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Symplectic factorization, Darboux theorem and ellipticity

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Abstract

This manuscript identifies a maximal system of equations which renders the classical Darboux problem elliptic, thereby providing a selection criterion for its well posedness. Let f be a symplectic form close enough to ω_m, the standard symplectic form on $\mathbb{R}^{2m}$. We prove existence of a diffeomorphism φ, with optimal regularity, satisfying

$$\varphi^*(\omega_m) = f \quad \text{and} \quad \left\langle d\varphi^\flat; \omega_m \right\rangle = 0.$$

We establish uniqueness of φ when the system is coupled with a Dirichlet datum. As a byproduct, we obtain, what we term *symplectic factorization of vector fields*, that any map u, satisfying appropriate assumptions, can be factored as:

$$u = \chi \circ \psi \quad \text{with} \quad \psi^*(\omega_m) = \omega_m\,, \quad \left\langle d\chi^\flat; \omega_m \right\rangle = 0 \quad \text{and} \quad \nabla\chi + (\nabla\chi)^t > 0;$$

moreover there exists a closed 2-form Φ such that $\chi = (\delta\Phi \lrcorner\, \omega_m)^\sharp$. Here, $\sharp$ is the musical isomorphism and $\flat$ its inverse. We connect the above result to an L^2-projection problem.

Keywords: Symplectic decomposition; Darboux theorem for symplectic forms; Elliptic systems; Optimal transport of symplectic forms

1. Introduction

In the current manuscript we pursue our discussion on the transport of differential forms (cf. [11] and [12]). The focus here will be on Darboux Theorem for symplectic forms, which foundational character has been recognized since the pioneer work of Darboux [13]. The current state of the theory allows to assert that, given two smooth enough symplectic forms f and g, there exist infinitely many diffeomorphisms that pull f back to g; at least one of these symplectomorphisms can be shown to have optimal regularity properties (cf. [3]). A natural question is: "is there

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a maximal system of equations which renders Darboux problem well posed?" In this manuscript we address this issue by identifying additional structural properties to be satisfied by these symplectomorphisms, which allow to select a unique solution without compromising on a loss of optimal regularity.

For the sake of simplicity, let us assume that $g = \omega_m$ is the standard symplectic form on $\mathbb{R}^{2m}$ and ρ_{2m} is the associated volume form

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i} \quad \text{and} \quad \rho_{2m} = \frac{1}{m!}(\omega_m)^m = dx^1 \wedge \cdots \wedge dx^{2m}.$$

We shall make use of the musical isomorphism $\sharp$ which turns 1-forms into vector fields; while $\flat$ will denote its inverse.

We prove (in Corollary 8) that if f is a symplectic form close enough to ω_m in a Hölder norm, then there exists a diffeomorphism φ such that

$$\varphi^*(\omega_m) = f \quad \text{and} \quad \left\langle d\varphi^\flat; \omega_m \right\rangle = 0 \tag{1}$$

and, when coupled with appropriate boundary and monotonicity conditions, such a φ is unique. The map φ is also shown to have optimal regularity properties in Hölder spaces. Our uniqueness result in Corollary 8 yields a stability property namely: if (φ^s, f^s) satisfy (1) and the sequence $\{f^s\}$ converges to f in the Hölder norm $C^{r,a}$, then the sequence $\{\varphi^s\}$ converges in the $C^{r+1,b}$ norm, with $0 < b < a$, to a $\varphi \in C^{r+1,a}$ that satisfy (1). This stability result with a gain of regularity of φ holds provided that f is close to ω_m.

In Cartesian coordinates $(x_1, \cdots, x_n)$, the identity $\varphi^*(g) = f$ is the system of $\left[n(n-1)/2\right]$ equations

$$\sum_{1 \le p < q \le n} g^{pq}(\varphi)\left(\frac{\partial \varphi^p}{\partial x_i}\frac{\partial \varphi^q}{\partial x_j} - \frac{\partial \varphi^p}{\partial x_j}\frac{\partial \varphi^q}{\partial x_i}\right) = f^{ij}, \quad 1 \le i < j \le n.$$

The new complete system is obtained by adding the single equation $\left\langle d\varphi^\flat; g\right\rangle = 0$, totaling $\left[n(n-1)/2\right] + 1$ equations, the latter one reading off (cf. Section 2 (ix))

$$\sum_{1 \le i < j \le n} \left(\varphi^j_{x_i} - \varphi^i_{x_j}\right) g^{ij} = 0.$$

The novelty here is that the additional requirement $\left\langle d\varphi^\flat; \omega_m\right\rangle = 0$ makes (1) an elliptic system of equations (cf. Proposition 14) according to the definition we propose in the appendix. Coincidentally, as explained at the end of this introduction, the concept of ellipticity possesses an underlying variational feature.

When Ω is contractible, Corollary 10 asserts that under appropriate boundary conditions, there exists an unique 2-form $\Phi = \sum \Phi^{ij} dx^i \wedge dx^j$ satisfying

$$\left((\delta\Phi \lrcorner\, \omega_m)^\sharp\right)^*(\omega_m) = f \quad \text{and} \quad d\Phi = 0$$

meaning, in Cartesian coordinates, that

$$\begin{cases} \displaystyle\sum_{l=1}^{m}\sum_{s,t=1}^{2m}\left[\Phi^{s(2l-1)}_{x_s x_i}\Phi^{t(2l)}_{x_t x_j} - \Phi^{s(2l-1)}_{x_s x_j}\Phi^{t(2l)}_{x_t x_i}\right] = f^{ij} & 1 \le i < j \le n \\ \Phi^{ij}_{x_k} - \Phi^{ik}_{x_j} + \Phi^{jk}_{x_i} = 0 & 1 \le i < j < k \le n. \end{cases}$$

Note that when $n = 2$ the second equation is trivially fulfilled, while the first one reduces to the Monge–Ampère equation.

The above considerations should be compared with a discovery made in the 90's by Brenier [5] (cf. also [15,19,20]) on a given probability volume form ρ on a convex set Ω. It asserts that if φ is the $\|\cdot\|_{L^2(\rho_n)}$-projection of the identity map onto the set of maps that pull ρ back to the restriction of ρ_n to Ω, then $d\varphi^\flat = 0$. Similar conclusions can be reached if one replaces the $\|\cdot\|_{L^2(\rho_n)}$-projection by any $\|\cdot\|_{L^p(\rho_n)}$-projection for $p \in [1, \infty)$ (cf. [6,7,14,16]). Beside the variational analogy between the pull-back of volume forms and that of symplectic forms, there is a striking algebraic coincidence in the number of equations which makes either system elliptic. Indeed, in the case of volume forms the original problem at hand (resolved in the seminal work by Moser [21]; for an account of some of the further developments see [4,8,10]) was to find a map φ satisfying

$$\det \nabla\varphi = f. \tag{2}$$

The equation (2) is classically augmented with $d\varphi^\flat = 0$, which means additional $\left[n(n-1)/2\right]$ equations. The final system of interest

$$\det \nabla\varphi = f \quad \text{and} \quad d\varphi^\flat = 0 \tag{3}$$

totals $\left[n(n-1)/2\right] + 1$ equations. According to the definition in Appendix A, it is elliptic (see Example 31 (ii)). If we further assume that Ω is simply connected, one recovers, (since then $\varphi = \nabla\Phi$), the Monge–Ampère equation

$$\det \nabla^2\Phi = f.$$

The convexity condition on Φ simply expresses the fact that φ and the identity map are isotopic.

Our contribution includes understanding the importance of ellipticity as a way to address the *regularity* and *uniqueness* issues for Darboux problem. As the notion of *ellipticity* is not easy to find in the literature under the general form given here, we provide it in Appendix A. The starting point of the application of ellipticity in our context is the conclusion reached for the following system: if (ν being the exterior unit normal to $\partial\Omega$)

$$\begin{cases} u^*(\omega_m) = v^*(\omega_m) \quad \text{and} \quad \left\langle du^\flat; \omega_m\right\rangle = \left\langle dv^\flat; \omega_m\right\rangle & \text{in } \Omega \\ \left\langle u^\flat \wedge \nu^\flat; \omega_m\right\rangle = \left\langle v^\flat \wedge \nu^\flat; \omega_m\right\rangle & \text{on } \partial\Omega, \end{cases}$$

then $u \equiv v$ (cf. Proposition 16). In particular, we have uniqueness for the Dirichlet problem (i.e. when $u = v$ on $\partial\Omega$) or for the weaker Dirichlet problem when $u^\flat \wedge \nu^\flat = v^\flat \wedge \nu^\flat$ on $\partial\Omega$.

Theorem 6, the backbone of our work, interestingly enough yields Corollary 21 (while Corollary 18 provides a local version) which we refer to as the *symplectic factorization of vector fields*. Our statement is that any map u, close enough to a linear map (in particular the identity map), can be factored as

$$u = \chi \circ \psi \quad \text{with} \quad \psi^*(\omega_m) = \omega_m\,, \quad \left\langle d\chi^\flat; \omega_m\right\rangle = 0 \quad \text{and} \quad \nabla\chi + (\nabla\chi)^t > 0. \tag{4}$$

Moreover, if Ω is contractible, there exists a closed 2-form Φ such that $\chi = (\delta\Phi \lrcorner\, \omega_m)^\sharp$.

The factorization (4) can be viewed as a nonlinear Hodge factorization (cf. Subsection 4.3) and is reminiscent but different of the so-called *polar factorization of vector fields*, a result by Brenier [5] which has had profound influence on many other fields of mathematics. The polar factorization states that a vector field u can be written as

$$u = \chi \circ \psi \quad \text{with} \quad \det \nabla\psi = 1, \quad \operatorname{curl}\chi = 0 \quad \text{and} \quad \nabla\chi + (\nabla\chi)^t > 0.$$

The condition $\psi^*(\omega_m) = \omega_m$ is sufficient, but much stronger, to ensure that ψ preserves Lebesgue measure (i.e. $\det \nabla\psi = 1$ which is equivalent to $\psi^*\left((\omega_m)^m\right) = (\omega_m)^m$), while the condition $\left\langle d\chi^\flat; \omega_m\right\rangle = 0$ is necessary, but much weaker, to ensure that $\operatorname{curl}\chi = 0$.

Observe that (4), which resulted from a system of elliptic equations, is related to an orthogonal projection problem. Indeed, define the 2-form f by $u^*(f) = \omega_m$, then Proposition 23 (with $h = \omega_m$) shows that, would χ be a $\|\cdot\|_{L^2(\rho_{2m})}$-projection of the identity map onto the set of diffeomorphisms pulling back f to ω_m, then χ must satisfy $\nabla\chi + (\nabla\chi)^t \geq 0$ and $\left\langle d\chi^\flat; \omega_m\right\rangle = 0$. Furthermore letting $\psi = \chi^{-1} \circ u$, we have $\psi^*(\omega_m) = \omega_m$.

We would like to end up this introduction with some questions that remain open. It would be very enlightening to have a more geometrical proof of our results. In our theorems, which are essentially perturbative, can the smallness assumption be removed, or, in other words, under what more stringent conditions our results can be global?

2. Notation

We refer to [8] for this section and adopt the following notations. In the sequel the dimension is always even, i.e. $n = 2m$.

(i) To any $f = \sum_{i<j} f^{ij} dx^i \wedge dx^j \in \Lambda^2(\mathbb{R}^n)$, we associate, in a bijective way, a skew symmetric matrix $F \in \mathbb{R}^{n \times n}$ in the natural way. We also sometimes denote it by $\overline{f}$. Explicitly the coefficient at the i-th column and the j-th row of F (or $\overline{f}$) is f^{ij}, i.e.

$$F_i^j = f^{ij}.$$

The rank of the 2-form f is the rank of the matrix F and is therefore necessarily even. Note that

$$\det F = \left(\frac{1}{m\,!}\left|f^m\right|\right)^2$$

where $f^m = \underbrace{f \wedge \cdots \wedge f}_{m-times}$; in particular $\det F \geq 0$.

(ii) The standard symplectic form is

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}$$

and the associate standard symplectic matrix is

$$J_m = \begin{pmatrix} J_1 & 0 & \cdots & 0 & 0 \\ 0 & J_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & J_1 & 0 \\ 0 & 0 & \cdots & 0 & J_1 \end{pmatrix} \quad \text{where} \quad J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(iii) To any symplectic 2-form f, we write $f^{-1} \in \Lambda^2(\mathbb{R}^n)$ for the 2-form associated to the skew symmetric matrix $F^{-1} \in \mathbb{R}^{n\times n}$. Note that

$$J_m^{-1} = -J_m\,, \quad \omega_m^{-1} = -\omega_m \quad \text{and} \quad \overline{\omega}_m = J_m\,.$$

(iv) If $u \in \Lambda^1(\mathbb{R}^n)$ and $f \in \Lambda^2(\mathbb{R}^n)$, then

$$u \lrcorner f = \sum_{j=1}^{n}\left[\sum_{i=1}^{n} f^{ij}\,u_i\right] dx^j \in \Lambda^1\left(\mathbb{R}^n\right).$$

(v) Note that if $u \in \Lambda^1(\mathbb{R}^n)$, then

$$u \lrcorner f = v \quad \Leftrightarrow \quad Fu = v$$

in particular

$$u \lrcorner \omega_m = v \quad \Leftrightarrow \quad v \lrcorner \omega_m = -u.$$

(vi) As classical, we denote by d the exterior derivative, by $*$ the Hodge star operator and by δ the interior derivative (or co-differential). So that if f is a k-form then

$$\delta f = (-1)^{n(k-1)} * (d\,(*f))\,.$$

(vii) Let $\varphi \in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$, then

$$\varphi^*(f) = \sum_{1\leq i<j\leq n} f^{ij}(\varphi)\,d\varphi^i \wedge d\varphi^j.$$

In particular if $\varphi_D(x) = Dx$ and f is a constant 2-form, then

$$\varphi_D^*(f) = \sum_{1\leq i<j\leq n} f^{ij}\,D^i \wedge D^j$$

and thus the skew symmetric matrix associated to $\varphi_D^*(f)$ is $D^t F D$.

(viii) Recall that for $u \in \Lambda^1(\mathbb{R}^n)$, $f \in \Lambda^k(\mathbb{R}^n)$ and $h \in \Lambda^{k+1}(\mathbb{R}^n)$

$$\langle u \wedge f; h\rangle = \langle f; u \lrcorner h\rangle \tag{5}$$

where $\langle\cdot;\cdot\rangle$ denotes the scalar product of forms (on the left hand side of $(k+1)$-forms and on the right hand side of k-forms).

(ix) Observe that the musical isomorphism $\sharp : T^*M = \mathbb{R}^n \times \mathbb{R}^n \to TM = \mathbb{R}^n \times \mathbb{R}^n$ can be in the current situation viewed as the identity map $\sharp : \mathbb{R}^n \to \mathbb{R}^n$. Similarly, the inverse map $\flat : TM \to T^*M$ will be viewed as the identity map $\flat : \mathbb{R}^n \to \mathbb{R}^n$. Therefore throughout the article we identify a map $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with its associated 1-form $\varphi^\flat \in C^1(\mathbb{R}^n; \Lambda^1)$, so that $d\varphi$ can be seen as the 2-form $d\varphi^\flat \in C^0(\mathbb{R}^n; \Lambda^2)$ and therefore $d\varphi \lrcorner f$ is the scalar product of the 2-forms $d\varphi$ and $f \in C^0(\mathbb{R}^n; \Lambda^2)$. The notations $\sharp$ and $\flat$ are used only in the introduction in order not to burden too much the notations. In Cartesian coordinates we thus have that

$$d\varphi \lrcorner f = \langle d\varphi^\flat ; f \rangle = \langle d\varphi ; f \rangle = \sum_{1 \le i < j \le n} \left(\varphi^j_{x_i} - \varphi^i_{x_j} \right) f^{ij} = \sum_{i,j=1}^{n} \varphi^j_{x_i} f^{ij} = \langle \nabla \varphi ; F \rangle$$

where F is the skew symmetric matrix associated to f.

(x) Remark (cf. Theorem 3.5 in [8]) that, for $f \in C^1(\mathbb{R}^n; \Lambda^k)$ and $g \in C^1(\mathbb{R}^n; \Lambda^l)$, then

$$\delta(f \lrcorner g) = (-1)^{k+l} df \lrcorner g - f \lrcorner \delta g. \tag{6}$$

In particular if $k = 1$, $l = 2$ and g is constant (or more generally $\delta g = 0$), we have

$$\delta(f \lrcorner g) = -df \lrcorner g.$$

(xi) The integration by parts formula (cf. Theorem 3.28 in [8]) says that, if $1 \le k \le n$, $f \in C^1(\overline{\Omega}; \Lambda^{k-1})$ and $g \in C^1(\overline{\Omega}; \Lambda^k)$, then

$$\int_\Omega \langle df ; g \rangle + \int_\Omega \langle f ; \delta g \rangle = \int_{\partial\Omega} \langle \nu \wedge f ; g \rangle = \int_{\partial\Omega} \langle f ; \nu \lrcorner g \rangle . \tag{7}$$

(xii) We also let

$$\mathcal{H}_N\left(\Omega; \Lambda^2\right) = \left\{ \chi \in C^\infty\left(\overline{\Omega}; \Lambda^2\right) : d\chi = 0,\ \delta\chi = 0 \text{ and } \nu \lrcorner \chi = 0 \text{ on } \partial\Omega \right\} \tag{8}$$

$$\mathcal{H}_T\left(\Omega; \Lambda^2\right) = \left\{ \chi \in C^\infty\left(\overline{\Omega}; \Lambda^2\right) : d\chi = 0,\ \delta\chi = 0 \text{ and } \nu \wedge \chi = 0 \text{ on } \partial\Omega \right\}. \tag{9}$$

When Ω is contractible, then $\mathcal{H}_N(\Omega; \Lambda^2) = \mathcal{H}_T(\Omega; \Lambda^2) = \{0\}$.

3. Darboux theorem as an elliptic system

In this section we couple the classical Darboux theorems (local and global) with a natural constraint so as to get an elliptic system. For that we first state some preliminary results and we then show the existence and regularity of a solution for a system of first order equations of Cauchy–Riemann type.

3.1. Some preliminary results

The proofs of the results in this subsection are straightforward to obtain and, so, will be skipped.

Proposition 1. *Let $n = 2m$ and $\Omega \subset \mathbb{R}^n$ be an open set. Let $f \in C^1(\Omega; \Lambda^2)$ be a 2-form such that $f^m \neq 0$. Then*

$$(*f^m)\, f^{-1} = *\left(m f^{m-1}\right).$$

Therefore $d\left[f^{m-1}\right] = 0$ if and only if $\delta\left((*f^m)\, f^{-1}\right) = 0$ in Ω.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n$ be convex and $u \in C^1(\overline{\Omega}; \mathbb{R}^n)$ verifying for a certain $\gamma > 0$*

$$\langle \nabla u(x)\, \xi ; \xi \rangle \ge \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega},$$

then $u \in \mathrm{Diff}^1\left(\overline{\Omega}; u\left(\overline{\Omega}\right)\right)$.

Remark 3. The lemma is however false as soon as Ω is not convex. Examples of the type (in the complex plane) $z^{1+\epsilon}$ with $\epsilon > 0$ small or a very similar example written in polar coordinates as

$$u(r, \theta) = (r \cos((1+\epsilon)\theta), r \sin((1+\epsilon)\theta))$$

show that the conclusion of the lemma is, in general, false.

3.2. *Existence theorem for a linear elliptic system*

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set with exterior unit normal ν. Let $r \geq 0$ be an integer and $0 < a < 1$ be a real number. Let $g \in C^{r,a}\left(\overline{\Omega}\right)$ and $f \in C^{r,a}\left(\overline{\Omega}; \Lambda^2\right)$ be such that

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega \langle f; \chi \rangle = 0, \ \forall \chi \in \mathcal{H}_N\left(\Omega; \Lambda^2\right).$$

In the sequel we say that the symmetric part of $A \in C^0\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ is *definite with constant* $e = e(A) > 0$, if

$$\frac{1}{e}|\xi|^2 \geq |\langle A(x)\xi; \xi\rangle| \geq e|\xi|^2 \quad \text{for every } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^n.$$

Throughout this subsection we suppose that $A \in C^{r+1,a}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ and $B \in C^{r,a}\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ are invertible and the symmetric part of BA^{-1} is definite with constant $e = e\left(BA^{-1}\right)$. Let $C \in C^{r,a}\left(\overline{\Omega}; \mathbb{R}^n\right)$.

The following theorem is an intermediary and central step toward the proof of Theorem 6, the heart of Subsection 3.3. It can be expressed in terms of differential forms when A and B are skew symmetric matrices and thus $n = 2m$ (cf. Remark 5).

Theorem 4 *(A first order elliptic system). (I) Under the above assumptions, there exists $u \in C^{r+1,a}\left(\overline{\Omega}; \Lambda^1\right)$ such that, in Ω,*

$$d(Au) = f \quad \text{and} \quad \left\langle B^t; \nabla u\right\rangle + \langle C; u\rangle = g \tag{10}$$

and the system (10) is elliptic. Furthermore, there exists a constant $c = c(r, a, e, \Omega)$ such that

$$\|u\|_{C^{r+1,a}} \leq c\left(\|f\|_{C^{r,a}} + \|g\|_{C^{r,a}}\right). \tag{11}$$

(II) If moreover $\int_\Omega g = 0$ and

$$\operatorname{div}\left(B_j\right) = \sum_{i=1}^n \left(B_j^i\right)_{x_i} \in C^{r,a}\left(\overline{\Omega}\right), \quad 1 \leq j \leq n \tag{12}$$

then there exists $u \in C^{r+1,a}\left(\overline{\Omega}; \Lambda^1\right)$ satisfying (11) and

$$\begin{cases} d(Au) = f \quad \text{and} \quad \delta(Bu) = g & \text{in } \Omega \\ \qquad \nu \lrcorner (Bu) = 0 & \text{on } \partial\Omega; \end{cases}$$

if, in addition, Ω is simply connected then such a u is unique.

Remark 5. Further assume A and B are skew symmetric matrices and let $\alpha \in C^{r+1,a}\left(\overline{\Omega}; \Lambda^2\right)$, $\beta \in C^{r,a}\left(\overline{\Omega}; \Lambda^2\right)$ be respectively the 2-forms associated to A and B so that $u \lrcorner \alpha = Au$ and $u \lrcorner \beta = Bu$. Assume that $\delta\beta \in C^{r,a}\left(\overline{\Omega}; \Lambda^1\right)$. Since

$$\delta(u \lrcorner \beta) = -du \lrcorner \beta - u \lrcorner \delta\beta = \left\langle B^t; \nabla u\right\rangle + \langle C; u\rangle, \tag{13}$$

where we have set $C = -\delta\beta$, the theorem asserts the existence of $u \in C^{r+1,a}\left(\overline{\Omega}; \Lambda^1\right)$, such that

$$d(u \lrcorner \alpha) = f \quad \text{and} \quad \delta(u \lrcorner \beta) = g \text{ in } \Omega.$$

– For example, if $\beta = \alpha$ or $\beta = (*\alpha^m)\alpha^{-1}$, then the symmetric part of BA^{-1} is automatically definite. Note also that, in view of Proposition 1 and (6), if $\beta = (*\alpha^m)\alpha^{-1}$ and α is closed then β is co-closed.

– The boundary condition $\nu \lrcorner (Bu) = 0$ reads off

$$\nu \lrcorner (u \lrcorner \beta) = (u \wedge \nu) \lrcorner \beta = \langle u \wedge \nu; \beta \rangle = 0 \quad \text{on } \partial\Omega.$$

Proof of Theorem 4. *Step 1.* (i) Theorem 8.3 in [8] provides us with $F \in C^{r+1,a}\left(\overline{\Omega}; \Lambda^1\right)$ and a constant $c = c(r, a, \Omega)$ such that

$$dF = f \quad \text{in } \Omega \tag{14}$$

and

$$\|F\|_{C^{r+1,a}} \le c\, \|f\|_{C^{r,a}} .$$

(ii) We claim that the operator $L : C^2\left(\overline{\Omega}\right) \to C^0\left(\overline{\Omega}\right)$ defined by

$$LV = \left\langle B^t; \nabla \left(A^{-1}\nabla V\right)\right\rangle + \left\langle C; A^{-1}\nabla V\right\rangle$$

is elliptic. Indeed let $A^{-1} = \left(a^i_j\right)$ and $B = \left(b^i_j\right)$ and observe that the leading term in L is

$$\sum_{i,j,k=1}^{n} b^j_i a^i_k v_{x_j x_k}$$

and thus the ellipticity follows from the fact that the symmetric part of BA^{-1} is definite.

(iii) We then set

$$u = A^{-1}\nabla V + A^{-1}F$$

where $V \in C^{r+2,a}\left(\overline{\Omega}\right)$ (cf. Theorems 6.6 and 6.8 in [17]) is the solution of

$$\begin{cases} LV = g - \left\langle B^t; \nabla\left(A^{-1}F\right)\right\rangle - \left\langle C; A^{-1}F\right\rangle & \text{in } \Omega \\ \qquad V = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

The map u satisfies then the conclusions of the theorem.

(iv) We now prove the ellipticity of (10), cf. Definition 26. First observe that the leading term (in terms of derivatives) in the operator $\left\langle B^t; \nabla u\right\rangle + \langle C; u\rangle$ is of the form $\operatorname{div}(Bu) = \delta(Bu)$, we have therefore to show that the following algebraic system

$$\xi \wedge (A\sigma) = 0 \quad \text{and} \quad \xi \lrcorner (B\sigma) = 0$$

has, for any $\xi \neq 0$, $\sigma = 0 \in \mathbb{R}^n$ as the only solution. The first equation leads to the existence of $s \in \mathbb{R}$ such that $A\sigma = s\xi$, or equivalently $\sigma = sA^{-1}\xi$. Plugging this in the second equation we obtain

$$s\left\langle BA^{-1}\xi; \xi\right\rangle = 0$$

which implies, since the symmetric part of BA^{-1} is definite, $s = 0$ and hence $\sigma = 0$.

Step 2. We next discuss (II) except the uniqueness that is dealt with in Step 3. Note that

$$\delta(Bu) = \operatorname{div}(Bu) = \left\langle B^t; \nabla u\right\rangle + \langle C; u\rangle$$

where $C = (C_1, \cdots, C_n)$ with $C_j = \operatorname{div}\left(B_j\right)$. Hence the operator L takes the following form

$$LV = \delta\left(BA^{-1}dV\right).$$

We then solve (14), as in Step 1 and instead of solving (15), we replace the boundary datum $V = 0$ by $\left\langle \nu; BA^{-1}dV\right\rangle = -\left\langle \nu; BA^{-1}F\right\rangle$; or in other words we find a solution of

$$\begin{cases} \delta\left(BA^{-1}dV\right) = g - \delta\left(BA^{-1}F\right) & \text{in } \Omega \\ \left\langle \nu; BA^{-1}dV\right\rangle = -\left\langle \nu; BA^{-1}F\right\rangle & \text{on } \partial\Omega. \end{cases}$$

This is a Neumann type problem which is solvable since $\int_\Omega g = 0$. We then proceed exactly as in Step 1.

Step 3. We finally deal with the uniqueness issue. So let v satisfy

$$\begin{cases} d(Av) = 0 \quad \text{and} \quad \delta(Bv) = 0 & \text{in } \Omega \\ \nu \lrcorner (Bv) = 0 & \text{on } \partial\Omega \end{cases}$$

and set $w = Av$. Since Ω is simply connected and $dw = 0$, we can find W so that $w = \nabla W$. The function W therefore satisfies

$$\begin{cases} \operatorname{div}\left(BA^{-1}\nabla W\right) = 0 & \text{in } \Omega \\ \left\langle \nu; BA^{-1}\nabla W\right\rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

The divergence theorem implies

$$\int_\Omega \operatorname{div}\left(W\left(BA^{-1}\nabla W\right)\right) = \int_{\partial\Omega} W\left\langle \nu; BA^{-1}\nabla W\right\rangle = 0.$$

We therefore have

$$0 = \int_\Omega \operatorname{div}\left(W\left(BA^{-1}\nabla W\right)\right) = \int_\Omega \left\langle BA^{-1}\nabla W; \nabla W\right\rangle \geq \gamma \int_\Omega |\nabla W|^2$$

(provided $\left\langle BA^{-1}\xi;\xi\right\rangle \geq \gamma |\xi|^2$, if the other sign prevails, we just reverse the inequality above) and hence $\nabla W = 0$ which implies that $v = 0$ as claimed. □

3.3. The main theorem and some corollaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set with exterior unit normal ν. Recall that $\mathcal{H}_N\left(\Omega;\Lambda^2\right)$ has been defined in (8) and if Ω is contractible, then $\mathcal{H}_N\left(\Omega;\Lambda^2\right) = \{0\}$.

Throughout this subsection $r \geq 0$ and $n = 2m$ are integers, $0 < a < 1$ and ω_m is the standard symplectic form and its associated skew symmetric matrix is J_m. Let $f \in C^{r,a}\left(\overline{\Omega};\Lambda^2\right)$ be such that

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega \langle f;\chi\rangle = 0,\ \forall \chi \in \mathcal{H}_N\left(\Omega;\Lambda^2\right).$$

Given a matrix $D \in \mathbb{R}^{n\times n}$ we denote by σ_D the linear map which to $x \in \mathbb{R}^n$ associates $Dx \in \mathbb{R}^n$.

Theorem 6 *(A global theorem under a smallness assumption). Let $D \in \mathbb{R}^{n\times n}$ and $B \in C^{r,a}\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$ be both invertible and such that the symmetric part of DBJ_m is definite with constant $e = e(DBJ_m)$. Let $C \in C^{r,a}\left(\overline{\Omega};\mathbb{R}^n\right)$. Then there exist $\epsilon, \gamma, c > 0$ depending only on (r, a, e, Ω) such that if*

$$\left\| f - \sigma_D^*(\omega_m)\right\|_{C^{0,a/2}} \leq \epsilon$$

then there exists $\varphi \in \operatorname{Diff}^{r+1,a}\left(\overline{\Omega};\varphi\left(\overline{\Omega}\right)\right)$ satisfying, in Ω,

$$\varphi^*(\omega_m) = f \quad \text{and} \quad \left\langle B^t; \nabla(\varphi - \sigma_D)\right\rangle + \langle C; (\varphi - \sigma_D)\rangle = 0 \tag{16}$$

and

$$\begin{cases} \|\varphi - \sigma_D\|_{C^{r+1,a}} \leq c\left\| f - \sigma_D^*(\omega_m)\right\|_{C^{r,a}} \\ \|\varphi - \sigma_D\|_{C^{1,a/2}} \leq c\left\| f - \sigma_D^*(\omega_m)\right\|_{C^{0,a/2}} \end{cases} \tag{17}$$

$$|\langle[\nabla\varphi(x)\, B(x)\, J_m]\xi;\xi\rangle| \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega}. \tag{18}$$

Furthermore the system (16), restricted to maps satisfying (18), is elliptic. If, moreover,

$$\operatorname{div}\left(B_j\right) = \sum_{i=1}^n \left(B_j^i\right)_{x_i} \in C^{r,a}\left(\overline{\Omega}\right), \quad 1 \leq j \leq n \tag{19}$$

then there exists $\varphi \in \operatorname{Diff}^{r+1,a}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ *satisfying (17), (18) and*

$$\begin{cases} \varphi^*(\omega_m) = f \quad \textit{and} \quad \delta\left(B\left(\varphi - \sigma_D\right)\right) = 0 & \textit{in } \Omega \\ \nu \lrcorner \left(B\left(\varphi - \sigma_D\right)\right) = 0 & \textit{on } \partial\Omega; \end{cases}$$

such a φ *is unique if* Ω *is simply connected.*

Remark 7. Theorem 6 is particularly interesting if we further assume that B is skew symmetric. Indeed, in that case let $\beta \in C^{r,a}\left(\overline{\Omega}; \Lambda^2\right)$ be the differential form associated to B and assume $\delta\beta \in C^{r,a}\left(\overline{\Omega}; \Lambda^1\right)$. If $C = -\delta\beta$, then (13) holds and the theorem asserts the existence of φ such that

$$\begin{cases} \varphi^*(\omega_m) = f \quad \text{and} \quad \delta\left(\left(\varphi - \sigma_D\right) \lrcorner \beta\right) = 0 & \text{in } \Omega \\ \nu \lrcorner \left(\left(\varphi - \sigma_D\right) \lrcorner \beta\right) = 0 & \text{on } \partial\Omega. \end{cases}$$

Corollary 8. *There exist* $\epsilon, \gamma, c > 0$ *depending only on* (r, a, Ω) *such that if*

$$\|f - \omega_m\|_{C^{0,a/2}} \leq \epsilon,$$

then there exists $\varphi \in \operatorname{Diff}^{r+1,a}\left(\overline{\Omega}; \varphi\left(\overline{\Omega}\right)\right)$ *satisfying*

$$\begin{cases} \varphi^*(\omega_m) = f \quad \textit{and} \quad d\varphi \lrcorner \omega_m = 0 & \textit{in } \Omega \\ \nu \lrcorner \left(\left(\varphi - \mathrm{id}\right) \lrcorner \omega_m\right) = 0 & \textit{on } \partial\Omega; \end{cases} \tag{20}$$

and such that

$$\begin{cases} \|\varphi - \mathrm{id}\|_{C^{r+1,a}} \leq c\,\|f - \omega_m\|_{C^{r,a}} \\ |\langle[\nabla\varphi(x)]\xi; \xi\rangle| \geq \gamma\,|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \textit{ and } \forall x \in \overline{\Omega}. \end{cases} \tag{21}$$

Furthermore the system (20) is elliptic when restricted to maps satisfying the second inequality in (21) and if, in addition, Ω *is simply connected, then such a* φ *is unique.*

Remark 9. The following variant of Corollary 8 can easily be proved. There is φ such that

$$\begin{cases} \varphi^*(\omega_m) = f \quad \text{and} \quad d\varphi \lrcorner f^{-1} = 0 & \text{in } \Omega \\ \nu \lrcorner \left(\left(\varphi - \mathrm{id}\right) \lrcorner f^{-1}\right) = 0 & \text{on } \partial\Omega. \end{cases}$$

Corollary 8, as well as Theorem 12 below, can be written as second order systems, which is the counterpart of Monge–Ampère equation when $n = 2$ and so, f is a volume form.

Corollary 10 *(Second order Darboux theorem). Let* $\Omega \subset \mathbb{R}^n$ *be a bounded contractible open smooth set. Then there exists* $\epsilon = \epsilon(r, a, \Omega)$ *such that if* $f \in C^{r,a}\left(\overline{\Omega}; \Lambda^2\right)$ *is closed and*

$$\|f - \omega_m\|_{C^{0,a/2}} \leq \epsilon,$$

then there exists an unique $\Phi \in C^{r+2,a}\left(\overline{\Omega}; \Lambda^2\right)$ *satisfying the elliptic system*

$$\begin{cases} \left(\delta\Phi \lrcorner \omega_m\right)^*(\omega_m) = f \quad \textit{and} \quad d\Phi = 0 & \\ \nabla\left(\delta\Phi \lrcorner \omega_m\right) + \left(\nabla\left(\delta\Phi \lrcorner \omega_m\right)\right)^t > 0 & \textit{in } \Omega \\ \nu \lrcorner \Phi = -\nu \lrcorner H & \textit{on } \partial\Omega. \end{cases}$$

Here H *is such that* $\delta H = \mathrm{id} \lrcorner \omega_m$.

Remark 11. (i) Writing

$$\Phi = \sum_{i<j} \Phi^{ij} dx^i \wedge dx^j$$

and similarly for f we have that $(\delta\Phi \lrcorner \omega_m)^*(\omega_m) = f$ reads as (recalling that $\Phi^{ij} = -\Phi^{ji}$)

$$\sum_{l=1}^{m}\sum_{s,t=1}^{2m}\left[\Phi^{s(2l-1)}_{x_s x_i}\Phi^{t(2l)}_{x_t x_j} - \Phi^{s(2l-1)}_{x_s x_j}\Phi^{t(2l)}_{x_t x_i}\right] = f^{ij}, \quad 1 \le i < j \le n \tag{22}$$

while $d\Phi = 0$ means that

$$\Phi^{ij}_{x_k} - \Phi^{ik}_{x_j} + \Phi^{jk}_{x_i} = 0, \quad 1 \le i < j < k \le n.$$

Note that when $n = 2$ the equation $d\Phi = 0$ is trivially fulfilled, while (22) is exactly Monge–Ampère equation.

(ii) The form H can be taken, for example, as

$$H = \sum_{i=1}^{m}\frac{(x_{2i-1})^2 + (x_{2i})^2}{2}dx^{2i-1}\wedge dx^{2i}.$$

Proof of Corollary 10. *Step 1.* Using Corollary 8, we have

$$\begin{cases} \varphi^*(\omega_m) = f, \quad \delta(\varphi \lrcorner \omega_m) = 0, \quad \nabla\varphi + (\nabla\varphi)^t \text{ definite} & \text{in } \Omega \\ \nu \lrcorner (\varphi \lrcorner \omega_m) = \nu \lrcorner (\mathrm{id} \lrcorner \omega_m) = \nu \lrcorner \delta H & \text{on } \partial\Omega. \end{cases}$$

Since Ω is contractible (thus $\mathcal{H}_N\left(\Omega;\Lambda^1\right) = \{0\}$) and $\delta(\varphi \lrcorner \omega_m) = 0$, we can find Φ verifying (cf. Theorem 7.4 in [8])

$$\begin{cases} \delta\Phi = -(\varphi \lrcorner \omega_m) \quad \text{and} \quad d\Phi = 0 & \text{in } \Omega \\ \nu \lrcorner \Phi = -\nu \lrcorner H & \text{on } \partial\Omega \end{cases}$$

and thus $\varphi = \delta\Phi \lrcorner \omega_m$, showing the existence part. For the uniqueness we let Φ and Ψ be two solutions. From Proposition 16 (applied to $\alpha = \beta = \omega_m$, $u = \delta\Phi \lrcorner \omega_m$ and $v = \delta\Psi \lrcorner \omega_m$, recalling that $\nu \lrcorner \Phi = \nu \lrcorner \Psi$ implies $\nu \lrcorner \delta\Phi = \nu \lrcorner \delta\Psi$), we get that $\delta\Phi = \delta\Psi$ and hence

$$\begin{cases} \delta(\Phi - \Psi) = 0 \quad \text{and} \quad d(\Phi - \Psi) = 0 & \text{in } \Omega \\ \nu \lrcorner (\Phi - \Psi) = 0 & \text{on } \partial\Omega \end{cases}$$

and since $\mathcal{H}_N\left(\Omega;\Lambda^2\right) = \{0\}$, we get that $\Phi \equiv \Psi$.

Step 2. We now discuss the ellipticity of the system. Strictly speaking our system does not fit into the definition of ellipticity we give in the Appendix, because the equations are not of the same order (some being of the second and some of the first order). To make it of the same order we consider the equivalent system

$$(\delta\Phi \lrcorner \omega_m)^*(\omega_m) = f \quad \text{and} \quad \nabla(d\Phi) = 0.$$

We first linearize the system $(\delta\Phi \lrcorner \omega_m)^*(\omega_m) = f$. From

$$(\delta\Phi \lrcorner \omega_m)^*(\omega_m) = \sum_{p=1}^{m}\left[d\left[(\delta\Phi \lrcorner \omega_m)^{2p-1}\right] \wedge d\left[(\delta\Phi \lrcorner \omega_m)^{2p}\right]\right]$$

we get that the algebraic system that should lead to $\lambda = 0 \in \Lambda^2$ is

$$\sum_{p=1}^{m}\begin{bmatrix} \left[\xi \wedge ((\xi \lrcorner \lambda) \lrcorner \omega_m)^{2p-1}\right] \wedge d\left[(\delta\Phi \lrcorner \omega_m)^{2p}\right] \\ + d\left[(\delta\Phi \lrcorner \omega_m)^{2p-1}\right] \wedge \left[\xi \wedge ((\xi \lrcorner \lambda) \lrcorner \omega_m)^{2p}\right] \end{bmatrix} = 0.$$

This can be rewritten as

$$\xi \wedge \left(\sum_{p=1}^{m}\begin{bmatrix} ((\xi \lrcorner \lambda) \lrcorner \omega_m)^{2p-1}\, d\left[(\delta\Phi \lrcorner \omega_m)^{2p}\right] \\ -((\xi \lrcorner \lambda) \lrcorner \omega_m)^{2p}\, d\left[(\delta\Phi \lrcorner \omega_m)^{2p-1}\right] \end{bmatrix}\right) = 0$$

or equivalently

$$\xi \wedge \left(\sum_{p=1}^{m} \left[(\xi \lrcorner \lambda)^{2p} \, d\,(\delta \Phi \lrcorner \omega_m)^{2p} + ((\xi \lrcorner \lambda))^{2p-1} \, d\,(\delta \Phi \lrcorner \omega_m)^{2p-1} \right] \right) = 0.$$

This last equation just means that

$$\xi \wedge \left[(\delta \Phi \lrcorner \omega_m)^* \, (\xi \lrcorner \lambda) \right] = \xi \wedge \left[(\nabla \, (\delta \Phi \lrcorner \omega_m))^t \cdot (\xi \lrcorner \lambda) \right] = 0.$$

Therefore there exists $s \in \mathbb{R}$ such that

$$(\nabla \, (\delta \Phi \lrcorner \omega_m))^t \cdot (\xi \lrcorner \lambda) = s\,\xi \quad \Leftrightarrow \quad \left[(\nabla \, (\delta \Phi \lrcorner \omega_m))^t \, \Lambda \right] \xi = s\,\xi$$

where Λ is the skew symmetric matrix associated to λ. Since the symmetric part of $\nabla \, (\delta \Phi \lrcorner \omega_m)$ is definite, $\nabla \, (\delta \Phi \lrcorner \omega_m)$ is invertible and we deduce that

$$\Lambda \, \xi = s \, (\nabla \, (\delta \Phi \lrcorner \omega_m))^{-t} \, \xi.$$

Λ being a skew symmetric matrix we obtain

$$0 = \langle \Lambda \, \xi; \xi \rangle = s \left\langle (\nabla \, (\delta \Phi \lrcorner \omega_m))^{-t} \, \xi; \xi \right\rangle.$$

Using again that the symmetric part of $\nabla \, (\delta \Phi \lrcorner \omega_m)$ is definite and that $\xi \neq 0$, we infer that $s = 0$ and thus

$$\Lambda \, \xi = \xi \lrcorner \lambda = 0.$$

The equation $d\Phi = 0$ (or equivalently $\nabla \, (d\Phi) = 0$) leads to

$$\xi \wedge \lambda = 0$$

and thus, if $\xi \neq 0$, we get that $\lambda = 0$ and the ellipticity is proved. $\square$

3.4. Proof of the main theorem

We now deal with the proof of Theorem 6.

Proof. Steps 1 to 4 deal with the main statement, while Step 5 handles the extra result.

Step 1. We linearize the equation around σ_D and we let $\varphi = \sigma_D + u$. We immediately find

$$f = \varphi^* \, (\omega_m) = (\sigma_D + u)^* \, (\omega_m) = \sigma_D^* \, (\omega_m) + d \left(D^t \, (u \lrcorner \omega_m) \right) + u^* \, (\omega_m) \, .$$

Step 2. We solve by fixed point (more precisely Theorem 18.1 of [8]) the problem, in Ω,

$$\begin{cases} d \left[D^t \, (u \lrcorner \omega_m) \right] = d \left[\left(D^t \, J_m \right) u \right] = f - \sigma_D^* \, (\omega_m) - u^* \, (\omega_m) \\ \left\langle B^t; \nabla u \right\rangle + \langle C; u \rangle = 0. \end{cases}$$

Let us check all the hypotheses of that theorem.

1) Set

$$X_1 = \left\{ u \in C^{1,a/2} \left(\overline{\Omega}; \mathbb{R}^n \right) : \left\langle B^t; \nabla u \right\rangle + \langle C; u \rangle = 0 \right\}$$

$$X_2 = \left\{ u \in C^{r+1,a} \left(\overline{\Omega}; \mathbb{R}^n \right) : \left\langle B^t; \nabla u \right\rangle + \langle C; u \rangle = 0 \right\}$$

$$Y_1 = \left\{ b \in C^{0,a/2} \left(\overline{\Omega}; \Lambda^2 \right) : db = 0 \text{ in } \Omega \text{ and } \int_\Omega \langle b; \chi \rangle = 0, \; \forall \chi \in \mathcal{H}_N \right\}$$

$$Y_2 = \left\{ b \in C^{r,a} \left(\overline{\Omega}; \Lambda^2 \right) : db = 0 \text{ in } \Omega \text{ and } \int_\Omega \langle b; \chi \rangle = 0, \; \forall \chi \in \mathcal{H}_N \right\}.$$

It is easy to see (cf. Proposition 16.23 in [8]) that Hypothesis (H_{XY}) of Theorem 18.1 in [8] is satisfied.

2) Consider next the linear operator $L : X_i \to Y_i$ defined by

$$Lu = d\left[\left(D^t J_m\right) u\right].$$

Noting that the symmetric part of $B\left(D^t J_m\right)^{-1}$ is definite if and only if that of DBJ_m is definite, we can apply Theorem 4 (with $A = D^t J_m$). Therefore the operator has a right inverse $L^{-1} : Y_2 \to X_2$ verifying, where $C_1 = C_1(r, a, e, \Omega)$ is a constant,

$$\left\| L^{-1} b \right\|_{X_i} \le C_1 \|b\|_{Y_i} \quad \text{for every } b \in Y_2 \text{ and } i = 1, 2.$$

The Hypothesis (H_L) of Theorem 18.1 in [8] is then also satisfied.

3) Let $Q : X_2 \to Y_2$ be defined by $Q(u) = u^*(\omega_m)$. Note that $Q(0) = 0$ and $dQ(u) = 0$ in Ω. Moreover

$$\int_\Omega \langle Q(u); \chi \rangle = 0, \quad \forall \chi \in \mathcal{H}_N\left(\Omega; \Lambda^2\right)$$

since $Q(u)$ is exact independently of the topology of Ω. This follows from (7) and the fact that

$$\omega_m = d\left(\sum_{i=1}^m x_{2i-1} dx^{2i}\right) \quad \Rightarrow \quad Q(u) = d\left[u^*\left(\sum_{i=1}^m x_{2i-1} dx^{2i}\right)\right].$$

It is easily verified (using Theorem 16.28 in [8]) that there exists a constant $C_2 = C_2(r, a, \Omega)$ such that, for every $u, v \in C^{r+1,a}(\overline{\Omega}; \mathbb{R}^n)$, the following estimates hold

$$\|Q(u) - Q(v)\|_{C^{0,a/2}} \le C_2(\|u\|_{C^{1,a/2}} + \|v\|_{C^{1,a/2}}) \|u - v\|_{C^{1,a/2}}$$

and

$$\|Q(u)\|_{C^{r,a}} \le C_2 \|u\|_{C^{1,a/2}} \|u\|_{C^{r+1,a}} .$$

The Hypothesis $\left(H_Q\right)$ of Theorem 18.1 in [8] is therefore verified for $\rho = 1$,

$$c_1(r, s) = C_2(r + s) \quad \text{and} \quad c_2(r, s) = C_2 r s.$$

4) Our problem then becomes

$$Lu = f - \sigma_D^*(\omega_m) - Q(u).$$

Step 3. Theorem 18.1 in [8] gives us the existence of a $u \in X_2$ satisfying

$$Lu = f - \sigma_D^*(\omega_m) - Q(u)$$

provided

$$\left\| f - \sigma_D^*(\omega_m) \right\|_{C^{0,a/2}} \le \epsilon = \frac{1}{2C_1 \max\{4C_1 C_2, 1\}}.$$

Moreover there exists a constant $c = c(r, a, e, \Omega)$ such that

$$\|u\|_{C^{r+1,a}} \le c \left\| f - \sigma_D^*(\omega_m) \right\|_{C^{r,a}} \quad \text{and} \quad \|u\|_{C^{1,a/2}} \le c \left\| f - \sigma_D^*(\omega_m) \right\|_{C^{0,a/2}} .$$

Step 4. Let $R > 0$ be sufficiently large so that $\overline{\Omega} \subset B_R$, the ball centered at 0 and of radius R. Using Theorem 16.11 in [8], we extend u to $\overline{B}_R$ so that the extension $\widetilde{u}$ satisfies

$$\|\widetilde{u}\|_{C^{1,a/2}(\overline{B}_R)} \le \widetilde{c} \|u\|_{C^{1,a/2}(\overline{\Omega})} \le \widetilde{c} c \epsilon$$

where $\widetilde{c} = \widetilde{c}(r, \Omega) > 0$. From now on we ignore the difference between u and $\widetilde{u}$. By choosing ϵ smaller, if necessary, we can also ensure that there exists a constant $\gamma > 0$ such that (recall that $\varphi = \sigma_D + u$)

$$|\langle [\nabla\varphi(x) B(x) J_m] \xi; \xi \rangle| \ge \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{B}_R .$$

We therefore deduce (cf. Lemma 2) that $\varphi \in \mathrm{Diff}^{r+1,a}\left(\overline{B}_R; \varphi\left(\overline{B}_R\right)\right)$ restricted to $\overline{\Omega}$ verifies the conclusions of the theorem; while the ellipticity follows from Proposition 14.

Step 5. The statement with boundary datum follows as before considering now the spaces

$$X_1=\left\{u\in C^{1,a/2}\left(\overline{\Omega};\mathbb{R}^n\right):\delta\left(Bu\right)=0\text{ and }\nu\lrcorner\left(Bu\right)=0\right\}$$

$$X_2=\left\{u\in C^{r+1,a}\left(\overline{\Omega};\mathbb{R}^n\right):\delta\left(Bu\right)=0\text{ and }\nu\lrcorner\left(Bu\right)=0\right\}$$

while Y_1 and Y_2 are unchanged; the proof being then identical. The uniqueness is shown in Proposition 16. □

3.5. Local Darboux theorem

We continue our analysis with the local case. The theorem is a refinement of a result of Bandyopadhyay–Dacorogna [3] for Darboux theorem. The new statement here is that, not only there exists a solution to $\varphi^*(\omega_m)=f$ with optimal regularity, but there is one which satisfies the additional constraint $d\varphi\lrcorner\omega_m=0$ (which renders the system elliptic) and still preserves the optimal regularity.

In this subsection $r\geq 0$ and $n=2m$ are integers, $0<a<1$ and $x_0\in\Omega$ where $\Omega\subset\mathbb{R}^n$ is a bounded open set. Recall that (cf. Theorem 19 in [9]) since $n=2m$, the condition $\operatorname{rank}\left[f\left(x_0\right)\right]=n$ is equivalent to the existence of an invertible symmetric matrix $D\in\mathbb{R}^{n\times n}$ such that

$$\sigma_D^*\left(\omega_m\right)=f\left(x_0\right)$$

where $\sigma_D\left(x\right)=D\left(x-x_0\right)+x_0$.

Theorem 12. *Let $f\in C^{r,a}\left(\Omega;\Lambda^2\right)$ be closed and $D\in\mathbb{R}^{n\times n}$ symmetric and invertible such that*

$$\sigma_D^*\left(\omega_m\right)=f\left(x_0\right).$$

Let $\lambda\in C^{r,a}\left(\Omega;\Lambda^2\right)$ with the symmetric part of $D\Lambda J_m$ definite, where Λ is the skew symmetric matrix associated to λ. Then there exist a neighborhood $U\subset\Omega$ of x_0 and $\varphi\in\operatorname{Diff}^{r+1,a}\left(U;\varphi\left(U\right)\right)$ such that $\varphi\left(x_0\right)=x_0$,

$$\varphi^*\left(\omega_m\right)=f\text{ in }U\quad\text{and}\quad d\varphi\lrcorner\lambda=0\text{ in }U.\tag{23}$$

Moreover there exists a constant $\gamma>0$ such that

$$\left\langle\left[\nabla\varphi\left(x\right)\Lambda\left(x\right)J_m\right]\xi;\xi\right\rangle\geq\gamma\left|\xi\right|^2,\quad\forall\xi\in\mathbb{R}^n\text{ and }\forall x\in U\tag{24}$$

and the system (23) is elliptic when restricted to φ satisfying (24).

Remark 13. (i) The proof below will show that for every $\delta>0$, we can choose the neighborhood U depending on δ so that

$$\|\nabla\varphi-D\|_{C^0(\overline{U})}\leq\delta\quad\text{if }D\Lambda J_m>0$$

$$\|\nabla\varphi+D\|_{C^0(\overline{U})}\leq\delta\quad\text{if }D\Lambda J_m<0.$$

(ii) If $D=I$ (meaning that $f\left(x_0\right)=\omega_m$) and $\lambda=\omega_m$, then the theorem reads as follows

$$\varphi^*\left(\omega_m\right)=f,\quad d\varphi\lrcorner\omega_m=0\quad\text{and}\quad\nabla\varphi+\left(\nabla\varphi\right)^t>0.$$

Recall that

$$d\varphi\lrcorner\omega_m=\left\langle\nabla\varphi;J_m\right\rangle=\sum_{j=1}^m\left(\varphi_{x_{2j-1}}^{2j}-\varphi_{x_{2j}}^{2j-1}\right)=0$$

and, since ω_m is a constant form, we also have

$$\delta\left[\varphi\lrcorner\omega_m\right]=-d\varphi\lrcorner\omega_m=0.$$

(iii) Note that, in general, one cannot assume that $d\varphi=0$ (i.e. $\varphi=\nabla\Phi$) as in the case of volume forms. Indeed a slight change in Proposition 12 of [9] shows that if

$$f=\left(1+x_3\right)dx^1\wedge dx^2+x_2\,dx^1\wedge dx^3+dx^3\wedge dx^4,$$

then there exists no $\Phi \in C^3(\mathbb{R}^4)$ such that near 0

$$(\nabla\Phi)^*(\omega_m) = f$$

although there exists a local C^∞ diffeomorphism φ such that

$$\varphi^*(\omega_m) = f.$$

Our theorem nevertheless proves that the map φ can be chosen so that

$$\varphi^1_{x_2} - \varphi^2_{x_1} + \varphi^3_{x_4} - \varphi^4_{x_3} = 0.$$

Proof of Theorem 12 and Remark 13 (i). *Step 1 (Theorem 12).* We can assume, without loss of generality, that $x_0 = 0$. Let W be the unit ball centered at 0. For every $0 < \epsilon < 1$ sufficiently small define

$$f^\epsilon(x) = f(\epsilon x) \quad \text{and} \quad \lambda^\epsilon(x) = \lambda(\epsilon x).$$

Observe that $\lambda^\epsilon \in C^{r,a}\left(\overline{W};\Lambda^2\right)$ with the symmetric part of $D\Lambda^\epsilon J_m$ definite, $f^\epsilon \in C^{r,a}\left(\overline{W};\Lambda^2\right)$, $df^\epsilon = 0$, $f^\epsilon(0) = f(0)$ and

$$\left\| f^\epsilon - f(0)\right\|_{C^{0,a/2}(\overline{W})} \to 0 \quad \text{as } \epsilon \to 0.$$

Applying Theorem 6 (to $B^t = \Lambda^\epsilon$ and $C = 0$) we find, for ϵ small enough, $\psi_\epsilon \in \operatorname{Diff}^{r+1,a}(\overline{W};\psi_\epsilon\left(\overline{W}\right))$ satisfying

$$\psi_\epsilon^*(\omega_m) = f^\epsilon \text{ in } W, \quad \left\langle \Lambda^\epsilon;\nabla(\psi_\epsilon - \sigma_D)\right\rangle = 0 \text{ in } W$$

and

$$\left|\left\langle\left[\nabla\psi_\epsilon(x)\,\Lambda^\epsilon(x)\,J_m\right]\xi;\xi\right\rangle\right| \geq \gamma\,|\xi|^2, \quad \forall\,\xi \in \mathbb{R}^n \text{ and } \forall\,x \in W.$$

Since D is symmetric and Λ^ϵ is skew symmetric, we obtain

$$0 = \left\langle\Lambda^\epsilon;\nabla(\psi_\epsilon - \sigma_D)\right\rangle = \left\langle\Lambda^\epsilon;\nabla\psi_\epsilon - D\right\rangle = \left\langle\Lambda^\epsilon;\nabla\psi_\epsilon\right\rangle = d\psi_\epsilon \lrcorner \lambda^\epsilon.$$

Let

$$\chi_\epsilon(x) = \frac{x}{\epsilon} \quad \text{and} \quad \varphi = \epsilon\,\psi_\epsilon \circ \chi_\epsilon.$$

The map $x \to \varphi(x) - \varphi(0)$ and the set $U = \epsilon\,W$ have all the desired properties with (24) replaced by

$$|\langle[\nabla\varphi(x)\,\Lambda(x)\,J_m]\,\xi;\xi\rangle| \geq \gamma\,|\xi|^2, \quad \forall\,\xi \in \mathbb{R}^n \text{ and } \forall\,x \in U.$$

Finally note that, since U is connected, we have either $\langle[\nabla\varphi(x)\,\Lambda(x)\,J_m]\,\xi;\xi\rangle > 0$ or $\langle[\nabla\varphi(x)\,\Lambda(x)\,J_m]\,\xi;\xi\rangle < 0$. In the first case we have right away the claim, while in the second one we let

$$\overline{\varphi}(x) = -\varphi(x)$$

and observe that $\overline{\varphi}$ has all the properties claimed in the theorem. The ellipticity following from Proposition 14.

Step 2 (Remark 13 (i)). It is clearly enough to prove only the case $D\Lambda J_m > 0$. Theorem 6 also gives that

$$\|\psi_\epsilon - \sigma_D\|_{C^{1,a/2}(\overline{W})} \leq c\left\| f^\epsilon - \sigma_D^*(\omega_m)\right\|_{C^{0,a/2}(\overline{W})}.$$

Note that

$$\|\nabla\varphi - D\|_{C^0(\epsilon\overline{W})} = \|\nabla\psi_\epsilon - D\|_{C^0(\overline{W})} \leq c\left\| f^\epsilon - \sigma_D^*(\omega_m)\right\|_{C^{0,a/2}(\overline{W})}$$

and therefore, since

$$\left\| f^\epsilon - \sigma_D^*(\omega_m)\right\|_{C^{0,a/2}(\overline{W})} = \left\| f^\epsilon - f(0)\right\|_{C^{0,a/2}(\overline{W})} \to 0 \quad \text{as } \epsilon \to 0,$$

we have proved Remark 13 (i). $\square$

3.6. Ellipticity and uniqueness

We now prove that the system considered in Darboux theorem, cf. Theorems 6 and 12, is indeed elliptic (see the appendix).

Proposition 14 *(Ellipticity of the Darboux system). Let $n = 2m$ and $\Omega \subset \mathbb{R}^n$ be a connected open set. Let $\alpha \in C^1\left(\mathbb{R}^n; \Lambda^2\right)$ be such that the associated skew symmetric matrix A is invertible. Let $B \in C^1\left(\Omega; \mathbb{R}^{n\times n}\right)$ be invertible and $C \in C^1(\Omega; \mathbb{R}^n)$. Let $f \in C^1\left(\Omega; \Lambda^2\right)$ and $g \in C^1(\Omega)$. Let $\mathcal{S}$ be the set of maps $u \in C^1(\Omega; \mathbb{R}^n)$ satisfying*

$$\left\langle \left[\nabla u(x)\, B(x)\, A^{-1}(u(x))\right]\xi; \xi\right\rangle \neq 0 \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \forall\, x \in \Omega.$$

Then the system

$$u^*(\alpha) = f \quad \text{and} \quad \left\langle B^t; \nabla u\right\rangle + \langle C; u\rangle = g$$

is elliptic (over Ω and over $\mathcal{S}$).

Remark 15. (i) Let $\beta \in C^1\left(\mathbb{R}^n; \Lambda^2\right)$ be such that the associated skew symmetric matrix B is invertible. The proposition then includes as special cases the following two systems

$$u^*(\alpha) = f \quad \text{and} \quad \delta\left[u \lrcorner \beta\right] = g$$
$$u^*(\alpha) = f \quad \text{and} \quad du \lrcorner \beta = g.$$

(ii) When $\alpha = \beta$ are constant forms, then $BA^{-1} = I$ and we require therefore that

$$\langle \nabla u(x)\,\xi; \xi\rangle \neq 0 \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \forall\, x \in \Omega.$$

Proof of Proposition 14. *Step 1.* Our system is

$$\begin{cases} u^*(\alpha) = \sum_{p<q} \alpha^{pq}(u)\, du^p \wedge du^q = f \\ \left\langle B^t; \nabla u\right\rangle = g - \langle C; u\rangle. \end{cases}$$

Differentiating with respect to x_k the first equation we get, up to lower order terms that we write as $h(u, \nabla u)$,

$$\begin{cases} \sum_{p<q} \alpha^{pq}(u)\left[du^p_{x_k} \wedge du^q + du^p \wedge du^q_{x_k}\right] = f_{x_k} + h(u, \nabla u) \\ \left(\left\langle B^t; \nabla u\right\rangle\right)_{x_k} = (g - \langle C; u\rangle)_{x_k} \end{cases}$$

and therefore the algebraic problem becomes (recalling that $\alpha^{pq} = -\alpha^{qp}$)

$$\begin{cases} \xi_k\left\{\sum_{p<q} \alpha^{pq}(u)\left[\lambda^p \xi \wedge du^q - \lambda^q \xi \wedge du^p\right]\right\} = \xi_k\left\{\xi \wedge \left[u^*(\lambda \lrcorner \alpha)\right]\right\} = 0 \\ \xi_k\left(\left\langle B^t; \lambda \otimes \xi\right\rangle\right) = 0, \end{cases}$$

where $\lambda \in \Lambda^1$. Noting that $\left\langle B^t; \lambda \otimes \xi\right\rangle = \langle \xi; B\lambda\rangle$, we get after simplification by ξ_k (recall that $\xi \neq 0$),

$$\xi \wedge \left[u^*(\lambda \lrcorner \alpha)\right] = 0 \quad \text{and} \quad \langle \xi; B\lambda\rangle = 0.$$

Step 2. Note that

$$u^*(\lambda \lrcorner \alpha) = [\nabla u(x)]^t\, (\lambda \lrcorner \alpha(u(x))) = [\nabla u(x)]^t\, A(u(x))\,\lambda.$$

The equation $\xi \wedge \left[u^*(\lambda \lrcorner \alpha)\right] = 0$ and the fact that $\xi \neq 0$ lead to the existence of $s \in \mathbb{R} \setminus \{0\}$ such that

$$\xi = s\,[\nabla u(x)]^t\, A(u(x))\,\lambda.$$

Therefore the equation $\langle \xi; B\lambda\rangle = 0$ implies that

$$s\left\langle [\nabla u(x)]^t\, A(u(x))\,\lambda; B(x)\,\lambda\right\rangle = s\,\langle A(u(x))\,\lambda; \nabla u(x)\, B(x)\,\lambda\rangle = 0.$$

Since $s \neq 0$ and the symmetric part of $\nabla u(x)\, B(x)\, A^{-1}(u(x))$ is definite, we deduce that indeed $\lambda = 0$ and the ellipticity is proved. $\square$

Proposition 16 *(An uniqueness result). Let $n=2m$ and $\Omega\subset\mathbb{R}^n$ be a bounded open smooth simply connected set. Let α be a constant 2-form and $\beta\in C^1\left(\overline{\Omega};\Lambda^2\right)$. Let $A\in\mathbb{R}^{n\times n}$ and $B\in C^1\left(\overline{\Omega};\mathbb{R}^{n\times n}\right)$ be the associated skew symmetric matrices to α and β, which are assumed to be invertible. Let $u,v\in C^1\left(\overline{\Omega};\mathbb{R}^n\right)$ satisfy, $\forall\,\xi\neq 0$ and $\forall\,x\in\overline{\Omega}$,*

$$\left\langle\left[\nabla u\left(x\right)B\left(x\right)A^{-1}\right]\xi;\xi\right\rangle\cdot\left\langle\left[\nabla v\left(x\right)B\left(x\right)A^{-1}\right]\xi;\xi\right\rangle>0 \tag{25}$$

and such that

$$\begin{cases} u^*\left(\alpha\right)=v^*\left(\alpha\right) \quad \text{and} \quad \delta\left[u\lrcorner\beta\right]=\delta\left[v\lrcorner\beta\right] & \text{in } \Omega\\ \nu\lrcorner\left(u\lrcorner\beta\right)=\nu\lrcorner\left(v\lrcorner\beta\right) & \text{on } \partial\Omega.\end{cases}$$

Then

$$u\equiv v.$$

Remark 17. (i) The condition $\nu\lrcorner\left(u\lrcorner\beta\right)=\nu\lrcorner\left(v\lrcorner\beta\right)$ is equivalent (cf. Proposition 2.16 in [8]) to

$$\left(u\wedge\nu\right)\lrcorner\beta=\left(v\wedge\nu\right)\lrcorner\beta \quad\Leftrightarrow\quad \left\langle u\wedge\nu;\beta\right\rangle=\left\langle v\wedge\nu;\beta\right\rangle.$$

Therefore the uniqueness is obtained under a much weaker hypothesis than the Dirichlet condition $u\wedge\nu=v\wedge\nu$; which is precisely the condition under which uniqueness is obtained for the Monge–Ampère equation (written under a system of first order equations, cf. Example 31 (ii)).

(ii) In fact the hypothesis (25) can be weakened, it suffices to require that, if $z=u+v$, then

$$\left|\left\langle\left[\nabla z\left(x\right)B\left(x\right)A^{-1}\right]\xi;\xi\right\rangle\right|>0,\quad \forall\,\xi\neq 0 \text{ and } \forall\,x\in\overline{\Omega}.$$

This is interesting, because we get then uniqueness if, for example,

$$\left\langle\left[\nabla u\,BA^{-1}\right]\xi;\xi\right\rangle>0 \quad\text{and}\quad \left\langle\left[\nabla v\,BA^{-1}\right]\xi;\xi\right\rangle\geq 0.$$

Proof of Proposition 16. *Step 1.* Set

$$w=\frac{v-u}{2}\quad\text{and}\quad z=\frac{v+u}{2}.$$

The equation $u^*\left(\alpha\right)=v^*\left(\alpha\right)$ becomes then

$$\sum_{p<q}\alpha^{pq}\left[\left(dz^p-dw^p\right)\wedge\left(dz^q-dw^q\right)\right]=\sum_{p<q}\alpha^{pq}\left[\left(dz^p+dw^p\right)\wedge\left(dz^q+dw^q\right)\right].$$

After simplification we obtain

$$\sum_{p<q}\alpha^{pq}\left[\left(dz^p\wedge dw^q\right)+\left(dw^p\wedge dz^q\right)\right]=0$$

and thus, since $\alpha^{pq}=-\alpha^{qp}$, we find

$$0=\sum_{p,q}\alpha^{pq}\left(dw^p\wedge dz^q\right)=d\left[\sum_{p,q}\alpha^{pq}w^p dz^q\right]=d\left[\sum_{p,q,r}\alpha^{pq}w^p z^q_{x_r}dx^r\right]$$
$$=d\left[\left(\nabla z\right)^t\left(w\lrcorner\alpha\right)\right]=d\left[\left(\nabla z\right)^t\,A\,w\right].$$

Step 2. The problem under consideration is then

$$\begin{cases} d\left[\left(\nabla z\right)^t\,A\,w\right]=0 \quad\text{and}\quad \delta\left[B\,w\right]=0 & \text{in } \Omega\\ \nu\lrcorner\left(B\,w\right)=0 & \text{on } \partial\Omega.\end{cases}$$

Observe next that since (25) holds, then $\left|\left\langle\left[\nabla z\,B\,A^{-1}\right]\xi;\xi\right\rangle\right|>0$ and thus

$$\left|\left\langle\left[B\,A^{-1}\left(\nabla z\right)^{-t}\right]\xi;\xi\right\rangle\right|>0.$$

The uniqueness follows then from Theorem 4. $\square$

4. Corollaries: the symplectic factorization

Recall that the polar factorization (cf. [5] or [15]) tells us that if u is a map satisfying some non-degeneracy condition, then u can be written as the composition of $\chi = \nabla\Phi$, where Φ is a convex function, with a measure preserving map ψ, i.e.

$$u = \chi \circ \psi .$$

A natural question is to see if such a factorization exists in the symplectic context (we then call it the *symplectic factorization*), i.e. try to prove that a map $u : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ with appropriate regularity assumptions can be written as $u = \chi \circ \psi$ where

$$\nabla\chi + (\nabla\chi)^t > 0, \quad d\chi \lrcorner \omega_m = 0, \quad \psi^*(\omega_m) = \omega_m$$

where ω_m is the standard symplectic form, namely

$$\omega_m = \sum_{i=1}^{m} dx^{2i-1} \wedge dx^{2i}.$$

Or, more generally, replacing ω_m by any general symplectic form ω try to write u as $u = \chi \circ \psi$ where (note that $\omega_m^{-1} = -\omega_m$)

$$\nabla\chi + (\nabla\chi)^t > 0, \quad d\chi \lrcorner \omega^{-1} = 0 \quad \text{and} \quad \psi^*(\omega) = \omega .$$

We indeed give some evidences of the existence of the symplectic factorization. We prove it in two cases (cf. Corollaries 18 and 21) and formally in Remark 24 (i). We also formally show that the symplectic factorization implies the Hodge decomposition.

But before entering into details, we would like to point out that in the linear case, i.e. when $u(x) = Ax$ for some constant matrix A, the situation is better, since (cf. Theorem 20 in [9]) for any invertible matrix $A \in \mathbb{R}^{n\times n}$, there exists $S \in \mathbb{R}^{n\times n}$ such that $S^t J_m S = J_m$ (i.e. S preserves the standard symplectic matrix J_m) and a symmetric matrix B such that

$$A = SB.$$

By inversion, one directly obtains the factorization in the reverse order. Using the language of differential forms, the previous result reads as follows: any map $u(x) = Ax$ can be written as $u = \chi \circ \psi$ where $\psi(x) = Sx$ preserves ω_m and where $\chi(x) = Bx$ is the gradient of the function

$$x \to \frac{\langle Bx; x\rangle}{2}.$$

4.1. The local case

We have the following local result. In this subsection we let $r \geq 1$ and $n = 2m$ be integers, $0 < a < 1$, $\Omega \subset \mathbb{R}^n$ be open and $x_0 \in \Omega$. We recall that for any invertible skew symmetric matrix F there exists an invertible symmetric matrix $D \in \mathbb{R}^{n\times n}$ (cf. Theorem 19 in [9]) such that

$$F = DJ_m D.$$

We also recall that

$$d\chi \lrcorner \omega_m = \langle \nabla\chi; J_m\rangle = \sum_{j=1}^{m} \left(\chi^{2j}_{x_{2j-1}} - \chi^{2j-1}_{x_{2j}}\right) = 0.$$

Corollary 18 *(Local symplectic factorization). Let $D \in \mathbb{R}^{n\times n}$ be symmetric and definite. Let $u \in \operatorname{Diff}^{r,a}(\Omega; u(\Omega))$ be such that*

$$\left(u^{-1}\right)^*(\omega_m)(u(x_0)) = \sigma_D^*(\omega_m) \quad \textit{with } \sigma_D(y) = D(y - u(x_0)) + u(x_0).$$

Then there exist a neighborhood U of x_0,

$$\psi \in \operatorname{Diff}^{r,a}(U;\psi(U)) \quad \text{and} \quad \chi \in \operatorname{Diff}^{r,a}(\psi(U);\chi(\psi(U)))$$

such that $\psi(x_0) = u(x_0)$ and for $x \in U$

$$\psi^*(\omega_m)(x) = \omega_m\,, \quad \left[d\chi \lrcorner \omega_m\right](\psi(x)) = 0 \quad \text{and} \quad u = \chi \circ \psi.$$

Moreover there exists a constant $\gamma > 0$ such that

$$\langle \nabla\chi(\psi(x))\xi;\xi\rangle \geq \gamma\,|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in U.$$

Remark 19. (i) The condition $\left(u^{-1}\right)^*(\omega_m)(u(x_0)) = \sigma_D^*(\omega_m)$ is equivalent to

$$(\sigma_D \circ u)^*(\omega_m)(x_0) = \omega_m\,.$$

So, in particular, if

$$u^*(\omega_m)(x_0) = \omega_m$$

we can take $D = I$ and the factorization holds.

(ii) In a completely analogous way, we can prove the symplectic factorization in the reverse order, namely there exist a neighborhood U of x_0,

$$\chi \in \operatorname{Diff}^{r,a}(U;\chi(U)) \quad \text{and} \quad \psi \in \operatorname{Diff}^{r,a}(\chi(U);\psi(\chi(U)))$$

such that $\chi(x_0) = x_0$ and for $x \in U$

$$\psi^*(\omega_m)(\chi(x)) = \omega_m\,, \quad \left[d\chi \lrcorner \omega_m\right](x) = 0 \quad \text{and} \quad u = \psi \circ \chi.$$

Moreover there exists a constant $\gamma > 0$ such that

$$\langle \nabla\chi(x)\xi;\xi\rangle \geq \gamma\,|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in U.$$

Proof of Corollary 18. With no loss of generality we can assume that $D > 0$, since

$$\sigma_D^*(\omega_m) = \sigma_{-D}^*(\omega_m)\,.$$

We let $f(y) = \left(u^{-1}\right)^*(\omega_m)(y)$ (and F be its associated skew symmetric matrix) which is a $C^{r-1,a}$ symplectic form. Note that

$$\sigma_D^*(\omega_m) = f(u(x_0))$$

and therefore

$$DF^{-1}(u(x_0))\,J_m = D\,[DJ_mD]^{-1}\,J_m = J_m^t D^{-1} J_m\,.$$

Moreover $DF^{-1}(u(x_0))\,J_m > 0$, since the right-hand side is positive definite since $D > 0$ (and thus $D^{-1} > 0$). Hence, by continuity, we have that $DF^{-1}J_m > 0$ in a small enough neighborhood of $u(x_0)$. Appealing to Theorem 12, we find a neighborhood $V \subset u(\Omega)$ of $u(x_0)$ and $\varphi \in \operatorname{Diff}^{r,a}(V;\varphi(V))$ such that $\varphi(u(x_0)) = u(x_0)$ and, for $y \in V$,

$$\varphi^*(\omega_m)(y) = f(y)\,, \quad \left[d\varphi \lrcorner f^{-1}\right](y) = 0$$

and, in view of Remark 13 (i) restricting, if necessary, the neighborhood V, we have that $\nabla\varphi + (\nabla\varphi)^t > 0$ on V. We then claim that $\chi = \varphi^{-1}$, $\psi = \varphi \circ u$ and $U = u^{-1}(V)$ have all the desired properties. Indeed we immediately have $\psi^*(\omega_m) = \omega_m$. The condition $d\chi \lrcorner \omega_m = 0$ follows from Lemma 20, since

$$\begin{aligned} 0 &= \left[d\varphi \lrcorner f^{-1}\right](y) = \left[d\varphi \lrcorner (\varphi^*(\omega_m))^{-1}\right](y) = -\left[d\left(\varphi^{-1}\right) \lrcorner \omega_m^{-1}\right](\varphi(y)) \\ &= \left[d\chi \lrcorner \omega_m\right]\left(\chi^{-1}(y)\right). \end{aligned}$$

Finally the conclusion $\nabla\chi + (\nabla\chi)^t > 0$ holds, since $\nabla\varphi + (\nabla\varphi)^t > 0$. This completes the proof. $\square$

In the above considerations, as well as in (30) of Proposition 23, we use the following lemma.

Lemma 20. *Let $n = 2m$ and $\omega \in C^0\left(\mathbb{R}^n; \Lambda^2\right)$ be such that*

$$\operatorname{rank}[\omega(x)] = n, \quad \textit{for every } x.$$

Let $\varphi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$. Then, for every $x \in \mathbb{R}^n$, the following holds

$$\left[d\varphi \lrcorner (\varphi^*(\omega))^{-1}\right](x) = -\left[d\left(\varphi^{-1}\right) \lrcorner\, \omega^{-1}\right](\varphi(x)).$$

Proof of Lemma 20. We let $\overline{\omega} \in \mathbb{R}^{n\times n}$ be the skew symmetric matrix associated to $\omega \in \Lambda^2$. On one hand we have

$$\begin{aligned}\left[d\varphi \lrcorner (\varphi^*(\omega))^{-1}\right](x) &= \left\langle \nabla\varphi(x); \left\{(\nabla\varphi)^t(x)\cdot\overline{\omega}(\varphi(x))\cdot\nabla\varphi(x)\right\}^{-1}\right\rangle \\ &= \left\langle \nabla\varphi(x); (\nabla\varphi)^{-1}(x)\cdot(\overline{\omega})^{-1}(\varphi(x))\cdot(\nabla\varphi)^{-t}(x)\right\rangle.\end{aligned}$$

On the other hand we get

$$\begin{aligned}\left[d\left(\varphi^{-1}\right)\lrcorner\,\omega^{-1}\right](\varphi(x)) &= \left\langle \nabla\left(\varphi^{-1}\right)(\varphi(x)); (\overline{\omega})^{-1}(\varphi(x))\right\rangle \\ &= \left\langle (\nabla\varphi)^{-1}(x); (\overline{\omega})^{-1}(\varphi(x))\right\rangle.\end{aligned}$$

Observe that, for any invertible matrices $A, B \in \mathbb{R}^{n\times n}$ with B skew symmetric, we have

$$\left\langle ABA^t; A^{-1}\right\rangle = \left\langle A; A^{-1}AB^t\right\rangle = -\langle A; B\rangle.$$

Taking $A = (\nabla\varphi)^{-1}(x)$ and $B = (\overline{\omega})^{-1}(\varphi(x))$, we obtain the lemma. □

4.2. The global case for small data

We now show the symplectic factorization under a smallness assumption. We let $r \geq 1$ and $n = 2m$ be integers, $0 < a < 1$, $\Omega \subset \mathbb{R}^n$ be a bounded connected open smooth set and ω_m be the standard symplectic form. For $D \in \mathbb{R}^{n\times n}$ we set

$$\sigma_D(x) = Dx.$$

Corollary 21 *(Global symplectic factorization for small data). Let $D \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Then there exists $\delta = \delta(r, a, D, \Omega) > 0$ such that for every $u \in \operatorname{Diff}^{r,a}\left(\overline{\Omega}; u\left(\overline{\Omega}\right)\right)$ with*

$$\|u - \sigma_D\|_{C^{1,a/2}} \leq \delta$$

there exist

$$\psi \in \operatorname{Diff}^{r,a}\left(\overline{\Omega}; \psi\left(\overline{\Omega}\right)\right) \quad \textit{and} \quad \chi \in \operatorname{Diff}^{r,a}\left(\psi\left(\overline{\Omega}\right); u\left(\overline{\Omega}\right)\right)$$

such that $u = \chi \circ \psi$ and

$$\nabla\chi + (\nabla\chi)^t > 0, \quad d\chi \lrcorner\, \omega_m = 0, \quad \psi^*(\omega_m) = \omega_m.$$

Moreover if Ω is contractible, there exists a closed 2-form Φ such that $\chi = \delta\Phi \lrcorner\, \omega_m$.

Remark 22 *(Reverse order symplectic factorization).* The proof below can be adapted to prove a variant of the above symplectic factorization in the reverse order, namely $u = \psi \circ \chi$ where

$$\nabla\chi + (\nabla\chi)^t > 0, \quad d\chi \lrcorner\, \omega_m = 0, \quad \psi^*(\omega_m) = \omega_m.$$

Proof of Corollary 21. *Step 1.* Let f be the $C^{r-1,a}$ symplectic form defined on $u\left(\overline{\Omega}\right)$ by $f=\left(u^{-1}\right)^{*}(\omega_m)$ and let F be its associated skew symmetric matrix. Observe that

$$F^{-t}(u(x))=\left[\left(\nabla u^{-1}(u(x))\right)^{t} J_m\left(\nabla u^{-1}(u(x))\right)\right]^{-t}=(\nabla u(x))\,J_m\,(\nabla u(x))^{t}$$

and thus (recalling that D is symmetric)

$$D^{-1}F^{-t}J_m=D^{-1}(\nabla u)\,J_m\,(\nabla u)^{t}J_m=J_m D^{t} J_m+O(\delta)$$

where we used the fact that $\|\nabla u-D\|_{C^0}\leq\delta$ with δ small.

Step 2. We apply Theorem 6 to D^{-1}, $B^{t}=F^{-1}$, $C=0$ and f. Choosing δ small enough, we can ensure (by Step 1) that the symmetric part of $D^{-1}BJ_m$ is definite (since $D>0$) and

$$\left\|f-\sigma_{D^{-1}}^{*}(\omega_m)\right\|_{C^{0,a/2}}=\left\|\left(u^{-1}\right)^{*}(\omega_m)-\sigma_{D^{-1}}^{*}(\omega_m)\right\|_{C^{0,a/2}}\leq\epsilon$$

where ϵ is as in Theorem 6 (note that the condition of orthogonality with the harmonic forms is here automatically satisfied since f is exact). We therefore find $\varphi\in\operatorname{Diff}^{r,a}\left(\overline{\Omega};\varphi\left(\overline{\Omega}\right)\right)$ satisfying

$$\varphi^{*}(\omega_m)=f,\quad\left\langle F^{-1};\nabla\left(\varphi-\sigma_{D^{-1}}\right)\right\rangle=0$$

which implies, since D^{-1} is symmetric and F^{-1} is skew symmetric, that

$$\varphi^{*}(\omega_m)=f,\quad d\varphi\lrcorner f^{-1}=0.$$

We furthermore have

$$\left\|\varphi-\sigma_{D^{-1}}\right\|_{C^{1,a/2}}\leq c\left\|f-\sigma_{D^{-1}}^{*}(\omega_m)\right\|_{C^{0,a/2}}.$$

Choosing δ even smaller if necessary, we deduce from the previous inequality that $\nabla\varphi+(\nabla\varphi)^{t}>0$, since $D^{-1}>0$. Letting $\chi=\varphi^{-1}$ and $\psi=\varphi\circ u$ we have shown the corollary (appealing to Lemma 20). □

4.3. Hodge decomposition via symplectic factorization

We now prove that whenever the symplectic factorization is true, we can then derive, at least formally, the classical Hodge decomposition (cf., for example, [8]) by linearization. Indeed, let h be any regular map. Formally at least, for any real ϵ, the symplectic factorization reads off $\operatorname{id}+\epsilon\,h=\chi_\epsilon\circ\psi_\epsilon$ where

$$\psi_\epsilon^{*}(\omega_m)=\omega_m\quad\text{and}\quad d\chi_\epsilon\lrcorner\omega_m=0. \tag{26}$$

We can assume (since we are only at the formal level) that χ_ϵ and ψ_ϵ can be written as

$$\chi_\epsilon=\operatorname{id}+\epsilon\,v+o(\epsilon)\quad\text{and}\quad\psi_\epsilon=\operatorname{id}+\epsilon\,w+o(\epsilon)$$

for some regular maps v and w. Note that

$$\begin{aligned}\operatorname{id}+\epsilon\,h=\chi_\epsilon\circ\psi_\epsilon&=\psi_\epsilon+\epsilon\,v\circ\psi_\epsilon+o(\epsilon)\\&=\operatorname{id}+\epsilon\,(v+w)+o(\epsilon)\end{aligned}$$

which implies $h=v+w$ or equivalently

$$h\lrcorner\omega_m=v\lrcorner\omega_m+w\lrcorner\omega_m\,. \tag{27}$$

An immediate calculation (as in Step 1 of the proof of Theorem 6) tells us that the first equation in (26) reads as

$$d\,(w\lrcorner\omega_m)=0. \tag{28}$$

Moreover the second equation in (26) directly gives $dv\lrcorner\omega_m=0$, or, which amounts to the same (cf. (6)),

$$\delta\,(v\lrcorner\omega_m)=0. \tag{29}$$

Combining (27), (28) and (29) we obtain (here we assume that Ω is contractible) the Hodge decomposition for $h \lrcorner \omega_m$, namely

$$h \lrcorner \omega_m = d\alpha + \delta\beta.$$

We therefore have obtained the same decomposition for any form g, choosing $h = -g \lrcorner \omega_m$ in the above equation and recalling that $g = (-g \lrcorner \omega_m) \lrcorner \omega_m$.

5. Optimal transport and the symplectic factorization

Let $n = 2m$, Ω be a bounded connected smooth open set in $\mathbb{R}^{2m}$ and $u \in \mathrm{Diff}^1\left(\overline{\Omega}; u\left(\overline{\Omega}\right)\right)$. Let $h \in C^1\left(\overline{\Omega}; \Lambda^2\right)$ be a symplectic form on $\overline{\Omega}$ and f be the symplectic form on $u\left(\overline{\Omega}\right)$ defined by $u^*(f) = h$. Set

$$\mathcal{S} = \left\{\chi \in \mathrm{Diff}^1\left(\overline{\Omega}; u\left(\overline{\Omega}\right)\right) : \chi^*(f) = h\right\}$$

and consider

$$(P) \qquad \inf_{\chi \in \mathcal{S}} \left\{ \int_{\Omega} |\chi(y) - y|^2 \rho(y)\, dy \right\}$$

where ρ is the volume form (which, by abuse of notations, is identified with a function) associated to h i.e.

$$\rho\, dx^1 \wedge \cdots \wedge dx^{2m} = \frac{1}{m!} h^m \quad \Leftrightarrow \quad \rho = \frac{1}{m!}\left(*h^m\right).$$

Proposition 23. *Assume that there exists a minimizer $\chi \in \mathcal{S}$ of (P). Then*

$$d\chi \lrcorner h^{-1} = 0, \quad \text{in } \overline{\Omega} \tag{30}$$

and

$$\left\langle \left[\nabla \chi(x)\right] \xi; \xi \right\rangle \geq 0, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega}. \tag{31}$$

Remark 24. (i) Proposition 23 is then related to the discussion made in the introduction of Section 4. Indeed we have our desired factorization (with semi-definiteness of the symmetric part of $\nabla\chi$ instead of definiteness), namely, letting $\psi = \chi^{-1} \circ u$, we obtain

$$\psi^*(h) = h, \quad d\chi \lrcorner h^{-1} = 0 \quad \text{and} \quad u = \chi \circ \psi. \tag{32}$$

(ii) Note that, by (6) and Proposition 1,

$$\delta\left[\chi \lrcorner h^{-1} \rho\right] = -d\chi \lrcorner h^{-1} \rho.$$

Hence (30) means that $\chi \lrcorner h^{-1}\rho$ is co-closed. In particular, assuming that Ω is contractible, we have if $h = \omega_m$, since $\omega_m^{-1} = -\omega_m$ and $\rho \equiv 1$, that there exists a closed 2-form Φ such that $\chi = \delta\Phi \lrcorner \omega_m$. By (31), (32) and the above remark, we get

$$u = (\delta\Phi \lrcorner \omega_m) \circ \psi, \quad \psi^*(\omega_m) = \omega_m, \quad d\Phi = 0 \quad \text{and} \quad \nabla(\delta\Phi \lrcorner \omega_m) + (\nabla(\delta\Phi \lrcorner \omega_m))^t \geq 0.$$

(iii) We could have proceeded in a completely analogous way considering the minimization problem

$$(P) \qquad \inf_{\psi \in \mathcal{S}_h} \left\{ \int_{\Omega} |\psi(x) - u(x)|^2 \rho(x)\, dx \right\}$$

where

$$\mathcal{S}_h = \left\{\psi \in \mathrm{Diff}^1\left(\overline{\Omega}; \overline{\Omega}\right) : \psi^*(h) = h\right\}.$$

Note that every $\psi \in \mathcal{S}_h$ satisfies

$$\psi^*\left(h^m\right)=h^m \quad \text{or equivalently} \quad \rho\left(\psi\right)\det\nabla\psi=\rho.$$

We would have found that $\chi=u\circ\psi^{-1}\in\mathcal{S}$ and satisfies (31) and (32).

Proof of Proposition 23. *Step 1.* For every $v:\Omega\to\mathbb{R}^n$ of the form

$$v=\nabla a\lrcorner h^{-1}$$

where $a\in C_0^\infty\left(\Omega\right)$ define S_t^v by

$$\frac{d}{dt}S_t^v=v\left(S_t^v\right) \quad \text{and} \quad S_0^v=\mathrm{id}.$$

Since $v\in C_0^\infty\left(\Omega;\mathbb{R}^n\right)$ classical results show that

$$S_t^v\in\mathrm{Diff}^\infty\left(\overline{\Omega};\overline{\Omega}\right) \quad \text{and} \quad \mathrm{supp}\left(S_t^v-\mathrm{id}\right)\subset\Omega\,.$$

Moreover noticing that

$$d\left(v\lrcorner h\right)=0 \quad \text{in } \Omega$$

then, again by classical methods (cf. Theorems 12.5 and 12.7 in [8]),

$$\left(S_t^v\right)^*\left(h\right)=h\,, \quad \text{for every } t.$$

This in turn implies that, for every t,

$$\left(S_t^v\right)^*\left(h^m\right)=h^m \quad \Leftrightarrow \quad \rho\left(S_t^v\right)\det\nabla S_t^v=\rho. \tag{33}$$

Let χ be a minimizer of (P). Since, for every t, $\left(\chi\circ S_t^v\right)^*\left(f\right)=h$, the functional $J:\mathbb{R}\to\mathbb{R}$ defined by

$$J\left(t\right)=\int_\Omega\left|\chi\circ S_t^v-\mathrm{id}\right|^2\rho$$

attains its minimum at $t=0$. Changing the variables and using (33) we get

$$J\left(t\right)=\int_\Omega\left|\chi-S_{-t}^v\right|^2\rho=\int_\Omega\left(|\chi|^2+|\mathrm{id}|^2\right)\rho-\int_\Omega 2\left\langle\chi;S_{-t}^v\right\rangle\rho.$$

Hence J is smooth and thus $J'\left(0\right)=0$ and $J''\left(0\right)\geq 0$. A direct computation leads to

$$J'\left(t\right)=2\int_\Omega\left\langle\chi;v\left(S_{-t}^v\right)\right\rangle\rho=2\int_\Omega\left\langle\chi\left(S_t^v\right);v\right\rangle\rho$$

and

$$J''\left(t\right)=2\int_\Omega\left\langle\nabla\chi\left(S_t^v\right)\cdot v\left(S_t^v\right);v\right\rangle\rho.$$

Therefore $J'\left(0\right)=0$ and $J''\left(0\right)\geq 0$ read as

$$\int_\Omega\left\langle\chi;v\right\rangle\rho=0 \quad \text{and} \quad \int_\Omega\left\langle\nabla\chi\cdot v;v\right\rangle\rho\geq 0.$$

Step 2. We now prove (30). From the definition of v, $J'\left(0\right)=0$ can be rewritten as

$$\int_\Omega\left\langle\chi;\nabla a\lrcorner h^{-1}\right\rangle\rho=0.$$

Since $\overline{h^{-1}}$ is skew-symmetric the above equation is equivalent to

$$\int_\Omega \left\langle \chi \lrcorner \rho h^{-1}; \nabla a \right\rangle = 0,$$

yielding, since $a \in C_0^\infty(\Omega)$ is arbitrary,

$$\delta\left(\chi \lrcorner \rho h^{-1}\right) = 0.$$

Using (6) and Proposition 1 we directly deduce (30) since $\rho > 0$ in $\overline{\Omega}$.

Step 3. We finally show (31). From $J''(0) \geq 0$ we know that, for every $a \in C_0^\infty(\Omega)$,

$$\int_\Omega \left\langle \nabla\chi \cdot \nabla a \lrcorner h^{-1}; \nabla a \lrcorner h^{-1} \right\rangle \rho \geq 0$$

and get at once (31) from Lemma 25. This ends the proof. □

In the proof of the previous proposition we have used the following lemma.

Lemma 25. *Let $n = 2m$, $\Omega \subset \mathbb{R}^n$ be a bounded open set, $\rho \in C^0\left(\overline{\Omega}\right)$ with $\rho > 0$, $A \in C^0\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$ and $\omega \in C^0\left(\overline{\Omega}; \Lambda^2\right)$ be such that*

$$\operatorname{rank}[\omega(x)] = n, \quad \text{for every } x \in \overline{\Omega}.$$

Assume that

$$\int_\Omega \left\langle A \cdot (\nabla a \lrcorner \omega^{-1}); \nabla a \lrcorner \omega^{-1} \right\rangle \rho \geq 0, \quad \text{for every } a \in C_0^\infty(\Omega).$$

Then

$$\langle A(x)\, \xi; \xi \rangle \geq 0, \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega}. \tag{34}$$

Proof. Let $B = (\overline{\omega})^{-1} \in C^0\left(\overline{\Omega}; \mathbb{R}^{n\times n}\right)$, where $\overline{\omega}$ is the skew symmetric matrix associated to ω. We have by assumption that

$$\int_\Omega \left\langle \left(B^t \cdot A \cdot B\right) \nabla a; \nabla a \right\rangle \rho \geq 0, \quad \text{for every } a \in C_0^\infty(\Omega). \tag{35}$$

Fix $a \in C_0^\infty(\Omega)$ and $x_0 \in \Omega$. Using the test functions (extending a by 0 outside Ω and taking ϵ small enough)

$$a_\epsilon(x) = \frac{1}{\epsilon^{m-1}}\, a\left(x_0 + (x - x_0)/\epsilon\right)$$

in (35) and letting ϵ tend to 0, a simple calculation gives

$$\int_\Omega \left\langle \left[\left(B^t \cdot A \cdot B\right)(x_0)\right] \cdot \nabla a(y); \nabla a(y) \right\rangle \rho(x_0)\, dy \geq 0.$$

Finally using test functions of the form

$$a_\nu(x) = g(x) \sin\left(\nu \langle \xi; x\rangle\right)$$

in the previous inequality where $\xi \in \mathbb{R}^n$ and $g \in C_0^\infty(\Omega)$ is not identically 0 are both fixed, a direct calculation gives that, for ν large enough,

$$\left\langle \left[\left(B^t \cdot A \cdot B\right)(x_0)\right] \xi; \xi \right\rangle \geq 0.$$

Since ξ is arbitrary and $B(x_0)$ is invertible, the last inequality implies directly that

$$\langle A(x_0)\, \xi; \xi \rangle \geq 0, \quad \text{for every } \xi \in \mathbb{R}^n$$

which implies (34) up to the boundary by continuity. □

Conflict of interest statement

There is no conflict of interest.

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Appendix A. On the definition of ellipticity

Although the definition of ellipticity that we use is standard (see [1,2,17,18]) and is, in some more geometrical context, called right-ellipticity, we think that, for the ease of the reader and for fixing the notations, it might be appropriate to recall it. We proceed in three steps, first discussing the linear case, then the quasilinear one and finally the fully nonlinear case.

A.1. The linear case

Let us first introduce some notations.

(i) Let n, N, m and M be integers and let $\Omega \subset \mathbb{R}^n$ be open and connected.

(ii) Let

$$u : \Omega \to \mathbb{R}^N, \quad u = u(x) = u(x_1, \cdots, x_n) = \left(u^1, \cdots, u^N\right)$$

and thus $\nabla^m u \in \mathbb{R}^{N \times n^m}$ (seen as a vector) i.e.

$$\nabla^m u = \left(u^j_{x_{i_1} \cdots x_{i_m}}\right)^{1 \le j \le N}_{1 \le i_1, \cdots, i_m \le n} \quad \text{where} \quad u^j_{x_{i_1} \cdots x_{i_m}} = \frac{\partial^m u^j}{\partial x_{i_1} \cdots \partial x_{i_m}}.$$

(iii) Let $A \in \mathbb{R}^{M \times (N \times n^m)}$ be a matrix, that can also depend on x, be such that

$$A = A(x) = \left(A^{j;k}_{i_1 \cdots i_m}(x)\right)^{1 \le j \le N; 1 \le k \le M}_{1 \le i_1, \cdots, i_m \le n}$$

with

$$A^{j;k}(x) \neq 0, \quad \text{for every } 1 \le j \le N, 1 \le k \le M \text{ and for every } x \in \Omega.$$

Consider the linear operator

$$\mathcal{A} : C^m\left(\Omega; \mathbb{R}^N\right) \to C^0\left(\Omega; \mathbb{R}^M\right)$$

be defined, for $x \in \Omega$, by

$$\mathcal{A}(u)(x) = \left(\sum_{i_1, \cdots, i_m = 1}^{n} \sum_{j=1}^{N} A^{j;1}_{i_1 \cdots i_m} u^j_{x_{i_1} \cdots x_{i_m}}, \cdots, \sum_{i_1, \cdots, i_m = 1}^{n} \sum_{j=1}^{N} A^{j;M}_{i_1 \cdots i_m} u^j_{x_{i_1} \cdots x_{i_m}}\right)$$

which we write, for $x \in \Omega$, as

$$\mathcal{A}(u)(x) = A(x) \cdot \nabla^m u(x)$$

where the right-hand side can be seen as a $\left[M \times (N \times n^m)\right]$ matrix A applied to the $\left[N \times n^m\right]$ vector $\nabla^m u$ leading to a M vector.

(iv) We denote by

$$\xi^{\otimes^m} = \underbrace{\xi \otimes \cdots \otimes \xi}_{m \text{ times}} = \left(\xi_{i_1} \cdots \xi_{i_m}\right)_{1 \leq i_1, \cdots, i_m \leq n},$$

where $\otimes$ stands for the tensor product of vectors in $\mathbb{R}^n$; for example

$$\xi \otimes \xi = \left(\xi_i \xi_j\right)_{1 \leq i,j \leq n} = \begin{pmatrix} (\xi_1)^2 & \xi_1\xi_2 & \cdots & \xi_1\xi_n \\ \xi_1\xi_2 & (\xi_2)^2 & \cdots & \xi_2\xi_n \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1\xi_n & \xi_2\xi_n & \cdots & (\xi_n)^2 \end{pmatrix}.$$

To the differential operator $\mathcal{A}$ we associate in a natural way a linear (algebraic) operator. Namely for $x \in \Omega$ and $\xi \in \mathbb{R}^n$ fixed, we define

$$\mathcal{A}_{x,\xi} : \mathbb{R}^N \to \mathbb{R}^M$$

through

$$\mathcal{A}_{x,\xi}(\lambda) = A(x) \cdot \left(\lambda \otimes \xi^{\otimes^m}\right).$$

Definition 26. We say that the system

$$\mathcal{A}(u)(x) = A(x) \cdot \nabla^m u(x) = f(x), \quad x \in \Omega$$

is *elliptic* (over Ω) if $\forall x \in \Omega$ and $\forall \xi \in \mathbb{R}^n \setminus \{0\}$, then $\lambda = 0 \in \mathbb{R}^N$ is the only solution of

$$\mathcal{A}_{x,\xi}(\lambda) = 0.$$

Remark 27. In other words, the definition states that, for every $x \in \Omega$ and $\xi \neq 0$, the operator $\mathcal{A}_{x,\xi}$ is one to one. This definition is, sometimes, called "*right-ellipticity*". Classically one also specifies the range of the operator, which is usually the whole of $\mathbb{R}^M$, as in Example 28 (i) and (ii); however this is not always the case as seen in Example 28 (iii).

– Let us first examine this particular example. Indeed in this last case the range of the operator cannot be the whole of $\mathbb{R}^M \simeq \Lambda^{k+1} \times \Lambda^{k-1}$. Since $dd = 0$ and $\delta\delta = 0$, it is necessary that

$$df^1 = 0 \quad \text{and} \quad \delta f^2 = 0.$$

The range of the operator is then

$$X = \left\{\mu = \left(\mu^1, \mu^2\right) \in \Lambda^{k+1} \times \Lambda^{k-1} : \xi \wedge \mu^1 = 0 \text{ and } \xi \lrcorner \mu^2 = 0\right\}.$$

– More generally when there are natural conditions that restricts the space, this should be taken into account. Indeed if we consider the system

$$A \cdot \nabla^m u = f$$

where the data satisfy the constraint

$$B \cdot \nabla^l f = 0$$

where $B \in \mathbb{R}^{L \times (M \times n^l)}$, the range of the operator is then

$$X = \left\{\mu \in \mathbb{R}^M : B \cdot \left(\mu \otimes \xi^{\otimes^l}\right) = 0\right\}.$$

Therefore we require that $\forall \xi \in \mathbb{R}^n \setminus \{0\}$ and $\forall \mu \in X$, there exists a unique $\lambda \in \mathbb{R}^N$ such that

$$A \cdot \left(\lambda \otimes \xi^{\otimes^m}\right) = \mu.$$

But in order not to burden too much the technicalities, we have adopted the less restrictive definition of ellipticity where we do not specify the range.

Here are some examples showing that the definition corresponds to the classical ones. We start with a single equation.

Example 28. (i) The equation (here $N = M = 1$ and $m = 2$)

$$\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} = f$$

is elliptic if, for every $\xi \neq 0$,

$$\sum_{i,j=1}^{n} a_{ij}\, \xi_i \xi_j \neq 0.$$

(ii) Let (here $N = M$ and $m = 2$)

$$\sum_{i=1}^{N} \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}^{ij}\, u_{x_\alpha x_\beta}^{i} = f^{j}, \quad j = 1, \cdots, N$$

our definition of ellipticity gives (which is essentially the Legendre–Hadamard condition)

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}^{ij}\, \eta^i \eta^j \xi_\alpha \xi_\beta \neq 0, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \ \forall \eta \in \mathbb{R}^N \setminus \{0\}.$$

(iii) Let u be a k-form, d denotes the exterior derivative and δ the co-differential (here $N = \binom{n}{k}$, $M = \binom{n}{k+1} + \binom{n}{k-1}$ and $m = 1$). Consider the system

$$du = f^1 \quad \text{and} \quad \delta u = f^2$$

which is indeed easily seen to be elliptic.

A.2. The nonlinear case

We now turn to quasilinear systems. Besides the above notations we adopt the following ones.

(i) Let $\mathcal{S} \subset C^m\left(\Omega; \mathbb{R}^N\right)$, for instance

$$\mathcal{S} = \left\{ u \in C^1\left(\Omega; u\left(\Omega\right)\right) : \langle \nabla u\left(x\right) \xi; \xi \rangle \neq 0, \ \forall \xi \in \mathbb{R}^n \setminus \{0\}, \ \forall x \in \Omega \right\}$$

as in Proposition 14 (when $A = B$ are constant matrices) or, as in Example 31 (i),

$$\mathcal{S} = \left\{ u \in C^2\left(\Omega\right) : \left\langle \nabla^2 u\left(x\right) \xi; \xi \right\rangle \neq 0, \ \forall \xi \in \mathbb{R}^n \setminus \{0\}, \ \forall x \in \Omega \right\}.$$

(ii) For $u \in C^{m-1}\left(\Omega; \mathbb{R}^N\right)$, we let

$$U\left(x\right) = \left(x, u\left(x\right), \nabla u\left(x\right), \cdots, \nabla^{m-1} u\left(x\right)\right) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{N \times n} \times \cdots \times \mathbb{R}^{N \times n^{m-1}}.$$

(iii) Let $A \in \mathbb{R}^{M \times (N \times n)}$ be such that

$$A = A\left(U\left(x\right)\right) = \left(A_{i_1 \cdots i_m}^{j;k}\left(U\left(x\right)\right)\right)_{1 \leq i_1, \cdots, i_m \leq n}^{1 \leq j \leq N; 1 \leq k \leq M}$$

with, for every $x \in \Omega$ and $u \in \mathcal{S}$,

$$A^{j;k}\left(U\left(x\right)\right) \neq 0, \quad \text{for every } 1 \leq j \leq N, \ 1 \leq k \leq M.$$

We then let

$$\mathcal{A}\left(u\right)\left(x\right) = A\left(U\left(x\right)\right) \cdot \nabla^m u\left(x\right).$$

Note that

$$\mathcal{A}(u)(x) \in \mathbb{R}^M.$$

(iv) We let $f \in \mathbb{R}^M$ that may also depend on $U(x)$ and

$$f[u](x) = f(U(x)) \in \mathbb{R}^M.$$

(v) To the differential operator $\mathcal{A}$ we associate, as above, a linear (algebraic) operator. Namely for $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $u \in \mathcal{S}$ fixed, we define

$$\mathcal{A}_{x,\xi,u} : \mathbb{R}^N \to \mathbb{R}^M$$

through

$$\mathcal{A}_{x,\xi,u}(\lambda) = A(U(x)) \cdot \left(\lambda \otimes \xi^{\otimes^m}\right).$$

Definition 29. The system

$$\mathcal{A}(u)(x) = f[u](x), \quad x \in \Omega$$

is said to be *elliptic* (over Ω and over $\mathcal{S}$) if $\forall\, x \in \Omega$, $\forall\, \xi \in \mathbb{R}^n \setminus \{0\}$ and $\forall\, u \in \mathcal{S}$, then $\lambda = 0 \in \mathbb{R}^N$ is the only solution of

$$\mathcal{A}_{x,\xi,u}(\lambda) = 0.$$

In the fully nonlinear case we proceed in the classical way. We reduce the system to a quasilinear one by differentiating with respect to all variables according to the definition below. Here we add the following notations.

(i) Let

$$F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \times \cdots \times \mathbb{R}^{N\times n^{m-1}} \times \mathbb{R}^{N\times n^m} \to \mathbb{R}^M$$

with

$$F = \left(F^1(U,z), \cdots, F^M(U,z)\right)$$

where $U \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \times \cdots \times \mathbb{R}^{N\times n^{m-1}}$ and $z = \left(z^j_{i_1\cdots i_m}\right) \in \mathbb{R}^{N\times n^m}$.

(ii) Let, for $k = 1, \cdots, M$ and $\alpha = 1, \cdots, n$,

$$\widetilde{F}(x,z) = F(U(x), z) \quad \text{and} \quad \widetilde{F}^k_{x_\alpha} = \frac{\partial \widetilde{F}^k}{\partial x_\alpha}.$$

(iii) We consider the system

$$F[u](x) = F\left(U(x), \nabla^m u(x)\right) = 0, \quad x \in \Omega.$$

Differentiating the system with respect to x_α (in case one of the equations is already in a quasilinear form, there is no need to differentiate it, since the algebraic system is the same before or after differentiation), we get

$$\sum_{i_1,\cdots,i_m=1}^{n} \sum_{j=1}^{N} \left[F^k_{z^j_{i_1\cdots i_m}}\left(U(x), \nabla^m u(x)\right) \frac{\partial^{m+1} u^j}{\partial x_\alpha \partial x_{i_1} \cdots \partial x_{i_m}} \right] = -\widetilde{F}^k_{x_\alpha}\left(x, \nabla^m u(x)\right)$$

for every $k = 1, \cdots, M$ and $\alpha = 1, \cdots, n$. It is therefore a quasilinear problem of the type considered in Definition 29. Hence the algebraic problem for which we have to prove that $\lambda = 0$ is the only solution is

$$\sum_{i_1,\cdots,i_m=1}^{n} \sum_{j=1}^{N} \left[F^k_{z^j_{i_1\cdots i_m}}\left(U(x), \nabla^m u(x)\right) \lambda^j \xi_\alpha \xi_{i_1} \cdots \xi_{i_m} \right] = 0.$$

Since $\xi \neq 0$, we deduce that the algebraic problem has been reduced to

$$\sum_{i_1,\cdots,i_m=1}^{n}\sum_{j=1}^{N}\left[F^k_{z^j_{i_1\cdots i_m}}\left(U(x),\nabla^m u(x)\right)\lambda^j\,\xi_{i_1}\cdots\xi_{i_m}\right]=0$$

for every $1\leq k\leq M$.

(iv) We then set $A=A\left(U(x),\nabla^m u(x)\right)$ be defined by

$$A=\left(A^{j;k}_{i_1\cdots i_m}=F^k_{z^j_{i_1\cdots i_m}}\right)^{1\leq j\leq N;1\leq k\leq M}_{1\leq i_1,\cdots,i_m\leq n}\in\mathbb{R}^{M\times(N\times n^m)}$$

and we make the hypothesis that

$$A^{j;k}=F^k_{z^j}\neq 0,\quad \text{for every } 1\leq j\leq N \text{ and } 1\leq k\leq M.$$

(v) We finally define, for $x\in\Omega$, $\xi\in\mathbb{R}^n$ and $u\in\mathcal{S}$ fixed,

$$\mathcal{A}_{x,\xi,u}:\mathbb{R}^N\to\mathbb{R}^M$$

through

$$\mathcal{A}_{x,\xi,u}(\lambda)=A\cdot\left(\lambda\otimes\xi^{\otimes^m}\right).$$

Definition 30. The system

$$F[u](x)=0,\quad x\in\Omega$$

is said to be *elliptic* (over Ω and over $\mathcal{S}$) if $\forall\, x\in\Omega$, $\forall\,\xi\in\mathbb{R}^n\setminus\{0\}$ and $\forall\, u\in\mathcal{S}$, then $\lambda=0\in\mathbb{R}^N$ is the only solution of

$$\mathcal{A}_{x,\xi,u}(\lambda)=0.$$

We now give two examples, starting with the Monge–Ampère equation in dimension 2.

Example 31. *(i)* Consider (here $N=M=1$ and $n=m=2$)

$$u_{x_1x_1}u_{x_2x_2}-\left(u_{x_1x_2}\right)^2=f.$$

We find that the algebraic system is reduced to

$$\lambda\left(u_{x_2x_2}\xi_1^2-2u_{x_1x_2}\xi_1\xi_2+u_{x_1x_1}\xi_2^2\right)=0$$

which has $\lambda=0$ as the unique solution, for every $\xi\neq 0$, if and only if

$$u_{x_2x_2}\xi_1^2-2u_{x_1x_2}\xi_1\xi_2+u_{x_1x_1}\xi_2^2\neq 0\quad\Leftrightarrow\quad\left\langle\nabla^2u(x)\,\xi;\xi\right\rangle\neq 0. \tag{36}$$

Therefore if

$$\mathcal{S}=\left\{u\in C^2(\Omega):\left\langle\nabla^2u(x)\,\xi;\xi\right\rangle\neq 0,\ \forall\,\xi\in\mathbb{R}^n\setminus\{0\},\ \forall\, x\in\Omega\right\}$$

we see that the equation is elliptic (over Ω and over $\mathcal{S}$). Note that on $\mathcal{S}$ the functions are either strictly convex or strictly concave.

(ii) It is interesting to make again the above computation but considering the problem as a first order system, namely

$$\det\nabla v=v^1_{x_1}v^2_{x_2}-v^1_{x_2}v^2_{x_1}=f\quad\text{and}\quad\operatorname{curl} v=v^1_{x_2}-v^2_{x_1}=0.$$

After differentiation we get (here $N=n=M=m=2$) that the algebraic problem is then

$$\begin{cases}\xi_1\left[\lambda^1\left(\xi_1v^2_{x_2}-\xi_2v^2_{x_1}\right)+\lambda^2\left(\xi_2v^1_{x_1}-\xi_1v^1_{x_2}\right)\right]=0\\ \xi_2\left[\lambda^1\left(\xi_1v^2_{x_2}-\xi_2v^2_{x_1}\right)+\lambda^2\left(\xi_2v^1_{x_1}-\xi_1v^1_{x_2}\right)\right]=0\\ \quad\xi_1\left(\lambda^1\xi_2-\lambda^2\xi_1\right)=\xi_2\left(\lambda^1\xi_2-\lambda^2\xi_1\right)=0.\end{cases}$$

We therefore obtain that

$$\lambda^1 \left[\xi_1^2 v_{x_2}^2 - \xi_1\xi_2 \left(v_{x_1}^2 + v_{x_2}^1\right) + \xi_2^2 v_{x_1}^1\right] = 0 \quad \text{and} \quad \lambda^2 \left[\xi_1^2 v_{x_2}^2 - \xi_1\xi_2 \left(v_{x_1}^2 + v_{x_2}^1\right) + \xi_2^2 v_{x_1}^1\right] = 0.$$

The system thus has $\lambda = 0$ as a unique solution, for every $\xi \neq 0$, if and only if

$$\xi_1^2 v_{x_2}^2 - \xi_1\xi_2 \left(v_{x_1}^2 + v_{x_2}^1\right) + \xi_2^2 v_{x_1}^1 \neq 0 \quad \Leftrightarrow \quad \langle \nabla v(x)\xi;\xi\rangle \neq 0.$$

Therefore if

$$\mathcal{S} = \left\{v \in C^1\left(\Omega;\mathbb{R}^2\right) : \langle \nabla v(x)\xi;\xi\rangle \neq 0,\ \forall \xi \in \mathbb{R}^n \setminus \{0\},\ \forall x \in \Omega\right\}$$

we see that the system is elliptic (over Ω and over $\mathcal{S}$). Note that on $\mathcal{S}$ the symmetric part of the gradient of the map is definite (either positive or negative).

We now turn to the most general nonlinear equation.

Example 32. Consider a single equation (here $N = M = 1$)

$$F[u](x) = F\left(U(x), \nabla^m u(x)\right) = 0 \quad \text{in } \Omega$$

where

$$U(x) = \left(x, u(x), \nabla u(x), \cdots, \nabla^{m-1} u(x)\right) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \cdots \times \mathbb{R}^{n^{m-1}}.$$

The condition $\mathcal{A}_{x,\xi,u}(\lambda) = 0$ becomes now

$$\lambda \sum_{i_1,\cdots,i_m=1}^{n} F_{z_{i_1\cdots i_m}} \xi_{i_1}\cdots\xi_{i_m} = 0 \quad \text{where} \quad F_{z_{i_1\cdots i_m}} = F_{z_{i_1\cdots i_m}}\left(U(x), \nabla^m u(x)\right).$$

It has $\lambda = 0$, for $\xi \neq 0$, as the only solution if and only if

$$\sum_{i_1,\cdots,i_m=1}^{n} F_{z_{i_1\cdots i_m}} \xi_{i_1}\cdots\xi_{i_m} \neq 0, \quad \forall x \in \Omega \text{ and } \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{37}$$

Therefore if

$$\mathcal{S} = \left\{u \in C^m(\Omega) \text{ verifying } (37)\right\}$$

we see that the equation is elliptic (over Ω and over $\mathcal{S}$). In the special case $m = 2$ (which is the one of Monge–Ampère equation, in this case $F(z) = \det z$) we get that (37) reads as

$$\langle F_z[u](x)\cdot\xi;\xi\rangle = \sum_{i,j=1}^{n} F_{z_{ij}}\left(x, u(x), \nabla u(x), \nabla^2 u(x)\right)\xi_i\xi_j \neq 0, \tag{38}$$

$\forall x \in \Omega$ and $\forall \xi \in \mathbb{R}^n \setminus \{0\}$. This is indeed the classical definition of ellipticity for fully nonlinear equations of the second order. When we consider Monge–Ampère equation in dimension 2, we see that (38) is nothing else than (36).

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