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On a free boundary problem and minimal surfaces

Yong Liu [a], Kelei Wang [b], Juncheng Wei [c,*]

[a] *School of Mathematics and Physics, North China Electric Power University, Beijing, China*
[b] *School of Mathematics and Statistics, Wuhan University, China*
[c] *Department of Mathematics, University of British Columbia, Vancouver, B.C., V6T 1Z2, Canada*

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Abstract

From minimal surfaces such as Simons' cone and catenoids, using refined Lyapunov–Schmidt reduction method, we construct new solutions for a free boundary problem whose free boundary has two components. In dimension 8, using variational arguments, we also obtain solutions which are global minimizers of the corresponding energy functional. This shows that the theorem of Valdinoci et al. [41,42] is optimal.

Keywords: Free boundary problems; Minimal surfaces; Global minimizers; Allen–Cahn equation; Reduction method

1. Introduction

In this paper, we are interested in the following free boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega := \{-1 < u < 1\}, \\ |\nabla u| = 1 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here the domain $\Omega \subset \mathbb{R}^n$ is a priori unspecified and $\partial\Omega$ is the free boundary. Solutions of (1) arise formally as critical points of the energy functional:

$$J(u) := \int \left[|\nabla u|^2 + \chi_{(-1,1)}(u) \right]. \tag{2}$$

In this variational formulation, the boundary condition $|\nabla u| = 1$ should be understood in some weak sense if the free boundary $\partial\Omega$ is not regular enough. Problem (1) can be regarded as a simplified version of the classical one-phase free boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega := \{u > 0\}, \\ |\nabla u| = 1 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

* Corresponding author.
E-mail addresses: liuyong@ncepu.edu.cn (Y. Liu), wangkelei@whu.edu.cn (K. Wang), jcwei@math.ubc.ca (J. Wei).

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The regularity of the free boundary problems actually has been a subject of extensive studies, pioneered by the work of Caffarelli (see [2,3,6–8] and the references therein). It is now known that in dimension $n \le 4$, the free boundary of a solution to (5) has no singularity, provided that it is an energy *minimizer* [2,9,27]. In fact, it is conjectured that for $n \le 6$, minimizers should have smooth free boundary. On the other hand, in higher dimensions $n \ge 7$, an energy minimizing free boundary may have singularities. To explain this, let us mention that by blow up analysis, the regularity of the free boundary is essentially related to the existence or nonexistence of minimizing cone. Let us consider the cone in $\mathbb{R}^n$ given by (see [9])

$$|x_n| < \alpha_n \sqrt{x_1^2 + \ldots + x_n^2}, \tag{4}$$

where α_n is a dimensional constant chosen such that on this cone there is a solution to the one-phase free boundary problem. It has been proved [10] that in dimension $n = 7$ (actually also for $n = 9, 11, 13, 15$ and hopefully for all $n \ge 7$), the solution to (5) corresponding to the cone (4) is a minimizer for the energy functional. For $3 \le n \le 6$, this solution is already known to be unstable, thanks to the work of [9]. On the other hand, if a solution to (5) is a minimizer and if the free boundary is a priori a graph, then by the result of [11], this free boundary will be real analytic. It is worth pointing out that all these regularity results are in many respect analogous to that of the minimal surface theory, and these two subjects are closely related.

In $\mathbb{R}^2$, Traizet [39,40] proved that there is a one-to-one correspondence between solutions to (1) and (5) and certain type of minimal surfaces in $\mathbb{R}^3$. Hence at least in dimension two, this problem is well understood, although even for the minimal surfaces in $\mathbb{R}^3$, many questions remain unanswered up to now. We also refer to [25,26,33] for related existence and classification results for other types of free boundary problems. Now we emphasize that in higher dimensions, the explicit correspondence between minimal surfaces and free boundary problem is not available. However, in $\mathbb{R}^9$, it is proved by Kamburov [29] using sub and super solution method that there exists a solution to (1) where the free boundary is close to two copies of the famous Bombieri–De Giorgi–Giusti minimal graph. His result indicates that there should be deeper relation between minimal surface and the free boundary problem (1). Here in this paper we would like to further explore this relation by constructing solutions to (1) based on minimal surfaces.

Notice that problem (1) can be considered as a special case of over-determined problems. In recent years the following so-called Serrin's overdetermined problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega := \{u > 0\}, \\ u = 0, \ |\nabla u| = \mathit{Constant} & \text{on } \partial\Omega. \end{cases} \tag{5}$$

has also received much attention. We refer to [17–19,13,34,33,37,43] and the references therein.

Another motivation for studying (1) is related to De Giorgi's conjecture. In 1978 De Giorgi conjectured that the only bounded solution to

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^n \tag{6}$$

which is monotone in x_n must be one dimensional (up to rotation and translation) at least in dimension $n \le 8$. De Giorgi's conjecture is a natural, parallel statement to Bernstein theorem for minimal graphs, which in its most general form, due to Simons [36], states that any minimal hypersurface in $\mathbb{R}^n$, which is also a graph of a function of $n-1$ variables, must be a hyperplane if $n \le 8$. Strikingly, Bombieri, De Giorgi and Giusti [5] proved that this fact is false in dimension $n \ge 9$.

Great advance in De Giorgi's conjecture has been achieved in recent years, having been fully established in dimensions $n = 2$ by Ghoussoub and Gui [23] and for $n = 3$ by Ambrosio and Cabré [4]. A celebrated result by Savin [35] established its validity for $4 \le n \le 8$ under the following additional assumption

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1$$

(See Savin–Sciunzi–Valdinoci [42] and Farina–Valdinoci [20,21] for generalizations.) Del Pino, Kowalczyk and Wei [15] constructed a counterexample in dimensions $n \ge 9$.

Replacing the monotonicity assumption by global minimality of energy, Savin proved that in dimensions $n \le 7$ all global minimizers to (6) are one-dimensional. We proved that Savin's result is optimal by constructing global minimizers in dimensional 8 [31]. (Stable solutions are constructed in Pacard–Wei [32].)

In [41,42], Valdinoci et al. also extended the De Giorgi type conjecture result to problems with more general nonlinearities including

$$\Delta u = W_u(u)$$

where $W = (1-u^2)^{\alpha}$, $\alpha \geq 0$. $\alpha = 0$ intuitively corresponds to the problem (1). In particular they proved global minimizers of (1) must be one-dimensional if $n \leq 7$. One of our results below shows that this is optimal.

The purpose of this paper is to establish the connection between minimal surfaces and problem (1). In particular we shall construct new solutions to (1) by developing new gluing methods for overdetermined problems. We know very little information about the solutions of (1) in dimensions $n \geq 3$. In dimension 2 Traizet's characterization [39] reduces the problem to singly periodic minimal surfaces in $\mathbb{R}^3$. In dimension 9, Kambrunov's solution [29] is a monotone solution whose two components are approximately Bombieri–De Giorgi–Giusti graphs. For $3 \leq n \leq 8$ we know no solutions to (1). In this paper we establish a connection between minimal surfaces and solutions to (1) and thereby provide plenty of new solutions to (1). In addition, we shall prove the existence of global minimizers in $\mathbb{R}^8$ and execute the Jerison–Monneau program for problem (1).

Rather than considering the most general minimal surfaces, we shall focus on two types of classical minimal surfaces. The first type of minimal surfaces are the area minimizing cones (minimizing hypersurfaces) in $\mathbb{R}^n$ ($n \geq 8$). As an example, let us consider the famous Simons' cone:

$$S := \left\{(x_1, \dots, x_8) \in \mathbb{R}^8 : \Sigma_{i=1}^4 x_i^2 = \Sigma_{i=5}^8 x_i^2\right\}.$$

This is a minimal surface with one singularity at the origin. The fact that Simons' cone is area minimizing has been proved in the classical work of Bombieri–De Giorgi–Giusti [5]. Using the minimizing property, Hardt–Simon [24] was able to show that there exists a family of foliated minimal surfaces S_δ^+ lying on one side of the cone and is asymptotic to the cone at infinity. Similarly, the other side of the cone is also foliated by a family of minimal surfaces S_δ^-. Due to scaling invariance, this family of surfaces $S_\delta^{\pm}$ can be obtained simply as homothety of $S_1^{\pm}$, that is $S_\delta^{\pm} = \delta^{-1} S_1^{\pm}$. Actually, Hardt–Simon proved more. They showed that the Simons' cone is strictly area minimizing which implies that each surface $S_\delta^{\pm}$ approaches the cone at the slowest possible rate.

As we mentioned before, there should be similarities between the minimal surfaces and free boundary problem. A natural question is whether there are analogous solutions for the free boundary problem (1) as the Simons' cone and its associated foliation. We answer this affirmatively.

Theorem 1. *For each ε small enough, there exists domain Ω^{ε} close to the radius one tubular neighborhood of S_ε^+ and solution u_ε to the free boundary problem (1). Moreover, u_ε is stable in the sense that there exists a function $\Phi > 0$ in Ω^{ε}, and*

$$\begin{cases} \Delta\Phi = 0, & \text{in } \Omega^{\varepsilon}, \\ \Phi_\nu + H\Phi = 0, & \text{on } \partial\Omega^{\varepsilon}. \end{cases} \tag{7}$$

Here ν is the outward normal to $\partial\Omega^{\varepsilon}$ and H is the mean curvature of $\partial\Omega^{\varepsilon}$.

By this theorem, there are solutions whose nodal set is close to S_ε^+ *for ε small*. It is well known that the family of minimal surfaces $S_\delta^+, \delta \in \mathbb{R}$, are all area minimizing. Therefore, it is natural to ask that whether the solutions u_ε are also minimizers of the energy functional J. We believe this is true, but here in this paper we shall only give the following.

Theorem 2. *There exists a nontrivial solution (not one dimensional) U to the free boundary problem (1) in $\mathbb{R}^8$ which is also energy minimizing.*

With additional efforts, one can actually prove that for each S_δ^+, there exists an energy minimizer whose nodal set is asymptotic to S_δ^+ at infinity. We will not pursue this in this paper. One can compare this result with a similar result for the Allen–Cahn equation [31].

Using the variational method of Jerison–Monneau [28], we can construct monotone solutions in $\mathbb{R}^9$ using this minimizer U. This complements the result of Kamburov [29], where the existence of monotone solutions is established by sub and super solution method.

Theorem 3. *There is a family of solutions in $\mathbb{R}^9$ to (1) which are monotone in the x_9 direction.*

Our second type of minimal surfaces will be the catenoids, which is a family of classical minimal surfaces with finite total curvature. They are rotationally symmetric and given explicitly by the equation

$$x_1^2 + x_2^2 = \frac{1}{\varepsilon^2} \cosh^2 (\varepsilon x_3).$$

Here $\varepsilon > 0$ is a parameter. In higher dimensions, we have analogous codimension one minimal submanifold which we call higher dimensional catenoids. To be more precise, let $(x_1, \ldots, x_n)$ be the coordinate in $\mathbb{R}^n$ $(n > 3)$. Let ω be the solution of

$$\begin{cases} \frac{\omega''}{1+\omega'^2} - \frac{n-2}{\omega} = 0, \\ \omega(0) = 1, \omega'(0) = 0. \end{cases}$$

Then the surface $\mathcal{C}_1$ in $\mathbb{R}^n$ given by

$$r := \sqrt{x_1^2 + \ldots + x_{n-1}^2} = \omega(x_n)$$

is a minimal surface, called catenoid. We can also write it as

$$x_n = \bar{\omega}(r), r \in [r_0, +\infty).$$

Then there are constants c_n, c_n' such that

$$x_n \sim c_n - c_n' r^{3-n}.$$

Actually a homothety of $\mathcal{C}_1$ is also a minimal surface, which we denoted by $\mathcal{C}_\varepsilon$, which is then described by

$$x_n = \bar{\omega}_\varepsilon(r) := \frac{1}{\varepsilon} \bar{\omega}(\varepsilon r).$$

We refer to [38] for more detailed properties on catenoids, including their Morse index. Here we are interested in $\mathcal{C}_\varepsilon$ with ε small. In this case, the catenoid has a large waist.

Theorem 4. *For ε small enough, there exists a rationally symmetric domain Ω^ε close to radius one tubular neighborhood of $\mathcal{C}_\varepsilon$ and a solution u_ε to the free boundary problem (1).*

Now let us explain the main ideas of the proof. The proofs of Theorems 1 and 4 are based on the infinite dimensional gluing methods developed in [12,13]. In [1,12], entire solutions for the Allen–Cahn equation have been constructed. The zero level sets of the solutions lie close to certain nondegenerate minimal surfaces. To construct these solutions, they used the method of infinite dimensional Lyapunov–Schmidt reduction. More recently, in [13], an over-determined problem was investigated using similar method. Here we develop new gluing methods for (1). There are two main difficulties in performing gluing methods for (1). The first one is that the one-dimensional solution, which is given by

$$u_0(x_1) = \begin{cases} -1, & x_1 \le -1; \\ x_1, & -1 < x_1 < 1; \\ 1, & x_1 \ge 1, \end{cases} \tag{8}$$

is only continuous and is not differentiable. This means that one can not linearize the problem around this one dimensional profile. This is quite different from [1,12,13]. The second difficulty is that this is an over-determined problem and we have to adjust two interfaces.

To solve the problem (1), we introduce a pair of unknown functions (h_1, h_2) on a rescaled minimal surface. Using these two functions, we define a perturbed domain Ω_h which will be very close to the radius one tubular neighborhood $\mathcal{N}_1$ of the minimal surface. The functions h_1 and h_2 measures the deviation of Ω_h to $\mathcal{N}_1$. Next, we define suitable approximate solutions for (1) on Ω_h. We analyze in detail the differences between this approximate solution and the harmonic function in Ω with Dirichlet boundary condition. In the last step, we use fixed point argument to show that one can find functions h_1 and h_2 such that our problem is solvable and we can get a solution u. In this step, we show that to match the required Neumann boundary condition, we need to analyze the solvability and a priori estimate of a system of equations for the function h_1, h_2. (See (22).) It turns out that one of them reduces to the analysis of the Jacobi operator on the minimal surface

$$\Delta_M h + |A|^2 h = f \tag{9}$$

but the other problem is similar to that of the fractional differential operator

$$(-\Delta_M + 1)^{\frac{1}{2}} h = f. \tag{10}$$

We remark that the family of solutions constructed from the Simons' cone are ordered and hence stable, while the solutions arising from catenoids are unstable.

To prove Theorems 2 and 3, we first extend the construction of Jerison–Monneau [28] and follow the variational approach in [31] to construct minimizers in $\mathbb{R}^8$ and monotone solutions in $\mathbb{R}^9$. The main difficulty is the regularity of the solutions. To this end, we use axial symmetry of the solutions and also make use of classical regularity result of Weiss [44,45] as well as recent regularity results of Jerison–Savin [27].

2. Solutions from Simons' cone

2.1. Preliminary on Simons' cone and the associated foliation

Let us first of all recall some basic facts about the geometry of the Simons' cone. Throughout the paper we shall use $S^k(\rho)$ to denote the radius ρ sphere in $\mathbb{R}^{k+1}$. In the manifold $S^7(1)$, we shall consider the codimension one submanifold

$$\Lambda := S^3(\rho) \times S^3(\rho),$$

where

$$\rho = \sqrt{\frac{1}{2}}.$$

The induced metric on Λ is given by $g^* := \rho^2 g_1 + \rho^2 g_2$, where g_1, g_2 are the metric on the two copies $S^3(1)$. The Simons cone is defined to be

$$S := \left\{ rX \in \mathbb{R}^8 : r \in (0, +\infty), X \in \Lambda \right\}.$$

One can verify that this is a minimal hypersurface in $\mathbb{R}^8$. The induced metric tensor on S is then given by

$$dr^2 + r^2 g^*.$$

For a codimension one submanifold M in $\mathbb{R}^n$, with the induced metric, we shall use J_M to denote its Jacobi operator, which explicitly has the form

$$J_M = \Delta_M + |A|^2,$$

where $|A|^2 = \Sigma_{i=1}^{n-1} k_i^2$ is the squared norm of the second fundamental form of M, with k_i being the principle curvatures of M. The Jacobi operator about S is then given by

$$J_S = \Delta_S + |A|^2 = \partial_r^2 + \frac{6}{r}\partial_r + \frac{\Delta_{g^*} + 6}{r^2}.$$

The set $\mathbb{R}^8 \backslash S$ has two components. Each component is foliated by a family of smooth minimal hypersurfaces $S_\varepsilon^\pm$ which are asymptotic to S at infinity. We can choose S_1 to be the surface having the form

$$S_1 \backslash B_{r_0} = \{X + \eta_0(X)\nu, X \in S\},$$

where ν is a choice of the unit normal at S, and $\eta_0(X) = |X|^{-2} + o\left(|X|^{-2}\right)$. Then $S_\varepsilon = \varepsilon^{-1} S_1$.

Let $x = \sqrt{x_1^2 + \ldots + x_4^2}$, $y = \sqrt{x_5^2 + \ldots + x_8^2}$. We can write the standard metric on $\mathbb{R}^8$ in the polar coordinate as

$$dx^2 + x^2 d\theta^2 + dy^2 + y^2 d\bar{\theta}^2,$$

where $d\theta^2$ and $d\bar{\theta}^2$ represents the metric tensor on the unit three-dimensional sphere $S^3(1)$. Suppose in the (x, y) coordinate S_δ is described by $y = \varphi_\delta(x)$ for a monotone function φ_δ, then the metric tensor on S_δ is

$$\left[1 + \varphi_\delta'^2(x)\right] dx^2 + \varphi_\delta^2(x)\, d\bar{\theta}^2 + x^2 d\theta^2.$$

Let us introduce the arc length variable l by the formula

$$l = \int_0^x \sqrt{1 + \varphi_\delta'^2(t)}\, dt.$$

Then the metric g_δ on S_δ also read as

$$dl^2 + \varphi_\delta^2(x)\, d\theta^2 + x^2 d\bar{\theta}^2.$$

Note that $\det \mathrm{g}_\delta = \varphi_\delta^6(x)\, x^6$. Let η be a function on S_δ which is invariant under the action of the group $O(4) \times O(4)$. The Laplacian operator on S_δ acting on function η has the form

$$\begin{aligned}
\Delta_{S_\delta}\eta &= \frac{1}{\sqrt{\det \mathrm{g}_\delta}} \partial_i \left(\sqrt{\det \mathrm{g}_\delta}\, \mathrm{g}_\delta^{i,j} \partial_j \eta \right) \\
&= \frac{d^2\eta}{dl^2} + \frac{\frac{d\left[\varphi_\delta^3(x) x^3\right]}{dl}}{\varphi_\delta^3(x)\, x^3} \frac{d\eta}{dl} \\
&= \frac{d^2\eta}{dl^2} + \left(\frac{3}{x} + \frac{3\varphi_\delta'}{\varphi_\delta} \right) \frac{dx}{dl} \frac{d\eta}{dl}.
\end{aligned} \tag{11}$$

2.2. Analysis of the approximate solutions

We will construct solutions based on the minimal hypersurfaces S_ε where $\varepsilon > 0$ is sufficiently small. Let us choose a unit normal ν for the codimension one manifold S_ε. Let $h_1, h_{-1} \in C^{2,\alpha}_{loc}(S_\varepsilon)$, small in certain sense. For each function η defined on S_ε, we set

$$\Gamma_\eta := \{X + \eta(X)\, \nu(X) : X \in S_\varepsilon\}.$$

Although Γ_η depends also on ε, we will not make this dependence explicit in the notation. We establish a Fermi coordinate in a tubular neighborhood of S_ε. By s we denote the signed distance of a point to S_ε. Slightly abusing the notation, define

$$\Gamma_s := \{X + s\nu(X) : X \in S_\varepsilon\}.$$

Note that for ε small, this is well defined and Γ_s is smooth, for all $|s| < 1$.

Let us consider the region Ω trapped between the surfaces $\Gamma_{-1+h_{-1}}$ and Γ_{1+h_1}. For each pair of functions $h = (h_{-1}, h_1)$, we shall define an approximate solution w_h in Ω:

$$w_h(s, l) = \frac{s - g(l)}{1 + f(l)},$$

where

$$f = \frac{h_1 - h_{-1}}{2},$$
$$g = \frac{h_1 + h_{-1}}{2}.$$

Note that in the current situation, the range of l is $[0, +\infty)$. With this definition, w_h satisfies the boundary condition:

$$w_h = \begin{cases} -1, & \text{on } \Gamma_{-1+h_{-1}}, \\ 1, & \text{on } \Gamma_{1+h_1}. \end{cases}$$

It will be convenient for us to introduce a new variable

$$t = \frac{s - g(l)}{1 + f(l)}.$$

Then the domain Ω_h can be parameterized by (l, t) with $t \in [-1, 1]$.

Let us use H_M to denote the mean curvature of a codimension one submanifold M. The formula of Laplacian operator in the Fermi coordinate (see [14]) tells us that

$$\begin{aligned} \Delta w_h(s, l) &= \Delta_{\Gamma_s} w_h + \partial_s^2 w_h - H_{\Gamma_s} \partial_s w_h \\ &= \Delta_{\Gamma_s} w_h - \frac{H_{\Gamma_s}}{1 + f}. \end{aligned}$$

We need to understand the main order of these terms.

Lemma 5. *We have*

$$\Delta_{\Gamma_0} w_h = -\Delta_{\Gamma_0} g - t\Delta_{\Gamma_0} f + E_1,$$

where

$$E_1 = -tf\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}(fg) - g\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left[(s - g)\frac{f^2}{1 + f}\right].$$

Remark 6. E_1 can be regarded as a perturbation term.

Proof. Having in mind that f, g are small, we write

$$\begin{aligned} w_h = \frac{s - g(l)}{1 + f(l)} &= (s - g)\left(1 - f + \frac{f^2}{1 + f}\right) \\ &= s - g - sf + gf + (s - g)\frac{f^2}{1 + f}. \end{aligned}$$

We then compute

$$\Delta_{\Gamma_0} w_h = -\Delta_{\Gamma_0} g - s\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}(fg) + \Delta_{\Gamma_0}\left[(s - g)\frac{f^2}{1 + f}\right].$$

Inserting the relation $s = t(1 + f) + g$ into the left hand side, we get

$$\begin{aligned} \Delta_{\Gamma_0} w_h &= -\Delta_{\Gamma_0} g - [t(1 + f) + g]\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}(fg) + \Delta_{\Gamma_0}\left[(s - g)\frac{f^2}{1 + f}\right] \\ &= -\Delta_{\Gamma_0} g - t\Delta_{\Gamma_0} f - tf\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}(fg) - g\Delta_{\Gamma_0} f + \Delta_{\Gamma_0}\left[(s - g)\frac{f^2}{1 + f}\right]. \end{aligned}$$

This finishes the proof. □

Let us use k_i, $i = 1, \ldots, 6$ to denote the principle curvatures of S_ε.

Lemma 7. *We have the following formula:*

$$\frac{H_{\Gamma_s}}{1 + f} = t|A|^2 + g|A|^2 + E_2,$$

where

$$E_2 = \frac{1}{1 + f}\sum_{i=1}^{6}\frac{s^2 k_i^3}{1 - sk_i} - \frac{fg|A|^2}{1 + f}.$$

Proof. By a well known formula (see [14]),

$$H_{\Gamma_s}=\sum_{i=1}^{6}\frac{k_i}{1-sk_i}=\sum_{i=1}^{6}k_i+\sum_{i=1}^{6}sk_i^2+\sum_{i=1}^{6}\frac{s^2k_i^3}{1-sk_i}.$$

Recall that $\sum_{i=1}^{6}k_i=H_{\Gamma_0}=0$. Hence

$$\frac{H_{\Gamma_s}}{1+f}=\frac{|A|^2}{1+f}\left[(1+f)t+g\right]+\frac{1}{1+f}\sum_{i=1}^{6}\frac{s^2k_i^3}{1-sk_i}$$

$$=t\,|A|^2+g\,|A|^2-\frac{fg\,|A|^2}{1+f}+\frac{1}{1+f}\sum_{i=1}^{6}\frac{s^2k_i^3}{1-sk_i}.$$

The proof is thus completed. □

We seek a solution u to the free boundary problem (1) in the form $u=w_h+\phi$. Here we require $\phi=0$ on $\partial\Omega_h$. Let us now analyze the boundary condition $|\nabla u|=1$ on $\partial\Omega_h$. Suppose in the $\left(l,\theta,\bar{\theta},s\right)$ coordinate the metric tensor $\mathfrak{g}$ in a tubular neighborhood of S_ε has matrix with entries $\mathfrak{g}_{i,j}$ and its inverse matrix has entries $\mathfrak{g}^{i,j}$. Since we are working in the Fermi coordinate, the entries in the last column and row are all zero, except the rightmost entry on the last row. We omit the subscript h in w_h and write it as w.

Lemma 8. *The condition $|\nabla u|=1$ on Γ_{i+h_i} is equivalent to*

$$\partial_t\phi-f=E_{3,i}.$$

Here for $i=-1,1$, $E_{3,i}$ is defined on Γ_{i+h_i} to be

$$-\frac{1}{2}\left(1+\mathfrak{g}^{1,1}h_i'^2\right)(\partial_t\phi)^2+\frac{\mathfrak{g}^{1,1}h_i'}{1+f}\partial_t\phi+\frac{1}{2}f^2-\frac{1}{2}\mathfrak{g}^{1,1}\left(g'+tf'\right)^2.$$

Proof. We compute the norm of the gradient in the (s,l) coordinate and get the following equation to be satisfied on the boundary $\partial\Omega_h$:

$$|\nabla(w+\phi)|^2=(\partial_s w+\partial_s\phi)^2+\mathfrak{g}^{1,1}(\partial_l w+\partial_l\phi)^2=1. \tag{12}$$

Direct computation yields

$$\partial_s w=\frac{1}{1+f},$$

and

$$\partial_l w=\frac{-g'}{1+f}-\frac{(s-g)f'}{(1+f)^2}.$$

On the other hand, differentiating the identity $\phi(-1+h_1,l)=0$ with respect to l, we obtain

$$\partial_l\phi=-\partial_s\phi h_1' \text{ on } \Gamma_{-1+h_{-1}}.$$

On $\Gamma_{-1+h_{-1}}$, the right hand side of (12) is equivalent to

$$\left(1+\mathfrak{g}^{1,1}h_1'^2\right)(\partial_s\phi)^2+\left(2\partial_s w-2\mathfrak{g}^{1,1}h_1'\right)\partial_s\phi+(\partial_s w)^2+\mathfrak{g}^{1,1}(\partial_l w)^2=1. \tag{13}$$

Inserting the equation

$$\partial_s\phi=\frac{\partial_t\phi}{1+f}$$

into (13), we get

$$\left(1+\mathfrak{g}^{1,1}h_1'^2\right)(\partial_t\phi)^2+\left(2-2\frac{\mathfrak{g}^{1,1}h_1'}{1+f}\right)\partial_t\phi-2f-f^2+\mathfrak{g}^{1,1}\left(g'+\frac{(s-g)f'}{1+f}\right)^2=0.$$

This completes the proof. □

The function ϕ should also satisfy

$$\Delta\phi=-\Delta w=J_{\Gamma_0}g+\left(\Delta_{\Gamma_0}f+|A|^2\right)t-E_1+E_2+\Delta_{\Gamma_0}w-\Delta_{\Gamma_s}w,\ \text{ in }\Omega_h.$$

Here we recall that by J_{Γ_0} we denote the Jacobi operator of Γ_0. Therefore, we are lead to solve the following nonlinear problem for the unknown functions (f, g, ϕ).

$$\begin{cases}\Delta\phi=J_{\Gamma_0}g+\left(\Delta_{\Gamma_0}f+|A|^2\right)t-E_1+E_2+\Delta_{\Gamma_0}w-\Delta_{\Gamma_s}w, & \text{in } \Omega_h,\\ \phi=0 \text{ and } \partial_t\phi-f=E_{3,i}, & \text{on } \partial\Omega_h.\end{cases}\tag{14}$$

Lemma 9. *We have the following estimate for the Laplacian operator acting on functions depending on s and l:*

$$\Delta_{\Gamma_0}\eta-\partial_l^2\eta-\frac{3}{l}\partial_l\eta=O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l\eta,$$

and

$$\Delta_{\Gamma_s}\eta-\Delta_{\Gamma_0}\eta=O\left(\frac{\varepsilon^2}{(1+\varepsilon l)^2}\right)\partial_l\eta+O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l^2\eta.\tag{15}$$

Proof. By (11), we have

$$\Delta_{\Gamma_0}\eta-\frac{d^2\eta}{dl^2}-\frac{3}{l}\frac{d\eta}{dl}=\left[\left(\frac{3}{x}+\frac{3\varphi_\varepsilon'}{\varphi_\varepsilon}\right)\frac{dx}{dl}-\frac{3}{l}\right]\frac{d\eta}{dl}.$$

We compute

$$\begin{aligned}\left(\frac{3}{x}+\frac{3\varphi_\varepsilon'}{\varphi_\varepsilon}\right)\frac{dx}{dl}-\frac{3}{l}&=\frac{1}{\sqrt{1+\varphi_\varepsilon'^2}}\left(\frac{3}{x}+\frac{3\varphi_\varepsilon'}{\varphi_\varepsilon}\right)-\frac{3}{l}\\&=\frac{1}{\sqrt{1+\left(\varphi_1'(\varepsilon x)\right)^2}}\left(\frac{3}{x}+\frac{3\varepsilon\varphi_1'(\varepsilon x)}{\varphi_1(\varepsilon x)}\right)-\frac{3}{l}\\&=3\frac{l-x\sqrt{1+\left(\varphi_1'(\varepsilon x)\right)^2}}{lx\sqrt{1+\left(\varphi_1'(\varepsilon x)\right)^2}}+\varepsilon O\left(\frac{1}{1+\varepsilon l}\right)\\&=O\left(\frac{\varepsilon}{1+\varepsilon l}\right).\end{aligned}$$

Next we prove (15). Let us denote by $\mathfrak{g}_s$ the metric tensor of Γ_s. Explicitly, $\mathfrak{g}_s\left(l,\theta,\bar\theta\right)=\mathfrak{g}\left(l,\theta,\bar\theta,s\right)$. From the calculation in [14], we know that

$$\sqrt{\det\mathfrak{g}_s}=\sqrt{\det\mathfrak{g}_0}\prod_{i=1}^{6}(1-k_is),$$

where k_i are the principle curvatures of $\Gamma_0=S_\varepsilon$. Hence, for a function η depending on s and l,

$$\begin{aligned}\Delta_{\Gamma_s}\eta&=\frac{1}{\sqrt{\det\mathfrak{g}_s}}\partial_i\left(\sqrt{\det\mathfrak{g}_s}\mathfrak{g}_s^{i,j}\partial_j\eta\right)\\&=\partial_l\left(\ln\left(\sqrt{\det\mathfrak{g}_0}\prod_{i=1}^{6}(1-k_is)\right)\right)\mathfrak{g}_s^{1,1}\partial_l\eta+\partial_l\left(\mathfrak{g}_s^{1,1}\partial_l\eta\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\Delta_{\Gamma_s}\eta-\Delta_{\Gamma_0}\eta=&\ \partial_l\left(\ln\left(\prod_{i=1}^{6}(1-k_is)\right)\right)\mathfrak{g}_s^{1,1}\partial_l\eta\\&+\partial_l\left(\ln\sqrt{\det\mathfrak{g}_0}\right)\left(\mathfrak{g}_s^{1,1}-\mathfrak{g}_0^{1,1}\right)\partial_l\eta\\&+\partial_l\left(\left(\mathfrak{g}_s^{1,1}-\mathfrak{g}_0^{1,1}\right)\partial_l\eta\right).\end{aligned}$$

Then the desired estimate follows from the fact that

$$\left|\frac{dk_i}{dl}\right|\leq C\frac{\varepsilon^2}{(1+\varepsilon l)^2}.\quad\square$$

By the previous computations, the term $-E_1+E_2+\Delta_{\Gamma_0}w-\Delta_{\Gamma_s}w$ will be small and can be regarded as perturbation terms.

To get a solution (f,g,ϕ) for the original problem, let us introduce the functional framework to work with. Let $\alpha\in(0,1)$ be a fixed constant. Note that the functions f and g are both defined on the minimal surface S_ε. However, we shall work both in functional spaces defined on S_ε and S_1. Hence we introduce the following

Definition 10. For $\mu=0,1,2$, $\beta\geq 0$, $\delta>0$, the space $\mathcal{B}_{\beta,\mu;\delta}$ consists of those functions η defined on S_δ such that

$$\|\eta\|_{\beta,\mu;\delta}:=\sup_{l,|z|=l}\left[(1+\delta l)^\beta\|\eta\|_{C^{\mu,\alpha}(S_\delta\cap B_1(z))}\right]<+\infty.$$

Definition 11. The space $\bar{\mathcal{B}}_{\beta,2;\delta}$ consists of those functions η defined on S_δ such that

$$\begin{aligned}\|\eta\|_{\beta,2;\delta,\hat{}}:=&\sup_{l,|z|=l}\left[(1+\delta l)^\beta\|\eta\|_{C^{0,\alpha}(S_\delta\cap B_1(z))}+(1+\delta l)^{\beta+1}\left\|\eta'\right\|_{C^{0,\alpha}(S_\delta\cap B_1(z))}\right]\\&+\sup_{l,|z|=l}\left[(1+\delta l)^{\beta+2}\left\|\eta''\right\|_{C^{0,\alpha}(S_\delta\cap B_1(z))}\right]<+\infty.\end{aligned}$$

With the above definition, we shall assume a priori $f\in\mathcal{B}_{2,2;\varepsilon}$. We also assume the rescaled function $\bar{g}(\cdot)=g\left(\frac{\cdot}{\varepsilon}\right)\in\bar{\mathcal{B}}_{\beta_0,2;1}$, where $\beta_0>2$ is a fixed constant with β_0-2 small. On the other hand, the function ϕ is defined on Ω_h, which depends on f and g. This turns out to be not very convenient for our later purpose. Hence slightly abusing the notation, we also regard ϕ as the restriction of a function $\mathcal{T}(\phi)$ on $\Xi:=[-1,1]\times[0,+\infty)$, where $\mathcal{T}(\phi)$ is a function of t and l defined for $(t,l)\in\bar{\Xi}:=[-1,1]\times\mathbb{R}$, even in the variable l.

Definition 12. For $\mu=0,1,2$, $\beta\geq 0$, the space $\mathcal{B}_{\beta,\mu;*}$ consists of those functions ϕ such that

$$\|\phi\|_{\beta,\mu;*}:=\sup_{l\in\mathbb{R};z\in\bar{\Xi},|z|=|l|}\left[(1+\varepsilon|l|)^\beta\|\mathcal{T}(\phi)\|_{C^{\mu,\alpha}\left(\bar{\Xi}\cap B_1(z)\right)}\right]<+\infty.$$

We shall assume $\phi\in\mathcal{B}_{2,2;*}$. The following invertibility property of the Jacobi operator on S_1 will play an important role in our analysis.

Lemma 13. *For each function $\xi\in\mathcal{B}_{\beta_0+2,0;1}$, there is a solution $\eta\in\bar{\mathcal{B}}_{\beta_0,2;1}$ such that*

$$J_{S_1}(\eta)=\xi.$$

Moreover, it satisfies

$$\|\eta\|_{\beta_0,2;1,\hat{}}\leq C\|\xi\|_{\beta_0+2,0;1}.$$

Proof. The proof of this lemma goes in a similar fashion as that of [32], we omit the details. $\square$

We would like to solve the nonlinear problem (14) using fixed point arguments.

Lemma 14. *For each $\eta \in \mathcal{B}_{\beta,0;*}$, there exists a unique solution $\phi \in \mathcal{B}_{\beta,2;*}$, to the problem*

$$\begin{cases} \partial_t^2 \phi + \partial_l^2 \phi + \frac{3}{l} \partial_l \phi = \eta, & \text{in } \Omega_h, \\ \phi = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{16}$$

with $\|\phi\|_{\beta,2;*} \leq C \|\eta\|_{\beta,0;*}$. This solution will be denoted by $L_1(\eta)$.

Remark 15. In terms of the (t, l) coordinate, the first equation in (16) actually should be considered in the region $(t, l) \in [-1, 1] \times [0, +\infty)$. However, for the sake of notational simplicity, we just write it as in Ω_h. Similarly, we use the notation $\partial\Omega_h$ in the second equation of (16).

The proof of Lemma 14 follows from standard arguments.

Next, given two functions γ_1 and γ_{-1} defined on $\mathcal{S}_\varepsilon$, we consider

$$\begin{cases} \partial_t^2 \phi + \partial_l^2 \phi + \frac{3}{l} \partial_l \phi = J_{\Gamma_0} g + \left(\Delta_{\Gamma_0} f + |A|^2\right) t, & \text{in } \Omega_h, \\ \phi(\pm 1, l) = 0, & \\ \partial_t \phi - f = \gamma_{-1}, & \text{for } t = -1, \\ \partial_t \phi - f = \gamma_1, & \text{for } t = 1. \end{cases} \tag{17}$$

To find the explicit form of the solution ϕ of this problem, we need to introduce some notations. For each fixed $\xi \in \mathbb{R}^4$, let us use $p_{1,\xi}(\cdot)$ to denote the solution of the problem

$$\begin{cases} p''_{1,\xi}(t) - |\xi|^2 p_{1,\xi}(t) = 1, \\ p_{1,\xi}(-1) = p_{1,\xi}(1) = 0. \end{cases}$$

We use $p_{2,\xi}(\cdot)$ to denote the solution of

$$\begin{cases} p''_{2,\xi}(t) - |\xi|^2 p_{2,\xi}(t) = t, \\ p_{2,\xi}(-1) = p_{2,\xi}(1) = 0. \end{cases}$$

Note that $p_{1,\xi}$ is even, while $p_{2,\xi}$ is odd. For convenience, we collect properties of $p_{i,\xi}$ in the following

Lemma 16. *Explicitly,*

$$p_{1,\xi}(t) = \frac{\cosh(|\xi| t)}{|\xi|^2 \cosh|\xi|} - \frac{1}{|\xi|^2},$$
$$p_{2,\xi}(t) = \frac{\sinh(|\xi| t)}{|\xi|^2 \sinh|\xi|} - \frac{t}{|\xi|^2}.$$

Moreover,

$$\frac{1}{p'_{1,\xi}(1)} - |\xi| = \frac{|\xi|}{\tanh|\xi|} - |\xi| = O\left(e^{-\frac{|\xi|}{2}}\right), \text{ as } |\xi| \to +\infty,$$

and

$$|\xi|^2 p'_{2,\xi}(1) = \frac{|\xi|}{\tanh|\xi|} - 1.$$

Proof. This follows from direct computation. $\square$

In the following, we shall use the following Fourier type transform

$$\hat{\eta}(t, \xi) := \int_{\mathbb{R}^4} e^{-i(\xi_1 z_1 + \ldots + \xi_4 z_4)} \eta(t, l)\, dz_1 \ldots dz_4,$$

where $l=\sqrt{z_1^2+\ldots+z_4^2}$, $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)$. Note that this actually corresponds to the usual Fourier transform in $\mathbb{R}^4$. We denote by $(\cdot)^\vee$ the inverse Fourier transform. Define a new function f_0 by

$$f_0=-\left(\frac{\left(|A|^2\right)^\wedge}{|\xi|^2-\frac{1}{p'_{2,\xi}(1)}}\right)^\vee .$$

By the discussion in the next proposition, this definition makes sense.

Proposition 17. *Suppose $\gamma_1-\gamma_{-1}\in\mathcal{B}_{\beta_0+2,1;\varepsilon}$, $\gamma_1+\gamma_{-1}\in\mathcal{B}_{\beta_0,1;\varepsilon}$. Then the system (17) has a solution $(f,\bar g)$ with*

$$\|f-f_0\|_{\beta_0,2;\varepsilon}\le C\,\|\gamma_1+\gamma_{-1}\|_{\beta_0,1;\varepsilon}\,, \tag{18}$$

and

$$\|\bar g\|_{\beta_0,2;1,\hat{}}\le C\varepsilon^{-2}\,\|\gamma_1-\gamma_{-1}\|_{\beta_0+2,1;\varepsilon}\,. \tag{19}$$

This solution $(f,\bar g)$ will be denoted by $L_2(\gamma_{-1},\gamma_1)$.

Proof. We are lead to the problem

$$\begin{cases}\partial_t^2\hat\phi-|\xi|^2\hat\phi=\left(J_{\Gamma_0}g\right)^\wedge+\left(\Delta_{\Gamma_0}f+|A|^2\right)^\wedge t, \quad t\in[-1,1],\\ \hat\phi(-1,\xi)=\hat\phi(1,\xi)=0,\\ \partial_t\hat\phi(-1,\xi)-\hat f(\xi)=\hat\gamma_{-1}(\xi),\\ \partial_t\hat\phi(1,\xi)-\hat f(\xi)=\hat\gamma_1(\xi).\end{cases} \tag{20}$$

The solution $\hat\phi$ of the first equation in (20) can be written in the form

$$\hat\phi(t,\xi)=\left(J_{\Gamma_0}g\right)^\wedge p_{1,\xi}(t)+\left(\Delta_{\Gamma_0}f+|A|^2\right)^\wedge p_{2,\xi}(t).$$

Therefore, to get a solution for (20), it suffices for us to solve the following problem:

$$\begin{cases}\left(J_{\Gamma_0}g\right)^\wedge p'_{1,\xi}(-1)+\left(\Delta_{\Gamma_0}f+|A|^2\right)^\wedge p'_{2,\xi}(-1)-\hat f(\xi)=\hat\gamma_{-1}(\xi),\\ \left(J_{\Gamma_0}g\right)^\wedge p'_{1,\xi}(1)+\left(\Delta_{\Gamma_0}f+|A|^2\right)^\wedge p'_{2,\xi}(1)-\hat f(\xi)=\hat\gamma_1(\xi).\end{cases} \tag{21}$$

Due to the symmetry of $p_{1,\xi}$ and $p_{2,\xi}$, (21) is equivalent to

$$\begin{cases}\left(J_{\Gamma_0}g\right)^\wedge=\frac{\hat\gamma_1(\xi)-\hat\gamma_{-1}(\xi)}{2p'_{1,\xi}(1)},\\ \left(\Delta_{\Gamma_0}f+|A|^2\right)^\wedge=\frac{2\hat f(\xi)+\hat\gamma_{-1}(\xi)+\hat\gamma_1(\xi)}{2p'_{2,\xi}(1)}.\end{cases} \tag{22}$$

One can perform inverse Fourier transform for the first equation in this system and then use Lemma 13 to get a solution g.

We proceed to estimate the norm of $\bar g(\cdot)=g\left(\frac{\cdot}{\varepsilon}\right)$. Put $\rho=\gamma_1-\gamma_{-1}$. We would like to show

$$\left\|\left(\frac{\hat\rho(\xi)}{p'_{1,\xi}(1)}\right)^\vee\right\|_{\beta_0+2,0;\varepsilon}\le C\,\|\rho\|_{\beta_0+2,1;\varepsilon}\,.$$

Once this is proved, the estimate (19) follows from the invertibility property of the Jacobi operator J_{S_1}. Observe that $\frac{1}{p'_{1,\xi}(1)}$ is real analytic in $|\xi|$. By Lemma 16,

$$\frac{1}{p'_{1,\xi}(1)}=|\xi|+O\left(e^{-\frac{|\xi|}{2}}\right),\ \text{as}\ |\xi|\to+\infty.$$

Let us now estimate the inverse Fourier transform of $|\xi|\,\hat\rho(\xi)$. Using the fact that in $\mathbb{R}^4$, inverse Fourier transform of $|\xi|$ is equal to $c_0\,|x|^{-5}$, where c_0 is a constant (see for instances, [22] Theorem 2.4.6, or [16]), we get

$$\left(|\xi|\,\hat{\rho}(\xi)\right)^{\vee}(z)=c_0\,\text{P.V.}\int_{\mathbb{R}^4}\frac{\rho(|z|)-\rho(|y|)}{|z-y|^5}dy.$$

For $|z|$ large, we have

$$\begin{aligned}\left|\int_{|z-y|>\frac{|z|}{2}}\frac{\rho(|z|)-\rho(|y|)}{|z-y|^5}dy\right| &\le C\,|\rho(|z|)|+\int_{|z-y|>\frac{|z|}{2}}\frac{|\rho(|y|)|}{|z-y|^5}dy\\ &\le C\,|\rho(|z|)|+\frac{C}{|z|^5}\int_{|z-y|>\frac{|z|}{2}}|\rho(|y|)|\,dy\\ &\le C\,|\rho(|z|)|+C\frac{\|\rho\|_{\beta_0+2,1;\varepsilon}}{1+\varepsilon^5\,|z|^5}.\end{aligned}\tag{23}$$

On the other hand,

$$\begin{aligned}\left|\int_{1<|z-y|<\frac{|z|}{2}}\frac{\rho(|z|)-\rho(|y|)}{|z-y|^5}dy\right| &\le\frac{C\,\|\rho\|_{\beta_0+2,1;\varepsilon}}{(1+\varepsilon\,|z|)^{\beta_0+2}}\int_{1<|z-y|<\frac{|z|}{2}}\frac{dy}{|z-y|^5}\\ &\le\frac{C\,\|\rho\|_{\beta_0+2,1;\varepsilon}}{(1+\varepsilon\,|z|)^{\beta_0+2}}.\end{aligned}\tag{24}$$

Furthermore, using the fact that $\rho\in C^{1,\alpha}$, we get

$$\left|\text{P.V.}\int_{0<|z-y|<1}\frac{\rho(|z|)-\rho(|y|)}{|z-y|^5}dy\right|\le C\,\|\rho\|_{C^{1,\alpha}(B_1(z))}\,.\tag{25}$$

Inequalities (23), (24), (25) give us the required weighted C^0 estimate of $\left(|\xi|\,\hat{\rho}(\xi)\right)^{\vee}(z)$. Similarly, one can also get corresponding estimate for the Holder norm. Hence the desired estimate (19) follows.

To find the solution f for the second equation in (22), we first consider the equation

$$\left(f''+\frac{3}{l}f'+|A|^2\right)^{\wedge}=\frac{2\hat{f}(\xi)+\hat{\gamma}_{-1}(\xi)+\hat{\gamma}_1(\xi)}{2p'_{2,\xi}(1)}.\tag{26}$$

This can be written as

$$\hat{f}(\xi)=-\frac{\left(|A|^2\right)^{\wedge}}{|\xi|^2-\frac{1}{p'_{2,\xi}(1)}}+\frac{\hat{\gamma}_{-1}(\xi)+\hat{\gamma}_1(\xi)}{2\left(|\xi|^2\,p'_{2,\xi}(1)-1\right)}.\tag{27}$$

We may take inverse Fourier transform on both sides of (27). Let

$$K_1=\left(\frac{1}{|\xi|^2-\frac{1}{p'_{2,\xi}(1)}}\right)^{\vee},\;K_2=\left(\frac{1}{|\xi|^2\,p'_{2,\xi}(1)-1}\right)^{\vee}.$$

In view of the explicit formula of $p'_{2,\xi}(1)$, we know $|\xi|^2-\frac{1}{p'_{2,\xi}(1)}$ and $|\xi|^2\,p'_{2,\xi}(1)-1$ are positive and real analytic. This implies that K_1 and K_2 decay fast enough at infinity. On the other hand,

$$\frac{1}{|\xi|^2-\frac{1}{p'_{2,\xi}(1)}}\sim\frac{1}{|\xi|^2},\;\frac{1}{|\xi|^2\,p'_{2,\xi}(1)-1}\sim\frac{1}{|\xi|},\;\text{as }|\xi|\to+\infty.$$

Observe that the inverse Fourier transform of $|\xi|^{-1}$ is $c_1\,|x|^{-3}$ (see [22]). It follows that K_2 has a singularity of the order $O\left(|x|^{-3}\right)$ near origin. The estimate (18) for solution f of (27) then follows from routine calculation in potential theory. Since by Lemma 9, $\Delta_{\Gamma_0} f$ is a small perturbation of $f'' + \frac{3}{l} f'$, then we can use a perturbation argument to show the same estimate for solution f of the second equation in (22). This finishes the proof. □

With the model linear problem understood, we proceed to solve the nonlinear problem. Let ϕ_0 be the solution of the problem

$$\begin{cases} \partial_t^2 \phi_0 + \partial_l^2 \phi_0 + \frac{3}{l} \partial_l \phi_0 = t\,|A|^2, & \text{in } \Omega_h, \\ \phi_0 = 0 & \text{on } \partial\Omega_h. \end{cases}$$

Lemma 18. *Suppose* $\|f - f_0\|_{\beta_0,2;\varepsilon} \le C\varepsilon^2$, $\|\bar{g}\|_{\beta_0,2;1,\hat{}} \le C\varepsilon$, *and* $\|\phi - \phi_0\|_{\beta_0,2;*} \le C\varepsilon^2$. *There holds*

$$\left\| E_{3,1} - E_{3,-1} \right\|_{\beta_0+2,1;\varepsilon} \le C\varepsilon^3,$$
$$\left\| E_{3,1} + E_{3,-1} \right\|_{\beta_0,1;\varepsilon} \le C\varepsilon^3.$$

Proof. Recall that

$$E_{3,i} = -\frac{1}{2}\left(1 + \mathfrak{g}^{1,1} h_i'^2\right)(\partial_t \phi)^2 + \frac{\mathfrak{g}^{1,1} h_i'}{1+f} \partial_t \phi + \frac{1}{2} f^2 - \frac{1}{2} \mathfrak{g}^{1,1}\left(g' + t f'\right)^2.$$

Using the boundedness of $\mathfrak{g}^{1,1}$, taking into account of the fact that

$$\left\|g'\right\|_{3,1;\varepsilon} \le C\varepsilon^2,\ \|f\|_{2,2;\varepsilon} \le C\varepsilon^2,\ \|\partial_t \phi(\pm 1, l) - \partial_t \phi_0(\pm 1, l)\|_{\beta_0,2;\varepsilon} \le C\varepsilon^2,$$

we find that

$$\left\|\mathfrak{g}^{1,1} h_i'^2 (\partial_t \phi)^2\right\|_{\beta_0+2,1;\varepsilon} + \left\|\mathfrak{g}^{1,1}\left(g'\right)^2\right\|_{\beta_0+2,1;\varepsilon} + \left\|\mathfrak{g}^{1,1} g' f\right\|_{\beta_0+2,1;\varepsilon} \le C\varepsilon^3.$$

Now we subtract $E_{3,1}$ with $E_{3,-1}$, the term f^2 will be canceled. Additionally, using the asymptotic expansion of $\mathfrak{g}^{1,1}$, we know

$$\left\|\left(t^2 \mathfrak{g}^{1,1} f'^2\right)|_{t=-1} - \left(t^2 \mathfrak{g}^{1,1} f'^2\right)|_{t=1}\right\|_{\beta_0+2,1;\varepsilon} \le C\varepsilon^3.$$

Furthermore, observing that $\left\|f_0'\right\|_{3,1;\varepsilon} \le C\varepsilon^2$, we get

$$\begin{aligned} &\left\|\left(\mathfrak{g}^{1,1} h_{-1}' \partial_t \phi\right)|_{t=-1} - \left(\mathfrak{g}^{1,1} h_1' \partial_t \phi\right)|_{t=1}\right\|_{\beta_0+2,1;\varepsilon} \\ &\le C\varepsilon^3 + C\left\|\left(f_0' \partial_t \phi_0\right)|_{t=-1} + \left(f_0' \partial_t \phi_0\right)|_{t=1}\right\|_{\beta_0+2,1;\varepsilon} \\ &\le C\varepsilon^3. \end{aligned}$$

Hence we get

$$\begin{aligned} \left\|E_{3,1} - E_{3,-1}\right\|_{\beta_0+2,1;\varepsilon} &\le C\varepsilon^3 + \frac{1}{2}\left\|\partial_t \phi_0(-1, l)^2 - \partial_t \phi_0(1, l)^2\right\|_{\beta_0+2,1;\varepsilon} \\ &\le C\varepsilon^3. \end{aligned}$$

The proof of $\left\|E_{3,1} + E_{3,-1}\right\|_{\beta_0,1;\varepsilon} \le C\varepsilon^3$ is similar. □

To proceed, let us consider the nonlinear problem

$$\begin{cases} \Delta\phi = J_{\Gamma_0} g + \left(\Delta_{\Gamma_0} f + |A|^2\right) t - E_1 + E_2 + \Delta_{\Gamma_0} w - \Delta_{\Gamma_s} w, & \text{in } \Omega_h, \\ \phi = 0 & \text{on } \partial\Omega_h. \end{cases} \tag{28}$$

Let us introduce the notation

$$P(f, \bar{g}, \phi) := -E_1 + E_2 + \Delta_{\Gamma_0} w - \Delta_{\Gamma_s} w + \partial_t^2 \phi + \partial_l^2 \phi + \frac{3}{l} \partial_l \phi - \Delta\phi. \tag{29}$$

We will investigate the Lipschitz dependence of P on f and $\bar{g}$.

Lemma 19. *For* $f_i \in \mathcal{B}_{2,2;\varepsilon}$, $\bar{g}_i \in \bar{\mathcal{B}}_{\beta_0,2;1}$, *with* $\|f_i - f_0\|_{\beta_0,2;\varepsilon} \le C\varepsilon^2$, $\|\bar{g}_i\|_{\beta_0,2;1,\hat{}} \le C\varepsilon$, $i = 1, 2$, *we have*

$$\|P(f_1, \bar{g}_1, \phi) - P(f_2, \bar{g}_2, \phi)\|_{\beta_0+2,0;\varepsilon} = O\left(\varepsilon^2\right)\|f_1 - f_2\|_{\beta_0,2;\varepsilon} + O\left(\varepsilon^3\right)\|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2;1,\hat{}}\,.$$

Proof. Let us consider the terms in (29). Recall that

$$E_1(f,\bar{g}) = -tf\Delta_{\Gamma_0}f + \Delta_{\Gamma_0}(fg) - g\Delta_{\Gamma_0}f + \Delta_{\Gamma_0}\left[(s-g)\frac{f^2}{1+f}\right].$$

We compute directly that

$$f_1\Delta_{\Gamma_0}f_1 - f_2\Delta_{\Gamma_0}f_2 = f_1\Delta_{\Gamma_0}(f_1 - f_2) + \Delta_{\Gamma_0}f_2(f_1 - f_2)\,. \tag{30}$$

Next, since

$$\Delta_{\Gamma_0}(fg) - g\Delta_{\Gamma_0}f = 2f'g' + f\Delta_{\Gamma_0}g,$$

we have

$$\begin{aligned}&\left[\Delta_{\Gamma_0}(f_1g_1) - g_1\Delta_{\Gamma_0}f_1\right] - \left[\Delta_{\Gamma_0}(f_2g_2) - g_2\Delta_{\Gamma_0}f_2\right]\\&= 2\left(f_1' - f_2'\right)g_1' + 2f_2'\left(g_1' - g_2'\right)\\&+ \Delta_{\Gamma_0}g_1(f_1 - f_2) + f_2\Delta_{\Gamma_0}(g_1 - g_2)\,.\end{aligned} \tag{31}$$

Now combining (30), (31) and performing a similar computation for the term $\Delta_{\Gamma_0}\left[(s-g)\frac{f^2}{1+f}\right]$, we obtain

$$\|E_1(f_1,\bar{g}_1) - E_1(f_2,\bar{g}_2)\|_{\beta_0+2,0;\varepsilon} = O\left(\varepsilon^2\right)\|f_1 - f_2\|_{\beta_0,2;\varepsilon} + O\left(\varepsilon^3\right)\|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2;1}\,.$$

For the term

$$E_2(f,g) = \frac{1}{1+f}\sum_{i=1}^{6}\frac{s^2k_i^3}{1-sk_i} - \frac{fg\,|A|^2}{1+f},$$

we have

$$E_2(f_1,g_1) - E_2(f_2,g_2) = -|A|^2\left(\frac{f_1g_1}{1+f_1} - \frac{f_2g_2}{1+f_2}\right) + \frac{f_2 - f_1}{(1+f_1)(1+f_2)}\sum_{i=1}^{6}\frac{s^2k_i^3}{1-sk_i}.$$

Since $|A|^2 = O\left(\frac{\varepsilon^2}{(1+\varepsilon l)^2}\right)$, we obtain

$$\|E_2(f_1,\bar{g}_1) - E_2(f_2,\bar{g}_2)\|_{\beta_0+2,0;\varepsilon} = O\left(\varepsilon^2\right)\|f_1 - f_2\|_{\beta_0,2;\varepsilon} + O\left(\varepsilon^3\right)\|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2;1}\,.$$

It remains to analyze the term $\Delta_{\Gamma_0}w - \Delta_{\Gamma_s}w$. To handle it, we simply note that by Lemma 9 the following expansion holds:

$$\begin{aligned}\Delta_{\Gamma_0}w - \Delta_{\Gamma_s}w &= O\left(\frac{\varepsilon^2}{(1+\varepsilon l)^2}\right)\partial_l w + O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l^2 w\\&= O\left(\frac{\varepsilon^2}{(1+\varepsilon l)^2}\right)\left(\frac{-g'(1+f) - (s-g)f'}{(1+f)^2}\right)\\&+ O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\left(\frac{-g'(1+f) - (s-g)f'}{(1+f)^2}\right)',\end{aligned}$$

which yields the desired estimate:

$$\begin{aligned}&\left\|\left(\Delta_{\Gamma_0}w - \Delta_{\Gamma_s}w\right)|_{(f_1,g_1)} - \left(\Delta_{\Gamma_0}w - \Delta_{\Gamma_s}w\right)|_{(f_2,g_2)}\right\|_{\beta_0+2,0;\varepsilon}\\&= O\left(\varepsilon^2\right)\|f_1 - f_2\|_{\beta_0,2;\varepsilon} + O\left(\varepsilon^3\right)\|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2;1}\,.\end{aligned}$$

The proof is thus completed. □

Lemma 20. *Given f, $\bar{g}$, with $\|f-f_0\|_{\beta_0,2;\varepsilon}\le C\varepsilon^2$, $\|\bar{g}\|_{\beta_0,2;1,\hat{}}\le C\varepsilon$, problem (28) has a unique solution ϕ with*

$$\|\phi-\phi_0\|_{\beta_0+1,2;*}\le C\varepsilon^2.$$

If we write this solution as $\Phi(f,\bar{g})$, then

$$\|\Phi(f_1,\bar{g}_1)-\Phi(f_2,\bar{g}_2)\|_{\beta_0+1,2;*}\le C\|f_1-f_2\|_{\beta_0,2;\varepsilon}+C\varepsilon^2\|\bar{g}_1-\bar{g}_2\|_{\beta_0,2;1,\hat{}}.$$

Proof. We may recast (28) as

$$\phi=L_1\left[J_{\Gamma_0}g+\left(\Delta_{\Gamma_0}f+|A|^2\right)t\right]+L_1\left[P(f,\bar{g},\phi)\right],$$

where $\phi=\phi_0+\phi^*$, $\phi^*\in\mathcal{B}_{\beta_0+1,2;*}$. In other words,

$$\phi^*=\bar{L}_1\left(f,\bar{g},\phi^*\right):=L_1\left[J_{\Gamma_0}g+\left(\Delta_{\Gamma_0}f+|A|^2\right)t\right]+L_1\left[P\left(f,\bar{g},\phi_0+\phi^*\right)\right]-\phi_0.$$

We regard it as a fixed point problem of ϕ^* for the map $\bar{L}_1$. Observe that although ϕ_0 only belongs to $\mathcal{B}_{2,2;*}$, the function $P(f,\bar{g},\phi_0+\phi^*)$ actually lies in $\mathcal{B}_{\beta_0+1,0;*}$. Now we show $\bar{L}_1$ is a contraction map. Indeed, by Lemma 9,

$$\begin{aligned}\Delta\phi&=\partial_s^2\phi+\Delta_{\Gamma_s}\phi-H_{\Gamma_s}\partial_s\phi\\&=\frac{1}{(1+f)^2}\partial_t^2\phi+\Delta_{\Gamma_0}\phi+O\left(\frac{\varepsilon}{(1+\varepsilon l)^2}\right)\partial_l\phi\\&+O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l^2\phi+O\left(\sum k_i^2\right)\partial_t\phi.\end{aligned}$$

Using this expansion, we can verify that

$$\left\|\bar{L}_1\left(f,\bar{g},\phi_1^*\right)-\bar{L}_1\left(f,\bar{g},\phi_2^*\right)\right\|_{\beta_0+1,2;*}\le C\varepsilon\left\|\phi_1^*-\phi_2^*\right\|_{\beta_0+1,2;*}.$$

This implies that $\bar{L}_1$ is a contraction mapping provided that ε is small enough. It follows that (28) has a solution.

To see the Lipschitz dependence of Φ on f, $\bar{g}$, we subtract the equations satisfied by $\Phi(f_1,\bar{g}_1)$ and $\Phi(f_2,\bar{g}_2)$. Then one can use the explicit expression for E_1, E_2 to get the desired estimate. □

If we write $\Phi(f,\bar{g})=\phi_1+L_1(P(f,\bar{g},\Phi(f,\bar{g})))$, then our original nonlinear problem will be transformed into

$$\begin{cases}\partial_t^2\phi_1+\partial_l^2\phi_1+\frac{3}{l}\partial_l\phi_1=J_{\Gamma_0}g+\left(\Delta_{\Gamma_0}f+|A|^2\right)t+P(f,\bar{g},\phi_1), & \text{in } \Omega_h,\\ \phi_1=0 \text{ and } \partial_t\phi_1-f=E_{3,i}-\partial_t\left[L_1\left(P\left(f,\bar{g},\Phi(f,\bar{g})\right)\right)\right], & \text{on } \Gamma_{i+h_i}.\end{cases}\tag{32}$$

With all these preparations, we are now ready to prove Theorem 1.

Proof of Theorem 1. Let us set $f=f_0+\tilde{f}$. Using Proposition 17, we find that to solve (32), it suffices to get a solution for the following fixed point problem for $(\tilde{f},g)$:

$$\left(\tilde{f},\bar{g}\right)=\bar{L}_2\left(\tilde{f},\bar{g}\right):=L_2\left(\Upsilon_{-1},\Upsilon_1\right)-\left(f_0,0\right),$$

where

$$\Upsilon_i=E_{3,i}-\partial_t\left[L_1\left(P\left(f,\bar{g},\Phi(f,\bar{g})\right)\right)\right]|_{t=i},\ i=\pm1.$$

Let us define the space

$$\mathcal{B}:=\left\{\left(\tilde{f},\bar{g}\right)|,\left(\tilde{f},\bar{g}\right)\in\mathcal{B}_{\beta_0,2;\varepsilon}\times\mathcal{B}_{\beta_0,2;1,\hat{}}\right\},$$

equipped with the norm

$$\left\|\left(\tilde{f},\bar{g}\right)\right\|:=\varepsilon\left\|\tilde{f}\right\|_{\beta_0,2;\varepsilon}+\varepsilon^2\|\bar{g}\|_{\beta_0,2;1,\hat{}}.$$

We claim that $\bar{L}_2$ is a contraction mapping in the set

$$B_1:=\left\{\left(\tilde{f},\bar{g}\right)\in\mathcal{B}:\left\|\left(\tilde{f},\bar{g}\right)\right\|\le C_0\varepsilon^3\right\},$$

where C_0 is a fixed large constant. Indeed, let

$$\eta_{\pm}(f,\bar{g}) := \partial_t [L_1 (P(f,\bar{g},\Phi(f,\bar{g})))]|_{t=-1} \pm \partial_t [L_1 (P(f,\bar{g},\Phi(f,\bar{g})))]|_{t=1},$$

and

$$f_1 = f_0 + \widetilde{f}_1,\ f_2 = f_0 + \widetilde{f}_2.$$

Using Proposition 17, we can show

$$\begin{aligned}&\|\eta_+(f_1,\bar{g}_1) - \eta_+(f_2,\bar{g}_2)\|_{\beta_0,2;\varepsilon} + \|\eta_-(f_1,\bar{g}_1) - \eta_-(f_2,\bar{g}_2)\|_{\beta_0,2;\varepsilon}\\ &= O\left(\varepsilon^2\right)\|f_1 - f_2\|_{\beta_0,2;\varepsilon} + O\left(\varepsilon^3\right)\|\bar{g}_1 - \bar{g}_2\|_{\beta_0,2;1,\hat{}}.\end{aligned}$$

It then follows from Proposition 17, Lemma 19 and Lemma 20 that

$$\left\|\bar{L}_2\left(\widetilde{f}_1,\bar{g}_1\right) - \bar{L}_2\left(\widetilde{f}_2,\bar{g}_2\right)\right\| \le C\varepsilon\left\|\left(\widetilde{f}_1,\bar{g}_1\right) - \left(\widetilde{f}_2,\bar{g}_2\right)\right\|.$$

This proves the claim.

To prove the existence of a fixed point for $\bar{L}_2$, it remains to show that $\bar{L}_2(B_1) \subset B_1$. Since $\left(\widetilde{f},\bar{g}\right) \in B_1$, we have $\|\widetilde{f}\|_{\beta_0,2;\varepsilon} \le C_0\varepsilon^2$, $\|\bar{g}\|_{\beta_0,2;1,\hat{}} \le C_0\varepsilon$. Observe that due to the presence of the term $|A|^2 t$ and $t^3\sum k_i^3$, the function $L_1(P(f,\bar{g},\Phi(f,\bar{g})))|_{\pm 1}$ does not have enough decay and only belongs to $\mathcal{B}_{2,2;\varepsilon,*}$. However, since these two terms are odd, their contribution to the boundary derivative at $t = \pm 1$ cancel and therefore

$$\|\eta_+\|_{\beta_0,2;\varepsilon} \le C\varepsilon^2,\ \|\eta_-\|_{\beta_0+2,2;\varepsilon} \le C\varepsilon^3.$$

Hence by Proposition 17,

$$\bar{L}_2\left(\widetilde{f},\bar{g}\right) \le C\varepsilon^3,$$

which implies that $\bar{L}_2(B_1) \subset B_1$, provided that C_0 is chosen large enough.

The solution $w_h + \phi$ depends smoothly on ε. Let us take the derivatives of $w_h + \phi$ with respect to ε. Note that the main order of $w_h + \phi$ is $\frac{s-g}{1+f}$, where s is the Fermi coordinate around the minimal hypersurface S_ε. Using the fact that S_ε is a minimal foliation associated to the Simons' cone, we find that $\frac{d(w_h+\phi)}{d\varepsilon}$ is positive and satisfy the system (7) (see [27]). This proves that our solution of the free boundary problem is stable. This finishes the proof of Theorem 1. □

3. Existence of an energy minimizer in $\mathbb{R}^8$ — Proof of Theorem 2

In the previous section, we have shown that if $\varepsilon_0 > 0$ is small enough, then for each $\varepsilon < \varepsilon_0$, we have a solution for the free boundary problem whose nodal set is asymptotic to S_ε^+. By symmetry, one also has solutions whose nodal sets are asymptotic to S_ε^-. We denote these two continuous families of solutions by u_ε^+ and u_ε^-, with $u_\varepsilon^- < u_\varepsilon^+$. In this section, we will use variational arguments to show the existence of an energy minimizer U in $\mathbb{R}^8$, lying between $u_{\varepsilon_0}^+$ and $u_{\varepsilon_0}^-$. The arguments in this section are very similar to that of [31], where the global minimizers of the Allen–Cahn equation in dimension $n \ge 8$ are constructed.

We use B_a to denote the open ball of radius a in $\mathbb{R}^8$. Choose a Lipschitz function b_a which is invariant under the natural $O(4) \times O(4)$ action on $\mathbb{R}^8$ and

$$u_{\varepsilon_0}^- < b_a < u_{\varepsilon_0}^+ \text{ on } \partial B_a.$$

Let us consider the minimizing problem

$$\min_{\eta - b_a \in H_0^1(B_a)} J(\eta). \tag{33}$$

Lemma 21. *The minimizing problem (33) has a solution u_a which is invariant under $O(4) \times O(4)$.*

Proof. The existence of a minimizer u for (33) follows from standard arguments. The point is that we need to prove the existence of a minimizer which is additionally invariant under $O(4) \times O(4)$.

Since u solves the free boundary problem, it is continuous. We define

$$w_1(x) = \min\{u(gx) : g \in O(4) \times O(4)\},$$
$$w_2(x) = \max\{u(gx) : g \in O(4) \times O(4)\}.$$

Then w_1 and w_2 are invariant under $O(4) \times O(4)$. We claim that w_1 and w_2 are also minimizers. Indeed, for each $k \in \mathbb{N}$ and a finite set $\{g_1, \cdots, g_k\} \in O(4) \times O(4)$, let

$$\bar{w}_k = \min\{u(g_i x) : g_i \in O(4) \times O(4), i = 1, \ldots, k\}.$$

Then $\bar{w}_k$ is a minimizer. We cover $O(4) \times O(4)$ by finitely many balls with radius ε. Denote by n_ε the number of balls. In each ball, let us choose a $g_i \in O(4) \times O(4)$. We will define

$$q_\varepsilon(x) := \min\{u(g_i x) : i = 1, \ldots, n_\varepsilon\}.$$

Then q_ε is also a minimizer. We observe that by the continuity of a minimizer,

$$w_1(x) = \lim_{\varepsilon \to 0} q_\varepsilon(x).$$

On the other hand, let $\{\varepsilon_k\}$ be a sequence converge to 0. Then standard arguments yield that $q_{\varepsilon_k}(x)$ converges a.e. to minimizer q. This q must be w_1. This proves that w_1 is also a minimizer. Similarly, w_2 is also a minimizer. □

3.1. Regularity of the free boundary

We would like to analyze the regularity property of the free boundary of the solution u_a.

Lemma 22. *The free boundary of u_a is smooth in $B_a \setminus \{0\}$.*

Proof. We shall use the standard arguments in the regularity theory: Blow up analysis around a free boundary point, cf. [44,45]. Let $x_0 \in B_a$ be a point on the free boundary of u. Suppose $x_0 \neq 0$ and $u_a(x_0) = 1$. We distinguish three cases.

Case 1. x_0 is not on the x axis and not on y axis.

In this case, standard arguments, based on Weiss monotonicity formula [44,45], tell us that the sequence $w_k := \frac{u_a(x_0+\rho_k\cdot)-1}{\rho_k}$, with $\rho_k \to 0$, has a subsequence converges in suitable sense to a minimizing cone $\mathfrak{C}$ in $\mathbb{R}^8$. We observe that u_a is invariant under $O(4) \times O(4)$. Hence $\mathfrak{C}$ reduces to a minimizing cone in $\mathbb{R}^2$. Therefore it must be a trivial cone. This implies that around x_0, the free boundary is flat and the regularity theory implies that actually it is smooth (analytic).

Case 2. x_0 is on the x or y axis.

In this case, the cone $\mathfrak{C}$ reduces to a minimizing cone in $\mathbb{R}^5$ which is invariant under the $O(4)$ action of the last four coordinates. If this cone were not trivial, it would be unstable, due to the classification of stable cones by Jerison and Savin in the axial symmetric case (see [27]). This contradicts with the fact that u_a is a minimizer. □

With this regularity at hand, we now want to prove that these minimizers are bounded by $u^+_{\varepsilon_0}$ and $u^-_{\varepsilon_0}$, by sweeping the family of ordered solutions u^+_ε and u^-_ε, similarly as in [31]. By our previous construction, for ε sufficiently small, we have

$$\begin{cases} u_a \le u^+_\varepsilon, & \text{in } B_a, \\ u_a < u^+_\varepsilon, & \text{in } \Lambda := \{X : |u_a(X)| < 1\}. \end{cases} \tag{34}$$

We show that actually (34) holds for all $\varepsilon \le \varepsilon_0$. To see this, we continuously increase the value of ε. Assume to the contrary that there existed a $\delta < \varepsilon_0$, which were the first value where we have

$$u_a \le u^+_\delta \text{ in } B_a, \text{ and } u_a(X) = u^+_\varepsilon(X) \text{ for some } X \in \bar{\Lambda}. \tag{35}$$

Maximum principle tells us that this X must be on ∂B_a. By the results in [30], the free boundary approaches the fixed boundary tangentially, this contradicts with the choice of δ, which is the smallest value satisfying (35). This finishes the proof.

Proof of Theorem 2. For each a large, we have a solution u_a with $u^-_{\varepsilon_0} < u_a < u^+_{\varepsilon_0}$. Sending a to infinity, we can find a subsequence of u_a which converges to a nontrivial solution U of (1). This solution U must be an energy minimizer of J, since each u_a is minimizing. $\square$

4. From minimizers in $\mathbb{R}^8$ to monotone solutions in $\mathbb{R}^9$ — Proof of Theorem 3

We have obtained a minimizer of the energy functional in dimension 8. Now we would like to construct monotone solutions in $\mathbb{R}^9$ from U, following the arguments of Jerison–Monneau [28]. We use $\left(x', x_9\right)$ to denote the coordinate of a point in $\mathbb{R}^9$, where $x' \in \mathbb{R}^8$. We will still use minimizing argument and work directly in the class of functions which is invariant w.r.p.t $O(4) \times O(4)$ action on the first eight variables.

We denote by v_1 the global minimizer in $\mathbb{R}^8$ we constructed in the last section. We also consider the solution v_2 which in the (x, y) coordinate is given by

$$v_2(x, y) = -v_1(y, x).$$

Since v_1 is constructed using minimizing argument, we can assume without loss of generality that $v_1 \le v_2$.

Proposition 23. *Either there exists a nontrivial solution $u : \mathbb{R}^9 \to \mathbb{R}$ monotone in the x_9 direction, or for each $\delta \in [v_1(0), v_2(0)]$, there exists a nontrivial global minimizer v in $\mathbb{R}^8$ with $v(0) = \delta$.*

Proof. Let ρ be a smooth decreasing cutoff function which satisfies

$$\rho(s) = \begin{cases} 1, & s < 1, \\ 0, & s > 2. \end{cases}$$

Define the function $w\left(x', x_9\right) = \rho(x_9)\, v_1\left(x'\right) + (1 - \rho(x_9))\, v_2\left(x'\right)$. For each cylinder $C_{R',l} = B_{R'} \times [-l, l]$, consider the minimization problem which equals w on $\partial B_{R'} \times [-l, l]$ and equals v_1 on $B_{R'} \times \{-l\}$, equals v_2 on $B_{R'} \times \{l\}$, in the class of functions which are invariant under $O(4) \times O(4)$ with respect to the first eight variables. We can find a minimizer $u_{R',l}$ that is monotone in the x_9 direction with this boundary condition. By the gradient bound of De Silva–Jerison [11], the free boundary is smooth in the interior of the cylinder.

Let $l \to +\infty$, we get a solution $u_{R'}$ on the whole cylinder $B_{R'} \times \mathbb{R}$, still monotone in x_9 and invariant under $O(4) \times O(4)$. We observe that

$$\lim_{x_9 \to +\infty} u_{R'} = v_2, \quad \lim_{x_9 \to -\infty} u_{R'} = v_1, \tag{36}$$

otherwise it will contradict with the fact that v_1 and v_2 are global minimizer. Now fix an $a \in (v_1(0), v_2(0))$. By (36), there exists $h_{R'}$ such that

$$u_{R'}\left(x', h_{R'}\right) = a.$$

Let $\bar{u}_{R'}\left(x', x_9\right) = u_{R'}\left(x', x_9 - h_{R'}\right)$. Then $\bar{u}_{R'}\left(x', 0\right) = a$. Let $R' \to +\infty$, we get a solution u monotone in x_9, invariant under $O(4) \times O(4)$, and

$$u(0) = a, v_1 \le u \le v_2.$$

If u is independent on x_9, then u is a global minimizer in $\mathbb{R}^8$. This proves the proposition. $\square$

Finally we are ready to prove Theorem 3.

Theorem 24. *There exists a solution u to our free boundary problem such that u is invariant w.r.p.t $O(4) \times O(4)$, monotone in x_9 and u is not one dimensional.*

Proof. Suppose the second possibility of Proposition 23 occurs. Then we can assume there is a global minimizer v in $\mathbb{R}^8$, invariant under $O(4) \times O(4)$ and $-1 < v(0) < 1$.

By Θ we shall denote the standard one dimensional solution to our free boundary problem:

$$\Theta(x)=\begin{cases} x, & x\in[-1,1], \\ 1, & x>1, \\ -1, & x<-1. \end{cases}$$

Note that Θ is monotone, but not strictly monotone. We would like to pose suitable boundary condition on the cylinder $C_{R',l}$. For each $t\in[0,1]$, let

$$\Theta_t\left(x',x_9\right)=\Theta\left(tv\left(x'\right)+(1-t)\,x_9\right).$$

Then $\Theta_1\left(x',x_9\right)=\Theta\left(v\left(x'\right)\right)=v\left(x'\right)$. Θ_t is a connection between Θ and v. Certainly, $\Theta_t\left(x',x_9\right)\in[-1,1]$. We check that Θ_t is continuous and monotone in the x_9 direction, since Θ itself is monotone. Consider those points where

$$tv\left(x'\right)+(1-t)\,x_9=1. \tag{37}$$

For each fixed x', there is a unique point x_9 satisfying (37).

Let $U_{t,R',l}$ be the minimizer of J in the symmetric (invariant under $O(4)\times O(4)$ action) class of functions defined on $C_{R',l}$ with boundary condition

$$U_t|_{\partial C_{R',l}}=\Theta_t|_{\partial C_{R',l}}.$$

After a possible translation in the x_9 direction, we can assume that

$$U_{t,R',l}(0)=v(0).$$

For each R', letting $l\to+\infty$, $U_{t,R',l}$ converges pointwisely to a solution $U_{t,R'}$, defined on the infinite cylinder $C_{R',+\infty}$. $U_{t,R'}$ is monotone in x_9 on the boundary of $C_{R',+\infty}$. Then one can show that $U_{t,R'}$ is monotone in x_9 in $C_{R',+\infty}$, with

$$U_{t,R'}(0)=v(0).$$

We claim that the map $t\to\partial_{x_9}U_{t,R'}(0)$ is a continuous map. We first show that it is continuous at the points where $t\neq1$. In this case, let $t_n\to t$. Then the sequence $U_{t_n,R'}$ converges to a monotone solution W. This W must be equal to $U_{t,R'}$. Indeed, since w and $U_{t,R'}$ are equal to each other on the boundary of the cylinder and the boundary value are monotone in the x_9 direction, we can infer that $W\geq U_{t,R'}$ and $W\leq U_{t,R'}$ by the sliding method.

The continuity at $t=1$ also follows from similar arguments as that of Jerison–Monneau [28]. The proof is thus completed. □

5. Solutions from catenoids

In this section, we shall construct solutions of the free boundary problem starting from another type of minimal surfaces — Catenoids. Since most of the arguments are similar to the Simons' cone case, we will only sketch the proof and point out the difference if necessary.

We remark that it is possible to do the construction for more general minimal surfaces, but this is beyond the scope of this paper.

5.1. The geometry of the catenoids

To begin with, let us choose an "arc-length" parametrization for the catenoid, this choice of coordinate will simplify the computation. Let $(x_1,\ldots,x_n)$ be the coordinate in $\mathbb{R}^n$. Let (r,θ) be the polar coordinate in $\mathbb{R}^{n-1}$, where θ is the coordinate on the unit sphere S^{n-2} in $\mathbb{R}^{n-1}$. As we mentioned before, the generalized catenoid $\mathcal{C}_\varepsilon$ in $\mathbb{R}^n$ can be described by

$$x_n=\bar{\omega}_\varepsilon(r),\, r\in[r_0,+\infty).$$

Introduce

$$l = l(r) := \int_{r_0}^{r} \sqrt{1 + \bar{\omega}_\varepsilon'(s)^2}\, ds.$$

Then locally the catenoid can also be described by the coordinate (l, θ). We would like to write the Laplacian–Beltrami operator $\Delta_{\mathcal{C}_\varepsilon}$ on $\mathcal{C}_\varepsilon$ in this coordinate. In the (r, θ) variable, the metric tensor on $\mathcal{C}$ is given by

$$\left[1 + \bar{\omega}_\varepsilon'(r)^2\right] dr^2 + r^2 d\theta^2.$$

It follows that the metric $\mathfrak{g}$ in the (l, θ) coordinate is $dl^2 + r^2 d\theta^2$. Observe that $\det \mathfrak{g} = r^{2(n-2)}$. For rotationally symmetric function $\varphi = \varphi(l)$, the Laplacian–Beltrami operator is given by

$$\begin{aligned}
\Delta_{\mathcal{C}_\varepsilon}\varphi &= \frac{1}{\sqrt{\det \mathfrak{g}}}\partial_i\left(\sqrt{\det \mathfrak{g}}\,\mathfrak{g}^{ij}\partial_j\varphi\right) \\
&= \varphi''(l) + \frac{n-2}{r}\varphi'(l) \\
&= \varphi''(l) + O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\varphi'(l).
\end{aligned} \tag{38}$$

Using s to denote the signed distance of a point P to $\mathcal{C}_\varepsilon$. Then we can write

$$P = X + s\nu(X),$$

where $X = X(l, \theta)$ designates a point on the $\mathcal{C}_\varepsilon$, $\nu(\cdot)$ is the unit normal of $\mathcal{C}_\varepsilon$ at X. We also put

$$\Gamma_s := \{X + s\nu(X) : X \in \mathcal{C}_\varepsilon\}.$$

Note that actually Γ_s depends on ε, although it is not explicit in the notation. To understand the Laplacian–Beltrami operator Δ_{Γ_s}, we need to analyze the metric on the surface Γ_s. Let $\nu_1 = \partial_l \nu$, $\nu_2 = \partial_\theta \nu$, and $X_1 = \partial_l X$, $X_2 = \partial_\theta X$. Define the matrix $B_0 = [X_1 + s\nu_1, X_2 + s\nu_2]$ and

$$B := [X_1 + s\nu_1, X_2 + s\nu_2, \nu].$$

Then the matrix of the induced metric $\mathfrak{g}$ in a tubular neighborhood of $\mathcal{C}$ in (l, θ, s) coordinate has the form

$$B^T B = \begin{bmatrix} B_0^T B_0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For more details, we refer to [14].

5.2. *Proof of Theorem 4*

In this part, we sketch the proof of Theorem 4.

Let $h_{-1}, h_1 \in C^{2,\alpha}_{loc}(\mathcal{C}_\varepsilon)$, small in certain sense. As before, define an approximate solution w_h in Ω_h, which is a region trapped between $\Gamma_{-1+h_{-1}}$ and Γ_{1+h_1}:

$$w_h(s, l) = \frac{s - g(l)}{1 + f(l)},$$

where

$$f = \frac{h_1 - h_{-1}}{2}, g = \frac{h_1 + h_{-1}}{2}.$$

Still set

$$t = \frac{s - g(l)}{1 + f(l)}.$$

The solution u we are looking for will have the form $u = w_h + \phi$.

We have the same formulas as in Lemma 5, Lemma 7 and Lemma 8 and will not restate them in this section again.

Lemma 25. *We have the following estimate for the Laplacian operator acting on functions depending on s and l:*

$$\Delta_{\Gamma_0}\eta - \partial_l^2\eta = O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l\eta,$$

and

$$\Delta_{\Gamma_s}\eta - \Delta_{\Gamma_0}\eta = O\left(\frac{\varepsilon^2}{(1+\varepsilon l)^2}\right)\partial_l\eta + O\left(\frac{\varepsilon}{1+\varepsilon l}\right)\partial_l^2\eta.$$

Proof. The first equation has already been proved in (38). The proof of the second equation is same as that of Lemma 9. □

Let us introduce the functional framework to work with. Let $\alpha \in (0,1)$ be a fixed constant.

Definition 26. For $\mu = 0, 1, 2$, $\beta \geq 0$, $\delta > 0$, the space $\mathcal{E}_{\beta,\mu;\delta}$ consists of those functions η defined on $\mathcal{C}_\delta$ such that

$$\sup_{l,|z|=l}\left[(1+\delta l)^\beta \|\eta\|_{C^{\mu,\alpha}(S_\delta\cap B_1(z))}\right] < +\infty.$$

Same as before, we also regard ϕ as the restriction of a function $\mathcal{T}(\phi)$ on $\Xi := [-1,1]\times[0,+\infty)$, where $\mathcal{T}(\phi)$ is a function of t and l defined for $(t,l) \in \bar{\Xi} := [-1,1]\times\mathbb{R}$, even in the variable l.

Definition 27. For $\mu = 0, 1, 2$, $\beta \geq 0$, the space $\mathcal{E}_{\beta,\mu;*}$ consists of those functions ϕ such that

$$\|\phi\|_{\beta,\mu;*} := \sup_{l\in\mathbb{R};z\in\bar{\Xi},|z|=|l|}\left[(1+\varepsilon|l|)^\beta \|\mathcal{T}(\phi)\|_{C^{\mu,\alpha}(\bar{\Xi}\cap B_1(z))}\right] < +\infty.$$

Let $v(\cdot)$ be an even smooth function such that

$$v(l) = \begin{cases} |l|^{3-n}, & |l| > 2, \\ 0, & |l| < 1. \end{cases}$$

The one dimensional space spanned by this function will be denoted by $\mathcal{D}$. Let $\bar{g}(\cdot) = g\left(\frac{\cdot}{\varepsilon}\right)$. If $n \geq 4$, we shall assume a priori $\bar{g} \in \mathcal{E}_{2n-6,2;1} \oplus \mathcal{D}$, $f \in \mathcal{E}_{2n-4,2;\varepsilon}$, with $\|\bar{g}\|_{\mathcal{E}_{2n-6,2;1}\oplus\mathcal{D}} \leq C\varepsilon$, $\|f\|_{2n-4,2;\varepsilon} \leq C\varepsilon^2$. For notational simplicity, the norm of $\mathcal{E}_{2n-6,2;1}\oplus\mathcal{D}$ will be denoted by $\|\cdot\|$. In the case $n=3$, we assume $\bar{g} \in \mathcal{E}_{2,2;1}\oplus\mathcal{D}$, $f \in \mathcal{E}_{4,2;\varepsilon}$, with $\|\bar{g}\|_{\mathcal{E}_{2,2;1}\oplus\mathcal{D}} \leq C\varepsilon$, $\|f\|_{4,2;\varepsilon} \leq C\varepsilon^2$, and in this case, the norm of $\mathcal{E}_{2,2;1}\oplus\mathcal{D}$ will also be denoted by $\|\cdot\|$.

With these choice of function spaces, we can verify that $\|\Delta w\|_{2n-4,2;*} \leq C\varepsilon^2$ if $n \geq 4$; while $\|\Delta w\|_{4,2;*} \leq C\varepsilon^2$ if $n = 3$.

Recall that the Jacobi operator on $\mathcal{C}_\delta$ is given by

$$J_{\mathcal{C}_\delta}(\eta) = \Delta_{\mathcal{C}_\delta}\eta + |A|^2\eta.$$

Here $|A|^2 = \sum k_i^2$ is the squared norm of the second fundamental form. Using the asymptotic behavior of $\bar{\omega}$, we deduce $|A|^2 = O\left(\frac{1}{(1+l)^{2n-2}}\right)$ as $l \to +\infty$. We need the following lemma, which states that the Jacobi operator on the catenoid $\mathcal{C}_1$ is invertible in suitable functional spaces.

Lemma 28. *For each function $\xi \in \mathcal{E}_{2n-4,2;1}$, there is a solution $\eta \in \mathcal{E}_{2n-6,2;1}\oplus\mathcal{D}$ such that*

$$J_{\mathcal{C}_1}(\eta) = \xi,$$

with

$$\|\eta\| \leq C\|\xi\|_{2n-4,0;1}.$$

Proof. Detailed analysis of the Jacobi operator on the higher dimensional catenoid can be found in [1]. The proof of this Lemma follows from similar arguments there. The basic idea is using variation of parameter formula to get the desired estimates. □

With this functional framework at hand, we now deal with the corresponding linear theory for our nonlinear problem. Given functions γ_1, γ_{-1}, consider the problem

$$\begin{cases} \partial_t^2 \phi + \partial_l^2 \phi = J_{\Gamma_0} g + \left(\Delta_{\Gamma_0} f + |A|^2\right) t, & \text{in } \Omega_h, \\ \phi = 0, & \text{on } \partial\Omega_h, \\ \partial_t \phi - f = \gamma_{-1}, & \text{on } \Gamma_{-1+h_{-1}}, \\ \partial_t \phi - f = \gamma_1, & \text{on } \Gamma_{1+h_1}. \end{cases} \tag{39}$$

Proposition 29. *Suppose $\gamma_1 \pm \gamma_{-1}$ is in $\mathcal{E}_{2n-4,1;\varepsilon}$ for $n \geq 4$ and in $\mathcal{E}_{4,1;\varepsilon}$ for $n = 3$. Then the system (39) has a solution $(f, \bar{g})$ such that*

$$\|f\|_{2n-4,2;\varepsilon} \leq C \|\gamma_1 + \gamma_{-1}\|_{2n-4,1;\varepsilon} + C \left\| |A|^2 \right\|_{2n-4,1;\varepsilon}, n \geq 4,$$
$$\|f\|_{2n-4,2;\varepsilon} \leq C \|\gamma_1 + \gamma_{-1}\|_{4,1;\varepsilon} + C \left\| |A|^2 \right\|_{4,1;\varepsilon}, n = 3,$$

and

$$\|\bar{g}\| \leq C\varepsilon^{-2} \|\gamma_1 - \gamma_{-1}\|_{2n-4,1;\varepsilon}, n \geq 4,$$
$$\|\bar{g}\| \leq C\varepsilon^{-2} \|\gamma_1 - \gamma_{-1}\|_{4,1;\varepsilon}, \ n = 3.$$

Proof. By even reflection, we can regard (39) as a problem in $(t, l) \in [-1, 1] \times \mathbb{R}$. Take the Fourier transform

$$\hat{\eta}(t, \xi) := \int_{\mathbb{R}} e^{-i\xi l} \eta(t, l)\, dl.$$

It is worth mentioning that here $\xi \in \mathbb{R}$, unlike the Simons' cone case where the Fourier transform is taken in $\mathbb{R}^4$. We are lead to the problem

$$\begin{cases} \partial_t^2 \hat{\phi} - |\xi|^2 \hat{\phi} = \left(J_{\mathcal{C}_\varepsilon} g\right)^{\hat{}} + \left(\Delta_{\mathcal{C}_\varepsilon} f + |A|^2\right)^{\hat{}} t, \quad t \in [-1, 1], \\ \hat{\phi}(-1, \xi) = \hat{\phi}(1, \xi) = 0, \\ \partial_t \hat{\phi}(-1, \xi) - \hat{f}(\xi) = \hat{\gamma}_{-1}(\xi), \\ \partial_t \hat{\phi}(1, \xi) - \hat{f}(\xi) = \hat{\gamma}_1(\xi). \end{cases} \tag{40}$$

The solution $\hat{\phi}$ of the first equation in (40) can be written in the form

$$\hat{\phi}(t, \xi) = \left(J_{\mathcal{C}_\varepsilon} g\right)^{\hat{}} p_{1,\xi}(t) + \left(\Delta_{\mathcal{C}_\varepsilon} f + |A|^2\right)^{\hat{}} p_{2,\xi}(t).$$

This implies that

$$\begin{cases} \left(J_{\mathcal{C}_\varepsilon} g\right)^{\hat{}} = \frac{\hat{\gamma}_1(\xi) - \hat{\gamma}_{-1}(\xi)}{2p'_{1,\xi}(1)}, \\ \left(\Delta_{\mathcal{C}_\varepsilon} f + |A|^2\right)^{\hat{}} = \frac{2\hat{f}(\xi) + \hat{\gamma}_{-1}(\xi) + \hat{\gamma}_1(\xi)}{2p'_{2,\xi}(1)}. \end{cases}$$

Observe that $\frac{1}{p'_{1,\xi}(1)} - \xi \tanh \xi$ is real analytic and of the order $O\left(e^{-\frac{|\xi|}{2}}\right)$ as $|\xi| \to +\infty$. According to the proof of Lemma 17, one need to estimate the inverse Fourier transform of $\xi \tanh \xi \left[\hat{\gamma}_1(\xi) - \hat{\gamma}_{-1}(\xi)\right]$. To do this, we can apply the fact that the Fourier transform of $x \tanh(\pi x)$ is equal to $-\frac{\cosh(\xi/2)}{2\sinh^2(\xi/2)}$, which has a singularity of order $O\left(\xi^{-2}\right)$ near the origin. The estimate of f is similar as before. □

Once we have established the functional framework and the linear solvability theory, we can proceed in the same way as the Simons' cone case.

Conflict of interest statement

There is no conflict of interest.

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