Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 1375–1407

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Schauder estimates for nonlocal fully nonlinear equations ☆

Tianling Jin [a], Jingang Xiong [b,*]

[a] *Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA*
[b] *School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China*

Received 22 September 2014; received in revised form 24 April 2015; accepted 13 May 2015

Available online 1 June 2015

Abstract

In this paper, we establish pointwise Schauder estimates for solutions of nonlocal fully nonlinear elliptic equations by perturbative arguments. A key ingredient is a recursive Evans–Krylov theorem for nonlocal fully nonlinear translation invariant equations.

MSC: 35R09; 35B65

Keywords: Integro-differential equations; Schauder estimates; Recursive Evans–Krylov theorem

1. Introduction

Integro-differential equations, which are usually called nonlocal equations nowadays, appear naturally when studying discontinuous stochastic process. In a series papers of Caffarelli and Silvestre [6–8], regularities of solutions of nonlocal fully nonlinear elliptic equations such as Hölder estimates, Cordes–Nirenberg type estimates and Evans–Krylov theorem were established. In this paper, we shall prove Schauder estimates for nonlocal fully nonlinear elliptic equations of the type:

$$\inf_{a\in\mathcal{A}}\left\{\int_{\mathbb{R}^n}\delta u(x,y)K_a(x,y)\,\mathrm{d}y\right\}=f(x)\quad\text{in } B_5, \tag{1.1}$$

where $\delta u(x,y)=u(x+y)+u(x-y)-2u(x)$, $\mathcal{A}$ is an index set, and each K_a is a positive kernel. We will restrict our attention to symmetric kernels which satisfy

$$K(x,y)=K(x,-y). \tag{1.2}$$

☆ T. Jin was supported in part by NSF grant DMS-1362525. J. Xiong was supported in part by the First Class Postdoctoral Science Foundation of China (No. 2012M520002) and Beijing Municipal Commission of Education for the Supervisor of Excellent Doctoral Dissertation (20131002701).
* Corresponding author.
E-mail addresses: tj@math.uchicago.edu (T. Jin), jx@bnu.edu.cn (J. Xiong).

http://dx.doi.org/10.1016/j.anihpc.2015.05.004
0294-1449/

We also assume that the kernels are uniformly elliptic

$$\frac{(2-\sigma)\lambda}{|y|^{n+\sigma}} \le K(x,y) \le \frac{(2-\sigma)\Lambda}{|y|^{n+\sigma}} \tag{1.3}$$

for some $0<\lambda\le\Lambda<\infty$, which is an essential assumption leading to local regularizations. Finally, we suppose that the kernels are C^2 away from the origin and satisfy

$$|\nabla_y^i K(x,y)| \le \frac{\Lambda}{|y|^{n+\sigma+i}}, \quad i=1,2. \tag{1.4}$$

We say that a kernel $K\in\mathscr{L}_0(\lambda,\Lambda,\sigma)$ if K satisfies (1.2) and (1.3), and $K\in\mathscr{L}_2(\lambda,\Lambda,\sigma)$ if K satisfies (1.2), (1.3) and (1.4). In this paper, all the solutions of nonlocal equations are understood in the viscosity sense, where the definitions of such solutions can be found in [6].

One way to obtain Schauder estimates is that first we prove high regularity for solutions of translation invariant (or "constant coefficients") equations, and then use perturbative arguments or approximations. In our case, the regularities for translation invariant equations should be the Evans–Krylov theorem for nonlocal fully nonlinear equations proved in [8], which states that: If u is a bounded solution of

$$\inf_{a\in\mathcal{A}}\left\{\int_{\mathbb{R}^n}\delta u(x,y)K_a(y)\,dy\right\}=0 \quad \text{in } B_5,$$

where every $K_a(y)\in\mathscr{L}_2(\lambda,\Lambda,\sigma)$ with $\sigma\ge\sigma_0>0$. Then, $u\in C^{\sigma+\bar\alpha}(B_1)$ for some $\bar\alpha>0$. Moreover,

$$\|u\|_{C^{\sigma+\bar\alpha}(B_1)} \le N_{ek}\|u\|_{L^\infty(\mathbb{R}^n)}, \tag{1.5}$$

where both $\bar\alpha$ and N_{ek} are positive constants depending only on $n,\sigma_0,\lambda,\Lambda$. Note that $\bar\alpha$ and N_{ek} do not depend on σ, and thus, do not blow up as $\sigma\to2$. The result becomes most interesting when σ is close to 2 and $\sigma+\bar\alpha>2$. If we let $\sigma\to2$, then it recovers the theorem of Evans and Krylov about the regularity of solutions to concave uniformly elliptic PDEs of second order.

Throughout the paper, we will always denote $\bar\alpha$ as the one in (1.5) without otherwise stated.

In the step of approximations to obtain Schauder estimates at $x=0$, it usually requires that the coefficients of the equations, which in our case are $K(x,y)$ and $f(x)$, are Hölder continuous at $x=0$ in some sense. For the right-hand side $f(x)$, we assume f satisfies the standard Hölder condition that

$$|f(x)-f(0)|\le M_f|x|^\alpha \quad\text{and}\quad |f(x)|\le M_f \tag{1.6}$$

for all $x\in B_5$, where M_f is a nonnegative constant.

For the kernel K, one may impose different types of Hölder conditions. Here, we focus on the (most delicate, as explained below) case that $\sigma+\bar\alpha-2\ge\gamma_0>0$, and we will assume the kernels satisfy

$$\int_{\mathbb{R}^n}|K(x,y)-K(0,y)|\min(|y|^2,r^2)dy\le\Lambda|x|^\alpha r^{2-\sigma} \tag{1.7}$$

for all $r\in(0,1]$, $x\in B_5$.

For $s\in\mathbb{R}$, $[s]$ denotes the largest integer that is less than or equals to s. Our main result is the following pointwise Schauder estimates for solutions of (1.1). Recall that $\bar\alpha$ is the one in (1.5).

Theorem 1.1. *Assume every $K_a(x,y)\in\mathscr{L}_2(\lambda,\Lambda,\sigma)$ satisfies (1.7) with $\alpha\in(0,\bar\alpha)$, $\sigma+\bar\alpha-2\ge\gamma_0>0$ and $|\sigma+\alpha-2|\ge\varepsilon_0>0$. Suppose that f satisfies (1.6). If u is a bounded viscosity solution of (1.1), then there exists a polynomial $P(x)$ of degree $[\sigma+\alpha]$ such that for $x\in B_1$,*

$$\begin{aligned}|u(x)-P(x)|&\le C\left(\|u\|_{L^\infty(\mathbb{R}^n)}+M_f\right)|x|^{\sigma+\alpha};\\ |\nabla^jP(0)|&\le C\left(\|u\|_{L^\infty(\mathbb{R}^n)}+M_f\right),\ j=0,\cdots,[\sigma+\alpha],\end{aligned} \tag{1.8}$$

where C is a positive constant depending only on $\lambda,\Lambda,n,\bar\alpha,\alpha,\varepsilon_0$ and γ_0.

Roughly speaking, Theorem 1.1 states that if K and f are of C^α at $x = 0$ in the sense of (1.7) and (1.6), respectively, then the solution u of (1.1) is precisely of $C^{\sigma+\alpha}$ at $x = 0$. Moreover, the constant C in (1.8) does not depend on σ, and hence, does not blow up as $\sigma \to 2$.

Various Schauder estimates for solutions of some nonlocal linear equations were obtained before by R.F. Bass [3], R. Mikulevicius and H. Pragarauskas [21], H. Dong and D. Kim [14], B. Barrera, A. Figalli and E. Valdinoci [2], D. Kriventsov [18], as well as the authors [16]. The results in [2] contain bootstrap regularity and applications to nonlocal minimal surfaces. The equations considered in [3,21,14,18] are of rough kernels, i.e., without the assumption (1.4). Also in [18], D. Kriventsov proved $C^{1+\alpha}$ estimates for nonlocal fully nonlinear equations with rough kernels when the order of the equation $s > 1$ by perturbative arguments. Later, J. Serra [23] extended this result in [18] to parabolic equations and used a different method. In [17], M. Kassmann, M. Rang and R.W. Schwab proved Hölder regularity results for those nonlocal equations whose ellipticity bounds are strongly directionally dependent. Recently, X. Ros-Oton and J. Serra [22] studied boundary regularity for nonlocal fully nonlinear equations. One may see, e.g., [1,13,15] for more regularity results on nonlocal elliptic equations.

For the Hölder condition (1.7) on the kernels, one can check that it will hold if the kernels satisfy the pointwise Hölder continuous condition $|K(x, y) - K(0, y)| \le \Lambda(2-\sigma)|x|^\alpha|y|^{-n-\sigma}$. In the case of $\sigma + \bar{\alpha} < 2$, all of our arguments still work except that one needs to change the condition (1.7) to (3.16) or (3.17), since the approximation solutions will be of only $C^{\sigma+\bar{\alpha}}$; see Remark 3.3 and Corollary 3.4.

In the case of second order partial differential equations $F(\nabla^2 u, x) = 0$, to show that $u \in C^{2+\alpha}$, we usually use second order polynomials $p(x)$ to approximate u (see [4,5]), in which one implicit convenience is that $\nabla^2 p(x)$ is a constant function. In the nonlocal case, to prove $C^{\sigma+\alpha}$ estimates of solutions to (1.1) for $\sigma + \alpha > 2$, second order polynomial approximation does not seem to work directly, since first of all, for a second order polynomial $p(x)$, it grows too fast at infinity so that $\delta p(x, y)K(y)$ is not integrable; and secondly, in general $\int_{\mathbb{R}^n} \delta \tilde{p}(x, y)K(y)\mathrm{d}y$ will not be a constant function for any cut-off $\tilde{p}(x)$ of $p(x)$ so that we cannot apply Evans–Krylov theorem during the approximation and will lose control of the error. Another common difficulty in approximation arguments to obtain regularities for nonlocal equations is to control the error outside of the balls in the iteration, which may results in a slight loss of regularity as in [7] compared to second order equations. Instead of polynomials, we will approximate the genuine solution by solutions of "constant coefficients" equations, which is inspired by [4,20]. In this way, we do not need to worry about either polynomials or the errors coming from the infinity. But a new difficulty arises for fully nonlinear equations (which does not appear in the case of linear equations): the Evans–Krylov theorem in [8] cannot be applied to obtain the uniform estimates for the sequence of approximation solutions to those "constant coefficients" equations; see also Remark 3.2. This leads us to establish a recursive Evans–Krylov theorem in Theorem 2.2 to overcome this difficulty.

Our paper is organized as follows. In Section 2, we prove Theorem 2.2, a recursive Evans–Krylov theorem for nonlocal fully nonlinear equations, where we adapt the proofs in [8] with delicate decomposition and cut-offs arguments. In Section 3, we will use Theorem 2.2 and perturbative arguments to prove the Schauder estimates in Theorem 1.1. In Appendix A, we recall some definitions and notions of nonlocal operators from [7], and establish two approximation lemmas for our own purposes, which are variants of that in [7].

After we finished our paper, we learned from Joaquim Serra that he has a preprint [24] on estimates for concave nonlocal fully nonlinear elliptic equations with rough kernels, where interior Schauder estimates are obtained independently and by a very different method. Interior estimates of $\|u\|_{C^{\sigma+\alpha}(B_{1/2})}$ were obtained in [24] when the kernels are rough for $u \in C^\alpha(\mathbb{R}^n)$, and when the kernels are C^α in y for $u \in L^\infty(\mathbb{R}^n)$. The proof in [24] uses compactness arguments, combining a Liouville theorem and a blow up procedure.

2. A recursive Evans–Krylov theorem

We start with our motivation of the recursive Evans–Krylov theorem in Theorem 2.2. Recall that the idea of our proof of Theorem 1.1 is to find a sequence of suitable functions approximating u in a desirable way.

Let u be a bounded viscosity solution of (1.1). We first approximate u by w_0 which solves

$$\inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta w_0(x, y)K_a(0, y)\,\mathrm{d}y = f(0) \quad \text{in } B_4$$

$$w_0 = u \quad \text{in } \mathbb{R}^n \setminus B_4.$$

Then the error estimate $\|u-w_0\|_{L^\infty(B_4)}$ can be controlled by the approximation lemmas in Appendix A. We need to further estimate the error $u-w_0$ in the $C^{\sigma+\alpha}$ norm near 0. To do this, we scale the error:

$$W_1(x)=\rho^{-(\sigma+\alpha)}(u-w_0)(\rho x),$$

where ρ will be universally chosen, and look for a desirable approximation for W_1 in B_4. It follows from (1.1) that W_1 satisfies

$$\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta W_1(x,y)+\rho^{-(\sigma+\alpha)}\delta w_0(\rho x,\rho y)\right)K_a^{(1)}(x,y)\mathrm{d}y=\rho^{-\alpha}f(\rho x),$$

where

$$K_a^{(i)}(x,y)=\rho^{(n+\sigma)i}K_a(\rho^i x,\rho^i y),\quad i\in\mathbb{N}.$$

The correct approximation (see Remark 3.2) of W_1 will be v_1, which solves

$$\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta v_1(x,y)+\rho^{-(\sigma+\alpha)}\delta w_0(\rho x,\rho y)\right)K_a^{(1)}(0,y)\mathrm{d}y=\rho^{-\alpha}f(0)\quad\text{in } B_4$$

$$v_1=W_1\quad\text{in }\mathbb{R}^n\setminus B_4.$$

From the approximation lemmas, we know that $|v_1|\le 1$ in $\mathbb{R}^n$. If we can get a desired estimate for $\|v_1\|_{C^{\sigma+\bar\alpha}(B_1)}$, then for $w_1(x)=\rho^{\sigma+\alpha}v_1(\rho^{-1}x)$, w_0+w_1 approximates u better than w_0 does near 0.

We do one more step to illustrate the essential difficulty. Now we need to estimate the error $u-w_0-w_1$ in the $C^{\sigma+\alpha}$ norm near 0. We again further scale the error:

$$W_2(x)=\rho^{-2(\sigma+\alpha)}(u-w_0-w_1)(\rho^2 x).$$

Based on the equation of W_2, the correct approximation of W_2 will be v_2, which satisfies

$$\begin{aligned}&\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta v_2(x,y)+\rho^{-(\sigma+\alpha)}\delta v_1(\rho x,\rho y)+\rho^{-2(\sigma+\alpha)}\delta v_0(\rho^2x,\rho^2y)\right)K_a^{(2)}(0,y)\mathrm{d}y\\&\quad=\rho^{-2\alpha}f(0)\quad\text{in } B_4\end{aligned}$$

such that $v_2=W_2$ in $\mathbb{R}^n\setminus B_4$, where $v_0\equiv w_0$. We will know that $|v_2|\le 1$ in $\mathbb{R}^n$. If we can get a desired estimate for $\|v_2\|_{C^{\sigma+\bar\alpha}(B_1)}$, then for $w_2(x)=\rho^{2(\sigma+\alpha)}v_2(\rho^{-2}x)$, $w_0+w_1+w_2$ approximates u better than w_0+w_1 does near 0.

Continuing this process, we can find a sequence of function $\{v_\ell\}_{\ell=1}^\infty$, where each v_m is defined recursively through $v_0, v_1, \cdots, v_{m-1}$ by solving an equation like (2.1).

We know that $|v_\ell|\le 1$ in $\mathbb{R}^n$ for all ℓ. If one applies the estimate (1.4) in [8] to v_ℓ directly, one will get their $C^{\sigma+\bar\alpha}$ estimates depending on ℓ, which blows up as $\ell\to\infty$. What we need is to find a universal ρ and obtain the estimate $\|v_\ell\|_{C^{\sigma+\bar\alpha}}$ uniformly, i.e., independent of ℓ. This is what we call the recursive Evans–Krylov estimate.

2.1. Statements and ideas of the proof

If we re-examine the proof of the nonlocal Evans–Krylov theorem in [8], we can show the following theorem with few modification.

Theorem 2.1. *Assume that every $K_a(y)\in\mathscr{L}_2(\lambda,\Lambda,\sigma)$ with $2>\sigma\ge\sigma_0>1$ and every b_a is a constant. If w is a bounded solution of*

$$\inf_{a\in\mathcal{A}}\left\{\int_{\mathbb{R}^n}\delta w(x,y)K_a(y)\,\mathrm{d}y+b_a\right\}=0\quad\textit{in } B_5,$$

then, $w\in C^{\sigma+\bar\alpha}(B_1)$, and there holds

$$\|w\|_{C^{\sigma+\bar\alpha}(B_1)} \le N_{ek}(\|w\|_{L^\infty(\mathbb{R}^n)} + |\inf_a b_a|),$$

where both $\bar\alpha$ and N_{ek} are the same as those in (1.5).

The recursive Evans–Krylov theorem we are going to show is the following.

Theorem 2.2. *Assume that every b_a is a constant, $K_a(y) \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$ with $2 > \sigma \ge \sigma_0 > 1$. For each $m \in \mathbb{N} \cup \{0\}$, let $\{v_\ell\}_{\ell=0}^m$ be a sequence of functions satisfying*

$$\inf_{a\in\mathcal{A}} \left\{ \int_{\mathbb{R}^n} \sum_{\ell=0}^{j} \rho^{-(j-\ell)(\sigma+\alpha)} \delta v_\ell(\rho^{j-\ell}x, \rho^{j-\ell}y) K_a^{(j)}(y)\,\mathrm{d}y + \rho^{-j\alpha} b_a \right\} = 0 \quad \text{in } B_5 \tag{2.1}$$

in viscosity sense for all $0 \le j \le m$, where $K_a^{(j)}(x) = \rho^{j(n+\sigma)} K_a(\rho^j x)$, $\rho \in (0,1)$, $\alpha \in (0, \bar\alpha)$. Suppose that $\|v_\ell\|_{L^\infty(\mathbb{R}^n)} \le 1$ for all ℓ and $|\inf_{a\in\mathcal{A}} b_a| \le 1$. Then, $v_\ell \in C^{\sigma+\bar\alpha}(B_1)$, and there exist constants $C > 0$ and $\rho_0 \in (0, 1/100)$, both of which depend only on $n, \sigma_0, \lambda, \Lambda, \bar\alpha$ and α, such that if $\rho \le \rho_0$ then we have

$$\|v_\ell\|_{C^{\sigma+\bar\alpha}(B_1)} \le C \quad \forall\, \ell = 0, 1, \ldots, m. \tag{2.2}$$

The rest of this section will be devoted to proving Theorem 2.2. The regularity of v_{i+1} follows from the Evans–Krylov theorem in [8]. But if one applies the estimate (1.4) in [8] to v_ℓ directly, one will get their $C^{\sigma+\bar\alpha}$ estimates depending on ℓ and ρ. Our goal is to prove the estimate (2.2) which is independent of both ℓ and ρ.

A constant C is said to be a universal constant if C only depends on $n, \sigma_0, \lambda, \Lambda, \alpha$ and $\bar\alpha$. Throughout this section, all the constants denoted as C will be universal constants, and it may vary from lines to lines.

Let $M >> 1$ be a universal constant which will be fixed later. Replacing v_ℓ by v_ℓ/M, we may assume that

$$\|v_\ell\|_{L^\infty(\mathbb{R}^n)} \le 1/M \quad \text{and} \quad |\inf_{a\in\mathcal{A}} b_a| \le 1/M.$$

Then our goal is to show that

$$\|v_\ell\|_{C^{\sigma+\bar\alpha}(B_1)} \le 1 \quad \forall\, \ell = 0, 1, \cdots, m.$$

The proof is by induction on m. When $m = 0$, then by Theorem 2.1, (2.2) holds for $M = 2N_{ek}$. We assume that Theorem 2.2 holds up to $m = i$ for some $i \ge 0$, and we are going to show that it holds for $i + 1$ as well.

It follows from the induction hypothesis and the $i + 1$ equations for $v_0, \ldots, v_i$ that

$$\|v_\ell\|_{C^{\sigma+\bar\alpha}(B_1)} \le 1, \quad \forall\, \ell = 0, 1, \ldots, i.$$

We are going to show

$$\|v_{i+1}\|_{C^{\sigma+\bar\alpha}(B_1)} \le 1. \tag{2.3}$$

To illustrate the idea of our proof, let us first consider the second order fully nonlinear elliptic equations

$$F(D^2u) := \inf_{k\in\mathcal{K}} a_{ij}^{(k)} u_{ij} = 0 \quad \text{in } B_5, \tag{2.4}$$

where $\mathcal{K}$ is an index set, and $\lambda I \le (a_{ij}^{(k)}) \le \Lambda I$ for all $k \in \mathcal{K}$. By the Evans–Krylov theorem, for every viscosity solution u of (2.4), we have

$$\|u\|_{C^{2+\bar\alpha}(B_1)} \le N_{ek}\|u\|_{L^\infty(B_5)}.$$

Suppose that there exists a sequence of functions $\{v_\ell\}_{\ell=0}^m$ satisfying

$$F\Big(\sum_{\ell=0}^{j} D^2\big(\rho^{-(j-\ell)(2+\alpha)} v_\ell(\rho^{j-\ell}x)\big)\Big) = 0 \quad \text{in } B_5$$

in viscosity sense for all $0 \le j \le m$, and $\|v_\ell\|_{L^\infty(B_5)} \le 1/M$ for all ℓ. Suppose that up to $m = i$ for some $i \ge 0$ there holds

$$\|v_\ell\|_{C^{2+\bar\alpha}(B_1)} \le 1 \quad \text{for all } \ell = 0, 1, \dots, m.$$

We are going to show this holds for $m = i+1$ as well. For $\ell = 0, \dots, i$, we let P_ℓ be the second order Taylor expansion polynomial of v_ℓ at $x = 0$. Let

$$\tilde v_{i+1} = v_{i+1} + \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(2+\alpha)}(v_\ell - P_\ell)(\rho^{i+1-\ell}x).$$

Then

$$G(D^2\tilde v_{i+1}) := F\Big(D^2\tilde v_{i+1} + \sum_{\ell=0}^{i} D^2\big(\rho^{-(i+1-\ell)(2+\alpha)}P_\ell(\rho^{i+1-\ell}x)\big)\Big) = 0 \quad \text{in } B_5.$$

It is clear that $G(\cdot)$ is uniformly elliptic and concave. Since,

$$\sum_{\ell=0}^{i} D^2\big(\rho^{-(i+1-\ell)(2+\alpha)}P_\ell(\rho^{i+1-\ell}x)\big) \text{ is a constant matrix,} \tag{2.5}$$

and

$$F\Big(\sum_{\ell=0}^{i} D^2\big(\rho^{-(i+1-\ell)(2+\alpha)}P_\ell(\rho^{i+1-\ell}x)\big)\Big) = 0, \tag{2.6}$$

we have $G(0) = 0$. By the Evans–Krylov theorem,

$$\|\tilde v_{i+1}\|_{C^{2+\bar\alpha}(B_1)} \le N_{ek}\|\tilde v_{i+1}\|_{L^\infty(B_5)}.$$

Since

$$\Big\|\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(2+\alpha)}(v_\ell - P_\ell)(\rho^{i+1-\ell}x)\Big\|_{L^\infty(B_5)} \le 5^{2+\bar\alpha}\sum_{\ell=0}^{i}\rho^{(i+1-\ell)(\bar\alpha-\alpha)} \le \frac{5^3\rho^{\bar\alpha-\alpha}}{1-\rho^{\bar\alpha-\alpha}}$$

$$\begin{aligned}\Big\|\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(2+\alpha)}(v_\ell - P_\ell)(\rho^{i+1-\ell}x)\Big\|_{C^{2+\bar\alpha}(B_1)} &\le 4\cdot 5^{2+\bar\alpha}\sum_{\ell=0}^{i}\rho^{(i+1-\ell)(\bar\alpha-\alpha)}\\ &\le \frac{5^4\rho^{\bar\alpha-\alpha}}{1-\rho^{\bar\alpha-\alpha}},\end{aligned}$$

it follows that

$$\|v_{i+1}\|_{C^{2+\bar\alpha}(B_1)} \le N_{ek}\Big(1/M + \frac{5^3}{1-\rho^{\bar\alpha-\alpha}}\rho^{\bar\alpha-\alpha}\Big) + \frac{5^4}{1-\rho^{\bar\alpha-\alpha}}\rho^{\bar\alpha-\alpha} \le 1$$

if we choose M sufficiently large and ρ_0 sufficiently small.

From this proof for the second order case, we see that the idea is to decompose v_ℓ as $(v_\ell - P_\ell) + P_\ell$, and apply Evans–Krylov theorem to the equation for $\tilde v_{i+1}$ which is v_{i+1} plus those rescaled $(v_\ell - P_\ell)$. In this step, we used (2.5) and (2.6).

In the nonlocal fully nonlinear case (2.1), we are going to use the same idea of decomposing v_ℓ and studying the equation of $\tilde v_{i+1}$. However, there is a difficulty that $\delta P_\ell(x,y)K(y)$ is not integrable and $\int_{\mathbb{R}^n}\delta\tilde P_\ell(x,y)K(y)\,dy$ will never be a constant for any cut-off $\tilde P_\ell$ of P_ℓ. Thus, we are not be able to use the Evans–Krylov theorem proved in [8]. Instead, we are going to employ the proofs in [8] to prove the $C^{\sigma+\bar\alpha}$ estimate for v_{i+1}. A delicate part is that we need to decompose v_ℓ in an appropriate way. We start with some preliminaries in the following.

2.2. Preliminaries

For a kernel $K(y)$, we denote

$$Lu(x) = \int_{\mathbb{R}^n} \delta u(x, y) K(y)\, \mathrm{d}y,$$

We will also say $L \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$ (or $\mathscr{L}_0(\lambda, \Lambda, \sigma)$) if $K \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$ (or $\mathscr{L}_0(\lambda, \Lambda, \sigma)$).

Lemma 2.3. *Suppose that $u \in C^4(B_2) \cap L^\infty(\mathbb{R}^n)$ and $K(y) \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$. Then*

$$\|Lu\|_{C^2(B_1)} \le C(\|u\|_{C^4(B_2)} + \|u\|_{L^\infty(\mathbb{R}^n)}),$$

where C is a positive constant depending only on α, σ_0 and Λ.

Proof. Let $\eta \in C_c^\infty(B_{3/2})$ and $\eta \equiv 1$ in $B_{5/4}$. Then

$$Lu = L(\eta u) + L((1-\eta)u).$$

It is clear that $\partial_{ij}(L(\eta u)) = L(\partial_{ij}(\eta u))$, from which it follows that

$$\|L(\eta u)\|_{C^2(B_1)} \le C(\|u\|_{C^4(B_2)} + \|u\|_{L^\infty(\mathbb{R}^n)}).$$

For the second term, we have $1 - \eta(x) = 0$ if $x \in B_1$, and thus

$$L((1-\eta)u)(x) = \int_{\mathbb{R}^n} (1 - \eta(x+y))u(x+y)K(y)\mathrm{d}y = \int_{\mathbb{R}^n \setminus B_{5/4}} (1 - \eta(y))u(y)K(y - x)\mathrm{d}y.$$

The lemma follows immediately since $K(y) \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$. □

Lemma 2.4. *Suppose that $u \in C^{\sigma+\alpha}(\mathbb{R}^n)$, $0 \le K(y) \le (2-\sigma)\Lambda|y|^{-n-\sigma}$ and $K(y) = K(-y)$. Then*

$$\|Lu\|_{C^\alpha(\mathbb{R}^n)} \le C\|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)},$$

and C is a positive constant depending only on α, σ_0 and Λ.

Proof. First of all, it is clear that

$$\|Lu\|_{L^\infty(\mathbb{R}^n)} \le C\|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)}.$$

In the following, we are going to estimate the C^α norm of Lu. We first consider that $\sigma + \alpha \ge 2$, which is the most difficult case. Since

$$Lu(x) = 2\int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x)y)K(y)\mathrm{d}y$$

$$Lu(0) = 2\int_{\mathbb{R}^n} (u(y) - u(0) - \nabla u(0)y)K(y)\mathrm{d}y,$$

we have that, for $r = |x|$

$$\begin{aligned}\frac{Lu(x) - L(0)}{2} &= \int_{B_r} ((u(x+y) - u(x) - \nabla u(x)y) - (u(y) - u(0) - \nabla u(0)y))K(y)\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^n \setminus B_r} ((u(x+y) - u(x) - \nabla u(x)y) - (u(y) - u(0) - \nabla u(0)y))K(y)\mathrm{d}y \\ &= I_1 + I_2.\end{aligned}$$

For I_1, we have that

$$I_1 = \int_{B_r} (u(x+y) - u(x) - \nabla u(x)y - \frac{1}{2}y^T\nabla^2 u(x)y)K(y)\mathrm{d}y$$
$$- \int_{B_r} (u(y) - u(0) - \nabla u(0)y - \frac{1}{2}y^T\nabla^2 u(0)y)K(y)\mathrm{d}y$$
$$+ \frac{1}{2}\int_{B_r} (y^T\nabla^2 u(x)y - y^T\nabla^2 u(0)y)K(y)\mathrm{d}y,$$

and thus

$$|I_1| \le 2\int_{B_r} \|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)}|y|^{\sigma+\alpha}K(y)\mathrm{d}y + \|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)} r^{\sigma+\alpha-2}\int_{B_r} |y|^2 K(y)\mathrm{d}y$$
$$\le (4\alpha^{-1} + \Lambda)\|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)} r^\alpha.$$

For I_2, it follows from mean value theorem that

$$|(u(x+y) - u(x) - \nabla u(x)y) - (u(y) - u(0) - \nabla u(0)y)| \le \|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)}|x||y|^{\sigma+\alpha-1}.$$

Thus,

$$|I_2| \le \|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)}|x| \int_{\mathbb{R}^n\setminus B_r} |y|^{\sigma+\alpha-1}K(y)\mathrm{d}y \le (1-\alpha)^{-1}\|u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)}|x|^\alpha.$$

For the case $\sigma + \alpha < 2$, one can prove them similarly and we omit its proof here. □

Lemma 2.5. *Suppose that $u \in C^{\sigma+\alpha}(B_2) \cap L^\infty(\mathbb{R}^n)$, $0 \le K(y) \le (2-\sigma)\Lambda|y|^{-n-\sigma}$, $K(y) = K(-y)$ and $|\nabla K(y)| \le \Lambda|y|^{-n-\sigma-1}$. Then*

$$\|Lu\|_{C^\alpha(B_1)} \le C(\|u\|_{C^{\sigma+\alpha}(B_2)} + \|u\|_{L^\infty(\mathbb{R}^n)}),$$

where

$$Lu = \int_{\mathbb{R}^n} \delta u(x,y)K(y)\mathrm{d}y,$$

and C is a positive constant depending only on α, σ_0 and Λ.

Proof. Let $\eta \in C_c^\infty(B_{3/2})$ and $\eta \equiv 1$ in $B_{5/4}$. Then

$$Lu = L(\eta u) + L((1-\eta)u).$$

It follows from Lemma 2.4 that

$$\|L(\eta u)\|_{C^\alpha(B_1)} \le C\|\eta u\|_{C^{\sigma+\alpha}(\mathbb{R}^n)} \le C(\|u\|_{C^{\sigma+\alpha}(B_2)} + \|u\|_{L^\infty(\mathbb{R}^n)}).$$

For the second term, we have $1 - \eta(x) = 0$ if $x \in B_1$, and thus

$$L((1-\eta)u)(x) = \int_{\mathbb{R}^n} (1-\eta(x+y))u(x+y)K(y)\mathrm{d}y = \int_{\mathbb{R}^n\setminus B_{5/4}} (1-\eta(y))u(y)K(y-x)\mathrm{d}y.$$

The lemma follows immediately since $|\nabla K(y)| \le \Lambda|y|^{-n-\sigma-1}$. □

Lemma 2.6. *Let $v \in C_c^{\sigma+\bar{\alpha}}(B_{1/2})$ such that $\|v\|_{C_c^{\sigma+\bar{\alpha}}(B_{1/2})} \le 1$, and $p(x)$ be the Taylor expansion polynomial of v at $x=0$ with degree $[\sigma+\bar{\alpha}]$. For every $L \in \mathscr{L}_0(\lambda,\Lambda,\sigma)$, there exists $P \in C_c^\infty(B_{1/2})$ such that $P(x)=p(x)$ in $B_{1/4}$, $\|P\|_{C^4(B_{1/2})} \le C$ and*

$$LP(0)=Lv(0),$$

where C is a positive constant depending only on $n,\lambda,\Lambda,\sigma_0$ and $\bar{\alpha}$.

Proof. Let $\eta \in C_c^\infty(B_{1/3})$ be such that $\eta \equiv 1$ in $B_{1/4}$. Let $h(x) \in C_c^4(B_{1/2} \setminus \bar{B}_{1/3})$ be such that $h(x)=1$ for $B_{11/24} \setminus B_{9/24}$ and $0 \le h \le 1$ in $B_{1/2}$. Let $P(x)=\eta(x)p(x)+t\cdot h(x)$, where $t=L(v-\eta p)(0)/Lh(0)$. Then we are left to show that $|t| \le C$, which depends only on $n,\lambda,\Lambda,\sigma_0$ and $\bar{\alpha}$. On one hand, it is clear that

$$Lh(0) \ge (2-\sigma)C^{-1}.$$

On the other hand, since $|v(x)-p(x)| \le C|x|^{\sigma+\bar{\alpha}}$ for $x \in B_{1/4}$, we have

$$\begin{aligned}|L(v-\eta p)(0)| &= \int_{B_{1/4}} |v(y)-p(y)|K(y)\mathrm{d}y + \int_{B_{1/2}\setminus B_{1/4}} |v(y)-\eta(y)p(y)|K(y)\mathrm{d}y \\ &\le C(2-\sigma)\int_{B_{1/4}} |y|^{\sigma+\bar{\alpha}-n-\sigma}\mathrm{d}y + C(2-\sigma) \\ &\le C(2-\sigma),\end{aligned}$$

from which it follows that $|t| \le C$. □

2.3. Decompositions

We shall adapt the proofs in [8] with delicate decomposition and cut-off arguments indicated in Section 2.1 to prove Theorem 2.2. Recall that we are left to show (2.3).

For a function v, we denote $v_\rho(x)=\rho^{-(\sigma+\alpha)}v(\rho x)$. Set

$$R(x)=\sum_{\ell=0}^{i} \rho^{-(i-\ell)(\sigma+\alpha)} v_\ell(\rho^{i-\ell}x).$$

By (2.1),

$$\inf_{a\in\mathcal{A}}\{L_a^{(i+1)}R_\rho(x)+\rho^{-(i+1)\alpha}b_a\}=0 \quad \text{in } B_{5/\rho},$$

where $L_a^{(i+1)}$ is the linear operator with kernel $K_a^{(i+1)} \in \mathscr{L}_2(\lambda,\Lambda,\sigma)$. Hence, there exists an $\bar{a} \in \mathcal{A}$ such that

$$0 \le L_{\bar{a}}^{(i+1)}R_\rho(0)+\rho^{-(i+1)\alpha}b_{\bar{a}} < \rho^{\bar{\alpha}-\alpha}. \tag{2.7}$$

Let $\eta_0(x)=1$ in $B_{1/4}$ and $\eta_0 \in C_c^\infty(B_{1/2})$ be a fixed cut-off function. Set

$$v_\ell(x)=v_\ell\eta_0+v_\ell(1-\eta_0)=: v_\ell^{(1)}+v_\ell^{(2)}.$$

Let $p_\ell(x)$ be the Taylor expansion polynomial of $v_\ell^{(1)}(x)$ at $x=0$ with degree $[\sigma+\bar{\alpha}]$. By Lemma 2.6, there exists $P_\ell \in C_c^\infty(B_{1/2})$ such that $P_\ell(x)=p_\ell(x)$ in $B_{1/4}$, $\|P_\ell\|_{C^4(B_{1/2})} \le c_0$ (a universal constant, independent of ℓ) and

$$L_{\bar{a}}^{(\ell)}P_\ell(0)=L_{\bar{a}}^{(\ell)}v_\ell^{(1)}(0). \tag{2.8}$$

Set

$$v_\ell=(v_\ell^{(1)}-P_\ell)+(v_\ell^{(2)}+P_\ell)=: V_\ell^{(1)}+V_\ell^{(2)}.$$

We have

$$\begin{aligned}
&\|V_\ell^{(1)}\|_{L^\infty(\mathbb{R}^n)}+\|V_\ell^{(2)}\|_{L^\infty(\mathbb{R}^n)}\le c_0+1,\quad V_\ell^{(1)}(0)=0,\\
&V_\ell^{(1)}\in C_c^{\sigma+\bar\alpha}(B_{1/2}),\quad \|V_\ell^{(1)}\|_{C^{\sigma+\bar\alpha}(\mathbb{R}^n)}+\|V_\ell^{(2)}\|_{C^{\sigma+\bar\alpha}(B_1)}\le 4^4(c_0+1),\\
&V_\ell^{(1)}=v_\ell-p_\ell \text{ in } B_{1/4},\quad V_\ell^{(2)}=p_\ell \text{ in } B_{1/4},\quad |V_\ell^{(1)}(x)|\le 4^4(c_0+1)|x|^{\sigma+\bar\alpha} \text{ in } \mathbb{R}^n.
\end{aligned}\tag{2.9}$$

Decompose $R(x)$ as

$$R(x)=R^{(1)}(x)+R^{(2)}(x),$$

where

$$\begin{aligned}
R^{(1)}(x)&=\sum_{\ell=0}^{i}\rho^{-(i-\ell)(\sigma+\alpha)}V_\ell^{(1)}(\rho^{i-\ell}x)\\
R^{(2)}(x)&=\sum_{\ell=0}^{i}\rho^{-(i-\ell)(\sigma+\alpha)}V_\ell^{(2)}(\rho^{i-\ell}x).
\end{aligned}$$

By change of variables, we have that for each $a\in\mathcal{A}$,

$$\begin{aligned}
L_a^{(i+1)}R_\rho^{(1)}(x)&=\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)\alpha}(L_a^{(\ell)}V_\ell^{(1)})(\rho^{i+1-\ell}x),\\
L_a^{(i+1)}R_\rho^{(2)}(x)&=\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)\alpha}(L_a^{(\ell)}V_\ell^{(2)})(\rho^{i+1-\ell}x).
\end{aligned}\tag{2.10}$$

By (2.7) and (2.8), we have

$$\begin{aligned}
&L_{\bar a}^{(i+1)}R_\rho^{(1)}(0)=0,\\
&0\le L_{\bar a}^{(i+1)}R_\rho^{(2)}(0)+\rho^{-(i+1)\alpha}b_{\bar a}=L_{\bar a}^{(i+1)}R_\rho(0)+\rho^{-(i+1)\alpha}b_{\bar a}\le\rho^{\bar\alpha-\alpha}.
\end{aligned}\tag{2.11}$$

It follows from Lemma 2.4, (2.10), (2.11) and (2.9) that

$$\begin{aligned}
|L_{\bar a}^{(i+1)}R_\rho^{(1)}(x)|&=|L_{\bar a}^{(i+1)}R_\rho^{(1)}(x)-L_{\bar a}^{(i+1)}R_\rho^{(1)}(0)|\\
&\le\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)\alpha}|(L_a^{(\ell)}V_\ell^{(1)})(\rho^{i+1-\ell}x)-(L_a^{(\ell)}V_\ell^{(1)})(0)|\\
&\le C|x|^{\bar\alpha}\sum_{\ell=0}^{i}\rho^{(i+1-\ell)(\bar\alpha-\alpha)}\|V_\ell^{(1)}\|_{C^{\sigma+\bar\alpha}(\mathbb{R}^n)}\\
&\le C|x|^{\bar\alpha}\rho^{\bar\alpha-\alpha}\sum_{\ell=0}^{\infty}\rho^{\ell(\bar\alpha-\alpha)}\\
&\le C\rho^{\bar\alpha-\alpha}|x|^{\bar\alpha}\quad\text{for } x\in\mathbb{R}^n.
\end{aligned}\tag{2.12}$$

Similarly, it follows from Lemma 2.5, (2.10)and (2.9) that

$$\begin{aligned}
&|L_{\bar a}^{(i+1)}R_\rho^{(2)}(x)-L_{\bar a}^{(i+1)}R_\rho^{(2)}(0)|\\
&\quad\le\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)\alpha}|(L_a^{(\ell)}V_\ell^{(2)})(\rho^{i+1-\ell}x)-(L_a^{(\ell)}V_\ell^{(2)})(0)|\\
&\quad\le C|x|^{\bar\alpha}\sum_{\ell=0}^{i}\rho^{(i+1-\ell)(\bar\alpha-\alpha)}(\|V_\ell^{(2)}\|_{C^{\sigma+\bar\alpha}(B_1)}+\|V_\ell^{(2)}\|_{L^\infty(\mathbb{R}^n)})\\
&\quad\le C\rho^{\bar\alpha-\alpha}|x|^{\bar\alpha}\quad\text{for } x\in B_5.
\end{aligned}\tag{2.13}$$

Thus, by (2.11), we have

$$|L_{\bar{a}}^{(i+1)} R_\rho^{(2)}(x) + \rho^{-(i+1)\alpha} b_{\bar{a}}| \le C\rho^{\bar{\alpha}-\alpha}(|x|^{\bar{\alpha}} + 1) \quad \text{for } x \in B_5. \tag{2.14}$$

Let

$$\tilde{v}_{i+1} = v_{i+1} + R_\rho^{(1)}.$$

Hence, the equation of (2.1) involving v_{i+1} is

$$\inf_a \{L_a^{(i+1)}(v_{i+1} + R_\rho) + \rho^{-(i+1)\alpha} b_a\} = 0,$$

which is equivalent to

$$\inf_a \{L_a^{(i+1)}(\tilde{v}_{i+1} + R_\rho^{(2)}) + \rho^{-(i+1)\alpha} b_a\} = 0 \quad \text{in } B_5. \tag{2.15}$$

It follows from (2.12) and (2.14) that

$$\begin{aligned} &L_{\bar{a}}^{(i+1)} v_{i+1}(x) \ge -C\rho^{\bar{\alpha}-\alpha} \quad \text{in } B_5, \\ &L_{\bar{a}}^{(i+1)} \tilde{v}_{i+1}(x) \ge -C\rho^{\bar{\alpha}-\alpha} \quad \text{in } B_5, \end{aligned} \tag{2.16}$$

where C is a universal positive constant.

2.4. C^σ estimates

Define the maximal operators

$$\mathcal{M}_0^+ u(x) = \sup_{K \in \mathscr{L}_0(\lambda,\Lambda,\sigma)} \int_{\mathbb{R}^n} \delta u(x,y) K(y) \mathrm{d}y,$$

$$\mathcal{M}_2^+ u(x) = \sup_{K \in \mathscr{L}_2(\lambda,\Lambda,\sigma)} \int_{\mathbb{R}^n} \delta u(x,y) K(y) \mathrm{d}y.$$

And one can define the extremal operators $\mathcal{M}_0^-$ and $\mathcal{M}_2^-$ similarly. Let $\eta_1 \in C_c^\infty(B_4)$ be a smooth cut-off function such that $\eta_1 \equiv 1$ in B_3. We write (2.15) as

$$\inf_{a \in \mathcal{A}} \{L_a^{(i+1)} \tilde{v}_{i+1} + h_a(x) + \rho^{-(i+1)\alpha} b_a\} = 0 \quad \text{in } B_3, \tag{2.17}$$

where

$$h_a(x) := \eta_1(x) L_a^{(i+1)} R_\rho^{(2)}(x).$$

Lemma 2.7. *Let K be a symmetric kernel satisfying $0 \le K(y) \le (2-\sigma)\Lambda|y|^{-n-\sigma}$. Then for every bump function η such that*

$$\begin{aligned} 0 \le \eta(x) \le 1 \quad &\text{in } \mathbb{R}^n, \\ \eta(x) = \eta(-x) \quad &\text{in } \mathbb{R}^n, \\ \eta(x) = 0 \quad &\text{in } \mathbb{R}^n \setminus B_{3/2}, \end{aligned}$$

we have

$$\mathcal{M}_2^+ \left(\int_{\mathbb{R}^n} \delta \tilde{v}_{i+1}(x,y) K(y) \eta(y) \, \mathrm{d}y \right) \ge -C\rho^{2-\alpha} \quad \text{in } B_{3/2}.$$

Proof. Let ϕ_k be the L^1 function $\phi_k = \chi_{\mathbb{R}^n \setminus B_{1/k}} K(y)\eta(y)$, where χ_E is the characteristic function of a set E. For every $a \in \mathcal{A}$, we know from (2.17) that

$$L_a^{(i+1)} \tilde{v}_{i+1}(x) + h_a(x) + \rho^{-(i+1)\alpha} b_a \geq 0 \quad \forall\, x \in B_3.$$

It follows that for all $x \in B_{3/2}$,

$$\begin{aligned} 0 &\leq (L_a^{(i+1)} \tilde{v}_{i+1} + h_a + \rho^{-(i+1)\alpha} b_a) * \phi_k(x) \\ &\leq L_a^{(i+1)} (\tilde{v}_{i+1} * \phi_k)(x) + h_a * \phi_k(x) + \rho^{-(i+1)\alpha} b_a \|\phi_k\|_{L^1}. \end{aligned}$$

It also follows from (2.17) that

$$\inf_{a\in\mathcal{A}} \{\|\phi_k\|_{L^1} (L_a^{(i+1)} \tilde{v}_{i+1}(x) + h_a(x) + \rho^{-(i+1)\alpha} b_a)\} = 0 \quad \forall\, x \in B_3.$$

This implies that for all $x \in B_{3/2}$,

$$\sup_{a\in\mathcal{A}} L_a^{(i+1)} (\tilde{v}_{i+1} * \phi_k - \|\phi_k\|_{L_1} \tilde{v}_{i+1})(x) + \sup_{a\in\mathcal{A}} \{h_a * \phi_k(x) - \|\phi_k\|_{L_1} h_a(x)\} \geq 0.$$

For any $x \in B_{3/2}$, any $a \in \mathcal{A}$, by using (2.10) and change of variables we have

$$\begin{aligned} &2|h_a * \phi_k(x) - \|\phi_k\|_{L_1} h_a(x)| \\ &\quad\leq \Big| \int\limits_{B_{3/2}\setminus B_{1/k}} \delta(L_a^{(i+1)} R_\rho^{(2)})(x,y) K(y)\eta(y)\,\mathrm{d}y \Big| \\ &\quad\leq \sum_{\ell=0}^{i} \rho^{(\ell-1-i)\alpha} \int\limits_{B_{3/2}\setminus B_{1/k}} |\delta(L_a^{(\ell)} V_\ell^{(2)})(\rho^{i+1-\ell}x, \rho^{i+1-\ell}y)| K(y)\eta(y)\,\mathrm{d}y \\ &\quad\leq \sum_{\ell=0}^{i} \rho^{(i+1-\ell)(\sigma-\alpha)} \int\limits_{B_{3\rho^{i+1-\ell}/2}\setminus B_{\rho^{i+1-\ell}/k}} |\delta(L_a^{(\ell)} V_\ell^{(2)})(\rho^{i+1-\ell}x, y)| K^{-(i+1-\ell)}(y)\,\mathrm{d}y \\ &\quad\leq \sum_{\ell=0}^{i} \rho^{(i+1-\ell)(\sigma-\alpha)} \int\limits_{B_{3\rho^{i+1-\ell}/2}} \|L_a^{(\ell)} V_\ell^{(2)}\|_{C^2(B_{1/8})} |y|^2 K^{-(i+1-\ell)}(y)\,\mathrm{d}y \\ &\quad\leq \sum_{\ell=0}^{i} \rho^{(i+1-\ell)(\sigma-\alpha)} \|L_a V_\ell^{(2)}\|_{C^2(B_{1/8})} \int\limits_{B_{3\rho^{i+1-\ell}/2}} \frac{\Lambda(2-\sigma)}{|y|^{n+\sigma-2}}\,\mathrm{d}y \\ &\quad\leq C \sum_{\ell=0}^{i} \rho^{(i+1-\ell)(\sigma-\alpha)} (\|V_\ell^{(2)}\|_{C^4(B_{1/4})} + \|V_\ell^{(2)}\|_{L^\infty(\mathbb{R}^n)}) \rho^{(i+1-\ell)(2-\sigma)} \\ &\quad\leq C\rho^{2-\alpha} (\|V_\ell^{(2)}\|_{C^4(B_{1/4})} + \|V_\ell^{(2)}\|_{L^\infty(\mathbb{R}^n)}) \sum_{\ell=0}^{\infty} \rho^{\ell(2-\alpha)} \\ &\quad\leq C\rho^{2-\alpha}, \end{aligned}$$

where $K^{-(i+1-\ell)}(y) = \rho^{-(i+1-\ell)(n+\sigma)} K(\rho^{-(i+1-\ell)} y)$, and Lemma 2.3 was used since $V_\ell^{(2)}(x) = p_\ell(x)$ in $B_{1/4}$. Consequently,

$$\mathcal{M}_2^+ (\tilde{v}_{i+1} * \phi_k - \|\phi_k\|_{L_1} \tilde{v}_{i+1})(x) \geq -C\rho^{2-\alpha}.$$

The result follows from Lemma 5 in [7] by taking the limit as $k \to \infty$. □

Lemma 2.8. *Let K be a symmetric kernel satisfying $0 \le K(y) \le (2-\sigma)\Lambda|y|^{-n-\sigma}$. Then for every smooth bump function η such that*

$$0 \le \eta(x) \le 1 \quad in\ \mathbb{R}^n, \qquad \eta(x) = \eta(-x) \quad in\ \mathbb{R}^n,$$
$$\eta(x) = 0 \quad in\ \mathbb{R}^n \setminus B_{4/5}, \qquad \eta(x) = 1 \quad in\ B_{3/4},$$

we have

$$\mathcal{M}_2^+\left(\eta(x)\int_{B_1}\delta\tilde{v}_{i+1}(x,y)K(y)\,\mathrm{d}y\right) \ge -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}) \quad in\ B_{3/5}.$$

Proof. Define

$$Tv(x) = \int_{B_1}\delta v(x,y)K(y)\,\mathrm{d}y.$$

It follows from Lemma 2.7 that

$$\mathcal{M}_2^+(T\tilde{v}_{i+1})(x) \ge -C\rho^{2-\alpha} \quad \text{in } B_{3/2}. \tag{2.18}$$

Let $\bar{L}$ be any operator with kernel $\bar{K} \in \mathscr{L}_2(\lambda,\Lambda,\sigma)$. For $x \in B_{3/5}$, we have

$$\begin{aligned}\bar{L}(\eta T\tilde{v}_{i+1})(x) &= \int_{\mathbb{R}^n}\delta(T\tilde{v}_{i+1})(x,y)\bar{K}(y)\,\mathrm{d}y - \int_{\mathbb{R}^n}\delta((1-\eta)T\tilde{v}_{i+1})(x,y)\bar{K}(y)\,\mathrm{d}y \\ &\ge \bar{L}(T\tilde{v}_{i+1})(x) - 2\int_{\mathbb{R}^n}(1-\eta(x-y))T\tilde{v}_{i+1}(x-y)\bar{K}(y)\,\mathrm{d}y. \end{aligned}\tag{2.19}$$

Now we estimate the second term in the last inequality. Recall that $\tilde{v}_{i+1} = v_{i+1} + R_\rho^{(1)}$. It is clear that

$$\begin{aligned}&\int_{\mathbb{R}^n} Tv_{i+1}(x-y)(1-\eta(x-y))\bar{K}(y)\,\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^n} v_{i+1}(x-y)T((1-\eta(x-\cdot))\bar{K}(\cdot))(y)\,\mathrm{d}y \le C\|v_{i+1}\|_{L^\infty} \le C/M.\end{aligned}$$

By change of variables, we have for all $x \in \mathbb{R}^n$,

$$\begin{aligned}|TR_\rho^{(1)}(x)| &= |\int_{B_1}\delta R_\rho^{(1)}(x,y)K(y)\,\mathrm{d}y| \\ &= |\sum_{\ell=0}^{i}\int_{B_{\rho^{i+1-\ell}}}\rho^{-(i+1-\ell)\alpha}\delta V_\ell^{(1)}(\rho^{i+1-\ell}x,y)K^{-(i+1-\ell)}(y)\,\mathrm{d}y|,\end{aligned}$$

where $K^{-(i+1-\ell)}(y) = \rho^{-(i+1-\ell)(n+\sigma)}K(\rho^{-(i+1-\ell)}y)$.

By triangle inequality, we have

$$\begin{aligned}|TR_\rho^{(1)}(x)| \le &\sum_{l=0}^{i}\rho^{-(i+1-\ell)\alpha}|\int_{B_{\rho^{i+1-\ell}}}(\delta V_\ell^{(1)}(\rho^{i+1-\ell}x,y) - \delta V_\ell^{(1)}(0,y))K^{-(i+1-\ell)}(y)\,\mathrm{d}y| \\ &+ \sum_{l=0}^{i}\rho^{-(i+1-\ell)\alpha}|\int_{B_{\rho^{i+1-\ell}}}\delta V_\ell^{(1)}(0,y)K^{-(i+1-\ell)}(y)\,\mathrm{d}y|\end{aligned}$$

$$\le C\sum_{\ell=0}^{i}\|V_\ell^{(1)}\|_{C^{\sigma+\bar\alpha}(\mathbb{R}^n)}|x|^{\bar\alpha}\rho^{(i+1-\ell)(\bar\alpha-\alpha)}+C\sum_{l=0}^{i}\rho^{-(i+1-\ell)\alpha}\int_{B_{\rho^{i+1-\ell}}}\frac{(2-\sigma)\Lambda|y|^{\sigma+\bar\alpha}}{|y|^{n+\sigma}}\,d\zeta$$

$$\le C\rho^{\bar\alpha-\alpha}(1+|x|^{\bar\alpha})\quad \text{for all } x\in\mathbb{R}^n, \tag{2.20}$$

where we used Lemma 2.4 and (2.9) in the second inequality.

It follows that for $x\in B_{3/5}$,

$$\begin{aligned}&\int_{\mathbb{R}^n}(1-\eta(x-y))TR_\rho^{(1)}(x-y)\bar K(y)\,dy\\ &=\int_{\mathbb{R}^n}(1-\eta(y))TR_\rho^{(1)}(y)\bar K(x-y)\,dy\\ &=\int_{\mathbb{R}^n\setminus B_{3/4}}(1-\eta(y))TR_\rho^{(1)}(y)\bar K(x-y)\,dy\\ &\le C\rho^{\bar\alpha-\alpha}\int_{|y|>1/64}\frac{(2-\sigma)}{|y|^{n+\sigma-\bar\alpha}}\le C\rho^{\bar\alpha-\alpha},\end{aligned} \tag{2.21}$$

where we used that $\sigma\ge\sigma_0>1>\bar\alpha$. Taking the supremum of all $\bar K$ in $\mathscr{L}_2(\lambda,\Lambda,\sigma)$ in (2.19) and using (2.18), we complete the proof. □

Lemma 2.9. *We have*

$$|L_{\bar a}^{(i+1)}v_{i+1}(x)|\le C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})\quad \text{in } B_{1/2}.$$

Proof. Let $\eta_1(x)\ge 0$ be a smooth cutoff function in B_2 with $\eta_1\equiv 1$ in B_1. Then

$$\int_{\mathbb{R}^n}L_{\bar a}^{(i+1)}v_{i+1}\eta_1=\int_{\mathbb{R}^n}v_{i+1}L_{\bar a}^{(i+1)}\eta_1\le C\|v_{i+1}\|_{L^\infty(\mathbb{R}^n)}\le C/M.$$

By (2.16), $L_{\bar a}^{(i+1)}v_{i+1}\ge -C\rho^{\bar\alpha-\alpha}$ in B_4, we have

$$\int_{B_1}|L_{\bar a}^{(i+1)}v_{i+1}|\le C(\rho^{\bar\alpha-\alpha}+\frac{1}{M}).$$

Let

$$T_{\bar a}^{(i+1)}v=\int_{B_1}\delta v(x,y)K_{\bar a}^{(i+1)}(y)\,dy.$$

It is easy to see that

$$\int_{B_1}|T_{\bar a}^{(i+1)}v_{i+1}|\le\int_{B_1}|L_{\bar a}^{(i+1)}v_{i+1}|+C\|v_{i+1}\|_{L^\infty}\le C(\rho^{\bar\alpha-\alpha}+\frac{1}{M}).$$

It follows from (2.20) that for all $x\in\mathbb{R}^n$,

$$|T_{\bar a}^{(i+1)}R_\rho^{(1)}(x)|\le C\rho^{\bar\alpha-\alpha}(1+|x|^{\bar\alpha}). \tag{2.22}$$

Since $\tilde v_{i+1}=v_{i+1}+R_\rho^{(1)}$, we obtain

$$\int_{B_1}|T_{\bar a}^{(i+1)}\tilde v_{i+1}|\le C(\rho^{\bar\alpha-\alpha}+\frac{1}{M}). \tag{2.23}$$

Let η be the cut-off function in Lemma 2.8, and denote $v(x) := \eta(x)T_{\bar a}^{(i+1)}\tilde v_{i+1}(x)$. It follows from Lemma 2.8 that

$$\mathcal{M}_2^+ v(x) \ge -C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}) \quad \text{in } B_{3/5}.$$

It follows from (2.23) and Theorem 5.1 in [8] that $v \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M})$ in $B_{1/2}$. But $v = T_{\bar a}^{(i+1)}\tilde v_{i+1}$ in $B_{1/2}$, so we have proved that

$$T_{\bar a}^{(i+1)}\tilde v_{i+1} \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}) \quad \text{in } B_{1/2}.$$

By (2.22), we have $T_{\bar a}^{(i+1)} v_{i+1}(x) \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M})$ in $B_{1/2}$, and thus,

$$L_{\bar a}^{(i+1)} v_{i+1}(x) \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}) \quad \text{in } B_{1/2}.$$

We complete the proof together with (2.16). □

Lemma 2.10. *There is a universal constant C such that for every operator L with a symmetric kernel K satisfying $0 \le K(y) \le (2-\sigma)\Lambda|y|^{n+\sigma}$, we have*

$$|Lv_{i+1}(x)| \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}) \quad \text{in } B_1.$$

Proof. We will prove the estimate in $B_{1/6}$, and the general estimate follows from scaling and translation arguments. By Lemma 2.9 we have

$$\|L_{\bar a}^{(i+1)} v_{i+1}\|_{L^2(B_{1/2})} \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}).$$

Note that $\|v_{i+1}\|_{L^1(\mathbb{R}^n, 1/(1+|y|^{n+\sigma}))} \le C\|v_{i+1}\|_{L^\infty} \le C/M$. From Theorem 4.3 of [8], we have L^2 estimate for every linear operator L with kernel $K \in \mathscr{L}_0(\lambda, \Lambda, \sigma)$,

$$\|Lv_{i+1}\|_{L^2(B_{1/3})} \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}).$$

We split the integral of Lv_{i+1} as

$$Lv_{i+1}(x) = \int_{B_1} + \int_{B_1^c} \delta v_{i+1}(x,y)K(y)\,dy.$$

It is clear that

$$|\int_{B_1^c} \delta v_{i+1}(x,y)K(y)\,dy| \le C\|v_{i+1}\|_{L^\infty} \le C/M.$$

Hence, we have L^2 estimates for the first one

$$\left\| \int_{B_1} \delta v_{i+1}(x,y)K(y)\,dy \right\|_{L^2(B_{1/3})} \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}).$$

It follows from (2.20) that

$$|\int_{B_1} \delta R_\rho^{(1)}(x,y)K(y)\,dy| \le C\rho^{\bar\alpha-\alpha} \quad \text{for } x \in B_1. \tag{2.24}$$

By triangle inequality, we have

$$\left\| \int_{B_1} \delta \tilde v_{i+1}(x,y)K(y)\,dy \right\|_{L^2(B_{1/3})} \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}). \tag{2.25}$$

For a smooth cut-off function $c(x) \in C_c^\infty(B_{1/3})$, $c(x) = c(-x)$, and $c(x) = 1$ in $B_{1/4}$, we define

$$v(x) := c(x) \int_{B_1} \delta \tilde{v}_{i+1}(x, y) K(y) \, dy.$$

It follows from the proof of Lemma 2.8 that $\mathcal{M}_2^+ v \geq -C(1/M + \rho^{\bar{\alpha}-\alpha})$ in $B_{1/5}$. By (2.25) and Theorem 5.1 in [8] we have $v \leq C(1/M + \rho^{\bar{\alpha}-\alpha})$ in $B_{1/6}$, and thus

$$\int_{B_1} \delta \tilde{v}_{i+1}(x, y) K(y) \, dy \leq C(1/M + \rho^{\bar{\alpha}-\alpha}) \quad \text{in } B_{1/6}.$$

Since (2.24) holds for $x \in B_1$, we have that

$$\int_{B_1} \delta v_{i+1}(x, y) K(y) \, dy \leq C(1/M + \rho^{\bar{\alpha}-\alpha}) \quad \text{in } B_{1/6}.$$

Consequently,

$$L v_{i+1} \leq C(1/M + \rho^{\bar{\alpha}-\alpha}) \quad \text{in } B_{1/6}.$$

Consider the kernel

$$K_d = \frac{2}{\lambda} K_{\bar{a}}^{(i+1)} - \frac{1}{\Lambda} K$$

and the corresponding linear operator L_d, where $0 \leq K \leq (2-\sigma)\Lambda |y|^{-n-\sigma}$. Then K_d satisfies the ellipticity condition $(2-\sigma)|y|^{-n-\sigma} \leq K_d(y) \leq (2-\sigma)(2\Lambda/\lambda)|y|^{-n-\sigma}$. The same proof as above yields that

$$L_d v_{i+1} \leq C(1/M + \rho^{\bar{\alpha}-\alpha}) \quad \text{in } B_{1/6}.$$

Since $L_{\bar{a}}^{(i+1)} v_{i+1}$ is lower bounded by (2.16), we obtain a bound from below for L in $B_{1/6}$

$$L v_{i+1} = 2\frac{\Lambda}{\lambda} L_{\bar{a}}^{(i+1)} v_{i+1} - \Lambda L_d v_{i+1} \geq -C(1/M + \rho^{\bar{\alpha}-\alpha}) \quad \text{in } B_{1/6}.$$

Similarly, if we consider $\tilde{K}_d = \frac{2}{\lambda} K_{\bar{a}}^{(i+1)} + \frac{1}{\Lambda} K$, we obtain that $L v_{i+1} \leq C(1/M + \rho^{\bar{\alpha}-\alpha})$. In conclusion, we obtained that $|L v_{i+1}| \leq C(1/M + \rho^{\bar{\alpha}-\alpha})$ in $B_{1/6}$. □

The above lemma immediately gives

Corollary 2.11. *$\mathcal{M}_0^+ v_{i+1}$ and $\mathcal{M}_0^- v_{i+1}$ are bounded by $C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})$ in B_1. In particular,*

$$\|\nabla v_{i+1}\|_{L^\infty(B_{1/2})} \leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}), \tag{2.26}$$

and consequently,

$$\|\nabla \tilde{v}_{i+1}\|_{L^\infty(B_{1/2})} \leq C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M}). \tag{2.27}$$

Proof. The first conclusion is clear, from which (2.26) also follows immediately since $\sigma \geq \sigma_0 > 1$. To prove (2.27), we notice that $V_\ell^{(1)} = v_\ell^{(1)} - P_\ell \in C_c^{\sigma+\bar{\alpha}}(B_{1/2})$, and $V_\ell^{(1)} = v_\ell^{(1)} - p_\ell$ in $B_{1/4}$ where p_ℓ is the Taylor expansion polynomial of $v_\ell^{(1)}$ at $x = 0$ with degree $[\sigma + \bar{\alpha}]$. Hence, $|\nabla V_\ell^{(1)}(x)| \leq C|x|^{\sigma+\bar{\alpha}-1}$ in $B_{1/2}$. Thus, for all $x \in B_{1/2}$,

$$\begin{aligned} |\nabla R_\rho^{(1)}(x)| &= |\nabla \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} V_\ell^{(1)}(\rho^{i+1-\ell} x)| \\ &\leq C \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha-1)} |\rho^{i+1-\ell} x|^{\sigma+\bar{\alpha}-1} \leq C\rho^{\bar{\alpha}-\alpha}. \end{aligned}$$

Thus, (2.27) follows immediately. □

Theorem 2.12. *We have*

$$\int_{\mathbb{R}^n} |\delta v_{i+1}(x,y)| \frac{(2-\sigma)}{|y|^{n+\sigma}}\,\mathrm{d}y \le C(\rho^{\bar\alpha-\alpha} + \frac{1}{M}) \quad \text{in } B_1.$$

Proof. Given Lemma 2.10 and Corollary 2.11, it follows from the same proof as that of Theorem 7.4 in [8]. □

2.5. $C^{\sigma+\bar\alpha}$ estimates

For brevity, we write

$$u = v_{i+1} \quad \text{and} \quad \tilde u = \tilde v_{i+1}$$

in this subsection.

Let η be a bump function as in Lemma 2.8. For each measurable set A with $-A = A$, we write

$$w_A(x) = \eta(x) \int_{B_1} (\delta\tilde u(x,y) - \delta\tilde u(0,y)) K_A(y)\,\mathrm{d}y,$$

where

$$K_A(y) = \frac{(2-\sigma)}{|y|^{n+\sigma}} \chi_A(y).$$

For $x \in B_1$, by Lemma 2.4 and change of variables, we have

$$\begin{aligned}
&|\int_{B_1} (\delta R_\rho^{(1)}(x,y) - \delta R_\rho^{(1)}(0,y)) K_A(y)\,\mathrm{d}y| \\
&= |\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} \int_{B_1} (\delta V_\ell^{(1)}(\rho^{i+1-\ell}x, \rho^{i+1-\ell}y) - \delta V_\ell^{(1)}(0, \rho^{i+1-\ell}y)) K_A(y)\,\mathrm{d}y| \\
&= |\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)\alpha} \int_{B_{\rho^{i+1-\ell}}} (\delta V_\ell^{(1)}(\rho^{i+1-\ell}x, y) - \delta V_\ell^{(1)}(0,y)) K_A^{(\ell-1-i)}(y)\,\mathrm{d}y| \\
&\le \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)\alpha} \|V_\ell^{(1)}\|_{C^{\sigma+\bar\alpha}(\mathbb{R}^n)} \rho^{(i+1-\ell)\bar\alpha} |x|^{\bar\alpha} \\
&\le C\rho^{\bar\alpha-\alpha}|x|^{\bar\alpha}.
\end{aligned} \tag{2.28}$$

Then it follows from Theorem 2.12 that

$$|w_A| \le C(\rho^{\bar\alpha-\alpha} + 1/M) \quad \text{in} \quad \mathbb{R}^n. \tag{2.29}$$

Also, it follows from Lemma 2.10 as well as (2.20) that

$$|\int_{B_1} \delta\tilde u(0,y) K_A(y)\,\mathrm{d}y| \le C(\rho^{\bar\alpha-\alpha} + 1/M).$$

Together with Lemma 2.8, we have

$$\mathcal{M}_2^+ w_A \ge -C(\rho^{\bar\alpha-\alpha} + 1/M) \quad \text{in } B_{3/5} \text{ uniformly in } A.$$

As in [8], we define

$$N^+(x) := \sup_A w_A(x) = \eta(x) \int_{B_1} (\delta\tilde u(x,y) - \delta\tilde u(0,y))^+ \frac{(2-\sigma)}{|y|^{n+\sigma}}\,\mathrm{d}y,$$

$$N^-(x) := \sup_A -w_A(x) = \eta(x) \int_{B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y))^- \frac{(2-\sigma)}{|y|^{n+\sigma}}\, dy.$$

Lemma 2.13. *For all $x \in B_{1/4}$, we have*

$$\frac{\lambda}{\Lambda} N^-(x) - C(\rho^{\bar{\alpha}-\alpha} + 1/M)|x| \le N^+(x) \le \frac{\Lambda}{\lambda} N^-(x) + C(\rho^{\bar{\alpha}-\alpha} + 1/M)|x|.$$

Proof. For some $x \in B_{1/2}$, let $\tilde{u}_x(z) = \tilde{u}(x+z)$. It follows from (2.17) that

$$\mathcal{M}_2^+(\tilde{u}_x - \tilde{u})(0) \ge -\sup_a (h_a(x) - h_a(0)), \quad \mathcal{M}_2^-(\tilde{u}_x - \tilde{u})(0) \le \sup_a (h_a(0) - h_a(x)).$$

Note that for $x \in B_3$,

$$h_a(x) = L_a^{(i+1)} R_\rho^{(2)}(x) = \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)\alpha} (L_a^{(\ell)} V_\ell^{(2)})(\rho^{i+1-\ell} x)$$

and thus for $\rho x \in B_{1/4}$

$$\begin{aligned}
|h_a(x) - h_a(0)| &= |\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)\alpha} (L_a^{(\ell)} V_\ell^{(2)}(\rho^{(i+1-\ell)} x) - L_a^{(\ell)} V_\ell^{(2)}(0))| \\
&\le C \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)\alpha} (\|V_\ell^{(2)}\|_{C^4(B_{1/2})} + \|V_\ell^{(2)}\|_{L^\infty(\mathbb{R}^n)}) |\rho^{(i+1-\ell)} x| \\
&\le C\rho^{1-\alpha} \sum_{\ell=0}^{\infty} \rho^{\ell(1-\alpha)} |x|,
\end{aligned}$$

where Lemma 2.3 was used in the first inequality. Hence, we have

$$\mathcal{M}_2^+(\tilde{u}_x - \tilde{u})(0) \ge -C\rho^{1-\alpha}|x|, \quad \mathcal{M}_2^-(\tilde{u}_x - \tilde{u})(0) \le C\rho^{1-\alpha}|x|. \tag{2.30}$$

For every kernel $K \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$, we have

$$\begin{aligned}
L(\tilde{u}_x - \tilde{u})(0) &= \int_{\mathbb{R}^n} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y)) K(y)\, dy \\
&= \int_{B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y)) K(y)\, dy + \int_{\mathbb{R}^n \setminus B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y)) K(y)\, dy.
\end{aligned}$$

Now we estimate the second term of right hand side: for $x \in B_{1/4}$

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^n \setminus B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y)) K(y)\, dy \\
&= \int_{\mathbb{R}^n} \tilde{u}(y)(K(y-x)\chi_{B_1^c}(y-x) - K(y)\chi_{B_1^c}(y))\, dy - (\tilde{u}(x) - \tilde{u}(0)) \int_{\mathbb{R}^n \setminus B_1} K(y)\, dy \\
&\le \int_{\mathbb{R}^n \setminus B_{1+|x|}} |\tilde{u}(y)||K(y-x) - K(y)|\, dy + \|\tilde{u}\|_{L^\infty(B_{1+|x|})} \int_{B_{1+|x|} \setminus B_{1-|x|}} K(y)\, dy + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x| \\
&\le C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|,
\end{aligned} \tag{2.31}$$

where in the first inequality we have used (2.27), and in the last we used that $|\nabla K(y)| \le (2-\sigma)\Lambda |y|^{-n-\sigma-1}$ and

$$
\begin{aligned}
|\tilde{u}(y)| &\le \|v_{i+1}\|_{L^\infty(\mathbb{R}^n)} + |R_\rho^{(1)}(y)| \\
&\le \frac{1}{M} + C\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} |V_\ell^{(1)}(\rho^{i+1-\ell} y)| \\
&\le \frac{1}{M} + C\sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} |\rho^{i+1-\ell} y|^{\sigma+\bar{\alpha}} \\
&\le \frac{1}{M} + C\rho^{\bar{\alpha}-\alpha}|y|^{\sigma+\bar{\alpha}}.
\end{aligned}
$$

Therefore, for every kernel $K \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$, we have

$$
\int_{\mathbb{R}^n} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y))K(y)\,\mathrm{d}y \le \int_{B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y))K(y)\,\mathrm{d}y + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|.
$$

Taking the supremum and using (2.30), we obtain

$$
-C\rho^{1-\alpha}|x| \le \mathcal{M}_2^+(\tilde{u}_x - \tilde{u}) \le \sup_K \int_{B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y))K(y)\,\mathrm{d}y + C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|.
$$

In particular, if we take the supremum over all kernels $K \in \mathscr{L}_0(\lambda, \Lambda, \sigma)$, we still have

$$
\sup_{\frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} \le K \le \frac{\Lambda(2-\sigma)}{|y|^{n+\sigma}}} \int_{B_1} (\delta\tilde{u}(x,y) - \delta\tilde{u}(0,y))K(y)\,\mathrm{d}y \ge -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|,
$$

which is equivalent to

$$
\Lambda N^+(x) - \lambda N^-(x) \ge -C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|.
$$

The same computation with $\mathcal{M}_2^-(\tilde{u}_x - \tilde{u})(0) \le C(\rho^{\bar{\alpha}-\alpha} + \frac{1}{M})|x|$ provides the other inequality. □

One may consider $\bar{w}_A = (C(1/M + \rho^{\bar{\alpha}-\alpha}))^{-1} w_A(rx)$, where C is the constant in (2.29). For every ε_1 small, we can choose r smaller so that

$$
\begin{aligned}
&\text{for every set } A: \ |w_A| \le 1 \quad \text{in } \mathbb{R}^n, \\
&\text{for every set } A: \ \mathcal{M}_2^+ w_A \ge -\varepsilon_1 \quad \text{in } B_1 \\
&\frac{\lambda}{\Lambda} N^-(x) - \varepsilon_1|x| \le N^+(x) \le \frac{\Lambda}{\lambda} N^-(x) + \varepsilon_1|x|.
\end{aligned} \tag{2.32}
$$

Note that w_A and $\bar{w}_A$ share the same Hölder exponent.

Lemma 2.14. *We have for $x \in B_{1/4}$,*

$$
N^+(x) \le C(1/M + \rho^{\bar{\alpha}-\alpha})|x|^{\bar{\alpha}}.
$$

Proof. It follows from exactly the same proof of Lemma 9.2 in [8]. □

Proof of Theorem 2.2. For $x \in B_{1/4}$, we have

$$
\begin{aligned}
&|-\Delta)^{\sigma/2}\tilde{v}_{i+1}(x) - (-\Delta)^{\sigma/2}\tilde{v}_{i+1}(0)| \\
&= C\left| N^+(x) - N^-(x) + \int_{\mathbb{R}^n \setminus B_1} (\delta\tilde{v}_{i+1}(x,y) - \delta\tilde{v}_{i+1}(0,y))K(y)\,\mathrm{d}y \right|
\end{aligned}
$$

$$\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})|x|^{\bar\alpha}+C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})|x|$$
$$\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})|x|^{\bar\alpha},$$

where in the first inequality we used Lemma 2.14, Lemma 2.13 and (2.31).

On the other hand, it follows from the computations in (2.28) and Lemma 2.4 that

$$|(-\Delta)^{\sigma/2}R^{(1)}_\rho(x)-(-\Delta)^{\sigma/2}R^{(1)}_\rho(0)|\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})|x|^{\bar\alpha}.$$

Thus,

$$|(-\Delta)^{\sigma/2}v_{i+1}(x)-(-\Delta)^{\sigma/2}v_{i+1}(0)|\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M})|x|^{\bar\alpha}.$$

It follows from Lemma 2.10, standard translation arguments and Schauder estimates for $(-\Delta)^{\sigma/2}$ that

$$\|v_{i+1}\|_{C^{\sigma+\bar\alpha}(B_1)}\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M}).$$

This finishes the proof of Theorem 2.2 provided that $\rho_0^{\bar\alpha-\alpha}\leq 1/(2C)$ and $M\geq 2C$. □

Lastly, let us discuss the case $0<\sigma_0\leq\sigma\leq 1$. In this case, the Evans–Krylov theorem in [8] does not provide any improvement with respect to the $C^{1,\alpha}$ estimate in [6]. However, we do not know how to use the incremental quotients method as in [6] to prove our Theorem 2.2. But we still can find some $\bar\alpha>0$ so that Theorem 2.2 holds. Recall that in the proof of Theorem 2.2 above, there are two places where we used $\sigma>1$:

(i): In (2.21), we used $\sigma\geq\sigma_0>1>\bar\alpha$ so that the integral there is universally bounded.
(ii): In (2.26), we have the gradient estimate for v_{i+1} when $\sigma\geq\sigma_0>1$. This was used in proving (2.31) in the proof of Lemma 2.13 and (2.32).

It is clear that the use in (i) is not essential, since we can assume that $\bar\alpha<\sigma_0$ when $0<\sigma_0\leq\sigma\leq 1$. The use in (ii) is not essential, either, since we can proceed using the Hölder estimates in [6] that

$$\|v_{i+1}\|_{C^\beta(B_{1/2})}\leq C(\rho^{\bar\alpha-\alpha}+\frac{1}{M}) \tag{2.33}$$

instead of (2.26), where $\beta\in(0,1)$ is a constant depending only on $n,\sigma_0,\lambda,\Lambda$. Consequently, the statement of Lemma 2.13 becomes

$$\frac{\lambda}{\Lambda}N^-(x)-C(\rho^{\bar\alpha-\alpha}+1/M)|x|^\beta\leq N^+(x)\leq\frac{\Lambda}{\lambda}N^-(x)+C(\rho^{\bar\alpha-\alpha}+1/M)|x|^\beta\ \forall\, x\in B_{1/4},$$

and (2.32) becomes

$$\frac{\lambda}{\Lambda}N^-(x)-\varepsilon_1|x|^\beta\leq N^+(x)\leq\frac{\Lambda}{\lambda}N^-(x)+\varepsilon_1|x|^\beta.$$

The same proof of Lemma 9.2 in [8] will give that there exists some $\bar\beta>0$ depending only on $\sigma_0,n,\lambda,\Lambda$ such that

$$N^+(x)\leq C(1/M+\rho^{\bar\alpha-\alpha})|x|^{\bar\beta}\quad\forall\, x\in B_{1/4},$$

and we will choose $\bar\alpha=\bar\beta$ (which might be smaller than the one in (1.5) when $\sigma<1$ if one consider the best possible one due to the $C^{1,\alpha}$ estimates in [6] even for σ very small).

Thus, we can prove that

Theorem 2.15. *For $\sigma_0\in(0,2)$ and $\sigma\in[\sigma_0,2)$, there exists a constant $\bar\alpha\in(0,1)$ depending only on n,σ_0,λ and Λ so that Theorem 2.2 holds.*

3. Schauder estimates

In this section, we will prove the Schauder estimates in Theorem 1.1. We start with a lemma. It follows quickly from comparison principles and we omit the proof here.

Lemma 3.1. *Suppose that every $K_a(y) \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$ with $\sigma \geq \sigma_0 > 0$, c_0 is a constant. Let u be the viscosity solution of*

$$\inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta u(x,y) K_a(y)\,\mathrm{d}y = c_0 \quad \text{in } B_1$$
$$u = g \quad \text{in } \mathbb{R}^n \setminus B_1.$$

Then there exists a constant C depending only on λ, Λ, n and σ_0 such that

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C(\|g\|_{L^\infty(\mathbb{R}^n\setminus B_1)} + |c_0|).$$

Proof of Theorem 1.1. The strategy of the proof is to find a sequence of approximation solutions which are sufficiently regular, and the error between the genuine solution and the approximation solutions can be controlled in a desired rate. We divide the proof into four steps.

Step 1: Normalization and rescaling.

Let w_0 be the viscosity solution of

$$I_0 w_0(x) := \inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta w_0(x,y) K_a(0,y)\,\mathrm{d}y - f(0) = 0 \quad \text{in } B_4$$
$$w_0 = u \quad \text{in } \mathbb{R}^n \setminus B_4.$$

Then by Lemma 3.1 we have that

$$\|w_0\|_{L^\infty(\mathbb{R}^n)} \leq C(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_5)}).$$

Thus by normalization, we may assume that

$$\|w_0\|_{L^\infty(\mathbb{R}^n)} \leq 1/2, \ \ \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_5)} \leq 1/2.$$

For some universal small positive constant $\gamma < 1$, which will be chosen later in (3.15), we may also assume that $|f(x) - f(0)| \leq \gamma |x|^\alpha$ and

$$\int_{\mathbb{R}^n} |K_a(x,y) - K_a(0,y)| \min(|y|^2, r^2) dy \leq \gamma |x|^\alpha r^{2-\sigma} \tag{3.1}$$

for all $a \in \mathcal{A}$, $r \in (0,1]$, $x \in B_5$. This can be achieved by the scaling for $s < 1$ small that if we let

$$\begin{aligned}
\tilde{K}_a(x,y) &= s^{n+\sigma} K_a(sx, sy) \in \mathscr{L}_2(\lambda, \Lambda, \sigma),\\
\tilde{u}(x) &= u(sx),\\
\tilde{f}(x) &= s^\sigma f(sx),
\end{aligned} \tag{3.2}$$

then we see that

$$\tilde{I}\tilde{u}(x) = \inf_{a\in\mathcal{A}} \tilde{L}_a \tilde{u}(x) = \tilde{f}(x) \quad \text{in } B_5,$$

where

$$\tilde{L}_a \tilde{u}(x) := \int_{\mathbb{R}^n} \delta\tilde{u}(x,y) \tilde{K}_a(x,y)\,\mathrm{d}y.$$

It follows that if we choose s sufficiently small, then

$$|\tilde{f}(x)-\tilde{f}(0)|\le M_f s^{\sigma+\alpha}|x|^\alpha\le\gamma|x|^\alpha\le 5\gamma,$$

and

$$\int_{\mathbb{R}^n}|\tilde{K}_a(x,y)-\tilde{K}_a(0,y)|\min(|y|^2,r^2)\mathrm{d}y\le 2\Lambda s^\alpha|x|^\alpha r^{2-\sigma}\le\gamma|x|^\alpha r^{2-\sigma}$$

for all $a\in\mathcal{A}$, $r\in(0,1]$, $x\in B_5$. Thus, we may consider the equation of $\tilde{u}$ instead.

Consequently, it follows from (3.1) that ($\|\cdot\|_*$ is defined in (A.1) in Appendix A

$$\|I-I_0\|_*\le 25\gamma.$$

Indeed, if $x\in B_5$, $h\in C^2(x)$, $\|h\|_{L^\infty(\mathbb{R}^n)}\le M$, $|h(y)-h(x)-(y-x)\cdot\nabla h(x)|\le\frac{M}{2}|x-y|^2$ for every $y\in B_1(x)$, we have

$$\begin{aligned}\|I-I_0\|_*&\le\sup_{x,a,h}\frac{1}{1+M}\int_{\mathbb{R}^n}|\delta h(x,y)||K_a(x,y)-K_a(0,y)|\,\mathrm{d}y\\&\le\sup_a\frac{M}{1+M}\left(\int_{B_1}|y|^2|K_a(x,y)-K_a(0,y)|+4\int_{\mathbb{R}^n\setminus B_1}|K_a(x,y)-K_a(0,y)|\right)\\&<5\gamma|x|^\alpha\le 25\gamma.\end{aligned}\tag{3.3}$$

Step 2: From now on, we denote

$\rho=\rho_0$ as the one in Theorem 2.2, which is a universal constant.

We claim that we can find a sequence of functions w_i, $i=0,1,2,\cdots$, such that for all i,

$$\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\sum_{\ell=0}^{i}\delta w_\ell(x,y)K_a(0,y)\mathrm{d}y=f(0)\quad\text{in }B_{4\cdot\rho^i},\tag{3.4}$$

and

$$(u-\sum_{l=0}^{i}w_\ell)(\rho^i x)=0\quad\text{for all }x\in\mathbb{R}^n\setminus B_4,\tag{3.5}$$

and

$$\begin{aligned}\|w_i\|_{L^\infty(\mathbb{R}^n)}&\le\rho^{(\sigma+\alpha)i},\\\|Dw_i\|_{L^\infty(B_{(4-\tau)\cdot\rho^i})}&\le c_2\rho^{(\sigma+\alpha-1)i}\tau^{-1},\\\|D^2w_i\|_{L^\infty(B_{(4-\tau)\cdot\rho^i})}&\le c_2\rho^{(\sigma+\alpha-2)i}\tau^{-2},\\{}[D^2w_i]_{C^{\sigma+\bar{\alpha}-2}(B_{(4-\tau)\cdot\rho^i})}&\le c_2\rho^{(\alpha-\bar{\alpha})i}\tau^{-4},\end{aligned}\tag{3.6}$$

and

$$\|u-\sum_{\ell=0}^{i}w_\ell\|_{L^\infty(\mathbb{R}^n)}\le\rho^{(\sigma+\alpha)(i+1)},\tag{3.7}$$

and

$$[u-\sum_{\ell=0}^{i}w_\ell]_{C^{\alpha_1}(B_{(4-3\tau)\cdot\rho^i})}\le 8c_1\rho^{(\sigma+\alpha-\alpha_1)i}\tau^{-4},\tag{3.8}$$

where τ is an arbitrary constant in $(0, 1]$, α_1 and c_1 are positive constants depending only on n, λ, Λ, γ_0 and $\bar{\alpha}$, and c_2 is the constant in (2.2).

Then Theorem 1.1 will follow from this claim and standard arguments. Indeed, we have, when $1 < \sigma + \alpha < 2$ and for $\rho^{i+1} \le |x| < \rho^i$,

$$
\begin{aligned}
&|u(x,0) - \sum_{\ell=0}^{\infty} w_\ell(0,0) - \sum_{\ell=0}^{\infty} \nabla_x w_\ell(0,0)\cdot x| \\
&\quad \le |u(x,0) - \sum_{\ell=0}^{i} w_\ell(x,0)| + |\sum_{\ell=0}^{i} w_\ell(x,0) - \sum_{\ell=0}^{i} w_\ell(0,0) - \sum_{\ell=0}^{i} \nabla_x w_\ell(0,0)\cdot x| \\
&\qquad + |\sum_{\ell=i+1}^{\infty} w_\ell(0,0)| + |\sum_{\ell=i+1}^{\infty} \nabla_x w_\ell(0,0)\cdot x| \\
&\quad \le \rho^{(\sigma+\alpha)(i+1)} + c_2|x|^2 \sum_{\ell=0}^{i} \rho^{(\sigma+\alpha-2)\ell} + \sum_{\ell=i+1}^{\infty} \rho^{(\sigma+\alpha)\ell} + |x| \sum_{\ell=i+1}^{\infty} c_2 \rho^{(\sigma+\alpha-1)\ell} \\
&\quad \le C_2|x|^{\sigma+\alpha}.
\end{aligned}
$$

When $\sigma + \alpha > 2$ and for $\rho^{i+1} \le |x| < \rho^i$,

$$
\begin{aligned}
&|u(x) - \sum_{\ell=0}^{\infty} w_\ell(0) - \sum_{\ell=0}^{\infty} Dw_\ell(0)\cdot x - \sum_{\ell=0}^{\infty} \frac{1}{2} x^T D^2 w_\ell(0)x| \\
&\quad \le |\sum_{\ell=0}^{i} w_\ell(x) - \sum_{\ell=0}^{i} w_\ell(0) - \sum_{\ell=0}^{i} Dw_\ell(0)\cdot x - \sum_{\ell=0}^{i} \frac{1}{2} x^T D^2 w_\ell(0)x| \\
&\qquad + |u(x) - \sum_{\ell=0}^{i} w_\ell(x)| + |\sum_{\ell=i+1}^{\infty} w_\ell(0)| + |\sum_{l=i+1}^{\infty} Dw_\ell(0)\cdot x| + \frac{1}{2}|\sum_{\ell=i+1}^{\infty} x^T D^2 w_\ell(0)x| \\
&\quad \le \rho^{(\sigma+\alpha)(i+1)} + 2c_2|x|^{\sigma+\bar{\alpha}} \sum_{\ell=0}^{i} \rho^{(\alpha-\bar{\alpha})\ell} + \sum_{\ell=i+1}^{\infty} \rho^{(\sigma+\alpha)\ell} + |x| \sum_{\ell=i+1}^{\infty} c_2 \rho^{(\sigma+\alpha-1)\ell} \\
&\qquad + |x|^2 \sum_{\ell=i+1}^{\infty} c_2 \rho^{(\sigma+\alpha-2)\ell} \\
&\quad \le C_3|x|^{\sigma+\alpha}.
\end{aligned}
$$

This proves the estimate (1.8).

Now we are left to prove this claim. Before we provide the detailed proof, we would like to first mention the idea and the structure of (3.4)–(3.8):

- Solving (3.4) and (3.5) inductively is how we construct this sequence of functions $\{w_i\}$.
- (3.7) will follow from the approximation lemmas in Appendix A, where (3.8) will be used.
- (3.6) will follow from (3.7), maximum principles and the recursive Evans–Krylov theorem, Theorem 2.2.

Step 3: Prove the claim for $i = 0$.

Let u be a viscosity solution of (1.1). It follows from the Hölder estimates in [6], standard scaling and covering (contributing at most a factor of $4/\tau$) arguments that there exist constants $\alpha_1 \in (0, 1)$, $c_1 > 0$, depending only on $n, \lambda, \Lambda, \gamma_0, \bar{\alpha}$, such that for $\tau \in (0, 1]$

$$
\|u\|_{C^{\alpha_1}(B_{4-\tau})} \le c_1 \tau^{-1-\alpha_1} \left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_4)}\right). \tag{3.9}
$$

Let w_0 be the one in Step 1 and c_2 be the constant in (2.2). Then by Theorem 2.2, standard scaling, translation and covering arguments that

$$\begin{aligned} &\|w_0\|_{L^\infty(\mathbb{R}^n)} \le 1, && \|Dw_0\|_{L^\infty(B_{4-\tau})} \le c_2\tau^{-1}, \\ &\|D^2 w_0\|_{L^\infty(B_{4-\tau})} \le c_2\tau^{-2}, && [D^2 w_0]_{C^{\sigma+\bar\alpha-2}(B_{4-\tau})} \le c_2\tau^{-4}. \end{aligned} \tag{3.10}$$

Let us set up to apply the approximation lemma, Lemma A.1, in Appendix A. Let $\varepsilon = \rho^3 \le \rho^{\sigma+\alpha}$ and $M = 1$. Let us fixed a modulus continuity $\omega_1(r) = r^{\alpha_1}$. Then for these ω_1, ε, M, there exist η_1 (small) and R (large) so that Lemma A.1 holds. We can assume that the rescaling in (3.2) make the equation hold in a very large ball containing B_{2R} and $|u(x) - u(y)| \le \omega_1(|x - y|)$ for every $x \in B_R \setminus B_4$ and $y \in \mathbb{R}^n \setminus B_4$. The latter one can be done due to (3.9). We will choose $\gamma < \eta_1/25$ in (3.15). Then by the rescaling in Step 1, we can conclude from Lemma A.1 that

$$\|u - w_0\|_{L^\infty(B_4)} \le \varepsilon \le \rho^{\sigma+\alpha},$$

and thus,

$$\|u - w_0\|_{L^\infty(\mathbb{R}^n)} \le \|u - w_0\|_{L^\infty(B_4)} \le \varepsilon \le \rho^{\sigma+\alpha}.$$

This proves that (3.4), (3.5), (3.6) and (3.7) hold for $i = 0$.

Let $v(x) = u(x) - w_0(x)$. Since $w_0 \in C^{\sigma+\bar\alpha}$, v is a solution of

$$\begin{aligned} I^{(0)} v &:= \inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta v(x, y) K_a(x, y) + \delta w_0(x, y) K_a(x, y) \mathrm{d}y - f(0) \\ &= f(x) - f(0) \quad \text{in } B_4. \end{aligned}$$

It is clear that $I^{(0)}$ is elliptic with respect to $\mathscr{L}_0(\lambda, \Lambda, \sigma)$. Moreover, for $x \in B_{4-2\tau}$,

$$\begin{aligned} |I^{(0)} 0| &:= |\inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta w_0(x, y) K_a(x, y) \mathrm{d}y - f(0)| \\ &= |\inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta(w_0(x, y)) K_a(x, y) \mathrm{d}y - \inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \delta(w_0(x, y)) K_a(0, y) \mathrm{d}y| \\ &\le \sup_{a\in\mathcal{A}} \int_{\mathbb{R}^n} |\delta w_0(x, y)||K_a(x, y) - K_a(0, y)| \mathrm{d}y \\ &\le \sup_{a\in\mathcal{A}} \left(\int_{B_\tau} c_2\tau^{-2}|y|^2 |K_a(x, y) - K_a(0, y)| \mathrm{d}y + 4 \int_{\mathbb{R}^n\setminus B_\tau} |K_a(x, y) - K_a(0, y)| \mathrm{d}y \right) \\ &\le \gamma\, (c_2 + 4)\, |x|^\alpha \tau^{-\sigma} \le \gamma 4\, (c_2 + 4)\, \tau^{-\sigma} \le \tau^{-\sigma}, \end{aligned} \tag{3.11}$$

where (3.10) was used in the second inequality, and (3.1) was used in the third inequality, and (3.15) was used in the last inequality. It follows from Hölder estimates established in [6], standard scaling and covering arguments (contributing at most a factor of $4/\tau$) we have

$$\|v\|_{C^{\alpha_1}(B_{4-3\tau})} \le c_1\tau^{-\alpha_1-1}(\tau^{-\sigma} + 4\gamma + 1) \le 8c_1\tau^{-4},$$

and thus,

$$[u - w_0]_{C^{\alpha_1}(B_{4-3\tau})} \le 8c_1\tau^{-4}.$$

This finishes the proof of (3.8) for $i = 0$.

Step 4: We assume all of (3.4), (3.5), (3.6), (3.7) and (3.8) hold up to $i \ge 0$, and we will show that they all hold for $i + 1$ as well.

Let

$$W(x)=\rho^{-(i+1)(\sigma+\alpha)}\left(u-\sum_{\ell=0}^{i}w_\ell\right)(\rho^{i+1}x),$$
$$v_\ell=\rho^{-(\sigma+\alpha)\ell}w_\ell(\rho^\ell x),$$

and

$$K^{(i+1)}(x,y)=\rho^{(n+\sigma)(i+1)}K(\rho^{i+1}x,\rho^{i+1}y).$$

Since $w_\ell\in C^{\sigma+\bar\alpha}$ for each ℓ, then W is a solution of

$$I^{(i+1)}W=\rho^{-(i+1)\alpha}f(\rho^{i+1}x)-\rho^{-(i+1)\alpha}f(0)\quad\text{in } B_{4/\rho},$$

where

$$\begin{aligned}I^{(i+1)}W&:=\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta W(x,y)+\sum_{\ell=0}^{i}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)\right)K_a^{(i+1)}(x,y)\mathrm{d}y-\rho^{-(i+1)\alpha}f(0)\\&=\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta W(x,y)+\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)(\sigma+\alpha)}\delta v_\ell(\rho^{i+1-\ell}x,\rho^{i+1-\ell}y)\right)K_a^{(i+1)}(x,y)\mathrm{d}y-\rho^{-(i+1)\alpha}f(0).\end{aligned}$$

It is clear that $I^{(i+1)}$ is elliptic with respect to $\mathscr{L}_0(\lambda,\Lambda,\sigma)$. Denote

$$\begin{aligned}I_0^{(i+1)}v&:=\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta v(x,y)+\sum_{\ell=0}^{i}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)\right)K_a^{(i+1)}(0,y)\mathrm{d}y-\rho^{-(i+1)\alpha}f(0)\\&=\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\left(\delta v(x,y)+\sum_{\ell=0}^{i}\rho^{-(i+1-\ell)(\sigma+\alpha)}\delta v_\ell(\rho^{i+1-\ell}x,\rho^{i+1-\ell}y)\right)K_a^{(i+1)}(0,y)\mathrm{d}y-\rho^{-(i+1)\alpha}f(0),\end{aligned}$$

which is also elliptic with respect to $\mathscr{L}_0(\lambda,\Lambda,\sigma)$. Let v_{i+1} be the solution of

$$\begin{aligned}I_0^{(i+1)}v_{i+1}&=0\quad\text{in } B_4\\v_{i+1}&=W\quad\text{in } \mathbb{R}^n\setminus B_4.\end{aligned}$$

It follows that

$$\|v_{i+1}\|_{L^\infty(\mathbb{R}^n)}\le\|W\|_{L^\infty(\mathbb{R}^n)}\le 1. \tag{3.12}$$

Indeed, we first know from the nonlocal Evans–Krylov theorem that $v_{i+1}\in C^{\sigma+\bar\alpha}$ and thus $I_0^{(i+1)}v_{i+1}$ can be calculated point-wisely. Since $I_0^{(i+1)}0=0$ which follows from (3.4), we have for $x\in B_4$,

$$\inf_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\delta v_{i+1}(x,y)K_a^{(i+1)}(0,y)\mathrm{d}y\le I_0^{(i+1)}v_{i+1}(x)\le\sup_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\delta v_{i+1}(x,y)K_a^{(i+1)}(0,y)\mathrm{d}y.$$

We also know from then boundary regularity in [7] that $v_{i+1}\in C(\overline{B}_4)$. Suppose that there exists $x_0\in B_4$ so that $v_{i+1}(x_0)=\max_{\overline{B}_4}v_{i+1}>\|W\|_{L^\infty(\mathbb{R}^n\setminus B_4)}$. Then

$$\sup_{a\in\mathcal{A}}\int_{\mathbb{R}^n}\delta v_{i+1}(x_0,y)K_a^{(i+1)}(0,y)\mathrm{d}y<0,$$

which is a contradiction to $I_0^{(i+1)}v_{i+1}(x_0)=0$. It follows from similar arguments that $v_{i+1}(x)\ge-\|W\|_{L^\infty(\mathbb{R}^n\setminus B_4)}$ for $x\in B_4$. This proves (3.12).

Again, by our induction hypothesis (3.4), it follows that for all $m = 0, 1, \cdots, i$,

$$\inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \left(\sum_{\ell=0}^{m} \rho^{-(m-\ell)(\sigma+\alpha)} \delta v_\ell(\rho^{m-\ell}x, \rho^{m-\ell}y) \right) K_a^{(m)}(0, y)\mathrm{d}y = \rho^{-m\alpha} f(0) \quad \text{in } B_4.$$

It follows from Theorem 2.2 and standard scaling arguments that

$$\begin{aligned} \|Dv_{i+1}\|_{L^\infty(B_{4-\tau})} &\le c_2\tau^{-1}, \\ \|D^2 v_{i+1}\|_{L^\infty(B_{4-\tau})} &\le c_2\tau^{-2}, \\ [D^2 v_{i+1}]_{C^{\sigma+\bar\alpha-2}(B_{4-\tau})} &\le c_2\tau^{-4}. \end{aligned}$$

We want to apply Lemma A.2 to the equations of W and v_{i+1} so that we have $|W - v_{i+1}| \le \rho^{\sigma+\alpha}$ in B_4.

First of all, $|W| \le 1$ in $\mathbb{R}^n$, $W \equiv 0$ in $\mathbb{R}^n \setminus B_{4/\rho}$, and $[W]_{C^{\alpha_1}(B_{(4-3\tau)/\rho})} \le 8c_1\rho^{\alpha_1-\sigma-\alpha}\tau^{-4} \le 8c_1\rho^{-3}\tau^{-4}$. Secondly, it follows from similar computations in (2.28), and making use of (3.6) and Lemma 2.5 that

$$[L_a^{(i+1)} R_\rho]_{C^{\bar\alpha}(B_4)} \le M_0 \quad \forall\, a \in \mathcal{A},$$

where M_0 is a universal constant independent of i,

$$L_a^{(i+1)} v = \int_{\mathbb{R}^n} \delta v(x, y) K_a^{(i+1)}(0, y)\,\mathrm{d}y, \qquad R_\rho(x) = \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} v_\ell(\rho^{i+1-\ell}x).$$

Lastly, we are going to show that we can choose γ sufficiently small so that

$$\|I^{(i+1)} - I_0^{(i+1)}\|_* \le \eta_2 \quad \text{in } B_4 \tag{3.13}$$

and we can apply Lemma A.2, where η_2 is the one in (A.2) with $\varepsilon = \rho^3 \le \rho^{\sigma+\alpha}$, M_0 as above, $M_1 = 1$, $M_2 = 8c_1\rho^{-3}$, $M_3 = c_2$.

For $x \in B_4$, $h \in C^2(x)$, $\|h\|_{L^\infty(\mathbb{R}^n)} \le M$, $|h(y) - h(x) - (y - x)\cdot\nabla h(x)| \le \frac{M}{2}|x - y|^2$ for every $y \in B_1(x)$, we have

$$\begin{aligned} &\|I^{(i+1)} - I_0^{(i+1)}\|_* \\ &\quad \le \sup_{a,h,x} \Big|\int_{\mathbb{R}^n} \delta h(x, y)(K_a^{(i+1)}(x, y) - K_a^{(i+1)}(0, y))\mathrm{d}y\Big| \\ &\qquad + \sum_{\ell=0}^{i} \sup_{a\in\mathcal{A}} \Big|\int_{\mathbb{R}^n} \rho^{-(i+1)(\sigma+\alpha)} \delta w_\ell(\rho^{i+1}x, \rho^{i+1}y)(K_a^{(i+1)}(x, y) - K_a^{(i+1)}(0, y))\mathrm{d}y\Big| \\ &\quad = I_1 + I_2. \end{aligned}$$

It follows from the same computations in (3.3) that

$$|I_1| \le 25\gamma.$$

For $a \in \mathcal{A}$, $\ell = 0, 1, \cdots, i$ and for $x \in B_{(4-2\tau)/\rho}$, we have, similar to (3.11),

$$\begin{aligned} &\Big|\int_{\mathbb{R}^n} \delta w_\ell(\rho^{i+1}x, \rho^{i+1}y)(K_a^{(i+1)}(0, y) - K_a^{(i+1)}(x, y))\mathrm{d}y\Big| \\ &\quad \le \rho^{\sigma(i+1)} \int_{\mathbb{R}^n} |\delta w_\ell(\rho^{i+1}x, y)||K_a(0, y) - K_a(\rho^{i+1}x, y)|\mathrm{d}y \\ &\quad \le \rho^{\sigma(i+1)} \int_{B_{\rho^\ell\tau}} c_2\rho^{(\sigma+\alpha-2)\ell}\tau^{-2}|y|^2|K_a(0, y) - K_a(\rho^{i+1}x, y)|\mathrm{d}y \end{aligned}$$

$$+\rho^{\sigma(i+1)} \int\limits_{\mathbb{R}^n\setminus B_{\rho^\ell\tau}} \rho^{(\sigma+\alpha)\ell} 4|K_a(0,y)-K_a(\rho^{i+1}x,y)|\mathrm{d}y$$

$$\le \rho^{(\sigma+\alpha)(i+1)}\gamma(c_2+4)\rho^{\alpha\ell}\tau^{-\sigma}|x|^\alpha, \tag{3.14}$$

where we used (3.6) in the second inequality. We choose γ such that

$$\left(25+(c_2+4)4\sum_{\ell=0}^{\infty}\rho^{\alpha\ell}\right)\gamma \le \min(\eta_1/25,\eta_2). \tag{3.15}$$

It follows that (3.13) holds (here we can choose $\tau=1$). By Lemma A.2 we have that

$$\|W-v_{i+1}\|_{L^\infty(\mathbb{R}^n)} = \|W-v_{i+1}\|_{L^\infty(B_4)} \le \varepsilon \le \rho^{\sigma+\alpha}.$$

Let

$$w_{i+1}(x) = \rho^{(\sigma+\alpha)(i+1)} v_{i+1}(\rho^{-(i+1)}x).$$

Thus, we have shown in the above that all of (3.4), (3.5), (3.6), (3.7) hold for $i+1$. In the following, we shall show that (3.8) hold for $i+1$ as well. Let

$$V = W - v_{i+1} = \rho^{-(i+1)(\sigma+\alpha)}\left(u-\sum_{\ell=0}^{i+1} w_\ell\right)(\rho^{i+1}x).$$

Thus, for $x\in B_4$

$$I^{(i+1)}V := \inf_{a\in\mathcal{A}} \int\limits_{\mathbb{R}^n} [\delta V(x,y) + \sum_{\ell=0}^{i+1}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)]K_a^{(i+1)}(x,y)\mathrm{d}y - \rho^{-(i+1)\alpha}f(0)$$

$$= \rho^{-(i+1)\alpha} f(\rho^{i+1}x) - \rho^{-(i+1)\alpha} f(0).$$

Moreover, for $x\in B_{4-2\tau}$,

$$|I^{(i+1)}0| = |\inf_{a\in\mathcal{A}} \int\limits_{\mathbb{R}^n} [\sum_{\ell=0}^{i+1}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)]K_a^{(i+1)}(x,y)\mathrm{d}y - \rho^{-(i+1)\alpha}f(0)|$$

$$= |\inf_{a\in\mathcal{A}} \int\limits_{\mathbb{R}^n} [\sum_{\ell=0}^{i+1}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)]K_a^{(i+1)}(x,y)\mathrm{d}y$$

$$- \inf_{a\in\mathcal{A}} \int\limits_{\mathbb{R}^n} [\sum_{\ell=0}^{i+1}\rho^{-(i+1)(\sigma+\alpha)}\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)]K_a^{(i+1)}(0,y)\mathrm{d}y|$$

$$\le \sup_{a\in\mathcal{A}} \sum_{l=0}^{i+1} \int\limits_{\mathbb{R}^n} \rho^{-(i+1)(\sigma+\alpha)}|\delta w_\ell(\rho^{i+1}x,\rho^{i+1}y)||K_a^{(i+1)}(x,y)-K_a^{(i+1)}(0,y)|\mathrm{d}y$$

$$\le \eta_2\tau^{-\sigma},$$

where in the last inequality we have used (3.14) and the choice of η_2 in (3.15). Thus, by standard scaling and covering arguments,

$$[V]_{C^{\alpha_1}(B_{4-3\tau})} \le 8c_1\tau^{-4}.$$

Hence, (3.8) holds for $i+1$.

This finishes the proof of the claim in Step 2. Therefore, the proof of Theorem 1.1 is completed. □

Remark 3.2. In the step of approximation, one cannot use

$$\tilde{I}_0^{(i+1)}v := \inf_{a\in\mathcal{A}} \int_{\mathbb{R}^n} \left(\delta v(x,y) + \sum_{\ell=0}^{i} \rho^{-(i+1-\ell)(\sigma+\alpha)} \delta v_\ell(0, \rho^{i+1-\ell} y) \right) K_a^{(i+1)}(0,y)\mathrm{d}y - \rho^{-(i+1)\alpha} f(0)$$

to approximate $I^{(i+1)}W$, since one can check that $\tilde{I}_0^{(i+1)}$ will not be close to $I^{(i+1)}$. This is the main reason why we need Theorem 2.2.

Remark 3.3. In the case of $\sigma \geq \sigma_0 > 0$ and $\sigma + \bar{\alpha} \leq 2 - \gamma_0$ for some $\gamma_0 > 0$, our approximation solutions $\{w_\ell\}$ are of only $C^{\sigma+\bar{\alpha}}$ but may not be C^2. Thus, instead of (1.7), we need the following (stronger) assumption on K_a:

$$\int_{\mathbb{R}^n} |K_a(x,y) - K_a(0,y)| \min(|y|^{\sigma+\bar{\alpha}}, r^{\sigma+\bar{\alpha}}) dy \leq \Lambda |x|^\alpha r^{\bar{\alpha}}, \tag{3.16}$$

which will be used in (3.11) and (3.14). Then, with the help of Theorem 2.15, for $|\sigma + \bar{\alpha} - 1| \geq \gamma_0$, $\alpha \in (0, \bar{\alpha})$ and $|\sigma + \alpha - 1| \geq \varepsilon_0$, the same proof shows that the Schauder estimate (1.8) holds under the conditions (3.16) and (1.6), where the constant C there will additionally depend on σ_0.

Let $\sigma_0 \in (0,2)$. A unified Hölder condition on the kernels K for all $\sigma \in [\sigma_0, 2)$, which is slightly stronger than both (1.7) and (3.16), would be

$$\int_{B_{2r}\setminus B_r} |K(x,y) - K(0,y)| dy \leq (2-\sigma)\Lambda |x|^\alpha r^{-\sigma} \tag{3.17}$$

for all $r > 0$, $x \in B_5$.

Combining Theorem 1.1 and Remark 3.3, we have this corollary.

Corollary 3.4. *Let $\sigma_0 \in (0,2)$. There exists $\bar{\alpha} \in (0,1)$ depending only on n, λ, Λ and σ_0 such that the following statement holds: Assume every $K_a(x,y) \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$ satisfies (3.17) with $\sigma \in [\sigma_0, 2)$, $\alpha \in (0, \bar{\alpha})$, $|\sigma + \bar{\alpha} - j| \geq \gamma_0 > 0$ and $|\sigma + \alpha - j| \geq \varepsilon_0 > 0$ for $j = 1, 2$. Suppose that f satisfies (1.6). If u is a bounded viscosity solution of (1.1), then there exists a polynomial $P(x)$ of degree $[\sigma + \alpha]$ such that (1.8) holds for $x \in B_1$, where C in (1.8) is a positive constant depending only on $\lambda, \Lambda, n, \sigma_0, \bar{\alpha}, \alpha, \varepsilon_0$ and γ_0.*

An application of our Schauder estimates is another proof of the following Evans–Krylov type estimates for viscosity solutions of nonlocal fully nonlinear parabolic equations:

$$u_t(x,t) = \inf_{a\in\mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta u(x,y;t) K_a(y)\,\mathrm{d}y \right\} \quad \text{in } B_2 \times (-2, 0], \tag{3.18}$$

where $\delta u(x,y;t) = u(x+y,t) + u(x-y,t) - 2u(x,t)$, $\mathcal{A}$ is an index set, and each $K_a \in \mathscr{L}_2(\lambda, \Lambda, \sigma)$. These estimates for more general nonlocal parabolic equations have been established by H. Chang Lara and G. Davila [12]. The definition of viscosity solutions to nonlocal parabolic equations and their many properties can be found in [9,10].

Theorem 3.5. *Let $u : \mathbb{R}^n \times [-2, 0] \to \mathbb{R}$ be a viscosity solution of (3.18). Suppose that u is Lipschitz continuous in t in $(\mathbb{R}^n \setminus B_2) \times [-2, 0]$ and $\|\mathcal{M}_0^{\pm} u(\cdot, -2)\|_{L^\infty(\mathbb{R}^n)} \leq C_0$. Then there exists $\bar{\beta} \in (0,1)$ depending only on n, λ, Λ such that for $\sigma + \bar{\beta} - 2 \geq \gamma_0 > 0$ we have*

$$\|u_t\|_{C^{\bar{\alpha}}_{x,t}(B_1\times[-1,0])} + \|\nabla_x^2 u\|_{C^{\bar{\alpha}}_{x,t}(B_1\times[-1,0])} \leq C(\|u\|_{L^\infty(\mathbb{R}^n\times[-2,0])} + \|u_t\|_{L^\infty((\mathbb{R}^n\setminus B_2)\times[-2,0])} + C_0), \tag{3.19}$$

where $\bar{\alpha} = \gamma_0\bar{\beta}/2$ and C is a positive constant depending only on n, λ, Λ and γ_0.

Proof. It follows from Theorem 6.2 in [9] and Theorem 4.1 in [10] that there exists some $\bar\beta \in (0,1)$ depending only on n, λ, Λ such that

$$\nabla_{x,t} u \in C^{\bar\beta}_{x,t}(B_1 \times [-1,0]).$$

In particular, the right hand side of (3.18) is Hölder in x. By the Schauder estimates in Theorem 1.1 (and adjusting $\bar\beta$ if necessary), when $\sigma + \bar\beta - 2 \ge \gamma_0 > 0$, we have for all $t \in [-1,0]$

$$\nabla_x^2 u(\cdot,t) \in C_x^{\sigma+\bar\beta-2}(B_1).$$

By Lemma 3.1 on page 78 in [19], we have for all $x \in B_1$,

$$\nabla_x^2 u(x,\cdot) \in C_t^{\bar\beta(\sigma+\bar\beta-2)/(\sigma+\bar\beta-1)}([-1,0]) \subset C_t^{\bar\alpha}([-1,0]).$$

Thus, $\nabla_x^2 u \in C^{\bar\alpha}_{x,t}(B_1 \times [-1,0])$, and the estimate (3.19) follows from the estimates in Theorem 6.2 in [9], Theorem 4.1 in [10] and the Schauder estimates we proved. This finishes the proof. □

The estimate (3.19) is not written in the scaling invariant form for the purpose of convenience in its proof. Note that Example 2.4.1 in [10] shows that the assumption of the Lipschitz continuity on u in $(\mathbb{R}^n \setminus B_2) \times [-2,0]$ is necessary to obtain Hölder continuity of u_t in $B_1 \times [-1,0]$. The constant C in (3.19) does not depend on σ, and thus, does not blow up as $\sigma \to 2$.

One also can replace the condition on the initial data $u(\cdot,-2)$ in Theorem 3.5 by the following global Lipschitz type assumption:

$$[u]_{C^{0,1}((t_1,t_2];L^1(\omega_\sigma))} := \sup_{(t-\tau,t]\subset(t_1,t_2]} \frac{\|u(\cdot,t)-u(\cdot,t-\tau)\|_{L^1(\omega_\sigma)}}{\tau} < \infty, \tag{3.20}$$

where $\|v\|_{L^1(\omega_\sigma)} = \int_{\mathbb{R}^n} |v(y)| \min(1,|y|^{-n-\sigma}) \mathrm{d}y$.

Theorem 3.6. *Let $u : \mathbb{R}^n \times [-2,0] \to \mathbb{R}$ be a viscosity solution of (3.18) and satisfy (3.20). Then there exists $\bar\beta \in (0,1)$ depending only on n, λ, Λ such that for $\sigma + \bar\beta - 2 \ge \gamma_0 > 0$ we have*

$$\|u_t\|_{C^{\bar\alpha}_{x,t}(B_1\times[-1,0])} + \|\nabla_x^2 u\|_{C^{\bar\alpha}_{x,t}(B_1\times[-1,0])} \le C(\|u\|_{L^\infty(\mathbb{R}^n\times[-2,0])} + [u]_{C^{0,1}((t_1,t_2];L^1(\omega_\sigma))}),$$

where $\bar\alpha = \gamma_0\bar\beta/2$ and C is a positive constant depending only on n, λ, Λ and γ_0.

Proof. It is the same as the proof of Theorem 3.5, except that we use Corollary 7.2 and Corollary 7.4 in [11] instead of Theorem 6.2 in [9] and Theorem 4.1 in [10], □

Conflict of interest statement

We declare that there is no conflict of interest.

Acknowledgements

We would like to thank Professor Luis Silvestre for many useful discussions and suggestions. We also thank Professor YanYan Li for his interests and constant encouragement.

Appendix A. Approximation lemmas

Our proof of Schauder estimates uses perturbative arguments, and we need the following two approximation lemmas, which are variants of Lemma 7 in [6]. We will do a few modifications for our own purposes, and we decide to include them in this appendix for completeness and convenience.

To start with, we recall some definitions and notations about nonlocal elliptic operators, which can be found in [6,7]. Let $\sigma_0 \in (0,2)$ be fixed, and $\omega(y) = (1+|y|^{n+\sigma_0})^{-1}$. We say $u \in L^1(\mathbb{R}^n,\omega)$ if $\int_{\mathbb{R}^n} |u(y)|\omega(y)\mathrm{d}y < \infty$. Let Ω be an open subset of $\mathbb{R}^n$. Let us recall Definition 21 in [7] for nonlocal operators. A nonlocal operator I in Ω is a rule that assigns a function u to a value $I(u,x)$ at every point $x \in \Omega$ satisfying the following assumptions:

- $I(u,x)$ is well-defined as long as $u \in C^2(x)$ and $u \in L^1(\mathbb{R}^n, \omega)$.
- If $u \in C^2(\Omega) \cap L^1(\mathbb{R}^n, \omega)$, then $I(u,x)$ is continuous in Ω as a function of x.

Here $u \in C^2(x)$ we mean that there is a quadratic polynomial p such that $u(y) = p(y) + o(|y-x|^2)$ for y close to x. An operator is translation invariant if $\tau_z I u = I(\tau_z u)$ where τ_z is the translation operator $\tau_z u(x) = u(x-z)$.

Given such a nonlocal operator I, one can defined a norm $\|I\|$ as in Definition 22 in [7]. We also define a (weaker) norm $\|I\|_*$ for our own purpose:

$$\begin{aligned}\|I\|_* := \sup\{&|I(u,x)|/(1+M) : x \in \Omega, u \in C^2(x), \|u\|_{L^\infty(\mathbb{R}^n)} \le M,\\ &|u(y) - u(x) - (y-x)\cdot\nabla u(x)| \le \frac{M}{2}|x-y|^2 \text{ for every } y \in B_1(x)\}.\end{aligned} \tag{A.1}$$

We say that a nonlocal operator I is uniformly elliptic with respect to $\mathscr{L}_0(\lambda, \Lambda, \sigma)$, which will be written as $\mathscr{L}_0(\sigma)$ for short, if

$$\mathcal{M}^-_{\mathscr{L}_0(\sigma)} v(x) \le I(u+v, x) - I(u,x) \le \mathcal{M}^+_{\mathscr{L}_0(\sigma)} v(x),$$

where

$$\mathcal{M}^-_{\mathscr{L}_0(\sigma)} v(x) = \inf_{L \in \mathscr{L}_0(\sigma)} Lv(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\lambda \delta v(x,y)^+ - \Lambda \delta v(x,y)^-}{|y|^{n+\sigma}} \mathrm{d}y$$

$$\mathcal{M}^+_{\mathscr{L}_0(\sigma)} v(x) = \sup_{L \in \mathscr{L}_0(\sigma)} Lv(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda \delta v(x,y)^+ - \lambda \delta v(x,y)^-}{|y|^{n+\sigma}} \mathrm{d}y.$$

It is also convenient to define the limit operators when $\sigma \to 2$ as

$$\mathcal{M}^-_{\mathscr{L}_0(2)} v(x) = \lim_{\sigma \to 2} \mathcal{M}^-_{\mathscr{L}_0(\sigma)} v(x)$$
$$\mathcal{M}^+_{\mathscr{L}_0(2)} v(x) = \lim_{\sigma \to 2} \mathcal{M}^+_{\mathscr{L}_0(\sigma)} v(x).$$

It has been explained in [7] that $\mathcal{M}^+_{\mathscr{L}_0(2)}$ is a second order uniformly elliptic operator, whose ellipticity constants $\tilde{\lambda}$ and $\tilde{\Lambda}$ depend only λ, Λ and the dimension n. Moreover, $\mathcal{M}^+_{\mathscr{L}_0(2)} v \le \mathcal{M}^+(\nabla^2 v)$, where $\mathcal{M}^+(\nabla^2 v)$ is the second order Pucci operator with ellipticity constants $\tilde{\lambda}$ and $\tilde{\Lambda}$. Similarly, we also have corresponding relations for $\mathcal{M}^-_{\mathscr{L}_0(2)}$.

Our approximation lemmas will be proved by compactness arguments, where we need the concepts of the weak convergence of nonlocal operators in Definition 41 in [7]. We say that a sequence of nonlocal operators $I_k \rightharpoonup I$ weakly in Ω if, for every $x_0 \in \Omega$ and for every function v of the form

$$v(x) = \begin{cases} p(x) & \text{if } |x - x_0| \le r; \\ u(x) & \text{if } |x - x_0| > r, \end{cases}$$

where p is a polynomial of degree two and $u \in L^1(\mathbb{R}^n, \omega)$, we have $I_k(v,x) \to I(v,x)$ uniformly in $B_{r/2}(x_0)$.

Lemma A.1. *For some $\sigma \ge \sigma_0 > 0$ we consider nonlocal operators I_0, I_1 and I_2 uniformly elliptic with respect to $\mathscr{L}_0(\sigma)$. Assume also that I_0 is translation invariant and $I_0(0) = 1$.*

Given $M > 0$, a modulus of continuity ω_1 and $\varepsilon > 0$, there exists η_1 (small, independent of σ) and R (large, independent of σ) so that if u, v, I_0, I_1 and I_2 satisfy

$$I_0(v,x) = 0, \quad I_1(u,x) \ge -\eta_1, \quad I_2(u,x) \le \eta_1 \quad \textit{in } B_1$$

in viscosity sense, and

$$\|I_1 - I_0\|_* \le \eta_1, \quad \|I_2 - I_0\|_* \le \eta_1 \quad \textit{in } B_1,$$

and

$$u = v \quad \text{in } \mathbb{R}^n \setminus B_1,$$
$$|u(x)| \le M \quad \text{in } \mathbb{R}^n,$$
$$|u(x) - u(y)| \le \omega_1(|x-y|) \quad \text{for every } x \in B_R \setminus B_1 \text{ and } y \in \mathbb{R}^n \setminus B_1,$$

then $|u - v| \le \varepsilon$ *in* B_1.

Proof. It follows from the proof of Lemma 7[1] in [7] with modifications. We argue by contradiction. Suppose the above lemma was false. Then there would be sequences σ_k, $I_0^{(k)}$, $I_1^{(k)}$, $I_2^{(k)}$, η_k, u_k, v_k such that $\sigma_k \to \sigma \in [\sigma_0, 2]$, $\eta_k \to 0$ and all the assumptions of the lemma are valid, but $\sup_{B_1} |u_k - v_k| \ge \varepsilon$.

Since $I_0^{(k)}$ is a sequence of uniformly elliptic translation invariant operators with respect to $\mathscr{L}(\sigma_k)$, by Theorem 42 in [7] that we can take a subsequence, which is still denoted as $I_0^{(k)}$, that converges weakly to some nonlocal operator I_0, and I_0 is also translation invariant, and uniformly elliptic with respect to the class $\mathscr{L}_0(\sigma)$.

It follows from the boundary regularity Theorem 32 in [7] that u_k and v_k have a modulus of continuity, uniform in k, in the closed unit ball $\overline{B}_1$. Thus, u_k and v_k have a uniform (in k) modulus of continuity on B_{R_k} with $R_k \to \infty$. We can subsequences of $\{u_k\}$ and $\{v_k\}$, which will be still denoted as $\{u_k\}$ and $\{v_k\}$, which converges locally uniformly in $\mathbb{R}^n$ to u and v, respectively. Moreover, $u = v$ in $\mathbb{R}^n \setminus B_1$, and $\sup_{B_1} |u - v| \ge \varepsilon$.

In the following, we are going to show that

$$I_0(u, x) = 0 = I_0(v, x) \quad \text{in } B_1, \tag{A.2}$$

from which we can conclude that $u \equiv v$ in B_1, since I_0 is translation invariant. But we know that $\sup_{B_1} |u - v| \ge \varepsilon$. This reaches a contradiction.

The second equality of (A.2) follows from Lemma 5 in [7]. The first equality actually follows almost identically from the proof of Lemma 5 in [7]: we only need to notice that the sequence $\{u_k\}$ is uniformly bounded by M, and thus the conditions that $I_1^{(k)}(u_k, x) \ge -\eta_k$, $I_2^{(k)}(u_k, x) \le \eta_k$, $\|I_1^{(k)} - I_0^{(k)}\|_* \to 0$ and $\|I_2^{(k)} - I_0^{(k)}\|_* \to 0$ are sufficient to show $I_0(u, x) = 0$ in B_1 as in the proof of Lemma 5 in [7]. □

Lemma A.2. *For some* $\sigma \ge \sigma_0 > 0$ *we consider nonlocal operators* I_0, I_1 *and* I_2 *uniformly elliptic with respect to* $\mathscr{L}_0(\sigma)$. *Assume also that*

$$I_0 v(x) := \inf_{a \in \mathcal{A}} \left\{ \int_{\mathbb{R}^n} \delta v(x, y) K_a(y) \mathrm{d}y + h_a(x) \right\} \quad \text{in } B_4,$$

where each $K_a \in \mathscr{L}_2(\sigma)$ *and for some constant* $\beta \in (0, 1)$,

$$[h_a]_{C^\beta(B_4)} \le M_0 \quad \text{and} \quad \inf_{a \in \mathcal{A}} h_a(x) = 0 \; \forall\, x \in B_4.$$

Given $M_0, M_1, M_2, M_3 > 0$, $R_0 > 5$, $\beta, \nu \in (0, 1)$, *and* $\varepsilon > 0$, *there exists* η_2 *(small, independent of* σ*) so that if* u, v, I_0, I_1 *and* I_2 *satisfy*

$$I_0(v, x) = 0, \quad I_1(u, x) \ge -\eta_2, \quad I_2(u, x) \le \eta_2 \quad \text{in } B_4,$$

in viscosity sense, and

$$\|I_1 - I_0\|_* \le \eta_2, \quad \|I_2 - I_0\|_* \le \eta_2 \quad \text{in } B_4,$$

and

$$u = v \quad \text{in } \mathbb{R}^n \setminus B_4,$$
$$u \equiv 0 \quad \text{in } \mathbb{R}^n \setminus B_{R_0},$$

[1] The statements of Lemma 7 and Lemma 8 in [7] should be read under the condition that I_0 is translation invariant (see [25]), which does not affect their applications in [7].

$$
\begin{aligned}
|u| &\le M_1 \quad \text{in } \mathbb{R}^n,\\
[u]_{C^{\nu}(B_{R_0-\tau})} &\le M_2\tau^{-4} \quad \forall\, \tau\in(0,1),\\
\|v\|_{C^{\sigma+\beta}(B_{4-\tau})} &\le M_3\tau^{-4} \quad \forall\, \tau\in(0,1),
\end{aligned}
$$

then $|u-v|\le\varepsilon$ *in* B_4.

Proof. This lemma can be proved similarly to Lemma A.1. Suppose the above lemma was false. Then there would be sequences σ_k, $I_0^{(k)}$, $I_1^{(k)}$, $I_2^{(k)}$, η_k, u_k, v_k such that $\sigma_k\to\sigma\in[\sigma_0,2]$, $\eta_k\to 0$ and all the assumptions of the lemma are valid, but $\sup_{B_1}|u_k-v_k|\ge\varepsilon$.

By our assumptions, it is clear that, up to a subsequence, u_k converges locally uniformly in B_{R_0}. Since $u_k\equiv 0$ in $\mathbb{R}^n\setminus B_{R_0}$, it converges almost everywhere to some function u in $\mathbb{R}^n$. Since v_k is bounded and has a modulus continuity on $B_5\setminus B_4$, then by the boundary regularity Theorem 32 in [7], there is another modulus continuity that extends to the closed unit ball $\overline{B}_4$, and thus, v_k converges uniformly in $\overline{B}_4$, as well as in $C_{loc}^{\sigma+\beta-\mu}(B_4)$ for any arbitrarily small $\mu>0$. Therefore, v_k converges to some function $v\in C_{loc}^{\sigma+\beta-\mu}(B_4)$ almost everywhere in $\mathbb{R}^n$. Moreover, $u=v$ in $\mathbb{R}^n\setminus B_4$, and $\sup_{B_4}|u-v|\ge\varepsilon$.

We are going to show that there exists a subsequence of $\{I_0^{(k)}\}$, which is still denoted as $I_0^{(k)}$, that converges weakly in B_4 to some nonlocal operator I_0, and I_0 is uniformly elliptic with respect to the class $\mathscr{L}_0(\sigma)$. Then it follows from the proof of (A.2) that u and v solve the same equation $I_0(u,x)=I_0(v,x)=0$ in B_4 in viscosity sense. Since $v\in C_{loc}^{\sigma+\beta-\mu}(B_4)$ is a classical solution and $u=v$ in $\mathbb{R}^n\setminus B_4$, we have $u=v$ in B_4, which is a contradiction.

The proof of that there exists a subsequence of $\{I_0^{(k)}\}$ weakly converges in B_4 will basically follow from the proofs of Lemma 6 and Theorem 42 in [7].

Claim 1. Let φ be a function

$$
\varphi(x)=\begin{cases} p(x) & \text{in } B_r\\ \Phi(x) & \text{in } \mathbb{R}^n\setminus B_r,\end{cases}
$$

where $r>0$, $p(x)$ is a second order polynomial, and $\Phi\in L^1(\mathbb{R}^n,\omega)$. Then there exists a subsequence $\{I_0^{(k_j)}\}$ such that $f_{k_j}(x):=I_0^{(k_j)}\varphi(x)$ converges uniformly in $B_{r/2}$.

Proof of Claim 1. Since $I_0^{(k)}(0)=0$, by uniformly ellipticity, f_k is uniformly bounded in $\overline{B}_{r/2}$. We are going to find a uniform modulus of continuity for f_k in $\overline{B}_{r/2}$ so that Claim 1 follows from Arzela–Ascoli theorem.

Recall $\tau_z\varphi(x)=\varphi(x+z)$. Given $x,y\in B_{r/2}$ with $|x-y|<r/8$, we have

$$
f_k(x)-f_k(y)\le \mathcal{M}^{+}_{\mathscr{L}(\sigma_k)}(v-\tau_{y-x}v,x)+M_0|x-y|^{\beta},
$$

where the first term has a modulus of continuity depends on φ but not $I_0^{(k)}$ as shown in the proof of Lemma 6 in [7]. This finishes the proof of Claim 1.

As long as we have Claim 1, it follows from the proof of Theorem 42 in [7] identically that there exists a subsequence of $\{I_0^{(k)}\}$, which is still denoted as $I_0^{(k)}$, that converges weakly in B_4 to some nonlocal operator I_0, and I_0 is uniformly elliptic with respect to the class $\mathscr{L}_0(\sigma)$. $\square$

References

[1] G. Barles, E. Chasseigne, C. Imbert, Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations, J. Eur. Math. Soc. 13 (2011) 1–26.

[2] B. Barrera, A. Figalli, E. Valdinoci, Bootstrap regularity for integro-differential operators, and its application to nonlocal minimal surfaces, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 13 (2014) 609–639.

[3] R.F. Bass, Regularity results for stable-like operators, J. Funct. Anal. 257 (2009) 2693–2722.

[4] L.A. Caffarelli, Interior a priori estimates for solutions of fully nonlinear equations, Ann. Math. (2) 130 (1989) 189–213.

[5] L.A. Caffarelli, X. Cabré, Fully Nonlinear Elliptic Equations, Am. Math. Soc. Colloq. Publ., vol. 43, American Mathematical Society, Providence, RI, 1995.
[6] L.A. Caffarelli, L. Silvestre, Regularity theory for fully nonlinear integro-differential equations, Commun. Pure Appl. Math. 62 (2009) 597–638.
[7] L.A. Caffarelli, L. Silvestre, Regularity results for nonlocal equations by approximation, Arch. Ration. Mech. Anal. 200 (2011) 59–88.
[8] L.A. Caffarelli, L. Silvestre, The Evans–Krylov theorem for non local fully non linear equations, Ann. Math. (2) 174 (2011) 1163–1187.
[9] H.A. Chang Lara, G. Davila, Regularity for solutions of non local parabolic equations, Calc. Var. Partial Differ. Equ. 49 (2014) 139–172.
[10] H.A. Chang Lara, G. Davila, Regularity for solutions of non local parabolic equations II, J. Differ. Equ. 256 (2014) 130–156.
[11] H.A. Chang Lara, G. Davila, Hölder estimates for non-local parabolic equations with critical drift, arXiv:1408.0676.
[12] H.A. Chang Lara, G. Davila, $C^{\sigma+\alpha}$ estimates for concave, non-local parabolic equations with critical drift, arXiv:1408.5149.
[13] H. Dong, D. Kim, On L_p-estimates for a class of nonlocal elliptic equations, J. Funct. Anal. 262 (3) (2012) 1166–1199.
[14] H. Dong, D. Kim, Schauder estimates for a class of non-local elliptic equations, Discrete Contin. Dyn. Syst. 33 (2013) 2319–2347.
[15] N. Guillen, R.W. Schwab, Aleksandrov–Bakelman–Pucci type estimates for integro-differential equations, Arch. Ration. Mech. Anal. 206 (1) (2012) 111–157.
[16] T. Jin, J. Xiong, Schauder estimates for solutions of linear parabolic integro-differential equations, arXiv:1405.0755.
[17] M. Kassmann, M. Rang, R.W. Schwab, Hölder regularity for integro-differential equations with nonlinear directional dependence, Indiana Univ. Math. J. 63 (2014) 1467–1498.
[18] D. Kriventsov, $C^{1,\alpha}$ interior regularity for nonlinear nonlocal elliptic equations with rough kernels, Commun. Partial Differ. Equ. 38 (2013) 2081–2106.
[19] O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and Quasilinear Equations of Parabolic Type, translated from the Russian by S. Smith, Transl. Math. Monogr., vol. 23, American Mathematical Society, Providence, RI, 1968.
[20] Y.Y. Li, L. Nirenberg, Estimates for elliptic system from composition material, Commun. Pure Appl. Math. 56 (2003) 892–925.
[21] R. Mikulevicius, H. Pragarauskas, On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem, Potential Anal. 40 (4) (2014) 539–563.
[22] X. Ros-Oton, J. Serra, Boundary regularity for fully nonlinear integro-differential equations, arXiv:1404.1197.
[23] J. Serra, Regularity for fully nonlinear nonlocal parabolic equations with rough kernels, arXiv:1401.4521.
[24] J. Serra, $C^{\sigma+\alpha}$ regularity for concave nonlocal fully nonlinear elliptic equations with rough kernels, arXiv:1405.0930.
[25] L. Silvestre, Personal communications.