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An improvement on the Brézis–Gallouët technique for 2D NLS and 1D half-wave equation

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Abstract

We revise the classical approach by Brézis–Gallouët to prove global well-posedness for nonlinear evolution equations. In particular we prove global well-posedness for the quartic NLS on general domains Ω in $\mathbb{R}^2$ with initial data in $H^2(\Omega) \cap H_0^1(\Omega)$, and for the quartic nonlinear half-wave equation on $\mathbb{R}$ with initial data in $H^1(\mathbb{R})$.

Keywords: Half-wave equation; Nonlinear Schrödinger equation; Energy estimates; Global existence

The main aim of this paper is to revise the technique developed by Brézis and Gallouët to study the global well-posedness of Cauchy problems associated with some nonlinear evolution equations. More precisely we prove that by using the Brézis–Gallouët inequality in conjunction with suitable higher order energies that we shall introduce along the paper, then the standard theory, developed in [4] and [16] respectively for NLS and half-wave equation with cubic nonlinearity, has an improvement to quartic nonlinearity.

Our first result concerns an extension to higher order nonlinearities of the very classical result in [4]. More precisely the first family of problems that we shall address is the following one:

$$\begin{cases} i\partial_t u + \Delta u = \lambda u|u|^3, \ (t,x) \in \mathbb{R} \times \Omega, \\ u(t,x) = 0, \ (t,x) \in \mathbb{R} \times \partial\Omega, \\ u(0) = \varphi, \end{cases} \tag{0.1}$$

where $\lambda = \pm 1$, $\Omega \subset \mathbb{R}^2$ is open and satisfies the following hypotheses:

$(H1)$ $\exists L \in \mathcal{L}(H^2(\Omega), H^2(\mathbb{R}^2)) \cap \mathcal{L}(H^1(\Omega), H^1(\mathbb{R}^2))$ s.t. $(Lu)_{|\Omega} = u$ a.e. in Ω;
$(H2)$ $L^2(\Omega) \supset H^2(\Omega) \cap H_0^1(\Omega) \ni u \mapsto \Delta u \in L^2(\Omega)$ is self-adjoint.

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By the celebrated Brézis–Gallouët inequality it follows that if Ω satisfies $(H1)$, then the following logarithmic Sobolev embedding occurs:

$$\|v\|_{L^\infty(\Omega)} \lesssim \|v\|_{H^1(\Omega)}\sqrt{\ln\left(2+\|v\|_{H^2(\Omega)}\right)}+1,\ \forall v \in H^2(\Omega). \tag{0.2}$$

There has been a growing interest in the last decades on the Cauchy problem associated with NLS on domains, starting from the pioneering paper [4]. In this paper the authors prove global well-posedness for the defocusing cubic NLS on domains $\Omega \subset \mathbb{R}^2$, by combining (0.2) with the conservation of the energy. The first extension of the result by Brézis and Gallouët, up to the fourth order nonlinearity, was obtained in [19] under some restrictive conditions on the initial data φ. More precisely it is assumed that $\varphi|\varphi| \in H^3(\Omega) \cap H_0^1(\Omega)$, $\Delta\varphi \in H_0^1(\Omega)$. A fundamental tool to treat NLS on domains, with higher order nonlinearities, are the so-called Strichartz inequalities (see [7] and the bibliography therein for the case $\Omega = \mathbb{R}^2$). In [5] it is proved a suitable version of Strichartz inequalities with loss, on general compact manifolds. Beside other results in this paper it is studied the Cauchy problem associated with NLS on 2D compact manifolds for every nonlinearity $u|u|^p$. The results in [5] have been extended to NLS on domains $\Omega \subset \mathbb{R}^2$, under suitable assumptions. In particular the cases of bounded domains and external domains have been widely investigated in the literature. Just to quote a few results we mention [1,3,6,13], etc.

Due to the huge literature devoted to NLS on 2D domains, Theorem 0.1 below could be considered somewhat weaker compared with the known results, however we prefer to keep its statement along this paper for three reasons. First of all our argument is exclusively based on integration by parts and energy estimates, and hence it is independent of the use of Strichartz estimates. The second reason is that the proof of Theorem 0.1 can help to understand the idea behind the more involved proof of our second result concerning the nonlinear half-wave equation, where as far as we know our result is a novelty in the literature. The third reason is that as far as we know it is unclear whether or not the aforementioned Strichartz estimates are available under the rather general assumptions $(H1)$, $(H2)$.

Let us recall that by the usual energy estimates, in conjunction with the classical Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$, one can prove that the Cauchy problem (0.1) is well-posed locally in time provided that $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. More precisely there exists one unique solution $u \in \mathcal{C}([0, T_{max}); H^2(\Omega) \cap H_0^1(\Omega))$ of (0.1), where $T_{max} > 0$. Moreover, we have the alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and $\lim_{t\to T_{max}^-} \|u(t)\|_{H^2(\Omega)} = \infty$.

The first result of the paper is the following.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set that satisfies $(H1)$, $(H2)$, $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ and let $u \in \mathcal{C}([0, T_{max}); H^2(\Omega) \cap H_0^1(\Omega))$ be the unique local solution of (0.1). Then we have the following alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and $\sup_{[0,T_{max})} \|u(t)\|_{H^1(\Omega)} = \infty$.*

Next we give some concrete conditions on the initial data φ in order to guarantee global well-posedness of (0.1). We need to introduce the energy preserved along (0.1) for $\lambda = \pm 1$:

$$E_{NLS,\pm}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx \pm \frac{1}{5}\int_\Omega |u|^5 dx. \tag{0.3}$$

We also introduce the ground state $Q(|x|)$ defined as the unique solution to

$$-\Delta Q + Q = Q^4, \quad Q \in H^1(\mathbb{R}^2), \quad Q > 0.$$

We are now in a position to state the following global well-posedness result.

Corollary 0.1. *Let Ω be as in Theorem 0.1 and $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$.*

If $\lambda = 1$ then (0.1) has one unique global solution $u \in \mathcal{C}([0, \infty); H^2(\Omega) \cap H_0^1(\Omega))$.

If $\lambda = -1$ and φ satisfies:

$$\begin{aligned} &E_{NLS,-}(\varphi)\|\varphi\|_{L^2}^4 < E_{NLS,-}(Q)\|Q\|_{L^2}^4 \text{ and} \\ &\|\nabla\varphi\|_{L^2}\|\varphi\|_{L^2(\Omega)}^2 < \|\nabla Q\|_{L^2}\|Q\|_{L^2}^2, \end{aligned} \tag{0.4}$$

then (0.1) has one unique global solution $u \in \mathcal{C}([0, \infty); H^2(\Omega) \cap H_0^1(\Omega))$.

The proof of Corollary 0.1 follows by Theorem 0.1 in conjunction with the conservation of the energy (0.3). In fact, in the defocusing case, since the energy is positive definite, it prevents blow-up of the H^1 norm. In the focusing case a combination of the conservation of the energy with conditions (0.4), prevents blow-up of the H^1-norm via a standard continuity argument (see [12] for details).

The second family of Cauchy problems that we consider in this paper is associated with the fourth order nonlinear half-wave equation:

$$\begin{cases} i\partial_t u - |D_x|u = \lambda u|u|^3, \ (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = \varphi \in H^1(\mathbb{R}), \end{cases} \tag{0.5}$$

where $|D_x| = \sqrt{-\partial_x^2}$ is the first order non-local fractional derivative, $\lambda = \pm 1$. Let us mention that evolution problems with non-local dispersion arise in various physical settings (see [8,17,10,15]). In the case of a cubic nonlinearity, the Cauchy problem (0.5) is strictly related with the Szegö model (see [11,18]).

We recall that by standard arguments one can prove the existence of one unique solution $u \in \mathcal{C}([0, T_{max}); H^1(\mathbb{R}))$ of (0.5), where $T_{max} > 0$. Moreover, we have the alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and

$$\lim_{t\to T_{max}^-} \|u(t)\|_{H^1(\mathbb{R})} = \infty.$$

We can state our second result.

Theorem 0.2. *Let $\varphi \in H^1(\mathbb{R})$ and $u \in \mathcal{C}([0, T_{max}); H^1(\mathbb{R}))$ be the unique local solution of (0.5). Then we have the following alternative: either $T_{max} = \infty$ or $T_{max} < \infty$ and $\sup_{[0,T_{max})} \|u(t)\|_{H^{\frac12}(\mathbb{R})} = \infty$.*

Next we give some concrete conditions on the initial data φ in order to guarantee global well-posedness of (0.5). We need to introduce the energy preserved along (0.5) for $\lambda = \pm 1$:

$$E_{HW,\pm}(u) = \frac{1}{2}\int_{\mathbb{R}} ||D_x|^{\frac12} u|^2 dx \pm \frac{1}{5}\int_{\mathbb{R}} |u|^5 dx. \tag{0.6}$$

We also introduce $R \in H^{\frac12}(\mathbb{R})$ as the unique (non-trivial) optimizer of the following Gagliardo–Nirenberg inequality

$$\|f\|_{L^5(\mathbb{R})} \le C_{GN} \||D_x|^{\frac12} f\|_{L^2(\mathbb{R})}^{\frac35} \|f\|_{L^2(\mathbb{R})}^{\frac25}, \tag{0.7}$$

that satisfies

$$|D_x|R + R = R^4, \quad R(x) = R(|x|) > 0. \tag{0.8}$$

The uniqueness of R defined as above is proved in [9] (concerning a general proof on the existence of optimizers for Gagliardo–Nirenberg inequalities see [2]).

The next result is a version of Corollary 0.1 in the context of the half-wave equation.

Corollary 0.2. *Assume $\lambda = 1$ then (0.5) has one unique global solution $u \in \mathcal{C}([0, \infty); H^1(\mathbb{R}))$.*

Assume $\lambda = -1$ and φ satisfies:

$$E_{HW,-}(\varphi)\|\varphi\|_{L^2}^4 < E_{HW,-}(R)\|R\|_{L^2}^4 \text{ and}$$
$$\||D_x|^{\frac12}\varphi\|_{L^2}\|\varphi\|_{L^2(\mathbb{R})}^2 < \||D_x|^{\frac12} R\|_{L^2}\|R\|_{L^2}^2, \tag{0.9}$$

then (0.5) has one unique global solution $u \in \mathcal{C}([0, \infty); H^1(\mathbb{R}))$.

Along the paper we shall present a proof of Corollary 0.2. Of course in the defocusing case it follows by Theorem 0.2 in conjunction with the fact that the energy $E_{HW,+}$ is positive definite. In the focusing case the proof is more involved and we need to adapt the argument in [12] in a non-local context.

The global well-posedness results above can be considered as an extension to the quartic half-wave equation of part of the results proved by Krieger, Lenzmann and Raphael in [16]. In this paper in fact the authors treat, beside very interesting blow-up results, the Cauchy theory for the half-wave equation with cubic nonlinearity via the classical approach in [4]. We should also notice that in [16] the authors work in $H^{\frac{1}{2}}(\mathbb{R})$, while in Theorem 0.2 we work in $H^1(\mathbb{R})$.

A basic tool along the proof of Theorem 0.2 will be the following version of (0.2):

$$\|v\|_{L^\infty(\mathbb{R})} \lesssim \|v\|_{H^{\frac{1}{2}}(\mathbb{R})} \sqrt{\ln\left(2 + \|v\|_{H^1(\mathbb{R})}\right) + 1}, \ \forall v \in H^1(\mathbb{R}). \tag{0.10}$$

Its proof follows by a straightforward adaptation of the argument in [4]. Hence we skip it and we shall make an extensive use of (0.10) without any further comment.

1. Proof of Theorem 0.1

Along this section we use the notations:

$$\nabla u = (\partial_x u, \partial_y u), \ \Delta = \partial_x^2 + \partial_y^2, \ (f, g) = \int_\Omega f \cdot \bar{g}\, dx, \ L^2 = L^2(\Omega), \ H^k = H^k(\Omega).$$

We also introduce the following energy:

$$\mathcal{E}(u) = \|\Delta u\|_{L^2}^2 - 2\lambda \mathcal{R}e(\Delta u, u|u|^3) - \frac{3}{4}\lambda(|\nabla |u|^2|^2, |u|).$$

Lemma 1.1. *Let u be as in Theorem 0.1, then we have the following identity:*

$$\frac{d}{dt}(\mathcal{E}(u) + \|u\|_{L^2}^2) = -2\lambda^2 \mathcal{I}m(\nabla u, u\nabla(|u|^6)) + \frac{3}{4}\lambda(|\nabla |u|^2|^2, \partial_t |u|)) + 2\lambda(|\nabla u|^2, \partial_t(|u|^3)). \tag{1.1}$$

Proof. Recall that $\frac{d}{dt}\|u\|_{L^2}^2 = 0$, hence we shall treat $\frac{d}{dt}\mathcal{E}(u)$. Next we assume that the solution is regular enough in order to justify all the computations. In the case that the solution u is only H^2, then one can proceed by a smoothing argument via the Yosida regularization (we skip this technical but standard regularization argument).

We start with the following computation:

$$\begin{aligned}\frac{d}{dt}\|\Delta u\|_{L^2}^2 &= 2\mathcal{R}e(\Delta\partial_t u, \Delta u) = 2\mathcal{R}e(\Delta\partial_t u, -i\partial_t u + \lambda u|u|^3) \\ &= 2\lambda\mathcal{R}e(\Delta\partial_t u, u|u|^3) = 2\lambda\frac{d}{dt}\mathcal{R}e(\Delta u, u|u|^3) - 2\lambda\mathcal{R}e(\Delta u, \partial_t(u|u|^3)),\end{aligned}$$

where we used the equation solved by u in the second equality. Next notice that

$$\begin{aligned}\mathcal{R}e(\Delta u, \partial_t(u|u|^3)) &= \mathcal{R}e(\Delta u, \partial_t u|u|^3) + \mathcal{R}e(\Delta u, u\partial_t(|u|^3)) \\ &= \mathcal{R}e(\Delta u, \partial_t u|u|^3) + \frac{1}{2}(\Delta|u|^2, \partial_t(|u|^3) - (|\nabla u|^2, \partial_t(|u|^3)) = I + II + III.\end{aligned}$$

By using the equation solved by u we get

$$I = \mathcal{R}e(\Delta u, -i\lambda u|u|^6) = \lambda\mathcal{I}m(\nabla u, u\nabla(|u|^6)).$$

Moreover, we have

$$\begin{aligned}II &= -\frac{1}{2}(\nabla|u|^2, \partial_t\nabla(|u|^3)) = -\frac{3}{4}(\nabla|u|^2, \partial_t(\nabla(|u|^2)|u|)) \\ &= -\frac{3}{4}\frac{d}{dt}(\nabla|u|^2, \nabla(|u|^2)|u|) + \frac{3}{4}(\partial_t\nabla|u|^2, \nabla(|u|^2)|u|) \\ &= -\frac{3}{4}\frac{d}{dt}(\nabla|u|^2, \nabla(|u|^2)|u|) + \frac{3}{8}(\partial_t|\nabla|u|^2|^2, |u|) \\ &= -\frac{3}{4}\frac{d}{dt}(\nabla|u|^2, \nabla(|u|^2)|u|) + \frac{3}{8}\frac{d}{dt}(|\nabla|u|^2|^2, |u|) - \frac{3}{8}(|\nabla|u|^2|^2, \partial_t|u|). \qquad \square\end{aligned} \tag{1.2}$$

Lemma 1.2. *Let u be as in Theorem 0.1 and $U = \sup_{[0,T_{Max})} \|u(t)\|_{H^1}$, then we have:*

$$\frac{d}{dt}(\mathcal{E}(u) + \|u\|_{L^2}^2) \lesssim U^8 \ln^3(2 + \|u\|_{H^2}) + U^3\|\Delta u\|_{L^2}^2 \ln(2 + \|u\|_{H^2}) + U^2 + U\|\Delta u\|_{L^2}^2, \quad \forall t \in [0, T_{max}). \tag{1.3}$$

Proof. Next we collect some useful inequalities satisfied by any solution u of (0.1):

$$|\mathcal{I}m(\nabla u, u\nabla(|u|^6))| \lesssim \int |\nabla u|^2 \cdot |u|^6 dx \lesssim \|u\|_{H^1}^2 \|u\|_{L^\infty}^6 \lesssim \|u\|_{H^1}^8 \ln^3(2 + \|u\|_{H^2}) + \|u\|_{H^1}^2,$$

where we used (0.2). We also have

$$\int |\nabla |u|^2|^2 \cdot |\partial_t |u||dx \lesssim \int |\nabla u|^2 \cdot |u|^6 dx + \int |\nabla u|^2 \cdot |\Delta u| \cdot |u|^2 dx, \tag{1.4}$$

where we used the diamagnetic inequality $|\partial_t |u|| \leq |\partial_t u|$ and the equation solved by u. By combining the Hölder inequality, the logarithmic Sobolev embedding (0.2) and the Gagliardo–Nirenberg inequality

$$\|\nabla u\|_{L^4} \lesssim \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{L^2}^{\frac{1}{2}}, \tag{1.5}$$

we can continue the estimate above as follows:

$$\ldots \lesssim \|u\|_{H^1}^8 \ln^3(2 + \|u\|_{H^2}) + \|u\|_{H^1}^2 + \|\Delta u\|_{L^2}^2 \|u\|_{H^1}^3 \ln(2 + \|u\|_{H^2}) + \|\Delta u\|_{L^2}^2 \|u\|_{H^1}.$$

Finally notice that (by using the equation solved by u)

$$\int |\nabla u|^2 \cdot \partial_t(|u|^3) dx \lesssim \int |\nabla u|^2 \cdot |u|^6 + \int |\nabla u|^2 \cdot |\Delta u| \cdot |u|^2 dx,$$

and we can continue as in (1.4). □

Proof of Theorem 0.1. Assume by the absurd that

$$T_{max} < \infty \text{ and } U = \sup_{t \in [0, T_{max})} \|u\|_{H^1} < \infty.$$

By elementary computations we get:

$$|(|\nabla |u|^2|^2, |u|)| \lesssim (\int |\nabla u|^4 dx)^{\frac{1}{2}} \cdot (\int |u|^6 dx)^{\frac{1}{2}} \lesssim U^4 \|\Delta u\|_{L^2},$$

where we used (1.5), and we also have

$$|(\Delta u, u|u|^3)| \lesssim \|\Delta u\|_{L^2} \|u\|_{L^8}^4 \lesssim U^4 \|\Delta u\|_{L^2}.$$

Hence

$$\|u\|_{H^2}^2 \lesssim \mathcal{E}(u) + \|u\|_{L^2}^2, \text{ for } \|u\|_{H^2} > R = R(U) > 0. \tag{1.6}$$

Next recall that by definition of T_{max} we have $\|u(t)\|_{H^2} > R, \forall t > \bar{T} \in (0, T_{max})$. Hence by combining (1.6) with (1.3) we get:

$$\|u(t)\|_{H^2}^2 \lesssim \|u(\bar{T})\|_{H^2}^2 + U^8 \int_{\bar{T}}^t \ln^3(2 + \|u\|_{H^2}) dt + U^3 \int_{\bar{T}}^t \|u\|_{H^2}^2 \ln(2 + \|u\|_{H^2}) dt + U \int_{\bar{T}}^t \|u\|_{H^2}^2 dt + U^2(t - \bar{T}), \quad \forall t \in [\bar{T}, T_{max}).$$

We are in a position to conclude, arguing as in [4], that $\sup_{t \in [0, T_{max})} \|u(t)\|_{H^2} < \infty$, and hence we get a contradiction with the definition of T_{max}. □

2. The half-wave equation

Along this section we use the notations:

$$|D_x|^s = (\sqrt{-\partial_x^2})^s,\ (f,g) = \int_{\mathbb{R}} f \cdot \bar{g}\, dx,\ L^p = L^p(\mathbb{R}),\ H^k = H^k(\mathbb{R}).$$

We also introduce the energy

$$\mathcal{F}(u) = \|\partial_x u\|_{L^2}^2 + 2\lambda \mathcal{R}e(|D_x|u, u|u|^3) - \frac{3}{4}\lambda(||D_x|^{\frac{1}{2}}(|u|^2)|^2, |u|) + \lambda(|D_x|^{\frac{1}{2}}|u|^2 - \bar{u}|D_x|^{\frac{1}{2}}u - u|D_x|^{\frac{1}{2}}\bar{u}, |D_x|^{\frac{1}{2}}(|u|^3)). \tag{2.1}$$

The following proposition from [14] will be crucial in the sequel.

Proposition 2.1. *We have the following estimate:*

$$\||D_x|^s(fg) - g|D_x|^s f - f|D_x|^s g\|_{L^p} \lesssim \||D_x|^{s_1} f\|_{L^q} \||D_x|^{s_2} f\|_{L^r},$$

where

$$\frac{1}{p} = \frac{1}{q} + \frac{1}{r},\ 1 < p, q, r < \infty,\ 1 > s = s_1 + s_2 > 0,\ s_i \geq 0.$$

Lemma 2.1. *Let u be as in Theorem 0.2. Then we have the following identity:*

$$\begin{aligned}\frac{d}{dt}(\mathcal{F}(u) + \|u\|_{L^2}^2) = &-2\lambda^2 \mathcal{I}m(|D_x|u, u|u|^6) + 2\lambda(||D_x|^{\frac{1}{2}}u|^2, \partial_t(|u|^3)) \\ &+ \lambda(|D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(\bar{u}|D_x|^{\frac{1}{2}}u) - \partial_t(u|D_x|^{\frac{1}{2}}\bar{u}), |D_x|^{\frac{1}{2}}(|u|^3)) \\ &+ 2\lambda \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}(u\partial_t(|u|^3)) - |D_x|^{\frac{1}{2}}u\partial_t(|u|^3) - u|D_x|^{\frac{1}{2}}\partial_t(|u|^3)) \\ &- \frac{3}{4}\lambda(||D_x|^{\frac{1}{2}}(|u|^2)|^2, \partial_t|u|) + \frac{3}{2}\lambda(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}|u|\partial_t(|u|^2)) \\ &+ \frac{3}{2}\lambda(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|) - |u||D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(|u|^2)|D_x|^{\frac{1}{2}}|u|).\end{aligned} \tag{2.2}$$

Proof. Recall that $\frac{d}{dt}\|u\|_{L^2}^2 = 0$, hence we shall compute $\frac{d}{dt}\mathcal{F}(u)$. In the sequel we assume that the solution is regular enough in order to justify the following computations. The proof in the case of lower regular solutions (i.e. H^1 solutions), can be done by a standard density argument. However we skip the details.

We start with the following computation:

$$\begin{aligned}\frac{d}{dt}\|\partial_x u\|_{L^2}^2 &= 2\mathcal{R}e(|D_x|\partial_t u, |D_x|u) = 2\mathcal{R}e(|D_x|\partial_t u, i\partial_t u - \lambda u|u|^3) \\ &= -2\lambda \mathcal{R}e(|D_x|\partial_t u, u|u|^3) = -2\lambda\frac{d}{dt}\mathcal{R}e(|D_x|u, u|u|^3) + 2\lambda \mathcal{R}e(|D_x|u, \partial_t(u|u|^3)),\end{aligned}$$

where we used the equation solved by u. Next notice that

$$\mathcal{R}e(|D_x|u, \partial_t(u|u|^3)) = \mathcal{R}e(|D_x|u, \partial_t u|u|^3) + \mathcal{R}e(|D_x|u, u\partial_t(|u|^3)) = I + II.$$

Concerning I we get (by using the equation solved by u)

$$I = -\lambda \mathcal{I}m(|D_x|u, u|u|^6),$$

and for II we have

$$\begin{aligned}II &= \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}(u\partial_t(|u|^3))) \\ &= \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}u\partial_t(|u|^3)) + \mathcal{R}e(|D_x|^{\frac{1}{2}}u, u|D_x|^{\frac{1}{2}}\partial_t(|u|^3)) \\ &+ \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}(u\partial_t(|u|^3)) - |D_x|^{\frac{1}{2}}u\partial_t(|u|^3) - u|D_x|^{\frac{1}{2}}\partial_t(|u|^3)),\end{aligned} \tag{2.3}$$

that can be written as (recall $\partial_t(|u|^3) = \frac{3}{2}\partial_t(|u|^2)|u|$)

$$\begin{aligned}
\ldots &= \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}u\partial_t(|u|^3)) + \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|)) \\
&- \frac{1}{2}(|D_x|^{\frac{1}{2}}(|u|^2) - \bar{u}|D_x|^{\frac{1}{2}}u - u|D_x|^{\frac{1}{2}}\bar{u}, |D_x|^{\frac{1}{2}}\partial_t(|u|^3)) \\
&+ \mathcal{R}e(|D_x|^{\frac{1}{2}}u, |D_x|^{\frac{1}{2}}(u\partial_t(|u|^3)) - |D_x|^{\frac{1}{2}}u\partial_t(|u|^3) - u|D_x|^{\frac{1}{2}}\partial_t(|u|^3)) \\
&= II_1 + II_2 + II_3 + II_4.
\end{aligned} \tag{2.4}$$

Next notice that

$$\begin{aligned}
II_2 &= \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |u||D_x|^{\frac{1}{2}}\partial_t(|u|^2)) + \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}|u|\partial_t(|u|^2)) \\
&+ \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|) - |u||D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(|u|^2)|D_x|^{\frac{1}{2}}|u|)
\end{aligned}$$

and hence

$$\begin{aligned}
\ldots &= \frac{3}{8}(\partial_t||D_x|^{\frac{1}{2}}(|u|^2)|^2, |u|) + \frac{3}{4}(|D_x|^{\frac{1}{2}}|u|^2, |D_x|^{\frac{1}{2}}|u|\partial_t(|u|^2)) \\
&+ \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|) - |u||D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(|u|^2)|D_x|^{\frac{1}{2}}|u|) \\
&= \frac{3}{8}\frac{d}{dt}(||D_x|^{\frac{1}{2}}(|u|^2)|^2, |u|) \\
&- \frac{3}{8}(||D_x|^{\frac{1}{2}}|u|^2|^2, \partial_t|u|) + \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}|u|\partial_t(|u|^2)) \\
&+ \frac{3}{4}(|D_x|^{\frac{1}{2}}(|u|^2), |D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|) - |u||D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(|u|^2)|D_x|^{\frac{1}{2}}|u|).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
II_3 &= -\frac{1}{2}\frac{d}{dt}(|D_x|^{\frac{1}{2}}(|u|^2) - \bar{u}|D_x|^{\frac{1}{2}}u - u|D_x|^{\frac{1}{2}}\bar{u}, |D_x|^{\frac{1}{2}}(|u|^3)) \\
&+ \frac{1}{2}(|D_x|^{\frac{1}{2}}\partial_t(|u|^2) - \partial_t(\bar{u}|D_x|^{\frac{1}{2}}u) - \partial_t(u|D_x|^{\frac{1}{2}}\bar{u}), |D_x|^{\frac{1}{2}}(|u|^3)). \qquad \square
\end{aligned}$$

Lemma 2.2. *Let u be as in Theorem 0.2 and let $U = \sup_{[0,T_{max})}\|u(t)\|_{H^{\frac{1}{2}}}$, then we have*

$$\frac{d}{dt}(\mathcal{F}(u) + \|u\|_{L^2}^2) \lesssim (1+U)^6\|u\|_{H^1}^2\ln(2 + \|u\|_{H^1}).$$

Proof. It follows by combining the estimates below with Lemma 2.1. More precisely we shall prove that all the terms on the r.h.s. in (2.2) can be estimated by $(1+U)^6\|u\|_{H^1}^2\ln(2+\|u\|_{H^1})$. First notice that

$$|\mathcal{I}m(|D_x|u, u|u|^6)| \lesssim \|u\|_{H^1}\|u\|_{L^{14}}^7 \lesssim \|u\|_{H^1}^2\|u\|_{H^{\frac{1}{2}}}^6 \lesssim \|u\|_{H^1}^2U^6.$$

On the other hand

$$|(||D_x|^{\frac{1}{2}}u|^2, \partial_t(|u|^3))| \lesssim \||D_x|^{\frac{1}{2}}u\|_{L^4}^2\|\partial_t u\|_{L^2}\|u\|_{L^\infty}^2,$$

that by (0.10) and the following Gagliardo–Nirenberg inequality

$$\||D_x|^{\frac{1}{2}}u\|_{L^4}^2 \lesssim \||D_x|u\|_{L^2}\||D_x|^{\frac{1}{2}}u\|_{L^2}, \tag{2.5}$$

implies

$$\ldots \lesssim \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}^3\|\partial_t u\|_{L^2}\ln(2+\|u\|_{H^1}) + \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}\|\partial_t u\|_{L^2}.$$

By looking at the equation solved by u

$$\dots \lesssim \|u\|_{H^1}\|u\|^3_{H^{\frac{1}{2}}}(\|u\|_{H^1}+\|u\|^4_{L^8})\ln(2+\|u\|_{H^1})+\|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}(\|u\|_{H^1}+\|u\|^4_{L^8}).$$

Next notice that if we develop by the classical Leibniz rule the derivative with respect to the time variable and we apply twice Proposition 2.1 (where $s=s_1=\frac{1}{2}$, $s_2=0$, $p=\frac{4}{3}$, $q=2$, $r=4$) we get:

$$\begin{aligned}&|(|D_x|^{\frac{1}{2}}\partial_t(|u|^2)-\partial_t(\bar u|D_x|^{\frac{1}{2}}u)-\partial_t(u|D_x|^{\frac{1}{2}}\bar u),|D_x|^{\frac{1}{2}}(|u|^3))|\\&\quad\lesssim \|\partial_t u\|_{L^2}\||D_x|^{\frac{1}{2}}u\|_{L^4}\||D_x|^{\frac{1}{2}}(|u|^3)\|_{L^4}\lesssim \|\partial_t u\|_{L^2}\|u\|^{\frac{1}{2}}_{H^1}\|u\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\||u|^3\|^{\frac{1}{2}}_{H^1}\||u|^3\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\\&\quad\lesssim \|\partial_t u\|_{L^2}\|u\|_{H^{\frac{1}{2}}}\|u\|_{H^1}\|u\|^2_{L^\infty}.\end{aligned}$$

Notice that we have used (2.5) and the property

$$\|v\cdot w\|_{H^s}\lesssim \|v\|_{H^s}\|w\|_{L^\infty}+\|w\|_{H^s}\|v\|_{L^\infty}. \tag{2.6}$$

We conclude by (0.10) and the equation solved by u.

Next we use again Proposition 2.1 (where $s=s_1=\frac{1}{2}$, $s_2=0$, $p=\frac{4}{3}$, $q=2$, $r=4$),

$$\begin{aligned}&|(|D_x|^{\frac{1}{2}}u,|D_x|^{\frac{1}{2}}(u\partial_t(|u|^3))-|D_x|^{\frac{1}{2}}u\partial_t(|u|^3)-u|D_x|^{\frac{1}{2}}\partial_t(|u|^3))|\\&\quad\lesssim \||D_x|^{\frac{1}{2}}u\|^2_{L^4}\|\partial_t(|u|^3)\|_{L^2}\lesssim \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}\|\partial_t u\|_{L^2}\|u\|^2_{L^\infty},\end{aligned}$$

where we used (2.5). We conclude by (0.10) and by using the equation solved by u.

By the Hölder inequality we get

$$\begin{aligned}|(||D_x|^{\frac{1}{2}}(|u|^2)|^2,\partial_t|u|)|&\lesssim \||D_x|^{\frac{1}{2}}(|u|^2)|\|^2_{L^4}\|\partial_t u\|_{L^2}\\&\lesssim \|u^2\|_{H^{\frac{1}{2}}}\|u^2\|_{H^1}\|\partial_t u\|_{L^2}\lesssim \|u\|_{H^{\frac{1}{2}}}\|u\|_{H^1}\|u\|^2_{L^\infty}\|\partial_t u\|_{L^2},\end{aligned}$$

where we have used (2.5) and (2.6). We conclude as above.

Next we have the estimate

$$\begin{aligned}|(|D_x|^{\frac{1}{2}}(|u|^2),|D_x|^{\frac{1}{2}}|u|\partial_t(|u|^2))|&\lesssim \||D_x|^{\frac{1}{2}}(|u|^2)\|_{L^4}\||D_x|^{\frac{1}{2}}|u|\|_{L^4}\|\partial_t u\|_{L^2}\|u\|_{L^\infty}\\&\lesssim \|u^2\|^{\frac{1}{2}}_{H^1}\|u^2\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\|u\|^{\frac{1}{2}}_{H^1}\|u\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\|\partial_t u\|_{L^2}\|u\|_{L^\infty},\end{aligned} \tag{2.7}$$

where we used (2.5). By (2.6) we get

$$\dots\lesssim \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}\|u\|^2_{L^\infty}\|\partial_t u\|_{L^2},$$

and we conclude by using the equation solved by u in conjunction with (0.10).

Finally by Proposition 2.1 and the Hölder inequality we get the following estimate:

$$\begin{aligned}&|(|D_x|^{\frac{1}{2}}(|u|^2),|D_x|^{\frac{1}{2}}(\partial_t(|u|^2)|u|)-|u||D_x|^{\frac{1}{2}}\partial_t(|u|^2)-\partial_t(|u|^2)|D_x|^{\frac{1}{2}}|u|)|\\&\quad\lesssim \|(|D_x|^{\frac{1}{2}}(|u|^2)\|_{L^4}\|\partial_t(|u|^2)\|_{L^2}\||D_x|^{\frac{1}{2}}u\|_{L^4},\end{aligned}$$

and by (2.5)

$$\dots\lesssim \|u^2\|^{\frac{1}{2}}_{H^1}\|u^2\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\|u\|^{\frac{1}{2}}_{H^1}\|u\|^{\frac{1}{2}}_{H^{\frac{1}{2}}}\|\partial_t u\|_{L^2}\|u\|_{L^\infty},$$

which is precisely the term in (2.7), hence we can conclude as above. □

Proof of Theorem 0.2. It is similar to the proof of Theorem 0.1, provided that we use Lemma 2.2 and we show that

$$|\mathcal{F}(u)+\|u\|^2_{L^2}-\|u\|^2_{H^1}|\lesssim C(U)(1+\|u\|_{H^1})\ln^{\frac{3}{2}}(2+\|u\|_{H^1}),$$

where $U = \sup_{[0,T_{max})} \|u(t)\|_{H^{\frac{1}{2}}}$. This fact follows by the following computations. First notice that

$$|(|D_x|u, u|u|^3)| \lesssim \||D_x|u\|_{L^2}\|u\|_{L^8}^4 \lesssim \|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}^4.$$

Moreover, we have

$$\begin{aligned}|(||D_x|^{\frac{1}{2}}(|u|^2)|^2, |u|)| &\lesssim \||D_x|^{\frac{1}{2}}(|u|^2)\|_{L^4}^2\|u\|_{L^2}\\ &\lesssim \|u^2\|_{H^{\frac{1}{2}}}\|u^2\|_{H^1}\|u\|_{L^2} \lesssim \|u\|_{L^\infty}^2\|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}^2,\end{aligned}$$

where we used (2.5) and (2.6). We conclude by (0.10). Finally notice that

$$\begin{aligned}&|(|D_x|^{\frac{1}{2}}|u|^2 - \bar{u}|D_x|^{\frac{1}{2}}u - u|D_x|^{\frac{1}{2}}\bar{u}, |D_x|^{\frac{1}{2}}(|u|^3))|\\ &\lesssim \||D_x|^{\frac{1}{2}}|u|^2\|_{L^2}\||D_x|^{\frac{1}{2}}(|u|^3)\|_{L^2} + \|u\|_{L^2}\||D_x|^{\frac{1}{2}}|u|\|_{L^4}\||D_x|^{\frac{1}{2}}(|u|^3)\|_{L^4}\end{aligned}$$

and hence by (2.5)

$$\begin{aligned}\ldots &\lesssim \|u^2\|_{H^{\frac{1}{2}}}\|u^3\|_{H^{\frac{1}{2}}} + \|u\|_{L^2}\|u\|_{H^{\frac{1}{2}}}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}}\|u^3\|_{H^{\frac{1}{2}}}^{\frac{1}{2}}\|u^3\|_{H^1}^{\frac{1}{2}}\\ &\lesssim \|u\|_{H^{\frac{1}{2}}}^2\|u\|_{L^\infty}^3 + \|u\|_{L^2}\|u\|_{H^1}\|u\|_{H^{\frac{1}{2}}}\|u\|_{L^\infty}^2,\end{aligned}$$

where we used (2.6). We conclude again by (0.10). □

3. Proof of Corollary 0.2

The case $\lambda = 1$ follows by combining the conservation of the energy $E_{HW,+}$ (which is positive definite) with Theorem 0.2.

Concerning the case $\lambda = -1$ it is sufficient to show that $\|u(t)\|_{\dot{H}^{\frac{1}{2}}}$ cannot blow up in finite time under the assumptions of Corollary 0.2.

Notice that by combining the conservation of the mass and the energy, with the assumption $E_{HW,-}(R)\|R\|_{L^2}^4 > E_{HW,-}(\varphi)\|\varphi\|_{L^2}^4$, we get

$$\begin{aligned}E_{HW,-}(R)\|R\|_{L^2}^4 &> E_{HW,-}(u(t))\|u(t)\|_{L^2}^4\\ &= \frac{1}{2}\||D_x|^{\frac{1}{2}}u(t)\|_{L^2}^2\|u(t)\|_{L^2}^4 - \frac{1}{5}\|u(t)\|_{L^5}^5\|u(t)\|_{L^2}^4.\end{aligned} \tag{3.1}$$

By the following Gagliardo–Nirenberg inequality

$$\|g\|_{L^5(\mathbb{R})} \le C_{GN}\||D_x|^{\frac{1}{2}}g\|_{L^2(\mathbb{R})}^{\frac{3}{5}}\|g\|_{L^2(\mathbb{R})}^{\frac{2}{5}} \tag{3.2}$$

we get

$$\ldots \ge \frac{1}{2}\big(\||D_x|^{\frac{1}{2}}u(t)\|_{L^2}\|u(t)\|_{L^2}^2\big)^2 - \frac{1}{5}C_{GN}^5\big(\||D_x|^{\frac{1}{2}}u(t)\|_{L^2}\|u(t)\|_{L^2}^2\big)^3.$$

Hence $\||D_x|^{\frac{1}{2}}u(t)\|_{L^2}\|u(t)\|_{L^2}^2$ belongs to the sublevel

$$\mathcal{A} = \{x \in \mathbb{R}^+ | f(x) < E_{HW,-}(R)\|R\|_{L^2}^4\},$$

where $f(x) = \frac{1}{2}x^2 - \frac{1}{5}C_{GN}^5x^3$. Next we denote by $x_{max} > 0$ the unique point where the maximum of f is achieved on $(0,\infty)$. We claim that

$$x_{max} = \||D_x|^{\frac{1}{2}}R\|_{L^2}\|R\|_{L^2}^2 \text{ and } f(x_{max}) = E_{HW,-}(R)\|R\|_{L^2}^4. \tag{3.3}$$

If this is the case then we get

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2,$$

where

$$\mathcal{A}_1 = (0, \| |D_x|^{\frac{1}{2}} R \|_{L^2} \|R\|_{L^2}^2) \text{ and } \mathcal{A}_2 = (\| |D_x|^{\frac{1}{2}} R \|_{L^2} \|R\|_{L^2}^2, \infty),$$

and we conclude by a continuity argument.

In order to prove (3.3) first notice that by the analysis of f' we get

$$x_{max} = \frac{5}{3C_{GN}^5}$$

and also since R is an optimizer for (3.2), then

$$E_{HW,-}(R)\|R\|_{L^2}^4 = \frac{1}{2}\big(\| |D_x|^{\frac{1}{2}} R\|_{L^2}\|R\|_{L^2}^2\big)^2 - \frac{1}{5}C_{GN}^5\big(\| |D_x|^{\frac{1}{2}} R\|_{L^2}\|R\|_{L^2}^2\big)^3.$$

Hence (3.3) follows provided that we prove

$$\frac{5}{3C_{GN}^5} = \| |D_x|^{\frac{1}{2}} R\|_{L^2}\|R\|_{L^2}^2. \tag{3.4}$$

To prove this fact notice that since R is an optimizer for (3.2) we get

$$\frac{d}{dt}\Big(\int |R+t\varphi|^5 dx - C_{GN}^5 \| |D_x|^{\frac{1}{2}}(R+t\varphi)\|_{L^2}^3 \|R+t\varphi\|_{L^2}^2\Big)_{t=0} = 0, \ \forall \varphi \in H^{\frac{1}{2}}$$

and hence by direct computations it implies

$$-(3C_{GN}^5\|R\|_{L^2}^2 \| |D_x|^{\frac{1}{2}} R\|_{L^2})|D_x|R - (2C_{GN}^5 \| |D_x|^{\frac{1}{2}} R\|_{L^2}^3)R + 5R^4 = 0.$$

Since R solves (0.8) we deduce that

$$3C_{GN}^5\|R\|_{L^2}^2 \| |D_x|^{\frac{1}{2}} R\|_{L^2} = 2C_{GN}^5 \| |D_x|^{\frac{1}{2}} R\|_{L^2}^3 = 5 \tag{3.5}$$

and hence we get (3.4). Notice that (3.5) follows by the fact that R cannot be a solution to $|D_x|R + aR = bR^4$ unless $a = b = 1$. In fact, if it not the case, then, since R solves (0.8), we would get $(b-1)R^4 = (a-1)R$ that implies R is a constant.

Conflict of interest statement

We confirm that the manuscript has been read and approved by all named authors and that there are no other persons who satisfied the criteria for authorship but are not listed. We further confirm that the order of authors listed in the manuscript has been approved by all of us.

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