

Boundary-degenerate elliptic operators and Hölder continuity for solutions to variational equations and inequalities

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Abstract

We prove local supremum bounds, a Harnack inequality, Hölder continuity up to the boundary, and a strong maximum principle for solutions to a variational equation defined by an elliptic operator which becomes degenerate along a portion of the domain boundary and where no boundary condition is prescribed, regardless of the sign of the Fichera function. In addition, we prove Hölder continuity up to the boundary for solutions to variational inequalities defined by this boundary-degenerate elliptic operator. © 2016 Elsevier Masson SAS. All rights reserved.

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1. Introduction

1.1. Overview

There is a distinguished history of research on local supremum estimates, Harnack inequalities, and local C^α estimates and C^α regularity for weak solutions to equations,

$$Au = f \quad \text{a.e. on } \mathcal{O}, \quad u = g \quad \text{on } \partial\mathcal{O},$$

defined by an elliptic partial differential operator,²

$$Au = -\bar{a}^{\mu\nu} u_{z_\mu z_\nu} - b^\mu u_{z_\mu} + cu, \tag{1.1}$$

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² We employ the Einstein summation convention with $1 \leq \mu, \nu \leq n$.

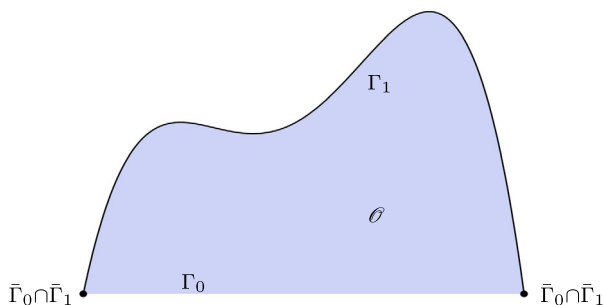


Fig. 1.1. Boundaries and corner points for the elliptic boundary value and obstacle problems.

whose coefficient matrix, $(\bar{a}^{\mu\nu})$, is Lipschitz but which fails to be strictly or uniformly elliptic on an open subset $\mathcal{O} \subset \mathbb{R}^n$ (for $n \geq 2$), in the sense of [29, p. 31]. For a selection of such results, see [4,15–17,28,35,40,41,44,47,53] and references contained therein. In those articles, Dirichlet boundary conditions are imposed on the *full* boundary, $\partial\mathcal{O}$, in order to obtain local supremum estimates and C^α regularity which hold up to $\partial\mathcal{O}$.

However, it is known from work of G. Fichera [26,27] and O. A. Oleĭnik and E. V. Radkevič [43,45,46], building on prior observations of M. V. Keldyš [33], that when A is *boundary-degenerate* — that is, $(\bar{a}^{\mu\nu})$ fails to be locally strictly elliptic along a non-empty open portion $\Gamma_0 \subseteq \partial\mathcal{O}$ of the boundary — then refined weak maximum principles imply that the boundary value problem or associated variational equation may have a unique solution, u in $C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ or $W^{1,2}(\mathcal{O})$ respectively, with Dirichlet boundary condition prescribed only along a *part* of the boundary, $\Gamma_1 := \partial\mathcal{O} \setminus \bar{\Gamma}_0$ (the ‘non-degenerate boundary’) and no boundary condition along Γ_0 (the ‘degenerate boundary’). However, the development of local supremum estimates, Harnack inequalities, and Hölder continuity up to Γ_0 for solutions to variational equations defined by boundary-degenerate elliptic partial differential operators — where *no* boundary condition is imposed along Γ_0 — is far less well developed and, with the exception of the Habilitation thesis of H. Koch [34] (about which we shall say more below), there are far fewer results despite the need from important applications.

We shall consider suitably defined *weak* solutions, u , to the elliptic boundary value problem,

$$Au = f \quad \text{on } \mathcal{O}, \quad u = g \quad \text{on } \Gamma_1, \tag{1.2}$$

and the elliptic obstacle problem with *partial* Dirichlet boundary condition (see Fig. 1.1),

$$\min\{Au - f, u - \psi\} = 0 \quad \text{a.e. on } \mathcal{O}, \quad u = g \quad \text{on } \Gamma_1, \tag{1.3}$$

where $\psi : \mathcal{O} \rightarrow \mathbb{R}$ is an obstacle function which is compatible with the Dirichlet boundary condition in the sense that

$$\psi \leq g \quad \text{on } \Gamma_1. \tag{1.4}$$

We note that obstacle problems are not considered by Koch in [34]. The purpose of this article is then to establish the following results for a variational equation corresponding to (1.2) defined by a class of boundary-degenerate operators that includes the Heston operator [31], which has wide application in mathematical finance:

- (1) Local supremum estimate up to $\partial\mathcal{O}$ for a subsolution;
- (2) A Harnack inequality for a non-negative solution on open subsets $\mathcal{O}' \Subset \mathcal{O} \cup \Gamma_0$ when $f = 0$ on \mathcal{O} ;
- (3) A strong maximum principle for a subsolution;

and, in the case of a solution, u , to a variational equation corresponding to (1.2) or variational inequality corresponding to (1.3),

- (4) C^α regularity up to $\partial\mathcal{O}$, including the ‘corner points’ where Γ_0 and Γ_1 meet, and a local C^α estimate;

where in each of these results, points in Γ_0 have the same role as those in the interior, \mathcal{O} , and *no* boundary condition is prescribed along Γ_0 . The supremum and C^α estimates for u are expressed in terms of integral norms of u , the source function, f , the boundary data, g , and, in the case of the variational inequality, the obstacle function, ψ .

Unlike the analogous classical results described by Gilbarg and Trudinger [29] for strictly elliptic operators — for example, local supremum estimates up to $\partial\mathcal{O}$ [29, Theorem 8.25] or local C^α estimates and regularity up to $\partial\mathcal{O}$ [29, Theorem 8.29] — or their analogues for degenerate-elliptic operators in the articles cited above (aside from [34]), we do not need to assume that u is bounded or C^α along Γ_0 : those properties are implied by the variational equation alone. In §1.3, we provide a detailed comparison with previous related results for solutions to variational equations defined by ‘degenerate elliptic’ operators. Our companion article [25] develops higher-order regularity properties up to Γ_0 for weak solutions.

Some of the motivation for developing these results can be inferred from the work of P. Daskalopoulos and R. Hamilton [11], C. L. Epstein and R. Mazzeo [13,14], H. Koch [34], and the authors [25,24,23], where one discovers that the imposition of a Dirichlet boundary condition along Γ_0 can limit the regularity of the solution, u , to be at most C^α up to Γ_0 , whereas employing suitable weighted Hölder or Sobolev spaces to facilitate solving the partial boundary problem (with Dirichlet boundary condition prescribed only along Γ_1) will yield a solution which is C^∞ up to Γ_0 (if the coefficients of A and source function f are also C^∞ up to Γ_0). Applications illustrate that the imposition of a boundary condition along Γ_0 is often not physically justified, as exemplified in work of Daskalopoulos and Hamilton and Koch on the porous medium equation, Daskalopoulos and the author [9,10] on stochastic volatility models in mathematical finance, E. Ekström and J. Tysk [12] on interest-rate models in mathematical finance, and Epstein and Mazzeo on Wright–Fisher diffusion models in mathematical biology, and many other examples. Instead, the relevant physical property sought is rather that the solution, u , be sufficiently smooth up to Γ_0 .

When the boundary-degenerate operator, A , can be expressed in both divergence and non-divergence forms (as we assume here), one has a choice of employing a Schauder approach to existence and regularity theory, as in [11,14,18, 24,23], or a variational approach as in [9,25,34]. However, for certain questions, the variational approach can have advantages over a Schauder approach. For example, it appears to be a challenging problem to use purely Schauder methods to prove that the solution, u , is C^α up to the ‘corner points’, where the degenerate and non-degenerate boundary portions, Γ_0 and Γ_1 , meet; see [18,25,24] for discussions of this difficulty. As shown by Daskalopoulos and one of the authors (Feehan) [9], a framework for solving a non-coercive variational equation defined with the aid of appropriate weighted Sobolev spaces is readily extended to include variational inequalities.

Furthermore, Daskalopoulos and Feehan use the Harnack inequality and continuity (up to Γ_0) developed in this article for a solution, u , to a variational inequality as important stepping stones in their proof of $C^{1,1}$ regularity (up to Γ_0) of a solution to an obstacle problem arising in mathematical finance. When A is as in (1.14) and $f = 0$, the solution, u , to the obstacle problem (1.3) can be interpreted as the value function for a perpetual American-style barrier option on a generalization of the Heston stochastic volatility asset price process [31], with payoff function ψ and barrier condition g on Γ_1 . The choice $\psi(x, y) = (K - e^x)^+$, for $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, yields the price of an American-style put, where x represents the asset log-price, y is the asset variance, and $K > 0$ is the strike.

1.2. Summary of main results

We shall state a selection of our main results here and then refer the reader to our guide to this article in §1.4 for more of our results on existence, uniqueness and regularity of solutions to variational equations and inequalities and corresponding obstacle problems.

1.2.1. Mathematical preliminaries

In this article, we shall study boundary-degenerate elliptic operators (1.1) of the specific form, for all $v \in C^\infty(\mathcal{O})$,

$$Av(z) = -ya^{\mu\nu}(z)v_{z_\mu z_\nu}(z) - b^\mu(z)v_{z_\mu}(z) + c(z)v(z), \quad \text{a.e. } z \in \mathcal{O}, \tag{1.5}$$

so $\bar{a}^{\mu\nu} = ya^{\mu\nu}$, where we denote $z = (z_1, \dots, z_n) = (x, y) \in \mathbb{H}$ with $x = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $x_n = y \in \mathbb{R}_+$. We require that the coefficients of the operator A satisfy

Assumption 1.1. There are positive constants, β , Λ and v_0 , with the following significance.

- (1) The coefficients $a^{\mu\nu}$ belong to $W^{1,\infty}(\mathcal{O})$ and c belongs to $L^\infty(\mathcal{O})$;
- (2) The coefficient matrix $(a^{\mu\nu}(z))$ is symmetric and strictly elliptic,

$$v_0|\xi|^2 \leq a^{\mu\nu}(z)\xi_\mu\xi_\nu, \quad \forall \xi \in \mathbb{R}^n, \quad \text{for a.e. } z \in \mathcal{O}; \tag{1.6}$$

(3) There are functions $\hat{b}_\mu \in L^\infty(\mathcal{O})$ such that

$$b^\mu = \beta a^{\mu n} - y \hat{b}^\mu, \quad 1 \leq \mu \leq n; \tag{1.7}$$

(4) The coefficients obey the bound

$$\max_{1 \leq \mu, \nu \leq n} \|a^{\mu\nu}\|_{W^{1,\infty}(\mathcal{O})} + \max_{1 \leq \mu \leq n} \|\hat{b}^\mu\|_{L^\infty(\mathcal{O})} + \|c\|_{L^\infty(\mathcal{O})} \leq \Lambda. \tag{1.8}$$

We shall consider variational solutions to (1.2) and (1.3), so we introduce our weighted Sobolev spaces. For $1 \leq q < \infty$, let

$$L^q(\mathcal{O}, \mathfrak{w}) := \{u \in L^1_{\text{loc}}(\mathcal{O}) : \|u\|_{L^q(\mathcal{O}, \mathfrak{w})} < \infty\}, \tag{1.9a}$$

$$H^1(\mathcal{O}, \mathfrak{w}) := \{u \in L^2(\mathcal{O}, \mathfrak{w}) : (1+y)^{1/2}u, y^{1/2}|Du| \in L^2(\mathcal{O}, \mathfrak{w})\}, \tag{1.9b}$$

$$H^2(\mathcal{O}, \mathfrak{w}) := \{u \in L^2(\mathcal{O}, \mathfrak{w}) : (1+y)^{1/2}u, (1+y)|Du|, y|D^2u| \in L^2(\mathcal{O}, \mathfrak{w})\}, \tag{1.9c}$$

where Du denotes the gradient of u , D^2u denotes the Hessian of u , with all derivatives of u being defined in the sense of distributions, and

$$\|u\|_{L^q(\mathcal{O}, \mathfrak{w})}^q := \int_{\mathcal{O}} |u|^q \mathfrak{w} \, dx \, dy, \tag{1.10a}$$

$$\|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} \left(y|Du|^2 + (1+y)u^2 \right) \mathfrak{w} \, dx \, dy, \tag{1.10b}$$

$$\|u\|_{H^2(\mathcal{O}, \mathfrak{w})}^2 := \int_{\mathcal{O}} \left(y^2|D^2u|^2 + (1+y)^2|Du|^2 + (1+y)u^2 \right) \mathfrak{w} \, dx \, dy, \tag{1.10c}$$

with weight function $\mathfrak{w} : \mathbb{H} \rightarrow (0, \infty)$ given by

$$\mathfrak{w}(x, y) := y^{\beta-1} e^{-\tau|x|-\mu y}, \quad \forall (x, y) \in \mathbb{H}, \tag{1.11}$$

where τ and μ are nonnegative constants. It will be convenient in our analysis to write A from (1.5), for all $v \in C^\infty(\mathcal{O})$, in the equivalent form,³

$$\begin{aligned} Av(z) = & -y \left(a^{ij}(z)v_{x_i x_j}(z) + 2a^{in}(z)v_{x_i y}(z) + a^{nn}(z)v_{yy}(z) \right) \\ & - b^i(z)v_{x_i}(z) - b^n(z)v_y(z) + c(z)v(z), \quad \text{a.e. } z \in \mathcal{O}. \end{aligned} \tag{1.12}$$

For all $u, v \in C^\infty_0(\mathbb{H})$, we define

$$\begin{aligned} \mathfrak{a}(u, v) := (Au, v)_{L^2(\mathcal{O}, \mathfrak{w})} = & \int_{\mathcal{O}} \left(a^{ij}u_{x_i}v_{x_j} + a^{in}(u_{x_i}v_y + u_yv_{x_i}) + a^{nn}u_yv_y \right) y\mathfrak{w} \, dx \, dy \\ & + \int_{\mathcal{O}} \left(\partial_{x_j}a^{ij} + \partial_y a^{in} + \hat{b}_i - \tau a^{ij} \frac{x_j}{|x|} - \mu a^{in} \right) u_{x_i}v_y \mathfrak{w} \, dx \, dy \\ & + \int_{\mathcal{O}} \left(\partial_{x_i}a^{in} + \partial_y a^{nn} + \hat{b}_n - \tau a^{in} \frac{x_i}{|x|} - \mu a^{nn} \right) u_yv_y \mathfrak{w} \, dx \, dy \\ & + \int_{\mathcal{O}} cuv \mathfrak{w} \, dx \, dy, \end{aligned} \tag{1.13}$$

and we call \mathfrak{a} the bilinear form associated with the operator A . The assumptions (1.6) and (1.8) ensure that the bilinear form $\mathfrak{a} : H^1(\mathcal{O}, \mathfrak{w}) \times H^1(\mathcal{O}, \mathfrak{w}) \rightarrow \mathbb{R}$ is continuous and satisfies the Gårding inequality and this motivates

³ We employ the Einstein summation convention with $1 \leq i, j \leq n-1$.

the definition (1.10b) of the weighted Sobolev space, $H^1(\mathcal{O}, \mathfrak{w})$. In definition (1.11) of the weight \mathfrak{w} , the power term $y^{\beta-1}$ is required in order to obtain a bilinear form \mathfrak{a} as in (1.13) that is continuous and satisfies the Gårding inequality. The role of the exponential term $e^{-\tau|x|-\mu y}$ is mainly to ensure that the measure of subsets $\mathcal{O} \subseteq \mathbb{H}$ is finite, when τ and μ are positive constants. Even though this property is used extensively in the results obtained in [9] and [20], it does not play any role in the proofs of the purely local results given in §2, §3, §4, §5 and §7, but we include the exponential term in the definition of the weight \mathfrak{w} for the sake of consistency with [9] and because positivity of τ is used in the proof of Claim 6.5 in §6.

Example 1.2 (Heston operator). A particular example of a degenerate operator as in (1.12) is the generator of the two-dimensional Heston stochastic volatility process with killing [31],

$$Av := -\frac{y}{2} \left(v_{xx} + 2\rho\sigma v_{xy} + \sigma^2 v_{yy} \right) - \left(r - q - \frac{y}{2} \right) v_x - \kappa(\theta - y)v_y + rv, \quad v \in C^\infty(\mathbb{H}), \tag{1.14}$$

where $\kappa > 0, \theta > 0, r \geq 0$, and $q \in \mathbb{R}$. We express the Heston operator A in (1.14) in divergence form as in (1.13) by choosing the weight \mathfrak{w} with

$$\beta := \frac{2\kappa\theta}{\sigma^2} \quad \text{and} \quad \mu := \frac{2\kappa}{\sigma^2}, \tag{1.15}$$

and τ is a positive constant; see [9, §1.1]. To ensure that the strict ellipticity condition (1.6) is satisfied, we assume that

$$\sigma \neq 0 \quad \text{and} \quad -1 < \rho < 1. \tag{1.16}$$

We notice that the condition (1.7) is satisfied only if

$$r - q - \frac{\kappa\theta\rho}{\sigma} = 0, \tag{1.17}$$

and this can be accomplished by using a simple affine change of variables on \mathbb{R}^2 which maps $(\mathbb{H}, \partial\mathbb{H})$ onto $(\mathbb{H}, \partial\mathbb{H})$, as described in [9, Lemma 2.2]. Then the bilinear form associated with the Heston operator, A , in (1.14) is given by

$$\begin{aligned} \mathfrak{a}(u, v) := & \frac{1}{2} \int_{\mathcal{O}} \left(u_x v_x + \rho\sigma u_y v_x + \rho\sigma u_x v_y + \sigma^2 u_y v_y \right) y \mathfrak{w} \, dx \, dy \\ & - \frac{1}{2} \int_{\mathcal{O}} \left(\tau \operatorname{sign}(x) + \mu\rho\sigma - 1 \right) u_x v_y \mathfrak{w} \, dx \, dy \\ & - \frac{1}{2} \int_{\mathcal{O}} \tau\rho\sigma \operatorname{sign}(x) u_y v_y \mathfrak{w} \, dx \, dy + \int_{\mathcal{O}} r u v \mathfrak{w} \, dx \, dy, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}). \end{aligned} \tag{1.18}$$

This completes our discussion of this example.

We now return to the general setting described prior to Example 1.2. Given a subset $T \subset \partial\mathcal{O}$ we let $H_0^1(\mathcal{O} \cup T, \mathfrak{w})$ be the closure in $H^1(\mathcal{O}, \mathfrak{w})$ of $C_0^\infty(\mathcal{O} \cup T)$. Given a source function $f \in L^2(\mathcal{O}, \mathfrak{w})$, we call a function $u \in H^1(\mathcal{O}, \mathfrak{w})$ a *solution* to the variational equation defined by the operator A in (1.12), if

$$\mathfrak{a}(u, v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \tag{1.19}$$

We call u a *subsolution* to (1.19) if $\mathfrak{a}(u, v) \leq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}$ for all nonnegative test functions, v , and call u a *supersolution* to (1.19) if $-u$ is a subsolution.

Given $g \in H^1(\mathcal{O}, \mathfrak{w})$, we say that u obeys an (inhomogeneous) Dirichlet boundary condition $u = g$ on Γ_1 in the sense of H^1 if

$$u - g \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}),$$

and, of course, a homogeneous Dirichlet boundary condition on Γ_1 if $g = 0$.

If $u \in H^2(\mathcal{O}, \mathfrak{w})$, we recall from [9] that u is a solution to (1.2) if and only if $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ and u is a solution to (1.19).

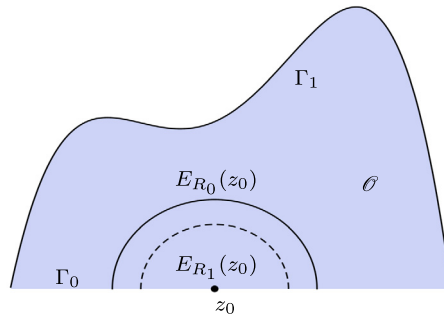


Fig. 1.2. Concentric half-balls centered at a ‘degenerate boundary interior’ point, $z_0 \in \Gamma_0$.

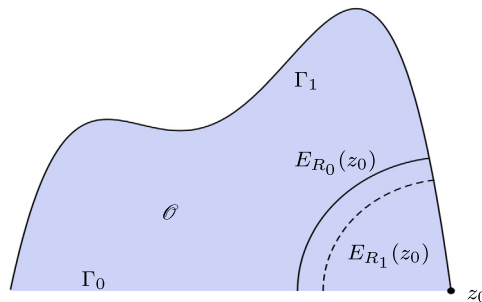


Fig. 1.3. Concentric half-balls centered at a ‘degenerate boundary corner point’, $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$.

Definition 1.3 (*Balls with respect to the Euclidean metric*). We let

$$\mathbb{E}_R(z_0) := \{z \in \mathbb{H} : |z - z_0| < R\}, \tag{1.20}$$

$$E_R(z_0) := \{z \in \mathcal{O} : |z - z_0| < R\}, \tag{1.21}$$

for any given $z_0 \in \bar{\mathbb{H}}$ and $R > 0$.

We say that an open subset, $U \subset \mathbb{H}$, obeys an *exterior cone condition relative to \mathbb{H} at a point $z_0 \in \partial U$* if there exists a finite, right circular cone $K = K_{z_0} \subset \bar{\mathbb{H}}$ with vertex z_0 such that $\bar{U} \cap K_{z_0} = \{z_0\}$ (compare [29, p. 203]). An open subset, $U \subset \mathbb{H}$, obeys a *uniform exterior cone condition relative to \mathbb{H} on $T \subset \partial U$* if U satisfies an exterior cone condition relative to \mathbb{H} at every point $z_0 \in T$ and the cones K_{z_0} are all congruent to some fixed finite cone, K (compare [29, p. 205]). Recall that Γ_0 is the interior of the portion, $\bar{\mathcal{O}} \cap \partial \mathbb{H}$, of the boundary, $\partial \mathcal{O}$, of the open subset $\mathcal{O} \subseteq \mathbb{H}$.

Definition 1.4 (*Interior and exterior cone conditions*). Let K be a finite, right circular cone. We say that \mathcal{O} obeys *interior and exterior cone conditions at $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ with cone K* if the open subsets \mathcal{O} and $\mathbb{H} \setminus \bar{\mathcal{O}}$ obey exterior cone conditions relative to \mathbb{H} at z_0 with cones congruent to K . We say that \mathcal{O} obeys *uniform interior and exterior cone conditions on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ with cone K* if the open subsets \mathcal{O} and $\mathbb{H} \setminus \bar{\mathcal{O}}$ obey exterior cone conditions relative to \mathbb{H} at each point $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ with cones congruent to K .

1.2.2. *Boundary local supremum bounds*

The volumes of bounded subsets in \mathbb{H} are finite with respect to the weight $y^{\beta-1} dx dy$, when $\beta > 0$, a fact which we repeatedly use in this article. We rely on the assumption that $\beta > 0$ in the statements and proofs of the local supremum estimates.

We have the following analogues of [34, Proposition 4.5.1] and [29, Theorem 8.15], but now for the cases of a ‘degenerate-boundary interior’ point, $z_0 \in \Gamma_0$, and a ‘degenerate boundary corner point’, $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$; see Figs. 1.2 and 1.3, respectively. Though Koch allows for points in the interior of Γ_0 , there is no analogue in [34] of our Theorem 1.6, which allows for corner points, while Gilbarg and Trudinger [29] only allow for boundary points where the elliptic partial differential operator is strictly elliptic.

Theorem 1.5 (Supremum estimates near points in Γ_0). *Let $s > n + \beta$ and let R_0 be a positive constant. Then there are positive constants, $C = C(\Lambda, n, \nu_0, R_0, s)$ and $R_1 = R_1(R_0) < R_0$, such that the following holds. Let $\mathcal{O} \subseteq \mathbb{H}$ be an open subset. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a subsolution (respectively, supersolution) to the variational equation (1.19) with source function $f \in L^2(\mathcal{O}, \mathfrak{w})$, and $z_0 \in \Gamma_0$ is such that $\mathbb{E}_{R_0}(z_0) \subset \mathcal{O}$, and f obeys*

$$f \in L^s(E_{R_0}(z_0), y^{\beta-1}), \tag{1.22}$$

then $u \in L^\infty(E_{R_1}(z_0))$, and

$$\operatorname{ess\,sup}_{E_{R_1}(z_0)} u(-u) \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u^+(u^-)\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right). \tag{1.23}$$

Theorem 1.6 (Supremum estimates near points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$). *Let K be a finite right circular cone, let $s > n + \beta$, and let $R_0 > 0$ be a positive constant. Then there are positive constants, $C = C(K, \Lambda, n, \nu_0, R_0, s)$ and $R_1 = R_1(K, \Lambda, n, \nu_0, R_0) < R_0$, such that the following holds. Let $\mathcal{O} \subsetneq \mathbb{H}$ be an open subset. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a subsolution (respectively, supersolution) of equation (1.19) with source function $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is such that \mathcal{O} obeys an interior cone condition at z_0 with cone K , and*

$$u = 0 \text{ on } \Gamma_1 \cap \bar{E}_{R_0}(z_0) \quad (\text{in the sense of } H^1),$$

and f obeys (1.22), then $\operatorname{ess\,sup}_{E_{R_1}(z_0)} u(-u) < \infty$ and the estimate (1.23) holds.

Remark 1.7 (Use of the weight $y^{\beta-1}$ versus \mathfrak{w} in Theorems 1.5 and 1.6). Notice that on the right-hand-side of estimate (1.23) we have $\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})}$ instead of $\|f\|_{L^s(E_{R_0}(z_0), \mathfrak{w})}$. This allows us to conclude that the constant C appearing in (1.23) is independent of the point $z_0 \in \bar{\Gamma}_0$. By (1.11), the weight \mathfrak{w} contains the term $e^{-\tau|x|}$, which means that the constant C will depend on the x -coordinate of the point $z_0 \in \bar{\Gamma}_0$, if we replace $\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})}$ by $\|f\|_{L^s(E_{R_0}(z_0), \mathfrak{w})}$ on the right-hand-side of (1.23).

For $g \in L^\infty_{\text{loc}}(\bar{\Gamma}_1)$ and $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ and $R_0 > 0$, we set

$$M := \operatorname{ess\,sup}_{\Gamma_1 \cap B_{R_0}(z_0)} g,$$

and define

$$u^M(z) := (u(z) \vee M)^+ \quad \text{for a.e. } z \in B_{R_0}(z_0).$$

We then have the following analogue of [29, Theorem 8.25] which applies to a variational equation defined by strictly elliptic operator and an inhomogeneous Dirichlet boundary condition.

Corollary 1.8 (Supremum estimates near points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ for variational subsolutions with inhomogeneous Dirichlet boundary condition). *Let $s > n + \beta$ and let R_0 be a positive constant. Then there are positive constants, $C = C(K, \Lambda, n, \nu_0, R_0, s)$ and $R_1 = R_1(K, \Lambda, n, \nu_0, R_0) < R_0$, such that the following holds. Let $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ is a subsolution of equation (1.19) with source function $f \in L^2(\mathcal{O}, \mathfrak{w})$ satisfying (1.22), and $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap L^\infty_{\text{loc}}(\bar{\Gamma}_1)$, in the sense that*

$$u - g \in H^1_0(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \tag{1.24}$$

then $u^M \in L^\infty(E_{R_1}(z_0))$, and

$$\operatorname{ess\,sup}_{E_{R_1}(z_0)} u^M \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} + \|g\|_{L^\infty(\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0))} \right). \tag{1.25}$$

Remark 1.9 (Supremum estimates near points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ for supersolutions with inhomogeneous Dirichlet boundary condition). Corollary 1.8 holds for supersolutions to equation (1.19) with the observation that in the estimate (1.25) we need to replace u^M with u^m where u^m is defined as follows. Let

$$m := \operatorname{ess\,inf}_{\Gamma_1 \cap B_{R_0}(z_0)} g,$$

and set

$$u^m(z) := (u(z) \wedge m)^- \quad \text{for a.e. } z \in B_{R_0}(z_0).$$

Remark 1.10 (*Inhomogeneous Dirichlet boundary conditions and variational equations*). Given a (non-zero) boundary-data function $g \in H^1(\mathcal{O}, \mathfrak{w})$ then, as an alternative to our proofs of [Corollaries 1.8, 1.16, and 1.17](#), we could replace u and $(f, v)_{L^2(\mathcal{O}, \mathfrak{w})}$ in [\(1.19\)](#) by $\tilde{u} := u - g \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ and the functional $F \in H^{-1}(\mathcal{O}, \mathfrak{w}) := (H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}))'$, where

$$F(v) := (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} - \mathfrak{a}(g, v), \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \tag{1.26}$$

and instead of [\(1.19\)](#), consider the variational equation,

$$\mathfrak{a}(\tilde{u}, v) = F(v), \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \tag{1.27}$$

This reduction would bring our arguments into closer alignment with those of Gilbarg and Trudinger [\[29, Chapter 8\]](#), but at the cost of a slightly more complicated proofs than those we employ in this article and little gain.

1.2.3. Hölder continuity up to the boundary for solutions to the variational equation

We recall the definition of the *Koch distance function*, $d(\cdot, \cdot)$, on \mathbb{H} introduced by Koch in [\[34, p. 11\]](#),

$$d(z, z_0) := \frac{|z - z_0|}{\sqrt{y + y_0 + |z - z_0|}}, \quad \forall z = (x, y), \quad z_0 = (x_0, y_0) \in \bar{\mathbb{H}}, \tag{1.28}$$

where $|z - z_0|^2 = (x - x_0)^2 + (y - y_0)^2$. The Koch distance function is equivalent to the *cycloidal distance function* introduced by Daskalopoulos and Hamilton in [\[11, p. 901\]](#) for the study of the porous medium equation.

Following [\[1, §1.26\]](#), for an open subset $U \subset \mathbb{H}$, we let $C(U)$ denote the vector space of continuous functions on U and let $C(\bar{U})$ denote the Banach space of functions in $C(U)$ which are bounded and uniformly continuous on U , and thus have unique bounded, continuous extensions to \bar{U} , with norm

$$\|u\|_{C(\bar{U})} := \sup_U |u|.$$

Given $\alpha \in (0, 1)$, we say that $u \in C_s^\alpha(\bar{U})$ if $u \in C(\bar{U})$ and

$$\|u\|_{C_s^\alpha(\bar{U})} < \infty,$$

where

$$\|u\|_{C_s^\alpha(\bar{U})} := [u]_{C_s^\alpha(\bar{U})} + \|u\|_{C(\bar{U})}, \tag{1.29}$$

and

$$[u]_{C_s^\alpha(\bar{U})} := \sup_{\substack{z_1, z_2 \in U \\ z_1 \neq z_2}} \frac{|u(z_1) - u(z_2)|}{d^\alpha(z_1, z_2)}. \tag{1.30}$$

Moreover, $C_s^\alpha(\bar{U})$ is a Banach space [\[11, §I.1\]](#) with respect to the norm [\(1.29\)](#). We say that $u \in C_s^\alpha(U)$ if $u \in C_s^\alpha(\bar{V})$ for all precompact open subsets $V \Subset U \cup \Gamma_0$.

When U may be *unbounded*, we let $C_{\text{loc}}(\bar{U})$ denote the linear subspace of functions $u \in C(U)$ such that $u \in C(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$; similarly, we let $C_{s, \text{loc}}^\alpha(\bar{U})$ denote the linear subspace of functions $u \in C_s^\alpha(U)$ such that $u \in C_s^\alpha(\bar{V})$ for every precompact open subset $V \Subset \bar{U}$.

We have the following analogues of [\[29, Theorem 8.27 and 8.29\]](#) and [\[34, Theorem 4.5.5 and 4.5.6\]](#), but again for the cases of a ‘degenerate-boundary interior’ point, $z_0 \in \Gamma_0$, and a ‘degenerate boundary corner point’, $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_0$; see [Figs. 1.2 and 1.3](#), respectively. Though Koch allows for points in the interior of Γ_0 , there is no analogue in [\[34\]](#) of our [Theorem 1.13](#), which allows for corner points; as before, Gilbarg and Trudinger [\[29\]](#) only allow for boundary points where the elliptic partial differential operator is strictly elliptic.

Theorem 1.11 (Hölder continuity near points in Γ_0 for solutions to the variational equation). Let $s > \max\{2n, n + \beta\}$ and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(R_0) < R_0$, and $C = C(\Lambda, n, \nu_0, R_0, s)$, and $\alpha = \alpha(\Lambda, n, \nu_0, R_0, s) \in (0, 1)$ such that the following holds. Let $\mathcal{O} \subseteq \mathbb{H}$ be an open subset. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ satisfies the variational equation (1.19) with source function $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $z_0 \in \Gamma_0$ is such that $\mathbb{E}_{R_0}(z_0) \subset \mathcal{O}$, and f obeys (1.22), then $u \in C_s^\alpha(\bar{E}_{R_1}(z_0))$, and

$$\|u\|_{C_s^\alpha(\bar{E}_{R_1}(z_0))} \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right). \tag{1.31}$$

Remark 1.12 (Hölder continuity up to Γ_0 and Sobolev embeddings). Hölder continuity of solutions does not follow by an embedding theorem for Sobolev weighted spaces, analogous to [29, Corollary 7.11], not even for functions $u \in H^2(\mathcal{O}, \mathfrak{w})$. For example, for any $\beta > 2$, let $q \in (0, (\beta - 2)/2)$ and

$$u(x, y) = y^{-q}, \quad \forall (x, y) \in \mathcal{O}.$$

Then, $u \in H^2(\mathcal{O}, \mathfrak{w})$, but $u \notin C_s^\alpha(\mathcal{O})$, for any $\alpha \in [0, 1]$, since, a fortiori, $u \notin C(\mathcal{O} \cup \Gamma_0)$.

Theorem 1.13 (Hölder continuity near points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ for solutions to the variational equation). Let K be a finite, right circular cone, let $s > \max\{2n, n + \beta\}$, and let R_0 be a positive constant. Then there are positive constants, $R_1 = R_1(K, \Lambda, n, \nu_0, R_0) < R_0$, and $C = C(K, \Lambda, n, \nu_0, R_0, s)$, and $\alpha = \alpha(K, \Lambda, n, \nu_0, R_0, s) \in (0, 1)$, such that the following holds. Let $\mathcal{O} \subsetneq \mathbb{H}$ be an open subset. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ satisfies the variational equation (1.19) with source function $f \in L^2(\mathcal{O}, \mathfrak{w})$ and $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is such that f obeys (1.22), and

$$u = 0 \text{ on } \Gamma_1 \cap \bar{E}_{R_0}(z_0) \quad (\text{in the sense of } H^1),$$

and \mathcal{O} obeys an interior and exterior cone condition with cone K at z_0 and a uniform exterior cone condition with cone K along $\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0)$, then $u \in C_s^\alpha(\bar{E}_{R_1}(z_0))$ and satisfies (1.31).

Remark 1.14 (Comparison with analysis near the non-degenerate boundary). The term $\sigma(\sqrt{R}R_0)$, where $\sigma(R) := \text{osc}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$, which appears in [29, Equation (8.72)] in the statement of [29, Theorem 8.27] does not appear in the statement of our Theorem 1.13. The reason is that unlike in [29, Equation (8.71)], the test functions defined in the proof of Theorem 1.13 do not need to involve $\text{ess sup}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$ or $\text{ess inf}_{\partial\mathcal{O} \cap \bar{B}_R(z_0)} u$ since no boundary condition is imposed on v along Γ_0 , in contrast to the Dirichlet boundary condition assumed for v in the proofs of [29, Theorem 8.18 and 8.26].

By constructing suitable weighted Sobolev spaces adapted both to the degeneracy of the operator and the geometry of the corners, we may be able to obtain improved regularity estimates in a neighborhood of the points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, similar to the ideas used for the study of strictly elliptic operators on polygonal domains described by Grisvard [30, §4.4.1]. We believe that this problem requires careful consideration and is best considered in a separate article.

Remark 1.15 (Counter-examples to higher-order regularity near corners for solutions to elliptic boundary value problems). It is worth recalling [36, §7.5] that the unique solution $u \in C^2(\mathcal{O}) \cap C(\bar{\mathcal{O}})$ to the Dirichlet problem, $\Delta u = 1$ on $\mathcal{O} := (0, \pi) \times (0, \pi)$ and $u = 0$ on $\partial\mathcal{O}$, belongs to $C^1(\bar{\mathcal{O}})$ but not $C^2(\bar{\mathcal{O}})$. (Following our customary sign convention, we denote $\Delta u = -\sum_{i=1}^n u_{x_i x_i}$.) This example illustrates that the question of regularity near corner points is delicate even for boundary value problems defined by strictly elliptic operators and thus can be expected to be even more so in the case of degenerate-elliptic operators.

We have the following analogue of [29, Theorem 8.27] which applies to a variational equation defined by a strictly elliptic operator on an open subset satisfying an exterior cone condition and an inhomogeneous Dirichlet boundary condition.

Corollary 1.16 (Hölder continuity near points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ for variational solutions with inhomogeneous Dirichlet boundary condition). Let K be a finite, right circular cone, let $s > \max\{2n, n + \beta\}$ and let R_0 be a positive constant. Assume $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C_{s, \text{loc}}^\gamma(\bar{\Gamma}_1)$, where $\gamma \in (0, 1]$. Then there are positive constants, $R_1 = R_1(K, \Lambda, n, \nu_0, R_0) < R_0$, and $C = C(K, \Lambda, n, \nu_0, R_0, s)$, and $\alpha = \alpha(\gamma, \Lambda, n, \nu_0, R_0, s) \in (0, 1)$ such that the following holds. Let

$z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. Assume that \mathcal{O} obeys an interior and exterior cone condition with cone K at z_0 and a uniform exterior cone condition with cone K along $\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0)$. If $u \in H^1(\mathcal{O}, \mathfrak{w})$ satisfies the variational equation (1.19) and (1.24), and the source function $f \in L^2(\mathcal{O}, \mathfrak{w})$ obeys (1.22), then $u \in C_s^\alpha(\bar{E}_{R_1}(z_0))$, and

$$\|u\|_{C_s^\alpha(\bar{E}_{R_1}(z_0))} \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} + \|g\|_{C_s^\gamma(\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0))} \right). \tag{1.32}$$

When $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C_{\text{loc}}(\bar{\Gamma}_1)$, then $u \in C(\bar{E}_{R_1}(z_0))$ and there is a positive constant $C = C(K, \Lambda, n, \nu_0, R_0, s)$ such that

$$\|u\|_{C(E_{R_1}(z_0))} \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} + \|g\|_{L^\infty(\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0))} \right). \tag{1.33}$$

For any $\delta > 0$, we let

$$\mathcal{O}_\delta := \mathcal{O} \cap (\mathbb{R} \times (0, \delta)). \tag{1.34}$$

We then have the

Corollary 1.17 (Hölder continuity up to $\bar{\Gamma}_0$ for solutions to the variational equation). *Let K be a finite, right circular cone, let $s > \max\{2n, n + \beta\}$, $\delta > 0$, and $\gamma \in (0, 1]$. Then there are constants $C = C(\delta, K, \Lambda, n, \nu_0, s) > 0$ and $\alpha_1 = \alpha_1(\delta, \gamma, K, \Lambda, n, \nu_0, s) \in (0, 1)$ such that the following hold. Assume that \mathcal{O} obeys a uniform interior and exterior cone condition with cone K on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ and a uniform exterior cone condition with cone K along $\Gamma_1 \cap \partial\mathcal{O}_\delta$. Let $f \in L^2(\mathcal{O}, \mathfrak{w})$, $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C_s^\gamma(\bar{\Gamma}_1 \cap \bar{\mathcal{O}}_\delta)$, and $u \in H^1(\mathcal{O}, \mathfrak{w})$ obey (1.19) and (1.24), and assume that f and u satisfy*

$$\sup_{z_0 \in \Gamma_0} \|f\|_{L^s(E_\delta(z_0), y^{\beta-1})} < \infty \quad \text{and} \quad \sup_{z_0 \in \Gamma_0} \|u\|_{L^2(E_\delta(z_0), y^{\beta-1})} < \infty. \tag{1.35}$$

Then $u \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2})$ and satisfies

$$\|u\|_{C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2})} \leq C \left(\sup_{z_0 \in \Gamma_0} \|f\|_{L^s(E_\delta(z_0), y^{\beta-1})} + \sup_{z_0 \in \Gamma_0} \|u\|_{L^2(E_\delta(z_0), y^{\beta-1})} + \|g\|_{C_s^\gamma(\bar{\Gamma}_1 \cap \bar{\mathcal{O}}_\delta)} \right). \tag{1.36}$$

When $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C(\bar{\Gamma}_1 \cap \bar{\mathcal{O}}_\delta)$, we have that $u \in C(\bar{\mathcal{O}}_{\delta/2})$, and there is a positive constant, $C = C(\delta, K, \Lambda, n, \nu_0, s)$, such that

$$\|u\|_{C(\bar{\mathcal{O}}_{\delta/2})} \leq C \left(\sup_{z_0 \in \Gamma_0} \|f\|_{L^s(E_\delta(z_0), y^{\beta-1})} + \sup_{z_0 \in \Gamma_0} \|u\|_{L^2(E_\delta(z_0), y^{\beta-1})} + \|g\|_{L^\infty(\bar{\Gamma}_1 \cap \bar{\mathcal{O}}_\delta)} \right) \tag{1.37}$$

Condition (1.35) on u is satisfied when $u \in L^2(\mathcal{O}, \mathfrak{w})$ and the open subset, \mathcal{O} , is bounded in the x -direction, as we can see from the definition (1.11) of the weight \mathfrak{w} .

1.2.4. Strong maximum principle

We also have the following analogue of [29, Theorem 8.19]. It is important to note that Theorem 1.18 is an analogue of the classical strong maximum principle, except that points in the degenerate-boundary portion, Γ_0 , play the same role as points in \mathcal{O} . We now assume that $\mathcal{O} \subseteq \mathbb{H}$ is domain, that is, a *connected*, open subset.

Theorem 1.18 (Strong maximum principle). *Let $\mathcal{O} \subseteq \mathbb{H}$ be a domain. Let $z_0 \in \mathcal{O} \cup \Gamma_0$, R_0 be a positive constant, and $u \in H^1(\mathcal{O}, \mathfrak{w})$ be a subsolution to equation (1.19) with $f = 0$. If the ball $E_{R_0}(z_0)$ as in (1.21) obeys $E_{R_0}(z_0) \subseteq \mathcal{O} \cup \Gamma_0$ and*

$$\text{ess sup}_{E_{R_0}(z_0)} u = \text{ess sup}_{\mathcal{O}} u,$$

then u is constant on \mathcal{O} .

Note that $\text{ess sup}_{E_{R_0}(z_0)} u < \infty$ by Theorem 1.5 when $z_0 \in \Gamma_0$, while [29, Theorem 8.17] yields this local boundedness result when $E_{R_0}(z_0) \subseteq \mathcal{O}$.

1.2.5. Hölder continuity up to the boundary for solutions to the variational inequality

Given a source function $f \in L^2(\mathcal{O}, \mathfrak{w})$, an (inhomogeneous) Dirichlet boundary condition $g \in H^1(\mathcal{O}, \mathfrak{w})$ on Γ_1 , and an obstacle function $\psi \in H^1(\mathcal{O}, \mathfrak{w})$ obeying (1.4) in the sense that

$$(\psi - g)^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \tag{1.38}$$

we call $u \in H^1(\mathcal{O}, \mathfrak{w})$ a solution to the variational inequality for the operator A defined in (1.12) with Dirichlet boundary condition along Γ_1 if

$$u - g \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad u \geq \psi \text{ a.e. on } \mathcal{O},$$

$$\alpha(u, v - u) \geq (f, v - u)_{L^2(\mathcal{O}, \mathfrak{w})}, \tag{1.39}$$

$$\forall v \in H^1(\mathcal{O}, \mathfrak{w}), \quad v - g \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad v \geq \psi \text{ a.e. on } \mathcal{O}.$$

Given additional mild conditions on f and ψ , it is proved in [9] that there is a unique solution, $u \in H^1(\mathcal{O}, \mathfrak{w})$, to (1.39), when A is the Heston operator defined in (1.14). For Theorem 1.20, we require

Hypothesis 1.19 (Conditions on the source and obstacle functions). For some $\delta > 0$,

$$f \in L^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta), \tag{1.40}$$

$$\psi \in H^2(\mathcal{O}_\delta, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta), \tag{1.41}$$

where \mathcal{O}_δ is defined in (1.34).

We then have

Theorem 1.20 (Hölder continuity up to $\bar{\Gamma}_0$ for solutions to the variational inequality with homogeneous boundary condition). Require that \mathcal{O} obeys a uniform interior and exterior cone condition on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ with cone K and a uniform exterior cone condition with cone K along $\Gamma_1 \cap \partial\mathcal{O}_\delta$, for some $\delta > 0$. Assume that f obeys (1.40) and $g = 0$ and ψ obeys (1.38) (with $g = 0$) and (1.41), and that

$$\text{ess sup}_{\mathcal{O}_\delta} (A\psi - f)^+ < \infty. \tag{1.42}$$

If \mathcal{O}_δ is bounded, require that $c \geq 0$ a.e. on \mathcal{O}_δ ; if \mathcal{O}_δ is unbounded, require in addition that $c \geq c_0 > 0$ a.e. on \mathcal{O}_δ for a positive constant c_0 . If $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ is a solution to (1.39) such that at least one of the following conditions holds,

$$\text{height}(\mathcal{O}) < \infty \quad \text{or} \quad u \in W^{1,\infty}(\mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}), \tag{1.43}$$

then

$$u \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2}),$$

where $\alpha_1 = \alpha_1(\delta, K, \Lambda, n, \nu_0, s) \in (0, 1)$.

Corollary 1.21 (Hölder continuity up to $\bar{\Gamma}_0$ for solutions to the variational inequality with inhomogeneous Dirichlet boundary condition). Assume the hypotheses of Theorem 1.20 and $g \in H^2(\mathcal{O}, \mathfrak{w}) \cap C_s^\gamma(\bar{\Gamma}_1 \cap \partial\mathcal{O}_{\delta/2})$, with $\gamma \in (0, 1]$. Let $u \in H^1(\mathcal{O}, \mathfrak{w})$ be a solution to (1.39) such that

$$\text{height}(\mathcal{O}) < \infty \quad \text{or} \quad u - g \in W^{1,\infty}(\mathcal{O}_\delta \setminus \mathcal{O}_{\delta/2}).$$

Then $u \in C_s^{\alpha_2}(\bar{\mathcal{O}}_{\delta/2})$, where $\alpha_2 = \alpha_1 \wedge \gamma$ and the constant α_1 is as in the conclusion of Theorem 1.20. If $g \in H^2(\mathcal{O}, \mathfrak{w}) \cap C(\bar{\Gamma}_1 \cap \partial\mathcal{O}_{\delta/2})$, then $u \in C(\bar{\mathcal{O}}_{\delta/2})$.

Remark 1.22 (Hypotheses on the solution to the variational inequality). The second condition in (1.43) in Theorem 1.20 is implied by the $W_{\text{loc}}^{2,p}(\mathcal{O})$ regularity result [9, Theorem 6.18] for $p > 2$ and corresponding $W^{2,p}(U)$ a priori estimates using the conditions (1.40) and (1.41), and the Sobolev embedding $W^{2,p}(U) \hookrightarrow C_b^1(U)$ for open subsets $U \Subset \mathbb{H}$ with the interior cone property [1, Theorem 5.4 (C)].

Remark 1.23 (*Inhomogeneous Dirichlet boundary conditions and variational inequalities*). Given a (non-zero) boundary-data function $g \in H^1(\mathcal{O}, \mathfrak{w})$ then, as an alternative to our proof of [Corollary 1.21](#), we could replace $u, v, \psi \in H^1(\mathcal{O}, \mathfrak{w})$ and $(f, v - u)_{L^2(\mathcal{O}, \mathfrak{w})}$ in [\(1.39\)](#) by $\tilde{u} := u - g, \tilde{v} := v - g, \tilde{\psi} := \psi - g$ in $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ and the functional $F \in H^{-1}(\mathcal{O}, \mathfrak{w})$ in [\(1.26\)](#) and, instead of [\(1.39\)](#), consider the variational inequality,

$$\alpha(\tilde{u}, w - \tilde{u}) \geq F(w - \tilde{u}), \quad \forall w \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad w \geq \tilde{\psi} \quad \text{a.e. on } \mathcal{O}. \tag{1.44}$$

This reduction would bring our arguments into closer alignment with those of Gilbarg and Trudinger [\[29, Chapter 8\]](#) and Troianiello [\[49, Chapter 4\]](#), but at the cost of a slightly more complicated proofs than those we employ in this article and little gain.

It is interesting to note that the bilinear form α given by [\(1.13\)](#) is coercive if the height of the domain \mathcal{O} is sufficiently small. Indeed, from the expression [\(1.13\)](#) for the bilinear form, we can write $\alpha(u, u)$ as a sum of four terms $I_1 + I_2 + I_3 + I_4$. If $\mathcal{O} \subseteq \mathbb{R}^{n-1} \times (0, \delta)$, for a constant $\delta \in (0, 1]$, the expression [\(1.13\)](#) yields a positive constant, $C = C(\Lambda, n)$, such that

$$|I_2| + |I_3| \leq \sqrt{\delta} C \|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2.$$

If in addition there is a positive constant, c_0 , such that $c \geq c_0$ on \mathcal{O} , the preceding inequality and the strict ellipticity condition [\(1.6\)](#) gives the inequality,

$$\alpha(u, u) \geq (v_0 - \sqrt{\delta} C) \|\sqrt{y} Du\|_{L^2(\mathcal{O}, \mathfrak{w})}^2 + (c_0 - \delta C) \|u\|_{L^2(\mathcal{O}, \mathfrak{w})}^2, \quad \forall u \in H^1(\mathcal{O}, \mathfrak{w}).$$

Hence, there are positive constants

$$\delta_0 = \delta_0(c_0, \Lambda, n, v_0) \quad \text{and} \quad C_0 = C_0(c_0, \Lambda, n, v_0), \tag{1.45}$$

such that

$$\alpha(u, u) \geq C_0 \|u\|_{H^1(\mathcal{O}, \mathfrak{w})}^2, \quad \forall u \in H^1(\mathcal{O}, \mathfrak{w}), \tag{1.46}$$

for all subdomains $\mathcal{O} \subseteq \mathbb{R}^{n-1} \times (0, \delta_0)$. Therefore, when $\mathcal{O} \subseteq \mathbb{R}^{n-1} \times (0, \delta_0)$ and $c \geq c_0 > 0$ a.e. on \mathcal{O} , the bilinear form $\alpha : H^1(\mathcal{O}, \mathfrak{w}) \times H^1(\mathcal{O}, \mathfrak{w}) \rightarrow \mathbb{R}$ is *coercive*. We use this observation in [Remark 6.7](#).

1.2.6. Harnack inequality for non-negative solutions to the variational equation

We also have the following analogue of [\[29, Theorem 8.20 and Corollary 8.21\]](#) and [\[34, Theorem 4.5.3\]](#); it is important to note that [Theorem 1.24](#) is a direct analogue of the classical *interior* Harnack inequality — with points in the degenerate-boundary portion, Γ_0 , playing the same role as points in \mathcal{O} — and *not* a ‘boundary Harnack inequality’ (compare, for example, [\[3, Theorem 1.1\]](#)).

Theorem 1.24 (*Harnack inequality near Γ_0*). *Let $\mathcal{O}' \subset \mathcal{O} \subseteq \mathbb{H}$ be open subsets such that $\mathcal{O}' \Subset \mathcal{O} \cup \Gamma_0$. Then there is a positive constant C , depending at most on $\text{diam}(\mathcal{O}')$, $\text{dist}(\partial \mathcal{O}' \cap \mathbb{H}, \partial \mathcal{O} \cap \mathbb{H})$, Λ , v_0 and n , such that for any non-negative $u \in H^1(\mathcal{O}, \mathfrak{w})$ obeying [\(1.19\)](#) with $f = 0$ on \mathcal{O} , we have*

$$\text{ess sup}_{\mathcal{O}'} u \leq C \text{ess inf}_{\mathcal{O}'} u. \tag{1.47}$$

Remark 1.25 (*Applications to the proof of optimal regularity for variational solutions to the obstacle problem*). Continuity up the ‘degenerate boundary’ ([Theorem 1.20](#)) and the Harnack inequality ([Theorem 1.24](#)) are among the results of this article which Daskalopoulos and Feehan apply in [\[10\]](#) to prove that a solution $u \in H^1(\mathcal{O}, \mathfrak{w})$ to [\(1.3\)](#) actually belongs to $C_{s, \text{loc}}^{1,1}(\mathcal{O} \cup \Gamma_0)$.

1.3. Connections with previous research

As noted in [§1.1](#), there is a long history of research on local L^∞ and C^α estimates and Hölder regularity and Harnack inequalities for weak solutions to degenerate-elliptic equations, so a reader may reasonably ask what is new in this article. Because our article builds most directly on work of Koch, we begin with a comparison of our methods

and results with those in [34]. We then contrast our work with that of S. Chanillo and R. L. Wheeden [4], E. B. Fabes, C. E. Kenig and R. P. Serapioni [15], J. J. Kohn and L. Nirenberg [35], and M. K. V. Murthy and G. Stampacchia [41], as well as a selection of later articles which further develop their ideas.

The arguments in our article are not straightforward adaptations of the proofs of the analogous classical results described by Gilbarg and Trudinger [29, Theorems 8.15, 8.20, 8.22 and 8.27], due to the fact that our Sobolev spaces are weighted, so the standard Sobolev, Poincaré, and John–Nirenberg inequalities do not apply. We rely on the Moser iteration technique and the most difficult step in making this technique work involves the selection of a suitable John–Nirenberg inequality. For this purpose, we use the so-called *abstract John–Nirenberg inequality*, due to Bombieri and Giusti [2, Theorem 4], which can be applied to any topological space endowed with a regular Borel measure satisfying some natural requirements. In order to verify the hypotheses of the abstract John–Nirenberg inequality in our weighted Sobolev space setting (Proposition 3.2), we prove a local version of the Poincaré inequality, Corollary 2.6, suitable for our weighted Sobolev spaces.

1.3.1. Connections with work of Koch

In [34], Koch considers weak solutions to a certain linear parabolic partial differential equation in divergence form and which arises in the study of the porous medium equation. He takes the spatial domain to be the whole upper half space, $\mathbb{H} = \mathbb{R}^{n-1} \times \mathbb{R}_+$, assumes a degeneracy similar to that in the operator (1.12) and obtains a local L^∞ bound [34, Proposition 4.5.1], a Harnack inequality [34, Theorem 4.5.3], and a C^α estimate and Hölder continuity [34, Theorem 4.5.5] up to the degenerate boundary ($y = 0$) for weak solutions. Koch uses the same Sobolev weights as ours, but whereas he uses potential theory and pointwise estimates for fundamental solutions to prove the Harnack inequality and Hölder continuity, our method of proof is based on Moser iteration and avoids any need for potential theory or pointwise estimates of fundamental solutions. We believe that this is an important distinction: in this article we establish results, for a broader class of degenerate elliptic operators, that would be difficult to achieve using potential theory.

While Koch takes the spatial domain to be the whole upper half-space, $\mathcal{O} = \mathbb{H}$, we consider the variational equation (1.19) on subdomains of the half-space, $\mathcal{O} \subsetneq \mathbb{H}$, with Dirichlet boundary condition along the non-degenerate boundary, Γ_1 . In [34], Koch does not need to analyze the regularity of solutions at the ‘corner points’ ($\bar{\Gamma}_0 \cap \bar{\Gamma}_1$), but in our article we establish local supremum bounds for weak subsolutions and C^α estimates and Hölder continuity up to $\bar{\Gamma}_0$ for weak solutions on neighborhoods of points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ (see our Theorems 1.6 and 1.13, and Corollaries 1.8 and 1.16) — results which appear difficult to obtain using pointwise estimates of the fundamental solution.

In [34], Koch uses Moser iteration but only to obtain the local L^∞ bound for a weak solution [34, Proposition 4.5.1]. In order to prove Hölder regularity of solutions along the boundary $\bar{\Gamma}_0$, we need the version of the Poincaré inequality for weighted Sobolev spaces that we prove in Corollary 2.6. Koch also obtains a version of the Poincaré inequality for weighted Sobolev spaces [34, Lemma 4.4.4] that applies to functions defined on the whole half-space. The Hölder regularity results we establish in this article are local and they are most easily proved using a local version of the Poincaré inequality, such as our Corollary 2.6. The proof of our Poincaré inequality — relying only on integration by parts and the Poincaré inequality for standard Sobolev spaces — appears simpler to us than the proof of [34, Lemma 4.4.4].

Our local version of the Poincaré inequality (Corollary 2.6) allows us to appeal to the ‘abstract John–Nirenberg inequality’ [2, Theorem 4] and employ Moser iteration to obtain, as we noted above, Hölder regularity for weak solutions up to the ‘corner points’ ($\bar{\Gamma}_0 \cap \Gamma_1$) and a Harnack inequality (on neighborhoods of points in Γ_0) for non-negative weak solutions without relying on pointwise estimates of fundamental solutions. In particular, Koch does not use a John–Nirenberg inequality for weighted Sobolev spaces to obtain the results we cited in [34].

Finally, Koch does not consider applications to Hölder regularity of solutions to variational inequalities as we do in our article.

1.3.2. Connections with other closely related work

Kohn and Nirenberg prove an *a priori* estimate, existence, and uniqueness of a solution in a certain weighted Sobolev space [35, Equation (1.6)] to a variational equation defined by a boundary-degenerate, linear, second-order elliptic operator [35, Theorem 1]. They assume that the domain boundary is smooth, while we allow the domain to have singularities (at points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$). Rather than exploit the regularity of the solution implied by a suitable choice

of weighted Sobolev space, they use the sign of the Fichera function⁴ to determine when to impose Dirichlet boundary condition on portions of the domain boundary. In the case of Heston operator A in (1.14), this implies a dichotomy, $0 < \beta < 1$ and $\beta \geq 1$, when applying a Dirichlet boundary condition along Γ_0 , whereas our choice of rather different weighted Sobolev spaces removes this undesirable dichotomy entirely and we never need to prescribe a Dirichlet boundary condition along Γ_0 ; see Appendix B in the earlier preprint version [19] of [20] for a detailed discussion. When $0 < \beta < 1$ (recall that $\beta = 2\kappa\theta/\sigma^2$ from (1.15)), Kohn and Nirenberg would require a homogeneous Dirichlet condition along the full boundary, $\partial\mathcal{O}$, in their main [35, Theorem 1]: while this is in accordance with the Fichera sign condition [35, pp. 798–801], a boundary condition along Γ_0 limits the regularity of the solution, u , to being at most continuous up to Γ_0 .

Even when $\beta \geq 1$, their additional technical conditions [35, (a)–(d), pp. 799–800] mean that their main result does not apply to the problem we consider in this article. For example, they use the Fichera condition to partition the boundary as $\partial\mathcal{O} = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and, when $\beta \geq 1$, $\Sigma_1 = \Gamma_0$, $\Sigma_2 = \emptyset$, and $\Sigma_3 = \Gamma_1$. They require that $\Sigma_2 \cup \Sigma_3$ be relatively closed, which means that Γ_1 should be relatively closed, which is not true in our problem. Moreover, the closures of the portions of the boundary with a Dirichlet condition, $\Sigma_2 \cup \Sigma_3$, and of the portion without any boundary condition, Σ_1 , are disjoint, while in our problem, they are allowed to intersect.

While [35, Theorem 1] provides a global *a priori* estimate (see [35, Inequality (1.7)]), along with existence and uniqueness of a solution in a certain weighted Sobolev space, it bounds the weighted Sobolev norm [35, Equation (1.6)] of $u \in W_{loc}^{2,k}(\mathcal{O})$ in terms of the same weighted Sobolev norm of $f \in W_{loc}^{2,k}(\mathcal{O})$, for any $k \geq 1$, and this regularity requirement on f is unusually strong. While we might try to extract global regularity for u (in terms of Hölder norms) up to $\partial\mathcal{O}$, that would require a suitable embedding theorem for weighted Sobolev spaces and, as far as we can tell (see, for example, [38]), such an embedding theorem is not available for the weighted Sobolev space defined in [35, Equation (1.6)]. Simple localization procedures, using cutoff functions, usually require appropriate interpolation inequalities and these are not developed in [35] and may not be straightforward. On the other hand, more advanced methods of developing local supremum or Hölder estimates usually require Sobolev, Poincaré, and John–Nirenberg inequalities and these are not developed in [35] and, again, may not be straightforward for the choices of weights selected in [35]. Indeed, the Sobolev weights in [35, Theorem 1] appear to have a technical motivation, while the weights used in our article are directly motivated by the discussion in [20, Section 8].

Murthy and Stampacchia [41,42] establish local supremum estimates, Hölder regularity, and global L^p estimates for solutions in weighted Sobolev spaces to a variational equation defined by a boundary-degenerate, linear, second-order elliptic operator. They assume that the (Lipschitz) coefficients $\bar{a}^{\mu\nu}$ in (1.1) obey

$$\langle \bar{a}(z)\xi, \xi \rangle \geq m(z)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } z \in \mathcal{O},$$

where the weight, $m \geq 0$ a.e. on a bounded domain \mathcal{O} , is required to obey [41, p. 1]

$$m \in L^s(\mathcal{O}) \quad \text{and} \quad m^{-1} \in L^t(\mathcal{O}),$$

for some $s, t \geq 1$ such that $1/s + 1/t < 2/n$. The operator A defined in (1.12) does not satisfy the Murthy–Stampacchia condition since we would need to choose $m(x, y) = v_0 y$ and clearly $m^{-1} \notin L^t(\mathcal{O})$ for any $t \geq 1$ whenever Γ_0 is non-empty (as we allow throughout our article).

Fabes, Kenig and Serapioni [15] consider operators of the form $Au = (a^{\mu\nu}u_{x_\mu})_{x_\nu}$, and Lipschitz coefficients $a^{\mu\nu}$ obeying [15, p. 78]

$$C^{-1}w(z)|\xi|^2 \leq \langle \bar{a}(z)\xi, \xi \rangle \leq Cw(z)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } z \in \mathcal{O},$$

where C is a positive constant, and w is a weight that belongs to the Muckenhoupt class, A_2 . They use Moser iteration to establish local supremum estimates and Hölder continuity for solutions and a Harnack inequality for non-negative solutions, $u \in H_0^1(\mathcal{O}, w)$, to the variational equation [15, p. 94]

$$\int_{\mathcal{O}} \bar{a}^{\mu\nu}u_{z_\mu}v_{z_\nu} dz = \int_{\mathcal{O}} f v dz, \quad \forall v \in C_0^\infty(\mathcal{O}),$$

⁴ Namely, $(b^\mu - a_{z_\nu}^{\mu\nu})\eta_\mu$, where (η_1, \dots, η_n) is the inward-pointing unit normal vector field along $\partial\mathcal{O}$ [45, Equation (1.1.3)].

given $f \in L^2(\mathcal{O})$ (by [15, p. 81]) and where they define [15, p. 91] (note the contrast with our definition (1.9b) of $H^1(\mathcal{O}, \mathfrak{w})$)

$$\|u\|_{H^1(\mathcal{O}, w)} := \left(\int_{\mathcal{O}} (|Du|^2 + u^2) w \, dx \right)^{1/2},$$

and $H^1(\mathcal{O}, w)$ is the completion of $C_0^\infty(\mathcal{O})$ in $H^1(\mathcal{O}, w)$. The Poincaré inequality holds in the case of A_2 weights [15, p. 95, Item (4)] and the Sobolev inequality holds in the case of A_p weights [15, Theorems 1.2, 1.3, 1.5 and 1.6]. A calculation shows that our choice of weight, $\mathfrak{w}(x, y) = y^{\beta-1} e^{-\tau|x|-\mu y}$ in (1.11) — or any of its variants which keep the important factor $y^{\beta-1}$ — is not contained in the A_p class when $\beta \geq p$, and therefore the crucial Sobolev and Poincaré inequalities established in [15] do not apply. Even if we restrict to the case $\beta < 2$, Fabes, Kenig and Serapioni only obtain results for solutions obeying a homogeneous Dirichlet boundary condition along the full boundary, $\partial\mathcal{O}$, whereas the essential feature of our article is that we impose no boundary condition along Γ_0 . Finally, the absence of the lower-order terms in (1.1) considerably simplifies the problem since, in a degenerate-elliptic operator, the term $b^\mu u_{z_\mu}$ may be as significant as $a^{\mu\nu} u_{z_\mu z_\nu}$.

The method of Moser iteration has also been extended to degenerate operators in divergence form in articles such as [6–8,5,28], where the properties of A_2 and $A_{n/2+1}$ weights are used to derive the Harnack inequality and Hölder regularity properties of solutions. We remark that the weight $y^{\beta-1}$ considered in our article does not belong to these classes of functions, when $\beta \geq 2$ and $\beta \geq n/2$, respectively. Moreover such a restriction would not be natural in the present context.

Chanillo and Wheeden [4] prove a Harnack inequality, extending that of [15] by allowing unequal weights,

$$w(z)|\xi|^2 \leq \langle \bar{a}(z)\xi, \xi \rangle \leq v(z)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \text{ and a.e. } z \in \mathcal{O}.$$

While they also relax the condition that $w \in A_2$, they require that v, w obey a doubling condition⁵ and Poincaré and Sobolev inequalities [4, §1.2]. However, their Harnack inequality has the traditional, interior form (compare [29, Theorem 8.21] for the case of a strictly elliptic operator) for a subdomain $\mathcal{O}' \Subset \mathcal{O}$. Mohammed [40] extends the work of Chanillo and Wheeden by allowing general, non-zero coefficients b^μ and c for A in (1.1). Pinggen also extends the work of Chanillo and Wheeden, but rather by considering quasilinear elliptic system in pure divergence form and no lower-order terms. He obtains an interior Harnack inequality and interior Hölder continuity, under suitable conditions on the structure of the quasilinearity and doubling conditions on the weights w and $z := v^2/w$. Di Fazio, Fanciullo, and Zamboni [16,17,53] and Stredulinsky [47] also obtain an interior Harnack inequality and interior Hölder continuity for quasi-linear degenerate elliptic equations in divergence form under related hypotheses.

Lierl and Saloff-Coste use Moser iteration to establish a parabolic Harnack inequality for time-dependent, non-symmetric, local Dirichlet forms [39, Theorem 3.14]. Their hypotheses, [39, Assumptions 0,1,2 and 4], are satisfied by the bilinear form (1.13) defined by the operator A in (1.12) on domains of finite height, for example, $\mathcal{O} \subseteq \mathbb{R}^{n-1} \times (0, y_0)$, where y_0 is a positive constant. The Poincaré inequality is a crucial ingredient in the proof of the Harnack inequality, which we prove in Corollary 2.10 by elementary methods. Lierl and Saloff-Coste state in [39, Theorem 3.11] a different version of the Poincaré inequality that involves the distance to the boundary of the ball, which in turn is proved in [48, Corollary 2.5]. In our Poincaré inequality, Corollary 2.10, we do not need to use the distance to the boundary of the ball.

Lierl and Saloff-Coste also prove Hölder continuity of solutions [39, Corollary 3.17] with zero source function. To prove the Hölder continuity of solutions with non-zero source function, f , we need the stronger *weak Harnack inequality* (compare [29, Theorem 8.18] for the case of a strictly elliptic operator), which is embedded in our proof of Theorem 1.11 in estimate (5.34). Since the weak Harnack inequality allows non-zero source functions (unlike the Harnack inequality), it enables us to establish Hölder continuity of solutions with non-zero source function. Because the Harnack inequality is an ‘interior estimate’ (recall that Γ_0 essentially plays the same role as the interior of \mathcal{O} in our article), it cannot be used to obtain Hölder continuity of solutions to the variational equation at corner points $(\bar{\Gamma}_0 \cap \bar{\Gamma}_1)$, as we do in our Theorem 1.13.

⁵ This is also true for our weight, \mathfrak{w} , by Lemma 2.4.

For variational inequalities defined by degenerate elliptic or parabolic operators, there has been little previous research. Vitanza and Zamboni [51,52] describe existence and uniqueness results for solutions in certain weighted Sobolev spaces, but do not consider boundary regularity of solutions or partial Dirichlet boundary conditions.

1.4. Mathematical highlights and guide to the article

For the convenience of the reader, we provide a brief outline of the article. We begin in §2 by describing a Sobolev inequality due to H. Koch [34] and prove a Poincaré inequality for our weighted Sobolev spaces. In §3, we recall the abstract John–Nirenberg inequality (Theorem 3.1) due to E. Bombieri and E. Giusti [2] and justify its application (via Proposition 3.2) in the setting of our weighted Sobolev spaces. The supremum estimate near $\bar{\Gamma}_0$ for solutions to the variational equation (1.19) (Theorems 1.5 and 1.6) is proved in §4 by adapting the Moser iteration technique employed in the proof of [29, Theorem 8.15] to the setting of our degenerate elliptic operators and weighted Sobolev spaces. Section 5 contains our proof of local Hölder continuity along $\bar{\Gamma}_0$ of solutions to the variational equation (1.19) (Theorems 1.11 and 1.13). The essential difference between the proofs of Theorems 1.11 and 1.13 and the proof of their classical analogue for variational solutions to non-degenerate elliptic equations [29, Theorems 8.27 and 8.29] consists in a modification of the methods of [29, §8.6, §8.9, and §8.10] when deriving our energy estimates (5.15), where we adapt the application of the John–Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. In this section we also prove the Strong Maximum Principle (Theorem 1.18). In §6, we apply the penalization method and techniques of [9], together with Theorems 1.11 and 1.13, to prove local Hölder continuity along $\bar{\Gamma}_0$ of solutions to the variational inequality (1.3) (Theorem 1.20). Finally, in §7 we prove the Harnack inequality (Theorem 1.24) for solutions to the variational equation (1.19). Appendix A contains the proofs of auxiliary results employed throughout the article whose proofs are sufficiently technical that they would have otherwise interrupted the logical flow of our article.

A longer, unpublished version of this article appeared as [21] and additional details for some lengthy but routine calculations are available there.

1.5. Notation and conventions

In the definition and naming of function spaces, including spaces of continuous functions, Hölder spaces, or Sobolev spaces, we follow Adams [1] and alert the reader to occasional differences in definitions between [1] and standard references such as Gilbarg and Trudinger [29] or Krylov [36,37]. We denote $\mathbb{R}_+ := (0, \infty)$, $\bar{\mathbb{R}}_+ := [0, \infty)$, $\mathbb{H} := \mathbb{R}^{n-1} \times \mathbb{R}_+$, and $\bar{\mathbb{H}} := \mathbb{R}^{n-1} \times \bar{\mathbb{R}}_+$, where $n \geq 2$. We let $\mathbb{N} := \{1, 2, 3, \dots\}$ denote the set of positive integers. For $x, y \in \mathbb{R}$, we denote $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$. Moreover, $x^+ := x \vee 0$ and $x^- := -(x \wedge 0)$, so $x = x^+ - x^-$ and $|x| = x^+ + x^-$, a convention which differs from that of [29, §7.4]. If $V \subset S$ is an open subset of a subset $S \subset \mathbb{R}^n$, we write $U \Subset S$ when \bar{U} is compact and $\bar{U} \subset S$.

When we label a condition an *Assumption*, then it is considered to be universal and in effect throughout this article and so not referenced explicitly in theorem and similar statements; when we label a condition a *Hypothesis*, then it is only considered to be in effect when explicitly referenced.

1.6. Acknowledgments

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2. Sobolev and Poincaré inequalities for weighted Sobolev spaces

The main result of this subsection is a Poincaré inequality (Lemma 2.5) for weighted Sobolev spaces. In addition, we review a Sobolev inequality (Lemma 2.2) due to H. Koch [34]. Recall from [34, Corollary 4.3.4] that the weight $y^{\beta-1}$ defines a doubling measure, $y^{\beta-1} dx dy$ on \mathbb{H} for any $\beta > 0$ (see, for example, [50, Definition 1.2.6]), where $dx dy$ is Lebesgue measure on \mathbb{H} . In the following Lemma 2.2 and the sequel, we will need the following

Definition 2.1. Throughout our article, we fix

$$p := \frac{2(n + \beta)}{n + \beta - 1}, \tag{2.1}$$

for any $\beta > 0$.

We recall the

Lemma 2.2 (Weighted Sobolev inequality). [34, Lemma 4.2.4] Let p be as in (2.1). Then there is a positive constant $C = C(n, p)$ such that

$$\int_{\mathbb{H}} |u|^p y^{\beta-1} dx dy \leq c \left(\int_{\mathbb{H}} |u|^2 y^{\beta-1} dx dy \right)^{\frac{p-2}{2}} \int_{\mathbb{H}} |\nabla u|^2 y^\beta dx dy, \tag{2.2}$$

for any $u \in L^2(\mathbb{H}, y^{\beta-1})$ such that $\nabla u \in L^2(\mathbb{H}, y^\beta)$.

For $R > 0$ and $z_0 \in \bar{\mathcal{O}}$, we denote

$$B_R(z_0) = \{z \in \mathcal{O} : d(z, z_0) < R\}, \tag{2.3}$$

$$\mathbb{B}_R(z_0) = \{z \in \mathbb{H} : d(z, z_0) < R\}, \tag{2.4}$$

while

$$\bar{B}_R(z_0) = \{z \in \bar{\mathcal{O}} : d(z, z_0) \leq R\} \quad \text{and} \quad \bar{\mathbb{B}}_R(z_0) = \{z \in \bar{\mathbb{H}} : d(z, z_0) \leq R\},$$

are the usual closures of $B_R(z_0)$ in $\bar{\mathcal{O}}$ and of $\mathbb{B}_R(z_0)$ in $\bar{\mathbb{H}}$. Using definition (1.28) of the cycloidal distance, we obtain the following inclusions. For all $R > 0$, we have

$$E_{R^2}(z_0) \subset B_R(z_0), \quad \forall z_0 \in \bar{\mathbb{H}}, \tag{2.5}$$

$$B_R(z_0) \subset E_{2R^2}(z_0), \quad \forall z_0 \in \partial\mathbb{H}. \tag{2.6}$$

Throughout the article we also use the following

Definition 2.3 (Volume of sets). If $S \subset \bar{\mathbb{H}}$ is a Borel measurable subset, we let $|S|_\beta$ denote the volume of S with respect to the measure $y^\beta dx dy$, and $|S|_{\mathfrak{w}}$ denote the volume of S with respect to the measure $\mathfrak{w} dx dy$.

We now recall

Lemma 2.4. [34, Lemma 4.3.3] There is a positive constant $c \geq 1$, depending only on n and β , such that, for any $R > 0$ and $z_0 \in \bar{\mathbb{H}}$,

$$c^{-1} R^n (R + \sqrt{y_0})^{n+2\beta} \leq |\mathbb{B}_R(z_0)|_\beta \leq c R^n (R + \sqrt{y_0})^{n+2\beta}. \tag{2.7}$$

Moreover, the following inclusions hold,

$$\mathbb{E}_{R_1}(z_0) \subseteq \mathbb{B}_R(z_0) \subseteq \mathbb{E}_{R_2}(z_0), \tag{2.8}$$

where $R_1 = R(R + \sqrt{y_0})/2000$ and $R_2 = R(R + 2\sqrt{y_0})$.

We have the following Poincaré inequalities, adapted to our weighted Sobolev spaces.

Lemma 2.5 (Poincaré inequality). Let $z_0 \in \partial\mathbb{H}$ and $R > 0$. Then there is a positive constant C , depending on β , n and R , such that for any $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$, we have

$$\inf_{c \in \mathbb{R}} \left(\int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \leq C \left(\int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^\beta dx dy \right)^{1/2}. \tag{2.9}$$

As a consequence of Lemma 2.5, we obtain

Corollary 2.6 (Poincaré inequality with scaling). *There is a positive constant C_0 , depending only on β and n , such that for any $z_0 \in \partial\mathbb{H}$, $R > 0$ and $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$ we have*

$$\begin{aligned} \inf_{c \in \mathbb{R}} \left(\frac{1}{|\mathbb{B}_R(z_0)|^{\beta-1}} \int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \\ \leq C_0 R^2 \left(\frac{1}{|\mathbb{B}_R(z_0)|^\beta} \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^\beta dx dy \right)^{1/2}. \end{aligned} \tag{2.10}$$

To prove Lemma 2.5 and Corollary 2.6, we make use of the following extension property.

Lemma 2.7 (Extension operator). *Let $z_0 \in \partial\mathbb{H}$ and $R > 0$. Let $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, for all $1 \leq i \leq n$, be such that $D = \prod_{i=1}^n (a_i, b_i)$ is a rectangle with the property that $\mathbb{B}_R(z_0) \subseteq\subseteq D$. Then, there is a continuous extension*

$$E : H^1(\mathbb{B}_R(z_0), \mathfrak{w}) \rightarrow H^1(D, \mathfrak{w}),$$

and there exists a positive constant C , depending on D , R , n and β , such that for any $u \in H^1(\mathbb{B}_R(z_0), \mathfrak{w})$ we have

$$\begin{aligned} \|Eu\|_{L^2(D, y^{\beta-1})} &\leq C \|u\|_{L^2(\mathbb{B}_R(z_0), y^{\beta-1})}, \\ \|\nabla Eu\|_{L^2(D, y^\beta)} &\leq C \|\nabla u\|_{L^2(\mathbb{B}_R(z_0), y^\beta)}. \end{aligned} \tag{2.11}$$

Remark 2.8. Without loss of generality, in the proofs of Lemmas 2.5 and 2.7 and Corollary 2.6 we may assume $z_0 = (0, 0)$.

Proof of Lemma 2.5. Let $a_i < b_i$, for all $1 \leq i \leq n - 1$, and let $\delta > 0$ be such that $\mathbb{B}_R(z_0) \subseteq\subseteq D_0 \times (0, \delta)$, where we denote $D_0 := \prod_{i=1}^{n-1} (a_i, b_i)$. Let $k > 1$ be such that

$$2k^{-\beta} = \frac{1}{2}. \tag{2.12}$$

Let $\hat{u} = Eu$ be the extension of u to D given by Lemma 2.7. Assuming that (2.9) holds for \hat{u} , we obtain that it holds for u also in the following way,

$$\begin{aligned} \inf_{c \in \mathbb{R}} \left(\int_{\mathbb{B}_R(z_0)} |u(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} &\leq \inf_{c \in \mathbb{R}} \left(\int_D |\hat{u}(z) - c|^2 y^{\beta-1} dx dy \right)^{1/2} \\ &\leq C \left(\int_D |\nabla \hat{u}(z)|^2 y^\beta dx dy \right)^{1/2} \\ &\leq C \left(\int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 y^\beta dx dy \right)^{1/2}. \end{aligned}$$

In the last inequality above, we made use of (2.11).

Therefore, we may assume $u \in H^1(D, \mathfrak{w})$. Our goal is to prove that (2.9) holds for $u \in H^1(D, \mathfrak{w})$. By [9, Corollary A.14], we may assume without loss of generality that $u \in C^1(\bar{D})$. Let $c \in \mathbb{R}$ and let $v = u - c$. Then, by the mean value theorem, we have for any $y \in (0, \delta)$ and $x \in D_0$,

$$v(x, y) = v(x, ky) + \int_{ky}^y v_y(x, t) dt.$$

Squaring both sides of the preceding equation and integrating in y with respect to $y^{\beta-1} dy$, we obtain

$$\int_0^\delta |v(x, y)|^2 y^{\beta-1} dy \leq 2 \int_0^\delta |v(x, ky)|^2 y^{\beta-1} dy + 2 \int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 y^{\beta-1} dy. \tag{2.13}$$

By applying the change of variable $y' = ky$, we see that

$$\int_0^\delta |v(x, ky)|^2 y^{\beta-1} dy = k^{-\beta} \int_0^{k\delta} |v(x, y')|^2 y'^{\beta-1} dy'. \tag{2.14}$$

Also, we have for $\beta \neq 1$,

$$\begin{aligned} \int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 y^{\beta-1} dy &= \int_0^\delta \left| \int_{ky}^y v_y(x, t) t^{\beta/2} t^{-\beta/2} dt \right|^2 y^{\beta-1} dy \\ &\leq \frac{1}{|1-\beta|} \int_0^\delta \int_y^{ky} |v_y(x, t)|^2 t^\beta dt \left| y^{-\beta+1} - (ky)^{-\beta+1} \right| y^{\beta-1} dy \\ &\leq \delta \frac{1+k^{-\beta+1}}{|1-\beta|} \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy. \end{aligned} \tag{2.15}$$

For $\beta = 1$, a similar calculation gives us

$$\int_0^\delta \left| \int_{ky}^y v_y(x, t) dt \right|^2 dy \leq \delta \log k \int_0^{k\delta} |v_y(x, y)|^2 y dy. \tag{2.16}$$

Define a positive constant $C_0 \equiv C_0(\beta, \delta)$ by $C_0 = 2\delta(1+k^{-\beta+1})/|1-\beta|$ when $\beta \neq 1$, and $C_0 = 2\delta \log k$ when $\beta = 1$. By combining equations (2.13), (2.14), (2.15) and (2.16), we obtain

$$\begin{aligned} \int_0^\delta |v(x, y)|^2 y^{\beta-1} dy &\leq 2k^{-\beta} \int_0^{k\delta} |v(x, y)|^2 y^{\beta-1} dy + C_0 \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy \\ &\leq 2k^{-\beta} \int_0^\delta |v(x, y)|^2 y^{\beta-1} dy + 2k^{-\beta} \int_\delta^{k\delta} |v(x, y)|^2 y^{\beta-1} dy \\ &\quad + C_0 \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy. \end{aligned}$$

Recall that $k > 1$ was chosen such that (2.12) is satisfied. Therefore, by integrating also in x , there exists $C = C(\beta, \delta)$ such that

$$\int_{D_0} \int_0^{k\delta} |v(x, y)|^2 y^{\beta-1} dy dx \leq C \int_{D_0} \int_\delta^{k\delta} |v(x, y)|^2 y^{\beta-1} dy dx + C \int_{D_0} \int_0^{k\delta} |v_y(x, y)|^2 y^\beta dy dx.$$

Since $v = u - c$, we have

$$\begin{aligned} & \inf_{c \in \mathbb{R}} \int_D |u(x, y) - c|^2 y^{\beta-1} dy dx \\ & \leq C \inf_{c \in \mathbb{R}} \int_{D_0} \int_{\delta}^{k\delta} |u(x, y) - c|^2 y^{\beta-1} dy dx + C \int_D |u_y(x, y)|^2 y^{\beta} dy dx. \end{aligned} \quad (2.17)$$

The rectangle $D' := D_0 \times (\delta, k\delta)$ is a convex domain and so we may apply the classical Poincaré inequality [29, Equation (7.45)] to give

$$\inf_{c \in \mathbb{R}} \int_{D_0} \int_{\delta}^{k\delta} |u(x, y) - c|^2 dy dx \leq C \int_{D_0} \int_{\delta}^{k\delta} |\nabla u(x, y)|^2 dy dx.$$

Let $C' := (k\delta)^{\beta-1}$ if $\beta \geq 1$, and $C' = \delta^{\beta-1}$ if $\beta < 1$. Then we see that $y^{\beta-1} \leq C'$, for all $y \in (\delta, k\delta)$, which gives

$$\int_{D_0} \int_{\delta}^{k\delta} |u(x, y) - c|^2 y^{\beta-1} dy dx \leq C' \int_{D_0} \int_{\delta}^{k\delta} |u(x, y) - c|^2 dy dx, \quad \forall c \in \mathbb{R}.$$

Using in addition the inequality,

$$\int_{D_0} \int_{\delta}^{k\delta} |\nabla u(x, y)|^2 dy dx \leq \delta^{-\beta} \int_{D_0} \int_{\delta}^{k\delta} |\nabla u(x, y)|^2 y^{\beta} dy dx,$$

and combining it with the preceding two inequalities, we obtain that

$$\inf_{c \in \mathbb{R}} \int_{D_0} \int_{\delta}^{k\delta} |u(x, y) - c|^2 y^{\beta-1} dy dx \leq C \int_{D_0} \int_{\delta}^{k\delta} |\nabla u(x, y)|^2 y^{\beta} dy dx,$$

where $C = C(\beta, D_0, \delta, k)$ is a positive constant. Because the domain D_0 and δ depend only on R , and k depends on β , the constant C in the preceding inequality depends only on β , n and R . Combining the preceding inequality with (2.17) yields (2.9). \square

Remark 2.9. Koch states a weighted Poincaré inequality on the half-space [34, Lemma 4.4.4], with weight $y^{\beta-1} e^{-\kappa\rho(z, z_0)}$, where κ is a positive constant, z_0 is a fixed point in \mathbb{H} , and $\rho(z, z_0)$ is equivalent to $d^2(z, z_0)$, in the sense that there exists a constant $c > 0$ such that

$$cd^2(z, z_0) \leq \rho(z, z_0) \leq \frac{1}{c}d^2(z, z_0), \quad \forall z \in \mathbb{H}.$$

The proof of this result is long and technical. So, rather than use this result to prove a weighted Poincaré inequality on a ball using an extension principle, we give a much simpler proof for balls and weights $y^{\beta-1}$ and y^{β} .

Remark 2.10 (Scaling under Koch metric). Using the definitions (1.28) for the cycloidal distance and (2.4) for the ball $\mathbb{B}_R(z_0)$, we obtain the following scaling property

$$\mathbb{B}_{R_1}(z_0) = \left(\frac{R_1}{R_2}\right)^2 \mathbb{B}_{R_2}(z_0), \quad \forall R_1, R_2 > 0 \text{ and } z_0 \in \partial\mathbb{H}, \quad (2.18)$$

since $d(z_0 + t^2(z - z_0), z_0) = td(z, z_0)$ for all $z \in \mathbb{H}$, $z_0 \in \partial\mathbb{H}$, and $t > 0$. Notice that (2.18) does not hold if $z_0 = (x_0, y_0)$ with $y_0 > 0$.

Proof of Corollary 2.6. Let $R > 0$ and $\bar{R} > 0$ and define v by rescaling

$$u(z) = v \left(z_0 + \left(\frac{\bar{R}}{R} \right)^2 (z - z_0) \right), \quad \forall z \in \mathbb{B}_R(z_0).$$

The rescaling map defined by $z \mapsto z_0 + (\bar{R}/R)^2(z - z_0)$ maps $\mathbb{B}_R(z_0)$ into $\mathbb{B}_{\bar{R}}(z_0)$ by Remark 2.10. By applying Lemma 2.5 to v on $\mathbb{B}_{\bar{R}}(z_0)$, there is a positive constant C_0 , depending only on \bar{R} , n and β , such that (2.9) holds. By changing variables, we obtain

$$\inf_{c \in \mathbb{R}} \left(\frac{\bar{R}}{R} \right)^{2(\beta-1)} \int_{\mathbb{B}_R(z_0)} |u - c|^2 y^{\beta-1} dx dy \leq C_0 \left(\frac{R}{\bar{R}} \right)^4 \left(\frac{\bar{R}}{R} \right)^{2\beta} \int_{\mathbb{B}_R(z_0)} |\nabla u|^2 y^\beta dx dy. \tag{2.19}$$

Using Lemma 2.4, we rewrite (2.19) in the following form

$$\inf_{c \in \mathbb{R}} \frac{|\mathbb{B}_{\bar{R}}(z_0)|^{\beta-1}}{|\mathbb{B}_R(z_0)|^{\beta-1}} \int_{\mathbb{B}_R(z_0)} |u - c|^2 y^{\beta-1} dx dy \leq C_0 \left(\frac{R}{\bar{R}} \right)^4 \frac{|\mathbb{B}_{\bar{R}}(z_0)|^\beta}{|\mathbb{B}_R(z_0)|^\beta} \int_{\mathbb{B}_R(z_0)} |\nabla u|^2 y^\beta dx dy,$$

from which (2.10) follows immediately by taking $\bar{R} = 1$. \square

3. John–Nirenberg inequality

In this section we recall the abstract John–Nirenberg inequality (Theorem 3.1) due to E. Bombieri and E. Giusti [2] and, in particular, provide a justification — via Proposition 3.2 — that its hypotheses hold in the setting of the problems described in §1.

We restrict the statement of [2, Theorem 4] to the framework of our problems, so in [2, Theorem 4] we choose \mathbb{H} to be the topological space and $d\mu = y^{\beta-1} dx dy$ to be the regular positive Borel measure on \mathbb{H} . Let S_r , $0 \leq r \leq 1$ be a family of non-empty open sets in \mathbb{H} such that

$$\begin{aligned} S_s \subseteq S_r, & \quad 0 \leq s \leq r \leq 1, \\ 0 < |S_r|_{\beta-1} < \infty, & \quad \forall r \in [0, 1]. \end{aligned} \tag{3.1}$$

Let w be a measurable positive function on S_1 . For $t \neq 0$ and $0 \leq r \leq 1$, we denote by

$$|w|_{t,r} = \left(\frac{1}{|S_r|_{\beta-1}} \int_{S_r} |w|^t y^{\beta-1} dx dy \right)^{1/t},$$

$$|w|_{\infty,r} = \operatorname{ess\,sup}_{S_r} w,$$

$$|w|_{-\infty,r} = \operatorname{ess\,inf}_{S_r} w.$$

We now recall the

Theorem 3.1 (Abstract John–Nirenberg inequality). [2, Theorem 4] Let $0 < \theta_0, \theta_1 \leq \infty$ and w be a measurable positive function on S_1 such that

$$|w|_{\theta_0,1} < \infty \text{ and } |w|_{\theta_1,1} > 0.$$

Suppose there exist constants $\gamma > 0$, $0 < t^* \leq \frac{1}{2} \min\{\theta_0, \theta_1\}$ and $Q > 0$ such that for all $0 \leq s < r \leq 1$ and $0 < t \leq t^*$,

$$\begin{aligned} |w|_{\theta_0,s} &\leq (Q(r-s)^\gamma)^{1/\theta_0-1/t} |w|_{t,r}, \\ |w|_{-\theta_1,s} &\geq (Q(r-s)^\gamma)^{1/t-1/\theta_1} |w|_{-t,r}. \end{aligned} \tag{3.2}$$

Assume further that

$$A := \sup_{0 \leq r \leq 1} \inf_{c \in \mathbb{R}} \frac{1}{|S_r|^{\beta-1}} \int_{S_r} |\log w - c| y^{\beta-1} dx dy < \infty. \tag{3.3}$$

Then, we have

$$|w|_{\theta_0,0} \leq \left(\frac{|S_1|^{\beta-1}}{|S_0|^{\beta-1}} \right)^{1/\theta_0+1/\theta_1} \exp \left\{ c_2 Q^{-2} (A + 1/t^*) \right\} |w|_{-\theta_1,0}, \tag{3.4}$$

where c_2 is a constant depending only on γ , but not on $Q, \theta_0, \theta_1, t^*, A$ and β .

In many of our proofs, we will make use of a sequence of cutoff functions, $\{\eta_N\}_{N \in \mathbb{N}}$. Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be a smooth function such that $\varphi(x) \equiv 1$ for $x < 0$, and $\varphi \equiv 0$ for $x > 1$. Let $z_0 \in \mathbb{H}$ and let $\{R_N\}_{N \geq 0}$ be a non-increasing sequence of positive numbers. We define

$$\eta_N(z) := \varphi \left(\frac{1}{R_{N-1}^2 - R_N^2} (d^2(z_0, z) - R_N^2) \right), \quad \forall z \in \mathbb{H}, \quad \forall N \in \mathbb{N}. \tag{3.5}$$

Then, the sequence $\{\eta_N\}_{N \geq 1}$ has the following properties,

$$\eta_N|_{B_{R_N}(z_0)} \equiv 1, \quad \eta_N|_{B_{R_{N-1}}^c(z_0)} \equiv 0, \tag{3.6}$$

$$|\nabla \eta_N| \leq \frac{C}{R_{N-1}^2 - R_N^2}, \tag{3.7}$$

where $B_{R_{N-1}}^c(z_0) := \mathbb{H} \setminus \bar{B}_{R_{N-1}}(z_0)$ and C is a positive constant independent of N and the sequence $\{R_N\}_{N \geq 0}$. The bound in (3.7) can be deduced from the calculation,

$$\nabla \eta_N = \varphi' \left(\frac{1}{R_{N-1}^2 - R_N^2} (d^2(z_0, z) - R_N^2) \right) \frac{1}{R_{N-1}^2 - R_N^2} \nabla d^2(z_0, z).$$

Also, we have that $|\nabla d^2(z_0, z)| \leq 5$, for all $z_0, z \in \mathbb{H}$. Since φ' is also uniformly bounded on \mathbb{R} , we obtain the forthcoming inequality (3.10).

Similarly, we can construct a sequence of cutoff functions, $\{\eta_N\}_{N \in \mathbb{N}}$, when $\{R_N\}_{N \geq 0}$ is a non-decreasing sequence of positive numbers.

We now provide a justification that the hypotheses of Theorem 3.1 hold in the setting of the problems discussed in this article.

Proposition 3.2 (Application of Theorem 3.1). *Let $z_0 \in \partial \mathbb{H}$ and $0 < 4R \leq 1$. Let $S_r = \mathbb{B}_{(2+r)R}(z_0)$, for all $0 \leq r \leq 1$. Let θ_0, θ_1 be as in Theorem 3.1 and set $t^* = \frac{1}{2} \min\{\theta_0, \theta_1\}$. Then, there exist positive constants Q and γ , independent of R and z_0 , such that (3.4) holds for any bounded positive function w on S_1 which satisfies the forthcoming energy estimates (5.15) or (7.4), where we recall that p is defined in (2.1).*

Proof. We begin by proving the first inequality in (3.2) by applying Moser iteration finitely many times. The second inequality in (3.2) can be proved by a similar technique. We give the proof when w satisfies the energy estimate (5.15), but the proof applies as well to positive bounded functions w satisfying the energy estimate (7.4).

As in the hypotheses of Theorem 3.1, we let $t \in (0, t^*]$. First, we consider the special case when θ_0 and t satisfy the requirement: There exists an integer $N^* \geq 1$ such that θ_0 can be written as

$$\theta_0 = t \left(\frac{p}{2} \right)^{N^*}. \tag{3.8}$$

Let $0 \leq s < r \leq 1$ and set $R_0 = (2 + r)R$. We denote

$$c := \sum_{k=1}^{\infty} \frac{1}{k^2}$$

and we let

$$R_N^2 := \left((2+r)^2 - (r-s)^2 \sum_{k=1}^N \frac{1}{ck^2} \right) R^2, \quad N = 1, \dots, N^*. \tag{3.9}$$

We observe that $(2+s)R < R_N < R_{N-1} \leq (2+r)R$. Let $\{\eta_N\}_{N \in \mathbb{N}}$ be a sequence of non-negative, smooth cutoff functions as constructed in (3.5), by choosing R_N as in (3.9). Then, (3.7) becomes

$$|\nabla \eta_N| \leq \frac{CN^2}{R^2(r-s)^2}. \tag{3.10}$$

Let $P_N := t(p/2)^N$, for $N = 1, \dots, N^*$, and $\alpha_N = p_N - 1$, for all $N = 0, \dots, N^* - 1$. We set

$$I(N) := \left(\int_{\mathbb{B}_{R_N}(z_0)} |w|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}, \tag{3.11}$$

From our hypothesis, w satisfies (5.15), that is,

$$\|\eta w^{(\alpha+1)/2}\|_{L^p(\mathbb{H}, y^{\beta-1})} \leq C_0(R, \alpha) \|w^{(\alpha+1)/2}\|_{L^2(\text{supp } \eta, y^{\beta-1})}, \tag{3.12}$$

where

$$C_0(R, \alpha) := (C|1 + \alpha|)^{(\xi+1)/p} \left(1 + \|\sqrt{y}\nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p}, \tag{3.13}$$

and ξ and C are positive constants, independent of w , α and η . We choose $\alpha = \alpha_{N-1}$ and $\eta = \eta_N$ in (3.12), so the definition (3.11) gives us, for all $N \geq 1$,

$$I(N) \leq C_1(R, r, s, N) I(N-1), \tag{3.14}$$

where

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(1 + \|\sqrt{y}\nabla \eta_N\|_{L^\infty(\mathbb{H})}^2 \right)^{1/p_N}.$$

From Lemma 2.4, we have $y \leq CR^2$ on $\mathbb{B}_{R_N}(z_0)$, where C is a positive constant independent of R and N . Using the bound (3.10), we obtain

$$C_1(R, r, s, N) := (C|p_{N-1}|)^{(\xi+1)/p_N} \left(\frac{CN^4}{R^2(r-s)^4} \right)^{1/p_N}.$$

By iterating inequality (3.14), we obtain

$$I(N^*) \leq C_2(R, r, s) I(0), \tag{3.15}$$

where

$$C_2(R, r, s) := \prod_{N=1}^{N^*} \left(C p_{N-1}^{\xi+1} N^4 R^{-2} (r-s)^{-4} \right)^{1/p_N}. \tag{3.16}$$

Next, we prove the

Claim 3.3. *There are positive constants Q and γ , independent of N^* , R , r and s , such that*

$$C_2(R, r, s) \leq (Q(r-s)^\gamma)^{1/\theta_0-1/t} R^{\frac{4}{p-2}(1/\theta_0-1/t)}. \tag{3.17}$$

Proof of Claim 3.3. We can rewrite the expression (3.16) for $C_2(R, r, s)$ to obtain

$$C_2(R, r, s) \leq \left(C t^{\xi+1} R^{-2} (r-s)^{-4} \right)^{\sum_{N=1}^{N^*} 1/p_N} \left(C \frac{p}{2} \right)^{\sum_{N=1}^{N^*} N/p_N}, \tag{3.18}$$

where we used in the last line that $N^4 \leq C(p/2)^N$, for some positive constant $C = C(p)$. Equation (3.8) leads to the identities

$$\sum_{N=1}^{N^*} \frac{1}{p_N} = \frac{2}{p-2} \left(\frac{1}{t} - \frac{1}{\theta_0} \right) \quad \text{and} \quad \sum_{N=1}^{N^*} \frac{N}{p_N} = \frac{4}{p(p-2)} \left(\frac{1}{t} - \frac{1}{\theta_0} \right).$$

Therefore, inequality (3.17) becomes

$$C_2(R, r, s) \leq \left(R^{-2}(r-s)^{-4} \right)^{\frac{2}{p-2} \left(\frac{1}{t} - \frac{1}{\theta_0} \right)} \left(C\theta_0^{\xi+1} \frac{P}{2} \right)^{\frac{4}{p(p-2)} \left(\frac{1}{t} - \frac{1}{\theta_0} \right)}, \tag{3.19}$$

which is equivalent to (3.17) with the choice of the constants $Q = \left(C\theta_0^{\xi+1} p/2 \right)^{-1}$ and $\gamma = 8/(p-2)$. This completes the proof of Claim 3.3. \square

From identity (2.1), we have that $4/(p-2) = 2(n+\beta-1)$, and so Lemma 2.4 (with constant $c_0 = c_0(n, \beta) > 1$) yields

$$\begin{aligned} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}} &\geq \frac{c_0^{-1/\theta_0} ((2+s)R)^{2(n+\beta-1)/\theta_0}}{c_0^{1/t} ((2+r)R)^{2(n+\beta-1)/t}} \\ &= c_0^{-1/\theta_0-1/t} \left(\frac{2+s}{2+r} \right)^{2(n+\beta-1)(1/\theta_0-1/t)} R^{2(n+\beta-1)(1/\theta_0-1/t)} \\ &\geq C^{1/\theta_0+1/t} R^{(4/(p-2))(1/\theta_0-1/t)}, \end{aligned}$$

for a positive constant $C = 1/c_0 < 1$ and recalling that $0 \leq s < r \leq 1$ and $\theta_0 > t$ by (3.8). Therefore, inequality (3.17) becomes

$$C_2(R, r, s) \leq C^{-1/\theta_0-1/t} \left(Q(r-s)^\gamma \right)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}}. \tag{3.20}$$

From the hypothesis of Proposition 3.2 that $t \leq t^* \leq \theta_0/2$, we have

$$3(1/\theta_0 - 1/t) \leq -1/\theta_0 - 1/t \leq 1/\theta_0 - 1/t,$$

and so, for a new positive constant Q , the inequality (3.20) leads to

$$C_2(R, r, s) \leq \left(Q(r-s)^\gamma \right)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}}. \tag{3.21}$$

By employing the inequalities (3.21) and (3.15) and the definition (3.11) of $I(N)$, we obtain

$$\begin{aligned} &\left(\int_{\mathbb{B}_{(2+s)R}(z_0)} |w|^{\theta_0} y^{\beta-1} dx dy \right)^{1/\theta_0} \\ &\leq \left(Q(r-s)^\gamma \right)^{1/\theta_0-1/t} \frac{|\mathbb{B}_{(2+s)R}(z_0)|_{\beta-1}^{1/\theta_0}}{|\mathbb{B}_{(2+r)R}(z_0)|_{\beta-1}^{1/t}} \left(\int_{\mathbb{B}_{(2+r)R}(z_0)} |w|^t y^{\beta-1} dx dy \right)^{1/t}, \end{aligned}$$

from which we readily obtain the first inequality in (3.2), in the special case where t and θ_0 satisfy (3.8) for some integer $N^* \geq 1$.

Next, we show that the first inequality in (3.2) holds for any $t \in (0, t^*)$. For this purpose, we choose an integer $N^* \geq 1$ such that

$$t \left(\frac{p}{2} \right)^{N^*-1} < \theta_0 < t \left(\frac{p}{2} \right)^{N^*}.$$

We denote $\theta_0^* = t(p/2)^{N^*}$ and we apply the previous analysis to t and θ_0^* , which now satisfy (3.8), to give

$$|w|_{\theta_0^*,s} \leq (Q(r-s)^\gamma)^{1/\theta_0^*-1/t} |w|_{t,r}.$$

Using Hölder’s inequality with $p = \theta_0^*/\theta_0 > 1$, we find that

$$|w|_{\theta_0,s} \leq |w|_{\theta_0^*,s},$$

and so

$$\begin{aligned} |w|_{\theta_0,s} &\leq (Q(r-s)^\gamma)^{1/\theta_0^*-1/t} |w|_{t,r} \\ &\leq (Q(r-s)^\gamma)^{\frac{1/\theta_0^*-1/t}{1/\theta_0-1/t}(1/\theta_0-1/t)} |w|_{t,r}. \end{aligned}$$

Notice that $2\theta_0^*/p \leq \theta_0 \leq \theta_0^*$ and $0 < t < \theta_0/2$. Then,

$$1 \leq \frac{1/\theta_0^* - 1/t}{1/\theta_0 - 1/t} \leq \frac{1/\theta_0^* - 1/t}{p/2\theta_0^* - 1/t} \leq \frac{(2/p)^{N^*} - 1}{(2/p)^{N^*+1} - 1} \leq \frac{p}{p-2}.$$

Consequently, we define \tilde{Q} to be $Q^{p/(p-2)}$ if $Q < 1$, and we leave Q unchanged if $Q \geq 1$ and, setting $\tilde{\gamma} := \gamma p/(p-2)$, the preceding estimate for $|w|_{\theta_0,s}$ becomes

$$|w|_{\theta_0,s} \leq (\tilde{Q}(r-s)^{\tilde{\gamma}})^{1/\theta_0-1/t} |w|_{t,r},$$

which is precisely the first inequality in (3.2). \square

4. Supremum estimates near the degenerate boundary

In this section, we prove Theorems 1.5 and 1.6 and Corollary 1.8, that is, local boundedness up to $\bar{\Gamma}_0$ for subsolutions (respectively, supersolutions), u , to the variational equation (1.19). Our choice of test functions when applying Moser iteration follows that employed in the proof of [29, Theorem 8.15]. However, the choice of test functions used in the proof of the classical local supremum estimates [29, Theorem 8.17] is not suitable in our case because the test functions in (1.19) are not required to satisfy a homogeneous Dirichlet boundary condition along $\bar{\Gamma}_0$. In addition, the method of deriving the energy estimate (4.5) is slightly different from [29, Theorem 8.18] because, instead of using the classical Sobolev inequalities [29, Theorem 7.10], we use Lemma 2.2.

We begin with the

Lemma 4.1. *Let K be a finite, right circular cone and \mathcal{O} be an open subset which obeys the uniform interior and exterior cone condition on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ with cone K . Then, there are positive constants \bar{R} and c depending on K , n and β such that, for all $R \in (0, \bar{R}]$, we have*

$$c^{-1}|\mathbb{B}_R(z_0)|_{\beta-1} \leq |B_R(z_0)|_{\beta-1} \leq c|\mathbb{B}_R(z_0)|_{\beta-1}, \quad \forall z_0 \in \bar{\Gamma}_0, \tag{4.1}$$

and also

$$c^{-1}|\mathbb{B}_R(z_0)|_{\beta-1} \leq |\mathbb{B}_R(z_0) \setminus B_R(z_0)|_{\beta-1} \leq c|\mathbb{B}_R(z_0)|_{\beta-1}, \quad \forall z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1. \tag{4.2}$$

An open subset, \mathcal{O} , which does not satisfy condition (4.1) can be created along the lines of [32, Example 4.2.17] (Lebesgue’s thorn); see [22, Example A.1].

Proof of Lemma 4.1. The proof of the lemma can be obtained just as in the case of the Euclidean distance function with the aid of Lemma 2.4. Complete details are provided in the proof of [21, Lemma 4.1]. \square

We can now proceed to the

Proof of Theorems 1.5 and 1.6. For the proof of Theorem 1.5, we choose $\bar{R} < \sqrt{R_0/2}$. For the proof of Theorem 1.6, we choose \bar{R} smaller than $\sqrt{R_0/2}$ and than the constant \bar{R} appearing in the conclusion of Lemma 4.1. Notice that (2.6) shows that $B_{\bar{R}}(z_0) \subset E_{R_0}(z_0)$.

Step 1 (Energy estimates). Let $\alpha \geq 1$ and let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function with support in $\bar{\mathbb{B}}_{2R}(z_0)$, where R is chosen such that $0 < 2R < \bar{R}$. We define

$$A := \|f\|_{L^s(\text{supp } \eta, y^{\beta-1})}. \tag{4.3}$$

We will apply the calculations in [Steps 1 and 2](#) to w defined by

$$w := u^+(u^-) + A. \tag{4.4}$$

For concreteness, we will illustrate our calculations with the choice $w = u^+ + A$ (when u is a subsolution), but they apply equally well to the choice $w = u^- + A$ (when u is a supersolution). Our goal in [Step 1](#) is to prove the following

Claim 4.2 (Energy estimate). *There are positive constants $C = C(\Lambda, \nu_0, n, s, \bar{R})$, and $\xi = \xi(n, \beta, s)$, such that*

$$\begin{aligned} & \left(\int_{\mathcal{O}} |\eta w^\alpha|^p y^{\beta-1} dx dy \right)^{1/p} \\ & \leq (C\alpha)^{\xi+1} \left(\|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta|_{\beta-1}^{1/p-1/2} \right) \left(\int_{\text{supp } \eta} w^{2\alpha} y^{\beta-1} dx dy \right)^{1/2}. \end{aligned} \tag{4.5}$$

Proof of Claim 4.2. We fix $k \in \mathbb{N}$. As in the proof of [\[29, Theorem 8.15\]](#), we consider the functions $H_k : \mathbb{R} \rightarrow [0, \infty)$,

$$H_k(t) := \begin{cases} 0, & t < A, \\ t^\alpha - A^\alpha, & A \leq t \leq k, \\ \alpha k^{\alpha-1}(t-k) + H_k(k), & t > k. \end{cases} \tag{4.6}$$

and

$$G_k(t) = \int_0^t |H'_k(s)|^2 ds. \tag{4.7}$$

Then,

$$v = G_k(w)\eta^2 \tag{4.8}$$

is a valid test function in $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ in [\(1.13\)](#) by [\[21, Lemma A.1\]](#). Using the strict ellipticity of the operator $y^{-1}A$, together with the fact that $\nabla v = G'_k(w)\eta^2\nabla w + 2G_k(w)\eta\nabla\eta$ and $G_k(w) = 0$ when $w \leq A$, we obtain as in the proof of [\[29, Theorem 8.15\]](#) that there is a positive constant, $C = C(\Lambda, n, \nu_0, \bar{R})$, such that

$$\begin{aligned} \int_{\mathcal{O}} |\nabla w|^2 \eta^2 G'_k(w) y^\beta dx dy & \leq C \left[\int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} dx dy \right. \\ & \quad \left. + \int_{\mathcal{O}} \left(\eta^2 + y|\nabla\eta|^2 \right) w^2 G'_k(w) y^{\beta-1} dx dy \right]. \end{aligned} \tag{4.9}$$

Hölder’s inequality applied to the conjugate pair (s, s^*) gives

$$\begin{aligned} & \int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} dx dy \\ & \leq \left(\int_{\text{supp } \eta} \frac{|f|^s}{A^s} y^{\beta-1} dx dy \right)^{1/s} \left(\int_{\mathcal{O}} |\eta^2 w^2 G'_k(w)|^{s^*} y^{\beta-1} dx dy \right)^{1/s^*}, \end{aligned}$$

and thus, by definition (4.3) of A ,

$$\int_{\mathcal{O}} \eta^2 \frac{|f|}{A} w^2 G'_k(w) y^{\beta-1} dx dy \leq \left(\int_{\mathcal{O}} |\eta^2 w^2 G'_k(w)|^{s^*} y^{\beta-1} dx dy \right)^{1/s^*}. \tag{4.10}$$

We need to justify first that the right-hand side in (4.10) is finite. First, we notice that the following identities hold

$$\begin{aligned} |\nabla H_k(w)|^2 &= |\nabla w|^2 |H'_k(w)|^2 = |\nabla w|^2 G'_k(w), \\ |w H'_k(w)|^2 &= |w|^2 G'_k(w), \end{aligned} \tag{4.11}$$

From the hypothesis $s > n + \beta$ in Theorems 1.5 and 1.6, we observe that $2 < 2s^* < p$, so we may apply the interpolation inequality [29, Inequality (7.10)]. For any $\varepsilon \in (0, 1)$, we have

$$\|\eta w H'_k(w)\|_{L^{2s^*}(\mathbb{H}, y^{\beta-1})} \leq \varepsilon \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} + \varepsilon^{-\xi} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}, \tag{4.12}$$

where

$$\xi \equiv \xi(p, s) := \frac{p(s^* - 1)}{p - 2s^*}. \tag{4.13}$$

We notice that $|H'_k(w)| \leq \alpha k^{\alpha-1}$ and $\eta w \in H^1(\mathcal{O}, \mathfrak{w})$ has compact support in $\bar{B}_{2R}(z_0)$. Therefore, we may apply Lemma 2.7 to build an extension \hat{w} of ηw to a rectangle D containing $\bar{B}_{2R}(z_0)$. Lemma 2.2, shows that $\hat{w} \in L^p(D, y^{\beta-1})$, which implies that

$$\|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} < \infty,$$

and so, the right-hand side of (4.10) is finite.

Inequalities (4.9) and (4.10), together with the identities (4.11) yield

$$\begin{aligned} \int_{\mathcal{O}} \eta^2 |\nabla H_k(w)|^2 y^\beta dx dy &\leq C \left[\left(\int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} \right. \\ &\quad \left. + \int_{\mathcal{O}} (\eta^2 + y |\nabla \eta|^2) |w H'_k(w)|^2 y^{\beta-1} dx dy \right]. \end{aligned} \tag{4.14}$$

From Lemma 2.2, we obtain

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq \left(\int_{\mathcal{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \int_{\mathcal{O}} |\nabla(\eta H_k(w))|^2 y^\beta dx dy \\ &\leq 2 \left(\int_{\mathcal{O}} \eta^2 |H_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \\ &\quad \times \left(\int_{\mathcal{O}} |\nabla \eta|^2 |H_k(w)|^2 y^\beta dx dy + \int_{\mathcal{O}} \eta^2 |\nabla H_k(w)|^2 y^\beta dx dy \right). \end{aligned} \tag{4.15}$$

Using $H_k(w) \leq w H'_k(w)$ and inequality (4.14) in (4.15), we see that

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq C \left[\left(1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \left(\int_{\text{supp } \eta} |w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{p/2} \right. \\ &\quad \left. + \left(\int_{\mathcal{O}} |\eta w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \left(\int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} \right], \end{aligned} \tag{4.16}$$

where $C = C(\Lambda, n, v_0, \bar{R}) > 0$. We rewrite the estimate for $\eta w H'_k(w)$ in (4.12) in the form

$$\begin{aligned} \left(\int_{\mathcal{O}} |\eta w H'_k(w)|^{2s^*} y^{\beta-1} dx dy \right)^{1/s^*} &= \|\eta w H'_k(w)\|_{L^{2s^*}(\mathbb{H}, y^{\beta-1})}^2 \\ &\leq 2\varepsilon^2 \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 + 2\varepsilon^{-2\xi} \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^2. \end{aligned}$$

Applying the preceding inequality in (4.16), we obtain

$$\begin{aligned} \|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C(1 + \varepsilon^{-2\xi}) \left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C\varepsilon^2 \|\eta w H'_k(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2} \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2. \end{aligned}$$

To estimate the last term in the preceding inequality, we apply Young’s inequality with the conjugate pair of exponents, $(p/2, p/(p - 2))$, to give

$$\begin{aligned} \|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p &\leq C \left(1 + (\varepsilon^2 + \varepsilon^{-2\xi})\right) \left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C\varepsilon^2 \|\eta w H'_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p. \end{aligned} \tag{4.17}$$

Employing the definition (4.6) of $H_k(w)$ gives $0 \leq w H'_k(w) \leq \alpha H_k(w) + \alpha A^\alpha$, and so

$$\int_{\mathcal{O}} |\eta w H'_k(w)|^p y^{\beta-1} dx dy \leq |2\alpha|^p \left[\int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy + |\text{supp } \eta|_{\beta-1} A^{\alpha p} \right],$$

and thus, applying inequality (4.17) yields

$$\begin{aligned} \int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy &\leq C \left(1 + (\varepsilon^2 + \varepsilon^{-2\xi})\right) \left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right) \|w H'_k(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p \\ &\quad + C|2\alpha|^p \varepsilon^2 \left(\|\eta H_k(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p + |\text{supp } \eta|_{\beta-1} A^{\alpha p}\right). \end{aligned}$$

By choosing $\varepsilon = 1/(2\sqrt{C(2\alpha)^p})$ and taking p -th order roots, we obtain

$$\begin{aligned} \left(\int_{\mathcal{O}} |\eta H_k(w)|^p y^{\beta-1} dx dy \right)^{1/p} &\leq (C\alpha)^\xi \left(\left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right)^{1/p} \left(\int_{\text{supp } \eta} |w H'_k(w)|^2 y^{\beta-1} dx dy \right)^{1/2} + |\text{supp } \eta|_{\beta-1}^{1/p} A^\alpha \right). \end{aligned}$$

Because the positive constants C and ξ are independent of k , we may take limit as k goes to ∞ , in the preceding inequality, and we obtain

$$\begin{aligned} \left(\int_{\mathcal{O}} |\eta w^\alpha|^p y^{\beta-1} dx dy \right)^{1/p} &\leq (C\alpha)^{\xi+1} \left(\left(1 + \|\sqrt{y}\nabla\eta\|_{L^\infty(\mathbb{H})}^2\right)^{1/p} \left(\int_{\text{supp } \eta} |w|^{2\alpha} y^{\beta-1} dx dy \right)^{1/2} \right. \\ &\quad \left. + |\text{supp } \eta|_{\beta-1}^{1/p} A^\alpha \right). \end{aligned}$$

We also have

$$A^\alpha \leq \left(\frac{1}{|\text{supp } \eta|_{\beta-1}} \int_{\text{supp } \eta} w^{2\alpha} y^{\beta-1} dx dy \right)^{1/2}.$$

Combining the last two inequalities gives (4.5). This completes the proof of Claim 4.2. \square

This completes [Step 1](#).

Step 2 (Moser iteration). The purpose of this step is to apply the Moser iteration technique to w in (4.4) with a suitable choice of $\alpha \geq 1$ and of a sequence of non-negative cutoff functions, $\{\eta_N\}_{N \geq 1} \subset C_0^1(\mathbb{H})$, with support in $\mathbb{B}_{2R}(z_0)$. We choose $\{\eta_N\}_{N \in \mathbb{N}}$ as in (3.5) with $R_N := R(1 + 1/(N + 1))$. Then, (3.6) and (3.7) become

$$\eta_N|_{B_{R_N}(z_0)} \equiv 1, \quad \eta_N|_{B_{R_{N-1}}(z_0)} \equiv 0, \quad |\nabla \eta_N| \leq \frac{cN^3}{R^2}, \tag{4.18}$$

where c is a positive constant independent of R and N . For each $N \geq 0$, we set $p_N := 2(p/2)^N$ and $\alpha_N := (p/2)^N$. Let $A_N := \|f\|_{L^s(\text{supp } \eta_N, y^{\beta-1})}$ and $w_N := u^+ + A_N$ or $w_N := u^- + A_N$. Define

$$I(N) := \left(\int_{B_{R_N}(z_0)} |w_N|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

Applying the energy estimate (4.5) with $w = w_N$, $\alpha = \alpha_{N-1}$, and $\eta = \eta_N$, we obtain for all $N \geq 1$ that

$$I(N) \leq C_0(R, N)I(N - 1), \tag{4.19}$$

where we denote

$$C_0(R, N) := (C|\alpha_{N-1}|)^{2(\xi+1)/p_{N-1}} \left(\|\sqrt{y}\nabla \eta_N\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta_N|_{\beta-1}^{1/p-1/2} \right)^{2/p_{N-1}}, \tag{4.20}$$

and $C = C(\Lambda, n, \nu_0, \bar{R})$. By applying (4.1) and (2.7), there is a constant $c > 0$ such that

$$c^{-1}R^{4/(p-2)} \leq |B_{2R}(z_0)|_{\beta-1} \leq cR^{4/(p-2)}, \quad \forall R \in (0, \bar{R}], \tag{4.21}$$

where we used the fact that $2(n + \beta - 1) = 4/(p - 2)$ by (2.1); the positive constant c depends only on n and β in the case of [Theorem 1.5](#), and on n , β and K , in the case of [Theorem 1.6](#). Moreover, by (2.6) we know that $0 \leq y \leq 2R^2$ on $B_R(z_0)$, for all $R \geq 0$. Consequently, we have

$$\|\sqrt{y}\nabla \eta_N\|_{L^\infty(\mathbb{H})}^{2/p} + |\text{supp } \eta_N|_{\beta-1}^{1/p-1/2} \leq cN^{6/p}R^{-2/p},$$

and so, using (4.21), we obtain

$$\prod_{N \geq 1} C_0(R, N) \leq C_1|B_{2R}(z_0)|_{\beta-1}^{-1/2},$$

where $C_1 = C_1(\Lambda, n, \nu_0, \bar{R}, s)$. In the case of [Theorem 1.6](#), the constant C_1 depends in addition on K . By iterating (4.19), we obtain, after using [1, [Theorem 2.8](#)],

$$\text{ess sup}_{B_R(z_0)} w = I(+\infty) \leq C_1 \left(\frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^2 y^{\beta-1} dx dy \right)^{1/2}. \tag{4.22}$$

Applying (4.22) to w as in (4.4) yields

$$\text{ess sup}_{B_R(z_0)} u^+(u^-) \leq C \left(|B_{2R}(z_0)|_{\beta-1}^{-1/2} \|u^+(u^-)\|_{L^2(B_{2R}(z_0), y^{\beta-1})} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right), \tag{4.23}$$

for all $0 < R < \bar{R}/2$, where $C = C(\Lambda, n, \nu_0, \bar{R}, s)$. In the case of [Theorem 1.6](#), the constant C_1 depends in addition on K . This completes [Step 2](#).

Step 3 (Completion of the proof of [Theorem 1.5](#)). Recall that we have chosen \bar{R} so that $R_0 > 2\bar{R}^2$ (we see by (2.6) that this implies $B_{\bar{R}}(z_0) \subset E_{R_0}(z_0)$). For any $R > 0$, we have by (2.5) that $E_R(z_0) \subset B_{\sqrt{R}}(z_0)$. Therefore, using (2.5), (2.6) and (4.23) we obtain, for all $R > 0$ obeying $2\sqrt{R} < \bar{R}$ or, equivalently, $R < R_0/8$,

$$\operatorname{ess\,sup}_{E_R(z_0)} u^+(u^-) \leq C \left(\|u^+(u^-)\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} + \|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} \right),$$

where $C = C(\Lambda, n, \nu_0, R_0, s)$. We obtain the desired inequality (1.23) by choosing $R_1 < R_0/8$ and setting $R = R_1$ in the preceding last inequality. This completes Step 3 and the proof of Theorem 1.5.

Step 4 (Completion of the proof of Theorem 1.6). The proof of Theorem 1.6 follows exactly in the same way as the proof of Theorem 1.5, with the only observation that all constants now also depend on the cone, K . (The dependence on K is due to the choice of \bar{R} via Lemma 4.1 at the start of the proof.) This completes Step 4 and the proof of Theorem 1.6.

This concludes the proofs of Theorems 1.5 and 1.6. \square

We now complete the

Proof of Corollary 1.8. Theorem 1.6 can be extended to the case of non-zero Dirichlet boundary condition given by a function $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap L^\infty_{\text{loc}}(\bar{\Gamma}_1)$, in the sense that

$$u - g \in H^1_0(\mathcal{O} \cup \Gamma_0, \mathfrak{w}),$$

with the aide of the following modifications to the proof of Theorem 1.6. Let

$$M := \operatorname{ess\,sup}_{\Gamma_1 \cap B_{2R}(z_0)} g \quad \text{and} \quad m := \operatorname{ess\,inf}_{\Gamma_1 \cap B_{2R}(z_0)} g,$$

and replace the definitions of the functions u^+ and u^- (the positive and negative part of the variational subsolution and supersolution, respectively) by

$$u^M(z) := (u(z) \vee M)^+ \quad \text{and} \quad u^m(z) := (u(z) \wedge m)^- \quad \text{for a.e. } z \in B_{2R}(z_0).$$

We also need to redefine the function H_k in (4.6) by

$$H_k(t) := \begin{cases} 0, & t < A + |M|, \\ t^\alpha - (A + |M|)^\alpha, & A + |M| \leq t \leq k, \\ \alpha k^{\alpha-1}(t - k) + H_k(k), & t > k, \end{cases}$$

when we apply Step 1 in the proof of Theorem 1.6 to the function $w = u^M + A$ (when u is assumed to be a subsolution), and by

$$H_k(t) := \begin{cases} 0, & t < A + |m|, \\ t^\alpha - (A + |m|)^\alpha, & A + |m| \leq t \leq k, \\ \alpha k^{\alpha-1}(t - k) + H_k(k), & t > k, \end{cases}$$

when we apply the same step to $w = u^m + A$ (when u is assumed to be a supersolution). Then, the argument used in the proof of Theorem 1.6 to obtain (4.22) now yields

$$\operatorname{ess\,sup}_{B_R(z_0)} u^M \leq C_1 \left[\left(\frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} |u^M|^2 y^{\beta-1} dx dy \right)^{1/2} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right],$$

$$\operatorname{ess\,sup}_{B_R(z_0)} u^m \leq C_1 \left[\left(\frac{1}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} |u^m|^2 y^{\beta-1} dx dy \right)^{1/2} + \|f\|_{L^s(B_{2R}(z_0), y^{\beta-1})} \right],$$

when u is assumed a subsolution and supersolution, respectively. The preceding estimates imply (1.25) and the statement in Remark 1.9, just as estimate (4.23) implies (1.23) in Step 4 of the proof of Theorem 1.6. \square

5. Hölder continuity for solutions to the variational equation

In this section, we prove [Theorems 1.11 and 1.13](#) and [Corollaries 1.16 and 1.17](#), that is, local Hölder continuity on a neighborhood of $\bar{\Gamma}_0$ for solutions u to the variational equation (1.19). We consider separately the case of the interior boundary points $z_0 \in \Gamma_0$ and of the ‘corner points’ $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. (While $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$ is a set of geometric corner points for the open subset, \mathcal{O} , the lesson of [11] is that the solution, u , along Γ_0 behaves, in many respects, just as it does in the interior of \mathcal{O} .) The proof of the second case, for corner points, is easier than the proof of the first case as it does not require an application of the John–Nirenberg inequality. The essential difference between the proofs of [Theorems 1.11 and 1.13](#) and the proof of its classical analogue for variational solutions to non-degenerate elliptic equations [29, [Theorems 8.27 and 8.29](#)] consists in a modification of the methods of [29, §8.6, §8.9, and §8.10] when deriving our energy estimates (5.15), where we adapt the application of the John–Nirenberg inequality and Poincaré inequality to our framework of weighted Sobolev spaces. Moreover, because the balls defined by the Koch metric, d , do not have good scaling properties unless they are centered at a point $z_0 \in \partial\mathbb{H}$ (see [Remark 2.10](#)), the Moser iteration technique applies only to such balls. Therefore, the estimate (5.5) holds only for points $z_0 \in \partial\mathbb{H}$, and in order to obtain the full Hölder continuity of solutions (1.31), we need to apply a rescaling argument which is outlined in the last steps of the arguments below. Therefore, boundary Hölder continuity does not follow in the same way as in [29]. We also prove [Theorem 1.18](#).

We now proceed to the proofs of [Theorems 1.11 and 1.13](#), first in §5.1 for the case of points $z_0 \in \Gamma_0$ and then in §5.2 for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. The proofs of [Corollaries 1.16 and 1.17](#) can be found in §5.2.

5.1. Local Hölder continuity in the interior the degenerate boundary

In this subsection, we prove [Theorem 1.11](#). Let $z_0 \in \Gamma_0$ and $R_0 > 0$ be as in the hypotheses of [Theorem 1.11](#), and let \bar{R} be small enough such that

$$B_{\bar{R}}(z_0) \subset E_{R_0}(z_0), \tag{5.1}$$

and for all $z_i = (x_i, y_i) \in B_{\bar{R}}(z_0)$, $i = 1, 2$, we have

$$0 < y_1 < 1, \quad 0 < y_2 < 1, \quad 0 \leq |z_1 - z_2| < 1, \quad \text{and} \quad 0 \leq d(z_1, z_2) < 1. \tag{5.2}$$

For $z_0 \in \bar{\mathcal{O}}$ and $0 < R < \bar{R}$, we denote

$$M_R := \text{ess sup}_{B_R(z_0)} u, \tag{5.3}$$

$$m_R := \text{ess inf}_{B_R(z_0)} u, \tag{5.4}$$

and we let

$$\text{osc}_{B_R(z_0)} u := M_R - m_R$$

denote the oscillation of u over the ball $B_R(z_0)$. From [Theorem 1.5](#), we know that M_R and m_R are finite quantities and $\text{osc}_{B_R(z_0)} u$ is well-defined. Before proceeding to the proof of [Theorem 1.11](#), we first establish the

Theorem 5.1 (*Oscillation estimate*). *There is a positive constant, C , depending at most on Λ , ν_0 , R_0 , n , s , and a constant $\alpha_0 \in (0, 1)$, depending at most on s , n and β , such that the following holds. For all R such that $0 < 4R \leq \bar{R}$, we have*

$$\text{osc}_{B_R(z_0)} u \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) R^{\alpha_0}. \tag{5.5}$$

Proof. We choose

$$q \in (n + \beta, s), \tag{5.6}$$

$$\omega \in (0, 2(n + \beta - 1)/q), \tag{5.7}$$

and define $k(R) > 0$ by

$$k \equiv k(R) := \|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})} + (|m_{\bar{R}}| + |M_{\bar{R}}|) R^\omega. \tag{5.8}$$

The remaining steps in the proof will apply to either of the following choices of functions w defined on $B_{4R}(z_0)$,

$$w = u - m_{4R} + k(R) \quad \text{or} \quad w = M_{4R} - u + k(R), \tag{5.9}$$

but, for concreteness, we choose

$$w = u - m_{4R} + k(R). \tag{5.10}$$

If $m_{\bar{R}} = M_{\bar{R}} = 0$ or $m_{4R} = M_{4R} = 0$, then we automatically have $u = 0$ on $B_{4R}(z_0)$ and (5.5) holds on $B_{4R}(z_0)$. Therefore, without loss of generality, we may assume

$$m_{4R} \neq 0 \quad \text{or} \quad M_{4R} \neq 0, \tag{5.11}$$

and $m_{\bar{R}} \neq 0$ or $M_{\bar{R}} \neq 0$. The last assumption implies that

$$k(R) \neq 0, \tag{5.12}$$

by (5.8). Therefore, we notice that both choices of w in (5.10) are bounded, positive functions.

Step 1 (Energy estimate for w). Let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function with $\text{supp } \eta \subseteq \bar{B}_{4R}(z_0)$. For any $\alpha \in \mathbb{R}$ with $\alpha \neq -1$, let

$$v := \eta^2 w^\alpha. \tag{5.13}$$

Then, v is a valid test function in $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ by [21, Lemma A.2]. Let

$$H(w) := w^{(\alpha+1)/2}, \tag{5.14}$$

and notice that Theorem 1.5 implies that $H(w)$ is a positive, bounded function, so the following operations are justified. The goal in this step is to prove

Claim 5.2 (Energy estimate). *There are positive constants, $C = C(\Lambda, \nu_0, n, \bar{R})$ and $\xi = \xi(n, \beta, q)$, such that*

$$\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} \leq C_0(R, \alpha) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}, \tag{5.15}$$

where the constant $C_0(R, \alpha)$ is defined by

$$C_0(R, \alpha) := (C|1 + \alpha|)^{(\xi+1)/p} \left(1 + \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^2\right)^{1/p}, \tag{5.16}$$

and the constant ξ is given by

$$\xi \equiv \xi(p, q) := \frac{p(q^* - 1)}{p - 2q^*}, \tag{5.17}$$

where q^* is the conjugate exponent for q in (5.6), that is, $1/q + 1/q^* = 1$.

The estimate (5.15) will be used in Moser iteration.

Proof of Claim 5.2. Notice that estimate (5.15) is similar to (4.5). The proofs of the two estimates are also very similar and we only outline the differences.

Substituting the choice (5.13) of v in (1.13), using $\nabla v = \alpha \eta^2 w^{\alpha-1} \nabla w + 2\eta \nabla \eta w^\alpha$ together with $\nabla H(w) = \frac{\alpha+1}{2} w^{(\alpha-1)/2} \nabla w$ (see (5.14)) and $w \geq k$ (by (5.10)), gives

$$\int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y^\beta dx dy \leq C|1 + \alpha| \left[\int_{\mathbb{H}} \left(\eta^2 + y|\nabla \eta|^2\right) w^{\alpha+1} y^{\beta-1} dx dy + \int_{\mathbb{H}} \eta^2 \frac{|f + c(k - m_{4R})|}{k} w^{\alpha+1} y^{\beta-1} dx dy \right], \tag{5.18}$$

where $C = C(\Lambda, \nu_0, \bar{R})$. By Hölder’s inequality, we have

$$\int_{\mathbb{H}} \eta^2 \frac{|f + c(k - m_{4R})|}{k} w^{\alpha+1} y^{\beta-1} dx dy \leq \left(\int_{\text{supp } \eta} \left| \frac{f + c(k - m_{4R})}{k} \right|^q y^{\beta-1} dx dy \right)^{1/q} \times \left(\int_{\mathbb{H}} |\eta w^{(\alpha+1)/2}|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*}. \tag{5.19}$$

From our definition of k in (5.8), the choice of ω in (5.7) and (2.7), we see that

$$\left(\int_{\text{supp } \eta} \left| \frac{f + c(k - m_{4R})}{k} \right|^q y^{\beta-1} dx dy \right)^{1/q} \leq 1 + c + \bar{R}^{2(n+\beta-1)/q-\omega}$$

and so, because ω was chosen such that $\omega < 2(n + \beta - 1)/q$ in (5.7), there is a positive constant, $C = C(\Lambda, \bar{R})$, such that

$$\left(\int_{\text{supp } \eta} \left| \frac{f + c(k - m_{4R})}{k} \right|^q y^{\beta-1} dx dy \right)^{1/q} \leq C. \tag{5.20}$$

From inequalities (5.18), (5.19) and (5.20), we obtain

$$\int_{\mathbb{H}} \eta^2 |\nabla H(w)|^2 y^\beta dx dy \leq C|1 + \alpha| \left[\int_{\mathbb{H}} (\eta^2 + y|\nabla \eta|^2) w^{\alpha+1} y^{\beta-1} dx dy + \left(\int_{\mathbb{H}} |\eta w^{(\alpha+1)/2}|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*} \right], \tag{5.21}$$

where $C = C(\Lambda, \nu_0, \bar{R})$.

Now, we can follow the argument used in the proof of estimate (4.5). We first apply Lemma 2.2 to $\eta H(w)$ which we combine with (5.21) to obtain

$$\int_{\mathbb{H}} |\eta H(w)|^p y^{\beta-1} dx dy \leq C|1 + \alpha| \left(1 + \|\sqrt{y}\nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \left(\int_{\text{supp } \eta} |H(w)|^2 y^{\beta-1} dx dy \right)^{p/2} + C|1 + \alpha| \left(\int_{\mathbb{H}} \eta^2 |H(w)|^2 y^{\beta-1} dx dy \right)^{(p-2)/2} \left(\int_{\mathbb{H}} |\eta H(w)|^{2q^*} y^{\beta-1} dx dy \right)^{1/q^*}. \tag{5.22}$$

Next, using the fact that $2 < 2q^* < p$ (by (5.6)), we apply the interpolation inequality [29, Inequality (7.10)], for any $\varepsilon > 0$, to give

$$\|\eta H(w)\|_{L^{2q^*}(\mathbb{H}, y^{\beta-1})} \leq \varepsilon \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})} + \varepsilon^{-\xi} \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})},$$

where ξ is given by (5.17). Applying the preceding inequality in (5.22), we obtain

$$\|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^p \leq C|1 + \alpha| \left(1 + \varepsilon^{-2\xi} \right) \left(1 + \|\sqrt{y}\nabla \eta\|_{L^\infty(\mathbb{H})}^2 \right) \|H(w)\|_{L^2(\text{supp } \eta, y^{\beta-1})}^p + C|1 + \alpha| \varepsilon^2 \|\eta H(w)\|_{L^p(\mathbb{H}, y^{\beta-1})}^2 \|\eta H(w)\|_{L^2(\mathbb{H}, y^{\beta-1})}^{p-2}.$$

To bound the last term in the preceding inequality, we apply Young's inequality with the conjugate exponents $(p/2, p/(p-2))$. By choosing $\varepsilon = 1/(2C|1+\alpha|)^{1/2}$ and taking roots of order p , we obtain (5.15) and (5.16). This concludes the proof of Claim 5.2. \square

This concludes Step 1.

Step 2 (Moser iteration with negative power). In this step we apply the Moser iteration technique starting with a suitable $\alpha = \alpha_0 < -1$ in (5.15) to functions w as in (4.4). Let $\{\eta_N\}_{N \in \mathbb{N}}$ be the sequence of cutoff functions considered in Step 2 in the proof of Theorem 1.5. Let $\alpha_0 < -1$, $p_0 := \alpha_0 + 1$, $p_N := p_0(p/2)^N$, where p is as in (2.1), and $\alpha_N + 1 := p_N$. We notice that $p_N \rightarrow -\infty$ as N increases. Set

$$I(N) := \left(\int_{B_{R_N}(z_0)} |w|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

Applying an argument very similar to that in Step 2 of the proof of Theorem 1.5, with the aid of (5.15) instead of (4.5), we find that

$$I(N) \geq C_1(R, N)I(N-1), \quad (5.23)$$

where $C_1(R, N)$ is given by

$$C_1(R, N) = \left(C|p_{N-1}|N^6 \right)^{(\xi+1)/p_N} R^{-2/p_N}, \quad (5.24)$$

and $C = C(\Lambda, \nu_0, \bar{R})$ is a positive constant, independent of R and N . Using (4.21), we obtain

$$\prod_{N \geq 1} C_1(R, N) \geq C_2 |B_{2R}(z_0)|_{\beta-1}^{1/|p_0|},$$

where $C_2 = C_2(\Lambda, \nu_0, \bar{R}, q)$. By iterating (5.23), we obtain $I(-\infty) \geq I(0) \prod_{N \geq 1} C_0(R, N)$, which gives us

$$\operatorname{ess\,inf}_{B_R(z_0)} w = I(-\infty) \geq C_2 \left(\frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^{p_0} y^{\beta-1} dx dy \right)^{1/p_0}. \quad (5.25)$$

This concludes Step 2.

Step 3 (Application of Theorem 3.1). The purpose of this step is to show that we may apply Theorem 3.1 to w with $S_r = B_{(2+r)R}(z_0)$, $0 \leq r \leq 1$, and $\theta_0 = \theta_1 = 1$. By Proposition 3.2, we find that w satisfies the inequalities (3.2), so it remains to show that (3.3) holds for $\log w$. For A as defined in (3.3) and $S_r = \mathbb{B}_{(2+r)R}(z_0) = B_{(2+r)R}(z_0)$, we have by Hölder's inequality that

$$A \leq \sup_{0 \leq r \leq 1} \inf_{c \in \mathbb{R}} \left(\frac{1}{|B_{(2+r)R}(z_0)|_{\beta-1}} \int_{B_{(2+r)R}(z_0)} |\log w - c|^2 y^{\beta-1} dx dy \right)^{1/2},$$

and so, Corollary 2.6 gives us

$$A \leq \sup_{0 \leq r \leq 1} ((2+r)R)^2 \left(\frac{1}{|B_{(2+r)R}(z_0)|_{\beta}} \int_{B_{(2+r)R}(z_0)} |\nabla \log w|^2 y^{\beta} dx dy \right)^{1/2}. \quad (5.26)$$

Let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function such that $\eta = 1$ on $B_{(2+r)R}(z_0)$, and $\eta = 0$ outside $B_{4R}(z_0)$, and $|\nabla \eta| \leq C/R^2$. We choose $v = \eta^2/w$, where w is given by (5.9), or (5.10) for concreteness, and notice that $v \in$

$H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$, which can be shown by modifying the corresponding argument in the proof of [29, Theorem 8.18]. With this choice of v as a test function in the variational equation (1.13) satisfied by u , using the strict ellipticity of the operator $y^{-1}A$ defined by (1.12) and Hölder’s inequality, we see that there is a positive constant $C = C(\Lambda, \nu_0, \bar{R})$, such that

$$\int_{\mathcal{O}} \eta^2 |\nabla \log w|^2 y^\beta dx dy \leq C \int_{\mathcal{O}} (|\nabla \eta|^2 + \eta^2) y^\beta dx dy + C \int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy. \tag{5.27}$$

From Lemma 2.4 and the fact that $|\nabla \eta| \leq C/R^2$, we have

$$\int_{\mathcal{O}} (|\nabla \eta|^2 + \eta^2) y^\beta dx dy \leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|_\beta. \tag{5.28}$$

Using the definition (5.8) of $k(R)$ and Hölder’s inequality, we obtain

$$\int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy \leq C \left(R^{2(n+\beta-1)/q^*} + R^{2(n+\beta-1)-\omega} \right). \tag{5.29}$$

The condition $q > n + \beta$ implies

$$2(n + \beta - 1)/q^* - 2(n + \beta) > -4, \tag{5.30}$$

since $1/q + 1/q^* = 1$. Also, because ω is chosen in $(0, 2(n + \beta - 1)/q)$ in (5.7) and $q > n + \beta$ in (5.6), we see that $\omega \in (0, 2)$, and we obviously have

$$-2 - \omega > -4. \tag{5.31}$$

Using (5.30) and (5.31), and $0 < R \leq \bar{R}$, we obtain in inequality (5.29) that there is a positive constant $C = C(\Lambda, \nu_0, \bar{R})$, such that

$$\int_{\mathcal{O}} \eta^2 \frac{|f| + |u|}{w} y^{\beta-1} dx dy \leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|_\beta. \tag{5.32}$$

In the last inequality, we used Lemma 2.4. By combining equations (5.27), (5.28) and (5.32), we obtain

$$\int_{B_{(2+r)R}(z_0)} |\nabla \log w|^2 y^\beta dx dy \leq C ((2+r)R)^{-4} |B_{(2+r)R}(z_0)|_\beta.$$

Then, it immediately follows that the right hand side of (5.26) is finite, and so, (3.3) holds for $\log w$. This concludes Step 3.

Step 4 (Proof of inequality (5.5)). In the previous step we showed that Theorem 3.1 applies to w with $\theta_0 = \theta_1 = 1$. Hence, there is a positive constant $C = C(\Lambda, \nu_0, \bar{R})$, independent of R and w , such that

$$\left(\frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w| y^{\beta-1} dx dy \right) \leq C \left(\frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w|^{-1} y^{\beta-1} dx dy \right)^{-1}. \tag{5.33}$$

From (5.25) and [1, Theorem 2.8], we obtain

$$\text{ess inf}_{B_R(z_0)} w = I(-\infty) \geq C \left(\frac{1}{|B_{2R}(z_0)|_{\beta-1}} \int_{B_{2R}(z_0)} |w| y^{\beta-1} dx dy \right). \tag{5.34}$$

We now choose $w = u - m_{4R} + k$ and $w = M_{4R} - u + k$ in (5.34). By adding the following two inequalities

$$m_R - m_{4R} + k(R) \geq \frac{C}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} (u - m_{4R})y^{\beta-1} dx dy,$$

$$M_{4R} - M_R + k(R) \geq \frac{C}{|B_{2R}(z_0)|^{\beta-1}} \int_{B_{2R}(z_0)} (M_{4R} - u)y^{\beta-1} dx dy,$$

we obtain

$$(M_{4R} - m_{4R}) - (M_R - m_R) + 2k(R) \geq C (M_{4R} - m_{4R}).$$

Without loss of generality, we may assume $C < 1$ (if not, we can make C smaller on the right-hand side of the preceding inequality). Therefore, the preceding inequality can be rewritten in the form

$$\operatorname{osc}_{B_R(z_0)} u \leq C \operatorname{osc}_{B_{4R}(z_0)} u + 2k(R). \tag{5.35}$$

Because $q \in (n + \beta, s)$ by (5.6) and $f \in L^s(B_{\bar{R}}(z_0), \mathfrak{w})$ for some $s > n + \beta$, by hypothesis in Theorem 1.11 and the assumption $B_{\bar{R}}(z_0) \subset E_{R_0}(z_0)$, Hölder’s inequality yields

$$\|f\|_{L^q(B_{4R}(z_0), y^{\beta-1})} \leq C R^{2(n+\beta-1)\frac{s-q}{sq}} \|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})}.$$

Let

$$\nu := \min \left\{ \omega, 2(n + \beta - 1) \frac{s - q}{sq} \right\}.$$

Consequently, from (5.8), we see that there is a positive constant $C = C(n, \beta)$, such that

$$k(R) \leq C \left(\|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})} + |m_{\bar{R}}| + |M_{\bar{R}}| \right) R^\nu. \tag{5.36}$$

Therefore, by applying [29, Lemma 8.23] to (5.35) and using the inequality (5.36), we find that there are positive constants, $C = C(\Lambda, \nu_0, \bar{R}, n, s)$ and $\alpha_0 = \alpha_0(s, n, \beta) \in (0, 1)$, such that

$$\operatorname{osc}_{B_R(z_0)} u \leq C \left(\|f\|_{L^s(B_{\bar{R}}(z_0), y^{\beta-1})} + \|u\|_{L^\infty(B_{\bar{R}}(z_0))} \right) R^{\alpha_0}, \quad \forall R \in (0, \bar{R}/4).$$

Without loss of generality, we may assume that $\bar{R} \leq R_1$, where R_1 is the constant appearing in the conclusion of Theorem 1.5. Then the preceding estimate together with (1.23) gives us (5.5). This concludes Step 4.

This concludes the proof of Theorem 5.1. \square

We can now conclude the

Proof of Theorem 1.11. Notice that if $z \in B_{\bar{R}/16}(z_0)$, then $B_{\bar{R}/16}(z) \subset B_{\bar{R}/4}(z_0) \subset E_{R_0}(z_0)$ (by (5.1)), and so inequality (5.5) applies in the form

$$\operatorname{osc}_{B_R(z)} u \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) R^{\alpha_0}, \tag{5.37}$$

for all $z \in B_{\bar{R}/16}(z_0)$ and $0 < R \leq \bar{R}/64$. In the remainder of the proof of Theorem 1.11, we assume that R obeys

$$0 < R \leq \bar{R}/64. \tag{5.38}$$

In particular, for any points⁶ $(x_1, y_1), (x_1, 0), (x_2, 0) \in \bar{B}_R(z_0)$, the estimate (5.37) gives

⁶ Here, we are using $x_1, x_2 \in \mathbb{R}^{n-1}$ to denote marked points rather than coordinates.

$$\begin{aligned}
 &|u(x_1, y_1) - u(x_1, 0)| \\
 &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0}((x_1, y_1), (x_1, 0)), \\
 &|u(x_1, 0) - u(x_2, 0)| \\
 &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0}((x_1, 0), (x_2, 0)).
 \end{aligned}
 \tag{5.39}$$

Notice that from (1.28) we have the simple identities,

$$\begin{aligned}
 d((x_1, y_1), (x_1, 0)) &= \sqrt{y_1/2}, \\
 d((x_1, 0), (x_2, 0)) &= \sqrt{|x_1 - x_2|},
 \end{aligned}
 \tag{5.40}$$

and so, we can rewrite (5.39) in the form

$$\begin{aligned}
 |u(x_1, y_1) - u(x_1, 0)| &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) |y_1|^{\alpha_0/2}, \\
 |u(x_1, 0) - u(x_2, 0)| &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) |x_1 - x_2|^{\alpha_0/2}.
 \end{aligned}
 \tag{5.41}$$

The proof of inequality (1.31) now follows the proofs of [11, Corollary I.9.7 and Theorem I.9.8], but with certain differences which we outline for clarity.

Claim 5.3. *There are constants $C = C(\Lambda, n, \nu_0, R_0, s) > 0$, and $\alpha = \alpha(\Lambda, n, \nu_0, R_0, s) \in (0, 1)$ such that*

$$|u(z_1) - u(z_2)| \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^\alpha(z_1, z_2),
 \tag{5.42}$$

for all points $z_1, z_2 \in B_{\bar{R}/16}(z_0)$.

Proof. Let $\varepsilon \in (0, 1/8)$ be fixed and consider the following two cases.

Case 1 (*Pairs of points in $B_R(z_0)$ obeying (5.43)*). Let $z_i = (x_i, y_i) \in B_R(z_0)$, for $i = 1, 2$, be such that

$$|z_1 - z_2| \geq \varepsilon(y_1^2 + y_2^2).
 \tag{5.43}$$

We want to show that (5.42) holds, for all points $z_1, z_2 \in B_R(z_0)$ satisfying (5.43).

From (5.2), we can find a positive constant C such that

$$|x_1 - x_2| \leq Cd(z_1, z_2).
 \tag{5.44}$$

Using our current assumption (5.43), in addition to (5.2), we also have

$$d(z_1, z_2) \geq \varepsilon C y_i^2, \quad i = 1, 2,$$

and so, there exists a positive constant C , depending on ε , such that

$$y_i \leq Cd^{1/2}(z_1, z_2), \quad i = 1, 2.
 \tag{5.45}$$

Denote $z'_i = (x_i, 0)$, for $i = 1, 2$. Applying (5.44) and (5.45) in (5.41), we obtain

$$\begin{aligned}
 |u(z_i) - u(z'_i)| &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0/4}(z_1, z_2), \quad i = 1, 2, \\
 |u(z'_1) - u(z'_2)| &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0/2}(z_1, z_2),
 \end{aligned}$$

and hence, using (5.2),

$$\begin{aligned}
 |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z'_1)| + |u(z'_1) - u(z'_2)| + |u(z_2) - u(z'_2)| \\
 &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0/4}(z_1, z_2).
 \end{aligned}$$

Therefore, the estimate (5.42) holds in the special case $|z_1 - z_2| \geq \varepsilon(y_1^2 + y_2^2)$.

Now we prove (5.42) for pairs of points obeying $|z_1 - z_2| < \varepsilon(y_1^2 + y_2^2)$.

Case 2 (Pairs of points in $B_R(z_0)$ obeying (5.46)). Now we consider points $z_i = (x_i, y_i) \in B_R(z_0)$, for $i = 1, 2$, such that

$$|z_1 - z_2| < \varepsilon(y_1^2 + y_2^2). \tag{5.46}$$

By scaling and using interior Hölder estimates [29, Theorem 8.22], we show that the estimate (1.31) also holds in this case. We proceed by analogy with the proofs of [11, Theorems I.9.1–4 and Corollary I.9.7]. We may assume without loss of generality that

$$1 > y_2 \geq y_1 \quad \text{and} \quad x_2 = 0. \tag{5.47}$$

Let $a = y_2$. We consider the function v defined by rescaling,

$$u(x, y) =: v(x/a, y/a).$$

The rescaling $z \mapsto z' = z/a$ maps $\mathbb{E}_{y_2/2}(z_2)$ into $\mathbb{E}_{1/2}(z'_2)$. Recall that $\mathbb{E}_\rho(z)$ denotes the Euclidean ball centered at z of radius ρ relative to \mathbb{H} (see (1.20)). From our assumptions (5.2), (5.46) and the choice of $\varepsilon \in (0, 1/8)$, we see that

$$|z'_1 - z'_2| \leq 2\varepsilon y_2 < 1/4, \tag{5.48}$$

and so $z'_1 \in \mathbb{E}_{1/4}(z'_2)$. From [9, Theorem 5.10], we know that $u \in H^2_{\text{loc}}(B_{\tilde{R}}(z_0))$, and so by direct calculation, we conclude that $v(z')$ solves

$$\tilde{A}v(z') = af(az') \quad \text{on } \mathbb{E}_{1/2}(z'_2),$$

where we define

$$\begin{aligned} \tilde{A}v(z') := & y' \left(a^{ij}(az')v_{x_i x_j}(z') + 2a^{in}(az')v_{x_i y}(z') + a^{nn}(az')v_{yy}(z') \right) \\ & + b^i(az')v_{x_i}(z') + b^n(az')v_y(z') - c(az')v(z'). \end{aligned}$$

On the ball $\mathbb{E}_{1/2}(z'_2)$, the operator \tilde{A} is strictly elliptic with bounded coefficients. For brevity, we denote $f_a(z') := af(az')$. By [29, Theorem 8.22], there are positive constants C and $\alpha_1 \in (0, 1)$, depending only on Λ, n, ν_0 and s , such that

$$\text{osc}_{\mathbb{E}_R(z'_2)} v \leq CR^{\alpha_1} \left(\|v\|_{L^\infty(\mathbb{E}_{1/2}(z'_2))} + \|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))} \right), \quad \forall R \in (0, 1/2], \tag{5.49}$$

because s was assumed to satisfy $s > 2n$. We see that

$$\|v\|_{L^\infty(\mathbb{E}_{1/2}(z'_2))} = \|u\|_{L^\infty(\mathbb{E}_{y_2/2}(z_2))} \leq \|u\|_{L^\infty(B_{\tilde{R}}(z_0))}, \tag{5.50}$$

where we used the fact that $\mathbb{E}_{y_2/2}(z_2) \subseteq B_{\tilde{R}}(z_0)$, which in turn follows from our assumption (5.38). We also have

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s = \int_{\mathbb{E}_{1/2}(z'_2)} |af(az')|^s dx' dy' = \int_{\mathbb{E}_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dx dy,$$

that is,

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s = \int_{\mathbb{E}_{y_2/2}(z_2)} |f(z)|^s a^{s-n} dx dy. \tag{5.51}$$

Using the fact that $y_2/2 \leq y \leq 3y_2/2$ for all $z = (x, y) \in \mathbb{E}_{y_2/2}(z_2)$, assumption (5.2), and the fact that $s > n + \beta$ by hypothesis of Theorem 1.11, the estimate (5.51) yields

$$\|f_a\|_{L^s(\mathbb{E}_{1/2}(z'_2))}^s \leq C \int_{B_{\tilde{R}}(z_0)} |f(z)|^s y^{\beta-1} dx dy, \tag{5.52}$$

where C is a positive constant depending only on β . Applying (5.50) and (5.52) in (5.49) yields

$$\operatorname{osc}_{\mathbb{B}_R(z'_2)} v \leq C \left(\|u\|_{L^\infty(B_{\bar{R}}(z_0))} + \|f\|_{L^s(B_{\bar{R}}(z_0))} \right) R^{\alpha_1}, \quad \forall R \in (0, 1/2].$$

In particular, because $z'_1 \in \mathbb{B}_{1/2}(z'_2)$, we see that

$$|v(z'_1) - v(z'_2)| \leq C \left(\|u\|_{L^\infty(B_{\bar{R}}(z_0))} + \|f\|_{L^s(B_{\bar{R}}(z_0))} \right) |z'_1 - z'_2|^{\alpha_1},$$

where the positive constant C depends on Λ , n , ν_0 and s . By rescaling back, we obtain

$$|u(z_1) - u(z_2)| \leq C \left(\|u\|_{L^\infty(B_{\bar{R}}(z_0))} + \|f\|_{L^s(B_{\bar{R}}(z_0))} \right) \left(\frac{|z_1 - z_2|}{y_2} \right)^{\alpha_1}. \tag{5.53}$$

Using (5.2) and the fact that $\varepsilon \in (0, 1/8)$, we see that

$$\frac{|z_1 - z_2|}{y_2} \leq C d^{1/2}(z_1, z_2), \tag{5.54}$$

for some positive constant C . Consequently, (5.53) and (1.23) give us

$$|u(z_1) - u(z_2)| \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_1/2}(z_1, z_2).$$

This implies estimate (5.42) in the special case $|z_1 - z_2| < \varepsilon(y_1^2 + y_2^2)$.

This completes the proof of Claim 5.3 by choosing $\alpha := \min\{\alpha_0/4, \alpha_1/2\}$. \square

By choosing R_1 smaller than $(\bar{R}/16)^2$ and than the constant R_1 in the conclusion of Theorem 1.5, we see by (2.5) that $E_{R_1}(z_0) \subset B_{\bar{R}/16}(z_0)$, and so estimates (5.42) and (1.23) now give us (1.31). This completes the proof of Theorem 1.11. \square

5.2. Hölder continuity on neighborhoods of the corner points of the degenerate boundary

We now have the

Proof of Theorem 1.13. Suppose $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. We let \bar{R} be as in the proof of Theorem 1.11, but in addition we require that \bar{R} be small enough so that the conclusion of Lemma 4.1 holds with the cone, K , given in the hypotheses of Theorem 1.13. From the standard theory of non-degenerate elliptic partial differential equations (for example, [29, Theorem 8.30]), we know that

$$u \in C(\bar{B}_{\bar{R}}(z_0) \cap \mathbb{H}) \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial B_{\bar{R}}(z_0) \cap \Gamma_1. \tag{5.55}$$

Recalling that $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ denote the positive and negative parts of u , respectively, we have that $u^\pm \in C(\bar{B}_{\bar{R}}(z_0) \cap \mathbb{H})$ and $u^\pm = 0$ along the portion of the boundary $\partial B_{\bar{R}}(z_0) \cap \Gamma_1$.

Our goal is first to prove that there are constants C , depending only on Λ , ν_0 , K , n , s , \bar{R} , and α_0 , depending only on n , s and β , such that

$$\operatorname{osc}_{B_R(z_0)} u^\pm \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) R^{\alpha_0}, \quad \forall R \in (0, \bar{R}/4], \tag{5.56}$$

which obviously implies that (5.5) holds for u , for possibly a different constant C with the same dependency as above.

Our proof uses the same method as in the case of points in Γ_0 but a choice of w which is different from that of (4.4), and a choice of test function v which is different from that of (5.13). Moreover, we do not need to appeal to the John–Nirenberg inequality. Since $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, however, it is important to make a distinction between $B_R(z_0)$ and $\mathbb{B}_R(z_0)$.

We denote

$$M_R^\pm := \operatorname{ess\,sup}_{B_R(z_0)} u^\pm. \tag{5.57}$$

Let $k \equiv k(R)$ be defined as in (5.8). Therefore, we now define w^\pm on $\mathbb{B}_{4R}(z_0)$ by

$$w^\pm(z) := k + \begin{cases} -u^\pm(z) + M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ +M_{4R}^\pm, & z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0). \end{cases} \tag{5.58}$$

As in the case of points in Γ_0 , we may assume without loss of generality that (5.11) and (5.12) hold. From (5.55), we notice that $M_{4R} \geq 0$ and $m_{4R} \leq 0$, and so it follows that $M_{4R} = M_{4R}^+$ and $m_{4R} = -M_{4R}^-$. Therefore, assumption (5.11) becomes

$$M_{4R}^+ \neq 0 \quad \text{or} \quad M_{4R}^- \neq 0.$$

If $M_{4R}^- = 0$, then $u = u^+$ on $B_{4R}(z_0)$, and it suffices to continue the following argument only for u^+ . The same remark applies to $M_{4R}^+ = 0$. Thus, we may assume without loss of generality that

$$M_{4R}^+ \neq 0 \quad \text{and} \quad M_{4R}^- \neq 0. \tag{5.59}$$

Let $\alpha < -1$, and let η be a smooth cutoff function such that $\text{supp } \eta \subseteq \mathbb{B}_{4R}(z_0)$. We now define

$$v^\pm := \eta^2 \left((w^\pm)^\alpha - (k + M_{4R}^\pm)^\alpha \right). \tag{5.60}$$

We notice that v^\pm is a well-defined function, for any choice of $\alpha \in \mathbb{R}$, by (5.59) and (5.12), and $v^\pm \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{m})$ is a valid test function in (1.13) by [21, Lemma A.3]. We observe that the function w^\pm obeys

$$k \leq w^\pm \leq k + M_{4R}^\pm \quad \text{on } \mathbb{B}_{4R}(z_0),$$

and, because α is non-positive, we also have

$$k^\alpha \geq (w^\pm)^\alpha \geq (k + M_{4R}^\pm)^\alpha \quad \text{on } \mathbb{B}_{4R}(z_0).$$

These inequalities are important in deriving the analogues of the energy estimates in the proof of Theorem 1.11 for points in Γ_0 . Steps 1 and 2 in the proof of Theorem 1.11 for points in Γ_0 apply to our current choice of w^\pm for points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, with the only exception that we now define $I(N)$ by

$$I(N) := \left(\int_{\mathbb{B}_{R_N}(z_0)} |w^\pm|^{p_N} y^{\beta-1} dx dy \right)^{1/p_N}.$$

Therefore, using the fact that

$$|\mathbb{B}_R(z_0) \setminus B_R(z_0)|_{\beta-1} \neq 0, \tag{5.61}$$

we obtain the analogue of (5.25),

$$\text{ess inf}_{\mathbb{B}_R} w^\pm \geq C \left(\frac{1}{|\mathbb{B}_{2R}(z_0)|_{\beta-1}} \int_{\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0)} |w^\pm|^{p_0} y^{\beta-1} dx dy \right)^{1/p_0}, \tag{5.62}$$

where p_0 is a negative power and $C = C(K, \Lambda, \nu_0, n, s)$. Condition (5.61) is implied by (4.2), which follows from the exterior cone condition on $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, by (4.2). Notice that (5.58) implies

$$w^\pm = k + M_{4R}^\pm \geq M_{4R}^\pm \quad \text{on } \mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0), \tag{5.63}$$

$$\text{ess inf}_{\mathbb{B}_R(z_0)} w^\pm = k - M_R^\pm + M_{4R}^\pm. \tag{5.64}$$

Using (5.64) on the left-hand-side of (5.62) and (5.63) on the right-hand-side of (5.62), we obtain

$$k(R) - M_R^\pm + M_{4R}^\pm \geq C M_{4R}^\pm. \tag{5.65}$$

Indeed, (5.65) follows because $p_0 < 0$ and

$$\frac{|\mathbb{B}_{2R}(z_0)|_{\beta-1}}{|\mathbb{B}_{2R}(z_0) \setminus B_{2R}(z_0)|_{\beta-1}} \geq 1.$$

We rewrite (5.65), using $\text{osc}_{B_R(z_0)} u^\pm = M_R^\pm$, as

$$\text{osc}_{B_R(z_0)} u^\pm \leq C \text{osc}_{B_{4R}(z_0)} u^\pm + k(R),$$

where $C \in (0, 1)$ is a constant independent of R . Just as in the proof of Theorem 1.11 for the case of points in Γ_0 , we can apply [29, Lemma 8.23] to conclude that (5.56) holds for u^\pm with positive constants $C = C(K, \Lambda, \nu_0, n, s, \bar{R})$, and $\alpha_0 = \alpha_0(s, n, \beta) \in (0, 1)$, which implies that (5.5) holds for u with $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, for possibly a different constant C with the same dependencies as before.

To establish (1.31), we proceed as in the proof of Theorem 1.11 for the case of points in Γ_0 . In order to adapt the argument for the case of points in Γ_0 to points in $\bar{\Gamma}_0 \cap \bar{\Gamma}_1$, we need analogues of the inequalities (5.39) to hold in a neighborhood in \mathcal{O} of $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. Given these analogues of the inequalities (5.39), we can apply the same argument as used in the proof of Theorem 1.11 for the case of points in Γ_0 , but instead of applying [29, Theorem 8.22], we now apply [29, Theorem 8.27]. As before, we assume (5.38) holds.

Without loss of generality, we may assume $z_0 = (0, 0)$. Let $z_1 = (x_1, 0)$, $z_2 = (x_2, 0)$, $z_3 = (x, y)$ and $z_4 = (x, 0)$ be points in $\bar{B}_R(z_0)$. We claim that the following analogues of the inequalities (5.39) (for points $z_0 \in \Gamma_0$) hold for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$,

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq C_3 \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_3}(z_1, z_2), \\ |u(z_3) - u(z_4)| &\leq C_3 \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_3}(z_3, z_4), \end{aligned} \tag{5.66}$$

for some positive constant C_3 and $\alpha_3 \in (0, 1)$ satisfying the same dependency conditions as in the statement of Theorem 1.13. For the first inequality in (5.66), we consider two cases.

Case 1 (Points $z_1, z_2 \in \bar{B}_R(z_0)$ obeying (5.67)). If

$$d(z_1, z_2) \geq \frac{1}{8} \max \{d(z_1, z_0), d(z_2, z_0)\}, \tag{5.67}$$

then we have

$$\begin{aligned} |u(z_1) - u(z_2)| &\leq |u(z_1) - u(z_0)| + |u(z_2) - u(z_0)| \\ &\leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0}(z_1, z_2) \quad (\text{by (5.5) and (5.67)}), \end{aligned}$$

and so the first inequality in (5.66) holds in this case.

Case 2 (Points $z_1, z_2 \in \bar{B}_R(z_0)$ obeying (5.68)). If

$$d(z_1, z_2) \leq \frac{1}{8} \max \{d(z_1, z_0), d(z_2, z_0)\}, \tag{5.68}$$

then, we apply (5.42) on the ball $B_{\tilde{R}}(z_2)$ with $\tilde{R} = d(z_1, z_2)$.

Combining the preceding two cases, we obtain the first inequality in (5.66).

Next, we consider the second inequality in (5.66). By (5.40), we have

$$d(z_3, z_4) = \sqrt{y/2} \quad \text{and} \quad d(z_4, z_0) = \sqrt{|x|}. \tag{5.69}$$

As in the proof of the first inequality in (5.66), we consider two possible cases.

Case 1 (Points $z_3, z_4 \in \bar{B}_R(z_0)$ obeying (5.70)). If

$$|x| \geq 32y, \tag{5.70}$$

then, by (5.69), we have $d(z_3, z_4) \leq (1/8)d(z_4, z_0)$. We may apply (5.42) on the ball $B_{\tilde{R}}(z_4)$ with $\tilde{R} = d(z_3, z_4)$, and we obtain the second inequality in (5.66).

Case 2 (Points $z_3, z_4 \in \bar{B}_R(z_0)$ obeying (5.71)). If

$$|x| < 32y, \tag{5.71}$$

then we have $d(z_4, z_0) \leq 8d(z_3, z_4)$. Also, a direct calculation gives us $d(z_3, z_0) \leq Cd(z_3, z_4)$, for some positive constant C . By (5.5), we obtain

$$\begin{aligned} |u(z_3) - u(z_4)| &\leq |u(z_3) - u(z_0)| + |u(z_4) - u(z_0)| \\ &\leq 2C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} \right) d^{\alpha_0}(z_3, z_4), \end{aligned}$$

and we obtain the second inequality in (5.66).

The proof of (5.66) is complete. We may now conclude that (1.31) holds at points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, by applying the same argument as in the proof of Theorem 1.11. \square

Proof of Corollary 1.16. Theorem 1.13 can now be extended to the case when we assume that the Dirichlet boundary condition along Γ_1 is defined by a function $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C_{s, \text{loc}}^\gamma(\bar{\Gamma}_1)$ with $\gamma \in (0, 1]$ or a function $g \in H^1(\mathcal{O}, \mathfrak{w}) \cap C_{\text{loc}}(\bar{\Gamma}_1)$, so that $u - g \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$. Corollary 1.8 and Remark 1.9 shows that the solutions are essentially bounded in neighborhoods of points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$. In the proof of Theorem 1.13 for points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, we need to make the following modifications. Let

$$M := \text{ess sup}_{\Gamma_1 \cap B_{4R}(z_0)} g \quad \text{and} \quad m := \text{ess inf}_{\Gamma_1 \cap B_{4R}(z_0)} g.$$

As in the proof of [29, Theorem 8.27], we replace our definitions of the functions u^\pm (the positive and negative part of the variational solution, respectively), w^\pm in (5.58) and v^\pm in (5.60) by

$$u_M(z) := u(z) \vee M \quad \text{and} \quad u_m(z) := u(z) \wedge m \quad \text{for a.e. } z \in B_{4R},$$

and

$$\begin{aligned} w_M(z) &:= k + \begin{cases} -u_M(z) + M_{4R}, & \text{for a.e. } z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ -M + M_{4R}, & \text{for a.e. } z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0), \end{cases} \\ w_m(z) &:= k + \begin{cases} u_m(z) - m_{4R}, & \text{for a.e. } z \in \mathbb{B}_{4R}(z_0) \cap B_{4R}(z_0), \\ m - m_{4R}, & \text{for a.e. } z \in \mathbb{B}_{4R}(z_0) \setminus B_{4R}(z_0), \end{cases} \end{aligned}$$

and

$$\begin{aligned} v_M &:= \eta^2 \left((w_M)^\alpha - (k + M_{4R} - M)^\alpha \right), \\ v_m &:= \eta^2 \left((w_m)^\alpha - (k + m - m_{4R})^\alpha \right). \end{aligned}$$

Inequality (5.62) applied to w_M and w_m now becomes

$$\begin{aligned} k + M_{4R} - M_R &\geq C(k + M_{4R} - M), \\ k + m_R - m_{4R} &\geq C(k + m - m_{4R}), \end{aligned}$$

and by adding, we obtain

$$(1 - C)(M_{4R} - m_{4R}) \geq 2(C - 1)k + (M_R - m_R) - C(M - m),$$

for a constant $C \in (0, 1)$. Therefore, instead of

$$\text{osc}_{B_R(z_0)} u \leq C \text{osc}_{B_{4R}(z_0)} u + k(R),$$

we now obtain

$$\text{osc}_{B_R(z_0)} u \leq C \text{osc}_{B_{4R}(z_0)} u + C \text{osc}_{\bar{\Gamma}_1 \cap \bar{B}_{4R}(z_0)} g + k(R). \tag{5.72}$$

Assuming that $g \in C_{s,\text{loc}}^\gamma(\bar{\Gamma}_1)$ with Hölder exponent $\gamma \in (0, 1]$, we see that

$$\text{osc}_{\bar{\Gamma}_1 \cap \bar{B}_{4R}(z_0)} g \leq C[g]_{C_s^\gamma(\bar{\Gamma}_1 \cap \bar{B}_{4R}(z_0))} R^\gamma,$$

for a positive constant $C = C(\Lambda, n, \nu_0)$. Applying [29, Lemma 8.23] and proceeding as in Step 4 in the proof of Theorem 1.11, we again obtain the following analogue of estimate (5.5),

$$\text{osc}_{B_R(z_0)} u \leq C \left(\|f\|_{L^s(E_{R_0}(z_0), y^{\beta-1})} + \|u\|_{L^2(E_{R_0}(z_0), y^{\beta-1})} + [g]_{C_s^\gamma(\bar{\Gamma}_1 \cap \bar{E}_{R_0}(z_0))} \right) R^{\alpha_0},$$

where the constants C and α_0 satisfy the same dependencies, with the exception that α_0 depends now in addition on γ . Then the argument in the proof of Theorem 1.13, following the oscillation estimate (5.5) at points $z_0 \in \bar{\Gamma}_0 \cap \bar{\Gamma}_1$, can be applied to show that u satisfies (1.32) with α depending now in addition on γ .

Set $\varphi(R) := (R\bar{R})^{1/2}$. When we assume $g \in C_{\text{loc}}(\bar{\Gamma}_1)$, [29, Lemma 8.23] applied to (5.72) with $\mu = 1/2$ gives

$$\text{osc}_{B_R(z_0)} u \leq C \left(R^\alpha \|u\|_{L^\infty(B_{4R}(z_0))} + \text{osc}_{\bar{\Gamma}_1 \cap \bar{B}_{4\varphi(R)}(z_0)} g + k(\varphi(R)) \right),$$

for some positive constants, C and $\alpha \in (0, 1)$, depending only on $K, \Lambda, n, \nu_0, R_0$ and s . Because the right-hand-side in the preceding inequality converges to 0 as R tends to 0, we see that u is continuous at z_0 . Therefore, using also Theorem 1.11 and [29, Theorem 8.27], we obtain that $u \in C(\bar{B}_{\bar{R}/4}(z_0))$. Letting $4R_1^2 = \bar{R}$, we see by (2.5) that $B_{\bar{R}/4}(z_0) \subset E_{R_1}(z_0)$, and so $u \in C(\bar{E}_{R_1}(z_0))$. \square

We now have the

Proof of Corollary 1.17. The proof of the corollary follows by a standard covering argument as in [23, Lemma 3.17] with the aid of Theorem 1.11 and Corollary 1.17 in place of [23, Theorem 3.8 and Proposition 3.13]. More details can be found in the proof of [21, Corollary 1.17]. \square

We conclude this section with the

Proof of Theorem 1.18. Suppose first that $E_{R_0}(z_0) \Subset \mathcal{O}$. Then the classical strong maximum principle [29, Theorem 8.19] implies that u is constant on \mathcal{O} , since the hypotheses [29, Equations (8.5), (8.6), and (8.8)] are obeyed on precompact open subdomains of \mathbb{H} , as one can easily see by examining the coefficients of our bilinear form (1.13) and this is sufficient for the proof of [29, Theorem 8.19].

Otherwise, by (2.6), we may assume that there is a constant $R > 0$ and a point $z'_0 \in \mathcal{O} \cup \Gamma_0$ such that $B_{4R}(z'_0) \Subset \mathcal{O} \cup \Gamma_0$ and

$$\text{ess sup}_{B_R(z'_0)} u = \text{ess sup}_{\mathcal{O}} u.$$

If $B_{4R}(z'_0) \Subset \mathcal{O}$, we can apply (2.5) to find a ball $E_{R_1}(z''_0) \Subset \mathcal{O}$ obeying the hypothesis of [29, Theorem 8.19] and the previous case applies. If $z'_0 \in \Gamma_0$, the argument in the proof of [29, Theorem 8.19] applies to show that u is constant on a ball centered at z'_0 , except that instead of using the classical weak Harnack inequality [29, Inequality (8.47)], we use estimate (5.34) applied to $w = M_{4R} - u$, where we recall that $M_{4R} = \text{ess sup}_{B_{4R}(z'_0)} u$. Notice that in the definition of $w = M_{4R} - u + k(R)$ in (5.9), we can take $k(R) = 0$ because u is a subsolution to equation (1.19) with $f = 0$. To complete the proof, we can use the argument employed in the proof of [29, Theorems 2.2 and 8.19], except that when $z'_0 \in \Gamma_0$, the role of the Euclidean ball is replaced by that of the ball defined by the cycloidal distance function. \square

6. Hölder continuity for solutions to the variational inequality

In this section, we use the penalization method and *a priori* estimates for solutions to the penalized equation derived in [9] together with Theorems 1.11 and 1.13 to prove local Hölder continuity on a neighborhood of $\bar{\Gamma}_0$ in $\bar{\mathcal{O}}$ for solutions u to the variational inequality (1.39) (Theorem 1.20).

6.1. Reduction to an open subset with finite-height

If $\text{height}(\mathcal{O}) = \infty$, we shall need to avail of the second condition in (1.43) to enable cutting off the solution and use localization to reduce to the case of an open subset with finite-height.⁷ Let $\mathcal{U} \subseteq \mathcal{O}$ be an open subset. Suppose we are given an open subset $\mathcal{V} \subset \mathcal{U}$ with $\mathcal{V} \setminus \partial\mathcal{O} \subset \mathcal{U}$ and

$$\text{dist}(\mathcal{O} \cap \partial\mathcal{V}, \mathcal{O} \cap \partial\mathcal{U}) > 0. \tag{6.1}$$

Let $\zeta \in C^\infty(\bar{\mathbb{H}})$ be a cutoff function such that $0 \leq \zeta \leq 1$ on \mathbb{H} , $\zeta = 1$ on \mathcal{V} , $\zeta > 0$ on \mathcal{U} , and $\zeta = 0$ on $\mathcal{O} \setminus \mathcal{U}$. By (6.1) and construction of ζ , there is a positive constant, C_0 , depending only on $\text{dist}(\mathcal{O} \cap \partial\mathcal{V}, \mathcal{O} \cap \partial\mathcal{U})$ such that

$$\|\zeta\|_{C^2(\bar{\mathbb{H}})} \leq C_0. \tag{6.2}$$

We obtain $\zeta\psi \in H^1(\mathcal{U}, \mathfrak{w})$ by (6.2) and the fact that $\psi \in H^1(\mathcal{O}, \mathfrak{w})$. Because $\zeta = 0$ on $\partial\mathcal{U} \setminus \partial\mathcal{O}$ and $\psi \leq 0$ on $\Gamma_1 = \partial\mathcal{O} \setminus \bar{\Gamma}_0$ (trace sense), then $\zeta\psi \leq 0$ on $\partial\mathcal{U} \setminus \bar{\Gamma}_0$ (trace sense). Similarly, as $\zeta = 0$ on $\partial\mathcal{U} \setminus \partial\mathcal{O}$ and $u = 0$ on $\partial\mathcal{O} \setminus \bar{\Gamma}_0$ (trace sense), then $\zeta u = 0$ on $\partial\mathcal{U} \setminus \bar{\Gamma}_0$ (trace sense) and therefore

$$\zeta u \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w}) \tag{6.3}$$

by [9, Lemma A.31].

Lemma 6.1 (Localization of solutions to variational inequalities). [9, Claim 6.16] If $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ is a solution to (1.39) with obstacle function, $\psi \in H^1(\mathcal{O}, \mathfrak{w})$ with $\psi^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$, and source function, $f \in L^2(\mathcal{O}, \mathfrak{w})$, then $\zeta u \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w})$ is a solution to the variational inequality (1.39) on \mathcal{U} with obstacle function, $\zeta\psi \in H^1(\mathcal{U}, \mathfrak{w})$ with $\zeta\psi^+ \in H_0^1(\mathcal{U} \cup \Gamma_0, \mathfrak{w})$, and source function,

$$f_\zeta := \zeta f + [A, \zeta]u \in L^2(\mathcal{U}, \mathfrak{w}), \tag{6.4}$$

where A is as in (1.12) and the commutator $[A, \zeta]u$ is given by

$$\begin{aligned} [A, \zeta]u = & -y \left(2a^{ij} \zeta_{x_i} u_{x_j} + 2a^{in} \zeta_y u_{x_i} + 2(a^{in} \zeta_{x_i} + a^{nn} \zeta_y) u_y \right) \\ & - y(a^{ij} \zeta_{x_i x_j} + 2a^{in} \zeta_{x_i y} + a^{nn} \zeta_{yy})u - b^i \zeta_{x_i} u - b^n \zeta_y u. \end{aligned}$$

Remark 6.2 (Reduction to the case of an open subset with finite-height). In order to reduce the case of an open subset $\mathcal{O} \subseteq \mathbb{H}$ with $\text{height}(\mathcal{O}) = \infty$ to the case of an open subset $\mathcal{O} \subseteq \mathbb{R} \times (0, \delta)$ with finite height $\delta > 0$, we can apply Lemma 6.1 to the choice

$$\zeta = \begin{cases} 1 & \text{on } \mathbb{R} \times (-\infty, \delta/2], \\ 0 & \text{on } \mathbb{R} \times [3\delta/4, \infty), \end{cases} \tag{6.5}$$

given by $\zeta(x, y) = \chi(y/\delta)$, for $(x, y) \in \mathbb{R}^2$, where $\chi \in C^\infty(\bar{\mathbb{R}})$ is a cutoff function with $0 \leq \chi \leq 1$ on \mathbb{R} , $\chi(t) = 1$ for $t \leq 1/2$, and $\chi(t) = 0$ for $t \geq 3/4$. Observe that $\text{supp}[A, \zeta]u \subset \mathbb{R} \times [\delta/2, 3\delta/4]$ in (6.4) and that, because u obeys (1.43), we obtain

$$f_\zeta \in L^2(\mathcal{O}_\delta, \mathfrak{w}) \cap L^\infty(\mathcal{O}_\delta),$$

and thus f_ζ obeys (1.40), while

$$\zeta u = u \quad \text{on } \mathcal{O}_{\delta/2}, \tag{6.6}$$

with \mathcal{O}_δ as in Hypothesis 1.19.

⁷ It is important to remember that we cannot use a cutoff function to localize solutions to the variational equation or inequality without assuming information about regularity of the solution u up to Γ_0 that is stronger than what we are trying to prove.

6.2. Proof of Hölder continuity up to $\bar{\Gamma}_0$ for solutions to the variational inequality

By Remark 6.2, we may assume without loss of generality for the remainder of this section that \mathcal{O} has finite height,

$$\mathcal{O} \subseteq \mathbb{R}^{n-1} \times (0, \delta), \tag{6.7}$$

where $\delta > 0$ is as in Hypothesis 1.19, with source function (relabelled if necessary), f , obeying (1.40) and obtain the desired Hölder continuity for u along the open subset $\mathcal{O}_{\delta/2}$ via (6.6).

We shall prove Theorem 1.20 using the method of penalization, following the pattern in [9], by first deriving an L^∞ bound on a penalization term, $\beta_\varepsilon(u_\varepsilon - \psi)$ in the semilinear penalized equation (6.10) corresponding to the variational inequality (1.39), which is uniform with respect to $\varepsilon \in (0, \varepsilon_0]$, for some sufficiently small positive constant ε_0 . We then appeal to Theorems 1.11 and 1.13 to conclude that the family of functions $\{u_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ solving the penalized equation is C^{α_0} -continuous up to $\bar{\Gamma}_0$ and hence, by passing to a subsequence and taking limits, via the convergence results in [9], that the same is true for a solution, $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$, to (1.39). Following [9, Equations (3.1) and (3.2)], we denote⁸

$$\mathfrak{a}_\lambda(u, v) := \mathfrak{a}(u, v) + \lambda((1 + y)u, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall u, v \in H^1(\mathcal{O}, \mathfrak{w}), \tag{6.8}$$

$$A_\lambda := A + \lambda(1 + y), \tag{6.9}$$

where $\lambda \geq 0$ and, as usual, $\mathfrak{a}(u, v)$ is given by (1.13) and A by (1.12).

Lemma 6.3 (Uniform bound on the penalization term). *Let $f \in L^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O})$ and $\psi \in H^2(\mathcal{O}, \mathfrak{w}) \cap L^\infty(\mathcal{O})$ obey (1.38). For $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ obeying $u \geq \psi$ a.e. on \mathcal{O} and $\lambda \geq 0$, and $\varepsilon > 0$, let $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \cap L^\infty(\mathcal{O})$ be a solution to the penalized equation,*

$$\mathfrak{a}_\lambda(u_\varepsilon, v) + (\beta_\varepsilon(u_\varepsilon - \psi), v)_{L^2(\mathcal{O}, \mathfrak{w})} = (f_\lambda, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \tag{6.10}$$

defined by the penalization function,

$$\beta_\varepsilon(t) := -\frac{1}{\varepsilon}t^-, \quad t \in \mathbb{R}, \tag{6.11}$$

where $t^- := -\min\{t, 0\}$, and⁹

$$f_\lambda := f + \lambda(1 + y)u \in L^2(\mathcal{O}, \mathfrak{w}). \tag{6.12}$$

If $\underline{c} := \text{ess inf}_{\mathcal{O}} c$ and $\lambda + \underline{c} > 0$, there is a positive constant ε_0 , depending only on n, λ, Λ and v_0 , such that

$$\|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^\infty(\mathcal{O})} \leq 2 \text{ess sup}_{\mathcal{O}} (A\psi - f)^+, \quad \forall \varepsilon \in (0, \varepsilon_0]. \tag{6.13}$$

Proof. We adapt an argument used in the proof of [49, Theorem 4.38]. Integration by parts [9, Lemma 2.23] with $\psi \in H^2(\mathcal{O}, \mathfrak{w})$ and $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ yields

$$\mathfrak{a}_\lambda(\psi, v) = (A_\lambda \psi, v)_{L^2(\mathcal{O}, \mathfrak{w})}. \tag{6.14}$$

Since $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ and $\psi^+ \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$, it follows that $\beta_\varepsilon(u_\varepsilon - \psi) \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ by the proof of [9, Lemma A.33]. In order to use $\beta_\varepsilon(u_\varepsilon - \psi)$ to construct suitable test functions, we need the forthcoming Claim 6.5 and that relies in turn on the

Claim 6.4 (Boundedness of the solution u to the variational inequality). *Assume that \mathcal{O} is bounded and $c \geq 0$ a.e. on \mathcal{O} or that \mathcal{O} is unbounded and $c \geq c_0 > 0$ a.e. on \mathcal{O} , for a positive constant c_0 , and $\tau > 0$ in (1.11). Then the solution u to the variational inequality (1.39) belongs to $L^\infty(\mathcal{O})$.*

⁸ We add a term $\lambda(1 + y)u$, rather than just λu , due to the presence of the factor $1 + y$ in our definition (1.10b) of the norm $H^1(\mathcal{O}, \mathfrak{w})$.

⁹ Not to be confused with f_ζ as defined in equation (6.4).

¹⁰ Recall that $\|c\|_{L^\infty(\mathcal{O})} \leq \Lambda$ by (1.8).

Proof of Claim 6.4. According to [9, Theorem 3.16], there exists a solution $w \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ to the inhomogeneous variational equation (1.19), namely

$$\alpha(w, v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \tag{6.15}$$

(While [9, Theorem 3.16] was proved for the Heston operator (1.14), the proof for the more general operator A in (1.12) is identical; moreover, as is clear from the proof of existence in [9, pp. 34–35], a condition such as $r > 0$ in (1.14) or more generally $c \geq c_0 > 0$ a.e. on \mathcal{O} in (1.12) is not required for existence even when \mathcal{O} is unbounded.) We rewrite the preceding variational equation as

$$\alpha(w, v - u) = (f, v - u)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$$

and subtract from the variational inequality (1.39) to give the equivalent variational inequality,

$$\alpha(u - w, v - u) \geq 0, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad v \geq \psi.$$

Set $u_0 := u - w$ and $\psi_0 := \psi - w$ and $v_0 = v - w$, and observe that $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ is a solution to the preceding variational inequality if and only if $u_0 \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ is a solution to the variational inequality,

$$\alpha(u_0, v_0 - u_0) \geq 0, \quad \forall v_0 \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}), \quad v_0 \geq \psi_0.$$

The bilinear form α given by (1.13) has the weak maximum principle property by [20, Theorem 8.7] when \mathcal{O} is a bounded domain and $c \geq 0$ a.e. on \mathcal{O} while α has the weak maximum principle property by [20, Theorem 8.14] when \mathcal{O} is a (finite height) unbounded domain and $c \geq c_0 > 0$ a.e. on \mathcal{O} and $\tau > 0$. Consequently, the weak maximum principle estimates [20, Proposition 7.9 (1) and (3)] for α imply that

$$0 \leq u_0 \leq 0 \vee \operatorname{ess\,sup}_{\mathcal{O}} \psi \quad \text{a.e. on } \mathcal{O},$$

where $x \vee y := \max\{x, y\}$, $\forall x, y \in \mathbb{R}$. In particular, $u_0 \in L^\infty(\mathcal{O})$ in either case. Moreover, the local supremum estimates provided by Theorems 1.5 and 1.6 for the solution $w \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ to the variational equation imply that $w \in L^\infty(\mathcal{O})$ and thus $u = u_0 + w \in L^\infty(\mathcal{O})$, as desired. This completes the proof of Claim 6.4. \square

Next, we have the key¹¹

Claim 6.5 (Boundedness of the penalization term). Assume that \mathcal{O} is bounded and $c \geq 0$ a.e. on \mathcal{O} or that \mathcal{O} is unbounded and $c \geq c_0 > 0$ a.e. on \mathcal{O} , for a positive constant c_0 , and $\tau > 0$ in (1.11). Then the penalization term, $\beta_\varepsilon(u_\varepsilon - \psi)$, belongs to $L^\infty(\mathcal{O})$.

Proof of Claim 6.5. Since $\beta_\varepsilon(u - \psi) \leq 0$ a.e. on \mathcal{O} , we have

$$\alpha_\lambda(u_\varepsilon, v) = (f_\lambda, v)_{L^2(\mathcal{O}, \mathfrak{w})} - (\beta_\varepsilon(u_\varepsilon - \psi), v)_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f_\lambda, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \tag{6.16}$$

for all $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ with $v \geq 0$ a.e. on \mathcal{O} . The bilinear form α_λ given by (6.8) has the weak maximum principle property by [20, Theorem 8.7], when \mathcal{O} is a bounded domain and $c \geq 0$ a.e. on \mathcal{O} , and by [20, Theorem 8.14], when \mathcal{O} is a (finite height) unbounded domain and $c \geq c_0 > 0$ a.e. on \mathcal{O} and $\tau > 0$. Hence, the *a priori* weak maximum principle estimate [20, Proposition 6.5 (4)] for α_λ implies that

$$u_\varepsilon \geq 0 \wedge \frac{1}{\lambda + \underline{c}} \operatorname{ess\,inf}_{\mathcal{O}} f_\lambda \quad \text{a.e. on } \mathcal{O},$$

where $x \wedge y := \min\{x, y\}$, for all $x, y \in \mathbb{R}$. Because $f_\lambda = f + \lambda(1 + y)u$ and $u \in L^\infty(\mathcal{O})$ by Claim 6.4 and \mathcal{O} has finite height and $f \in L^\infty(\mathcal{O})$ by hypothesis (1.40), then $f_\lambda \in L^\infty(\mathcal{O})$ and the preceding lower bound for u_ε is indeed finite. In particular,

$$(u_\varepsilon - \psi)^- \leq \left(\operatorname{ess\,sup}_{\mathcal{O}} \psi - 0 \wedge \frac{1}{\lambda + \underline{c}} \operatorname{ess\,inf}_{\mathcal{O}} f_\lambda \right)^+ \quad \text{a.e. on } \mathcal{O}.$$

¹¹ The hypothesis in Claim 6.5 that $\tau > 0$ can be removed by the alternative proof for that case in Remark 6.7.

Since $(u_\varepsilon - \psi)^- \geq 0$ and $\psi \in L^\infty(\mathcal{O})$ by hypothesis, it follows that $(u_\varepsilon - \psi)^- \in L^\infty(\mathcal{O})$ and thus $\beta_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\mathcal{O})$. This completes the proof of [Claim 6.5](#). \square

If $F(t) := t^{q-1}$, for $q > 2$, and $F'(t) = (q - 1)t^{q-2}$, for $t \in \mathbb{R}$, then the proofs of [\[29, Lemmas 7.5 and 7.6 and Theorem 7.8\]](#) (see [\[9, Lemma A.33\]](#) and its proof) and the fact that $\beta_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\mathcal{O})$ by [Claim 6.5](#) show that

$$v := |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}). \tag{6.17}$$

By subtracting [\(6.14\)](#) from [\(6.10\)](#) and choosing v as in [\(6.17\)](#), we obtain

$$\begin{aligned} & \mathfrak{a}_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) + (\beta_\varepsilon(u_\varepsilon - \psi), |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} \\ &= (f_\lambda - A_\lambda \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}. \end{aligned} \tag{6.18}$$

Since $u \geq \psi$ a.e. on \mathcal{O} by hypothesis, the term on the right-hand side of equation [\(6.18\)](#) obeys

$$(f_\lambda - A_\lambda \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}, \tag{6.19}$$

since $f_\lambda - A_\lambda \psi = f + \lambda(1 + y)(u - \psi) - A\psi \geq f - A\psi$ a.e. on \mathcal{O} by [\(6.9\)](#) and [\(6.12\)](#). Notice that

$$(\beta_\varepsilon(u_\varepsilon - \psi), |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} = - \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy, \tag{6.20}$$

and so [\(6.18\)](#), [\(6.19\)](#), and [\(6.20\)](#) yield

$$\begin{aligned} & \mathfrak{a}_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) - \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \\ & \geq (f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}. \end{aligned} \tag{6.21}$$

Observe that [\(6.17\)](#) and the fact that $|\beta_\varepsilon(u_\varepsilon - \psi)| = -\beta_\varepsilon(u_\varepsilon - \psi)$ by [\(6.11\)](#) gives

$$v_{x_i} = -(q - 1)|\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} \beta'_\varepsilon(u_\varepsilon - \psi)(u_\varepsilon - \psi)_{x_i}, \quad 1 \leq i \leq n - 1, \tag{6.22}$$

and similarly for v_y . By a straightforward calculation using the expression [\(6.8\)](#) for $\mathfrak{a}_\lambda(u, v)$ and [\(1.13\)](#) for $\mathfrak{a}(u, v)$ and the expressions [\(6.22\)](#) for v_{x_i} (and v_y), we find that

$$\begin{aligned} & \mathfrak{a}_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \\ &= - \int_{\mathcal{O}} \left(a^{ij}(u_\varepsilon - \psi)_{x_i}(u_\varepsilon - \psi)_{x_j} + 2a^{in}(u_\varepsilon - \psi)_{x_i}(u_\varepsilon - \psi)_y + a^{nn}((u_\varepsilon - \psi)_y)^2 \right) \\ & \quad \times (q - 1)|\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} \beta'_\varepsilon(u_\varepsilon - \psi) y \mathfrak{w} \, dx \, dy \\ &+ \int_{\mathcal{O}} \left(\partial_{x_j} a^{ij} + \partial_y a^{in} + \hat{b}_i - \tau a^{ij} \frac{x_j}{|x|} - \mu a^{in} \right) (u_\varepsilon - \psi)_{x_i} |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} y \mathfrak{w} \, dx \, dy \\ &+ \int_{\mathcal{O}} \left(\partial_{x_i} a^{in} + \partial_y a^{nn} + \hat{b}_n - \tau a^{in} \frac{x_i}{|x|} - \mu a^{nn} \right) (u_\varepsilon - \psi)_y |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} y \mathfrak{w} \, dx \, dy \\ &+ \int_{\mathcal{O}} (c + \lambda(1 + y))(u_\varepsilon - \psi) |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1} \mathfrak{w} \, dx \, dy. \end{aligned} \tag{6.23}$$

We write the sum of integrals on the right-hand side of [\(6.23\)](#) as $I_1 + I_2 + I_3 + I_4$. By the strict ellipticity of the operator $y^{-1}A$, we find that there exists a positive constant, $C_1 = C_1(\Lambda, v_0)$, such that

$$-I_1 \geq (q - 1)C_1 \int_{\mathcal{O}} |\nabla(u_\varepsilon - \psi)|^2 \beta'_\varepsilon(u_\varepsilon - \psi) |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy,$$

noting that $\beta'_\varepsilon(t) \geq 0$ a.e. $t \in \mathbb{R}$. Indeed, by (6.11) we have¹²

$$\beta'_\varepsilon(t) = \frac{1}{\varepsilon} 1_{\{t \leq 0\}} \leq \frac{1}{\varepsilon} \quad \text{a.e. } t \in \mathbb{R},$$

and so the identity,

$$\nabla \beta_\varepsilon(u_\varepsilon - \psi) = \beta'_\varepsilon(u_\varepsilon - \psi) \nabla(u_\varepsilon - \psi) = \frac{1}{\varepsilon} 1_{\{u_\varepsilon \leq \psi\}} \nabla(u_\varepsilon - \psi) \quad \text{a.e. on } \mathcal{O}, \tag{6.24}$$

yields

$$\begin{aligned} |\nabla(u_\varepsilon - \psi)|^2 \beta'_\varepsilon(u_\varepsilon - \psi) &= \frac{1}{\varepsilon} |\nabla(u_\varepsilon - \psi)|^2 1_{\{u_\varepsilon \leq \psi\}} \\ &= \varepsilon |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 1_{\{u_\varepsilon \leq \psi\}} \\ &= \varepsilon |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 \quad \text{a.e. on } \mathcal{O}. \end{aligned}$$

Hence, by combining the preceding inequality and identity, we see that

$$I_1 \leq -\varepsilon(q-1)C_1 \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy. \tag{6.25}$$

Using (6.24) and the fact that $\beta_\varepsilon(t) 1_{\{t \leq 0\}} = \beta_\varepsilon(t)$, we can write I_2 in the form

$$\begin{aligned} I_2 &= \varepsilon \int_{\mathcal{O}} \left(\partial_{x_j} a^{ij} + \partial_y a^{in} + \hat{b}_i - \tau a^{ij} \frac{x_j}{|x|} - \mu a^{in} \right) (\beta_\varepsilon(u_\varepsilon - \psi))_{x_i} \\ &\quad \times |\beta_\varepsilon(u_\varepsilon - \psi)|^{(q-2)/2} |\beta_\varepsilon(u_\varepsilon - \psi)|^{q/2} y \mathfrak{w} \, dx \, dy. \end{aligned}$$

Hence, there is a positive constant C_2 , depending only on Λ, ν_0 and τ , such that for any $\eta > 0$,

$$|I_2| \leq \varepsilon \eta \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy + C_2 \frac{\varepsilon}{\eta} \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy. \tag{6.26}$$

Similarly, we obtain for I_3 , for any $\eta > 0$,

$$|I_3| \leq \varepsilon \eta \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy + C_3 \frac{\varepsilon}{\eta} \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy, \tag{6.27}$$

where C_3 is a positive constant depending only on Λ and ν_0 . We can also estimate I_4 by

$$|I_4| \leq \varepsilon C_4 \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy, \tag{6.28}$$

where C_4 is a positive constant depending only on λ, Λ, ν_0 , and the height of the open subset \mathcal{O} . Substituting (6.25), (6.26), (6.27) and (6.28) in (6.23), we obtain

$$\begin{aligned} &\alpha_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \\ &\leq \varepsilon \left(\frac{C_2}{\eta} + \frac{C_3}{\eta} + C_4 \right) \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q y \mathfrak{w} \, dx \, dy \\ &\quad - \varepsilon((q-1)C_1 - 2\eta) \int_{\mathcal{O}} |\nabla \beta_\varepsilon(u_\varepsilon - \psi)|^2 |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-2} y \mathfrak{w} \, dx \, dy. \end{aligned}$$

Choose $\eta := C_1/2$ and, noting that $q > 2$, we have $(q-1)C_1 - 2\eta \geq 0$ and thus

¹² Recall that we define $t^- = 0 \vee (-t)$.

$$a_\lambda(u_\varepsilon - \psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1}) \leq \varepsilon C \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy, \tag{6.29}$$

where $C := 2C_2/C_1 + 2C_3/C_1 + C_4$. But (6.21) gives

$$\begin{aligned} & \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \\ & \leq -(f - A\psi, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})} + \varepsilon C \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy, \quad (\text{by (6.29)}) \end{aligned}$$

and thus,

$$(1 - \varepsilon C) \int_{\mathcal{O}} |\beta_\varepsilon(u_\varepsilon - \psi)|^q \mathfrak{w} \, dx \, dy \leq ((A\psi - f)^+, |\beta_\varepsilon(u_\varepsilon - \psi)|^{q-1})_{L^2(\mathcal{O}, \mathfrak{w})}.$$

Now choose $\varepsilon_0 = 1/(2C)$ and so $(1 - \varepsilon C) \geq 1/2$, for any $0 < \varepsilon \leq \varepsilon_0$. By applying the Hölder inequality on the right-hand side, we see that

$$\frac{1}{2} \|\beta_\varepsilon(u_\varepsilon - \psi)\|_{L^q(\mathcal{O}, \mathfrak{w})} \leq \|(A\psi - f)^+\|_{L^q(\mathcal{O}, \mathfrak{w})}, \quad \text{for } q > 2 \text{ and } 0 < \varepsilon \leq \varepsilon_0,$$

which yields, by taking the limit as $q \rightarrow \infty$ and applying [1, Theorem 2.8], the desired inequality (6.13). This completes the proof of Lemma 6.3. \square

Remark 6.6 (*Existence of solutions to the penalized equation*). When we specialize the variable-coefficient operator A in (1.12) to the Heston operator (1.14), then solutions to (6.10) exist by [9, Theorem 4.18] for all $\varepsilon > 0$ and $\lambda \geq \lambda_0$, where λ_0 is a positive constant depending only on Λ and ν_0 (see [9, Lemma 3.2]), chosen such that a_λ is coercive; the proof of the corresponding existence result for A in (1.12) is identical.

Remark 6.7 (*Alternative proof of Claim 6.5 when \mathcal{O} is unbounded and $c \geq c_0 > 0$*). When \mathcal{O} is unbounded but $c \geq c_0 > 0$ a.e. on \mathcal{O} , we may give an alternative proof of Claim 6.5. According to (1.46), the bilinear form \mathfrak{a} is coercive for $0 < \delta < \delta_0(c_0, \Lambda, n, \nu_0)$ and we may set $\lambda = 0$. The variational inequality (6.16) then simplifies to

$$a(u_\varepsilon, v) = (f, v)_{L^2(\mathcal{O}, \mathfrak{w})} - (\beta_\varepsilon(u_\varepsilon - \psi), v)_{L^2(\mathcal{O}, \mathfrak{w})} \geq (f, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \tag{6.30}$$

for all $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ with $v \geq 0$ a.e. on \mathcal{O} . Let $w \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ be as in the proof of Claim 6.4 and recall that $w \in L^\infty(\mathcal{O})$. Subtracting the variational equation (6.15) from (6.30) yields

$$a(u_\varepsilon - w, v) \geq 0, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \text{ with } v \geq 0 \text{ a.e. on } \mathcal{O}.$$

By choosing $v := (u_\varepsilon - w)^-$ in the preceding inequality, we obtain

$$-a((u_\varepsilon - w)^-, (u_\varepsilon - w)^-) \geq 0.$$

Coercivity of the bilinear form (1.46) implies that $C_0 \|v\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 \leq a(v, v)$. Hence, $-C_0 \|v\|_{H^1(\mathcal{O}, \mathfrak{w})}^2 \geq 0$ and consequently $v = 0$ a.e. on \mathcal{O} , and thus $u_\varepsilon \geq w$ a.e. on \mathcal{O} . Since $w, \psi \in L^\infty(\mathcal{O})$, it follows that $(u_\varepsilon - \psi)^- \in L^\infty(\mathcal{O})$ and therefore $\beta_\varepsilon(u_\varepsilon - \psi) \in L^\infty(\mathcal{O})$, as desired for this case. Note that this method does *not* require the hypothesis $\tau > 0$.

We can now proceed to the

Proof of Theorem 1.20. Fix $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ as in the hypothesis of Theorem 1.20 and, with f_λ as in (6.12) with this choice of u , set

$$f_{\lambda, \varepsilon} := f_\lambda - \beta_\varepsilon(u_\varepsilon - \psi) \in L^2(\mathcal{O}, \mathfrak{w}). \tag{6.31}$$

Since $f, \psi \in L^\infty(\mathcal{O})$ by (1.40) and (1.41) and u is a solution to the variational inequality (1.39) with $g = 0$ a.e. on \mathcal{O} , then u also solves

$$\alpha_\lambda(u, v - u) \geq (f_\lambda, v - u)_{L^2(\mathcal{O}, \mathfrak{w})} \quad \text{and} \quad u \geq \psi \text{ a.e. on } \mathcal{O},$$

$$\forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}) \text{ with } v \geq \psi \text{ a.e. on } \mathcal{O}.$$

For a Euclidean ball $E_\delta(z_0)$ with $z_0 \in \Gamma_0$, as in the statement of [Corollary 1.17](#), we observe that

$$\|f_{\lambda, \varepsilon}\|_{L^s(E_\delta(z_0))} \leq \text{vol}^{1/s}(E_\delta(z_0), \mathfrak{w}) \|f_{\lambda, \varepsilon}\|_{L^\infty(E_\delta(z_0))}, \quad \forall \varepsilon > 0, \tag{6.32}$$

where we take $s > 2n \vee (n + \beta)$. [Claim 6.4](#) implies that $u \in L^\infty(\mathcal{O})$ and the bound [\(6.13\)](#) for $\beta_\varepsilon(u_\varepsilon - \psi)$ and the definitions [\(6.12\)](#) for f_λ and [\(6.31\)](#) for $f_{\lambda, \varepsilon}$ imply that

$$\|f_{\lambda, \varepsilon}\|_{L^\infty(\mathcal{O})} \leq \|f\|_{L^\infty(\mathcal{O})} + \lambda(1 + \text{height}(\mathcal{O}))\|u\|_{L^\infty(\mathcal{O})} + 2 \text{ess sup}_{\mathcal{O}}(A\psi - f)^+, \quad \forall \varepsilon \in (0, \varepsilon_0],$$

where $\varepsilon_0 > 0$ is as in [Lemma 6.3](#). Because $f, u \in L^\infty(\mathcal{O})$ and $\text{ess sup}_{\mathcal{O}}(A\psi - f)^+ < \infty$ by [\(1.42\)](#), then [\(6.32\)](#) implies that $f_{\lambda, \varepsilon}$ in [\(6.31\)](#) obeys the hypothesis [\(1.35\)](#) of [Corollary 1.17](#) and so, by application to the solution $u_\varepsilon \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ to [\(6.10\)](#), that is

$$\alpha_\lambda(u_\varepsilon, v) = (f_{\lambda, \varepsilon}, v)_{L^2(\mathcal{O}, \mathfrak{w})}, \quad \forall v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w}),$$

we see that $u_\varepsilon \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2})$ satisfies estimate [\(1.36\)](#) with $g = 0$, where the Hölder exponent $\alpha_1 = \alpha_1(\delta, K, \Lambda, n, \nu_0, s) \in (0, 1)$ and the constant $C = C(\delta, K, \Lambda, n, \nu_0, s) > 0$ in [\(1.36\)](#) are independent of $\varepsilon \in (0, \varepsilon_0]$. By the Arzelá–Ascoli Theorem, we can find a subsequence which converges uniformly on compact subsets of $\bar{\mathcal{O}}_{\delta/2}$ to a function $u_0 \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2})$. But [\[9, Theorem 6.2\]](#) and the choice [\(6.12\)](#) of $f_\lambda = f + \lambda(1 + y)u$ imply that $u_\varepsilon \rightarrow u$ strongly in $L^2(\mathcal{O}, \mathfrak{w})$ (in fact, $H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$) as $\varepsilon \downarrow 0$ and thus, after passing to a subsequence, $u_\varepsilon \rightarrow u$ pointwise a.e. on \mathcal{O} as $\varepsilon \downarrow 0$. Therefore, by choosing a diagonal subsequence, we obtain $u = u_0$ a.e. on $\bar{\mathcal{O}}_{\delta/2}$, and the result follows. \square

Now we can give the

Proof of Corollary 1.21. We reduce the proof to the setting of [Theorem 1.20](#) by defining

$$\tilde{u} := u - g, \quad \tilde{\psi} := \psi - g, \quad \tilde{f} := f - Ag.$$

Notice that $\tilde{u}, \tilde{\psi}$ and \tilde{f} satisfy the assumptions of [Theorem 1.20](#) for u, ψ and f , respectively. Therefore, we obtain that $\tilde{u} \in C_s^{\alpha_1}(\bar{\mathcal{O}}_{\delta/2})$, for a constant $\alpha_1 = \alpha_1(\delta, K, \Lambda, \nu_0, n, s) \in (0, 1)$. When we assume $g \in C_s^\gamma(\bar{\Gamma}_1 \cap \partial \bar{\mathcal{O}}_{\delta/2})$ for $\gamma \in (0, 1]$, we see that $u \in C_s^{\alpha_2}(\bar{\mathcal{O}}_{\delta/2})$, where we choose $\alpha_2 := \alpha_1 \wedge \gamma$. When $g \in H^2(\mathcal{O}, \mathfrak{w}) \cap C(\bar{\Gamma}_1 \cap \partial \bar{\mathcal{O}}_{\delta/2})$, we see that $\alpha_2 = \alpha_1 \wedge 0$, and so $u \in C(\bar{\mathcal{O}}_{\delta/2})$. \square

7. Harnack inequality

In this section, we prove [Theorem 1.24](#), that is, the Harnack inequality for solutions $u \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ to the variational equation [\(1.19\)](#). The key differences from the proof of the classical Harnack inequality for variational solutions to non-degenerate elliptic equations [\[29, Theorem 8.20\]](#) are essentially those which we already outlined in [§5](#) and the proof follows the same pattern as that of [Theorem 1.11](#). Therefore, we only point out the major steps in the proof of [Theorem 1.24](#), as the details were explained in the preceding sections. We now proceed to the

Proof of Theorem 1.24. Let $\bar{R} := \text{dist}(\partial \mathcal{O} \cap \mathbb{H}, \partial \mathcal{O}' \cap \mathbb{H})$, and $R := \bar{R}/4$. We first show that there is a positive constant $C = C(\Lambda, \nu_0, n, \bar{R})$, such that for all $z_0 \in \Gamma_0 \cap \partial_0 \mathcal{O}'$, we have

$$\text{ess sup}_{\mathbb{B}_R(z_0)} u \leq C \text{ess inf}_{\mathbb{B}_R(z_0)} u. \tag{7.1}$$

For clarity, we split the proof into principal steps.

Step 1 (Energy estimates). Let $\eta \in C_0^1(\bar{\mathbb{H}})$ be a non-negative cutoff function with support in $\bar{\mathbb{B}}_{4R}(z_0)$. Let $\varepsilon > 0$ and

$$w = u + \varepsilon. \tag{7.2}$$

We consider $\alpha \in \mathbb{R}$, $\alpha \neq -1$. We set $H(w) = w^{(\alpha+1)/2}$ and

$$v = \eta^2 w^\alpha. \tag{7.3}$$

Then, $v \in H_0^1(\mathcal{O} \cup \Gamma_0, \mathfrak{w})$ is a valid test function in (1.13) by [21, Lemma A.4]. By applying the same arguments as in the proofs of Theorem 1.5 and Theorem 1.11, we obtain the following analogous energy estimate to (4.5) and (5.15), respectively

$$\begin{aligned} & \left(\int |\eta H(w)|^p y^{\beta-1} dx dy \right)^{1/p} \\ & \leq (C|1 + \alpha|)^{1/p} \|\sqrt{y} \nabla \eta\|_{L^\infty(\mathbb{H})}^{2/p} \left(\int_{\text{supp } \eta} |H(w)|^2 y^{\beta-1} dx dy \right)^{1/p}, \end{aligned} \tag{7.4}$$

where $C = C(\Lambda, \nu_0, n, \bar{R})$ is independent of ε .

Step 2 (Moser iteration). By applying Moser iteration as described in the proofs of Theorem 1.5, for $\alpha > 0$, and of Theorem 1.11, for $\alpha < 0$, we obtain

$$\begin{aligned} \text{ess sup}_{\mathbb{B}_R(z_0)} w & \leq C \left(\frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0)} w^2 y^{\beta-1} dx dy \right)^{1/2}, \\ \text{ess inf}_{\mathbb{B}_R(z_0)} w & \geq C^{-1} \left(\frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0)} w^{-2} y^{\beta-1} dx dy \right)^{-1/2}, \end{aligned} \tag{7.5}$$

where C satisfies the same dependencies as the constant in (7.4).

Step 3 (Application of Theorem 3.4). In this step, we verify that w satisfies the requirements of the abstract John–Nirenberg inequality (Theorem 3.1) with $\theta_0 = \theta_1 = 2$ and $S_r = \mathbb{B}_{(2+r)R}(z_0)$, for all $0 \leq r \leq 1$. From the hypotheses, we have that $0 < 4R < \text{dist}(z_0, \Gamma_1)$, and so $S_r = \mathbb{B}_{(2+r)R}(z_0)$, for all $0 \leq r \leq 1$, by (2.4) and (2.3). By Proposition 3.2, we see that w satisfies condition (3.2) of Theorem 3.1. Therefore, it remains to verify condition (3.3), which follows in precisely the same way as in the proof of Theorem 1.11.

Step 4 (Proof of the Harnack inequality (7.1) on a half-ball). Because w satisfies the conditions of Theorem 3.1 by the preceding step, there is a positive constant C , independent of ε , such that

$$\begin{aligned} & \left(\frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0)} w^2 y^{\beta-1} dx dy \right)^{1/2} \\ & \leq C \left(\frac{1}{|\mathbb{B}_{2R}(z_0)|^{\beta-1}} \int_{\mathbb{B}_{2R}(z_0)} w^{-2} y^{\beta-1} dx dy \right)^{-1/2}. \end{aligned} \tag{7.6}$$

Thus, combining inequalities (7.5) and (7.6) and recalling that $w = u + \varepsilon$, we obtain

$$\text{ess sup}_{\mathbb{B}_R(z_0)} (u + \varepsilon) \leq C \text{ess inf}_{\mathbb{B}_R(z_0)} (u + \varepsilon),$$

for all $\varepsilon > 0$. Taking the limit as $\varepsilon \downarrow 0$, we obtain the Harnack inequality (7.1) on a half-ball.

The proof of (1.47), the Harnack inequality on an open subset $\mathcal{O}' \Subset \mathcal{O} \cup \Gamma_0$, follows by a standard covering argument similar to that in the proof of [29, Corollary 8.21], with (7.1) replacing [29, Inequality (8.63)] on half-balls centered at boundary points. More details can be found in the proof of [21, Theorem 1.23]. \square

Conflict of interest statement

There is no conflict of interest.

Appendix A. Auxiliary results

In this section we give the proof of [Lemma 2.7](#). As in §2, we work under the assumption stated in [Remark 2.8](#) that $z_0 = (0, 0)$.

Proof of Lemma 2.7. By [\[9, Corollary A.14\]](#), it is enough to prove the existence of an extension operator for functions $u \in C^1(\mathbb{B}_R(z_0))$. Fix a point $z'_0 = (0, y'_0) \in \mathbb{B}_R(z_0)$, say with $y'_0 = R^2/100$. We consider two different cases depending on whether $0 < y \leq y'_0$ or $y > y'_0$.

First, we consider the points $z = (x, y) \in D \setminus \mathbb{B}_R(z_0)$ such that $0 < y \leq y'_0$. Let $z' = (x', y)$ be the intersection of $\partial\mathbb{B}_R(z_0)$ with the line through z and $(0, y)$. Then, we define $Eu(z)$ by reflection (with respect to the point z' in the hyperplane at level y):

$$Eu(z) := u\left(\frac{|x'|^2}{|x|^2}x, y\right), \quad \forall z = (x, y) \in (D \setminus \mathbb{B}_R(z_0)) \cap \{y < y'_0\}.$$

Next, we consider the case of points $z = (x, y) \in D \setminus \mathbb{B}_R(z_0)$ such that $y > y'_0$. Again let $z' = (x', y')$ be the intersection of $\partial\mathbb{B}_R(z_0)$ with the line through z and z'_0 . Then, we define $Eu(z)$ by reflection

$$Eu(z) := u\left(z'_0 + \frac{|z' - z'_0|^2}{|z - z'_0|^2}(z - z'_0)\right).$$

Therefore,

$$Eu(x, y) := u\left(\frac{|z' - z'_0|^2}{|z - z'_0|^2}x, y'_0 + \frac{|z' - z'_0|^2}{|z - z'_0|^2}(y - y'_0)\right), \quad \forall z = (x, y) \in (D \setminus \mathbb{B}_R(z_0)) \cap \{y > y'_0\},$$

and so it is clear that the function Eu is continuous on D and is equal to u on $\mathbb{B}_R(z_0)$. Because $\mathbb{H} \cap \partial\mathbb{B}_R(z_0)$ is a smooth surface, Eu has well-defined weak derivatives in D . Next, we show that [\(2.11\)](#) holds. For this purpose, we denote by

$$D_1 := (D \setminus \mathbb{B}_R(z_0)) \cap \{y < y'_0\},$$

$$D_2 := (D \setminus \mathbb{B}_R(z_0)) \cap \{y \geq y'_0\}.$$

To prove [\(2.11\)](#), it is enough to show there is a positive constant C , depending on R and D , such that

$$\begin{aligned} \int_{D_1} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \\ \int_{D_1} |\nabla Eu(x, y)|^2 y^\beta dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy, \\ \int_{D_2} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \\ \int_{D_2} |\nabla Eu(x, y)|^2 y^\beta dx dy &\leq C \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy, \end{aligned} \tag{A.1}$$

We begin by evaluating the integrals over D_1 in [\(A.1\)](#) and we denote by

$$(x'', y) = \varphi(x, y) := \left(\frac{|x'|^2}{|x|^2}x, y\right). \tag{A.2}$$

We notice that $\varphi(x, y) \in \mathbb{B}_R(z_0)$, for all $(x, y) \in D_1$, so $Eu(x, y)$ is well-defined on D_1 . The coordinate $x' = x'(y)$ is determined by the condition $d((x', y), z_0) = R$. Direct calculations give us that

$$x'(y) = \left(\left(\frac{R^2 + \sqrt{R^4 + 4R^2y}}{2} \right)^2 - y^2 \right)^{1/2} \frac{x}{|x|}.$$

We can find a positive constant C_1 , depending only on R , such that

$$|x| \geq |x''| \geq C_1, \quad \forall (x, y) \in D_1,$$

and using the fact that

$$\varphi^{-1}(x'', y) = \left(\frac{|x'|^2}{|x''|^2} x'', y \right),$$

we can find a positive constant C_2 , depending on R and D , such that

$$\begin{aligned} |\nabla\varphi(x, y)| &\leq C_2, \quad \forall (x, y) \in D_1, \\ |\nabla\varphi^{-1}(x'', y)| &\leq C_2, \quad \forall (x'', y) \in \varphi(D_1). \end{aligned} \tag{A.3}$$

Using the change of variable $(x'', y) = \varphi(x, y)$, we obtain

$$\begin{aligned} \int_{D_1} |Eu(x, y)|^2 y^{\beta-1} dx dy &\leq \int_{\varphi(D_1)} |u(x'', y)|^2 y^{\beta-1} |\det\nabla\varphi^{-1}(x'', y)| dx'' dy \\ &\leq C_2 \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 y^{\beta-1} dx dy, \quad (\text{by (A.3)}). \end{aligned} \tag{A.4}$$

Using

$$\begin{aligned} \partial_{x_i} Eu(x, y) &= \sum_{j=1}^{n-1} u_{x_j}(\varphi(x, y), y) \partial_{x_i} \varphi_j(x, y), \quad 1 \leq i \leq n-1, \\ \partial_y Eu(x, y) &= \sum_{j=1}^{n-1} u_{x_j}(\varphi(x, y), y) \partial_y \varphi_j(x, y) + u_y(\varphi(x, y), y), \end{aligned}$$

the change of variable $(x'', y) = \varphi(x, y)$ and the upper bound (A.3), we obtain for a positive constant C_3 , depending on R and D ,

$$\int_{D_1} |\nabla Eu(x, y)|^2 y^\beta dx dy \leq C \int_{\varphi(D_1)} |\nabla u(x'', y)|^2 |\nabla\varphi(x, y)|^2 |\det\nabla\varphi^{-1}(x'', y)| y^\beta dx'' dy,$$

and thus

$$\int_{D_1} |\nabla Eu(x, y)|^2 y^\beta dx dy \leq C_3 \int_{\mathbb{B}_R(z_0)} |\nabla u(x, y)|^2 y^\beta dx dy. \tag{A.5}$$

Therefore, (A.4) and (A.5) give us the first two inequalities in (A.1).

Next, we consider the last two integrals in (A.1). Notice that on D_2 we have $y \geq y'_0 > 0$ and so it is enough to show

$$\begin{aligned} \int_{D_2} |Eu(z)|^2 dz &\leq C_4 \int_{\mathbb{B}_R(z_0)} |u(z)|^2 dz, \\ \int_{D_2} |\nabla Eu(z)|^2 dz &\leq C_4 \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 dz, \end{aligned} \tag{A.6}$$

for some positive constant C_4 , depending on R and D . For all $z \in D_2$, we now denote

$$z'' = \varphi(z) := z'_0 + \frac{|z' - z'_0|^2}{|z - z'_0|^2} (z - z'_0).$$

Analogous to (A.3), we can find a positive constant C_5 , depending on R and D , such that for all $z \in D_2$,

$$\begin{aligned} |\nabla\varphi(z)| &\leq C_5, \quad \forall z \in D_2, \\ |\nabla\varphi^{-1}(z'')| &\leq C_5, \quad \forall z'' \in \varphi(D_2). \end{aligned} \tag{A.7}$$

We notice that $\varphi(z) \in \mathbb{B}_R(z_0)$, for all $z \in D_2$. Therefore, using the change of variable $z'' = \varphi(z)$, we obtain

$$\begin{aligned} \int_{D_2} |Eu(z)|^2 dz &\leq \int_{\varphi(D_2)} |u(z'')|^2 |\det\nabla\varphi^{-1}(z'')| dz'' \\ &\leq C_5 \int_{\mathbb{B}_R(z_0)} |u(x, y)|^2 dx dy \quad (\text{by (A.7)}). \end{aligned} \tag{A.8}$$

Using

$$\begin{aligned} \partial_{x_i} Eu(z) &= \sum_{j=1}^{n-1} u_{x_j}(\varphi(z)) \partial_{x_i} \varphi_j(z) + u_y(\varphi(z)) \partial_{x_i} \varphi_n(z), \quad 1 \leq i \leq n-1, \\ \partial_y Eu(z) &= \sum_{j=1}^{n-1} u_{x_j}(\varphi(z)) \partial_y \varphi_j(z) + u_y(\varphi(z)) \partial_y \varphi_n(z), \end{aligned}$$

we obtain

$$\begin{aligned} \int_{D_2} |\nabla Eu(z)|^2 dz &\leq C \int_{\varphi(D_2)} |\nabla u(z'')|^2 |\nabla\varphi(z)|^2 |\det\nabla\varphi^{-1}(z'')| dz'' \\ &\leq CC_5 \int_{\mathbb{B}_R(z_0)} |\nabla u(z)|^2 dz, \quad (\text{by (A.7)}). \end{aligned} \tag{A.9}$$

From (A.8) and (A.9), we obtain (A.6). This concludes the proof of Lemma 2.7. \square

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