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A sharp Cauchy theory for the 2D gravity-capillary waves

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Abstract

This article is devoted to the Cauchy problem for the 2D gravity-capillary water waves in fluid domains with general bottoms. Local well-posedness for this problem with Lipschitz initial velocity was established by Alazard–Burq–Zuily [1]. We prove that the Cauchy problem in Sobolev spaces is uniquely solvable for initial data $\frac{1}{4}$-derivative less regular than the aforementioned threshold, which corresponds to the gain of Hölder regularity of the semi-classical Strichartz estimate for the fully nonlinear system. In order to obtain this Cauchy theory, we establish global, quantitative results for the paracomposition theory of Alinhac [5].

Keywords: Water waves; Cauchy problem; Semi-classical Strichartz estimate; Paracomposition

1. Introduction

1.1. The equations

We consider an incompressible, inviscid fluid with unit density moving in a time-dependent domain

$$\Omega = \{(t, x, y) \in [0, T] \times \mathbf{R} \times \mathbf{R} : (x, y) \in \Omega_t\}$$

where each Ω_t is a domain located underneath a free surface

$$\Sigma_t = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \eta(t, x)\}$$

and above a fixed bottom $\Gamma = \partial\Omega_t \setminus \Sigma_t$. We make the following assumption on the domain:
Ω_t is the intersection of the half space

$$\Omega_{1,t} = \{(x, y) \in \mathbf{R} \times \mathbf{R} : y = \eta(t, x)\}$$

and an open connected set Ω_2 containing a fixed strip around Σ_t, i.e., there exists $h > 0$ such that

$$\{(x, y) \in \mathbf{R} \times \mathbf{R} : \eta(x) - h \leq y \leq \eta(t, x)\} \subset \Omega_2.$$

This assumption prevents the bottom from emerging or even from coming arbitrarily close to the free surface and thus avoids the emergence of contact lines, which are not the subject of this paper. In what follows, we shall prove that this assumption can be propagated in short time if it is satisfied initially.

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The velocity field v admits a harmonic potential $\phi : \Omega \to \mathbf{R}$, i.e., $v = \nabla_{x,y}\phi$ with $\Delta_{x,y}\phi = 0$. For the Zakharov formulation of irrotational water waves, we introduce the trace of ϕ on the free surface

$$\psi(t,x) = \phi(t,x,\eta(t,x)).$$

Then $\phi(t,x,y)$ is the unique variational solution of

$$\Delta_{x,y}\phi(t) = 0 \text{ in } \Omega_t, \quad \phi(t)|_\Sigma = \psi(t), \quad \partial_\nu \phi(t)|_\Gamma = 0, \tag{1.1}$$

ν being the outward normal to the bottom Γ. The Dirichlet–Neumann operator is then defined by

$$G(\eta)\psi = \sqrt{1+|\partial_x\eta|^2}\Big(\frac{\partial\phi}{\partial n}\Big|_\Sigma\Big) = (\partial_y\phi)(t,x,\eta(t,x)) - \partial_x\eta(t,x)(\partial_x\phi)(t,x,\eta(t,x)).$$

The gravity water wave problem with surface tension consists in solving the following system of (η, ψ) (see [14] or Chapter 9 of [20]):

$$\begin{cases} \partial_t\eta = G(\eta)\psi, \\ \partial_t\psi + g\eta + H(\eta) + \frac{1}{2}|\partial_x\psi|^2 - \frac{1}{2}\frac{(\partial_x\eta\partial_x\psi + G(\eta)\psi)^2}{1+|\partial_x\eta|^2} = 0 \end{cases} \tag{1.2}$$

where $H(\eta)$ is twice the mean curvature of the free surface:

$$H(\eta) = -\partial_x\left(\frac{\partial_x\eta}{\sqrt{1+|\partial_x\eta|^2}}\right).$$

Finally, we note that the vertical and horizontal components of the velocity can be expressed in terms of η and ψ as

$$B = (v_y)|_\Sigma = \frac{\partial_x\eta\partial_x\psi + G(\eta)\psi}{1+|\partial_x\eta|^2}, \quad V = (v_x)|_\Sigma = \partial_x\psi - B\partial_x\eta. \tag{1.3}$$

1.2. The problem

We are interested in the Cauchy problem for system (1.2) with *sharp Sobolev regularity for initial data*. For previous results on the Cauchy problem, we refer to the works of Yosihara [33], Coutand–Shkoller [12], Shatah–Zeng [26–28], Ming–Zhang [24] for sufficiently smooth solutions; see also the works of Nalimov [25], Craig [13], Wu [31] [32], Christodoulou–Lindblad [11], Lindblad [22], Lannes [19] for gravity waves without surface tension. In terms of regularity of initial data, the work of Alazard–Burq–Zuily [1] reached an important threshold: local wellposedness as long as the velocity field is Lipschitz, in terms of Sobolev embeddings, up to the free surface. More precisely, this corresponds to initial data (in view of the formula (1.3))

$$(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d), \quad s > 2 + \frac{d}{2}.$$

This is achieved by the energy method after reducing the system to a single quasilinear equation using a paradifferential calculus approach. However, observe that the linearization of (1.2) around the rest state $(0,0)$ reads

$$\partial_t\Phi + i\,|D|^{\frac{3}{2}}\,\Phi = 0, \quad \Phi = |D|^{\frac{1}{2}}\,\eta + i\psi$$

which is dispersive and enjoys the following Strichartz estimate with a gain of $\frac{3}{8}$ derivative

$$\|\Phi\|_{L^4_t W_x^{\sigma-\frac{1}{8},\infty}} \le C_\sigma\, \|\Phi|_{t=0}\|_{H^\sigma_x}, \quad \forall\sigma \in \mathbf{R}. \tag{1.4}$$

Therefore, one expects that the fully nonlinear system (1.2) is also dispersive and enjoys similar Strichartz estimates. Indeed, this is true and was first proved by Alazard–Burq–Zuily [2]: all solutions

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})), \quad s > 2 + \frac{1}{2} \tag{1.5}$$

possess the hidden regularity

$$(\eta, \psi) \in L^4([0,T]; W^{s+\frac{1}{4},\infty}(\mathbf{R}) \times W^{s-\frac{1}{4},\infty}(\mathbf{R})). \tag{1.6}$$

Compared to the classical (full) Strichartz estimate (1.4), the estimate (1.6) exhibits a loss of $\frac{1}{8}$ derivative and is called the *semi-classical Strichartz estimate*. This terminology comes from the work [9] for Schrodinger equations on manifolds. In fact, slightly earlier in [10] the same Strichartz estimate was obtained for the 2D gravity-capillary water waves under another formulation. We also refer to [18] for another proof of (1.6) and the semi-classical Strichartz estimate for 3D waves.

It is known, for instance from the works of Bahouri–Chemin [6] and Tataru [29] [30], that for dispersive PDEs, Strichartz estimates can be used to improve the Cauchy theory for data that are less regular than the one obtained merely via the energy method. We refer to [7], Chapter 9 for an expository presentation concerning quasilinear wave equations. Our aim is to proceed such a program for the gravity-capillary water waves system (1.2). For pure gravity water waves, this was considered by Alazard–Burq–Zuily [4]. Coming back to our system (1.2), from the semi-classical Strichartz estimate (1.6) for $s > 2 + \frac{1}{2}$ it is natural to ask:

Q: *Does the Cauchy problem for* (1.2) *have a unique solution for initial data*

$$(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d), \quad s > 2 + \frac{1}{2} - \frac{1}{4} = \frac{9}{4}?$$

In the previous joined work [16], we proved an "intermediate" result for $s > 2 + 1/2 - 3/20$ in 2D case (together with a similar result for 3D case), which asserts that water waves can still propagate starting from *non-Lipschitz velocity* (up to the free surface). See [4] for the corresponding result for vanishing surface tension. Our contribution in this work is to prove an affirmative answer for question **Q**.

Let us give an outline of the proof. In [15], using a paradifferential approach we reduced the system (1.2) to a single dispersive equation as follows. Assume that for some $s > r > 2$

$$(\eta, \psi) \in C^0([0,T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})) \cap L^4([0,T]; W^{r+\frac{1}{2},\infty}(\mathbf{R}) \times W^{r,\infty}(\mathbf{R})) \tag{1.7}$$

then after paralinearization and symmetrization, (1.2) reduces to the following equation of a complexed-valued unknown Φ

$$\partial_t \Phi + T_V \partial_x \Phi + i T_\gamma \Phi = f \tag{1.8}$$

for some paradifferential symbol $\gamma \in \Sigma^{3/2}$ and the source term $f(t)$ satisfies the *tame estimate*

$$\|f(t)\|_{H^s} \le \mathcal{F}\left(\left\|(\eta(t), \psi(t))\right\|_{H^{s+\frac{1}{2}} \times H^s}\right)\left(1 + \left\|(\eta(t), \psi(t))\right\|_{W^{r+\frac{1}{2},\infty} \times W^{r,\infty}}\right),$$

here $\mathcal{F}$ is some universal nonlinear function.

Such a reduction was first obtained in [1] for solutions at the "energy threshold" (1.5). Observe that the relation $s > r > 2$ exhibits a gap of $\frac{1}{2}$ derivative in view of the Sobolev embedding from H^s to $C_*^{s-\frac{1}{2}}$ (see Definition A.1). Having in hand the blow-up criterion and the contraction estimate in [15] at the regularity (1.7), the main difficulty in answering question **Q** is to prove the semi-classical Strichartz estimate for solution Φ to (1.8). Compared to the Strichartz estimates in [16] we remark that the semi-classical gain in [2] (when $s > 2 + \frac{1}{2}$) was achieved owing to the fact that in one spatial dimension, (1.8) can be further reduced to an equation whose highest order term is just the Fourier multiplier $|D_x|^{\frac{3}{2}}$:

$$\partial_t \widetilde{\Phi} + T_{\widetilde{V}} \partial_x \widetilde{\Phi} + i|D_x|^{\frac{3}{2}} \widetilde{\Phi} = \widetilde{f}. \tag{1.9}$$

This reduction is proceeded by means of the *paracomposition* of Alinhac [5]. Here, we shall see that in our case we need more precise paracomposition results for two purposes: (1) dealing with rougher functions and (2) deriving quantitative estimates. This will be the content of section 3 and can be of independent interest. After having (1.9) we show in section 4 that the method in [2] can be adapted to our lower regularity level to derive the semi-classical Strichartz estimate with an arbitrarily small ε loss of regularity.

1.3. Main results

Let us introduce the Sobolev norm and the Strichartz norm for solution (η, ψ) to the gravity-capillary system (1.2):

$$M_\sigma(T)=\|(\eta,\psi)\|_{L^\infty([0,T];H^{\sigma+\frac12}(\mathbf{R})\times H^\sigma(\mathbf{R}))},\qquad M_\sigma(0)=\|(\eta,\psi)|_{t=0}\|_{H^{\sigma+\frac12}(\mathbf{R})\times H^\sigma(\mathbf{R})},$$
$$N_\sigma(T)=\|(\eta,\psi)\|_{L^4([0,T];W^{\sigma+\frac12,\infty}(\mathbf{R})\times W^{\sigma,\infty}(\mathbf{R}))}.$$

Our first result concerns the semi-classical Strichartz estimate for system (1.2).

Theorem 1.1. *Assume that* (η,ψ) *is a solution to* (1.2) *with*

$$\begin{cases}(\eta,\psi)\in C^0([0,T];H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R}))\cap L^4([0,T];W^{r+\frac12,\infty}(\mathbf{R})\times W^{r,\infty}(\mathbf{R})),\\ s>r>\dfrac32+\dfrac12\end{cases}\tag{1.10}$$

and

$$\inf_{t\in[0,T]}\operatorname{dist}(\eta(t),\Gamma)\geq h>0.\tag{1.11}$$

For any $\mu<\frac14$ *there exists a nondecreasing function* $\mathcal{F}$ *independent of* (η,ψ) *such that*

$$N_{s-\frac12+\mu}(T)\leq\mathcal{F}\big(M_s(T)+N_r(T)\big).\tag{1.12}$$

As a consequence of Theorem 1.1 and the energy estimate in [15] we obtain a closed *a priori* estimate for the mixed norm $M_s(T)+N_r(T)$.

Theorem 1.2. *Assume that* (η,ψ) *is a solution to* (1.2) *and satisfies conditions* (1.10), (1.11) *with*

$$2<r<s-\frac12+\mu,\quad \mu<\frac14,\quad h>0.$$

There exists a nondecreasing function $\mathcal{F}$ *independent of* (η,ψ) *such that*

$$M_s(T)+N_r(T)\leq\mathcal{F}\Big(\mathcal{F}(M_s(0))+T\mathcal{F}\big(M_s(T)+N_r(T)\big)\Big).$$

Finally, we obtain a Cauchy theory for the gravity-capillary system (1.2) with initial data $\frac14$-derivative less regular than the energy threshold in [1].

Theorem 1.3. *Let* $\mu<\frac14$ *and* $2<r<s-\frac12+\mu$. *For any* $(\eta_0,\psi_0)\in H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R})$ *satisfying* $\operatorname{dist}(\eta_0,\Gamma)\geq h>0$, *there exists* $T>0$ *such that the gravity-capillary waves system* (1.2) *has a unique solution* (η,ψ) *in*

$$L^\infty([0,T];H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R}))\cap L^4([0,T];W^{r+\frac12,\infty}(\mathbf{R})\times W^{r,\infty}(\mathbf{R})).$$

Moreover, we have

$$(\eta,\psi)\in C^0([0,T];H^{s_0+\frac12}(\mathbf{R})\times H^{s_0}(\mathbf{R}))\quad\forall s_0<s,$$

and

$$\inf_{t\in[0,T]}\operatorname{dist}(\eta(t),\Gamma)>\frac{h}{2}.$$

Remark 1.4. The proof of Theorem 1.3 shows that for each $\mu<\frac14$ the existence time T can be chosen uniformly for data (η_0,ψ_0) lying in a bounded set of $H^{s+\frac12}(\mathbf{R})\times H^s(\mathbf{R})$ and the "fluid depth" h lying in a bounded set of $(0,+\infty)$.

Remark 1.5. We do not know yet if the semi-classical gain is optimal for solutions at the regularity (1.10). However, some remarks can be made as follows. On one hand, regarding Strichartz estimates for (1.8), since the symbol γ is x-dependent, trappings and thus loss of derivative may occur. As an example, the semi-classical Strichartz estimates are optimal for Schrödinger equations posed on spheres (see section 4, [9]). On the other hand, if one wishes to eliminate the geometry by making changes of variables, then as we shall see in Proposition 4.3 and Remark 4.4, there will appear a loss of $\frac{1}{2}$ derivative in the source term, which turns out to be the least allowable loss for the semi-classical Strichartz estimate (see the end of the proof of Theorem 4.14).

Remark 1.6. The linearization of (1.2) in 2 spatial dimensions ($\eta, \psi : \mathbf{R}^2 \to \mathbf{R}$) enjoys the semi-classical Strichartz estimate with a gain $\frac{1}{2}$ derivative (see [18]). It was proved in [18] that the same estimate holds for the nonlinear system (1.2) when

$$(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^2) \times H^s(\mathbf{R}^2)),\ s > \frac{5}{2} + 1.$$

If the preceding regularity could be improved to ($\frac{1}{2}$ derivative)

$$\begin{cases} (\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}} \times H^s) \cap L^2([0, T]; W^{r+\frac{1}{2},\infty} \times W^{r,\infty}), \\ s - \dfrac{1}{2} > r > 2, \end{cases}$$

the results in [15] would imply a Cauchy theory (see the proof of Theorem 1.3) with initial surface

$$\eta_0 \in H^{s+\frac{1}{2}}(\mathbf{R}^2),\ s > \frac{3}{2} + 1,$$

which is the lowest Sobolev regularity to ensure that the initial surface has *bounded curvature* (see the Introduction of [16]). Note that according to [4], in the absence of surface tension, the problem is well-posed even if the initial curvature is unbounded (in all spatial dimensions) or even not in L^2 (in 2 spatial dimensions).

2. Preliminaries on dyadic analysis

2.1. Dyadic partitions

Our analysis below is sensitive with respect to the underlying dyadic partition of $\mathbf{R}^d$. These partitions are constructed by using the cut-off functions given in the following lemma.

Lemma 2.1. *For every $n \in \mathbf{N}$, there exists $\phi_{(n)} \in C^\infty(\mathbf{R}^d)$ satisfying*

$$\phi_{(n)}(\xi) = \begin{cases} 1, & \text{if } |\xi| \le 2^{-n}, \\ 0, & \text{if } |\xi| > 2^{n+1}, \end{cases} \tag{2.1}$$

$$\forall(\alpha, \beta) \in \mathbf{N}^d \times \mathbf{N}^d, \exists C_{\alpha,\beta} > 0, \forall n \in \mathbf{N}, \left\| x^\beta \partial^\alpha \phi_{(n)}(x) \right\|_{L^1(\mathbf{R}^d)} \le C_{\alpha,\beta}. \tag{2.2}$$

We postpone the proofs of the results in this paragraph to Appendix B. In fact, to guarantee condition (2.2) we choose $\phi_{(n)}$ with support in a ball of size $2^{-n} + c$ for some $c > 0$.

We shall skip the subscript (n) and denote $\phi \equiv \phi_{(n)}$ for simplicity. Setting

$$\phi_k(\cdot) = \phi(\frac{\cdot}{2^k}),\ k \in \mathbf{Z}, \quad \varphi_0 = \phi = \phi_0,\ \varphi = \chi - \chi_{-1},\ \varphi_k = \phi_k - \phi_{k-1} = \varphi(\frac{\cdot}{2^k}),\ k \ge 1,$$

we see that

$$\begin{aligned} &\operatorname{supp} \varphi_0 \subset \mathcal{C}_0(n) := \{\xi \in \mathbf{R}^d : |\xi| \le 2^{n+1}\} \\ &\operatorname{supp} \varphi \subset \mathcal{C}(n) := \{\xi \in \mathbf{R}^d : 2^{-(n+1)} < |\xi| \le 2^{n+1}\} \\ &\operatorname{supp} \varphi_k \subset \mathcal{C}_k(n) := \{\xi \in \mathbf{R}^d : 2^{k-(n+1)} < |\xi| \le 2^{k+(n+1)}\},\ \forall k \ge 1. \end{aligned} \tag{2.3}$$

Observe also that with

$$N_0 := 2(n + 1)$$

we have

$$\mathcal{C}_j(n) \cap \mathcal{C}_k(n) = \emptyset \quad \text{if } |j - k| \ge N_0.$$

Definition 2.2. For every $\phi \equiv \phi_{(n)}$, define the following Fourier multipliers

$$\widehat{S_k u}(\xi) = \phi_k(\xi)\hat{u}(\xi),\ k \in \mathbf{Z}, \quad \widehat{\Delta_k u}(\xi) = \varphi_k(\xi)\hat{u}(\xi),\ k \geq 0.$$

Denoting $u_k = \Delta_k u$ we obtain a dyadic partition of unity

$$u = \sum_{p=0}^{\infty} u_p, \tag{2.4}$$

where n shall be called the size of this partition. Remark that with the notations above, there holds

$$\Delta_0 = S_0, \quad \sum_{p=0}^{q} \Delta_p = S_q, \quad S_{q+1} - S_q = \Delta_{q+1}.$$

Throughout this article, whenever $\mathbf{R}^d$ is equipped with a fixed dyadic partition, we always define the Zygmund-norm (see Definition A.1) of distributions on $\mathbf{R}^d$ by means of this partition.

To prove our paracomposition results we need to choose a particular size $n = n_0$, tailored to the diffeomorphism, in Proposition 2.9 below, whose proof requires uniform bounds for the norms of the operators $S_j,\ \Delta_j$ in Lebesgue spaces and Hölder spaces, with respect to the size n. This fact in turn stems from property (2.2) of $\phi_{(n)}$.

Lemma 2.3. *1. For every $\alpha \in \mathbf{N}^d$, there exists $C_\alpha > 0$ independent of n such that*

$$\forall j,\ \forall 1 \leq p \leq q \leq \infty, \quad \left\|\partial^\alpha S_j u\right\|_{L^q(\mathbf{R}^d)} + \left\|\partial^\alpha \Delta_j u\right\|_{L^q(\mathbf{R}^d)} \leq C_\alpha 2^{j(|\alpha| + \frac{d}{p} - \frac{d}{q})} \|u\|_{L^p(\mathbf{R}^d)}.$$

2. For every $\mu \in (0, \infty)$, there exists $M > 0$ independent of n such that

$$\forall j \in \mathbf{N},\ \forall u \in W^{\mu,\infty}(\mathbf{R}^d): \ \left\|\Delta_j u\right\|_{L^\infty(\mathbf{R}^d)} \leq M 2^{-j\mu} \|u\|_{W^{\mu,\infty}(\mathbf{R}^d)}.$$

As a consequence of this lemma, one can examine the proof of Proposition 4.1.16, [23] to have

Lemma 2.4. *Let $\mu > 0,\ \mu \notin \mathbf{N}$. There exists a constant C_μ independent of n, such that for any $u \in W^{\mu,\infty}(\mathbf{R}^d)$ we have*

$$\frac{1}{C_\mu} \|u\|_{W^{\mu,\infty}(\mathbf{R}^d)} \leq \|u\|_{C_*^\mu} \leq C_\mu \|u\|_{W^{\mu,\infty}(\mathbf{R}^d)}.$$

Moreover, when $\mu \in \mathbf{N}$ the second inequality still holds.

By virtue of Lemma 2.4, we shall identify $W^{\mu,\infty}(\mathbf{R}^d)$ with $C_*^\mu(\mathbf{R}^d)$ whenever $\mu > 0,\ \mu \notin \mathbf{N}$, regardless of the size n.

For very $j \geq 1$, the reverse estimates for Δ_j in Lemma 2.3 1. hold (see Lemma 2.1, [7]).

Lemma 2.5. *Let $\alpha \in \mathbf{N}^d$. There exists $C_\alpha(n) > 0$ such that for every $1 \leq p \leq \infty$ and every $j \geq 1$, we have*

$$\left\|\Delta_j u\right\|_{L^p(\mathbf{R}^d)} \leq C_\alpha(n) 2^{-j|\alpha|} \left\|\partial^\alpha \Delta_j u\right\|_{L^p(\mathbf{R}^d)}.$$

Applying the previous lemmas yields

Lemma 2.6. *1. Let $\mu > 0$. For every $\alpha \in \mathbf{N}^d$ there exists $C_\alpha > 0$ such that*

$$\forall v \in C_*^\mu(\mathbf{R}^d),\ \forall p \geq 0, \quad \left\|\partial^\alpha (S_p v)\right\|_{L^\infty} \leq \begin{cases} C_\alpha 2^{p(|\alpha| - \mu)} \|\partial^\alpha v\|_{C_*^{\mu - |\alpha|}}, & \text{if } |\alpha| > \mu \\ C_\alpha \|\partial^\alpha v\|_{L^\infty}, & \text{if } |\alpha| < \mu \\ C_\alpha p \|v\|_{C_*^\mu}, & \text{if } |\alpha| = \mu. \end{cases} \tag{2.5}$$

2. Let $\mu < 0$. For every $\alpha \in \mathbf{N}^d$ there exists $C_\alpha > 0$ such that

$$\forall v \in C_*^\mu(\mathbf{R}^d), \ \ \forall p \geq 0, \ \ \left\| \partial^\alpha (S_p v) \right\|_{L^\infty} \leq C_\alpha 2^{p(|\alpha|-\mu)} \left\| v \right\|_{C_*^\mu}. \tag{2.6}$$

3. Let $\mu > 0$. There exists $C(n) > 0$ such that for any $v \in \mathcal{S}'$ with $\nabla v \in C_*^{\mu-1}(\mathbf{R}^d)$ we have

$$\left\| v - S_p v \right\|_{L^\infty} \leq C(n) 2^{-p\mu} \left\| \nabla v \right\|_{C_*^{\mu-1}}. \tag{2.7}$$

2.2. On paradifferential operators

In this paragraph we clarify the choice of two cutoff functions χ and ψ appearing in the definition of paradifferential operators A.3 in accordance with the dyadic partitions above. Given a dyadic system of size n on $\mathbf{R}^d$, define

$$\chi(\eta, \xi) = \sum_{p=0}^{\infty} \phi_{p-N}(\eta) \varphi_p(\xi) \tag{2.8}$$

with $N = N(n) \gg n$ large enough. It is easy to check that the so defined χ satisfies (A.3) and (A.4). Plugging (2.8) into (A.2) gives

$$\begin{aligned} T_a u(x) &= \sum_{p=0}^{\infty} \int \int e^{i(\theta+\eta)x} \phi_{p-N}(\theta) \hat{a}(\theta, \eta) \varphi_p(\eta) \psi(\eta) \hat{u}(\eta) d\eta d\theta \\ &= \sum_{p=0}^{\infty} S_{p-N} a(x, D)(\psi \varphi_p)(D) u(x). \end{aligned}$$

Notice that for any $p \geq 1$ and $\eta \in \operatorname{supp} \varphi_p$ we have $|\eta| \geq 2^{-n}$. Choosing ψ (depending on n) verifying

$$\psi(\eta) = 1 \quad \text{if } |\eta| \geq 2^{-n}, \qquad \psi(\eta) = 0 \quad \text{if } |\eta| \leq 2^{-n-1}$$

gives

$$T_a u(x) = \sum_{p=1}^{\infty} S_{p-N} a(x, D) \Delta_p u(x) + S_{-N} a(x, D)(\psi \varphi_0)(D) u(x). \tag{2.9}$$

Defining the "truncated paradifferential operator" by

$$\dot{T}_a u = \sum_{p=1}^{\infty} S_{p-N} a \Delta_p u, \tag{2.10}$$

then the difference $T_a - \dot{T}_a$ is a smoothing operator in the following sense: if for some $\alpha \in \mathbf{N}^d$, $\partial^\alpha u \in H^{-\infty}$ then $(T_a - \dot{T}_a) u \in H^\infty$ since $\psi \varphi_0$ is supported away from 0. We thus can utilize the symbolic calculus Theorem A.5 for the truncated paradifferential operator $\dot{T}_a u$ when working on distributions u as above. The same remark applies to the paraproduct TP_a defined in (A.11). In general, smoothing remainders can be ignored in applications. However, to be precise in constructing abstract theories we decide to distinguish between these objects.

Definition 2.7. For $v, w \in \mathcal{S}'$ we define the truncated remainder

$$\dot{R}(v, w) = \dot{T}_v w - \dot{T}_w v.$$

Compared to the Bony's remainder $R(v, w)$ defined in (A.12), there holds

$$\dot{R}(v, w) = R(w, w) + \sum_{k=1}^{N} (S_{k-N} v \Delta_k w + S_{k-N} w \Delta_k v). \tag{2.11}$$

Remark 2.8. The relation (2.11) shows that the estimates (A.13), (A.14), (A.15) are valid for $\dot{R}$.

2.3. Choice of dyadic partitions

Let $\kappa : \mathbf{R}^d_1 \to \mathbf{R}^d_2$ be a diffeomorphism satisfying

$$\begin{gathered}\exists \rho > 0,\ \partial_x \kappa \in C^\rho_*(\mathbf{R}^d_1),\\ \exists m_0 > 0,\ \forall x \in \mathbf{R}^d_1,\ \left|\det \kappa'(x)\right| \geq m_0.\end{gathered}$$

We equip on $\mathbf{R}^d_2$ the dyadic partition (2.4) with $n = 0$ and on $\mathbf{R}^d_1$ the one with $n = n_0$ large enough as given in the next proposition.

Proposition 2.9. *Let $p, q, j \geq 0$. For $\varepsilon_0 > 0$ arbitrarily small, there exist $\mathcal{F}_1, \mathcal{F}_2$ nonnegative such that with*

$$n_0 = \mathcal{F}_1(m_0, \left\|\kappa'\right\|_{L^\infty}) \in \mathbf{N}, \quad p_0 = \mathcal{F}_2(m_0, \left\|\kappa'\right\|_{C^{\varepsilon_0}_*}) \in \mathbf{N},$$

and $N_0 = 2(n_0 + 1)$, we have

$$\begin{aligned}&\left|S_p\kappa'(y)\eta - \xi\right| \geq 1,\\ &\textit{if either } (\xi, \eta) \in \mathcal{C}_j(n) \times \mathcal{C}_q(1), \quad p \geq 0,\ j \geq q + N_0 + 1\\ &\textit{or } |\xi| \leq 2^{j+(n+1)}, \eta \in \mathcal{C}_q(1), \quad p \geq p_0,\ 0 \leq j \leq q - N_0 - 1.\end{aligned}$$

Proof. We consider 2 cases:

(i) $p \geq 0$, $j \geq q + N_0 + 1$. Using Lemma 2.3 we get for some constant $M_1 = M_1(d)$

$$\begin{aligned}\left|S_p\kappa'(y)\eta - \xi\right| &\geq |\xi| - |S_p\kappa'(y)\eta| \geq 2^{q+1}(2^{j-q-1-(n+1)} - M_1 \left\|\kappa'\right\|_{L^\infty})\\ &\geq 2^{N_0-(n+1)} - M_1 \left\|\kappa'\right\|_{L^\infty} \geq 2^{n+1} - M_1 \left\|\kappa'\right\|_{L^\infty}.\end{aligned}$$

We choose $n \geq \left[\log_2(M_1 \left\|\kappa'\right\|_{L^\infty} + 1)\right]$ to have $\left|S_p\kappa'(y)\eta - \xi\right| \geq 1$.

(ii) $j \leq q - N_0 - 1$. Note that for any $\varepsilon_0 > 0$, owing to the estimate (2.7), there is a constant $M_2 = M_2(d, \varepsilon_0)$ such that

$$|\kappa' - S_p\kappa'| \leq M_2 2^{-p\varepsilon_0} \left\|\kappa'\right\|_{C^{\varepsilon_0}_*}$$

and consequently, for some increasing function $\mathcal{F}$

$$\left|\det S_p\kappa'\right| \geq \left|\det \kappa'\right| - M_2 2^{-p\varepsilon_0} \mathcal{F}(\left\|\kappa'\right\|_{C^{\varepsilon_0}_*}) \geq \frac{m_0}{2} \tag{2.12}$$

if we choose

$$p \geq p_0 := \frac{1}{\varepsilon_0}\left[\ln\left(\frac{2M_2}{m_0}\mathcal{F}(\left\|\kappa'\right\|_{C^{\varepsilon_0}_*})\right)\right] + 1. \tag{2.13}$$

We then use the inverse formula with adjugate matrix $(S_p\kappa')^{-1} = \frac{1}{\det S_p\kappa'}\mathrm{adj}(S_p\kappa')$ when $d \geq 2$ to get for all $d \geq 1$,

$$\left|(S_p\kappa')^{-1}\right| \leq \frac{2}{m_0}\left(1 + C(d)\left\|\kappa'\right\|^{d-1}_{L^\infty}\right) := K.$$

It follows that

$$\begin{aligned}\left|S_p\kappa'(y)\eta - \xi\right| &\geq \frac{1}{K}|\eta| - |\xi| \geq 2^{j+n+1}\left(\frac{1}{K}\frac{2^{q-2-j-(n+1)}}{2} - 1\right)\\ &\geq \frac{1}{K}2^{N_0-1-(n+1)} - 1 \geq \frac{1}{K}2^n - 1.\end{aligned}$$

Choosing $n \geq [1 + \ln K] + 1$ leads to $\left|S_p\kappa'(y)\eta - \xi\right| \geq 1$. The Proposition then follows with p_0 as in (2.13) and

$$n_0 = \left[\log_2(M_1 \left\|\kappa'\right\|_{L^\infty} + 1)\right] + [1 + \ln K] + 1. \qquad \square$$

3. Quantitative and global paracomposition results

3.1. Motivations

The semi-classical Strichartz estimate [2] for solutions to (1.8) relies crucially on the fact that a *para-change of variables* can be performed to convert the highest order term T_γ to the simple Fourier multiplier $|D_x|^{\frac{3}{2}}$. This is achieved by using the theory of *paracomposition* of Alinhac [5]. Let us recall here the main features of this theory:

Theorem 3.1. *Let Ω_1, Ω_2 be two open sets in $\mathbf{R}^d$ and $\kappa : \Omega_1 \to \Omega_2$ be a diffeomorphism of class $C^{\rho+1}$, $\rho > 0$. Then, there exists a linear operator $\kappa_A^* : \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1)$ having the following properties:*
1. κ_A^* applies $H^s_{loc}(\Omega_2)$ to $H^s_{loc}(\Omega_1)$ for all $s \in \mathbf{R}$.
2. Assume that $\kappa \in H^{r+1}_{loc}$ with $r > \frac{d}{2}$. Let $u \in H^s_{loc}(\Omega_2)$ with $s > 1 + \frac{d}{2}$. Then we have

$$\kappa_A^* u = u \circ \kappa - T_{u' \circ \kappa} \kappa + \mathrm{R} \tag{3.1}$$

with $R \in H^{r+1+\varepsilon}_{loc}(\Omega_1)$, $\varepsilon = \min(s - 1 - \frac{d}{2}, r + 1 - \frac{d}{2})$.
3. Let $h \in \Sigma^m_\tau$. There exists $h^* \in \Sigma^m_\varepsilon$ with $\varepsilon = \min(\tau, \rho)$ such that

$$\kappa_A^* T_h u = T_{h^*} \kappa_A^* u + \mathrm{R}u \tag{3.2}$$

where R applies $H^s_{loc}(\Omega_2)$ to $H^{s-m+\varepsilon}_{loc}(\Omega_2)$ for all $s \in \mathbf{R}$. Moreover, the symbol h^* can be computed explicitly as in the classical pseudo-differential calculus (see Theorem 3.6 below).

Let $u \in \mathcal{E}'(\Omega_2)$, $\operatorname{supp} u = K$, $\psi \in C_0^\infty(\Omega_1)$, $\psi = 1$ near $\kappa^{-1}(K)$. The original definition of κ_A^* in [5] is given by

$$\kappa_A^* u = \sum_{p=0}^{\infty} \sum_{j=p-N_0}^{p+N_0} \widetilde{\Delta}_j (\psi \Delta_p u \circ \kappa) \tag{3.3}$$

for some $N_0 \in \mathbf{N}$ and some dyadic partition $1 = \sum \widetilde{\Delta}_j$ depending on κ, K.

This local theory was applied successfully by Alinhac in studying the existence and interaction of simple waves for nonlinear PDEs. The equation we have in hand is (1.8). More generally, let us consider the paradifferential equation

$$\partial_t u + Nu + iT_h u = f, \quad (t, x) \in (0, T) \times \mathbf{R}, \tag{3.4}$$

where u is the unknown, T_h is a paradifferential operator of order $m > 0$ and Nu is the lower order part. Assume furthermore that $h(x, \xi) = a(x)|\xi|^m$, $a(x) > 0$. We seek for a change of variables to convert T_h to the Fourier multiplier $|D_x|^m$. Set

$$\chi(x) = \int_0^x a^{-\frac{1}{m}}(y) \mathrm{d}y$$

and let κ be the inverse map of χ. Suppose that a global version of Theorem 3.1 were constructed then part 3. would yield

$$\kappa_A^* T_h u = T_{h^*} \kappa_A^* u + \mathrm{R}u$$

and the principal symbol of h^* (as in the case of classical pseudo-differential calculus) would be indeed $|\xi|^m$. However, to be rigorous we have to resolve the following technical difficulties.

Question 1. A global version of Theorem 1, that is, in all statements $H^s_{loc}(\mathbf{R})$ is replaced by $H^s(\mathbf{R})$.

Question 2. If the symbol h is elliptic: $a(x) \geq c > 0$ then the regularity condition $\kappa \in C^{\rho+1}(\mathbf{R})$ is violated for

$$\kappa'(x) = \frac{1}{\chi'(\kappa(x))} = a^{\frac{1}{m}}(\kappa(x)) \geq c^{\frac{1}{m}}.$$

So, we need a result without any regularity assumption on κ but only on its derivatives; in other words, only on the high frequency part of κ.

Assume now that equation (3.4) is quasilinear: $a(t,x) = F(u)(t,x)$. We then naturally consider for each t, the diffeomorphism

$$\chi_t(x) = \int_0^x F(u)^{-m}(t,y)\mathrm{d}y$$

and this gives rise to the following problem

Question 3. When one conjugates (3.4) with κ_A^* it is requisite to compute

$$\partial_t(\kappa_A^* u) = \kappa_A^*(\partial_t u) + \mathrm{R}. \tag{3.5}$$

This would be complicated in view of the original definition (3.3). In [2] the authors overcame this by using Theorem 3.1 2. as a new definition of the paracomposition:

$$\kappa^* u = u \circ \kappa - T_{u' \circ \kappa}\kappa.$$

For this purpose, we need to make use of part 2. of Theorem 3.1 to estimate the remainder $k_A^*(T_h u) - k^*(T_h u)$. This in turn requires $T_h u \in H^s$ with $s > 1 + \frac{d}{2}$ or $u \in H^s$ with $s > m + 1 + \frac{d}{2}$, which is not the case if one wishes to study the optimal Cauchy theory for (3.4) since we are always 1-derivative above the “critical index” $\mu = m + \frac{d}{2}$.

Question 4. Does a linearization result as in part 2. of Theorem 3.1 for $u \in H^s(\mathbf{R})$ with $s < 1 + \frac{d}{2}$ hold?

Let’s suppose that all the above questions can be answered properly. After conjugating (3.4) with κ^* the equation satisfied by $u^* := \kappa^* u$ reads

$$\partial_t u^* + M u^* + |D_x|^m u^* = \kappa^* f + g \tag{3.6}$$

where g contains all the remainders in Theorem 3.1 2., 3. and in (3.5).

To prove Strichartz estimates for (3.6), we need to control g, as a source term, in $L_t^p L_x^q$ norms, which in turns requires *tame estimates* for g. It is then crucial to have quantitative estimates for the remainders appearing in g and hence quantitative results for the paracomposition.

3.2. Statement of main results

Let $\kappa : \mathbf{R}_1^d \to \mathbf{R}_2^d$ be a diffeomorphism. We equip on $\mathbf{R}_2^d$ and $\mathbf{R}_1^d$ two dyadic partitions as in (2.4) with $n = 0$ and $n = n_0$, respectively, where n_0 is given in Proposition 2.9.

Notation 3.2. 1. For a fixed integer $\widetilde{N}$ sufficiently large (larger than N given in (2.8) and $N_0 = 2(n_0 + 1)$) to be chosen appropriately in the proof of Theorem 3.6, we set for any $v \in \mathcal{S}'(\mathbf{R}_1^d)$ the piece

$$[v]_p = \sum_{|j-p| \le \widetilde{N}} \Delta_j v. \tag{3.7}$$

2. For any positive real number μ we set $\mu_- = \mu$ if $\mu \notin \mathbf{N}$ and $\mu_- = \mu - \varepsilon$ if $s \in \mathbf{N}$ with $\varepsilon > 0$ arbitrarily small so that $\mu - \varepsilon \notin \mathbf{N}$.

Henceforth, we always assume the following assumptions on κ:

Assumption I.

$$\exists \rho > 0 : \ \partial_x \kappa \in C_*^\rho(\mathbf{R}_1^d); \quad \exists \alpha \in \mathbf{N}^d, r > -1 : \ \partial_x^{\alpha_0} \kappa \in H^{r+1-|\alpha_0|}(\mathbf{R}_1^d). \tag{3.8}$$

Assumption II.

$$\exists m_0 > 0, \forall x \in \mathbf{R}_1^d : \ \left|\det \kappa'(x)\right| \ge m_0. \tag{3.9}$$

Definition 3.3 *(Global paracomposition).* For any $u \in \mathcal{S}'(\mathbf{R}^d_2)$ we define formally

$$\kappa_g^* u = \sum_{p=0}^{\infty} [u_p \circ \kappa]_p.$$

We state now our precise results concerning the paracomposition operator κ_g^*.

Theorem 3.4 *(Operation). For every $s \in \mathbf{R}$ there exists $\mathcal{F}$ independent of κ such that*

$$\forall u \in C^s_*(\mathbf{R}^d_2), \quad \left\|\kappa_g^* u\right\|_{C^s_*} \le \mathcal{F}(m_0, \left\|\kappa'\right\|_{L^\infty}) \left\|u\right\|_{C^s_*},$$
$$\forall u \in H^s(\mathbf{R}^d_2), \quad \left\|\kappa_g^* u\right\|_{H^s} \le \mathcal{F}(m_0, \left\|\kappa'\right\|_{L^\infty}) \left\|u\right\|_{H^s}.$$

Theorem 3.5 *(Linearization). Let $s \in \mathbf{R}$. For all $u \in \mathcal{S}'(\mathbf{R}^d_2)$ we define*

$$\mathrm{R}_{line} u = u \circ \kappa - \left(\kappa_g^* u + \dot{T}_{u' \circ \kappa} \kappa\right). \tag{3.10}$$

(i) If $0 < \sigma < 1$, $\rho + \sigma > 1$ and $r + \sigma > 0$ then there exists $\mathcal{F}$ independent of κ, u such that

$$\left\|\mathrm{R}_{line} u\right\|_{H^{\tilde{s}}} \le \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho_*})\left(1 + \left\|\partial_x^{\alpha_0} \kappa\right\|_{H^{r+1-|\alpha_0|}}\right)\left(\left\|u'\right\|_{H^{s-1}} + \left\|u\right\|_{C^\sigma_*}\right)$$

where $\tilde{s} = \min(s + \rho, r + \sigma)$.
(ii) If $\sigma > 1$, set $\varepsilon = \min(\sigma - 1, \rho + 1)_-$ then there exists $\mathcal{F}$ independent of κ, u such that

$$\left\|\mathrm{R}_{line} u\right\|_{H^{\tilde{s}}} \le \mathcal{F}(m_0, \left\|\kappa\right\|_{C^\rho_*})\left(1 + \left\|\partial_x^{\alpha_0} \kappa\right\|_{H^{r+1-|\alpha_0|}}\right)\left(\left\|u'\right\|_{H^{s-1}} + \left\|u\right\|_{C^\sigma_*}\right)$$

where $\tilde{s} = \min(s + \rho, r + 1 + \varepsilon)$.

Theorem 3.6 *(Conjugation). Let m, $s \in \mathbf{R}$ and $\tau > 0$. Set $\varepsilon = \min(\tau, \rho)$. Then for every $h(x, \xi) \in \Gamma^m_\tau$, homogeneous in ξ there exist*

- *$h^* \in \Sigma^m_\varepsilon$,*
- *$\mathcal{F}$ nonnegative, independent of κ, h,*
- *$k_0 = k_0(d, \tau) \in \mathbf{N}$*

such that we have for all $u \in H^s(\mathbf{R}^d_2)$,

$$\kappa_g^* T_h u = T_{h^*} \kappa_g^* u + \mathrm{R}_{conj} u, \tag{3.11}$$
$$\left\|\mathrm{R}_{conj} u\right\|_{H^{s-m+\varepsilon}} \le \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho_*}) M^m_\tau(h; k_0) \left(1 + \left\|\partial^{\alpha_0} \kappa\right\|_{H^{+1-\alpha_0}}\right) \left\|u\right\|_{H^s} \tag{3.12}$$

(the semi-norm $M^m_\tau(h; k_0)$ is defined in (A.1)). Moreover, h^* is computed by the formula

$$h^*(x, \xi) = \sum_{j=0}^{[\rho]} h_j^* := \sum_{j=0}^{[\rho]} \frac{1}{j!} \partial_\xi^j D_y^j \left(h\left(\kappa(x), \mathrm{R}(x, y)^{-1} \xi\right) \frac{\left|\det \partial_y \kappa(y)\right|}{\left|\det \mathrm{R}(x, y)\right|}\right)\Big|_{y=x}, \tag{3.13}$$
$$\mathrm{R}(x, y) = {}^t\!\int_0^1 \partial_x \kappa(tx + (1-t)y) \mathrm{d}t.$$

Remark 3.7.

- The definition (3.10) of R_{line} involves $\dot{T}_{u \circ \kappa'} \kappa$ which does not require the regularity on the low frequency part of the diffeomorphism κ.
- Part (i) of Theorem 3.6 gives an estimate for the remainder of the linearization of $\kappa_g^* u$ where u is allowed to be less regular than Lipschitz.

- In part (ii) of Theorem 3.6, the possible loss of arbitrarily small regularity in $\varepsilon = \min(\sigma - 1, \rho + 1)_-$, according to Notation 3.2 2., is imposed to avoid the technical issue in the composition of two functions in Zygmund spaces (see the proof of Lemma 3.10). On the other hand, there is no loss in part (i) where $\sigma \in (0, 1)$.
- In the estimate (3.12), u is assumed to have Sobolev regularity. Therefore, in the conjugation formula (3.11) the paradifferential operators T_h and T_{h^*} can be replaced by their truncated operators $\dot{T}_h$ and $\dot{T}_{h^*}$, modulo a remainder bounded by the right-hand side of (3.12).

3.3. *Proof of the main results*

Notation 3.8. To simplify notations, we denote throughout this section $C^\mu = C^\mu_*(\mathbf{R}^d)$.

3.3.1. *Technical lemmas*

First, for every $u \in \mathcal{S}'(\mathbf{R}^d_2)$ we define formally

$$\mathrm{R}_g u = \kappa^*_g u - \sum_{p\geq 0}[u_p \circ S_p\kappa]_p. \tag{3.14}$$

The remainder R_g is ρ-regularized as to be shown in the following lemma.

Lemma 3.9. *For every $\mu \in \mathbf{R}$ there exists $\mathcal{F}$ independent of κ such that:*

$$\forall v \in H^\mu(\mathbf{R}^d_2),\quad \left\|\mathrm{R}_g v\right\|_{H^{\mu+\rho}} \leq \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho}) \left\|v'\right\|_{H^{\mu-1}} \left(1 + \left\|\partial^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}}\right).$$

Proof. By definition, we have

$$\mathrm{R}_g v = -\sum_{p\geq 0}[v_p \circ S_p\kappa - v_p\circ\kappa]_p = -\sum_{p\geq 0}[A_p]_p.$$

Each term A_p can be written using Taylor's formula:

$$A_p(x) = \int_0^1 v'_p(\kappa(x) + t(S_p\kappa(x) - \kappa(x)))\mathrm{d}t\,(S_p\kappa(x) - \kappa(x)).$$

1. Case 1: $p \geq p_0$. Setting $y(x) = \kappa(x) + t(S_p\kappa(x) - \kappa(x))$, one has as in (2.12) $\left|\det(y')\right| \geq \frac{m_0}{2}$, hence

$$\left\|\int_0^1 v'_p(\kappa(x) + t(S_p\kappa(x) - \kappa(x)))\mathrm{d}t\right\|_{L^2} \leq \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho}) \left\|v'_p\right\|_{L^2}. \tag{3.15}$$

Then by virtue of the estimate (2.7) we obtain

$$\forall p \geq p_0,\quad \left\|A_p\right\|_{L^2} \leq 2^{-p(\rho+1)}2^{-p(\mu-1)}\mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho})e_p = 2^{-p(\rho+\mu)}\mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho})e_p \tag{3.16}$$

with

$$\sum_{p=p_0}^{\infty} e_p^2 \leq \left\|v'\right\|^2_{H^{\mu-1}}.$$

2. Case 2: $0 \leq p < p_0$. We have by the Sobolev embedding $H^{d/2+1} \hookrightarrow L^\infty$

$$\left\|\int_0^1 v'_p(\kappa(x) + t(S_p\kappa(x) - \kappa(x)))\mathrm{d}t\right\|_{L^\infty} \leq \left\|v'_p\right\|_{L^\infty} \leq 2^{p(\frac{d}{2}-s+2)}\left\|v'\right\|_{H^{s-1}}.$$

Applying Lemma 2.5 we may estimate with $\sum_{p\geq 0} f_p^2 \leq \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho})\,\|\partial^{\alpha_0}\kappa\|^2_{H^{r+1-|\alpha_0|}}$

$$\begin{aligned}\left\|\kappa - S_p\kappa\right\|_{L^2} &\le \sum_{j=p+1}^{\infty} \left\|\Delta_j\kappa\right\|_{L^2} \le \sum_{j=p+1}^{\infty} 2^{-j|\alpha_0|}\left\|\Delta_j\partial^{\alpha_0}\kappa\right\|_{L^2}\\ &\le \sum_{j=p+1}^{\infty} 2^{-j|\alpha_0|}2^{-j(r+1-|\alpha_0|)}f_p \le \mathcal{F}(m_0,\left\|\kappa'\right\|_{C^\rho})\left\|\partial^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}},\end{aligned}$$

where we have used the assumption that $r+1>0$. Therefore,

$$\forall p < p_0,\quad \left\|A_p\right\|_{L^2} \le \mathcal{F}(m_0,\left\|\kappa'\right\|_{C^\rho})\left\|v'\right\|_{H^{\mu-1}}\left\|\partial^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}}. \tag{3.17}$$

3. Finally, noticing that the spectrum of $[A_p]_p$ is contained in an annulus $\{M^{-1}2^p \le |\xi| \le 2^p M\}$ with M depending on n_0, the lemma then follows from (3.16), (3.17). □

Lemma 3.10. *Let $\mu > 0$ and $\varepsilon = \min(\mu, \rho+1)_-$. For every $v \in C^\mu(\mathbf{R}^d_2)$, set*

$$r_p := S_p(v\circ\kappa) - (S_p v)\circ(S_p\kappa).$$

Then for every $\alpha\in\mathbf{N}$ there exists a non-decreasing function $\mathcal{F}_\alpha$ independent of κ and v such that

$$\left\|\partial_x^\alpha r_p\right\|_{L^\infty} \le 2^{p(|\alpha|-\varepsilon)}\mathcal{F}_\alpha(\left\|\kappa'\right\|_{C^\rho})\left\|v\right\|_{C^\mu}.$$

Proof. We first remark that by interpolation, it suffices to prove the estimate for $\alpha=0$ and all $|\alpha|$ large enough. By definition of ε we have $v\circ\kappa\in C^\varepsilon$ with norm bounded by $\mathcal{F}(\left\|\kappa'\right\|_{C^\rho})\left\|v\right\|_{C^\mu}$.

1. $\alpha=0$. One writes

$$r_p = (S_p(v\circ\kappa) - v\circ\kappa) + (v - S_p v)\circ\kappa + (S_p v\circ\kappa - S_p v\circ S_p\kappa)$$

and uses (2.7) to estimate the first two terms. For the last term, by Taylor's formula and (2.7) (consider $\mu>1, =1$ or <1) we have

$$\left\|S_p v\circ\kappa - S_p v\circ S_p\kappa\right\|_{L^\infty} \le \left\|S_p v'\right\|_{L^\infty}\left\|\kappa - S_p\kappa\right\|_{L^\infty} \le C2^{-p\varepsilon}\left\|v\right\|_{C^\mu}\left\|\kappa'\right\|_{C^\rho}.$$

Therefore,

$$\left\|r_p\right\|_{L^\infty} \le C2^{-p\varepsilon}\left(\left\|v\circ\kappa\right\|_{C^\varepsilon} + \left\|v\right\|_{C^\varepsilon} + \left\|v\right\|_{C^\mu}\left\|\kappa'\right\|_{C^\rho}\right).$$

2. $|\alpha| > \rho+1$. The estimate (2.5) implies

$$\left\|S_p(v\circ\kappa)^{(\alpha)}\right\|_{L^\infty} \le C_\alpha 2^{p(|\alpha|-\varepsilon)}\left\|v\circ\kappa\right\|_{C^\varepsilon}.$$

On the other hand, part 2. of the proof of Lemma 2.1.1, [5] gives

$$\left\|(S_p v\circ S_p\kappa)^{(\alpha)}\right\|_{L^\infty} \le C_\alpha 2^{p(|\alpha|-\varepsilon)}(1+\left\|\kappa'\right\|_{C^\rho})^{|\alpha|}\left\|v\right\|_{C^\mu}.$$

Consequently, we get the desired estimate for all $|\alpha| > 1+\rho$, which completes the proof. □

Lemma 3.11. *Let $v\in C^\infty(\mathbf{R}^d_2)$ with $\operatorname{supp}\hat{v}\in\mathcal{C}_q(0)$. Recall that $N_0 = 2(n_0+1)$ with n_0 given by Proposition 2.9.*
(i) For $p\ge 0$, $j\ge q+N_0+1$ and $k\in\mathbf{N}$ there exists $\mathcal{F}_k$ independent of κ, v such that

$$\left\|\left(v\circ S_p\kappa\right)_j\right\|_{L^2(\mathbf{R}^d)} \le 2^{-jk}2^{p(k-\rho)_+}\left\|v\right\|_{L^2}\mathcal{F}_k(m_0,\left\|\kappa'\right\|_{C^\rho}).$$

(ii) For $p\ge p_0$, $0\le\ell\le\ell'\le q-N_0-1$ and $k\in\mathbf{N}$ there exists $\mathcal{F}_k$ independent of κ, v such that

$$\left\|\sum_{j=\ell}^{\ell'}\left(v\circ S_p\kappa\right)_j\right\|_{L^2(\mathbf{R}^d)} \le 2^{-qk}2^{p(k-\rho)_+}\left\|v\right\|_{L^2}\mathcal{F}_k(m_0,\left\|\kappa'\right\|_{C^\rho}).$$

(iii) Set $R_p u = [u_p\circ S_p\kappa]_p - (u_p\circ S_p\kappa)$. For any $p\ge p_0$, there exists $\mathcal{F}_k$ independent of κ, u such that

$$\left\|R_p u\right\|_{L^2} \le 2^{-p\rho}\left\|u_p\right\|_{L^2}\mathcal{F}_k(m_0,\left\|\kappa'\right\|_{C^\rho}).$$

Proof. First, it is clear that (iii) is a consequence of (i) and (ii) both applied with $k > \rho$. The proof of (i) and (ii) follows *mutadis mutandis* that of Lemma 2.1.2, [5], using the technique of integration by parts with non-stationary phase. We only explain how to obtain the non-stationariness here. Let $\widetilde{\varphi} = 1$ on $\mathcal{C}(0)$ and $\operatorname{supp}\widetilde{\varphi} \subset \mathcal{C}(1)$. The phase of the integral (with respect to y) appearing in the expression of $(v \circ S_p\kappa)_j$ and $\sum_{j=\ell}^{\ell'}(v \circ S_p\kappa)_j$ is

$$S_p\kappa(y)\eta - y\xi$$

where,

- in case (i), $(\eta, \xi) \in \operatorname{supp}(\widetilde{\varphi}(2^{-q}\cdot) \times \operatorname{supp}\varphi(2^{-j}\cdot)$,
- in case (ii), $(\eta, \xi) \in \operatorname{supp}(\widetilde{\varphi}(2^{-q}\cdot) \times \operatorname{supp}\phi(2^{-\iota}\cdot)$ with $\iota = \ell$ or $\ell' + 1$, which comes from the fact that

$$\sum_{j=\ell'}^{\ell}\varphi(2^{-j}\xi) = \phi(2^{-\ell}\xi) - \phi(2^{-\ell'-1}).$$

In both cases,

$$\left|\partial_y(S_p\kappa(y)\eta - y\xi)\right| = \left|S_p\kappa'(y)\eta - \xi\right| \geq 1$$

by virtue of Proposition 2.9. $\square$

3.3.2. Proof of Theorem 3.4

By Definition 3.3 of the global paracomposition $\kappa_g^* u = \sum_{p=0}^{\infty}[u_p \circ \kappa]_p$. Since each $[u_p \circ \kappa]_p$ is spectrally localized in a dyadic cell depending on $n_0 = \mathcal{F}(m, \|\kappa'\|_{L^\infty})$, the theorem follows from Lemma 2.3 after making the change of variables $y = \kappa(x)$.

3.3.3. Proof of Theorem 3.5

Using the dyadic partition $u = \sum_{p\geq 0} u_p$ and the fact that $S_p \to Id$ in $\mathcal{S}'$ we have in $\mathcal{D}'(\mathbf{R}^d_1)$

$$u \circ \kappa = \sum_{p\geq 0} u_p \circ \kappa = \sum_{p\geq 0}\sum_{q\geq 0}\big(u_p \circ S_{q+1}\kappa - u_p \circ S_q\kappa\big) + \sum_{p\geq 0} u_p \circ S_0\kappa.$$

Denoting by S the first term on the right-hand side, one has by Fubini's theorem,

$$S = \sum_{q\geq 0}\sum_{0\leq p\leq q}(u_p \circ S_{q+1}\chi - u_p \circ S_q\kappa) + \sum_{q\geq 0}\sum_{p\geq q+1}(u_p \circ S_{q+1}\kappa - u_p \circ S_q\kappa) =: (I) + (II).$$

For (I) we take the sum in p first and notice that $S_0 = \Delta_0$ to get

$$(I) = \sum_{q\geq 0}(S_q u \circ S_{q+1}\kappa - S_q u \circ S_q\kappa).$$

For (II) we write

$$(II) = \sum_{p\geq 1}\sum_{0\leq q\leq p-1}(u_p \circ S_{q+1}\kappa - u_p \circ S_q\kappa) = \sum_{p\geq 1}(u_p \circ S_p\kappa - u_p \circ S_0\kappa).$$

Summing up, we derive

$$u \circ \kappa = \sum_{p\geq 0} u_p \circ S_p\kappa + \sum_{q\geq 0}(S_q u \circ S_{q+1}\kappa - S_q u \circ S_q\kappa) =: A + B. \tag{3.18}$$

Thanks to Lemma 3.9, there hold

$$A = \sum_{p\geq 0} u_p \circ S_p\kappa = \kappa_g^* u + \mathrm{R}_g u, \quad \text{with} \tag{3.19}$$

$$\left\|\mathrm{R}_g u\right\|_{H^{s+\rho}} \leq \mathcal{F}(m_0, \|\kappa'\|_{C^\rho})\left\|u'\right\|_{H^{s-1}}\left(1 + \left\|\partial^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}}\right). \tag{3.20}$$

On the other hand, $B = \sum_{q\ge 0} B_q$ with

$$B_q := S_q u \circ S_{q+1}\kappa - S_q u \circ S_q \kappa = r_{q+1}\kappa_{q+1} + S_{q-N+1}(u' \circ \kappa)\kappa_{q+1}$$

where

$$r_{q+1} = \int_0^1 (S_q u')(t S_{q+1}\kappa + (1-t)S_q\kappa)\mathrm{d}t - S_{q-N+1}(u' \circ \kappa).$$

By the definition of truncated paradifferential operators

$$\sum_{q\ge 0} S_{q-N+1}(u'\circ\kappa)\kappa_{q+1} = \sum_{p\ge 1} S_{p-N}(u'\circ\kappa)\kappa_p = \dot{T}_{u'\circ\kappa}\kappa. \tag{3.21}$$

Thus, it remains to bound

$$\sum_{q\ge 0} r_{q+1}\kappa_{q+1} = \sum_{q\ge 1} r_q\kappa_q.$$

(i) *Case 1*: $0<\sigma<1,\ \sigma+\rho>1$

In this case, we see that $u\circ\kappa \in C^\sigma$, hence $(u\circ\kappa)' \in C^{\sigma-1}$ with norm bounded by $\mathcal{F}(m_0, \|\kappa'\|_{C^\rho})\|u\|_{C^\sigma}$. Then, using (A.22) with $\alpha = 1-\sigma,\ \beta = \rho_-$ yields

$$\|u'\circ\kappa\|_{C^{\sigma-1}} = \left\|(\kappa')^{-1}(u\circ\kappa)'\right\|_{C^{\sigma-1}} \le \mathcal{F}(m_0, \|\kappa'\|_{C^\rho}) \left\|(\kappa')^{-1}\right\|_{C^{\rho-}} \|(u\circ\kappa)'\|_{C^{\sigma-1}}.$$

By writing $(\kappa')^{-1} = \frac{1}{\det(\kappa')}\mathrm{adj}(\kappa')$ we get easily that $\|(\kappa')^{-1}\|_{C^{\rho-}} \le \mathcal{F}(m_0, \|\kappa'\|_{C^\rho})$ and hence

$$\|u'\circ\kappa\|_{C^{\sigma-1}} \le \mathcal{F}(m_0, \|\kappa'\|_{C^\rho})\|u\|_{C^\sigma}.$$

Now, we claim that

$$\forall q\ge 1,\ \forall \alpha\in\mathbf{N}^d,\ \ \|\partial_x^\alpha r_p\|_{L^\infty} \le 2^{q(|\alpha|+1-\sigma)}\mathcal{F}_\alpha(m_0, \|\kappa'\|_{C^\rho})\|u\|_{C^\sigma}. \tag{3.22}$$

Since $\sigma-1<0$ it follows from (2.6) that

$$\|\partial_x^\alpha S_{q-N}(u'\circ\kappa)\|_{L^\infty} \le C_\alpha 2^{q(|\alpha|+1-\sigma)}\|u'\circ\kappa\|_{C^{\sigma-1}} \le 2^{q(|\alpha|+1-\sigma)}\mathcal{F}_\alpha(m_0, \|\kappa'\|_{C^\rho})\|u\|_{C^\sigma}.$$

Thus, to obtain (3.22) it remains to prove

$$\forall q\ge 1,\ \forall \alpha\in\mathbf{N}^d,\ \ \|\partial_x^\alpha (S_q u'(S_q\kappa))\|_{L^\infty} \le 2^{q(|\alpha|+1-\sigma)}\mathcal{F}_\alpha(m_0, \|\kappa'\|_{C^\rho})\|u\|_{C^\sigma}. \tag{3.23}$$

By interpolation, this will follow from the corresponding estimates for $\alpha=0$ and $|\alpha|>1+\rho$. Again, since $\sigma-1<0$ we have (3.23) for $\alpha=0$.

Now, consider $|\alpha|>1+\rho$. By the Faà-di-Bruno formula $((S_q u')\circ(S_q\kappa))^{(\alpha)}$ is a finite sum of terms of the following form

$$A = (S_q u')^{(m)}\prod_{j=1}^t [(S_q\kappa)^{(\gamma_j)}]^{s_j},$$

where $1\le |m|\le|\alpha|,\ |\gamma_j|\ge 1,\ |s_j|\ge 1,\ \sum_{j=1}^t |s_j|\gamma_j = \alpha,\ \sum_{j=1}^t s_j = m$.

By virtue of (2.5), one gets

$$\left\|(S_q\kappa)^{(\gamma_j)}\right\|_{L^\infty} = \left\|(S_q\kappa')^{(\gamma_j-1)}\right\|_{L^\infty} \le \begin{cases} C2^{q(|\gamma_j|-1-\rho)}\|\kappa'\|_{C^\rho}, & \text{if } |\gamma_j|-1>\rho \\ C\|\kappa'\|_{C^\rho}, & \text{if } |\gamma_j|-1<\rho \\ C_\eta 2^{q\eta}\|\kappa'\|_{C^\rho}, \forall\eta>0 & \text{if } |\gamma_j|-1=\rho. \end{cases}$$
$$\le C_\alpha 2^{q(|\gamma_j|-1)(1-\frac{\rho}{|\alpha|-1})}\|\kappa'\|_{C^\rho}.$$

Consequently,

$$\left\|\prod_{j=1}^{t}[(S_q\kappa)^{(\gamma_j)}]^{s_j}\right\|_{L^\infty} \le C_\alpha 2^{q(|\alpha|-|m|)(1-\frac{\rho}{|\alpha|-1})}\left\|\kappa'\right\|_{C^\rho}^{|m|}. \tag{3.24}$$

Combining (3.24) with the estimate (applying (2.5) since $m+1>\sigma$)

$$\left\|(S_q u')^{(m)}\right\|_{L^\infty} \le C_m 2^{q(m+1-\sigma)}\left\|u\right\|_{C^\sigma}$$

yields

$$\left\|\partial_x^\alpha(S_q u'(S_q\kappa))\right\|_{L^\infty} \le 2^{qM}\mathcal{F}_\alpha(\left\|\kappa'\right\|_{C^\rho})\left\|u\right\|_{C^\sigma}$$

with

$$M=(m+1-\sigma)+(|\alpha|-m)(1-\frac{\rho}{|\alpha|-1})\le|\alpha|+1-\sigma,$$

which concludes the proof the claim (3.23).

On the other hand, according to Lemma 2.5 (ii) for any $q\ge 1, \alpha\in\mathbf{N}$ there holds

$$\begin{aligned}\left\|\partial_x^\alpha\kappa_q\right\|_{L^2} &\le C_\alpha 2^{q|\alpha|}\left\|\kappa_q\right\|_{L^2}\le C_\alpha 2^{q(|\alpha|-|\alpha_0|)}\left\|\partial^{\alpha_0}\kappa_q\right\|_{L^2}\\ &\le C_\alpha 2^{q(|\alpha|-|\alpha_0|)}2^{-q(r+1-|\alpha_0|)}a_p = C_\alpha 2^{q(|\alpha|-r-1)}a_p,\end{aligned} \tag{3.25}$$

with

$$\sum_{q\ge1}a_q^2\le\mathcal{F}(m_0,\left\|\kappa'\right\|_{C^\rho})\left\|\partial_x^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}}^2.$$

We deduce from (3.23) and (3.25) that

$$\forall\alpha\in\mathbf{N}^d,\ \forall q\ge1,\ \left\|\partial_x^\alpha(r_q\kappa_q)\right\|_{L^2}\le 2^{q(|\alpha|-r-\sigma)}\mathcal{F}_\alpha(\left\|\kappa\right\|_{C^\rho})\left\|u\right\|_{C^\sigma}a_q.$$

By the assumption $r+\sigma>0$ we conclude

$$\|\sum_{q\ge1}r_q\kappa_q\|_{H^{r+\sigma}}\le\mathcal{F}(\left\|\kappa\right\|_{C^\rho})\left\|u\right\|_{C^\sigma}\left\|\partial_x^{\alpha_0}\kappa\right\|_{H^{r+1-\alpha_0}}. \tag{3.26}$$

Combining (3.19), (3.20), (3.21), (3.26) we obtain the assertion (i) of Theorem 3.5.
(ii) *Case 2*: $\sigma>1$. This case was studied in [5]. One writes

$$S_{q-N}(u'\circ\kappa)=t_q+S_{q-1}u'\circ S_{q-1}\kappa+s_q$$

with

$$t_q=S_{q-N}(u'\circ\kappa)-S_{q-1}(u'\circ\kappa),\ s_q=S_{q-1}(u'\circ\kappa)-S_{q-1}u'\circ S_{q-1}\kappa.$$

Plugging this into r_q gives

$$r_q=z_q-t_q-s_q$$

with

$$\begin{aligned}z_q&=\int_0^1(S_{q-1}u')(tS_q\kappa+(1-t)S_{q-1}\kappa)\mathrm{d}t-S_{q-1}u'\circ S_{q-1}\kappa\\ &=\kappa_q\int_0^1 t\int_0^1((S_{q-1}u'))'(S_{q-1}\kappa+st\kappa_q)\mathrm{d}s\mathrm{d}t.\end{aligned}$$

Now we estimate the L^∞-norm of derivatives of r_q. Since

$$t_q=-\sum_{j=q-N+1}^{q-1}(u'\circ\kappa)_j$$

we get with $\varepsilon = \min(\sigma - 1, \rho + 1)_-$

$$\forall q \geq 1, \forall \alpha \in \mathbf{N}, \ \left\| \partial_x^\alpha t_q \right\|_{L^\infty} \leq 2^{p(|\alpha|-\varepsilon)} \mathcal{F}_\alpha(\left\|\kappa'\right\|_{C^\rho}) \left\|u\right\|_{C^\sigma}. \tag{3.27}$$

Applying Lemma 3.10 we have the same estimate as (3.27) for s_q. Finally, following exactly part d) of the proof of Lemma 3.1, [5] with the use of Lemma 2.6 one obtains the same bound for z_q.

We conclude by using (3.25) that

$$\|\sum_{q\geq 1} r_q \kappa_q\|_{H^{r+1+\varepsilon}} \leq \left\| \partial_x^{\alpha_0} \kappa \right\|_{H^{r+1-\alpha_0}} \mathcal{F}(\left\|\kappa'\right\|_{C^\rho}) \left\|u\right\|_{C^\sigma},$$

which combined with (3.20) gives the assertion (ii) of Theorem 3.5.

3.3.4. Proof of Theorem 3.6

We recall first the following lemma on the boundedness of a class of Fourier integral operators.

Lemma 3.12. *([5, Lemme d), page 111]) Let $K \subset \mathbf{R}^d$ be a compact set. Let $a(x, y, \eta)$ be a bounded function satisfying the following properties: a is C^∞ in η and its support w.r.t. η is contained in K, all derivatives of a w.r.t. η are bounded. For every $p \in \mathbf{N}$, define the associated Fourier integral operator*

$$A_p v(x) = \int e^{i(x-y)\xi} a(x, y, 2^{-p\xi}) v(y) dy d\xi.$$

Then, there exist an integer $k_1 = k_1(d)$ and a positive constant C independent of a, p such that with

$$M = \sup_{|\alpha|\leq k_1} \left\| \partial_\eta^\alpha a \right\|_{L^\infty(\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R}^d)}$$

we have

$$\forall v \in L^2(\mathbf{R}^d), \ \left\| A_p v \right\|_{L^2} \leq CM \left\|v\right\|_{L^2}.$$

Now we quantify the proof of Lemma 3.3 in [5]. Let $m, \ s \in \mathbf{R}$, $\tau > 0$, $\varepsilon = \min(\tau, \rho)$ and $h(x, \xi) \in \Gamma_\tau^m$, homogeneous in ξ. We say that a quantity Q is controllable if $\|Q\|_{H^{s-m+\varepsilon}}$ is bounded by the right-hand side of (3.12) and therefore can be neglected. Also, by $A \sim B$ we mean that $A - B$ is controllable. Keep in mind that $\kappa : \mathbf{R}_1^d \to \mathbf{R}_2^d$, $u : \mathbf{R}_2^d \to \mathbf{C}$ where $\mathbf{R}_1^d$ and $\mathbf{R}_2^d$ are $\mathbf{R}^d$ equipped with a dyadic partition of size $n = n_0$ and $n = 0$ respectively.

Step 1. First, by Lemma 3.9 we have

$$\kappa_g^*(T_h u) \sim \sum_{p\geq 0} \left[\Delta_p T_h u \circ S_p \kappa\right]_p. \tag{3.28}$$

Then with $v^q := (S_{q-N} h)(x, D) u_q$,

$$\kappa_g^*(T_h u) \sim \sum_{p\geq 0} \sum_{q\geq 1} \left[\Delta_p v^q \circ S_p \kappa\right]_p.$$

One can see easily that if N is chosen sufficiently larger than n_0 then the spectrum of v^q is contained in the annulus

$$\left\{ \xi \in \mathbf{R}^d : 2^{q-M_1} \leq |\xi| \leq 2^{q+M_1} \right\}$$

with $M_1 = M_1(N, n_0)$. This implies

$$\Delta_p v^q = 0 \text{ if } |p - q| > M := M_1 + n_0 + 1 = M(N, n_0),$$

hence

$$\kappa_g^*(T_h u) = \sum_{|p-q|\leq M} \left[\Delta_p v^q \circ S_p \kappa\right]_p.$$

Set

$$S_1 = \sum_{|p-q|\le M} \Big(\big[\Delta_p v^q \circ S_p\kappa\big]_p - \big[\Delta_p v^q \circ S_p\kappa\big]_q \Big),$$
$$S_2 = \sum_{|p-q|\le M} \Big(\big[\Delta_p v^q \circ S_p\kappa\big]_q - \big[\Delta_p v^q \circ \kappa\big]_q \Big).$$

We shall prove that $S_1,\ S_2$ are controllable so that

$$\kappa_g^*(T_h u) \sim \sum_{p,q\ge 0} \big[\Delta_p v^q \circ \kappa\big]_q = \sum_{q\ge 0} \big[v^q \circ \kappa\big]_q = \sum_{p\ge 0} \big[(S_{p-N}h)\Delta_p u \circ \kappa\big]_p . \tag{3.29}$$

Following the proof of Lemma 3.9, it can be seen that S_2 is $(\rho - m)$-regularized and thus controllable. We now consider S_1. If we choose $\widetilde{N} \gg M + N_0$, $N_0 = 2(n_0+1)$, in the definition (3.7) of $[\cdot]_p$ then

$$S_1 = \sum_{\substack{p,q\\|p-q|\le M}} \ \sum_{\substack{j\\N_0<|j-p|\le \widetilde{N}}} \Delta_j(\Delta_p v^q \circ S_p\kappa) - \sum_{\substack{p,q\\|p-q|\le M}} \ \sum_{\substack{j\\N_0<|j-q|\le \widetilde{N}}} \Delta_j(\Delta_p v^q \circ S_p\kappa) = S_{1,1} - S_{1,2}.$$

Observe that the pieces $\Delta_j(\dots)$ with j close to both p and q canceled out in the above subtraction. $S_{1,1}$ and $S_{1,2}$ can be treated in the same way. Let us consider $S_{1,1} = \sum_j a_j^1 + \sum_j a_j^2$ with

$$a_j^1 = \sum_{\substack{p\\p<j-N_0\\|p-j|\le \widetilde{N}}} \ \sum_{\substack{q\\|q-p|\le M}} \Delta_j(\Delta_p v^q \circ S_p\kappa), \quad a_j^2 = \sum_{\substack{p\\p>j+N_0\\|p-j|\le \widetilde{N}}} \ \sum_{\substack{q\\|q-p|\le M}} \Delta_j(\Delta_p v^q \circ S_p\kappa).$$

For some positive constants C and k_0, where the later is an integer depending only on d, we have

$$\forall q, \quad \left\| v^q \right\|_{L^2} \le C M_0^m(h, k_0) \left\| \Delta_q u \right\|_{H^m} .$$

By virtue of Lemma 3.11 (i) applied with $k > \rho$,

$$\begin{aligned}
\left\| a_j^1 \right\|_{L^2} &\le \sum_{p<j-N_0, |p-j|\le \widetilde{N}, |q-p|\le M} C_k 2^{-jk} 2^{p(k-\rho)} M_0^m(h, k_0) \left\| \Delta_q u \right\|_{H^m} \mathcal{F}_k(m_0, \left\|\kappa'\right\|_{C^\rho}) \\
&\le \sum_{|q-j|\le M+\widetilde{N}} C' 2^{-j\rho+mq} M_0^m(h, k_0) \left\| \Delta_q u \right\|_{L^2} \mathcal{F}_k(m_0, \left\|\kappa'\right\|_{C^\rho}) \\
&\le C'' 2^{-j(\rho-m+s)} M_0^m(h, k_0) \mathcal{F}_k(m_0, \left\|\kappa'\right\|_{C^\rho}) \sum_{|q-j|\le M+\widetilde{N}} b_q
\end{aligned}$$

with $\|b\|_{\ell^2} \le C \|u\|_{H^s}$. Recall that m_0 is such that $|\det \kappa'(x)| \ge m_0$ for all $x \in \mathbf{R}_1^d$. Then, thanks to the spectral localization of a_j^1 we conclude that (for a different $\mathcal{F}_k$)

$$\| \sum_j a_j^1 \|_{H^{s-m+\rho}} \le M_0^m(h, k_0) \|u\|_{H^s} \mathcal{F}_k(m_0, \left\|\kappa'\right\|_{C^\rho}).$$

For the second sum $\sum a_j^2$ we apply Lemma 3.11 (ii).

Step 2. Recall from (3.29) that

$$\kappa_g^* T_h u = \sum_{p\ge 0} [A_p]_p, \quad A_p = \big((S_{p-N}h)(x, D)u_p\big) \circ \kappa.$$

One writes

$$A_p(y) = \int e^{i(\kappa(y)-y')\xi} \widetilde{\varphi}(2^{-p}\xi)(S_{p-N}h)(\kappa(y), \xi) u_p(y') \mathrm{d}y' \mathrm{d}\xi$$

where $\widetilde{\varphi}$ is a cut-off function analogous to φ and equal to 1 on the support of φ.

In the integral defining $A_p(y)$ we make two changes of variables

$$y' = \kappa(z), \quad \xi = {}^t\mathrm{R}^{-1}\eta, \quad \mathrm{R} = \mathrm{R}(y,z) := \int_0^1 \kappa'(ty + (1-t)z)\mathrm{d}t$$

to derive

$$A_p(y) = \int e^{i(y-z)\eta}\widetilde{\varphi}(2^{-p}\,{}^t\mathrm{R}^{-1}\eta)(S_{p-N}h)(\kappa(y), {}^t\mathrm{R}^{-1}\eta)u_p(\kappa(z))\frac{|\kappa'(z)|}{|\det \mathrm{R}|}\mathrm{d}z\mathrm{d}\eta.$$

The rest of the proof follows the same method as for pseudo-differential operators (see [17]) except that we shall regularize first the symbol $a_p(y,z,\eta)$ of A_p: set

$$b_p(y,z,\eta) = \widetilde{\varphi}(2^{-p}\,{}^t\mathrm{R}_p^{-1}\eta)(S_{p-N}h)(S_p\kappa(y), {}^t\mathrm{R}_p^{-1}\eta)\frac{|S_p\kappa'(z)|}{|\det \mathrm{R}_p|}$$

with

$$\mathrm{R}_p = \mathrm{R}_p(y,z) = \int_0^1 S_p\kappa'(ty + (1-t)z)\mathrm{d}t.$$

By the homogeneity of $S_{p-N}h$ we write $a_p(y,z,\eta) = 2^{pm}\widetilde{a}_p(y,z,2^{-p}\eta)$ and similarly for $\widetilde{b}_p$. Then due to the presence of the cut-off function $\widetilde{\varphi}$ one can prove without any difficulty that

$$\forall k \in \mathbf{N}, \quad \sup_{|\alpha|\le k}\left|\partial_\eta^\alpha(\widetilde{a}_p - \widetilde{b}_p)(y,z,\eta)\right| \le C_k 2^{-p\rho}\mathcal{F}_k(m_0, \|\kappa'\|_{C^\rho})M_0^m(h; k+1).$$

Therefore, in view of Lemma 3.12 we see that the replacement of a_p with b_p in $\kappa_g^* T_h u$ gives rise to a controllable remainder.

Step 3. Next, we expand $b_p(y,z,\eta)$ by Taylor's formula at $z = y$ up to order $\ell = [\rho]$ to have

$$b_p(y,z,\eta) = b_p^0(y,\eta) + b_p^1(y,\eta)(z-y) + \dots + b_p^\ell(y,\eta)(z-y)^\ell + r_p^{\ell+1}(y,z,\eta)(z-y)^{\ell+1}$$

where b^j is the jth-derivative of b_p with respect to z, taken at $z = y$ and

$$r_p^{\ell+1}(y,z,\eta) = C\int_0^1 b_p^{\ell+1}(y, y + t(z-y), \eta)\mathrm{d}t(z-y)^{\ell+1}.$$

In the pseudodifferential operator $\mathrm{R}_p^{\ell+1}$ with symbol $r_p^{\ell+1}$ we integrate by parts w.r.t. η $\ell+1$ times to obtain a sum of symbols of the form $2^{p(m-\ell-1)}\widetilde{r}_p(y,z,2^{-p}\eta)$,

$$\widetilde{r}_p(y,z,\eta) = C\int_0^1 \partial_z^\alpha\partial_\eta^\beta\widetilde{b}_p(y, y + t(z-y), \eta)\mathrm{d}t, \ |\alpha| = |\beta| = \ell+1.$$

For $|\alpha| = \ell+1, \ |\gamma| = \ell+1+k, \ k \in \mathbf{N}$ it holds

$$\left|\partial_z^\alpha\partial_\eta^\gamma\widetilde{b}_p(y,z,\eta)\right| \le C_k 2^{p(\ell+1-\rho)}M_0^m(h, [\rho]+1+k)\mathcal{F}_k(m_0, \|\kappa'\|_{C^\rho}).$$

Lemma 3.12 then gives for some $k_1 = k_1(d) \in \mathbf{N}$,

$$\left\|\mathrm{R}_p^{\ell+1}\right\|_{L^2} \le 2^{p(\ell+1-\rho)}2^{p(m-\ell-1)}\left\|u_p\right\|_{L^2}M_0^m(h,k_1)\mathcal{F}_k(m_0, \|\kappa'\|_{C^\rho}).$$

Therefore, the remainder $\sum_p [R_p^{\ell+1}]_p$ is $(\rho - m)$-regularized and thus controllable.

Step 4. We write

$$B_p^j u(y) = \int e^{i(y-z)\eta}b_p^j(y,\eta)(z-y)^j u_p(\kappa(z))\mathrm{d}z\mathrm{d}\eta$$

and integrate by parts j times w.r.t. η to get

$$B_p^j u = \int e^{i(y-z)\eta} c_p^j(y,\eta) u_p(\kappa(z)) \mathrm{d}z \mathrm{d}\eta.$$

The key point is that in the above expression above we shall replace $u_p \circ \kappa$, $p \geq 0$ by its approximation $[u_p \circ \kappa]_p$ which will contribute in $T_{h^*\kappa_g^*} u$. In order to do so, we will estimate the L^2-norm of the difference

$$W_p := u_p \circ \kappa - [u_p \circ \kappa]_p$$

as

$$\left\| W_p \right\|_{L^2} \leq 2^{-p\rho} \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho}) \left\| u_p \right\|_{L^2}.$$

Consider $0 \leq p < p_0$. We treat separately each term in W_p by making the change of variables $x \mapsto \kappa(x)$ to have

$$\left\| W_p \right\|_{L^2} \leq \mathcal{F}(m_0, \left\|\kappa'\right\|_{C^\rho}) \left\| u_p \right\|_{L^2}.$$

For $p \geq p_0$ we write

$$W_p = (u_p \circ \kappa - u_p \circ S_p\kappa) + (u_p \circ S_p\kappa - [u_p \circ S_p\kappa]_p) + ([u_p \circ S_p\kappa]_p - [u_p \circ \kappa]_p).$$

The second term is bounded using directly Lemma 3.11 (iii). The first and the last terms are treated as in the first part (case 1.) of the proof of Lemma 3.9 (see (3.16)).

Again, by virtue of Lemma 3.12 we conclude that: in $\kappa_g T_h u$ the replacement of $u_p \circ \kappa$ with $[u_p \circ \kappa]_p$ is $(\rho + j - m)$-regularized and controllable.

Step 5. Set

$$C_p^j u(y) = \int e^{i(y-z)\eta} c_p^j(y,\eta)[u_p \circ \kappa]_p(z) \mathrm{d}z \mathrm{d}\eta.$$

We observe that if the cut-off function $\widetilde{\varphi}$ is chosen appropriately then all the terms in c_p^j relating to $\partial^\alpha \widetilde{\varphi}$ are 1 if $\alpha = 0$ and are 0 if $\alpha \neq 0$, on the spectrum of $[u_p \circ \kappa]_p$. Therefore, compared to the classical calculus (3.13) for $S_{p-N} h$ we can prove that

$$\sup_{\substack{|\alpha| \leq k \\ 0 < c_1 \leq |\eta| \leq c_2}} \left| \partial_\eta^\alpha \left(c_p^j - S_{p-N} h_j^* \right)(y,\eta) \right| \leq C_k 2^{-p\varepsilon_j} M_\tau^m(h; k+j+1) \mathcal{F}_k(m_0, \left\|\kappa'\right\|_{C^\rho})$$

with $\varepsilon_j = \min(\tau, \rho - j)$.

Then, Lemma 3.12 implies that in $\kappa_g^* T_h u$ the replacement of $C_p^j u$ with

$$D_j u(y) := \int e^{i(y-z)\eta} (S_{p-N} h_j^*)(y,\eta)[u_p \circ \kappa]_p(z) \mathrm{d}z \mathrm{d}\eta$$

leaves a controllable remainder of order $m - j - \varepsilon_j \leq m - \varepsilon$.

Step 6. Summing up, we obtain

$$\kappa_g^* T_h u \sim \sum_{j=0}^{[\rho]} \sum_p \left[D_p^j u \right]_p = \sum_{j=0}^{[\rho]} \sum_p \left[(S_{p-N} h_j^*)(x,D)[u_p \circ \kappa]_p \right]_p.$$

Now, notice that if in the definition of $\kappa_g^* T_h u$ in (3.28) we had chosen instead of $[\cdot]_p$ a larger piece $[\cdot]_p'$ corresponding to $\overline{N} \gg \widetilde{N}$ (remark that such a replacement is controllable according to Lemma 3.9 and Lemma 3.11) we would have obtained

$$\kappa_g^* T_h u = \sum_{j=0}^{[\rho]} \sum_p \left[(S_{p-N} h_j^*)(x,D)[u_p \circ \kappa]_p \right]_p' = \sum_{j=0}^{[\rho]} \sum_p \sum_{|k-p| \leq \overline{N}} \Delta_k \left(S_{p-N} h_j^*)(x,D)[u_p \circ \kappa]_p \right).$$

The spectrum of $(S_{p-N} h_j^*)(x,D)[u_p \circ \kappa]_p$ is contained in the annulus

$$\left\{\xi \in \mathbf{R}^d : 2^{p-M} \le |\xi| \le 2^{p+M}\right\}$$

for some $M = M(\widetilde{N}, N) > 0$. Therefore, if we choose $\overline{N} \gg M(\widetilde{N}, N)$ then

$$\Delta_k\left(S_{p-N}h_j^*)(x, D)[u_p \circ \kappa]_p\right) = 0 \text{ if } |k-p| > \overline{N},$$

and thus

$$\kappa_g^* T_h u = \sum_{j=0}^{[\rho]} \sum_p (S_{p-N}h_j^*)(x, D)[u_p \circ \kappa]_p. \tag{3.30}$$

Finally, we write for $0 \le j \le [\rho]$

$$\begin{aligned} T_{h_j^*}\kappa^* u &= \sum_p (S_{p-N}h_j^*)(x, D)\Delta_p \sum_q [u_q \circ \kappa]_q \\ &= \sum_p (S_{p-N}h_j^*)(x, D)\Delta_p \sum_q \sum_{\substack{k \\ |k-q|\le \widetilde{N}}} \Delta_k(u_q \circ \kappa) \\ &= \sum_{\substack{p \\ k,q:|k-p|\le N_0, |k-q|\le \widetilde{N}}} (S_{p-N}h_j^*)(x, D)\Delta_p \Delta_k(u_q \circ \kappa). \end{aligned}$$

In the preceding expression, the replacement of $(S_{p-N}h_j^*)(x, D)$ with $(S_{q-N}h_j^*)(x, D)$ leaves a controllable remainder, so

$$\begin{aligned} T_{h_j^*}\kappa^* u &\sim \sum_{\substack{k \\ |p-k|\le N_0 \\ |q-k|\le \widetilde{N}}} (S_{q-N}h_j^*)(x, D)\Delta_p \Delta_k(u_q \circ \kappa) = \sum_{\substack{k,q \\ |q-k|\le \widetilde{N}}} (S_{q-N}h_j^*)(x, D)\Delta_k(u_q \circ \kappa) \\ &= \sum_q (S_{q-N}h_j^*)(x, D)[u_q \circ \kappa]_q. \end{aligned}$$

Therefore, we conclude in view of (3.30) that $\kappa_g^* T_h u \sim \sum_{j=0}^{[\rho]} T_{h_j^*}\kappa^* u$.

4. The semi-classical Strichartz estimate

4.1. Para-change of variable

First of all, let us recall the symmetrization of (1.2) into a paradifferential equation as performed in [15]. This symmetrization requires the introduction of the following symbols:

- $\gamma = \left(1 + (\partial_x \eta)^2\right)^{-\frac{3}{4}} |\xi|^{\frac{3}{2}}$,
- $\omega = -\frac{i}{2}\partial_x \partial_\xi \gamma$,
- $q = \left(1 + (\partial_x \eta)^2\right)^{-\frac{1}{2}}$,
- $p = \left(1 + (\partial_x \eta)^2\right)^{-\frac{5}{4}} |\xi|^{\frac{1}{2}} + p^{(-\frac{1}{2})}$, where $p^{(-\frac{1}{2})} = F(\partial_x \eta, \xi)\partial_x^2 \eta$, $F \in C^\infty(\mathbf{R} \times \mathbf{R} \setminus \{0\}; \mathbf{C})$ is homogeneous of order $-1/2$ in ξ.

Theorem 4.1. *([15, Proposition 4.1]) Assume that (η, ψ) is a solution to (1.2) and satisfies*

$$\begin{cases} (\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})) \cap L^4([0, T]; W^{r+\frac{1}{2},\infty}(\mathbf{R}) \times W^{r,\infty}(\mathbf{R})), \\ s > r > \dfrac{3}{2} + \dfrac{1}{2}. \end{cases} \tag{4.1}$$

Define

$$U := \psi - T_B\eta, \quad \Phi = T_p\eta + T_q U,$$

then Φ *solves the problem*

$$\partial_t\Phi + T_V\partial_x\Phi + iT_\gamma\Phi = f \tag{4.2}$$

where there exists a nondecreasing function $\mathcal{F}: \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$*, independent of* (η, ψ) *such that for a.e.* $t \in [0, T]$*,*

$$\|f(t)\|_{H^s} \le \mathcal{F}\left(\|(\eta(t), \psi(t))\|_{H^{s+\frac{1}{2}}\times H^s}\right)\left(1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|\psi(t)\|_{W^{r,\infty}}\right). \tag{4.3}$$

We assume throughout this section that (η, ψ) is a solution to (1.2) with regularity (4.1). We shall apply our results on the paracomposition in the preceding section to reduce further equation (4.2) by adapting the method in [2]. Define for every $(t, x) \in [0, T] \times \mathbf{R}$

$$\chi(t, x) = \int_0^x \sqrt{1 + \left(\partial_y\eta(t, y)\right)^2}\mathrm{d}y.$$

For each $t \in [0, T]$, the mapping $x \mapsto \chi(t, x)$ is a diffeomorphism from $\mathbf{R}$ to itself. Introduce then for each $t \in I$ the inverse $\kappa(t)$ of $\chi(t)$.

Concerning the underlying dyadic partitions, we shall write

$$\eta(t),\ \psi(t): \mathbf{R}_2 \to \mathbf{R}_1, \quad \kappa(t): \mathbf{R}_1 \to \mathbf{R}_2,$$

where, $\mathbf{R}_2$ is equipped with the dyadic partition (2.4) of size $n = 0$ and $\mathbf{R}_1$ is equipped with the one of size $n = n_0$ as in Proposition 2.9: $n_0 = \mathcal{F}_1(m_0, \left\|\kappa'\right\|_{L^\infty})$. Since

$$\kappa'(x) = \frac{1}{(\partial_x\chi)\circ\kappa} = \frac{1}{\sqrt{1 + (\partial_x\eta)\circ\kappa(x))^2}},$$

we get

$$m_0 := \left(1 + \|\partial_x\eta\|^2_{L^\infty_t L^\infty_x}\right)^{-1/2} \le \kappa'(x) \le 1, \quad \forall x \in \mathbf{R}. \tag{4.4}$$

Therefore, up to a multiplicative constant of the form $\mathcal{F}(\|\partial_x\eta\|_{L^\infty_t L^\infty_x})$ we will not distinguish between $\mathbf{R}_1$ and $\mathbf{R}_2$ in the rest of this article.

As mentioned in the introduction of our paracomposition results, we shall consider the linearized part of κ_g^* as a new definition of paracomposition. More precisely, we set

$$u = \kappa^*\Phi := \Phi\circ\kappa - \dot{T}_{(\partial_x\Phi)\circ\kappa}\kappa, \tag{4.5}$$

where, for any function $g: I \times \mathbf{R}_2 \to \mathbf{C}$ we have denoted

$$(g\circ\kappa)(t, x) = g(t, \kappa(t, x)), \quad \forall (t, x) \in I \times \mathbf{R}_1.$$

Let us first gather various estimates that will be used frequently in the sequel. To be concise, we denote

$$\mathcal{N} = \mathcal{F}\big(\|(\eta, \psi)\|_{L^\infty_t(H_x^{s+\frac{1}{2}}\times H^s_x)}\big)$$

where $\mathcal{F}$ is nondecreasing in each argument, independent of η, ψ and $\mathcal{F}$ may change from line to line.

Lemma 4.2. *The following estimates hold*

1. $\|\Phi\|_{L^\infty_t H^s_x} \le \mathcal{N}$,
2. $\|\Phi(t)\|_{C^r_{*,x}} \le \mathcal{N}\left(1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|\psi(t)\|_{W^{r,\infty}}\right)$,
3. $\|\partial_x\chi - 1\|_{L^\infty_t H_x^{s-\frac{1}{2}}} \le \mathcal{N}$,
4. $\|\partial_t\chi\|_{L^\infty_{t,x}} \le \mathcal{N}$,
5. $\|\partial_t\kappa\|_{L^\infty_{t,x}} \le \mathcal{N}$,

6. $\|\partial_x\kappa\|_{L_t^\infty W_x^{(s-1)_-,\infty}} \le \mathcal{N}$,
7. $\|\partial_x\kappa - 1\|_{L_t^\infty H_x^{s-\frac{1}{2}}} \le \mathcal{N}$,
8. $\|\partial_t\partial_x\chi(t)\|_{L_x^\infty} \le \mathcal{N}\left(1 + \|\psi(t)\|_{C_*^r}\right)$.

Proof. The estimates 1., 2., 3. can be deduced straightforwardly from the definition of Φ and the regularity of (η, ψ) given in (4.1).

4. By definition of χ,

$$\partial_t\chi(t,x) = \int_0^x \partial_t\partial_y\eta(t,y)\partial_y\eta(t,y)\left(1 + \left(\partial_y\eta(t,y)\right)^2\right)^{-\frac{1}{2}}\mathrm{d}y \tag{4.6}$$

so by Hölder's inequality we get

$$|\partial_t\chi(t,x)| \le \|\partial_x\partial_t\eta(t)\|_{L_x^2}\left\|F_0\left(\partial_y\eta(t)\right)\right\|_{L_x^2}$$

where $F_0(z) = \frac{z}{\sqrt{z^2+1}}$. Using (A.23) and Sobolev's embedding gives $\left\|F_0\left(\partial_y\eta(t)\right)\right\|_{L_x^2} \le \mathcal{N}$. On the other hand, using the first equation in (1.2) and the fact that $s > 2$ we obtain

$$\|\partial_x\partial_t\eta(t)\|_{L_x^2} \le \|G(\eta)\psi(t)\|_{H_x^1} \le \|G(\eta)\psi(t)\|_{H_x^{s-1}} \le \mathcal{N},$$

where we have used in the last inequality the continuity of the Dirichlet–Neumann operator from H^{s-1} to H^s (see Theorem 3.12, [3]).

5. This follows from 4. by using the formula $\partial_t\kappa = -\frac{\partial_t\chi}{\partial_x\chi}\circ\kappa$ and noticing that $\partial_x\chi \ge 1$.

6. With $F(z) = \frac{1}{\sqrt{1+z^2}} - 1$ and $G := F\circ(\partial_x\eta)$ we have

$$\partial_x\kappa = \frac{1}{(\partial_x\chi)\circ\kappa} = 1 + F\circ(\partial_x\eta)\circ\kappa = 1 + G\circ\kappa \tag{4.7}$$

From 3. and Sobolev's embedding, $\partial_x\eta \in L_t^\infty C_*^{s-1} \subset L_t^\infty W_x^{(s-1)_-,\infty}$ (recall Notation 3.2 2. for the notation μ_-). This together with the fact that $F \in C_b^\infty(\mathbf{R})$ implies $G \in L_t^\infty W_x^{(s-1)_-,\infty}$ and

$$\|G\|_{L_t^\infty W_x^{(s-1)_-,\infty}} \le \mathcal{N}. \tag{4.8}$$

Then, bootstrapping the recurrence relation (4.7) we deduce that $\partial_x\kappa \in L_t^\infty W_x^{[(s-1)_-],\infty}$ and

$$\|\partial_x\kappa\|_{L_t^\infty W_x^{[(s-1)_-],\infty}} \le \mathcal{N}. \tag{4.9}$$

Now, set $\mu = (s-1)_- - [(s-1)_-] \in (0,1)$. Again, by (4.7)

$$\partial_x^{[(s-1)_-]}(\partial_x\kappa) = \partial_x^{[(s-1)_-]}(G\circ\kappa) \tag{4.10}$$

is a finite combination of terms of the form

$$A = [(\partial^q G)\circ\kappa]\prod_{j=1}^m \partial_x^{\gamma_j}\kappa, \quad 1 \le q \le [(s-1)_-],\ \gamma_j \ge 1,\ \sum_{j=1}^m \gamma_j = [(s-1)_-]. \tag{4.11}$$

Using (4.9) and (4.8) it follows easily that A belongs to $W^{\mu,\infty}(\mathbf{R}^d)$ with norm bounded by $\mathcal{N}$ and thus 6. is proved.

7. First, the nonlinear estimate (A.23) implies that $G = F\circ\partial_x\eta$ defined in the proof of 6. satisfies

$$\|G\|_{L_t^\infty H_x^{s-\frac{1}{2}}} \le \mathcal{N}. \tag{4.12}$$

Then changing the variables $x \mapsto \chi(x)$ in (4.7) gives

$$\|\partial_x\kappa - 1\|_{L_t^\infty L_x^2} \le \|G\|_{L_t^\infty L_x^2}\left\|\chi'\right\|_{L_{t,x}^\infty}^{\frac{1}{2}} \le \mathcal{N}.$$

Now using (4.7), (4.9) and induction we arrive at

$$\|\partial_x\kappa - 1\|_{L^\infty_t H^{[(s-1)_-]}_x} \le \mathcal{N}. \tag{4.13}$$

Next, set $\mu = (s - \frac{1}{2}) - [(s-1)_-] \in [\frac{1}{2}, \frac{1}{2} + \varepsilon]$, ε arbitrarily small (so that $\mu \in [\frac{1}{2}, 1)$). To obtain 7. we are left with the estimate for the H^μ-norm of $\partial_x^{[(s-1)_-]}\partial_x\kappa$. This amounts to bound

$$\iint\limits_{\mathbf{R}^2} \frac{|\partial_x^{[(s-1)_-]}(G\circ\kappa)(x) - \partial_x^{[(s-1)_-]}(G\circ\kappa)(y)|^2}{|x-y|^{1+2\mu}} \mathrm{d}x\mathrm{d}y \tag{4.14}$$

where $\partial_x^{[(s-1)_-]}(G\circ\kappa)$ is a finite linear combination of terms of the form A in (4.11). Inserting A into (4.14) one estimates successively the difference of each factor in A under the double integral while the others are estimated in L^∞-norm. This is done using (4.12), (4.13) for Sobolev-norm estimates and (4.8), (4.9) for Hölder-norm estimates.

8. By definition of χ, it holds with $F_0(z) = \frac{z}{\sqrt{1+z^2}}$

$$\partial_t\partial_x\chi(t,x) = F_0(\partial_x\eta)\partial_x\partial_t\eta = F_0(\partial_x\eta)\partial_x G(\eta)\psi.$$

Then, applying the Hölder estimate for the Dirichlet–Neumann operator in Proposition 2.21, [15] leads to

$$\|\partial_x G(\eta)\psi\|_{L^\infty} \le \|\partial_x G(\eta)\psi\|_{C^{r-2}_*} \le \mathcal{N}\left(1 + \|\psi(t)\|_{C^r_*}\right),$$

hence 8. □

The main task here is to apply Theorem 3.5 and Theorem 3.6 to convert the highest order paradifferential operator T_γ to the Fourier multiplier $|D_x|^{\frac{3}{2}}$.

Proposition 4.3. *The function u defined by (4.5) satisfies the equation*

$$\left(\partial_t + T_W\partial_x + i|D_x|^{\frac{3}{2}}\right)u = f \tag{4.15}$$

where

$$W = (V\circ\kappa)(\partial_x\chi\circ\kappa) + \partial_t\chi\circ\kappa \tag{4.16}$$

and for a.e. $t\in[0,T]$,

$$\|f(t)\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}\Big(\|(\eta,\psi)\|_{L^\infty_t(H^{s+\frac{1}{2}}_x\times H^s_x)}\Big)\Big(1 + \|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|\psi(t)\|_{W^{r,\infty}}\Big). \tag{4.17}$$

Proof. We proceed in 4 steps. We shall say that A is controllable if for a.e. $t\in[0,T]$, $\|A(t)\|_{H^{s-\frac{1}{2}}}$ is bounded by the right-hand side of (4.17) denoted by RHS.

Step 1. Let us first prove that for some controllable remainder R_1,

$$\kappa^*(\partial_t\Phi) = \big(\partial_t + T_{(\partial_t\chi)\circ\kappa}\partial_x\big)u + R_1. \tag{4.18}$$

By definition of κ^* we have

$$\kappa^*(\partial_t\Phi) = \partial_t\Phi\circ\kappa - \dot{T}_{(\partial_x\partial_t\Phi)\circ\kappa}\kappa = \partial_t(\Phi\circ\kappa) - (\partial_x\Phi\circ k)\partial_t\kappa - \dot{T}_{(\partial_x\partial_t\Phi)\circ\kappa}\kappa.$$

Therefore,

$$\kappa^*(\partial_t\Phi) = \partial_t(\kappa^*\Phi) + A_1 + A_2 \tag{4.19}$$
$$A_1 = \dot{T}_{(\partial_x^2\Phi\circ\kappa)\partial_t\kappa}\kappa, \quad A_2 = \dot{T}_{(\partial_x\Phi)\circ\kappa}\partial_t\kappa - (\partial_x\Phi\circ\kappa)\partial_t\kappa.$$

1. Since the truncated paradifferential operator $\dot{T}_{(\partial_x^2\Phi\circ\kappa)\partial_t\kappa}\kappa$ involves only the high frequency part of κ we have

$$\|A_1\|_{H^{s+\frac{1}{2}}_x} \le \mathcal{N}\left\|(\partial_x^2\Phi\circ\kappa)\partial_t\kappa\right\|_{L^\infty_x}\left\|\partial_x^2\kappa\right\|_{H^{s-\frac{3}{2}}}. \tag{4.20}$$

From Lemma 4.2 2., 5. there holds

$$\left\|(\partial_x^2\Phi\circ\kappa)\partial_t\kappa\right\|_{L^\infty_x}\le\mathcal{N}\left(1+\|\eta(t)\|_{W^{r+\frac12,\infty}}+\|\psi(t)\|_{W^{r,\infty}}\right).$$

On the other hand, Lemma 4.2 7. gives $\left\|\partial_x^2\kappa\right\|_{H^{s-\frac32}}\le\mathcal{N}$, hence A_1 is controllable.

2. To study A_2, one uses $\partial_t\kappa=-ab$ with $a=(\partial_t\chi)\circ\kappa,\ b=\partial_x\kappa$. Set $c=(\partial_x\Phi)\circ\kappa$ then

$$\partial_x(\kappa^*\Phi)=bc-\dot T_c b-\dot T_{\partial_x c}\kappa,$$

hence

$$\begin{aligned}A_2&=-\dot T_c(ab)+abc=\dot T_{ab}c+\dot R(c,ab)=\dot T_a\dot T_b c+R_2+\dot R(c,ab)\\&=\dot T_a(bc-\dot T_c b)-\dot T_a\dot R(b,c)+R_2+\dot R(c,ab)\\&=\dot T_a(\partial_x(\kappa^*\Phi))+\dot T_a\dot T_{\partial_x c}\kappa-\dot T_a\dot R(b,c)+R_2+\dot R(c,ab)\end{aligned}$$

where $R_2=\dot T_{ab}c-\dot T_a\dot T_b c$.

(i) The symbolic calculus Theorem A.5 implies for a.e. $t\in[0,T]$

$$\|R_2(t)\|_{H^s}\le K\left(\|a(t)\|_{W^{1,\infty}}\|b(t)\|_{L^\infty}+\|a(t)\|_{L^\infty}\|b(t)\|_{W^{1,\infty}}\right)\|c(t)\|_{H^{s-1}}.$$

Now, from Lemma 4.2 6. and the fact that $s-1>1$ one gets $\|b\|_{L^\infty_t W^{1,\infty}_x}\le\mathcal{N}$. On the other hand, Lemma 4.2 4., 8. gives, respectively,

$$\|a(t)\|_{L^\infty}\le\mathcal{N},\quad \|a(t)\|_{W^{1,\infty}}\le\mathrm{RHS}.$$

Applying Lemma 3.2 in [1] and Lemma 4.2 1., 6. yields

$$\|c(t)\|_{L^\infty_t H^{s-1}_x}\le\mathcal{N}.\tag{4.21}$$

Therefore, $\|R_2(t)\|_{H^s}$ is controllable.

(ii) In view of Lemma 4.2 2., 4., 7. the term $\dot T_a\dot T_{\partial_x c}\kappa$ can be bounded as

$$\left\|\left(\dot T_a\dot T_{\partial_x c}\kappa\right)(t)\right\|_{H^s}\le\mathcal{N}\|a(t)\|_{L^\infty}\|\partial_x c(t)\|_{L^\infty}\left\|\partial_x^2\kappa(t)\right\|_{H^{s-2}}\le\mathrm{RHS}.$$

(iii) The estimate 7. in Lemma 4.2 and Sobolev's embedding imply that $\|b\|_{L^\infty_t C^{s-1}_*}\le\mathcal{N}$. Then according to (A.14) and the fact that $s>2$ we obtain

$$\left\|\dot T_a\dot R(b,c)(t)\right\|_{H^s}\le\mathcal{N}\|a(t)\|_{L^\infty}\|b(t)\|_{C^{s-1}_*}\|c(t)\|_{H^{s-1}}\lesssim\mathcal{N}.$$

By the same argument, to estimate $\|\dot R(ab,c)(t)\|_{H^s}$ it remains to bound $\|(ab)(t)\|_{C^1_*}$ which is in turn bounded by $\|(ab)(t)\|_{W^{1,\infty}}$. From Lemma 4.2 1. and 4. we have

$$\|a(t)\|_{L^\infty}+\|b(t)\|_{L^\infty}\le\mathcal{N}.$$

On the other hand, the estimate 6. (or 7.) of that lemma gives $\|\partial_x b\|_{L^\infty}\le\mathcal{N}$. Finally, we write $\partial_x a=[(\partial_t\partial_x\chi)\circ\kappa]\partial_x\kappa$ and use Lemma 4.2 8. to get $\|\partial_x a\|_{L^\infty}\le\mathrm{RHS}$.

We have proved that modulo a controllable remainder, $A_2=\dot T_{\partial_t\chi\circ\kappa}u$. Consequently, modulo a controllable remainder, $A_2=T_{\partial_t\chi\circ\kappa}u$. Then putting together this and (4.19), (4.20) we end up with the claim (4.18).

Step 2. With the definitions of R_{line} and R_{conj} in Theorem 3.5 and Theorem 3.6 we write for any $h\in\Gamma^m_\tau$

$$\kappa^*T_h\Phi=T_{h^*}\kappa^*\Phi-\mathrm{R}_{line}(T_h\Phi)+T_{h^*}\mathrm{R}_{line}\Phi+\mathrm{R}_{conj}\Phi.\tag{4.22}$$

It follows from Lemma 4.2 7. that

$$\|\partial_x\kappa-1\|_{L^\infty_t C^{s-1}_*}\le\|\partial_x\kappa-1\|_{L^\infty_t H^{s-\frac12}_x}\le\mathcal{N}.$$

Therefore, κ satisfies condition (3.8) with

$$\rho=1,\ r_1=s-\frac12,\ \alpha_0=2\tag{4.23}$$

where we have changed the notation in (3.8): $\partial_x^{\alpha_0}\kappa \in H^{r_1+1-|\alpha_0|}$ to avoid the r used in (4.1) for the Hölder regularity of ψ. On the other hand, we have seen from (4.4) that $\kappa' \geq m_0$ and thus the Assumptions I, II on κ are fulfilled.

For the transport term, the symbol is $h(x,\xi) = i\xi V(x)$.

(i) Now one can apply Theorem 3.6 with $\tau = \rho = 1$ (hence $\varepsilon = \min(\tau,\rho) = 1$) to have

$$h^*(x,\xi) = iV \circ \kappa(x)\frac{\xi}{\kappa'(x)} = i(V\circ\kappa)(\partial_x\chi\circ\kappa)\xi$$

and at a.e. $t \in [0,T]$,

$$\left\|\mathrm{R}_{conj}\Phi\right\|_{H^s} \leq \mathcal{F}(m_0, \|\kappa'\|_{C_*^\rho})M_1^1(h;k_0)\big(1 + \left\|\partial^2\kappa\right\|_{H^{s-\frac{3}{2}}}\big)\left\|\Phi\right\|_{H^s}.$$

Regarding the right-hand side, we bound

$$\|\kappa'\|_{C_*^\rho} + \|u\|_{H^s} + \left\|\partial^2\kappa\right\|_{H^{s-\frac{3}{2}}} \leq \mathcal{N}, \quad M_1^1(h;k_0) \leq \mathrm{RHS}$$

hence,

$$\left\|\mathrm{R}_{conj}\Phi(t)\right\|_{H^s} \leq \mathrm{RHS}.$$

(ii) The term $T_{h^*}\mathrm{R}_{line}\Phi$ is bounded as

$$\|T_{h^*}\mathrm{R}_{line}\Phi(t)\|_{H^s} \leq M_0^1(h^*)\left\|\mathrm{R}_{line}\Phi(t)\right\|_{H^{s+1}}$$

where $M_0^1(h^*) \leq \mathcal{N}$. Applying Theorem 3.5 (ii) with $\Phi(t) \in C_*^2$, $\sigma = r$, $\varepsilon = \min(\sigma-1, 1+\rho)_- \geq 1$ we have

$$\tilde{s} = \min(s+\rho, r_1+1+\varepsilon) = \min(s+1, s-\frac{1}{2}+1+\varepsilon) = s+1,$$
$$\|\mathrm{R}_{line}\Phi(t)\|_{H^{s+1}} \leq \mathcal{F}(m_0, \|\kappa'\|_{C_*^\rho})\big(1+\left\|\partial_x^2\kappa\right\|_{H^{s-\frac{3}{2}}}\big)\big(\left\|\Phi'(t)\right\|_{H^{s-1}} + \|\Phi(t)\|_{C_*^\sigma}\big) \leq \mathrm{RHS}.$$

(In the last inequality, we have used Lemma 4.2 1., 2.)

Therefore

$$\|T_{h^*}\mathrm{R}_{line}\Phi(t)\|_{H^s} \leq \mathrm{RHS}. \tag{4.24}$$

In (4.22) we are left with the estimate for $\mathrm{R}_{line}(T_h\Phi)$. Notice that since $M_0^1(h) \leq \mathcal{N}$, with $v = T_h\Phi$ one has

$$\|v(t)\|_{H^{s-1}} \leq \mathcal{N}, \quad \|v(t)\|_{C_*^{r-1}} \leq \mathrm{RHS}.$$

Then, by virtue of Theorem 3.5 (ii) applied to v and $\sigma = r-1$, $\varepsilon = \min(r-2, 2)_-$ we have

$$\tilde{s} = \min(s+1, s-\frac{1}{2}+1+\varepsilon) > s+\frac{1}{2},$$
$$\|\mathrm{R}_{line}v\|_{H^{s+\frac{1}{2}}} \leq \mathrm{RHS}.$$

Summing up, we conclude from (4.22) that

$$\kappa^*T_h\Phi = T_{h^*}\kappa^*\Phi + R_2, \quad \|R_2(t)\|_{H^s} \leq \mathrm{RHS}.$$

Step 3. We now conjugate the highest order term $T_\gamma\Phi$ with κ^*. This is the point where we really need Theorem 3.5 (i) for non-C^1 functions. Recall the formula (4.22) and the verifications of Assumptions I, II given by (4.23) and (4.4). With $c_0 = (1+(\partial_x\eta))^{-1/2}$, we have that $\gamma = c_0|\xi|^{3/2}$ satisfies $M_1^{\frac{3}{2}}(\gamma) \leq \mathcal{N}$. Theorem 3.6 applied with $m = 3/2$, $\tau = 1$ then yields

$$\gamma^*(x,\xi) = \gamma(\kappa(x), \frac{\xi}{\kappa'(x)}) = (c_0\circ\kappa)(x)\frac{|\xi|^{\frac{3}{2}}}{\kappa'(x)} = |\xi|^{\frac{3}{2}}$$

for $1/\kappa'(x) = (\chi'\circ\kappa)(x) = (c_0\circ\kappa)(x)$; and (at a.e. $t \in [0,T]$)

$$\left\|\mathrm{R}_{conj}\Phi\right\|_{H^{s-\frac{3}{2}+1}} \leq \mathcal{F}(m_0, \|\kappa'\|_{C_*^\rho})M_1^{\frac{3}{2}}(\gamma;k_0)\left(1+\left\|\partial^2\kappa\right\|_{H^{s-\frac{3}{2}}}\right)\|\Phi\|_{H^s} \leq \mathcal{N}.$$

The term $T_{\gamma^*}\mathrm{R}_{line}\Phi(t)$ is estimated exactly as in (4.24) noticing that γ^* now is of order 3/2 we get

$$\|T_{\gamma^*}\mathrm{R}_{line}\Phi(t)\|_{H^{s-\frac{1}{2}}} \le \mathrm{RHS}.$$

Consider the remaining term $\mathrm{R}_{line}T_\gamma\Phi(t)$. Since $T_\gamma\Phi(t)$ belongs to $C_*^{r-\frac{3}{2}}$ and $r-\frac{3}{2}$ can be smaller than 1, we have to use in this case Theorem 3.5 (i):

$$\sigma = \frac{1}{2},\ \rho+\sigma = \frac{3}{2} > 1,\ \tilde{s} = \min((s-\frac{3}{2})+1, (s-\frac{1}{2})+\frac{1}{2}) = s-\frac{1}{2},$$
$$\left\|\mathrm{R}_{line}T_\gamma\Phi(t)\right\|_{H^{s-\frac{1}{2}}} \le \mathcal{F}(m_0, \|\kappa'\|_{C_*^\rho})\big(1+\left\|\partial_x^2\kappa\right\|_{H^{s-\frac{3}{2}}}\big)\big(\left\|T_\gamma\Phi(t)\right\|_{H^{s-\frac{3}{2}}}+\left\|T_\gamma\Phi(t)\right\|_{C_*^\sigma}\big).$$

We conclude in this step that

$$\kappa^*T_\gamma\Phi = |D_x|^{\frac{3}{2}}\kappa^*\Phi + R_3, \quad \|R_3(t)\|_{H^{s-\frac{1}{2}}} \le \mathrm{RHS}.$$

Step 4. Since $\omega \in \Gamma_0^{\frac{1}{2}}$ with the semi-norms bounded by $\mathcal{N}$, one gets by virtue of Theorem 3.4 and Theorem 3.5 (ii)

$$\left\|\kappa^*T_\omega\Phi(t)\right\|_{H^{s-\frac{1}{2}}} \le \mathcal{N}.$$

Similarly, $f(t) \in H^s \hookrightarrow C_*^{s-\frac{1}{2}}$ with $s-\frac{1}{2} > \frac{3}{2}$ we also have

$$\left\|\kappa^*f(t)\right\|_{H^{s-\frac{1}{2}}} \le \mathrm{RHS}.$$

Putting together the results in the previous steps, we conclude the proof of Proposition 4.3. □

Remark 4.4. In fact, in the above proof, we have proved that

$$\kappa^*(\partial_t + T_V\partial_x)\Phi(t) = (\partial_t + T_W\partial_x)\kappa^*\Phi(t) + f_1(t)$$

with

$$\|f_1(t)\|_{H^s} \le \mathcal{N}\left(1+\|\eta(t)\|_{W^{r+\frac{1}{2},\infty}} + \|\psi(t)\|_{W^{r,\infty}}\right).$$

The loss of $\frac{1}{2}$ derivative only occurred in Step 3 and Step 4 when conjugating κ^* with $T_\gamma\Phi$ and T_ω, where in Step 3 we applied Theorem 3.6 with $\rho = 1, \tau = \frac{3}{2}$ and thus $\varepsilon = 1$. The reason is that we want to keep the right-hand side of (4.17) to be tame. On the other hand, if we apply the mentioned theorem with $\rho = \frac{3}{2}$ then it follows that

$$\kappa^*T_\gamma\Phi = |D_x|^{\frac{3}{2}}\kappa^*\Phi + R_3$$

with

$$\|R_3(t)\|_{H^s} \le \mathcal{F}_1\Big(\|(\eta,\psi)\|_{L_t^\infty(H^{s+\frac{1}{2}}\times H_x^s)}\Big)\mathcal{F}_2\Big(\|(\eta(t),\psi(t))\|_{W^{r+\frac{1}{2},\infty}\times W^{r,\infty}}\Big).$$

If we assume more regularity: $s > 2+\frac{1}{2}$ then by Sobolev's embedding $\|R_3(t)\|_{H^s} \le \mathcal{N}$ and thus we recover Proposition 3.3, [1] (after performing in addition another change of variables to suppress the $\frac{1}{2}$-order terms).

In the next paragraphs, we shall prove Strichartz estimates for u solution to (4.15). To have an independent result, let us restate the problem as follows. Let $I = [0, T]$, $s_0 \in \mathbf{R}$ and

$$\begin{aligned} &W \in L^\infty([0,T]; L^\infty(\mathbf{R})) \cap L^4([0,T]; W^{1,\infty}(\mathbf{R})),\\ &f \in L^4(I; H^{s_0-\frac{1}{2}}(\mathbf{R})). \end{aligned} \tag{4.25}$$

If $u \in L^\infty(I, H^{s_0}(\mathbf{R}))$ is a solution to the problem

$$\left(\partial_t + T_W\partial_x + i|D_x|^{\frac{3}{2}}\right)u = f \tag{4.26}$$

we shall derive the semi-classical Strichartz estimate for u (with a gain of $\frac{1}{4}-\varepsilon$ derivative). Remark that the same problem was considered in [2] at the following regularity level

$$W \in L^\infty([0,T]; H^{s-1}(\mathbf{R})), \quad f \in L^\infty(I; H^s(\mathbf{R})), \quad s > 2+\frac{1}{2}.$$

We shall in fact examine the proof in [2] to show that our regularity (4.25) is sufficient. It turns out that for the semi-classical Strichartz estimate, *the loss of* $\frac{1}{2}$ *derivative of the source term* f *is optimal.*

Recall that u is defined on $\mathbf{R}$ equipped with a dyadic partition of size n_0. Then as remarked before, up to a multiplicative constant of the form $\mathcal{F}(\|\partial_x\eta\|_{L^\infty_t L^\infty_x})$, which will appear in our final Strichartz estimate, we shall work as if $n_0 = 0$.

4.2. Frequency localization

In order to prove Strichartz estimates for equation (4.26), we will adapt the proof of Theorem 1.1 in [2]: microlocalize the solution into dyadic pieces using Littlewood–Paley theory and establish dispersive estimates for those dyadic pieces. The first step in realizing this strategy consists in conjugating (4.26) with the dyadic operator Δ_j to get the equation satisfied by $\Delta_j u$:

$$\Big(\partial_t + \frac{1}{2}(T_W\partial_x + \partial_x T_W) + i|D_x|^{\frac{3}{2}}\Big)\Delta_j u = \Delta_j f + \frac{1}{2}\Delta_j(T_{\partial_x W}u) + \frac{1}{2}\big([T_W,\Delta_j]\partial_x u + \partial_x[T_W,\Delta_j]u\big). \tag{4.27}$$

After localizing u at frequency 2^j one can replace the paradifferential operator T_W by the paraproduct with $S_{j-N}W$ as follows

Lemma 4.5. *([4, Lemma 4.9]) For all $j \geq 1$ and for some integer N, we have*

$$T_W\partial_x\Delta_j u = S_{j-N}W\partial_x\Delta_j u + R_j u$$
$$\partial_x T_W\Delta_j u == \partial_x S_{j-N}W\Delta_j u + R'_j u$$

where $R_j u$, $R'_j u$ have spectrum contained in an annulus $\{c_1 2^j \leq |\xi| \leq c_2 2^j\}$ and satisfy the following estimate for all $s_0 \in \mathbf{R}$:

$$\|R_j u\|_{H^{s_0}(\mathbf{R})} + \|R'_j u\|_{H^{s_0}(\mathbf{R}^d)} \leq C(s_0)\|W\|_{W^{1,\infty}(\mathbf{R}^d)}\|u\|_{H^{s_0}(\mathbf{R}^d)}.$$

From now on, we always consider the high frequency part of u, that is $\Delta_j u$ with $j \geq 1$. Combining (4.27) and Lemma 4.5 leads to

$$\Big(\partial_t + \frac{1}{2}(S_{j-N}W\partial_x + \partial_x S_{j-N}W) + i|D_x|^{\frac{3}{2}}\Big)\Delta_j u =$$
$$\Delta_j f + \frac{1}{2}\Delta_j(T_{\partial_x W}u) + \frac{1}{2}\big([T_W,\Delta_j]\partial_x u + \partial_x[T_W,\Delta_j]\big)u + R_j u + R'_j u. \tag{4.28}$$

Next, as in [6,29,4] we smooth out the symbols (see for instance Lemma 4.4, [4])

Definition 4.6. Let $\delta > 0$ and $U \in \mathcal{S}'(\mathbf{R})$. For any $j \in \mathbf{Z},\ j \geq -1$ we define

$$S_{\delta j}(U) = \psi(2^{-\delta j}D_x)U.$$

Let $\chi_0 \in C_0^\infty(\mathbf{R})$, supp $\chi \subset \{\frac{1}{4} \leq |\xi| \leq 4\}$, $\xi = 1$ in $\{\frac{1}{2} \leq |\xi| \leq 2\}$. Define

$$\begin{cases} a(\xi) = \chi_0(\xi)|\xi|^{\frac{3}{2}},\ h = 2^{-j}, \\ \mathcal{L}_\delta = \partial_t + \frac{1}{2}(S_{(j-N)\delta}W\cdot\partial_x + \partial_x\cdot S_{\delta(j-N)}W) + i\chi_0(h\xi)|D_x|^{\frac{3}{2}}. \end{cases} \tag{4.29}$$

Using (4.28), we have

$$\mathcal{L}_\delta\Delta_j u = F_j, \quad \text{where} \tag{4.30}$$

$$F_j = \Delta_j f + \frac{1}{2}\Delta_j(T_{\partial_x W}u) + \frac{1}{2}\big([T_W, \Delta_j]\partial_x u + \partial_x[T_W, \Delta_j]u\big) + R_j u + R'_j u + \frac{1}{2}\Big\{\big(S_{(j-N)\delta}W - S_{(j-N)}W\big)\partial_x \Delta_j u + \partial_x\big(S_{(j-N)\delta}W - S_{(j-N)}W\big)\Delta_j u\Big\}. \tag{4.31}$$

4.3. Semi-classical parametrix and dispersive estimate

Recall that φ is the cut-off function employed to defined the dyadic partition of size $n = 0$ in paragraph 2.1. To simplify the presentation, let us rescale the existence time to $T = 1$ and set $h = 2^{-j}$, $j \geq 1$,

$$E_0 = L^\infty([0, T]; L^\infty(\mathbf{R})), \quad E_1 = L^4([0, T]; W^{1,\infty}(\mathbf{R})).$$

The main result of this paragraph is the following semi-classical dispersive estimate for the operator $\mathcal{L}_\delta$.

Theorem 4.7. *Let $\delta < \frac{1}{2}$ and $t_0 \in \mathbf{R}$. For any $u_0 \in L^1(\mathbf{R}^d)$ set $u_{0,h} = \varphi(hD_x)u_0$. Denote by $S(t, t_0)u_{0,h}$ solution of the problem*

$$\mathcal{L}_\delta u_h(t, x) = 0, \quad u_h(t_0, x) = u_{0,h}(x).$$

Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|S(t, t_0)u_{0,h}\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\|W\|_{E_0})\, h^{-\frac{1}{4}}|t - t_0|^{-\frac{1}{2}}\|u_{0,h}\|_{L^1(\mathbf{R}^d)} \tag{4.32}$$

for all $0 < |t - t_0| \leq h^{\frac{1}{2}}$ and $0 < h \leq 1$.

We make the change of temporal variables $t = h^{\frac{1}{2}}\sigma$ (inspired by [21]) and set

$$W_h(\sigma, x) = S_{(j-N)\delta}W(\sigma h^{\frac{1}{2}}, x), \tag{4.33}$$

and denote the obtained semi-classical pseudo-differential operator by

$$L_\delta = h\partial_\sigma + h^{\frac{1}{2}}W_h(h\partial_x) + \frac{1}{2}h\partial_x W_h + ia(hD_x). \tag{4.34}$$

For this new differential operator, we shall prove the corresponding (classical) dispersive estimate:

Theorem 4.8. *Let $\delta < \frac{1}{2}$ and $\sigma_0 \in [0, 1]$. For any $u_0 \in L^1(\mathbf{R}^d)$ and $u_{0,h} = \varphi(hD_x)u_0$, denote by $\widetilde{S}(\sigma, \sigma_0)u_{0,h}$ solution of the problem*

$$L_\delta U_h(\sigma, x) = 0, \quad U_h(\sigma_0, x) = u_{0,h}(x).$$

Then there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|\widetilde{S}(\sigma, \sigma_0)u_{0,h}\|_{L^\infty(\mathbf{R}^d)} \leq \mathcal{F}(\|W\|_{E_0})\, h^{-\frac{1}{2}}|\sigma - \sigma_0|^{-\frac{1}{2}}\|u_{0,h}\|_{L^1(\mathbf{R}^d)} \tag{4.35}$$

for all $\sigma \in [0, 1]$.

Theorem 4.8 will imply Theorem 4.7. Indeed, the relation

$$L_\delta u_h(\sigma, x) = h^{\frac{3}{2}}\mathcal{L}_\delta u_h(\sigma h^{\frac{1}{2}}, x),$$

yields

$$\widetilde{S}(\sigma, \sigma_0)u_{0,h}(x) = S(h^{\frac{1}{2}}\sigma, h^{\frac{1}{2}}\sigma_0)u_{0,h}(x).$$

If Theorem 4.8 were proved then via the relation $t = \sigma h^{\frac{1}{2}}$,

$$\begin{aligned}\|S(t, t_0)u_{0,h}\|_{L^\infty_x} &= \|\widetilde{S}(\sigma, \sigma_0)u_{0,h}\|_{L^\infty_x} \\ &\leq \mathcal{F}(\|W\|_{E_0})\, h^{-\frac{1}{2}}|\sigma - \sigma_0|^{-\frac{1}{2}}\|u_{0,h}\|_{L^1(\mathbf{R}^d)} \\ &\leq \mathcal{F}(\|W\|_{E_0})\, h^{-\frac{1}{4}}|t - t_0|^{-\frac{1}{2}}\|u_{0,h}\|_{L^1(\mathbf{R}^d)}\end{aligned}$$

which proves Theorem 4.7.

For the proof Theorem 4.8, we use the WKB method to construct a parametrix of the following integral form

$$\widetilde{U}_h(\sigma,x)=\frac{1}{2\pi h}\iint e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)}\widetilde{b}(\sigma,x,z,\xi,h)u_{0,h}(z)dzd\xi \tag{4.36}$$

where
(i) the phase φ satisfies $\varphi(\sigma=0)=x\xi$,
(ii) the amplitude $\widetilde{b}$ has the form

$$\widetilde{b}(\sigma,x,\xi,h)=b(\sigma,x,\xi,h)\zeta(x-z-\sigma a'(\xi)) \tag{4.37}$$

with $\zeta\in C_0^\infty(\mathbf{R}),\ \zeta(s)=1$ if $|s|\le 1$ and $\zeta(s)=0$ if $|s|\ge 2$.
We shall work with the following class of symbols.

Definition 4.9. For small h_0 to be fixed, we set

$$\mathcal{O}=\left\{(\sigma,x,\xi,h)\in\mathbf{R}^4: h\in(0,h_0), |\sigma|<1, 1<|\xi|<3\right\}.$$

If $m\in\mathbf{R}$ and $\rho\in\mathbf{R}^+$, we denote by $S^m_\rho(\mathcal{O})$ the set of all functions f defined on $\mathcal{O}$ which are C^∞ with respect to (σ,x,ξ) and satisfy

$$|\partial_x^\alpha f(\sigma,x,\xi,h)|\le C_\alpha h^{m-\alpha\rho},\quad \forall\alpha\in\mathbf{N},\ \forall(\sigma,x,\xi,h)\in\mathcal{O}.$$

Remark 4.10. Recall that

$$W_h(\sigma,x)=S_{(j-N)\delta}(W)(\sigma h^{\frac12},x)\equiv\phi(2^{-(j-N)\delta}D_x)W(\sigma h^{\frac12},x).$$

Hence, for any $\alpha\in\mathbf{N}$, there hold

$$\begin{aligned}&|\partial_x^\alpha W_h(\sigma,x)|\le C_\alpha h^{-\delta\alpha}\|W(\sigma h^{\frac12},\cdot)\|_{L^\infty},\\ &|\partial_x^{\alpha+1} W_h(\sigma,x)|\le C_\alpha h^{-\delta\alpha}\|W(\sigma h^{\frac12},\cdot)\|_{W^{1,\infty}}.\end{aligned} \tag{4.38}$$

The following result for transport problems is elementary.

Lemma 4.11. *If v is a solution of the problem*

$$(\partial_\sigma+m(\xi)\partial_x+if)\,v(\sigma,x,\xi)=g(\sigma,x,\xi),\quad u|_{\sigma=0}=z\in\mathbf{C},$$

where f be real-valued, then v satisfies

$$|v(\sigma,x,\xi)|\le|z|+\int_0^\sigma\left|g(\sigma',x+(\sigma'-\sigma)a'(\xi),\xi)\right|\mathrm{d}\sigma'.$$

The existence of the parametrix is given in the following Proposition.

Proposition 4.12. *There exists a phase φ of the form*

$$\varphi(\sigma,x,\xi,h)=x\xi-\sigma a(\xi)+h^{\frac12}\psi(\sigma,x,\xi,h)$$

with $\partial_x\psi\in S^0_\delta(\mathcal{O})$ and there exists a symbol $b\in S^0_\delta(\mathcal{O})$ such that with the amplitude $\widetilde{b}$ defined by (4.37), we have

$$L_\delta\left(e^{\frac{i}{h}\phi}\widetilde{b}\right)=e^{\frac{i}{h}\phi}r_h, \tag{4.39}$$

where for any $N\in\mathbf{N}$ there holds

$$\sup_{\sigma\in[0,1]}\left\|\iint e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)}r(\sigma,x,z,\xi,h)u_{0,h}(z)dzd\xi\right\|_{H^1(\mathbf{R}_x)}\le h^N\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}\|_{L^1(\mathbf{R})}. \tag{4.40}$$

Proof. We proceed in several steps.
Step 1. Construction of the phase φ.

We find φ under the form

$$\varphi(t,x,\xi,h) = x\xi - \sigma a(\xi) + h^{\frac{1}{2}}\psi(\sigma,x,,\xi,h) \tag{4.41}$$

where ψ solves the following transport problem

$$\begin{cases} \partial_\sigma \psi + a'(\xi)\partial_x\psi = -\xi W_h, \\ \psi|_{\sigma=0} = 0. \end{cases} \tag{4.42}$$

Denote $m^+ = \max\{m, 0\}$. Differentiating (4.42) with respect to x and ξ then using Lemma 4.11 together with (4.38) and Hölder's inequality we derive

$$|\partial_\xi^k \partial_x^\alpha \psi(\sigma,x,\xi,h)| \le C_{k\alpha}|\sigma|^{\frac{3}{4}} h^{-\delta(\alpha+k-1)^+}\|W\|_{L^4([0,T],W_x^{1,\infty})}, \tag{4.43}$$

for $(\alpha,k)\in \mathbf{N}^2$ and $(\sigma,x,\xi,h)\in\mathcal{O}$.

Remark that in [2] where $W\in L^\infty([0,T],W^{1,\infty}(\mathbf{R}))$, one has the better estimate

$$|\partial_\xi^k \partial_x^\alpha \psi(\sigma,x,\xi,h)| \le C_{k\alpha}|\sigma| h^{-\delta(\alpha+k-1)^+}\|W\|_{L^\infty([0,T],W_x^{1,\infty})}. \tag{4.44}$$

However, (4.43) is enough to get $\partial_x\psi\in S^0_\delta(\mathcal{O})$ which is one of our main observations in making the argument of [2] work. Consequently, the estimates from (4.17) to (4.30) in [2] still hold and thus we have by (4.29), [2]

$$r = h\left(\partial_\sigma b + a'(\xi)\partial_x b + ifb + h^{\mu_0}\sum_{l=0}^{M_1} e_l(h^\delta\partial_x)^l b\right)\zeta + i\sum_{j=1}^{4} r_j \tag{4.45}$$

with $e_l\in S^0_\delta(\mathcal{O})$,

$$\mu_0 = \frac{1}{2}(\frac{1}{2}-\delta) > 0, \qquad f = W_h\partial_x\psi + a''(\xi)(\partial_x\psi)^2 \text{ (real valued);} \tag{4.46}$$

and with

$$\rho(x,y) = \int_0^1 \partial_x\varphi(\sigma,\lambda x + (1-\lambda)y,\xi,h)d\lambda,$$

the remainders $r_i's$ are then given by

$$r_1 = ch^{M-1}\iiint_0^1 e^{\frac{i}{h}(x-y)\eta}\kappa_0(\eta)(1-\lambda)^{M-1}\partial_y^M\left\{a^{(M)}(\lambda\eta + (\rho(x,y))\widetilde{b}(y)\right\}d\lambda dy d\eta, \tag{4.47}$$

$$r_2 = \sum_{k=0}^{M-1} c_{k,M}h^{M+k}\iint_0^1 z^M\hat{\kappa}_0(z)(1-\lambda)^{M-1}\partial_y^{M+k}\left\{a^{(k)}((\rho(x,y))\widetilde{b}(y)\right\}_{y=x-\lambda hz} d\lambda dz. \tag{4.48}$$

$$r_3 = \sum_{k=0}^{M-1}\sum_{j=1}^{k} c'_{j,k}h^k\partial_y^{k-j}\left\{(\partial_\xi^k a)(\rho(x,y))b(y)\right\}|_{y=x}\zeta^{(j)}. \tag{4.49}$$

$$r_4 = \frac{1}{i}h\left\{-a'(\xi) + h^{\frac{1}{2}}W_h\right\}b\zeta' \tag{4.50}$$

where $c, c_{k,M}, c'_{jk}$ are constants and $\kappa_0\in C_0^\infty(\mathbf{R})$, $\kappa = 1$ in a neighborhood of the origin.

Now, combining (4.43) with the fact that $W_h\in S^0_\delta(\mathcal{O})$ (by (4.38)) we obtain the following estimate for f

$$|\partial_\xi^k\partial_x^\alpha f(\sigma,x,\xi,h)| \le |\sigma|^{\frac{3}{4}} h^{-\delta(\alpha+k)}\mathcal{F}_{k\alpha}\left(\|W\|_{L^4([0,T],W_x^{1,\infty})}\right)\|W\|_{L^\infty([0,T],L_x^\infty)}, \tag{4.51}$$

$\forall(\alpha,k)\in\mathbf{N}^2$, $\forall(\sigma,x,\xi,h)\in\mathcal{O}$.

Step 2. Construction of the amplitude b.

According to the WKB method, one finds b under the form

$$b = \sum_{j=0}^{M-1} h^{j\mu_0} b_j \tag{4.52}$$

where b_0 solves

$$\begin{cases} \partial_\sigma b_0 + a'(\xi)\partial_x b_0 + if b_0 = 0, \\ b_0|_{\sigma=0} = \chi_1(\xi) \end{cases}$$

and b_j, $j \geq 1$, solves

$$\begin{cases} \partial_\sigma b_j + a'(\xi)\partial_x b_j + if b_j = -\sum_{l=0}^{M_1} e_l (h^\delta \partial_x)^l b_{j-1}, \\ b_j|_{\sigma=0} = 0. \end{cases}$$

Owing to Lemma 4.11 and the estimate (4.51), one can use induction for the preceding transport problems (see Lemma 4.7, [2]) to have

$$b_j(\sigma, x, \xi, h) = \chi_1(\xi) c_j(\sigma, x, \xi, h), \quad \forall 0 \leq j \leq J-1 \tag{4.53}$$

and the c_j satisfies $\forall(\alpha, k) \in \mathbf{N}^2$, $\forall(\sigma, x, \xi, h) \in \mathcal{O}$,

$$|\partial_\xi^k \partial_x^\alpha c_j(\sigma, x, \xi, h)| \leq h^{-\delta(\alpha+k)} \mathcal{F}_{jk\alpha}\left(\|W\|_{E_0} + \|W\|_{E_1}\right). \tag{4.54}$$

Step 3. Estimate for the remainder r.

Plugging (4.52) into (4.45) we obtain $r = \sum_{j=0}^5 r_j$ with $r_5 = h^{M\mu_0} b_{M-1}\zeta$. We want to prove (4.40), i.e., for a.e. $t \in [0, T]$ and for all $j = 1, ..5$,

$$\begin{aligned} &\left\| \iint e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)} r(\sigma, x, z, \xi, h) u_{0,h}(z) dz d\xi \right\|_{H^1(\mathbf{R}_x)} \\ &\qquad \leq h^N \mathcal{F}_N\left(\|W\|_{E_0} + \|W\|_{E_1}\right) \|u_{0,h}\|_{L^1(\mathbf{R})}. \end{aligned} \tag{4.55}$$

Let us denote the function inside the norm on the left-hand side by F_h^j. Using integration by parts, the proofs for $\|F_h^j\|_{H_x^1}$, $j = 1, 2, 3, 5$ remain unchanged compared to those in section 4.1.1, [2]. The only point that we need to take care of is the estimate for $\|F_4\|_{H^1}$ since r_4 contains W_h which is less regular than it was in [2]. Recall that

$$r_4 = \frac{1}{i} h \left\{ -a'(\xi) + h^{\frac{1}{2}} W_h \right\} b\zeta'.$$

On the support of all derivatives of ζ one has $|x - z - \sigma a'(\xi)| \geq 1$. Now, by (4.43)

$$h^{\frac{1}{2}} \partial_x \psi \leq C h^{\frac{1}{2}} |\sigma|^{\frac{3}{4}} \leq c h^{\frac{1}{2}}$$

hence using (4.41) we deduce that

$$|\partial_\xi(\varphi(\sigma, x, \xi, h) - z\xi)| = |x - z - \sigma a'(\xi) - h^{\frac{1}{2}} \partial_\xi \psi| \geq \frac{1}{2}$$

for h small enough. Therefore, we can integrate by parts N times in the integral defining F_4 using the vector filed

$$L = \frac{h}{i\partial_\xi(\varphi(\sigma, x, \xi, h) - z\xi)} \partial_\xi.$$

Taking into account the fact that for all $\alpha \in \mathbf{N}$, on the support of ζ, $\langle x - z - \sigma a'(\xi)\rangle \leq C$ and (due to (4.38), (4.54) and (4.43))

$$\begin{aligned} &|\partial_\xi^\alpha r_4(\sigma, x, \xi, h)| \leq C(1 + \|W_h(\sigma)\|_{L_x^\infty}) h^{1-\alpha\delta} \mathcal{F}_\alpha\left(\|W\|_{E_0} + \|W\|_{E_1}\right), \\ &|\partial_\xi^{\alpha+1}(\varphi(\sigma, x, \xi, h) - z\xi)| \leq C(1 + \|W\|_{E_1}) h^{-\alpha\delta} \end{aligned}$$

we obtain

$$\begin{aligned}\|F_h^4(\sigma)\|_{L_x^2} &\le h^{1+N(1-\delta)}(1+\|W_h(\sigma)\|_{L_x^\infty})\mathcal{F}_\alpha\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\times\\ &\quad\times\int|u_{0,h}(z)|dz\int|\chi_1(\xi)|d\xi\\ &\le h^{1+N(1-\delta)}\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}(z)\|_{L_x^1}.\end{aligned}$$

Similarly, one gets

$$\begin{aligned}\|\partial_x F_h^4(\sigma)\|_{L_x^2} &\le h^{1+N(1-\delta)}(1+\|\partial_x W_h(\sigma)\|_{L_x^\infty})\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}(z)\|_{L_x^1}\\ &\le h^{1+N(1-\delta)}(1+h^{-\delta}\|W(\sigma h^{\frac12})\|_{L_x^\infty})\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}(z)\|_{L_x^1}\\ &\le h^{(N+1)(1-\delta)}\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}(z)\|_{L_x^1}.\end{aligned}$$

Therefore, we end up with

$$\sup_{\sigma\in[0,1]}\|F_h^4(\sigma)\|_{H^1(\mathbf{R})}\le h^{N(1-\delta)}\mathcal{F}_N(\|W\|_{E_0}+\|W\|_{E_1})\|u_{0,h}(z)\|_{L_x^1},$$

which concludes the proof. □

With the preceding Proposition in hand, we turn to prove Theorem 4.8.

Proof of Theorem 4.8. Without loss of generality, we take $\sigma_0=0$. By a scaling argument, it suffices to prove the dispersive estimate (4.35) for the operator $\widetilde{S}$ for $\sigma=1$. Indeed, let $\sigma_1\in(0,1]$, making the following changes of variables

$$\tau=\frac{\sigma}{\sigma_1},\ \bar{x}=\frac{x}{\sigma_1},\ \bar{h}=\frac{h}{\sigma_1}$$

we see that the operator L_δ becomes

$$\bar{L}_\delta=\bar{h}\partial_\tau+\bar{h}^{\frac12}\bar{W}_h(\bar{h}\partial_{\bar{x}})+\frac12\bar{h}^{\frac32}(\partial_{\bar{x}}\bar{W}_h)+i|\bar{h}D_{\bar{x}}|^{\frac32}$$

where

$$\bar{W}_h(\tau,\bar{x})=\sigma_1^{\frac12}W_h(\sigma_1\tau,\sigma_1\bar{x}).$$

Observe that there exists $C>0$ independent of $\sigma_1\in(0,1]$ for which there holds

$$\left\|\bar{W}_h\right\|_{E_0}+\left\|\bar{W}_h\right\|_{E_1}\le C.$$

Suppose that the dispersive estimate (4.35) for L_δ were proved for $\sigma=1$, it then would imply the same estimate for $\bar{L}_\delta$ with $\tau=1$. Calling $\bar{S}$ the propagator of $\bar{L}_\delta$, we have for all $\sigma\in[0,1]$

$$\widetilde{S}(\sigma,0)u_0(x)=(\bar{S}(\frac{\sigma}{\sigma_1})\bar{u})(\frac{x}{\sigma_1}),\qquad \bar{u}(\frac{x}{\sigma_1})=u_0(x).$$

Taking $\sigma=\sigma_1$ then it would follow that

$$\left\|\widetilde{S}(\sigma_1)u_0\right\|_{L^\infty(\mathbf{R})}=\left\|\bar{S}(\frac{\sigma_1}{\sigma_1})\bar{u}\right\|_{L^\infty(\mathbf{R})}\le\frac{C}{\bar{h}^{\frac12}}\|\bar{u}_0\|_{L^1(\mathbf{R})}\le\frac{C\sigma_1^{\frac12}}{h^{\frac12}\sigma_1}\|u_0\|_{L^1(\mathbf{R})}\le\frac{C}{|h\sigma_1|^{\frac12}}\|u_0\|_{L^1(\mathbf{R})},$$

which is the estimate (4.35) for L_δ at $\sigma=\sigma_1$. Therefore, it suffices to prove (4.35) for $\sigma=1$.

Now, in view of (4.36) and Proposition 4.12 we have

$$L_\delta\widetilde{U}_h(\sigma,x)=F_h(\sigma,x)\tag{4.56}$$

with

$$\sup_{\sigma\in[0,1]}\|F_h(\sigma)\|_{H_x^1(\mathbf{R}))}\le C_Nh^N\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}\|_{L^1(\mathbf{R})}.\tag{4.57}$$

By integrating by parts we can show that $\widetilde{U}_h$ is a good parametrix at the initial time (see (4.53), [2]) in the following sense

$$\widetilde{U}_h(0,\cdot)=u_{0,h}+v_{0,h},\quad \|v_{0,h}\|_{H^1(\mathbf{R})}\le C_N h^N\|u_{0,h}\|_{H^1(\mathbf{R})}. \tag{4.58}$$

Combining (4.56), (4.58) and the Duhamel formula gives

$$S(\sigma,0)u_{0,h}=R_1+R_2+R_3$$

with

$$\begin{cases} R_1=\widetilde{U}_h(\sigma,x),\\ R_2=-S(\sigma,0)v_{0,h},\\ R_3=-\int_0^\sigma S(\sigma,r)[F_h(r,x)]dr.\end{cases}$$

We shall successively estimate R_i. First, by Sobolev's inequalities and (4.58),

$$\|R_2(\sigma)\|_{L^\infty_x}\le C\|S(\sigma,0)v_{0,h}\|_{H^1_x}=C\|v_{0,h}\|_{H^1_x}\le C_N h^N\|u_{0,h}\|_{L^1}.$$

Next, for R_3 we estimate

$$\|R_3(\sigma)\|_{L^\infty_x}\le\int_0^\sigma\|S(\sigma,r)[F_h(r,x)]\|_{H^1_x}dr\le\int_0^\sigma\|F_h(r,x)\|_{H^1_x}dr.$$

Then, by virtue of the estimate (4.57) we deduce that

$$\|R_3(\sigma)\|_{L^\infty_x}\le h^N\mathcal{F}_N\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}\|_{L^1(\mathbf{R})}.$$

Finally, from (4.36) we have

$$\widetilde{U}_h(\sigma,x)=\int K(\sigma,x,z,h)u_{0,h}(z)dz$$

with

$$K(\sigma,x,z,h)=\frac{1}{2\pi h}\int e^{\frac{i}{h}(\varphi(\sigma,x,\xi,h)-z\xi)}\widetilde{b}(\sigma,x,z,\xi,h)d\xi.$$

Because $\sigma=1$ is fixed, the proof of Proposition 4.8, [2] still works and we obtain for some $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ independent of all parameters

$$|K(1,x,z,h)|\le\frac{1}{h^{\frac12}}\mathcal{F}\left(\|W\|_{E_0}+\|W\|_{E_1}\right).$$

This gives

$$\|R_1(1)\|_{L^\infty_x}=\|\widetilde{U}_h(1)\|_{L^\infty_x}\le h^{-\frac12}\mathcal{F}\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|u_{0,h}\|_{L^1}.$$

The proof is complete. □

4.4. The semi-classical Strichartz estimate

Combining the dispersive estimate (4.32) with the usual TT^* argument and Duhamel's formula, we obtain the Strichartz estimate on the small time interval $[0,h^{\frac12}]$.

Corollary 4.13. *Let $I_h=[0,h^{\frac12}]$ and u_h be a solution to the problem*

$$\mathcal{L}_\delta u_h(t,x)=f(t,x),\quad u_h(0,x)=0$$

with supp $\hat{f}\subset\{c_1h^{-1}\le|\xi|\le c_2h^{-1}\}$. *Then there exists $\mathcal{F}:\mathbf{R}^+\to\mathbf{R}^+$ (independent of u_h, f, W, h) such that*

$$\|u_h\|_{L^4(I_h,L^\infty(\mathbf{R}))}\le h^{-\frac18}\mathcal{F}\left(\|W\|_{E_0}+\|W\|_{E_1}\right)\|f\|_{L^1(I_h,L^2(\mathbf{R}))}.$$

Finally, we glue these estimates together both in frequency and in time to obtain the semi-classical Strichartz estimate for u on $[0, T]$.

Theorem 4.14. *Let $I = [0, T]$ and $s_0 \in \mathbf{R}$. Let $W \in E_0 \cap E_1$ and $f \in L^4(I; H^{s_0-\frac{1}{2}}(\mathbf{R}))$. If $u \in L^\infty(I, H^{s_0}(\mathbf{R}))$ is a solution to the problem*

$$\left(\partial_t + T_W \partial_x + i|D_x|^{\frac{3}{2}}\right) u = f,$$

then for every $\varepsilon > 0$, there exists $\mathcal{F}_\varepsilon$ (independent of $u, \ f, \ W$) such that

$$\|u\|_{L^4(I;C_*^{s_0-\frac{1}{4}-\varepsilon}(\mathbf{R}))} \le \mathcal{F}_\varepsilon(\Xi)\left(\|f\|_{L^4(I;H^{s_0-\frac{1}{2}-\varepsilon}(\mathbf{R}))} + \|u\|_{L^\infty(I;H^{s_0}(\mathbf{R}))}\right), \tag{4.59}$$

where

$$\Xi = \|W\|_{E_0} + \|W\|_{E_1} + \|\partial_x \eta\|_{L^\infty_t L^\infty_x}.$$

Proof. Throughout this proof, we denote $\mathcal{F} = \mathcal{F}(\|W\|_{E_0} + \|W\|_{E_1})$ and RHS the right-hand side of (4.59). Remark first that by (4.30) we have $\mathcal{L}_\delta u_h = F_h$, where F_h is given by (4.31).
Step 1. Let $\chi \in C_0^\infty(0, 2)$ be equal to one on $[\frac{1}{2}, \frac{3}{2}]$. For $0 \le k \le [Th^{-1}] - 2$ define

$$I_{h,k} = [kh^{\frac{1}{2}}, (k+2)h^{\frac{1}{2}}], \quad \chi_{h,k}(t) = \chi\left(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\right), \quad u_{h,k} = \chi_{h,k}(t) u_h.$$

Then

$$\mathcal{L}_\delta u_{h,k} = \chi_{h,k} F_h + h^{-\frac{1}{2}} \chi'\left(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\right) u_h, \quad u_{h,k}(kh, \cdot) = 0.$$

Applying Corollary 4.13 to each $u_{h,k}$ on the interval $I_{h,k}$ we obtain, since $\chi_{h,k}(t) = 1$ for $(k+\frac{1}{2})h \le t \le (k+\frac{3}{2})h$,

$$\begin{aligned}
&\|u_h\|_{L^4((k+\frac{1}{2})h^{\frac{1}{2}}, (k+\frac{3}{2})h^{\frac{1}{2}}); L^\infty(\mathbf{R}))} \\
&\quad \le h^{-\frac{1}{8}} \mathcal{F}(\Xi)\left(\|F_h\|_{L^1(I_{h,k}; L^2(\mathbf{R}))} + h^{-\frac{1}{2}} \|\chi'\left(\frac{t - kh^{\frac{1}{2}}}{h^{\frac{1}{2}}}\right) u_h\|_{L^1(I_{h,k}; L^2(\mathbf{R}))}\right) \\
&\quad \le h^{-\frac{1}{8}} \mathcal{F}(\Xi)\left(h^{\frac{3}{8}} \|F_h\|_{L^4(I_{h,k}; L^2(\mathbf{R}))} + \|u_h\|_{L^\infty(I; L^2(\mathbf{R}))}\right).
\end{aligned}$$

Raising to the power 4 both sides of the preceding estimate, summing over k from 0 to $[Th^{-\frac{1}{2}}] - 2$ and then taking the power $1/4$ we get

$$\|u_h\|_{L^4(I; L^\infty(\mathbf{R}))} \le \mathcal{F}(\Xi)\left(h^{\frac{1}{4}} \|F_h\|_{L^4(I; L^2(\mathbf{R}))} + h^{-\frac{1}{4}} \|u_h\|_{L^\infty(I; L^2(\mathbf{R}))}\right). \tag{4.60}$$

Set $\nu = \frac{1}{2} - \delta$. Multiplying both sides of the above inequality by $h^{-s_0+\frac{1}{4}+\nu}$ and taking into account the fact that u_h and F_h are spectrally supported in annuli of size h^{-1}, it follows that

$$\|u_h\|_{L^4(I; L^\infty(\mathbf{R}))} h^{-s_0+\frac{1}{4}+\nu} \le \mathcal{F}(\Xi)\left(\|F_h\|_{L^4(I; H^{s_0-1+\delta}(\mathbf{R}))} + \|u_h\|_{L^\infty(I; H^{s_0-\nu}(\mathbf{R}))}\right). \tag{4.61}$$

Step 2. We now estimate $\|F_h\|_{L^4(I; H^{s_0-1+\delta}(\mathbf{R}))}$, where recall from (4.31) that

$$\begin{aligned}
F_h = \Delta_j f + \frac{1}{2} \Delta_j (T_{\partial_x W} u) + \frac{1}{2}\left([T_W, \Delta_j] \partial_x u + \partial_x [T_W, \Delta_j] u\right) + R_j u + R'_j u + \\
\frac{1}{2}\left\{\left(S_{(j-N)\delta} W - S_{(j-N)} W\right) \partial_x \Delta_j u + \partial_x \left(S_{(j-N)\delta} W - S_{(j-N)} W\right) \Delta_j u\right\}.
\end{aligned} \tag{4.62}$$

Since $W \in L^4(I, W^{1,\infty}(\mathbf{R}))$, we can apply the symbolic calculus Theorem A.5 (i), (ii) to have

$$\begin{aligned}&\|\Delta_j(T_{\partial_x W}u)\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}+\|[T_W\partial_x+\partial_x T_W,\Delta_j]u\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}\\&\quad\le C\|W\|_{L^4(I;W^{1,\infty}(\mathbf{R}))}\|u\|_{L^\infty(I;H^{s_0-1+\delta}(\mathbf{R}))}.\end{aligned}\tag{4.63}$$

Next, remarking that the spectrum of $\Lambda_j := \big(S_{(j-N)\delta}W - S_{j-N}W\big)\partial_x\Delta_j u$ is contained in a ball of radius $C2^j$ we can write for fixed t

$$\begin{aligned}\|\Lambda_j(t,\cdot)\|_{H^{s_0-1+\delta}(\mathbf{R})}&\le C2^{j(s_0-1+\delta)}\|(S_jW-S_{j\delta}W)\partial_x\Delta_j u_h)(t,\cdot)\|_{L^2(\mathbf{R})}\\&\le C2^{j(s_0-1+\delta)}\|(S_jW-S_{j\delta}W(t,\cdot)\|_{L^\infty(\mathbf{R})}2^{j(1-s_0)}\|u_h(t,\cdot)\|_{H^{s_0}(\mathbf{R})}.\end{aligned}$$

According to the convolution formula,

$$(S_jW-S_{j\delta}W)(t,x)=\int_{\mathbf{R}^d}\check{\phi}(z)\big(W(t,x-2^{-j}z)-W(t,x-2^{-j\delta}z)\big)\,dz,$$

where $\check{\phi}$ is the inverse Fourier transform of the Littlewood–Paley function ϕ. It follows that

$$\|(S_jW-S_{j\delta}W)(t,\cdot)\|_{L^\infty(\mathbf{R})}\le C2^{-j\delta}\|W(t,\cdot)\|_{W^{1,\infty}(\mathbf{R})}.$$

Therefore, we obtain

$$\|\big(S_{j\delta}W-S_jW\big)\partial_x\Delta_j u\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}\le C\,\|W\|_{E_1}\|u_h\|_{L^\infty(I;H^{s_0}(\mathbf{R}))}.\tag{4.64}$$

Similarly, it also holds that

$$\|\partial_x\big(S_{j\delta}W-S_jW\big)\Delta_j u\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}\le C\,\|W\|_{E_1}\|u_h\|_{L^\infty(I;H^{s_0}(\mathbf{R}))}.\tag{4.65}$$

Now, combining (4.63), (4.64), (4.65) and Lemma 4.5 and the fact that $0<\delta<\frac{1}{2}$ we conclude

$$\|F_h\|_{L^2(I;H^{s_0-1+\delta}(\mathbf{R}))}\le C\,\|f_h\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}+\|W\|_{E_1}\|u_h\|_{L^\infty(I;H^{s_0}(\mathbf{R}))}.\tag{4.66}$$

In view of (4.61) we arrive at

$$\|u_h\|_{L^4(I;L^\infty(\mathbf{R}))}h^{-s_0+\frac{1}{4}+\nu}\le\mathcal{F}(\Xi)\Big(\|f_h\|_{L^4(I;H^{s_0-1+\delta}(\mathbf{R}))}+\|u_h\|_{L^\infty(I;H^{s_0}(\mathbf{R}))}\Big).\tag{4.67}$$

Finally, for every given ε we choose $\delta=\frac{1}{2}-\varepsilon=\frac{1}{2}-\nu$ and end up with the desired estimate:

$$\|u\|_{L^4(I;C_*^{s_0-\frac{1}{4}-\varepsilon}(\mathbf{R}))}=\sup_h\|u_h\|_{L^4(I;L^\infty(\mathbf{R}))}h^{-s_0+\frac{1}{4}+\varepsilon}\le\text{RHS}.\qquad\square$$

5. Proof of Theorems 1.1, 1.2, 1.3

Throughout this section, we assume that (η,ψ) is a solution to (1.2) with the regularity given by (4.1):

$$\begin{cases}(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))\cap L^4([0,T];W^{r+\frac{1}{2},\infty}(\mathbf{R})\times W^{r,\infty}(\mathbf{R})),\\ s>r>\dfrac{3}{2}+\dfrac{1}{2}.\end{cases}$$

For any real number σ, let us denote the Sobolev-norm and the Strichartz-norm of the solution as

$$M_\sigma(T)=\|(\eta,\psi)\|_{L^\infty([0,T];H^{\sigma+\frac{1}{2}}\times H^\sigma)},\qquad M_\sigma(0)=\|(\eta,\psi)|_{t=0}\|_{H^{\sigma+\frac{1}{2}}\times H^\sigma},\tag{5.1}$$

$$N_\sigma(T)=\|(\eta,\psi)\|_{L^4([0,T];W^{\sigma+\frac{1}{2},\infty}\times W^{\sigma,\infty})}.\tag{5.2}$$

From the Strichartz estimate (4.59) we have for any $\varepsilon>0$

$$\|u\|_{L^4(I;W^{s-\frac{1}{4}-\varepsilon,\infty})}\le\mathcal{F}_\varepsilon\Big(\|W\|_{E_0}+\|W\|_{E_1}+\|\partial_x\eta\|_{L^\infty_tL^\infty_x}\Big)\Big(\|f\|_{L^4(I;H^{s-\frac{1}{2}})}+\|u\|_{L^\infty(I;H^s)}\Big).\tag{5.3}$$

We shall estimate the norms of W and u, which appear on the right-hand side of (5.3), in terms of M_s and N_s.

Lemma 5.1. *We have*

$$\|u\|_{L^\infty([0,T];H^s)} \le \mathcal{F}(M_s(T)).$$

Proof. By definition (4.5), u is given by

$$u = \Phi \circ \kappa - \dot{T}_{(\partial_x \Phi)\circ\kappa}\kappa = \kappa_g^*\Phi - \mathrm{R}_{line}\Phi.$$

Lemma 5.1 then follows from Theorem 3.4 and Theorem 3.5 (ii). □

Lemma 5.2. *We have*

$$\|W\|_{E_0} \le \mathcal{F}(M_s(T)), \quad \|W\|_{E_1} \le \mathcal{F}(M_s(T))(1 + N_r(T)).$$

Proof. Recall from (4.16) that W is given by

$$W = (V \circ \kappa)(\partial_x \chi \circ \kappa) + \partial_t \chi \circ \kappa.$$

First, by Sobolev's embedding and Lemma 4.2 4., $\|W\|_{L^\infty_t L^\infty_x} \le \mathcal{F}(M_s(T))$. To estimate $\|W\|_{E_1}$ we compute

$$\partial_x W = (\partial_x V \circ \kappa)(\partial_x \chi \circ \kappa)\partial_x \kappa + (V \circ \kappa)(\partial_x^2 \chi \circ \kappa)\partial_x \kappa + (\partial_t \partial_x \chi \circ \kappa)\partial_x \kappa.$$

Using the expression (1.3) for V together with the Hölder estimate for the Dirichlet–Neumann operator proved in Proposition 2.21, [15], we obtain for a.e. $t \in [0, T]$,

$$\|\partial_x V(t)\|_{L^\infty_x} \le \mathcal{F}(\|\eta(t)\|_{H^{s+\frac12}}, \|\psi(t)\|_{H^s}) \left(1 + \|\psi(t)\|_{W^{r,\infty}}\right). \tag{5.4}$$

On the other hand, Lemma 4.2 3. gives $\|\partial_x \chi\|_{L^\infty_t L^\infty_x} \le \mathcal{F}(M_s(T))$, hence

$$\|(\partial_x V \circ \kappa)(\partial_x \chi \circ \kappa)\partial_x \kappa\|_{L^4_t L^\infty_x} \le \mathcal{F}(M_s(T))(1 + N_r(T)). \tag{5.5}$$

The other two terms in the expression of $\partial_x W$ can be treated in the same way. □

Corollary 5.3. *For every $0 < \mu < \frac14$, there exists $\mathcal{F} : \mathbf{R}^+ \to \mathbf{R}^+$ such that*

$$\|u\|_{L^4(I;W^{s-\frac12+\mu,\infty}(\mathbf{R}))} \le \mathcal{F}(M_s(T) + N_r(T)). \tag{5.6}$$

Proof. In view of the Strichartz estimate (5.3) and Lemma 5.1, Lemma 5.2, there holds

$$\|u\|_{L^4(I;W^{s-\frac12+\mu,\infty}(\mathbf{R}))} \le \mathcal{F}(M_s(T) + N_r(T))\Big(\|f\|_{L^4(I;H^{s-\frac12}(\mathbf{R}))} + 1\Big). \tag{5.7}$$

On the other hand, from the estimate (4.17) we have

$$\|f\|_{L^4(I;H^{s-\frac12}(\mathbf{R}))} \le \mathcal{F}(M_s(T))(1 + N_r(T)),$$

which concludes the proof. □

Having established the estimate (5.6) for u, we now go back from u to the original unknown (η, ψ). To this end, we proceed in 2 steps:

$$u = k^*\Phi \longrightarrow \Phi \longrightarrow (\eta, \psi).$$

Fix $\mu \in (0, \frac14)$.

Step 1. By definition (4.5), $\Phi \circ \kappa = u + \dot{T}_{\partial_x \Phi\circ\kappa}\kappa$. It is easy to see that

$$\left\|\dot{T}_{\partial_x \Phi\circ\kappa}\kappa\right\|_{L^\infty_t H^{s+\frac12}_x} \le \mathcal{F}(M_s(T))$$

and thus by Sobolev's embedding and the estimate (5.6)

$$\|\Phi \circ \kappa\|_{L^4(I;W^{s-\frac12+\mu,\infty})} \le \mathcal{F}(M_s(T) + N_s(T)).$$

We then may estimate

$$\begin{aligned}\|\Phi(t)\|_{W^{s-\frac{1}{2}+\mu,\infty}} &= \|\Phi\circ\kappa\circ\chi(t)\|_{W^{s-\frac{1}{2}+\mu,\infty}}\\ &\le \|\Phi(t)\circ\kappa(t)\|_{W_x^{s-\frac{1}{2}+\mu,\infty}}\mathcal{F}(\|\chi'(t)\|_{W^{s-\frac{3}{2}+\mu,\infty}})\\ &\le \|\Phi(t)\circ\kappa(t)\|_{W^{s-\frac{1}{2}+\mu,\infty}}\mathcal{F}(M_s(T)),\end{aligned}$$

which implies

$$\|\Phi\|_{L^4(I;W^{s-\frac{1}{2}+\mu,\infty})} \le \mathcal{F}(M_s(T)+N_s(T)).$$

Step 2. By definition of Φ and the inequality $\|\cdot\|_{C^\sigma} \le C_\sigma\|\cdot\|_{W^{\sigma,\infty}}$ for any $\sigma>0$, the preceding estimate gives

$$\|T_p\eta\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} + \|T_q(\psi - T_B\eta)\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T)+N_s(T)). \tag{5.8}$$

1. Since

$$\sup_{t\in[0,T]} M_0^{-1/2}(p^{(-1/2)}(t)) + \sup_{t\in[0,T]} M_1^{1/2}(p^{(1/2)}(t)) \le \mathcal{F}(M_s(T))$$

it follows from (A.6) that

$$\|T_{p^{(-1/2)}}\eta\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T))\|\eta\|_{L^4(I;C_*^{s-1+\mu})} \le \mathcal{F}(M_s(T)).$$

Consequently, we have

$$\|T_{p^{(1/2)}}\eta\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T)+N_s(T)).$$

Since $p^{(1/2)} \in \Gamma_1^{1/2}$ is elliptic, applying (A.8) yields $\eta = T_{1/p^{(1/2)}}T_{p^{(1/2)}}\eta + R\eta$ where R is of order -1 and for any $\sigma\in\mathbf{R}$,

$$\sup_{t\in[0,T]} \|R(t)\|_{C_*^\sigma\to C_*^{\sigma+1}} \le \mathcal{F}(M_s(T)).$$

Thus,

$$\|\eta\|_{L^4(I;C_*^{s+\mu})} \le \mathcal{F}(M_s(T)+N_r(T)). \tag{5.9}$$

Likewise, we deduce from (5.8) that

$$\|\psi - T_B\eta\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T)+N_r(T)).$$

Owing to (5.9) and the fact that $\|B\|_{L_t^\infty L_x^\infty} \le \mathcal{F}(M_s(T))$, we obtain

$$\|\psi\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T)+N_r(T)).$$

In summary, we have proved that for all (η,ψ) solution to (1.2) with

$$\begin{cases}(\eta,\psi)\in C^0([0,T];H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R}))\cap L^4([0,T];W^{r+\frac{1}{2},\infty}(\mathbf{R})\times W^{r,\infty}(\mathbf{R})),\\ s>r>\dfrac{3}{2}+\dfrac{1}{2}\end{cases} \tag{5.10}$$

there holds for any $\mu<\frac{1}{4}$,

$$\|\eta\|_{L^4(I;C_*^{s+\mu})} + \|\psi\|_{L^4(I;C_*^{s-\frac{1}{2}+\mu})} \le \mathcal{F}(M_s(T)+N_r(T))$$

and thus (since $\mu<\frac{1}{4}$ is arbitrary)

$$N_{s-\frac{1}{2}+\mu}(T) \le \mathcal{F}(M_s(T)+N_r(T)), \tag{5.11}$$

where $M_\sigma(T)$, $N_\sigma(T)$ are respectively the Sobolev-norm and the Strichartz norm defined in (5.1). (5.11) is the semi-classical Strichartz estimate announced in Theorem 1.1.

Of course, (5.11) is meaningful only if $r < s - \frac{1}{2} + \mu$. Under this constraint, using an interpolation argument (see [4], page 88, for instance) we can make appear a factor T in front of $(M_s(T) + N_r(T))$:

$$N_r(T) \le \mathcal{F}\left(T\left(M_s(T) + N_r(T)\right)\right).$$

On the other hand, in Theorem 1.1 [15] it was proved that the following energy estimate at the regularity (5.10) holds

$$M_s(T) \le \mathcal{F}\Big(\mathcal{F}(M_s(0)) + T\mathcal{F}(M_s(T) + N_r(T))\Big).$$

Consequently, we end up with a closed a priori estimate for the mixed norm $M_s(T) + N_r(T)$ as announced in Theorem 1.2:

$$M_s(T) + N_r(T) \le \mathcal{F}\Big(\mathcal{F}(M_s(0)) + T\mathcal{F}(M_s(T) + N_r(T))\Big). \tag{5.12}$$

Finally, by virtue of the contraction estimate for two solutions (η_j, ψ_j) $j = 1, 2$ in the norm $M_{s-1,T} + N_{r-1,T}$ established in Theorem 5.9, [15] (whose proof actually makes use of Theorem 4.14) one can use the standard method of regularizing initial data (see section 6, [15]) to solve uniquely the Cauchy problem for system (1.2) with initial data $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R})$ with $s > 2 + \frac{1}{2} - \mu$ for any $\mu < \frac{1}{4}$. The proof of Theorem 1.3 is complete.

Conflict of interest statement

I declare I have no competing interests.

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Appendix A. Paradifferential calculus

Definition A.1. 1. (Zygmund spaces) Let

$$1 = \sum_{p=0}^{\infty} \Delta_p$$

be a dyadic partition of unity. For any real number s, we define the Zygmund class $C^s_*(\mathbf{R}^d)$ as the space of tempered distributions u such that

$$\|u\|_{C^s_*} := \sup_{q \ge 0} 2^{qs} \left\|\Delta_q u\right\|_{L^\infty} < +\infty.$$

2. (Hölder spaces) For $k \in \mathbf{N}$, we denote by $W^{k,\infty}(\mathbf{R}^d)$ the usual Sobolev spaces. For $\rho = k + \sigma$, $k \in \mathbf{N}$, $\sigma \in (0, 1)$ denote by $W^{\rho,\infty}(\mathbf{R}^d)$ the space of functions whose derivatives up to order k are bounded and Hölder continuous with exponent σ.

Let us review notations and results about Bony's paradifferential calculus (see [8,23]). Here we follow the presentation of Métivier in [23] and [3].

Definition A.2. 1. (Symbols) Given $\rho \in [0, \infty)$ and $m \in \mathbf{R}$, $\Gamma^m_\rho(\mathbf{R}^d)$ denotes the space of locally bounded functions $a(x, \xi)$ on $\mathbf{R}^d \times (\mathbf{R}^d \setminus 0)$, which are C^∞ with respect to ξ for $\xi \ne 0$ and such that, for all $\alpha \in \mathbf{N}^d$ and all $\xi \ne 0$, the function $x \mapsto \partial^\alpha_\xi a(x, \xi)$ belongs to $W^{\rho,\infty}(\mathbf{R}^d)$ and there exists a constant C_α such that,

$$\forall |\xi| \ge \frac{1}{2}, \quad \left\| \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)} \le C_\alpha (1+|\xi|)^{m-|\alpha|}.$$

Let $a \in \Gamma_\rho^m(\mathbf{R}^d)$, we define for every $n \in \mathbf{N}$ the semi-norm

$$M_\rho^m(a;n) = \sup_{|\alpha|\le n} \sup_{|\xi|\ge 1/2} \left\| (1+|\xi|)^{|\alpha|-m} \partial_\xi^\alpha a(\cdot,\xi) \right\|_{W^{\rho,\infty}(\mathbf{R}^d)}. \tag{A.1}$$

When $n = [d/2]+1$ we denote $M_\rho^m(a;n) = M_\rho^m(a)$.
2. (Classical symbols) For any $m \in \mathbf{R}$ and $\rho > 0$ we denote by $\Sigma_\rho^m(\mathbf{R}^d)$ the class of classical symbols $a(x,\xi)$ such that

$$a(x,\xi) = \sum_{0\le j\le[\rho]} a^{(m-j)}$$

where each $a^{(m-j)} \in \Gamma_{\rho-j}^{m-j}$ is homogeneous of degree $m-j$ with respect to ξ.

Definition A.3 *(Paradifferential operators).* Given a symbol a, we define the paradifferential operator T_a by

$$\widehat{T_a u}(\xi) = (2\pi)^{-d} \int \chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\,d\eta, \tag{A.2}$$

where $\widehat{a}(\theta,\xi) = \int e^{-ix\cdot\theta} a(x,\xi)\,dx$ is the Fourier transform of a with respect to the first variable; χ and ψ are two fixed C^∞ functions such that:
(i) ψ is identical to 0 near the origin and identical to 1 away from the origin,
(ii) there exists $0 < \varepsilon_1 < \varepsilon_2 < 1$ such that

$$\chi(\eta,\xi) = \begin{cases} 1 & \text{if } |\eta| \le \varepsilon_1(1+|\xi|), \\ 0 & \text{if } |\eta| \ge \varepsilon_2(1+|\xi|) \end{cases} \tag{A.3}$$

and for any $(\alpha,\beta) \in \mathbf{N}^2$ there exists $C_{\alpha,\beta} > 0$ such that

$$\forall (\eta,\xi) \in \mathbf{R}^d \times \mathbf{R}^d, \left| \partial_\eta^\alpha \partial_\xi^\beta \chi(\eta,\xi) \right| \le C_{\alpha,\beta}(1+|\xi|)^{-\alpha-\beta}. \tag{A.4}$$

Definition A.4. An operator T is said to be of order $m \in \mathbf{R}$, or equivalently $(-m)$-regularized, if for all $\mu \in \mathbf{R}$ it is bounded from H^μ to $H^{\mu-m}$ and from C_*^μ to $C_*^{\mu-m}$.

Symbolic calculus for paradifferential operators is summarized in the following theorem.

Theorem A.5 *(Symbolic calculus). ([23, Chapter 6]) Let $m \in \mathbf{R}$ and $\rho \in [0,\infty)$. Denote by $\overline{\rho}$ the smallest integer that is not smaller than ρ and $n_1 = [d/2] + \overline{\rho} + 1$.*
(i) If $a \in \Gamma_0^m(\mathbf{R}^d)$, then T_a is of order m. Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\|T_a\|_{H^\mu \to H^{\mu-m}} \le K M_0^m(a), \tag{A.5}$$

$$\|T_a\|_{C_*^\mu \to C_*^{\mu-m}} \le K M_0^m(a). \tag{A.6}$$

(ii) If $a \in \Gamma_\rho^m(\mathbf{R}^d)$, $b \in \Gamma_\rho^{m'}(\mathbf{R}^d)$ with $\rho > 0$. Then $T_aT_b - T_{a\sharp b}$ is of order $m+m'-\rho$ where

$$a\sharp b := \sum_{|\alpha|<\rho} \frac{(-i)^\alpha}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi).$$

Moreover, for all $\mu \in \mathbf{R}$ there exists a constant K such that

$$\left\| T_aT_b - T_{a\sharp b} \right\|_{H^\mu \to H^{\mu-m-m'+\rho}} \le K M_\rho^m(a;n_1) M_0^{m'}(b) + K M_0^m(a) M_\rho^{m'}(b;n_1), \tag{A.7}$$

$$\left\| T_aT_b - T_{a\sharp b} \right\|_{C_*^\mu \to C_*^{\mu-m-m'+\rho}} \le K M_\rho^m(a;n_1) M_0^{m'}(b) + K M_0^m(a) M_\rho^{m'}(b;n_1). \tag{A.8}$$

(iii) Let $a \in \Gamma^m_\rho(\mathbf{R}^d)$ with $\rho > 0$. Denote by $(T_a)^*$ the adjoint operator of T_a and by $\overline{a}$ the complex conjugate of a. Then $(T_a)^* - T_{a^*}$ is of order $m - \rho$ where

$$a^* = \sum_{|\alpha|<\rho} \frac{1}{i^{|\alpha|}\alpha!}\partial_\xi^\alpha \partial_x^\alpha \overline{a}.$$

Moreover, for all μ there exists a constant K such that

$$\left\|(T_a)^* - T_{\overline{a}}\right\|_{H^\mu \to H^{\mu-m+\rho}} \leq K M^m_\rho(a; n_1), \tag{A.9}$$

$$\left\|(T_a)^* - T_{\overline{a}}\right\|_{C^\mu_* \to C^{\mu-m+\rho}_*} \leq K M^m_\rho(a; n_1). \tag{A.10}$$

Definition A.6 *(Paraproducts and Bony's decomposition).* Let $1 = \sum_{j=0}^\infty \Delta_j$ be a dyadic partition of unity as in (2.4) and $N \in \mathbf{N}$ be sufficiently large such that the function χ defined in (2.8):

$$\chi(\eta, \xi) = \sum_{p=0}^\infty \phi_{p-N}(\eta)\varphi_p(\xi)$$

satisfies conditions (A.3) and (A.4).

Given $a,\ b \in \mathcal{S}'$ we define formally the paraproduct

$$TP_a u = \sum_{p=N+1}^\infty S_{p-N} a \Delta_p u \tag{A.11}$$

and the remainder

$$R(a, u) = \sum_{j,k\geq 0, |j-k|\leq N-1} \Delta_j a \Delta_k u \tag{A.12}$$

then we have (at least formally) the Bony's decomposition

$$au = TP_a u + TP_u a + R(a, u).$$

We shall use frequently various estimates about paraproducts (see Chapter 2, [7] and [3]) which are gathered here.

Theorem A.7.

1. *Let $\alpha, \beta \in \mathbf{R}$. If $\alpha + \beta > 0$ then*

$$\|R(a, u)\|_{H^{\alpha+\beta-\frac{d}{2}}(\mathbf{R}^d)} \leq K \|a\|_{H^\alpha(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)}, \tag{A.13}$$

$$\|R(a, u)\|_{H^{\alpha+\beta}(\mathbf{R}^d)} \leq K \|a\|_{C^\alpha_*(\mathbf{R}^d)} \|u\|_{H^\beta(\mathbf{R}^d)}, \tag{A.14}$$

$$\|R(a, u)\|_{C^{\alpha+\beta}_*(\mathbf{R}^d)} \leq K \|a\|_{C^\alpha_*(\mathbf{R}^d)} \|u\|_{C^\beta_*(\mathbf{R}^d)}. \tag{A.15}$$

2. *Let s_0, s_1, s_2 be such that $s_0 \leq s_2$ and $s_0 < s_1 + s_2 - \frac{d}{2}$, then*

$$\|TP_a u\|_{H^{s_0}} \leq K \|a\|_{H^{s_1}} \|u\|_{H^{s_2}}. \tag{A.16}$$

3. *Let $m > 0$ and $s \in \mathbf{R}$. Then*

$$\|TP_a u\|_{H^{s-m}} \leq K \|a\|_{C^{-m}_*} \|u\|_{H^s}, \tag{A.17}$$

$$\|TP_a u\|_{C^{s-m}_*} \leq K \|a\|_{C^{-m}_*} \|u\|_{C^s_*}. \tag{A.18}$$

Proposition A.8.

1. *If $u_j \in H^{s_j}(\mathbf{R}^d)$ $(j = 1, 2)$ with $s_1 + s_2 > 0$ then*

$$\|u_1 u_2\|_{H^{s_0}} \leq K \|u_1\|_{H^{s_1}} \|u_2\|_{H^{s_2}}, \tag{A.19}$$

if $s_0 \leq s_j$, $j = 1, 2$, and $s_0 < s_1 + s_2 - d/2$.

2. *If* $s \geq 0$ *then*
$$\|u_1u_2\|_{H^s} \leq K(\|u_1\|_{H^s}\|u_2\|_{L^\infty} + \|u_2\|_{H^s}\|u_1\|_{L^\infty}). \tag{A.20}$$
3. *If* $s \geq 0$ *then*
$$\|u_1u_2\|_{C^s_*} \leq K(\|u_1\|_{C^s_*}\|u_2\|_{L^\infty} + \|u_2\|_{C^s_*}\|u_1\|_{L^\infty}). \tag{A.21}$$
4. *Let* $\beta > \alpha > 0$. *Then*
$$\|u_1u_2\|_{C^{-\alpha}_*} \leq K\,\|u_1\|_{C^\beta_*}\|u_2\|_{C^{-\alpha}_*}. \tag{A.22}$$

Theorem A.9.

1. *Let* $s \geq 0$ *and consider* $F \in C^\infty(\mathbf{C}^N)$ *such that* $F(0) = 0$. *Then there exists a nondecreasing function* $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ *such that, for any* $U \in H^s(\mathbf{R}^d)^N \cap L^\infty(\mathbf{R}^d)$,
$$\|F(U)\|_{H^s} \leq \mathcal{F}\big(\|U\|_{L^\infty}\big)\,\|U\|_{H^s}. \tag{A.23}$$
2. *Let* $s \geq 0$ *and consider* $F \in C^\infty(\mathbf{C}^N)$ *such that* $F(0) = 0$. *Then there exists a nondecreasing function* $\mathcal{F}\colon \mathbf{R}_+ \to \mathbf{R}_+$ *such that, for any* $U \in C^s_*(\mathbf{R}^d)^N$,
$$\|F(U)\|_{C^s_*} \leq \mathcal{F}\big(\|U\|_{L^\infty}\big)\,\|U\|_{C^s_*}. \tag{A.24}$$

Theorem A.10 *(Paralinearization). ([7, Theorem 2.92]) Let* $r,\ \rho$ *be positive real numbers and* F *be a* C^∞ *function on* $\mathbf{R}$ *such that* $F(0) = 0$. *Assume that* ρ *is not an integer. There exists a nondecreasing function* $\mathcal{F}: \mathbf{R}^+ \to \mathbf{R}^+$ *such that for any* $u \in H^\mu(\mathbf{R}^d) \cap C^\rho_*(\mathbf{R}^d)$,
$$\big\|F(u) - TP_{F'(u)}u\big\|_{H^{\mu+\rho}(\mathbf{R}^d)} \leq \mathcal{F}(\|u\|_{L^\infty(\mathbf{R}^d)})\,\|u\|_{C^\rho_*(\mathbf{R}^d)}\,\|u\|_{H^\mu(\mathbf{R}^d)}.$$

Appendix B. Proof of some technical lemmas

B.1. Proof of Lemma 2.1

Let $f_n \in C(\mathbf{R}^d)$, $g \in C^\infty(\mathbf{R}^d)$ be two nonnegative functions satisfying
$$f_n(t) = \begin{cases} 1, \text{ if } |t| \leq 2^{-n} + \frac{1}{4}, \\ 0, \text{ if } |t| > 2^{n+1} - \frac{1}{4} \end{cases}$$
and
$$g(t) = 0, \text{ if } |t| \geq \frac{1}{4}, \quad \int_{\mathbf{R}^d} g(t)\mathrm{d}t = 1.$$
We then define $\phi_{(n)} = f_n * g$. It is easy to see that $\phi_{(n)} \geq 0$ and satisfies condition (2.1). To verify condition (2.2) we use $\partial^\alpha\phi_{(n)} = f_n * \partial^\alpha g$ to have
$$\begin{aligned} x^\beta\partial^\alpha\phi_{(n)}(x) &= \int_{\mathbf{R}^d} x^\beta f_n(x-y)\partial^\alpha g(y)\mathrm{d}y \\ &= \sum_{\beta_1+\beta_2=\beta}\int_{\mathbf{R}^d}(x-y)^{\beta_1}f_n(x-y)y^{\beta_2}\partial^\alpha g(y)\mathrm{d}y, \\ &= \sum_{\beta_1+\beta_2=\beta}\big((\cdot)^{\beta_1}f_n\big) * \big((\cdot)^{\beta_2}\partial^\alpha g\big)(x). \end{aligned}$$
Each term on the right-hand side is estimated by
$$\big\|\big((\cdot)^{\beta_1}f_n\big) * \big((\cdot)^{\beta_2}\partial^\alpha g\big)\big\|_{L^1} \leq \big\|(\cdot)^{\beta_1}f_n\big\|_{L^1}\big\|(\cdot)^{\beta_2}\partial^\alpha g\big\|_{L^1}$$

where $\left\|(\cdot)^{\beta_2}\partial^\alpha g\right\|_{L^1}$ is independent of n. It remains to have a uniformly bound with respect to n for $\left\|(\cdot)^{\beta_1} f_n\right\|_{L^1}$. To this end, one can choose the following piecewise affine functions

$$f_n(t) = \begin{cases} 1, \text{ if } |t| \le 2^{-n} + \frac{1}{4}, \\ 0, \text{ if } |t| > 2^{-n} + \frac{1}{2}, \\ -4(|t| - 2^{-n} - \frac{1}{2}), \text{ if } 2^{-n} + \frac{1}{4} \le |t| \le 2^{-n} + \frac{1}{2}. \end{cases}$$

B.2. Proof of Lemma 2.3

1. Let $1 \le p \le q \le \infty$. The estimates for Δ_j follow immediately from those of S_j since $\Delta_0 = S_0$ and $\Delta_j = S_j - S_{j-1},\ \forall j \ge 1$. By Definition 2.2 we have for each $n \in \mathbf{N}$, $S_j u = f_j * u$ where f_j is the inverse Fourier transform of ϕ_j, where $\phi \equiv \phi_{(n)}$. With r satisfying

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$$

we get by Young's inequality

$$\left\|\partial^\alpha S_j\right\|_{L^p \to L^q} \le \left\|\partial^\alpha f_j\right\|_{L^r}.$$

The problem then reduces to showing that

$$\left\|\partial^\alpha f_j\right\|_{L^r} \le C_\alpha 2^{j(|\alpha| + \frac{d}{p} - \frac{d}{q})}$$

which in turn reduces to

$$\left\|\partial^\alpha \mathfrak{F}^{-1}\phi_{(n)}\right\|_{L^r} \le C_\alpha,$$

which is true by virtue of (2.2).

2. The boundedness of the operators $2^{j\mu}\Delta_j,\ j \ge 1$ from $W^{\mu,\infty}(\mathbf{R}^d)$ to $L^\infty(\mathbf{R}^d)$ is proved in Lemma 4.1.8, [23]. Following that proof we see that

$$\left\|2^{j\mu}\Delta_j\right\|_{W^{\mu,\infty} \to L^\infty} \le 2^{j\mu} \int\limits_{\mathbf{R}^d} |x|^\mu |g_j(x)| dx := I,$$

where g_j is the inverse Fourier transform of $\varphi_j = \phi_j - \phi_{j-1}$. Owing to (2.2) it holds that

$$\forall \alpha \in \mathbf{N}^d,\ \exists C_\alpha > 0, \forall (j,n) \in \mathbf{N}^* \times \mathbf{N}, \int |x^\alpha g_j(x)| \mathrm{d}x \le C_\alpha 2^{-j|\alpha|}.$$

Thus, if $\mu \in \mathbf{N}$ we have the result. If $\mu = \delta n + (1-\delta)(n+1)$ for some $\delta \in (0,1),\ n \in \mathbf{N}$ we use Hölder's inequality to estimate

$$I \le 2^{j\mu} \left(\int |x|^n |g_j(x)| \mathrm{d}x\right)^\delta \left(\int |x|^{n+1} |g_j(x)| \mathrm{d}x\right)^{1-\delta} \le C_\mu 2^{j\mu} 2^{-jn\delta - j(n+1)(1-\delta)} = C_\mu$$

which concludes the proof.

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