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Regularity estimates for quasilinear elliptic equations with variable growth involving measure data [☆]

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Abstract

We investigate a quasilinear elliptic equation with variable growth in a bounded nonsmooth domain involving a signed Radon measure. We obtain an optimal global Calderón–Zygmund type estimate for such a measure data problem, by proving that the gradient of a very weak solution to the problem is as globally integrable as the first order maximal function of the associated measure, up to a correct power, under minimal regularity requirements on the nonlinearity, the variable exponent and the boundary of the domain.

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1. Introduction

There have been considerable theoretical advances in partial differential equations (PDEs) with variable exponent growth in recent years. The study of these problems has also become an important research field, and it represents various phenomena in applied sciences: for instance, electrorheological fluids [46], elasticity [52], flows in porous media [4], image restoration [18], thermo-rheological fluids [3], and magnetostatics [17].

In this paper, we consider the Dirichlet problem with measure data:

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du, x) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with nonsmooth boundary $\partial\Omega$, and μ is a signed Radon measure on Ω with finite mass. We can assume, by extending μ by zero to $\mathbb{R}^n \setminus \Omega$, that μ is defined in $\mathbb{R}^n$ with $|\mu|(\Omega) = |\mu|(\mathbb{R}^n) < \infty$. The vector field $\mathbf{a} = \mathbf{a}(\xi, x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in ξ and measurable in x, and it satisfies the following variable exponent growth and uniformly ellipticity conditions:

$$|\xi||D_\xi \mathbf{a}(\xi, x)| + |\mathbf{a}(\xi, x)| \leq \Lambda |\xi|^{p(x)-1}, \tag{1.2}$$

$$\lambda |\xi|^{p(x)-2} |\eta|^2 \leq \langle D_\xi \mathbf{a}(\xi, x)\eta, \eta \rangle, \tag{1.3}$$

for almost every $x \in \mathbb{R}^n$, every $\eta \in \mathbb{R}^n$, every $\xi \in \mathbb{R}^n \setminus \{0\}$, and appropriate constants $\lambda,\ \Lambda$. Here $D_\xi \mathbf{a}(\xi, x)$ is the Jacobian matrix of $\mathbf{a}$ with respect to ξ, $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$, and $p(\cdot)$ is a given continuous function in Ω satisfying

$$2 - \frac{1}{n} < \gamma_1 \leq p(\cdot) \leq \gamma_2 < \infty. \tag{1.4}$$

Note that (1.2) implies that $\mathbf{a}(0, x) = 0$ for $x \in \mathbb{R}^n$, and (1.3) yields the following monotonicity condition:

$$\langle \mathbf{a}(\xi_1, x) - \mathbf{a}(\xi_2, x), \xi_1 - \xi_2 \rangle \geq \begin{cases} \tilde{\lambda}\, |\xi_1 - \xi_2|^{p(x)} & \text{if} \quad p(x) \geq 2, \\ \tilde{\lambda} \left(|\xi_1|^2 + |\xi_2|^2 \right)^{\frac{p(x)-2}{2}} |\xi_1 - \xi_2|^2 & \text{if} \quad 1 < p(x) < 2 \end{cases} \tag{1.5}$$

for all $x, \xi_1, \xi_2 \in \mathbb{R}^n$ and for some constant $\tilde{\lambda} = \tilde{\lambda}(n, \lambda, \gamma_1, \gamma_2) > 0$.

If $\gamma_1 > n$, then μ belongs to the dual space of $W_0^{1,p(\cdot)}(\Omega)$ as a consequence of Morrey's inequality and a duality argument, and so the existence and uniqueness of a weak solution u to (1.1) are well understood from the monotone operator theory, see for instance [49]. In this case, regularity estimates for (1.1) have been extensively studied, see for example [1,2,12,13,26,28,37]. For this reason, we only consider the case that $\gamma_1 \leq n$ for which a solution u of (1.1) in the distributional sense does not necessarily become a weak solution in $W_0^{1,p(\cdot)}(\Omega)$. In this respect, we need to consider a more general class of solutions below the duality exponent.

Definition 1.1. $u \in W_0^{1,1}(\Omega)$ is a *SOLA (Solution Obtained by Limits of Approximations)* to the problem (1.1) under the assumptions (1.2)–(1.4) if the vector field $\mathbf{a}(Du, x) \in L^1(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \langle \mathbf{a}(Du, x), D\varphi \rangle \, dx = \int_\Omega \varphi \, d\mu$$

holds for all $\varphi \in C_c^\infty(\Omega)$, and moreover there exists a sequence of weak solutions $\{u_h\}_{h\geq 1} \in W_0^{1,p(\cdot)}(\Omega)$ of the Dirichlet problems

$$\begin{cases} -\operatorname{div} \mathbf{a}(Du_h, x) = \mu_h & \text{in } \Omega, \\ \qquad\qquad u_h = 0 & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

such that

$$u_h \to u \quad \text{in } W_0^{1,\max\{1, p(\cdot)-1\}}(\Omega) \quad \text{as } h \to \infty, \tag{1.7}$$

where $\{\mu_h\} \in L^\infty(\Omega)$ converges weakly to μ in the sense of measure and satisfies for each open set $V \subset \mathbb{R}^n$,

$$\limsup_{h\to\infty} |\mu_h|(V) \leq |\mu|(\overline{V}), \tag{1.8}$$

with μ_h defined in $\mathbb{R}^n$ by considering the zero extension to $\mathbb{R}^n$.

Throughout the paper, we consider $\mu_h := \mu * \phi_h$, where ϕ_h is the usual mollifier, and then $\mu_h \in C^\infty(\Omega)$ converges weakly to μ in the sense of measure satisfying (1.8) and the following uniform L^1-estimate:

$$\|\mu_h\|_{L^1(\Omega)} \leq |\mu|(\Omega). \tag{1.9}$$

With such μ_h, there exists a SOLA u of (1.1) belonging to $W_0^{1,q(\cdot)}(\Omega)$ for all $q(\cdot)$ with

$$1 \leq q(\cdot) < \min\left\{ \frac{n(p(\cdot)-1)}{n-1}, p(\cdot) \right\}.$$

This existence follows from a priori $L^{q(\cdot)}$ estimate of the gradient of solutions for the regularized problem of $p(\cdot)$-Laplace type and a proper approximation procedure, see [6,11] and the references therein. Moreover, the condition $p(\cdot) > 2 - \frac{1}{n}$ in (1.4) implies $\frac{n(p(\cdot)-1)}{n-1} > 1$, which ensures $u \in W_0^{1,1}(\Omega)$. On the other hand, if $p(\cdot) \le 2 - \frac{1}{n}$, then a solution does not belong to $W_0^{1,1}(\Omega)$, and so a new concept of solutions is needed. We refer to [7,47] for details, and we will no longer treat the case $p(\cdot) \le 2 - \frac{1}{n}$ here. It is worthwhile to mention that the existence of a solution of (1.1) can also be obtained by introducing the notion of renormalized solutions, see [6,21] and the references given there.

The uniqueness of a SOLA remains still an open problem except when $p(\cdot) \equiv 2$, see [44,48] for counterexamples. We also refer to [6–11,20,21,36,47] for a thorough discussion regarding the existence and uniqueness of measure data problems.

The aim of this paper is to establish a global Calderón–Zygmund type estimate for a SOLA u to the problem (1.1). More precisely, we want to prove that for all $q > 0$,

$$\int_\Omega |Du|^q \, dx \le c \left\{ \int_\Omega [\mathcal{M}_1(\mu)(x)]^{\frac{q}{p(x)-1}} \, dx + 1 \right\} \tag{1.10}$$

under optimal conditions on $p(\cdot)$, $\mathbf{a}$ and Ω. Here $\mathcal{M}_1$ is the fractional maximal function of order 1 for μ, defined as

$$\mathcal{M}_1(\mu)(x) := \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|} \quad \text{for } x \in \mathbb{R}^n,$$

where $|B_r(x)|$ is the n-dimensional Lebesgue measure of the open ball $B_r(x)$.

In [43], Phuc proved (1.10) for a renormalized solution u of (1.1) with the constant exponent, that is, $p(\cdot) \equiv p$. We generalize this result for a SOLA u in the setting of the variable exponent case. Indeed, from the point of regularity, there is little difference between a SOLA and a renormalized solution, as both are based on the approximation arguments. We would like to point out that Nguyen in [42] considered a problem to find a parabolic version of the result in [43] for the linear case.

The main difficulty in carrying out our result (1.10) is to establish comparison L^1-estimates and higher integrability for the variable exponent case, see Section 3. Moreover, unlike the constant exponent case, the problem (1.1) has no normalization property, and so it needs a delicate analysis and a very careful computation to obtain the standard L^1-estimates for measure data problems, see Remark 5.1 and Remark 5.2. The desired estimate (1.10) is obtained via the so-called *maximal function technique*, which has been previously used in [2,12,14,16,39,51]. A notable advantage of this approach is that it can completely avoid the use of explicit kernels and singular integrals. The basic tools in the maximal function technique are the maximal function, the Vitali covering lemma, and the integral identity formula

$$\int_\Omega \mathcal{M}(|Du|)^q \, dx = q \int_0^\infty \lambda^{q-1} |\{x \in \Omega : \mathcal{M}(|Du|)(x) > \lambda\}| \, d\lambda,$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator, see (2.6). For various regularity results for measure data problems, we refer to [5,11,24,25,29–32,38–41].

This paper is organized as follows. In Section 2, we introduce some notations, backgrounds, and assumptions on $(p(\cdot), \mathbf{a}, \Omega)$ to state the main theorem. In Section 3, we discuss comparison L^1-estimates of the problem (1.1) and the relevant regularized equations. In Section 4, we verify the hypotheses of the Vitali covering lemma (Lemma 2.4). Finally, Section 5 is devoted to proving our main result by controlling the upper-level sets of $\mathcal{M}(|Du|)$ for a given SOLA u.

2. Preliminaries and main results

2.1. Notations and main results

We start with notations, which will be used throughout the paper. Let us denote by $B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\}$ the open ball in $\mathbb{R}^n$ with center x and radius $r > 0$, $B_r^+(x) = B_r(x) \cap \{x \in \mathbb{R}^n : x_n > 0\}$, $B_r \equiv B_r(0)$, and $B_r^+ \equiv B_r^+(0)$. For $f \in L^1_{loc}(\mathbb{R}^n)$, $\overline{(f)}_U$ stands for the integral average of f over a bounded open set $U \subset \mathbb{R}^n$, that is,

$$\overline{(f)}_U \equiv \fint_U f(x)\,dx = \frac{1}{|U|}\int_U f(x)\,dx.$$

In what follows, we denote by c a universal constant that can be explicitly computed in terms of known quantities such as $n, \lambda, \Lambda, \gamma_1, \gamma_2, q$ and a modulus of continuity $\omega(\cdot)$, which will be explained below.

We recall here a brief overview of variable exponent Lebesgue and Sobolev spaces. Let a function $p(\cdot)$ satisfy (1.4). The *variable exponent Lebesgue spaces* $L^{p(\cdot)}(\Omega)$ are defined by

$$L^{p(\cdot)}(\Omega) := \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^{p(x)}\,dx < \infty \right\}$$

with the Luxemberg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\},$$

and the *variable exponent Sobolev spaces* $W^{1,p(\cdot)}(\Omega)$ are defined by

$$W^{1,p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega) : Df \in L^{p(\cdot)}(\Omega, \mathbb{R}^n) \right\}$$

equipped with the norm

$$\|f\|_{W^{1,p(\cdot)}(\Omega)} := \|f\|_{L^{p(\cdot)}(\Omega)} + \|Df\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^n)}.$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$ and $W^{-1,p'(\cdot)}(\Omega)$ the dual space of $W_0^{1,p(\cdot)}(\Omega)$. They are all separable reflexive Banach spaces.

We next introduce the log-Hölder continuity condition which is the correct condition for regularly varying exponents. Given a function $p(\cdot)$ satisfying (1.4), we say that $p(\cdot)$ is *log-Hölder continuous* in Ω if there exists a constant $L > 0$ such that for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$,

$$|p(x) - p(y)| \le \frac{L}{-\log|x-y|}.$$

We remark that $p(\cdot)$ is log-Hölder continuous in Ω if and only if $p(\cdot)$ has a modulus of continuity, that is, there exists a nondecreasing concave function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ and

$$|p(x) - p(y)| \le \omega(|x-y|) \quad \text{for } x, y \in \Omega,$$

and moreover

$$\sup_{0<r\le\frac{1}{2}} \omega(r)\log\left(\frac{1}{r}\right) \le \tilde{L}$$

for some constant $\tilde{L} > 0$. The log-Hölder continuity plays a crucial role in a systematic analysis of variable exponent Lebesgue and Sobolev spaces and PDEs with variable exponents; see the monographs [19,23].

We now introduce the main regularity assumptions on $p(\cdot)$, $\mathbf{a}$ and Ω.

Assumptions. Let $R > 0$ and $\delta \in (0, \frac{1}{8})$.

(AP) A function $p(\cdot)$ has a modulus of continuity $\omega : [0, \infty) \to [0, \infty)$, and it satisfies that

$$\sup_{0<r\leq R} \omega(r) \log\left(\frac{1}{r}\right) \leq \delta. \tag{2.1}$$

(AA) For a bounded open set $U \subset \mathbb{R}^n$, write

$$\theta\,(\mathbf{a}, U)\,(x) := \sup_{\xi\in\mathbb{R}^n\setminus\{0\}} \left| \frac{\mathbf{a}(\xi, x)}{|\xi|^{p(x)-1}} - \overline{\left(\frac{\mathbf{a}(\xi,\cdot)}{|\xi|^{p(\cdot)-1}}\right)_U} \right|. \tag{2.2}$$

Then, the vector field $\mathbf{a}$ satisfies

$$\sup_{0<r\leq R}\ \sup_{y\in\mathbb{R}^n} \fint_{B_r(y)} \theta\,(\mathbf{a}, B_r(y))\,(x)\,dx \leq \delta. \tag{2.3}$$

(AΩ) The domain Ω is (δ, R)-*Reifenberg flat*, that is, for each $x_0 \in \partial\Omega$ and each $r \in (0, R]$, there exists a coordinate system $\{y_1, \cdots, y_n\}$ such that in this new coordinate system, the origin is x_0 and

$$B_r \cap \{y_n > \delta r\} \subset B_r \cap \Omega \subset B_r \cap \{y_n > -\delta r\}. \tag{2.4}$$

We say $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-*vanishing* if **(AP), (AA)** and **(AΩ)** hold.

We are ready to state our main result.

Theorem 2.1. *Assume that (1.2)–(1.4) hold and let $0 < q < \infty$ and $\gamma_1 \leq n$. Then there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$ such that the following holds: if $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing for some $R \in (0, 1)$, then for any SOLA u of the problem (1.1) there exists a constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), q, R, \Omega) > 0$ such that*

$$\|Du\|_{L^q(\Omega)} \leq c K_s \left\{ \left\| \mathcal{M}_1(\mu)^{\frac{1}{p(\cdot)-1}} \right\|_{L^q(\Omega)} + 1 \right\} \tag{2.5}$$

for every constant s with $0 < s \leq \frac{1}{2}\left(\frac{n}{n-1} - \frac{1}{\gamma_1-1}\right) < 1$ depending only on n and γ_1, where

$$K_s := \left(|\mu|(\Omega) + |\mu|(\Omega)^{\frac{1}{(\gamma_1-1)(1-s)}} + 1 \right)^{n+1}.$$

Remark 2.2. We point out that the term K_s in the estimate (2.5) reflects a deficiency of the normalization property of the problem (1.1) from the presence of variable exponent $p(\cdot)$. On the other hand, in the constant exponent case, that is, $p(\cdot) \equiv p$, we can derive a more clean estimate than (2.5) by employing the normalization property. We would also like to note that the constant c goes to $+\infty$ when $s \searrow 0$, as we will see later in Remark 5.1 and Remark 5.2.

2.2. *Preliminary lemmas*

As we will see later in our proof of the main result, it is important to introduce here some analytic and geometric properties. We first begin with the *Hardy–Littlewood maximal function*. For $f \in L^1_{loc}(\mathbb{R}^n)$, we define

$$\mathcal{M}f(y) = \mathcal{M}(f)(y) := \sup_{r>0} \fint_{B_r(y)} |f(x)|\,dx \tag{2.6}$$

and

$$\mathcal{M}_U f := \mathcal{M}(\chi_U f)$$

if f is not defined outside a bounded open set $U \subset \mathbb{R}^n$. Here χ_U is the characteristic function of U. For simplicity, we drop the index U if $U = \Omega$. We will use the following *weak* $(1, 1)$ *estimates* and *strong* (p, p) *estimates*:

$$|\{x \in \Omega : \mathcal{M}f(x) > \alpha\}| \le \frac{c(n)}{\alpha} \int_{\Omega} |f|\,dx \quad \text{for all } \alpha > 0, \tag{2.7}$$

and for $1 < p < \infty$,

$$\|\mathcal{M}f\|_{L^p(\Omega)} \le c(n, p)\,\|f\|_{L^p(\Omega)}\,. \tag{2.8}$$

We also use an interior and an exterior measure density condition of the Reifenberg flat domains, which can be found in [14].

Lemma 2.3. *If Ω is (δ, R)-Reifenberg flat, then we have*

$$\sup_{0<r\le R}\ \sup_{y\in\Omega} \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \le \left(\frac{2}{1-\delta}\right)^n \le \left(\frac{16}{7}\right)^n, \tag{2.9}$$

and

$$\inf_{0<r\le R}\ \inf_{y\in\partial\Omega} \frac{|\Omega^c \cap B_r(y)|}{|B_r(y)|} \ge \left(\frac{1-\delta}{2}\right)^n \ge \left(\frac{7}{16}\right)^n. \tag{2.10}$$

The next lemma is a *Vitali type covering lemma*, which is a reformulation of *Calderón–Zygmund decomposition*.

Lemma 2.4. *(See [14].) Suppose Ω is (δ, R)-Reifenberg flat with $0 < R_0 \le R$. Let $C \subset D \subset \Omega$ be measurable sets and $0 < \epsilon < 1$ such that*

(i) $|C| \le \left(\frac{1}{1000}\right)^n \epsilon |B_{R_0}|$, *and*
(ii) *for $y \in \Omega$ and $r_0 \in (0, \frac{R_0}{1000}]$, if $|C \cap B_{r_0}(y)| \ge \epsilon |B_{r_0}(y)|$, then $B_{r_0}(y) \cap \Omega \subset D$.*

Then

$$|C| \le \left(\frac{10}{1-\delta}\right)^n \epsilon |D| \le \left(\frac{80}{7}\right)^n \epsilon |D|.$$

For a more detailed discussion on Reifenberg flat domains, we refer to [14,33,34,45,50] and the references therein. We end this section with the following standard measure theory.

Lemma 2.5. *(See [15].) Let f be a measurable function in a bounded open set $\Omega \subset \mathbb{R}^n$. Let $\theta > 0$ and $N > 1$ be constants. Then, for $0 < q < \infty$,*

$$f \in L^q(\Omega) \quad \Longleftrightarrow \quad S := \sum_{k\ge1} N^{qk} \left|\left\{x \in \Omega : |f(x)| > \theta N^k\right\}\right| < \infty \tag{2.11}$$

with the estimate

$$c^{-1}\theta^q S \le \int_{\Omega} |f|^q\,dx \le c\theta^q\left(|\Omega| + S\right), \tag{2.12}$$

where the constant $c = c(N, q) > 0$.

3. Comparison estimates in L^1 for regular problems

In this section, we assume that μ in the equation (1.1) is regular, which means that

$$\mu \in L^1(\Omega) \cap W^{-1,p'(\cdot)}(\Omega). \tag{3.1}$$

Then we derive comparison L^1-estimates for the gradient of the weak solution u to (1.1) in localized boundary and interior regions. Note that by the assumption (3.1), this weak solution u is well defined, that is, there exists a unique $u \in W_0^{1,p(\cdot)}(\Omega)$ satisfying

$$\int_\Omega \langle \mathbf{a}(Du,x), D\varphi\rangle \, dx = \int_\Omega \mu\varphi \, dx \quad \text{for all } \varphi \in W_0^{1,p(\cdot)}(\Omega). \tag{3.2}$$

We denote, for a measurable set $E \subset \mathbb{R}^n$,

$$|\mu|(E) := \int_E |\mu(x)| \, dx.$$

Throughout this section we assume that $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing.

3.1. Boundary comparisons

Let $0 < r \le \frac{R_0}{8}$ for small $R_0 > 0$, to be selected later. Assume that the following geometric setting:

$$B_{8r}^+ \subset \Omega_{8r} \subset B_{8r} \cap \{x_n > -16\delta r\}, \tag{3.3}$$

where $\Omega_{8r} := \Omega \cap B_{8r}$.

Let $w \in u + W_0^{1,p(\cdot)}(\Omega_{8r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw,x) = 0 & \text{in } \Omega_{8r}, \\ w = u & \text{on } \partial\Omega_{8r}. \end{cases} \tag{3.4}$$

In order to get the comparison result between the equations (1.1) and (3.4), it is helpful to define a new measure ν by

$$\nu(E) = |\mu|(E) + |E \cap \Omega| \tag{3.5}$$

for a measurable set $E \subset \mathbb{R}^n$, see Remark 3.2 for details. Hereafter in this subsection, we write

$$p_0 := p(0), \quad p_1 := \inf_{x\in\Omega_{8r}} p(x), \quad p_2 := \sup_{x\in\Omega_{8r}} p(x), \quad \text{and} \quad \chi_{\{p_0<2\}} := \begin{cases} 0 & \text{if} \quad p_0 \ge 2, \\ 1 & \text{if} \quad p_0 < 2. \end{cases}$$

Lemma 3.1. *Suppose that $R_0 > 0$ satisfies*

$$R_0 \le \min\left\{\frac{R}{2}, \frac{1}{\nu(\Omega)+1}, \frac{1}{\int_\Omega |Du|\,dx + 1}\right\}. \tag{3.6}$$

Let $0 < r \le \frac{R_0}{8}$ and assume that Ω_{8r} satisfies (3.3). If $w \in u + W_0^{1,p(\cdot)}(\Omega_{8r})$ is the weak solution of (3.4), then there exists a constant $c = c(n, \lambda, \gamma_1, \gamma_2) > 0$ such that

$$\fint_{\Omega_{8r}} |Du - Dw| \, dx \le c \left\{ \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}} + \chi_{\{p_0<2\}} \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right] \left(\fint_{\Omega_{8r}} |Du| \, dx\right)^{2-p_0} \right\}.$$

Proof. Since it has already been proved in the case $p_1 \ge 2$, see [11, Lemma 3.1], we only focus on the case $p_1 < 2$.

Step 1. Dimensionless estimates We first consider the case that $8r = 1$. We then claim that

$$\fint_{\Omega_1} |Du - Dw| \, dx \le c, \tag{3.7}$$

under the assumption that

$$|\mu|(\Omega_1) + |\mu|(\Omega_1) \left(\int_{\Omega_1} |Du| \, dx\right)^{2-p_1} \le c, \tag{3.8}$$

where the constants c depend on n, λ, γ_1, and γ_2. We will transfer back to the general case in Step 2.

Let us denote $\Omega_1^+ = \{x \in \Omega_1 : p(x) \geq 2\}$, $\Omega_1^- = \{x \in \Omega_1 : p(x) < 2\}$, $C_k^\pm = \{x \in \Omega_1^\pm : k < |u(x) - w(x)| \leq k+1\}$, and $D_k^\pm = \{x \in \Omega_1^\pm : |u(x) - w(x)| \leq k\}$ for $k \in \mathbb{N} \cup \{0\}$. We define the truncation operators

$$T_k(t) := \max\{-k, \min\{k, t\}\}, \quad \Phi_k(t) := T_1(t - T_k(t)) \quad \text{for } t \in \mathbb{R}. \tag{3.9}$$

Since u and w are the weak solutions of (1.1) and (3.4), respectively, it follows that

$$\int_{\Omega_1} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), D\varphi \rangle \, dx = \int_{\Omega_1} \mu\varphi \, dx \tag{3.10}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega_1)$.

First, substituting the test function $\varphi = T_k(u - w)$ in (3.10), and using (1.5) and (3.8), we obtain

$$\tilde{\lambda} \int_{D_k^+} |Du - Dw|^{p(x)} \, dx \leq \int_{\Omega_1} \langle \mathbf{a}(Du, x) - \mathbf{a}(Dw, x), Du - Dw \rangle \, dx \leq k|\mu|(\Omega_1) \leq ck.$$

Then, for $k \in \mathbb{N}$, we have

$$\int_{D_k^+} |Du - Dw| \, dx \leq \int_{D_k^+} (|Du - Dw| + 1)^{p(x)} \, dx \leq c(k+1). \tag{3.11}$$

Similarly, it follows that, for $k \in \mathbb{N}$,

$$\int_{C_k^+} |Du - Dw|^{p(x)} \, dx \leq c$$

by putting the test function $\varphi = \Phi_k(u - w)$ in (3.10). Since $p(x) \geq 2$ for $x \in C_k^+$, it follows from Hölder's inequality that

$$\int_{C_k^+} |Du - Dw| \, dx \leq |C_k^+|^{\frac{1}{2}} \left(\int_{C_k^+} (|Du - Dw| + 1)^{p(x)} \, dx \right)^{\frac{1}{2}} \leq c|C_k^+|^{\frac{1}{2}}.$$

From the definition of C_k^+, we find

$$|C_k^+| = \int_{C_k^+} 1 \, dx \leq \int_{C_k^+} \left(\frac{|u - w|}{k} \right)^{\frac{n}{n-1}} dx = k^{-\frac{n}{n-1}} \int_{C_k^+} |u - w|^{\frac{n}{n-1}} \, dx.$$

Therefore, we have

$$\int_{C_k^+} |Du - Dw| \, dx \leq ck^{-\frac{n}{2(n-1)}} \left(\int_{C_k^+} |u - w|^{\frac{n}{n-1}} \, dx \right)^{\frac{1}{2}}. \tag{3.12}$$

Then, by (3.11), (3.12), Hölder's inequality and Sobolev's inequality, we discover that for $k_0 \in \mathbb{N}$,

$$
\begin{aligned}
\int\limits_{\Omega_1^+} |Du-Dw|\,dx &= \int\limits_{D_{k_0}^+} |Du-Dw|\,dx + \sum_{k=k_0}^{\infty} \int\limits_{C_k^+} |Du-Dw|\,dx \\
&\le c(k_0+1) + c\sum_{k=k_0}^{\infty} k^{-\frac{n}{2(n-1)}} \left(\int\limits_{C_k^+} |u-w|^{\frac{n}{n-1}}\,dx \right)^{\frac12} \\
&\le c(k_0+1) + c\left[\sum_{k=k_0}^{\infty} k^{-\frac{n}{n-1}} \right]^{\frac12} \left(\sum_{k=k_0}^{\infty} \int\limits_{C_k^+} |u-w|^{\frac{n}{n-1}}\,dx \right)^{\frac12} \\
&\le c(k_0+1) + cH(k_0) \left(\int\limits_{\Omega_1} |Du-Dw|\,dx \right)^{\frac{n}{2(n-1)}},
\end{aligned}
\tag{3.13}
$$

where $H(k_0) := \left[\sum_{k=k_0}^{\infty} k^{-\frac{n}{n-1}}\right]^{\frac12}$.

For obtaining a comparison estimate in Ω_1^-, we now substitute the test function $\varphi = \Phi_k(u-w)$ in (3.10) and use (1.5), to find that

$$
\int\limits_{C_k^-} \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2\,dx \le c|\mu|(\Omega_1). \tag{3.14}
$$

On the other hand, from the fact that $p_1 \ge \gamma_1 > 2-\frac{1}{n}$, we can determine $\gamma = \gamma(n,\gamma_1) \in (0,1)$ such that $p_1 \ge \gamma_1 > 2-\frac{\gamma}{n}$, and so

$$
\frac{n(p_1-1)}{n-\gamma} \ge \frac{n(\gamma_1-1)}{n-\gamma} > 1. \tag{3.15}
$$

From Hölder's inequality and (3.14), we see that for $k \in \mathbb{N}\cup\{0\}$,

$$
\int\limits_{C_k^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx \le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}}\, |C_k^-|^{\frac{p_1-1}{p_1}}.
$$

For $k \in \mathbb{N}$, it follows from the definition of C_k^- that

$$
\begin{aligned}
\int\limits_{C_k^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx &\le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}} \left(\int\limits_{C_k^-} \left(\frac{|u-w|}{k}\right)^{\frac{n}{n-\gamma}} dx \right)^{\frac{p_1-1}{p_1}} \\
&\le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}} \frac{1}{k^{\frac{n(p_1-1)}{p_1(n-\gamma)}}} \left(\int\limits_{C_k^-} |u-w|^{\frac{n}{n-\gamma}}\,dx \right)^{\frac{p_1-1}{p_1}}.
\end{aligned}
$$

On the other hand, for $k=0$, we have that

$$
\int\limits_{C_0^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx \le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}}\, |B_1|^{\frac{\gamma_2-1}{\gamma_1}} \le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}}.
$$

From the two estimates above, Hölder's inequality, Sobolev's inequality, and (3.15), we discover that

$$
\begin{aligned}
I &:= \int_{\Omega_1^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx \\
&= \int_{C_0^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx \\
&\quad + \sum_{k=1}^{\infty} \int_{C_k^-} \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{2}} |Du-Dw|^2 \right)^{\frac{1}{p_1}} dx \\
&\le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(p_1-1)}{p_1(n-\gamma)}}} \left(\int_{C_k^-} |u-w|^{\frac{n}{n-\gamma}}\, dx \right)^{\frac{p_1-1}{p_1}} + 1 \right\} \\
&\le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}} \left\{ \left[\sum_{k=1}^{\infty} \frac{1}{k^{\frac{n(p_1-1)}{n-\gamma}}} \right]^{\frac{1}{p_1}} \left(\sum_{k=1}^{\infty} \int_{C_k^-} |u-w|^{\frac{n}{n-\gamma}}\, dx \right)^{\frac{p_1-1}{p_1}} + 1 \right\} \\
&\le c\,[|\mu|(\Omega_1)]^{\frac{1}{p_1}} \left\{ \left(\int_{\Omega_1} |Du-Dw|\, dx \right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} + 1 \right\}.
\end{aligned}
\tag{3.16}
$$

For $x \in \Omega_1^-$, we use Young's inequality to find that

$$
\begin{aligned}
|Du-Dw| &= \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw| \left(|Du|^2+|Dw|^2\right)^{\frac{2-p(x)}{4}} \\
&\le \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw| \left(|Du|^2+|Dw|^2+1\right)^{\frac{2-p_1}{4}} \\
&\le c \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw||Du-Dw|^{\frac{2-p_1}{2}} \\
&\quad + c \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw| \left(|Du|^2+1\right)^{\frac{2-p_1}{4}} \\
&\le \frac{1}{2}|Du-Dw| + c \left(\left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw| \right)^{\frac{2}{p_1}} \\
&\quad + c \left(|Du|^2+|Dw|^2\right)^{\frac{p(x)-2}{4}} |Du-Dw| \left(|Du|+1\right)^{\frac{2-p_1}{2}}.
\end{aligned}
$$

Then it follows from Hölder's inequality, (3.16), (3.8), and Young's inequality that

$$\int_{\Omega_1^-} |Du - Dw|\,dx \le cI + cI^{\frac{p_1}{2}} \left(\int_{\Omega_1} |Du| + 1\,dx \right)^{\frac{2-p_1}{2}}$$

$$\le c + c\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} + c\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n(p_1-1)}{2(n-\gamma)}} \tag{3.17}$$

$$\le c + c\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}}.$$

Combining (3.13) with (3.17), we have

$$\int_{\Omega_1} |Du - Dw|\,dx \le c + ck_0 + c\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n(p_1-1)}{p_1(n-\gamma)}} + cH(k_0)\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n}{2(n-1)}}.$$

Recall that $\frac{n(p_1-1)}{p_1(n-\gamma)} < \frac{p_1(n-1)}{p_1(n-\gamma)} < 1$. We then use Young's inequality to find

$$\int_{\Omega_1} |Du - Dw|\,dx \le c + ck_0 + cH(k_0)\left(\int_{\Omega_1} |Du - Dw|\,dx \right)^{\frac{n}{2(n-1)}}.$$

For $n > 2$, we know $0 < \frac{n}{2(n-1)} < 1$. Then there holds

$$\int_{\Omega_1} |Du - Dw|\,dx \le c \tag{3.18}$$

from Young's inequality and by taking $k_0 = 1$. For $n = 2$, we select $k_0 > 1$ sufficiently large in order to satisfy that $0 < cH(k_0) < 1$. Then the desired estimate (3.18) follows, and the claim (3.7) is now proved.

Step 2. Scaling and Normalization Let us define

$$\tilde{u}(y) = \frac{u(8ry)}{8Ar}, \quad \tilde{w}(y) = \frac{w(8ry)}{8Ar}, \quad \tilde{\mu}(y) = \frac{8r\mu(8ry)}{A^{p_0-1}}, \quad \tilde{p}(y) := p(8ry),$$

and

$$\tilde{\mathbf{a}}(\xi, y) = \frac{\mathbf{a}(A\xi, 8ry)}{A^{p_0-1}} \tag{3.19}$$

for $y \in \Omega_1$, $\xi \in \mathbb{R}^n$ and for some positive constant A, being determined below. We readily check that $\tilde{u}$ and $\tilde{w}$ are the weak solutions of

$$-\mathrm{div}\, \tilde{\mathbf{a}}(D\tilde{u}, y) = \tilde{\mu} \quad \text{in } \tilde{\Omega}_1 := \tilde{\Omega} \cap B_1, \tag{3.20}$$

where $\tilde{\Omega} := \{y \in \mathbb{R}^n : 8ry \in \Omega\}$, and

$$\begin{cases} -\mathrm{div}\, \tilde{\mathbf{a}}(D\tilde{w}, y) = 0 & \text{in } \tilde{\Omega}_1 \\ \tilde{w} = \tilde{u} & \text{on } \partial\tilde{\Omega}_1, \end{cases} \tag{3.21}$$

respectively. Moreover, we see that $p_1 \le \tilde{p}(y) \le p_2$ for $y \in \Omega_1$.

We next claim that $\tilde{\mathbf{a}}$ satisfies the corresponding growth condition (1.2) and uniformly ellipticity (1.3).

Indeed, if we set

$$A := \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}} + \chi_{\{p_0<2\}}\left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0}, \tag{3.22}$$

and denote $\bar{\xi} := A\xi$ and $x := 8ry$, then we discover that

$$|\xi||D_\xi \tilde{\mathbf{a}}(\xi, y)| + |\tilde{\mathbf{a}}(\xi, y)| \le A^{1-p_0}\left\{|\bar{\xi}||D_{\bar{\xi}}\mathbf{a}(\bar{\xi}, x)| + |\mathbf{a}(\bar{\xi}, x)|\right\} \le A^{p(x)-p_0}\Lambda|\bar{\xi}|^{\tilde{p}(y)-1}, \tag{3.23}$$

and

$$\begin{aligned}\left\langle D_\xi \tilde{\mathbf{a}}(\xi, y)\eta, \eta\right\rangle &= A^{2-p_0}\left\langle D_{\bar{\xi}}\mathbf{a}(\bar{\xi}, x)\eta, \eta\right\rangle \ge A^{2-p_0}\lambda\left|\bar{\xi}\right|^{p(x)-2}|\eta|^2\\ &\ge A^{p(x)-p_0}\lambda|\xi|^{p(x)-2}|\eta|^2 = A^{p(x)-p_0}\lambda|\xi|^{\tilde{p}(y)-2}|\eta|^2.\end{aligned} \tag{3.24}$$

In addition, it follows from (3.5) and (2.9) that

$$A \ge \left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}} \ge \left[\frac{8^n r|B_1||\Omega_{8r}|}{|B_{8r}|}\right]^{\frac{1}{p_0-1}} \ge \frac{1}{c} r^{\frac{1}{p_0-1}}. \tag{3.25}$$

On the other hand, we see from (2.9) and (3.6) that

$$\begin{aligned}A &\le [\nu(\Omega)+1]^{\frac{1}{\gamma_1-1}} r^{-\frac{n-1}{\gamma_1-1}} + c[\nu(\Omega)+1]\left(\int_\Omega |Du|\,dx + 1\right)^{2-\gamma_1} r^{-(n-1)-n(2-\gamma_1)}\\ &\le r^{-\frac{n}{\gamma_1-1}} + cr^{-n(3-\gamma_1)-(2-\gamma_1)}\\ &\le cr^{-\tilde{c}}\end{aligned} \tag{3.26}$$

for some $\tilde{c} = \tilde{c}(n, \gamma_1) > 1$. In the case that $p(x) - p_0 \ge 0$, we find from (2.1) and (3.25) that

$$A^{p(x)-p_0} \ge \left(\frac{1}{c}\right)^{\gamma_2-\gamma_1} r^{\frac{p(x)-p_0}{p_0-1}} \ge \frac{1}{c} r^{\frac{\omega(16r)}{\gamma_1-1}} \ge \frac{1}{c}, \tag{3.27}$$

and using (2.1) and (3.26), we discover

$$A^{p(x)-p_0} \le c\left(\frac{1}{r}\right)^{\omega(16r)\tilde{c}} \le c. \tag{3.28}$$

Similarly, in the case that $p(x) - p_0 < 0$, then we infer from (2.1), (3.25), and (3.26) that

$$\frac{1}{c} \le A^{p(x)-p_0} \le c \tag{3.29}$$

for some constant $c = c(n, \gamma_1, \gamma_2) > 0$. In light of (3.23), (3.24), (3.27), (3.28), and (3.29), we thus deduce

$$|\xi||D_\xi \tilde{\mathbf{a}}(\xi, y)| + |\tilde{\mathbf{a}}(\xi, y)| \le c\Lambda|\bar{\xi}|^{\tilde{p}(y)-1},$$

and

$$\left\langle D_\xi \tilde{\mathbf{a}}(\xi, y)\eta, \eta\right\rangle \ge \frac{\lambda}{c}|\xi|^{\tilde{p}(y)-2}|\eta|^2$$

for some constant $c = c(n, \gamma_1, \gamma_2) > 0$.

We next prove that (3.8) holds for $\tilde{u}$ and $\tilde{\mu}$, instead of u and μ, respectively. We recall (3.22) to see

$$|\tilde{\mu}|(\tilde{\Omega}_1) = A^{1-p_0}\frac{|\mu|(\Omega_{8r})}{(8r)^{n-1}} \le 1. \tag{3.30}$$

Moreover we note that

$$\begin{aligned} |\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1} |D\tilde{u}|\,dy\right)^{2-p_1} &\le cA^{p_1-p_0-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_1} \\ &\le cA^{-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_1}, \end{aligned} \tag{3.31}$$

as $A^{p_1-p_0} \le c$ by (3.25). But then we use (2.9), (3.6), and (2.1) to discover that

$$\begin{aligned} \left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_1} &= \left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{(2-p_0)+(p_0-p_1)} \\ &\le c\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0}\left(\left(\frac{16}{7}\right)^n \frac{1}{|B_{8r}|}\int_{\Omega} |Du|\,dx\right)^{p_0-p_1} \\ &\le c\left(\frac{1}{r}\right)^{\omega(16r)(n+1)}\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0} \\ &\le c\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0}. \end{aligned} \tag{3.32}$$

Combining (3.31) with (3.32), we find that, for $p_0 < 2$,

$$|\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1} |D\tilde{u}|\,dy\right)^{2-p_1} \le cA^{-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}}\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0} \le c.$$

On the other hand, for $p_0 \ge 2$, it follows from (2.9), (3.6), and (2.1) that

$$\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_1} \le \left(\fint_{\Omega_{8r}} |Du|\,dx + 1\right)^{(2-p_0)+(p_0-p_1)} \le c. \tag{3.33}$$

Then, from (3.31) and (3.33), we deduce that, for $p_0 \ge 2$,

$$|\tilde{\mu}|(\tilde{\Omega}_1)\left(\int_{\tilde{\Omega}_1} |D\tilde{u}|\,dy\right)^{2-p_1} \le cA^{p_1-p_0-1}\frac{|\mu|(\Omega_{8r})}{r^{n-1}} \le cA^{p_1-2}.$$

If $A > 1$, then $A^{p_1-2} \le 1$. If $A \le 1$, then $A^{p_0-1} \le A$, and so we have that $A^{p_1-2} = A^{p_1-p_0-1}A^{p_0-1} \le A^{p_1-p_0} \le c$. Thus, the property (3.8) holds for $\tilde{u}$ and $\tilde{\mu}$.

From the estimate (3.7) in Step 1, we obtain

$$\fint_{\tilde{\Omega}_1} |D\tilde{u} - D\tilde{w}|\, dx = \fint_{\Omega_{8r}} \frac{|Du - Dw|}{A}\, dx \le c$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2) > 0$, which completes the proof. $\square$

Remark 3.2. If $p(\cdot)$ is a constant, then in step 2 of Lemma 3.1, the vector field $\tilde{\mathbf{a}}$ directly satisfies the growth condition (1.2) and uniformly ellipticity (1.3), and we can readily derive the condition (3.8). We refer to [24,25] for details. In this case, we can prove Lemma 3.1 without introducing the measure ν. However, if $p(\cdot)$ is not a constant, then the log-Hölder continuity of $p(\cdot)$ and the property of ν are crucial to proving (1.2), (1.3) and (3.8) in step 2, see (3.25) and (3.26).

The following lemma yields some self-improving property for the homogeneous equation (3.4):

Lemma 3.3. *Let $M_1 > 1$. Suppose that $R_0 > 0$ satisfies*

$$R_0 \le \min\left\{\frac{R}{2}, \frac{1}{4}, \frac{1}{2M_1}\right\} \quad \text{and} \quad \omega(2R_0) \le \frac{1}{2n} < 1. \tag{3.34}$$

Then there exists a constant $\sigma_0 = \sigma_0(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$ such that the following holds: for any $r \in \left(0, \frac{R_0}{8}\right]$, if w is the weak solution of (3.4) with

$$\int_{\Omega_{8r}} |Dw|\, dx + 1 \le M_1, \tag{3.35}$$

then there is a constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, t) > 0$ such that for every $t \in (0, 1]$,

$$\left(\fint_{\Omega_{\tilde{r}}(\tilde{x}_0)} |Dw|^{p(x)(1+\sigma)}\, dx\right)^{\frac{1}{1+\sigma}} \le c\left\{\left(\fint_{\Omega_{2\tilde{r}}(\tilde{x}_0)} |Dw|^{p(x)t}\, dx\right)^{\frac{1}{t}} + 1\right\}, \tag{3.36}$$

whenever $0 < \sigma \le \sigma_0$ and $\Omega_{2\tilde{r}}(\tilde{x}_0) \subset \Omega_{8r}$ with $\tilde{r} \le 4r$.

Proof. To simplify notation, we write $B_{\tilde{r}} \equiv B_{\tilde{r}}(\tilde{x}_0)$, $B_{2\tilde{r}} \equiv B_{2\tilde{r}}(\tilde{x}_0)$, $\Omega_{\tilde{r}} \equiv \Omega_{\tilde{r}}(\tilde{x}_0)$, $\Omega_{2\tilde{r}} \equiv \Omega_{2\tilde{r}}(\tilde{x}_0)$, $\bar{p}_1 := \inf_{x \in \Omega_{2\tilde{r}}} p(x)$, and $\bar{p}_2 := \sup_{x \in \Omega_{2\tilde{r}}} p(x)$.

We first consider the interior case, that is, $\Omega_{2\tilde{r}} = B_{2\tilde{r}}$. We take $\eta^{\bar{p}_2}(w - \bar{w}_{B_{2\tilde{r}}})$ as a test function in (3.4), where $\eta \in C_0^\infty(B_{2\tilde{r}})$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\tilde{r}}$, and $|D\eta| \le \frac{2}{\tilde{r}}$. Then it follows from (1.5) and Young's inequality that

$$\fint_{B_{\tilde{r}}} |Dw|^{p(x)}\, dx \le c\left\{\fint_{B_{2\tilde{r}}} \left|\frac{w - \bar{w}_{B_{2\tilde{r}}}}{\tilde{r}}\right|^{\bar{p}_2} dx + 1\right\}. \tag{3.37}$$

Using Sobolev–Poincaré's inequality, we have

$$\left(\fint_{B_{2\tilde{r}}} \left|\frac{w - \bar{w}_{B_{2\tilde{r}}}}{\tilde{r}}\right|^{\bar{p}_2} dx\right)^{\frac{1}{\bar{p}_2}} \le c\left(\fint_{B_{2\tilde{r}}} |Dw|^{\frac{n\bar{p}_2}{n+\bar{p}_2}}\, dx\right)^{\frac{n+\bar{p}_2}{n\bar{p}_2}}. \tag{3.38}$$

From (2.1) and (3.34), we note that $\bar{p}_2 - \bar{p}_1 \le \omega(4\tilde{r}) \le \omega(2R_0) \le \frac{1}{2n}$. By setting $s := 1 + \frac{1}{2n}$, we find that

$$\frac{\bar{p}_2}{\bar{p}_2 - \bar{p}_1 + s} \ge \frac{n\bar{p}_2}{n+1} \ge \frac{n\bar{p}_2}{n+\bar{p}_2}.$$

Then, by (3.37), (3.38) and Hölder's inequality, we discover

$$\fint_{B_{\tilde r}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{B_{2\tilde r}} |Dw|^{\frac{n\bar p_2}{n+\bar p_2}}\,dx\right)^{\frac{n+\bar p_2}{n}} + 1\right\} \le c\left\{\left(\fint_{B_{2\tilde r}} |Dw|^{\frac{\bar p_2}{\bar p_2-\bar p_1+s}}\,dx\right)^{\bar p_2-\bar p_1+s} + 1\right\}.$$

But then the interpolation inequality yields

$$\left(\fint_{B_{2\tilde r}} |Dw|^{\frac{\bar p_2}{\bar p_2-\bar p_1+s}}\,dx\right)^{\frac{\bar p_2-\bar p_1+s}{\bar p_2}} \le \left(\fint_{B_{2\tilde r}} |Dw|^{\frac{\bar p_1}{s}}\,dx\right)^{\frac{s}{\bar p_2}} \left(\fint_{B_{2\tilde r}} |Dw|\,dx\right)^{\frac{\bar p_2-\bar p_1}{\bar p_2}}.$$

Moreover, it follows from (3.34), (3.35) and (2.1) that

$$\begin{aligned}\left(\fint_{B_{2\tilde r}} |Dw|\,dx\right)^{\bar p_2-\bar p_1} &\le \left(\int_{\Omega_{8r}} |Dw|\,dx\right)^{\bar p_2-\bar p_1} |B_{2\tilde r}|^{-n(\bar p_2-\bar p_1)} \le cM_1{}^{\omega(4\tilde r)}(2\tilde r)^{-n\omega(4\tilde r)} \\ &\le c\left(\frac{1}{2R_0}\right)^{\omega(2R_0)}\left(\frac{1}{4\tilde r}\right)^{\omega(4\tilde r)n} \le c.\end{aligned}$$

Consequently, we have

$$\fint_{B_{\tilde r}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{B_{2\tilde r}} |Dw|^{\frac{\bar p_1}{s}}\,dx\right)^{s} + 1\right\} \le c\left\{\left(\fint_{B_{2\tilde r}} |Dw|^{\frac{p(x)}{s}}\,dx\right)^{s} + 1\right\}. \tag{3.39}$$

We next consider the boundary case, that is, $\Omega_{2\tilde r} \ne B_{2\tilde r}$. Without loss of generality, one can assume that $\tilde x_0 \in \partial\Omega \cap B_{8r}(0)$. Taking a test function $\eta^{\bar p_2}u$ to (3.4), and using Sobolev–Poincaré's inequality along with the measure density condition (2.10), we have

$$\fint_{\Omega_{\tilde r}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde r}} |Dw|^{\frac{\bar p_1}{s}}\,dx\right)^{s}\left(\fint_{\Omega_{2\tilde r}} |Dw|\,dx\right)^{\bar p_2-\bar p_1} + 1\right\}.$$

Now it follows from (3.34), (3.35), (2.9) and (2.1) that

$$\left(\fint_{\Omega_{2\tilde r}} |Dw|\,dx\right)^{\bar p_2-\bar p_1} \le \left(\int_{\Omega_{8r}} |Dw|\,dx\right)^{\bar p_2-\bar p_1} |\Omega_{2\tilde r}|^{-n(\bar p_2-\bar p_1)} \le cM_1{}^{\omega(4\tilde r)}\left(\left(\frac{7}{16}\right)^{n}|B_{2\tilde r}|\right)^{-n\omega(4\tilde r)} \le c.$$

Therefore, we have

$$\fint_{\Omega_{\tilde r}} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde r}} |Dw|^{\frac{p(x)}{s}}\,dx\right)^{s} + 1\right\}. \tag{3.40}$$

Applying the modified version of Gehring's lemma [27, Remark 6.12] to the estimates (3.39) and (3.40), we finally reach the desired conclusion. □

Corollary 3.4. *Under the same assumptions and conclusion as in Lemma 3.3, we have*

$$\fint_{\Omega_{\tilde r}(\tilde x_0)} |Dw|^{p(x)}\,dx \le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} |Dw|\,dx\right)^{\bar p_2} + 1\right\} \tag{3.41}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof. From Hölder's inequality and Lemma 3.3, we have

$$\begin{aligned}
\fint_{\Omega_{\tilde r}(\tilde x_0)} |Dw|^{p(x)}\,dx &\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)^{\bar p_2 t}\,dx\right)^{\frac{1}{t}}+1\right\}\\
&\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)^{\frac{\bar p_2}{\gamma_2}}\,dx\right)^{\gamma_2}+1\right\}\\
&\le c\left\{\left(\fint_{\Omega_{2\tilde r}(\tilde x_0)} (|Dw|+1)\,dx\right)^{\bar p_2}+1\right\},
\end{aligned}$$

by taking $t=\frac{1}{\gamma_2}$ for the second inequality. Thus, the corollary follows. □

Now we will specifically derive the universal constant M_1 given in Lemma 3.3. Suppose that $R_0>0$ satisfies (3.6). Then from Lemma 3.1 and the measure density condition (2.9), we calculate

$$\begin{aligned}
\int_{\Omega_{8r}} |Dw|\,dx &\le \int_{\Omega_{8r}} |Du|\,dx + c|\Omega_{8r}|\left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0-1}}\\
&\qquad + c\chi_{\{p_0<2\}}|\Omega_{8r}|\left[\frac{\nu(\Omega_{8r})}{r^{n-1}}\right]\left(\fint_{\Omega_{8r}} |Du|\,dx\right)^{2-p_0}\\
&\le \int_{\Omega} |Du|\,dx + cr^{\alpha}\,[\nu(\Omega)]^{\frac{1}{p_0-1}} + c\chi_{\{p_0<2\}}r^{\beta}\nu(\Omega)\left(\int_{\Omega} |Du|\,dx\right)^{2-p_0}\\
&\le \int_{\Omega} |Du|\,dx + c\,diam(\Omega)^{\alpha}\,[\nu(\Omega)+1]^{\frac{1}{\gamma_1-1}}\\
&\qquad + c\chi_{\{p_0<2\}}\,diam(\Omega)^{\beta}\,[\nu(\Omega)+1]\left(\int_{\Omega} |Du|\,dx+1\right)^{2-\gamma_1}
\end{aligned}$$

for some $c=c(n,\lambda,\gamma_1,\gamma_2)>0$, where $\alpha:=n-\frac{n-1}{\gamma_1-1}>0$, $\beta:=\alpha(\gamma_1-1)>0$, and $diam(\Omega)$ is the diameter of Ω. We define

$$M:=\int_{\Omega} |Du|\,dx + c\,diam(\Omega)^{\alpha}\,[\nu(\Omega)+1]^{\frac{1}{\gamma_1-1}}+1.$$

Then we conclude that

$$\int_{\Omega_{8r}} |Dw|\,dx+1 \le (c\chi_{\{p_0<2\}}+1)M \le c_0M =: M_1 \tag{3.42}$$

for some $c_0=c_0(n,\lambda,\gamma_1,\gamma_2)>0$.

With this M_1, we obtain the following higher integrability result which is used later:

Lemma 3.5. *Suppose that $R_0 > 0$ satisfies (3.6), (3.34) with M_1 given in (3.42). Let w be the weak solution of (3.4) satisfying (3.3). Then we have $w \in W^{1,p_2}(\Omega_{3r})$ and the estimate*

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\,dx \le c\left\{\left(\fint_{\Omega_{8r}} |Dw|\,dx\right)^{p_2} + 1\right\} \tag{3.43}$$

for some constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Proof. We deduce from Lemma 3.3, (2.1), and [12, Section 3.3] that

$$\fint_{\Omega_{3r}} |Dw|^{p_2}\,dx \le c\left\{\fint_{\Omega_{4r}} |Dw|^{p(x)}\,dx + 1\right\}.$$

Applying Corollary 3.4 with $\tilde{r}$ and $\tilde{x}_0$ replaced by $4r$ and 0, we obtain the desired estimate (3.43). □

We next consider a new vector field $\mathbf{b} = \mathbf{b}(\xi, x) : \mathbb{R}^n \times \Omega_{8r} \to \mathbb{R}^n$ by

$$\mathbf{b}(\xi, x) = \mathbf{a}(\xi, x)|\xi|^{p_2 - p(x)}.$$

Then it satisfies the following growth and ellipticity conditions:

$$|\xi||D_\xi \mathbf{b}(\xi, x)| + |\mathbf{b}(\xi, x)| \le 3\Lambda|\xi|^{p_2-1}, \tag{3.44}$$

$$\frac{\lambda}{2}|\xi|^{p_2-2}|\eta|^2 \le \langle D_\xi \mathbf{b}(\xi, x)\eta, \eta\rangle \tag{3.45}$$

for all $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega_{8r}$ provided that

$$p_2 - p_1 \le \omega(16r) \le \omega(2R_0) \le \min\left\{1, \frac{\lambda}{2\Lambda}\right\}, \tag{3.46}$$

see [13] for details. Here λ and Λ are the constants given in (1.2) and (1.3), respectively. We denote by $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ the integral average of $\mathbf{b}(\xi, \cdot)$ on B_{8r}^+, as

$$\bar{\mathbf{b}}(\xi) = \fint_{B_{8r}^+} \mathbf{b}(\xi, x)\,dx.$$

Then $\bar{\mathbf{b}}$ also satisfies (3.44) and (3.45) with $\mathbf{b}(\xi, \cdot)$ replaced by $\bar{\mathbf{b}}(\xi)$. Moreover, we observe that

$$\sup_{\xi \in \mathbb{R}^n \setminus \{0\}} \frac{\left|\mathbf{b}(\xi, \cdot) - \bar{\mathbf{b}}(\xi)\right|}{|\xi|^{p_2-1}} = \theta(\mathbf{a}, B_{8r}^+)(x),$$

where θ is defined in (2.2). Then we recall (2.3) to discover that

$$\sup_{0<r\le R} \fint_{B_r^+} \theta(\mathbf{a}, B_r^+)(x)\,dx \le 4\delta.$$

We next let $v \in w + W_0^{1,p_2}(\Omega_{3r})$ be the weak solution of the homogeneous frozen problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(Dv) = 0 & \text{in } \Omega_{3r}, \\ v = w & \text{on } \partial\Omega_{3r}, \end{cases} \tag{3.47}$$

where w is the weak solution of (3.4), which belongs to $W^{1,p_2}(\Omega_{3r})$ from Lemma 3.5. By putting the test function $v - w$ into (3.47), we derive the standard energy estimate

$$\fint_{\Omega_{3r}} |Dv|^{p_2}\,dx \le c\fint_{\Omega_{3r}} |Dw|^{p_2}\,dx. \tag{3.48}$$

From Corollary 3.4, Lemma 3.5, and [12, Lemma 3.7], we obtain the comparison estimate between (3.4) and (3.47), as we now state

Lemma 3.6. *Suppose that $R_0 > 0$ satisfies (3.6), (3.34), (3.46) with M_1 given in (3.42), and*

$$p_2 - p_1 \le \omega(16r) \le \omega(2R_0) \le \frac{\sigma_0}{4}, \tag{3.49}$$

where σ_0 is given in Lemma 3.3. Let w be the weak solution of (3.4) satisfying (3.3), and let v be as in (3.47). Then there is a constant $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$ such that

$$\fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\, dx \le c\delta^{\frac{\sigma_0}{4+\sigma_0}} \left\{ \left(\fint_{\Omega_{8r}} |Dw|\, dx \right)^{p_2} + 1 \right\}. \tag{3.50}$$

We now consider a weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of the reference problem

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(D\bar{v}) = 0 & \text{in } B_{2r}^+, \\ \bar{v} = 0 & \text{on } B_{2r} \cap \{x_n = 0\}. \end{cases} \tag{3.51}$$

Then we have the following Lipschitz regularity of $\bar{v}$ up to the flat boundary:

Lemma 3.7. *(See [35].) For any weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of (3.51), we have $D\bar{v} \in L^\infty(B_r^+)$ and*

$$\|D\bar{v}\|_{L^\infty(B_r^+)} \le c \fint_{B_{2r}^+} |D\bar{v}|\, dx \tag{3.52}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

Note that the constant c given in (3.52) actually depends only on n, λ, Λ, and p_2; however, since $\gamma_1 \le p_2 \le \gamma_2$, we can choose c depending only on $n, \lambda, \Lambda, \gamma_1$, and γ_2.

We can now state the comparison estimate between (3.47) and (3.51).

Lemma 3.8. *(See [13].) For any $0 < \epsilon < 1$, there exists $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $v \in W^{1,p_2}(\Omega_{3r})$ is the weak solution of (3.47) with (3.3), then there exists a weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of (3.51) such that*

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}|^{p_2}\, dx \le \epsilon^{p_2} \fint_{\Omega_{3r}} |Dv|^{p_2}\, dx, \tag{3.53}$$

where $\bar{v}$ is extended by zero from B_{2r}^+ to Ω_{2r}.

We finally summarize the comparison L^1-estimates near a boundary region.

Lemma 3.9. *Suppose that $R_0 > 0$ satisfies (3.6), (3.34), (3.46), and (3.49) with M_1 given in (3.42). Let $\rho > 1$ and $0 < r \le \frac{R_0}{8}$. Suppose that Ω_{8r} satisfies (3.3). Then, for any $0 < \epsilon < 1$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing, $u \in W_0^{1,p(\cdot)}(\Omega)$, $w \in u + W_0^{1,p(\cdot)}(\Omega_{8r})$, and $v \in w + W_0^{1,p_2}(\Omega_{3r})$ are the weak solutions of (1.1), (3.4), and (3.47), respectively, with*

$$\fint_{\Omega_{8r}} |Du|\, dx \le \rho \quad \text{and} \quad \left[\frac{\nu(\Omega_{8r})}{r^{n-1}} \right]^{\frac{1}{p_0-1}} \le \delta\rho, \tag{3.54}$$

where ν is given in (3.5), then there exists a weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of (3.51) such that

$$\fint_{\Omega_{2r}} |Du - D\bar{v}|\,dx \le \epsilon\rho \quad \textit{and} \quad \|D\bar{v}\|_{L^\infty(\Omega_r)} \le c\rho \tag{3.55}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$. *Here* $\bar{v}$ *is extended by zero from* B_{2r}^+ *to* Ω_{2r}.

Proof. From Lemma 3.1, we have

$$\fint_{\Omega_{8r}} |Du - Dw|\,dx \le c\delta^{\min\{1,\gamma_1-1\}}\rho \ \text{ and } \fint_{\Omega_{8r}} |Dw|\,dx \le c\rho. \tag{3.56}$$

According to Hölder's inequality and Lemma 3.6, we observe

$$\fint_{\Omega_{3r}} |Dw - Dv|\,dx \le \left(\fint_{\Omega_{3r}} |Dw - Dv|^{p_2}\,dx \right)^{\frac{1}{p_2}} \le c\delta^{\frac{\sigma_0}{\gamma_2(4+\sigma_0)}}\rho, \tag{3.57}$$

and

$$\fint_{\Omega_{3r}} |Dv|\,dx \le c\rho. \tag{3.58}$$

At this point that by Lemma 3.8 with ϵ replaced by $\tilde{\epsilon}$, there is a weak solution $\bar{v} \in W^{1,p_2}(B_{2r}^+)$ of (3.51) such that

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}|^{p_2}\,dx \le \tilde{\epsilon}^{p_2} \fint_{\Omega_{3r}} |Dv|^{p_2}\,dx.$$

We also discover from Hölder's inequality, (3.48), and Lemma 3.5 that

$$\fint_{\Omega_{2r}} |Dv - D\bar{v}|\,dx \le c\tilde{\epsilon}\left\{ \fint_{\Omega_{8r}} |Dw|\,dx + 1 \right\} \le 2c\tilde{\epsilon}\rho \le \frac{\epsilon}{3}\rho \tag{3.59}$$

by choosing small $\tilde{\epsilon}$ such that $0 < \tilde{\epsilon} \le \frac{\epsilon}{6c}$, and it follows from (3.58) and (3.59) that

$$\fint_{\Omega_{2r}} |D\bar{v}|\,dx \le c\rho. \tag{3.60}$$

Then we combine (3.56), (3.57) and (3.59), to discover

$$\begin{aligned}\fint_{\Omega_{2r}} |Du - D\bar{v}|\,dx &\le \fint_{\Omega_{2r}} |Du - Dw| + |Dw - Dv| + |Dv - D\bar{v}|\,dx \\ &\le c\delta^{\min\{1,p_0-1\}}\rho + c\delta^{\frac{\sigma_0}{\gamma_2(4+\sigma_0)}}\rho + \frac{\epsilon}{3}\rho \\ &\le \epsilon\rho,\end{aligned}$$

by selecting δ sufficiently small.

On the other hand, according to Lemma 3.7, (3.59) and (3.60), we obtain

$$\|D\bar{v}\|_{L^\infty(\Omega_r)} \le c\rho,$$

which completes the proof. □

3.2. Interior comparisons

With the same spirit as in the boundary case, one can derive a comparison estimate in L^1 for the interior case, and we just sketch it here for the sake of simplicity. Let $0 < r \le \frac{R_0}{8}$ with $B_{8r}(x_0) \subset\subset \Omega$, where R_0 is selected so small that it satisfies (3.6), (3.34), (3.46), and (3.49) with M_1 given in (3.42). In this subsection, we denote

$$p_0 := p(x_0), \quad p_1 := \inf_{x \in B_{8r}(x_0)} p(x), \quad p_2 := \sup_{x \in B_{8r}(x_0)} p(x), \quad \text{and} \quad B_{kr} \equiv B_{kr}(x_0) \quad (k \in \mathbb{N}).$$

With the weak solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of (1.1), let $w \in u + W_0^{1,p(\cdot)}(B_{8r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \mathbf{a}(Dw, x) = 0 & \text{in } B_{8r}, \\ \qquad\qquad w = u & \text{on } \partial B_{8r}. \end{cases} \tag{3.61}$$

Then from the same argument for the boundary case, we have $w \in W^{1,p_2}(B_{3r})$.

Let $v \in w + W_0^{1,p_2}(B_{3r})$ be the weak solution of

$$\begin{cases} \operatorname{div} \bar{\mathbf{b}}(Dv) = 0 & \text{in } B_{3r}, \\ \qquad\qquad v = w & \text{on } \partial B_{3r}, \end{cases} \tag{3.62}$$

where $\bar{\mathbf{b}} = \bar{\mathbf{b}}(\xi) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\bar{\mathbf{b}}(\xi) = \fint_{B_{8r}} \mathbf{b}(\xi, x)\, dx = \fint_{B_{8r}} \mathbf{a}(\xi, x)|\xi|^{p_2 - p(x)}\, dx.$$

Then we have $Dv \in L^\infty(B_{2r})$ and

$$\|Dv\|_{L^\infty(B_{2r})} \le c \fint_{B_{3r}} |Dv|\, dx$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$, see [22] for details.

We now state the comparison L^1-estimates for the interior case.

Lemma 3.10. *Suppose that $R_0 > 0$ satisfies (3.6), (3.34), (3.46), and (3.49) with M_1 given in (3.42). Let $\rho > 1$ and $0 < r \le \frac{R_0}{8}$. Then, for any $0 < \epsilon < 1$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that if $p(\cdot)$ and $\mathbf{a}(\xi, x)$ satisfy the assumptions* **(AP)** *and* **(AA)** *in Section 2.1, respectively, and if $u \in W_0^{1,p(\cdot)}(\Omega)$, $w \in u + W_0^{1,p(\cdot)}(B_{8r})$, and $v \in w + W_0^{1,p_2}(B_{3r})$ are the weak solutions of (1.1), (3.61), and (3.62), respectively, with*

$$\fint_{B_{8r}} |Du|\, dx \le \rho \quad \text{and} \quad \left[\frac{\nu(B_{8r})}{r^{n-1}}\right]^{\frac{1}{p_0 - 1}} \le \delta\rho, \tag{3.63}$$

where ν is given in (3.5), then

$$\fint_{B_{3r}} |Du - Dv|\, dx \le \epsilon\rho \quad \text{and} \quad \|Dv\|_{L^\infty(B_{2r})} \le c\rho \tag{3.64}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$.

4. Covering arguments

Now, we consider a SOLA u of (1.1) and the weak solutions u_h, $h \in \mathbb{N}$, of (1.6), where $\mu_h = \mu * \phi_h$ with ϕ_h the usual mollifier. Suppose that $(p(\cdot), \mathbf{a}, \Omega)$ is (δ, R)-vanishing. Since $\mu_h \in C^\infty(\Omega)$, one can apply all the results obtained in Section 3 to $u = u_h$ and $\mu = \mu_h$. In this case, we denote by w_h, v_h, and $\bar{v}_h$ the weak solutions to (3.4) or (3.61), (3.47) or (3.62), and (3.51), respectively. Moreover, we assume that $R_0 > 0$ satisfies

$$R_0 \le \min\left\{\frac{R}{2}, \frac{1}{6M_1}, \frac{1}{\int_\Omega |Du|\,dx + 2}, \frac{1}{\nu(\Omega)+2}\right\}, \tag{4.1}$$

$$\omega(2R_0) \le \min\left\{\frac{\lambda}{2\Lambda}, \frac{1}{2n}, \frac{\sigma_0}{4}\right\}, \tag{4.2}$$

where ν, M_1, and σ_0 are given in (3.5), (3.42), and Lemma 3.3, respectively. Then, thanks to (1.7) and (1.8), we see that R_0 satisfies (3.6), (3.34), (3.46), and (3.49) with (u, μ) replaced by (u_h, μ_h) for sufficiently large h.

For any fixed $\epsilon \in (0, 1)$ and $N > 1$, we define

$$\lambda_0 := \frac{1}{\epsilon |B_{R_0}|}\left\{\int_\Omega |Du|\,dx + 1\right\} > 1 \tag{4.3}$$

and upper-level sets: for $k \in \mathbb{N} \cup \{0\}$,

$$C_k := \left\{x \in \Omega : \mathcal{M}(|Du|)(x) > N^{k+1}\lambda_0\right\},$$

$$D_k := \left\{x \in \Omega : \mathcal{M}(|Du|)(x) > N^k\lambda_0\right\} \cup \left\{x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^k \lambda_0\right\}.$$

Note that ϵ and N will be determined later as universal constants depending only on $n, \lambda, \Lambda, \gamma_1, \gamma_2$, and q.

We now verify two assumptions of the Vitali type covering lemma (Lemma 2.4).

Lemma 4.1. *There exists a constant $N_1 = N_1(n) > 1$ such that for any fixed $N \ge N_1$ and $k \in \mathbb{N} \cup \{0\}$,*

$$|C_k| \le \frac{\epsilon}{(1000)^n}|B_{R_0}|. \tag{4.4}$$

Proof. For each $k \in \mathbb{N} \cup \{0\}$, $|C_k| \le |C_0|$. Thus, we only need to show that (4.4) holds for $k = 0$. It follows from (2.7) and (4.3) that

$$|C_0| = |\{x \in \Omega : \mathcal{M}(|Du|)(x) > N\lambda_0\}| \le \frac{c}{N\lambda_0}\int_\Omega |Du|\,dx \le \frac{c\epsilon}{N}|B_{R_0}| \le \frac{\epsilon}{(1000)^n}|B_{R_0}|,$$

by selecting $N \ge N_1 = c(1000)^n > 1$. □

Lemma 4.2. *There is a constant $N_2 = N_2(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 1$ so that for any $\epsilon > 0$, there exists a small constant $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon) > 0$ such that for any fixed $N \ge N_2$, $k \in \mathbb{N} \cup \{0\}$, $y_0 \in \Omega$ and $r_0 \le \frac{R_0}{1000}$, if*

$$|C_k \cap B_{r_0}(y_0)| \ge \epsilon |B_{r_0}(y_0)|, \tag{4.5}$$

then $\Omega_{r_0}(y_0) \subset D_k$.

Proof. We simply write $\lambda_k := N^k\lambda_0 > 1$, where $N \ge N_2 > 1$. We argue by contradiction. Suppose that there exists $y_1 \in \Omega_{r_0}(y_0)$ such that $y_1 \notin D_k$. Then we have

$$\frac{1}{|B_r(y_1)|}\int_{\Omega_r(y_1)} |Du|\,dx \le \lambda_k \quad \text{and} \quad \left[\frac{\nu(B_r(y_1))}{r^{n-1}}\right]^{\frac{1}{p(y_1)-1}} \le c(n, \gamma_1)\delta\lambda_k \tag{4.6}$$

for all $r > 0$.

We divide the proof into two cases: an interior and a boundary case.

Case 1. The interior case $B_{10r_0}(y_1) \subset \Omega$.

Since $y_1 \in \Omega_{r_0}(y_0)$, we see that $\overline{B_{8r_0}(y_0)} \subset B_{10r_0}(y_1)$. We set

$$p_1 := \inf_{x \in B_{8r_0}(y_0)} p(x) \quad \text{and} \quad p_2 := \sup_{x \in B_{8r_0}(y_0)} p(x).$$

Then it follows that $p_2 - p_1 \le \omega(16r_0)$.

From (4.6), we have

$$\fint_{B_{8r_0}(y_0)} |Du|\,dx \le \frac{|B_{10r_0}(y_1)|}{|B_{8r_0}(y_0)|} \fint_{B_{10r_0}(y_1)} |Du|\,dx \le \left(\frac{5}{4}\right)^n \lambda_k,$$

and it follows from (1.7) that for any $\tilde{\epsilon} \in (0,1)$,

$$\fint_{B_{8r_0}(y_0)} |Du - Du_h|\,dx \le \tilde{\epsilon}\lambda_k \tag{4.7}$$

for h large enough. Then we discover

$$\fint_{B_{8r_0}(y_0)} |Du_h|\,dx \le \fint_{B_{8r_0}(y_0)} |Du|\,dx + \fint_{B_{8r_0}(y_0)} |Du - Du_h|\,dx \le \left(\left(\frac{5}{4}\right)^n + \tilde{\epsilon}\right)\lambda_k.$$

We next claim that

$$\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} \le c \tag{4.8}$$

for some constant $c = c(n) > 0$.

If $p(y_1) > p(y_0)$, then $p(y_1) - p(y_0) \le \omega(16r_0)$, and so we see from (4.1) and (2.1) that

$$\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} \le \left(\frac{1}{r_0}\right)^{(n-1)\omega(16r_0)} (\nu(\Omega)+1)^{\omega(16r_0)} \le c\left(\frac{1}{r_0}\right)^{n\omega(16r_0)} \le ce^{\delta n} \le c.$$

If $p(y_1) < p(y_0)$, then $p(y_0) - p(y_1) \le \omega(16r_0)$, and so we find from (2.1) and (3.5) that

$$\begin{aligned}\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{p(y_1)-p(y_0)} &= \left[\frac{8^n r_0 |B_1| \nu(\overline{B_{8r_0}(y_0)})}{|B_{8r_0}|}\right]^{p(y_1)-p(y_0)} \le \left[8^n r_0 |B_1|\right]^{p(y_1)-p(y_0)} \\ &\le c\left(\frac{1}{16r_0}\right)^{\omega(16r_0)} \le ce^{\delta} \le c.\end{aligned}$$

In any case, we obtain the inequality (4.8). We therefore have from (4.6) and (4.8) that

$$\begin{aligned}\left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} &= \left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{1}{p(y_1)-1}+\frac{p(y_1)-p(y_0)}{(p(y_0)-1)(p(y_1)-1)}} \\ &\le \left[\frac{\nu(B_{10r_0}(y_1))}{r_0^{n-1}}\right]^{\frac{1}{p(y_1)-1}} \left[\frac{\nu(\overline{B_{8r_0}(y_0)})}{r_0^{n-1}}\right]^{\frac{p(y_1)-p(y_0)}{(p(y_0)-1)(p(y_1)-1)}} \\ &\le c\delta\lambda_k.\end{aligned}$$

In addition, it follows from (1.8) that

$$\left[\frac{\nu_h(B_{8r_0}(y_0))}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le \left[\frac{\nu(\overline{B_{8r_0}(y_0)}) + \bar{\epsilon}}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le c_1\delta\lambda_k,$$

by selecting $\bar{\epsilon}$ small enough, the constant c_1 depending only on n, γ_1, and γ_2. Here ν_h is given in (3.5) with μ replaced by μ_h.

Consequently, we obtain

$$\fint_{B_{8r_0}(y_0)} |Du_h|\,dx \le c_2\lambda_k \quad \text{and} \quad \left[\frac{\nu_h(B_{8r_0}(y_0))}{r_0^{n-1}}\right]^{\frac{1}{p(y_0)-1}} \le c_2\delta\lambda_k, \tag{4.9}$$

where $c_2 := \max\left\{\left(\frac{5}{4}\right)^n + \tilde{\epsilon}, c_1\right\}$. Applying Lemma 3.10 with x_0, ρ, r, and ϵ replaced by y_0, $c_2\lambda_k$, r_0, and η, respectively, we can find $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \eta)$ such that

$$\fint_{B_{3r_0}(y_0)} |Du_h - Dv_h|\, dx \le c_2\eta\lambda_k \quad \text{and} \quad \|Dv_h\|_{L^\infty(B_{2r_0}(y_0))} \le cc_2\lambda_k =: c_3\lambda_k \tag{4.10}$$

for some $c_3 = c_3(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$. Thus, we have from (4.7) and (4.10) that

$$\fint_{B_{2r_0}(y_0)} |Du - Dv_h|\, dx \le \left(4^n + \left(\frac{3}{2}\right)^n\right) c_2\eta\lambda_k =: c_4\eta\lambda_k \tag{4.11}$$

by choosing sufficiently small $\tilde{\epsilon}$ with $\tilde{\epsilon} \le c_2\eta$.

Now we claim that

$$\begin{aligned} C_k \cap B_{r_0}(y_0) &= \left\{x \in B_{r_0}(y_0) : \mathcal{M}(|Du|)(x) > N\lambda_k\right\} \\ &\subset \left\{x \in B_{r_0}(y_0) : \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(x) > \lambda_k\right\} =: Q, \end{aligned} \tag{4.12}$$

provided $N \ge N_2 \ge \max\{3^n, 1 + c_3\}$.

Let $y \notin Q$. If $y \notin B_r(y_0)$, then it is done. Suppose $y \in B_r(y_0)$. If $\tilde{r} < r_0$, then $B_{\tilde{r}}(y) \subset B_{2r_0}(y_0)$. We have from (4.10) that

$$\begin{aligned} \fint_{B_{\tilde{r}}(y)} |Du|\, dx &\le \fint_{B_{\tilde{r}}(y)} \chi_{B_{2r_0}(y_0)}|Du - Dv_h|\, dx + \fint_{B_{\tilde{r}}(y)} |Dv_h|\, dx \\ &\le \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(y) + c_3\lambda_k \\ &\le (1 + c_3)\lambda_k. \end{aligned}$$

If $\tilde{r} \ge r_0$, then $B_{\tilde{r}}(y) \subset B_{2\tilde{r}}(y_0) \subset B_{3\tilde{r}}(y_1)$. We have from (4.6) that

$$\frac{1}{|B_{\tilde{r}}(y)|}\int_{\Omega_{\tilde{r}}(y)} |Du|\, dx \le \frac{3^n}{|B_{3\tilde{r}}(y)|}\int_{\Omega_{3\tilde{r}}(y_1)} |Du|\, dx \le 3^n\lambda_k.$$

Consequently, we have

$$\mathcal{M}(|Du|)(y) \le \max\left\{(1 + c_3)\lambda_k, 3^n\lambda_k\right\}.$$

Choosing $N_2 \ge \max\{1 + c_3, 3^n\}$, we have $y \notin C_k \cap B_{r_0}(y_0)$, that is, the claim (4.12) holds.

Using (4.12), (2.7), and (4.11), we discover

$$\begin{aligned} |C_k \cap B_{r_0}(y_0)| &\le \left|\left\{x \in B_{r_0}(y_0) : \mathcal{M}_{B_{2r_0}(y_0)}(|Du - Dv_h|)(x) > \lambda_k\right\}\right| \\ &\le \frac{c}{\lambda_k}\int_{B_{2r_0}(y_0)} |Du - Dv_h|\, dx \le cc_4\eta|B_{r_0}(y_0)| < \epsilon|B_{r_0}(y_0)|, \end{aligned}$$

by selecting η and δ that satisfy the last inequality above, which is a contradiction to (4.5).

Case 2. The boundary case $B_{10r_0}(y_1) \not\subset \Omega$.

At first we find a boundary point $\tilde{y}_1 \in \partial\Omega \cap B_{10r_0}(y_1)$. Since $640r_0 \le R_0 < \frac{R}{2}$ and the domain Ω is (δ, R)-Reifenberg flat, there exists a coordinate system, which we still denote by $x = (x_1, \cdots, x_n)$, with the origin at $\tilde{y}_1$, such that

$$B_{640r_0} \cap \{x_n > 640\delta r_0\} \subset \Omega_{640r_0} \subset B_{640r_0} \cap \{x_n > -640\delta r_0\}.$$

We select δ so small with $0 < \delta < \frac{1}{16}$. Then we see that $B_{480r_0}(640\delta r_0 e_n) \subset B_{640r_0}$, where $e_n = (0, \cdots, 0, 1)$. Translating this coordinate system to the x_n-direction $640\delta r_0$, still say x-coordinate, we observe

$$B^+_{480r_0} \subset \Omega_{480r_0} \subset B_{480r_0} \cap \{x_n > -1280\delta r_0\}. \tag{4.13}$$

Since $|y_1| \le |y_1 - \tilde{y}_1| + |\tilde{y}_1| \le 10r_0 + 640\delta r_0 \le 50r_0$ in the new coordinate, we have

$$\Omega_{2r_0}(y_0) \subset \Omega_{3r_0}(y_1) \subset \Omega_{60r_0} \quad \text{and} \quad \Omega_{480r_0} \subset \Omega_{640r_0}(y_1). \tag{4.14}$$

We denote

$$p_1 := \inf_{x\in\Omega_{480r_0}} p(x) \quad \text{and} \quad p_2 := \sup_{x\in\Omega_{480r_0}} p(x).$$

Then it follows that $p_2 - p_1 \le \omega(960r_0)$.

To obtain the corresponding estimates (4.9) in the boundary case, we deduce from (2.1), (4.6), (4.13), and (4.14) that

$$\fint_{\Omega_{480r_0}} |Du|\,dx \le c_5\lambda_k \quad \text{and} \quad \left[\frac{\nu(\overline{\Omega_{480r_0}})}{r_0^{n-1}}\right]^{\frac{1}{p(0)-1}} \le c_5\delta\lambda_k \tag{4.15}$$

for some constant $c_5 = c_5(n, \gamma_1) > 0$. Moreover, it follows from (1.7), (1.8) and (4.15) that, for any $\tilde{\epsilon} \in (0, 1)$,

$$\fint_{\Omega_{480r_0}} |Du_h|\,dx \le (c_5 + \tilde{\epsilon})\lambda_k =: c_6\lambda_k \quad \text{and} \quad \left[\frac{\nu_h(\Omega_{480r_0})}{r_0^{n-1}}\right]^{\frac{1}{p(0)-1}} \le c_6\delta\lambda_k$$

for h large enough. Applying Lemma 3.9 with ρ, r, and ϵ replaced by $c_6\lambda_k$, $60r_0$, and η, respectively, we can find $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \eta)$ such that

$$\fint_{\Omega_{120r_0}} |Du - D\bar{v}_h|\,dx \le c_6\eta\lambda_k \quad \text{and} \quad \|D\bar{v}_h\|_{L^\infty(\Omega_{60r_0})} \le cc_6\lambda_k =: c_7\lambda_k \tag{4.16}$$

for some $c_7 = c_7(n, \lambda, \Lambda, \gamma_1, \gamma_2) > 0$. Here we have chosen sufficiently small $\tilde{\epsilon}$ such that $\tilde{\epsilon} \le \frac{c_6\eta}{2}$.

Proceeding as in *Case 1*, we infer

$$\begin{aligned} C_k \cap B_{r_0}(y_0) &= \left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}(|Du|)(x) > N\lambda_k\right\} \\ &\subset \left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}_{\Omega_{2r_0}(y_0)}(|Du - D\bar{v}_h|)(x) > \lambda_k\right\} \end{aligned} \tag{4.17}$$

provided $N \ge N_2 \ge \max\{3^n, 1 + c_7\}$.

Thus, we have from (4.17), (2.7), (4.14) and (4.16) that

$$\begin{aligned} |C_k \cap B_{r_0}(y_0)| &\le \left|\left\{x \in \Omega_{r_0}(y_0) : \mathcal{M}_{\Omega_{2r_0}(y_0)}(|Du - D\bar{v}_h|)(x) > \lambda_k\right\}\right| \\ &\le \frac{c}{\lambda_k}\int_{\Omega_{2r_0}(y_0)} |Du - D\bar{v}_h|\,dx \le \frac{c|\Omega_{120r_0}|}{\lambda_k}\fint_{\Omega_{120r_0}} |Du - D\bar{v}_h|\,dx \\ &\le cc_6\eta|B_{r_0}(y_0)| < \epsilon|B_{r_0}(y_0)| \end{aligned}$$

by taking η sufficiently small, as a consequence $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, \epsilon)$ is also determined. This is a contradiction to (4.5). □

5. Global Calderón–Zygmund type estimates

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Choosing $N = \max\{N_1, N_2\}$ from Lemma 4.1 and Lemma 4.2, we can apply Lemma 2.4 to obtain

$$|C_k| \le \left(\frac{80}{7}\right)^n \epsilon|D_k| =: \epsilon_1|D_k| \quad \text{for } k \in \mathbb{N} \cup \{0\}.$$

As a consequence above and its iteration argument, we deduce the power decay estimates for the level sets of $\mathcal{M}(|Du|)$ to increasing levels, as follows:

$$\begin{aligned}&\left|\left\{x \in \Omega : \mathcal{M}(|Du|)(x) > N^k\lambda_0\right\}\right| \\ &\quad \le \epsilon_1^k \left|\{x \in \Omega : \mathcal{M}(|Du|)(x) > \lambda_0\}\right| + \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i}\lambda_0\right\}\right|.\end{aligned} \tag{5.1}$$

Now we write

$$S := \sum_{k=1}^{\infty} N^{qk} \left|\left\{x \in \Omega : \mathcal{M}(|Du|)(x) > N^k\lambda_0\right\}\right|.$$

Then we have from (5.1) and (2.12) that

$$\begin{aligned}S &\le \sum_{k=1}^{\infty} N^{qk}\epsilon_1^k \left|\{x \in \Omega : \mathcal{M}(|Du|)(x) > \lambda_0\}\right| + \sum_{k=1}^{\infty} N^{qk} \sum_{i=1}^{k} \epsilon_1^i \left|\left\{x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i}\lambda_0\right\}\right| \\ &\le |\Omega| \sum_{k=1}^{\infty} \left(N^q\epsilon_1\right)^k + \sum_{i=1}^{\infty} \left(N^q\epsilon_1\right)^i \sum_{k=i}^{\infty} N^{q(k-i)} \left|\left\{x \in \Omega : [\mathcal{M}_1(\nu)(x)]^{\frac{1}{p(x)-1}} > \delta N^{k-i}\lambda_0\right\}\right| \\ &\le \sum_{i=1}^{\infty} \left(N^q\epsilon_1\right)^i \left\{2|\Omega| + \frac{c}{(\delta\lambda_0)^q} \int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx\right\}.\end{aligned}$$

Now we select ϵ_1 with $N^q\epsilon_1 = \frac{1}{2}$, and then we can take ϵ and a corresponding $\delta = \delta(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$. Consequently, we find

$$S \le 2|\Omega| + \frac{c}{\lambda_0^q} \int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx. \tag{5.2}$$

According to (2.12) and (5.2), we have

$$\begin{aligned}\int_\Omega |Du|^q\, dx &\le \int_\Omega \mathcal{M}(|Du|)^q\, dx \le c\lambda_0^q\left(|\Omega| + S\right) \le c\left\{|\Omega|\lambda_0^q + \int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx\right\} \\ &\le c\left\{\frac{|\Omega|}{|B_{R_0}|^q}\left(\int_\Omega |Du|\, dx + 1\right)^q + \int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx\right\}\end{aligned} \tag{5.3}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, q) > 0$.

On the other hand, it follows from the estimate (5.7) in Remark 5.1 that

$$\int_\Omega |Du|\, dx \le c(n, \lambda, \gamma_1, \gamma_2, s, \Omega)\left\{|\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1-1)(1-s)}}\right\}.$$

Since R_0 satisfies (4.1) and (4.2) with M_1 given in (3.42), we see from the estimate above that

$$\frac{1}{R_0} \le \frac{c}{R}\left\{|\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1-1)(1-s)}} + 1\right\}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, \omega(\cdot), s, \Omega) > 0$ and for some $R < 1$. Thus, it follows that

$$\int_\Omega |Du|^q\, dx \le cK_s^q\left\{\int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx + 1\right\} \tag{5.4}$$

for some $c = c(n, \lambda, \Lambda, \gamma_1, \gamma_2, \omega(\cdot), q, R, s, \Omega) > 0$. Here we define

$$K_s := \left(|\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1-1)(1-s)}} + 1\right)^{n+1}.$$

Recalling the definition (3.5), we have that for $x \in \Omega$,

$$\mathcal{M}_1(\nu)(x) := \sup_{r>0} \frac{r\nu(B_r(x))}{|B_r(x)|} \le \sup_{r>0} \frac{r|\mu|(B_r(x))}{|B_r(x)|} + \sup_{r>0} \frac{r|B_r(x) \cap \Omega|}{|B_r(x)|}$$
$$\le \mathcal{M}_1(\mu)(x) + c(n)diam(\Omega) \le \mathcal{M}_1(\mu)(x) + c(n)|\Omega|^{\frac{1}{n}},$$

where $diam(\Omega)$ is the diameter of Ω. Then we have

$$\int_\Omega \mathcal{M}_1(\nu)^{\frac{q}{p(x)-1}}\, dx \le \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p(x)-1}}\, dx + c|\Omega|^{\frac{q}{n(\gamma_1-1)}+1}. \tag{5.5}$$

The estimates (5.4) and (5.5) yield the desired estimate (2.5). This completes the proof. □

Remark 5.1. We derive a standard estimate for measure data. Consider the regularized problem (1.6), we denote, for $k \in \mathbb{N}$,

$$D_k := \{x \in \Omega : |u_h(x)| \le k\} \quad \text{and} \quad C_k := \{x \in \Omega : k < |u_h(x)| \le k+1\}.$$

Then from (3.2), (1.5), and (1.9), we have

$$\int_{D_k} |Du_h|^{p(x)}\, dx \le ck|\mu|(\Omega) \quad \text{and} \quad \int_{C_k} |Du_h|^{p(x)}\, dx \le c|\mu|(\Omega)$$

by substituting test functions $\varphi = T_k(u)$ and $\varphi = \Phi_k(u)$ in (3.2), respectively. Here the functions T_k and Φ_k are defined as in (3.9). Then we discover

$$\int_{D_k} |Du_h|\, dx \le \int_{D_k} (|Du_h|+1)^{p(x)}\, dx \le ck\nu(\Omega),$$

where ν is given in (3.5). If $\frac{1}{\gamma_1-1} < t < \frac{n}{n-1}$, then it follows that

$$\int_{C_k} |Du_h|\, dx \le \left(\int_{C_k} |Du_h|^{\gamma_1}\, dx\right)^{\frac{1}{\gamma_1}} |C_k|^{\frac{1}{\gamma_1'}} \le c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \left(\int_{C_k} \left(\frac{|u_h|}{k}\right)^t dx\right)^{\frac{1}{\gamma_1'}}$$
$$\le c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \left(\frac{1}{k}\right)^{\frac{t}{\gamma_1'}} \left(\int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1'}},$$

where γ_1' is the Hölder conjugate of γ_1. Then we see that

$$\int_\Omega |Du_h|\, dx \le c\nu(\Omega) + c\,[\nu(\Omega)]^{\frac{1}{\gamma_1}} \underbrace{\sum_{k=1}^\infty \left(\frac{1}{k}\right)^{\frac{t}{\gamma_1'}} \left(\int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1'}}}_{(*)}.$$

Applying Hölder's inequality and Sobolev's inequality to $(*)$, we find that

$$(*) \le \left(\sum_{k=1}^\infty \left(\frac{1}{k}\right)^{\frac{t\gamma_1}{\gamma_1'}}\right)^{\frac{1}{\gamma_1}} \left(\sum_{k=1}^\infty \int_{C_k} |u_h|^t\, dx\right)^{\frac{1}{\gamma_1'}} \le c\left(\int_\Omega |u_h|^{\frac{n}{n-1}}\, dx\right)^{\frac{(n-1)t}{n\gamma_1'}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1'}}$$
$$\le c\left(\int_\Omega |Du_h|\, dx\right)^{\frac{t}{\gamma_1'}} |\Omega|^{\left(1-\frac{(n-1)t}{n}\right)\frac{1}{\gamma_1'}} \tag{5.6}$$

for some constant $c = c(n, \gamma_1, t) > 0$. Thus, we have

$$\int_\Omega |Du_h|\, dx \le c \left\{ \nu(\Omega) + [\nu(\Omega)]^{\frac{1}{\gamma_1}} \left(\int_\Omega |Du_h|\, dx \right)^{\frac{t}{\gamma_1'}} |\Omega|^{\left(1 - \frac{(n-1)t}{n}\right)\frac{1}{\gamma_1'}} \right\}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, t) > 0$. From the fact that $\gamma_1 \le n$, on the other hand, we see that $t < \frac{n}{n-1} \le \gamma_1'$. Then it follows from Young's inequality that

$$\int_\Omega |Du_h|\, dx \le c\nu(\Omega) + \frac{1}{2}\int_\Omega |Du_h|\, dx + c\,[\nu(\Omega)]^{\frac{1}{\gamma_1 - (\gamma_1 - 1)t}} |\Omega|^{\left(1 - \frac{(n-1)t}{n}\right)\frac{1}{\gamma_1' - t}}.$$

Then we have from (1.7) that

$$\int_\Omega |Du|\, dx \le c \left\{ \nu(\Omega) + [\nu(\Omega)]^{\frac{1}{\gamma_1 - (\gamma_1 - 1)t}} |\Omega|^{\left(1 - \frac{(n-1)t}{n}\right)\frac{1}{\gamma_1' - t}} \right\}.$$

Therefore we obtain from (3.5) that

$$\int_\Omega |Du|\, dx \le c \left\{ |\mu|(\Omega) + [|\mu|(\Omega)]^{\frac{1}{(\gamma_1 - 1)(1-s)}} \right\} \tag{5.7}$$

for some constant $c = c(n, \lambda, \gamma_1, \gamma_2, s, \Omega) > 0$, where we have selected $t := \frac{1}{\gamma_1 - 1} + s$ for small s with $0 < s \le \frac{1}{2}\left(\frac{n}{n-1} - \frac{1}{\gamma_1 - 1}\right) < 1$.

We clearly point out that this constant c goes to $+\infty$ as $s \searrow 0$, since the exponent $\frac{t\gamma_1}{\gamma_1'} = 1$ in the first inequality of (5.6).

Remark 5.2. If $p(\cdot)$ is a constant, then we infer from Remark 3.2 and (5.3) that

$$\int_\Omega |Du|^q\, dx \le c \left\{ \frac{|\Omega|}{R^{nq}} \left(\int_\Omega |Du|\, dx \right)^q + \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}}\, dx \right\} \tag{5.8}$$

for some $c = c(n, \lambda, \Lambda, p, q) > 0$. On the other hand, a standard estimate for measure data can be obtained by the normalization property for the problem (1.1), that is,

$$\int_\Omega |Du|\, dx \le c(n, \lambda, p) \int_\Omega \mathcal{M}_1(\mu)^{\frac{1}{p-1}}\, dx. \tag{5.9}$$

Indeed, the proof of (5.9) is similar to that of Lemma 3.1. Using (5.8) and (5.9), we derive

$$\int_\Omega |Du|^q\, dx \le c \int_\Omega \mathcal{M}_1(\mu)^{\frac{q}{p-1}}\, dx$$

for some constant $c = c(n, \lambda, \Lambda, p, q, R, \Omega) > 0$. This is the main estimate in [43]. However, in the case that $p(\cdot)$ is not a constant, the normalization property of (1.1) does not hold, and so (5.9) is no longer satisfied. See also Remark 5.1.

Conflict of interest statement

Authors declare that there is no conflict of interest.

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