2) to get

$$\int_T^{T+1} \max_{x\in[-1,1]} |u(t,x)|\mathrm{d}t + \int_T^{T+1}\int_{-1}^{1} |\partial_x u|(t,x)\mathrm{d}x \le \varepsilon \int_T^{T+1}\int_{-1}^{1} \rho\theta \mathrm{d}x\ \mathrm{d}t$$
$$+ c(\varepsilon)\left[\int_T^{T+1}\int_{-1}^{1} |\partial_x\sqrt{\theta}|^2 + \frac{1}{\theta}|\partial_x u|^2 \mathrm{d}x\ \mathrm{d}t\right] \le \varepsilon E_0 + c(\varepsilon)\Gamma(T), \tag{4.30}$$

where $\Gamma(T) \to 0$ for $T \to \infty$ as a consequence of (4.4). Thus we obtain (4.27) because (4.30) holds for every $\varepsilon > 0$.

Finally, by virtue of (4.2), both $h(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}h(t)$ are bounded and Arzelà–Ascoli theorem yields $h(T_n+t) \to \hat{h}_\infty(t)$ strongly in $C[0,1]$ up to a subsequence. As the velocity of the piston is determined by (1.6), we deduce from (4.27) that

$$\int_{T_n}^{T_n+1} \left|\frac{\mathrm{d}}{\mathrm{d}t}h(t)\,\mathrm{d}t\right| \le \int_{T_n}^{T_n+1} \max_{x\in[-1,1]} |u(t,x)| \to 0\,\mathrm{d}t,$$

and thus (4.28) also holds, where, in accordance with (4.11), we have that $\hat{h}_\infty \in (-1,1)$. □

4.3.2. Time shifts of ρ and θ

Our next result provides weak convergence of time-shifts of the state variables. To this end, we consider a sequence of time $T_n \to \infty$ and introduce the following notation for the time shifts of a function v:

$$v^n(t,x) = v(T_n + t, x).$$

Proposition 4.6. *Let $T_n \to \infty$ be a sequence of times such that*

$$h^n \to \hat{h}_\infty \text{ in } C[0,1]$$

(cf. (4.28)).

Then there exists a subsequence (not relabeled) such that

$$\rho^n \to \hat{\rho}_\infty \text{ weakly-(*) in } L^\infty((0,1)\times(-1,1))$$
$$\text{and in } C_{\text{weak}}([0,1]; L^q(-1,1)) \text{ for any } 1 \le q < \infty, \tag{4.31}$$
$$\theta^n \to \hat{\theta}_\infty \text{ weakly in } L^2((0,1)\times(-1,1)) \text{ as } n \to \infty, \tag{4.32}$$

where

$$\hat{\rho}_\infty = \begin{cases} \hat{\rho}_L \text{ for } -1 < x < \hat{h}_\infty, \\ \hat{\rho}_R \text{ for } \hat{h}_\infty < x < 1, \end{cases}$$
$$\hat{\theta}_\infty = \begin{cases} \hat{\theta}_L \text{ for } -1 < x < \hat{h}_\infty, \\ \hat{\theta}_R \text{ for } \hat{h}_\infty < x < 1, \end{cases}$$

and $\hat{\rho}_L,\ \hat{\rho}_R,\ \hat{\theta}_L,\ \hat{\theta}_R$ are the strictly positive constants satisfying

$$\hat{\rho}_L = \frac{m_L}{1+\hat{h}_\infty},\ \hat{\rho}_R = \frac{m_R}{1-\hat{h}_\infty},\ \hat{\rho}_L\hat{\theta}_L = \hat{\rho}_R\hat{\theta}_R = \frac{1}{2c_v}E_0. \tag{4.33}$$

Proof. As the velocity field u^n is continuous on the free surface, the equation of continuity (1.2) is satisfied on the whole set $(0,1)\times(-1,1)$ in the sense of distributions. Consequently, in accordance with (4.12), we may assume

$$\rho^n \to \hat{\rho}_\infty \text{ weakly-(*) in } L^\infty((0,1)\times(-1,1))$$
$$\text{and (strongly) in } C_{\text{weak}}([0,1]; L^q(-1,1)) \text{ for any } 1 \le q < \infty.$$

Moreover, Propositions 4.2 and 4.4 yield $\|p(\rho^n,\theta^n)\|_{L^2((0,1)\times(-1,1))} \le c$. We multiply (1.3) by u^n and integrate over (T_n, T_n+1) in order to deduce that $\|\partial_x u^n\|_{L^2(0,1;L^2(-1,1))} \le c$. This together with (4.27) imply

$$u^n \to 0 \text{ in } L^2(0,1; W_0^{1,q}(-1,1)) \text{ for any } 1 \le q < 2,$$

whence, passing to the limit in the weak formulation of (1.2), we may infer that $\partial_t \hat\rho_\infty = 0$ in the sense of distributions. Consequently, $\hat\rho_\infty = \hat\rho_\infty(x)$ depends only on the spatial variable, and we may set

$$\hat\rho_\infty = \begin{cases} \hat\rho_L = \hat\rho_L(x) \text{ for } -1 < x < \hat h_\infty, \\ \hat\rho = \hat\rho_R(x) \text{ for } \hat h_\infty < x < 1. \end{cases}$$

Next, by virtue of (4.2), (4.4), and (4.10) we obtain (4.32) with

$$\hat\theta_\infty = \begin{cases} \hat\theta_L = \hat\theta_L(t) \text{ for } -1 < \hat x < h_\infty, \\ \hat\theta_R = \hat\theta_R(t) \text{ for } \hat h_\infty < x < 1. \end{cases}$$

In addition, (4.16) implies that

$$0 < \underline{\theta} \leqslant \hat\theta_L, \hat\theta_R \leqslant \overline{\theta}. \tag{4.34}$$

Now, we rewrite the momentum balance in its weak formulation,

$$\begin{aligned} &\int_0^\infty \int_{-1}^1 \left[\rho u(\partial_t\varphi) + \rho u^2(\partial_x\varphi) + p(\partial_x\varphi)\right] \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\infty \int_{-1}^1 \mu\partial_x u\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty M\frac{\mathrm{d}^2 h}{\mathrm{d}t^2}(t)\varphi(t,h(t)) \,\mathrm{d}t \text{ for any } \varphi \in C_c^1((0,\infty)\times(-1,1)), \end{aligned} \tag{4.35}$$

where

$$\begin{aligned} \int_0^\infty M\frac{\mathrm{d}^2 h}{\mathrm{d}t^2}(t)\varphi(t,h(t)) \,\mathrm{d}t &= -\int_0^\infty M\frac{\mathrm{d}h}{\mathrm{d}t}(t)\left[\partial_t\varphi(t,h(t)) + \partial_x\varphi(t,h(t))\frac{\mathrm{d}h}{\mathrm{d}t}(t)\right] \,\mathrm{d}t \\ &= -\int_0^\infty Mu(t,h(t))\left[\partial_t\varphi(t,h(t)) + \partial_x\varphi(t,h(t))u(t,h(t))\right] \,\mathrm{d}t. \end{aligned}$$

Consequently, using (4.12), (4.27), we may perform the limit in the pressure time shifts p^n in (4.35) to conclude that

$$p^n = \rho^n\theta^n \to p_\infty \text{ weakly in } L^2((0,1)\times(-1,1)), \tag{4.36}$$

where $\nabla_x p_\infty = 0$ (in the weak sense); whence $p_\infty = p_\infty(t)$ is a function of t only.

The final observation is that

$$p_\infty = \hat\rho_\infty\hat\theta_\infty. \tag{4.37}$$

To see this, consider an arbitrary space–time cylinder $(0,1)\times(-1,r)$, $r < \hat h_\infty$. In view of the uniform bound on $\partial_x\theta^n$ established in (4.8), we get

$$\theta^n \to \hat\theta_L \text{ weakly in } L^2(0,1; W^{1,2}(-1,r)).$$

On the other hand, as we have already shown (4.31),

$$\rho^n \to \hat\rho_L \text{ (strongly) in } C([0,T]; W^{-1,2}(-1,r)),$$

and we may conclude that

$$\hat p_\infty = \hat\rho_L\hat\theta_L \text{ in } (0,1)\times(-1,r)$$

for any $r < h_\infty$. Similarly, we show that

$$\hat{p}_\infty = \hat{\rho}_R \hat{\theta}_R \text{ in } (0,1) \times (r,1)$$

for any $r > h_\infty$; whence (4.37) follows.

Thus, necessarily,

$$p_\infty = \begin{cases} \hat{\rho}_L \hat{\theta}_L \text{ for } -1 < x < \hat{h}_\infty, \\ \hat{\rho}_R \hat{\theta}_R \text{ for } \hat{h}_\infty < x < 1 \end{cases}$$

is a function of t only. Thus, necessarily, $\hat{\rho}_L$, $\hat{\rho}_R$ are positive constants, and

$$\hat{\rho}_L = \frac{m_L}{1 + \hat{h}_\infty}, \ \hat{\rho}_R = \frac{m_R}{1 - \hat{h}_\infty}, \ \hat{\rho}_L \hat{\theta}_L = \hat{\rho}_R \hat{\theta}_R = p_\infty.$$

Finally, we use the total energy balance (4.2) to deduce that

$$c_v p_\infty = \frac{1}{2} E_0 \ - \ \text{a positive constant;}$$

whence $\hat{\theta}_L$, $\hat{\theta}_R$ are also positive constants. □

4.3.3. Strong convergence of ρ^n, θ^n

Our ultimate goal will be to show that

$$\hat{h}_\infty = h_\infty, \ \hat{\rho}_\infty = \rho_\infty, \ \hat{\theta}_\infty = \theta_\infty, \tag{4.38}$$

where h_∞, ρ_∞, θ_∞ are the quantities appearing in Theorem 2.3. To this end, it is enough to show strong (a.e. pointwise) convergence of the time shifts ρ_n, θ_n, specifically,

$$\rho^n \to \hat{\rho}_\infty, \ \theta^n \to \hat{\theta}_\infty \text{ (strongly) in } L^1((0,1) \times (-1,1)). \tag{4.39}$$

Indeed, the strong convergence in (4.39) implies pointwise convergence of the entropy and since $\rho^n s(\rho^n, \theta^n)$ is bounded in $L^2((0,1) \times (-1,1))$ due to (4.5) and (4.12), Vitali's convergence theorem yields

$$\rho^n s(\rho^n, \theta^n) \to \hat{\rho}_\infty s(\hat{\rho}_\infty, \hat{\theta}_\infty) \text{ (strongly) in } L^1((0,1) \times (-1,1) \text{ and weakly in } L^2((0,1) \times (-1,1)).$$

In particular, we get

$$S_L = (1 + \hat{h}_\infty) \hat{\rho}_L s(\hat{\rho}_L, \hat{\theta}_L), \ S_R = (1 - \hat{h}_\infty) \hat{\rho}_R s(\hat{\rho}_R, \hat{\theta}_R), \tag{4.40}$$

which, together with (4.33) and uniqueness of the equilibrium solutions with the same limit energy and entropy implies (4.38). Finally, we realize that (4.38), (4.40) yields the desired conclusion (4.26).

In the remaining part of this section, we therefore focus on the proof of (4.39). We start by proving strong convergence of the density. To this end, we adapt certain ideas of the general existence theory proposed by P.-L. Lions [16]. In particular, we claim the *effective viscous flux identity*

$$\overline{p(\rho,\theta)\rho} - \overline{p(\rho,\theta)}\rho = \mu \left(\overline{\rho \partial_x u} - \rho \partial_x u \right) \text{ in } (0,1) \times (-1,1), \tag{4.41}$$

where the bars denote weak limits of compositions. Identity (4.41) was proved (in the general 3D setting and for a general pressure function) in [5, Chapter 6, Proposition 6.1]. The proof is based on local arguments and so it is applicable to the sequences ρ^n, θ^n, u^n as well. In the present context, (4.41) gives rise to

$$(\rho^n)^2 \theta^n \to (\hat{\rho}_\infty)^2 \hat{\theta}_\infty \text{ weakly in } L^2((0,1) \times (-1,1)). \tag{4.42}$$

At this stage, we recall that the equation of continuity is in fact satisfied (in the weak sense) for $x \in (-1,1)$. In particular, the densities ρ^n satisfy also the *renormalized* equation

$$\partial_t (\rho^n)^2 + \partial_x ((\rho^n)^2 u^n) + (\rho^n)^2 \partial_x u^n = 0 \text{ in } \mathcal{D}'((0,1) \times (-1,1)).$$

Consequently, similarly to the proof of Proposition 4.6, we have

$$(\rho^n)^2 \to \overline{\rho^2} \text{ in } C([0,1]; W^{-1,2}(-1,1)), \ (\theta^n - \hat{\theta}_\infty) \to 0 \text{ weakly in } L^2(0,1; W^{1,2}(I))$$

for any compact subinterval $I \subset (-1, \hat{h}_\infty) \cup (\hat{h}_\infty, 1)$. Thus (4.42) yields

$$(\rho^n)^2 \hat{\theta}_\infty \to (\hat{\rho}_\infty)^2 \hat{\theta}_\infty \text{ weakly in } L^2((0,1) \times (-1,1)), \tag{4.43}$$

which immediately implies the desired strong convergence of $\{\rho^n\}_{n=1}^\infty$.

In order to see the strong convergence of the temperature shifts, we modify the previous arguments to observe that

$$\theta_n \to \hat{\theta}_\infty \text{ weakly in } L^2(0,T; W^{1,2}(I))$$
$$\rho^n \theta^n \to \hat{\rho}_\infty \hat{\theta}_\infty \text{ in } C([0,T]; W^{-1,2}(I))$$

for any compact interval $I \subset (-1, \hat{h}_\infty) \cup (\hat{h}_\infty, 1)$, from which we deduce the desired conclusion

$$\int_0^1 \int_{-1}^1 \rho_n (\theta_n)^2 \; \mathrm{d}x \; \mathrm{d}t \to \int_0^1 \int_{-1}^1 \hat{\rho}_\infty (\hat{\theta}_\infty)^2 \; \mathrm{d}x \; \mathrm{d}t. \tag{4.44}$$

As we already know that ρ^n converges strongly, (4.44) yields strong convergence of $\{\theta^n\}_{n=1}^\infty$. We have shown (4.39).

4.3.4. Proof of Theorem 2.3: conclusion

As the sequence of time $T_n \to \infty$ introduced in Section 4.3.2 was arbitrary, we have

$$h(t) \to h_\infty \text{ as } t \to \infty.$$

Moreover, the convergence stated in (4.26), together with the coercivity properties of the relative energy functional stated in (4.23) imply the asymptotic behavior claimed in (2.10)–(2.12) of Theorem 2.3, together with

$$\rho(t,\cdot)\theta(t,\cdot) \to \rho_\infty \theta_\infty \text{ in } L^1(-1,1). \tag{4.45}$$

Obviously, (4.45) implies (2.13) and therefore completes the proof of Theorem 2.3 as soon as we are able to establish a lower bound for the density,

$$\rho(t,\cdot) \geqslant \underline{\rho} > 0 \text{ for all } t \geqslant 0.$$

In order to prove that we deduce from (4.18) that

$$\partial_t \left(\frac{1}{\rho\eta} \right) + u \partial_x \left(\frac{1}{\rho\eta} \right) \leqslant K_1 \theta - K_2 \left(\frac{1}{\rho\eta} \right)$$

and the proof can be completed similarly as the proof of Proposition 4.4. For more details we refer also to [8, Proposition 6.1].

Conflict of interest statement

The authors declare that there is no conflict of interest.

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