



Singularity formation for the incompressible Hall-MHD equations without resistivity

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Abstract

In this paper we show that the incompressible Hall-MHD system without resistivity is not globally in time well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ for any $m > \frac{7}{2}$. Namely, either the system is locally ill-posed in $H^m(\mathbb{R}^3)$, or it is locally well-posed, but there exists an initial data in $H^m(\mathbb{R}^3)$, for which the $H^m(\mathbb{R}^3)$ norm of solution blows-up in finite time if $m > \frac{7}{2}$. In the latter case we choose an axisymmetric initial data $u_0(x) = u_{0r}(r, z)e_r + b_{0z}(r, z)e_z$ and $B_0(x) = b_{0\theta}(r, z)e_\theta$, and reduce the system to the axisymmetric setting. If the convection term survives sufficiently long time, then the Hall term generates the singularity on the axis of symmetry and we have $\limsup_{t \rightarrow t_*} \sup_{z \in \mathbb{R}} |\partial_z \partial_r b_\theta(r = 0, z)| = \infty$ for some $t_* > 0$, which will also induce a singularity in the velocity field.

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1. Introduction and main results

In this paper, we are concentrated on the singularity formation for the incompressible Hall-MHD equations without resistivity. The incompressible Hall-MHD equations without resistivity take the following form:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = (\nabla \times B) \times B + \nu \Delta u, \\ \operatorname{div} u = 0, \\ \partial_t B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = 0, \end{cases} \quad (1.1)$$

where $u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ and $B(t, x) = (b_1(t, x), b_2(t, x), b_3(t, x))$, $(x, t) \in [0, \infty) \times \mathbb{R}^3$, are the fluid velocity and magnetic field. $\nu \geq 0$ is the viscosity, $\nu = 0$ and $\nu > 0$ correspond to the inviscid and viscous flow respectively. We will consider the Cauchy problem for (1.1), so we prescribe the initial data

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$$u(t = 0, x) = u_0(x), \quad B(t = 0, x) = B_0(x).$$

The initial data u_0 and B_0 satisfy the divergence free condition,

$$\operatorname{div} u_0(x) = \operatorname{div} B_0(x) = 0.$$

From the equations for the magnetic field B , it is easy to see that if one prescribes the divergence condition $\operatorname{div} B_0 = 0$ on the initial data B_0 , then $\operatorname{div} B = 0$ for later time.

Comparing with the well-known MHD system, the Hall term $\nabla \times ((\nabla \times B) \times B)$ is included due to the Ohm's law, which is believed to be a key issue for understanding magnetic reconnection. Note that the Hall term is quadratic in the magnetic field and involves the second order derivatives. Magnetic reconnection corresponds to a physical process in highly conducting plasmas in which the magnetic topology is rearranged and magnetic energy is converted to kinetic energy, thermal energy, and particle acceleration. During this process, the magnetic shear is large, the Hall term becomes dominant. Lighthill [15] started the systematic study of the application of Hall-MHD on plasma, which is followed by [2]. One may refer to [19] for a physical review of the background for Hall-MHD.

There are many mathematical results on MHD system, for the existence of global weak solutions [9,20], regularity criterion [12,13] and global smooth small solutions [18,24]. The Hall-MHD has received little attention from mathematicians. The paper [1] provided a derivation of Hall-MHD system from a two-fluids Euler–Maxwell system for electrons and ions, through a set of scaling limits. They also provided a kinetic formulation for the Hall-MHD, and proved the existence of global weak solutions for the incompressible viscous resistive Hall-MHD system. The authors in [6] obtained the local existence of smooth solutions for large data and global smooth solutions for small data to incompressible resistive, viscous or inviscid Hall-MHD model. Chae and Lee [4] also established the blow-up criterion for classical solutions to the incompressible resistive Hall-MHD system. Contrary to the usual MHD, the global regularity for solutions to the 3-D Hall-MHD which depends only on two variables (i.e. $2\frac{1}{2}$ dimensional Hall-MHD)

is still open. Note that $2\frac{1}{2}$ dimensional Hall-MHD solution has been used in [14] to investigate the influence of the Hall term on the width of the magnetic islands of the tearing-mode. The temporal decay estimates for weak solutions to Hall-MHD system was established by Chae and Schonbek [5]. They also obtained algebraic decay rates for higher order Sobolev norms of strong solutions to (1.1) with small initial data. It turned out that the Hall term does not affect the time asymptotic behavior, and the time decay rates behaved like those of the corresponding heat equation.

In this paper we investigate the singularity formation for (1.1). Dreher, Ruban and Grauer [8] have discussed the possible spontaneous development of shock-type singularities in axisymmetric solutions of the ideal Hall-MHD system and performed numerical simulation to support their claim. In the following we rigorously prove that for the incompressible Hall-MHD system (1.1) without resistivity the solution cannot preserve initial data regularity in $H^m(\mathbb{R}^3)$, $m > \frac{7}{2}$. Either the solution breakdown the initial data regularity or uniqueness at the initial instant of moment, or if the solution survives uniquely for a positive time, and if the convection term survives sufficiently long time, then a shock-type singularity in the magnetic field will develop in finite time, and this will also induce a singularity formation in the velocity field. As is well known, the global regularity problem for the 3-D incompressible Navier–Stokes equations is still widely open [11]. Unlike the compressible fluid case where one can show that the singularity will generate in finite time for both compressible Euler [22] and compressible Navier–Stokes [23], it is very difficult to exhibit any singularity formation in incompressible fluids. To our best knowledge, our blowup results on the incompressible Hall-MHD without resistivity seems to be the first physically interesting example, showing that there exists “singularity formation” in an incompressible system (i.e. satisfying the divergence free condition). Here by “singularity formation” we mean either the solution loses the regularity immediately or the solution blows up in finite time. From this point of view, we believe that the Hall-MHD system has its own interest and deserves more attention from mathematician.

Now we start the mathematical setup of our problem. We will choose a special class of smooth axisymmetric initial data with the form $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z \in (C_c^\infty(\mathbb{R}^3))^3$ and $B_0(x) = b_{0\theta}(r, z)e_\theta \in (C_c^\infty(\mathbb{R}^3))^3$, such that the corresponding solution $(u, B)(t, x)$ to (1.1) will develop a singularity for the magnetic field in finite time, which will also induce a singularity in the velocity field. Let us introduce the cylindrical coordinate

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3$$

and then investigate the axisymmetric solution to (1.1). In this case the velocity and magnetic field can be described as follows

$$\begin{aligned} u(t, x) &= u_r(t, r, z)e_r + u_\theta(t, r, z)e_\theta + u_z(t, r, z)e_z, \\ B(t, x) &= b_r(t, r, z)e_r + b_\theta(t, r, z)e_\theta + b_z(t, r, z)e_z, \\ p(t, x) &= p(t, r, z), \end{aligned}$$

where

$$e_r = (\cos \theta, \sin \theta, 0), \quad e_\theta = (-\sin \theta, \cos \theta, 0), \quad e_z = (0, 0, 1).$$

The Hall-MHD equation (1.1) can be written as the following equations in cylindrical coordinate

$$\begin{aligned} &\partial_t u_r + \left((u_r \partial_r + u_z \partial_z) u_r - \frac{u_\theta^2}{r} \right) + \partial_r \left(p + \frac{1}{2} (b_r^2 + b_\theta^2 + b_z^2) \right) \\ &= \left((b_r \partial_r + b_z \partial_z) b_r - \frac{b_\theta^2}{r} \right) + \nu (\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}) u_r, \\ &\partial_t u_\theta + \left((u_r \partial_r + u_z \partial_z) u_\theta + \frac{u_r u_\theta}{r} \right) \\ &= \left((b_r \partial_r + b_z \partial_z) b_\theta + \frac{b_r b_\theta}{r} \right) + \nu (\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}) u_\theta, \\ &\partial_t u_z + (u_r \partial_r + u_z \partial_z) u_z + \partial_z \left(p + \frac{1}{2} (b_r^2 + b_\theta^2 + b_z^2) \right) \\ &= (b_r \partial_r + b_z \partial_z) b_z + \nu (\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2) u_z, \\ &\partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0, \\ &\partial_t b_r + (u_r \partial_r + u_z \partial_z) b_r - (b_r \partial_r + b_z \partial_z) u_r - \frac{\partial}{\partial z} (j_z b_r - j_r b_z) = 0, \\ &\partial_t b_\theta + \left((u_r \partial_r + u_z \partial_z) b_\theta + \frac{b_r u_\theta}{r} \right) - \left((b_r \partial_r + b_z \partial_z) u_\theta + \frac{u_r b_\theta}{r} \right) \\ &\quad + \left(\frac{\partial}{\partial z} (j_\theta b_z - j_z b_\theta) - \frac{\partial}{\partial r} (j_r b_\theta - j_\theta b_r) \right) = 0, \\ &\partial_t b_z + (u_r \partial_r + u_z \partial_z) b_z - (b_r \partial_r + b_z \partial_z) u_z + \frac{1}{r} \frac{\partial}{\partial r} (r (j_z b_r - j_r b_z)) = 0, \\ &\partial_r b_r + \frac{1}{r} b_r + \partial_z b_z = 0. \end{aligned}$$

Here $j(t, x) = \nabla \times B = j_r(t, r, z)e_r + j_\theta(t, r, z)e_\theta + j_z(t, r, z)e_z$ and

$$j_r = -\partial_z b_\theta, \quad j_\theta = \partial_z b_r - \partial_r b_z, \quad j_z = \frac{1}{r} \partial_r (r b_\theta).$$

From these equations, one can easily find that for any smooth solution (u_r, u_θ, u_z) and (b_r, b_θ, b_z) , if initially one has

$$u_\theta(0, r, z) = b_r(0, r, z) = b_z(0, r, z) = 0, \tag{1.2}$$

then $u_\theta(t, r, z) = b_r(t, r, z) = b_z(t, r, z) \equiv 0$ for $t > 0$. Hence $j_\theta \equiv 0$ and

$$\begin{aligned} &\left(\frac{\partial}{\partial z} (j_\theta b_z - j_z b_\theta) - \frac{\partial}{\partial r} (j_r b_\theta - j_\theta b_r) \right) = -\partial_z (j_z b_\theta) - \partial_r (j_r b_\theta) \\ &= -(\partial_r j_r + \partial_z j_z) b_\theta - (j_r \partial_r + j_z \partial_z) b_\theta = -\frac{2b_\theta}{r} \partial_z b_\theta, \end{aligned}$$

where we have used the fact that $\operatorname{div}(\nabla \times B) = 0$, so $\partial_r j_r + \frac{1}{r} j_r + \partial_z j_z = 0$. Finally under the initial condition (1.2) the above equations reduce to

$$\begin{cases} \partial_t u_r + (u_r \partial_r + u_z \partial_z) u_r + \partial_r(p + \frac{1}{2} b_\theta^2) = -\frac{b_\theta^2}{r} + v(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}) u_r, \\ \partial_t u_z + (u_r \partial_r + u_z \partial_z) u_z + \partial_z(p + \frac{1}{2} b_\theta^2) = v(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2) u_z, \\ \partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0, \\ \partial_t b_\theta + (u_r \partial_r + u_z \partial_z) b_\theta - \frac{u_r b_\theta}{r} - \frac{2b_\theta}{r} \partial_z b_\theta = 0, \\ (u_r, u_z)(t = 0, r, z) = (u_{0r}, u_{0z})(r, z), \quad b_\theta(t = 0, r, z) = b_{0\theta}(r, z). \end{cases} \tag{1.3}$$

In this case, the vorticity $\omega(t, x) = \operatorname{curl} u(t, x) = \omega_\theta(t, r, z) e_\theta = (\partial_z u_r - \partial_r u_z)(t, r, z) e_\theta$ satisfies the following equation

$$\begin{aligned} \frac{\partial \omega_\theta}{\partial t} + (u_r \partial_r + u_z \partial_z) \omega_\theta + 2 \frac{b_\theta}{r} \partial_z b_\theta - \frac{u_r}{r} \omega_\theta &= v(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}) \omega_\theta, \\ \omega_\theta(t = 0, r, z) = \omega_{0\theta}(r, z) &= (\partial_z u_{0r} - \partial_r u_{0z})(r, z). \end{aligned}$$

Define the new unknowns $\Omega = \frac{\omega_\theta}{r}$ and $\Pi = \frac{b_\theta}{r}$, then one can check easily that Ω and Π satisfy the following equations

$$\frac{\partial \Omega}{\partial t} + (u_r \partial_r + u_z \partial_z) \Omega + 2 \Pi \partial_z \Pi = v(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega, \tag{1.4}$$

$$\frac{\partial \Pi}{\partial t} + (u_r \partial_r + u_z \partial_z) \Pi - 2 \Pi \partial_z \Pi = 0, \tag{1.5}$$

$$\Omega(t = 0, r, z) = \Omega_0(r, z) := \frac{\omega_{0\theta}(r, z)}{r}, \tag{1.6}$$

$$\Pi(t = 0, r, z) = \Pi_0(r, z) := \frac{b_{0\theta}(r, z)}{r}. \tag{1.7}$$

We refer two closely related results on the axisymmetric solution to the usual MHD or Hall-MHD system. Lei [17] has showed that the existence of global in time smooth solution to the incompressible viscous MHD without resistivity for some special axisymmetric data $u_0 = u_{0r} e_r + u_{0z} e_z$ and $B_0 = b_{0\theta} e_\theta$. The result in [10] established the existence of global smooth solution to the incompressible viscous, resistive Hall-MHD system with same initial data as in [17]. Our first main result is the formation singularity for the incompressible viscous Hall-MHD without resistivity.

Theorem 1.1 (Viscous case $v = 1$). *The incompressible viscous Hall-MHD system without resistivity (1.1) is not globally well-posed in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. There exists smooth initial data $u_0(x) = u_{0r}(r, z) e_r + u_{0z}(r, z) e_z \in C_c^\infty(\mathbb{R}^3)$, $B_0(x) = b_{0\theta}(r, z) e_\theta \in C_c^\infty(\mathbb{R}^3)$ with $\Pi_0(r, z) \in L^\infty(\mathbb{R}^3)$ such that if there is a local in time smooth solution $(u, B)(t, x)$ to (1.1) with initial data (u_0, B_0) , then (u, B) must blow up in finite time. Indeed, one can choose (u_0, B_0) such that $y_0 := \partial_z \Pi_0(0, 0) = \partial_{r,z}^2 b_{0\theta}(0, 0) \geq 10^4 C_*^2$, $t_0 = \frac{2}{y_0}$ and $J_0 := \Pi_0(0, 0) > 0$, where C_* depends only on $\|u_0\|_{H^2(\mathbb{R}^3)}$, $\|B_0\|_{H^1(\mathbb{R}^3)}$ and $\|\Pi_0\|_{L^\infty(\mathbb{R}^3)}$, then*

$$\limsup_{t \rightarrow t_0} \sup_{z \in \mathbb{R}} |\partial_z \Pi(t, 0, z)| = \infty.$$

Moreover, the velocity field also blows up

$$\limsup_{t \rightarrow t_0} \sup_{z \in \mathbb{R}} \left| \left(\partial_t \Omega + (u_r \partial_r + u_z \partial_z) \Omega - (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \Omega \right) (t, 0, z) \right| = \infty.$$

Remark 1.1. As will be shown in the proof below, the singularity occurs on the axis if the local well-posed is done. The blow-up happens on the second order derivative of b_θ and the third derivative of the velocity field. Whether the solution can blow-up off the axis is not clear yet.

Remark 1.2. Due to the Hall term it seems difficult to show that the local in time existence of smooth solution to (1.1). We could not rule out the possibility at this moment that (1.1) is locally ill-posed (see Remark 3.1 of [6]).

Remark 1.3. Note that in Theorem 1.1, C_* depends only on $\|u_0\|_{H^2(\mathbb{R}^3)}$, $\|B_0\|_{H^1(\mathbb{R}^3)}$ and $\|\Pi_0\|_{L^\infty(\mathbb{R}^3)}$, hence C_* depends only on the first order derivatives of B_0 and B_0 itself. So the condition $y_0 = \partial_{rz}^2 b_{0\theta}(0, 0) \geq 10^4 C_*^2$ can be guaranteed, the initial data $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z$, $B_0(x) = b_{0\theta}(r, z)e_\theta$ satisfying the conditions in Theorem 1.1 indeed exist.

Remark 1.4. If one consider Eqs. (1.1) with only partial viscosity in the z -direction, i.e. replace Δu by $\partial_z^2 u$, then Theorem 1.1 is still true. We will indicate the corresponding modification in the following section.

The second result concentrates on the singularity formation for the inviscid Hall-MHD system without resistivity.

Theorem 1.2 (Inviscid case $\nu = 0$). *The incompressible inviscid Hall-MHD system without resistivity (1.1) is not globally well-posedness in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. There exists smooth initial data $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z \in C_c^\infty(\mathbb{R}^3)$, $B_0(x) = b_{0\theta}(r, z)e_\theta \in C_c^\infty(\mathbb{R}^3)$ with $(\Omega_0, \Pi_0)(r, z) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ such that if there is a local in time smooth solution $(u, B)(t, x)$ to (1.1) with initial data (u_0, B_0) , then (u, B) must blow up in finite time. Indeed, one can choose (u_0, B_0) such that $y_0 := \partial_z \Pi_0(0, 0) = \partial_{rz}^2 b_{0\theta}(0, 0) \geq 4C_\sharp^{1/2}$, $t_\sharp = \frac{2}{y_0}$ and $J_0 := \Pi_0(0, 0) > 0$, where C_\sharp depends only on $\|\Omega_0\|_{L^1 \cap L^\infty} + \|\Pi_0\|_{L^1 \cap L^\infty}$, where $\|f\|_{L^1 \cap L^\infty} := \|f\|_{L^1(\mathbb{R}^3)} + \|f\|_{L^\infty(\mathbb{R}^3)}$, then*

$$\limsup_{t \rightarrow t_\sharp} \sup_{z \in \mathbb{R}} |\partial_z \Pi(t, 0, z)| = \infty.$$

Moreover, the velocity field also blows up

$$\limsup_{t \rightarrow t_\sharp} \sup_{z \in \mathbb{R}} |(\partial_t \Omega + (u_r \partial_r + u_z \partial_z)\Omega)(t, 0, z)| = \infty.$$

The paper will proceed as follows. In Section 2, we will give some a priori estimates on the smooth solutions to (1.3). Then we prove Theorem 1.1 and 1.2 in the last section.

2. Some a priori estimates for solutions to (1.3)

We first explain the difficulties and the key issues in our proof for Theorems 1.1 and 1.2. Since $\partial_t \Pi + (u_r \partial_r + u_z \partial_z)\Pi - 2\Pi \partial_z \Pi = 0$, we know that $\partial_z \Pi$ satisfies a Riccati type equation for

$$\partial_t \partial_z \Pi + (u_r \partial_r + u_z \partial_z - 2\Pi \partial_z)\partial_z \Pi - 2(\partial_z \Pi)^2 + \partial_z u_r \partial_r \Pi + \partial_z u_z \partial_z \Pi = 0. \tag{2.1}$$

From (2.1), we see that the convective term $(u_r \partial_r + u_z \partial_z)\Pi$ may prevent the blowup of $\partial_z \Pi$. To avoid the trouble caused by u_r , we first observe that $u_r(t, r = 0, z) \equiv 0$, so we have $\partial_z u_r(t, r = 0, z) \equiv 0$. Hence if we restrict Eq. (2.1) to $r = 0$, then we obtain

$$\partial_t \partial_z \Pi(t, 0, z) + (u_z - 2\Pi)\partial_z \partial_z \Pi(t, 0, z) - 2(\partial_z \Pi)^2(t, 0, z) + (\partial_z u_z \partial_z \Pi)(t, 0, z) = 0. \tag{2.2}$$

The remain thing is to get some strong enough estimate for $\partial_z u_z$, so that we can show that the quadratic $-2(\partial_z \Pi)^2$ will control the growth of $\partial_z \Pi$. Our idea is to use the divergence free condition to replace $\partial_z u_z(t, 0, z)$ by $\lim_{r \rightarrow 0} \frac{u_r(t, r, z)}{r}$

$$\partial_z u_z(t, 0, z) = - \lim_{r \rightarrow 0^+} (\partial_r u_r + \frac{1}{r} u_r)(t, r, z) = -2 \lim_{r \rightarrow 0} \frac{u_r(t, r, z)}{r}.$$

Hence we obtain

$$\partial_t \partial_z \Pi(t, 0, z) + (u_z - 2\Pi)\partial_z \partial_z \Pi(t, 0, z) - 2\left(\partial_z \Pi(t, 0, z) + \lim_{r \rightarrow 0} \frac{u_r(t, r, z)}{r}\right)\partial_z \Pi(t, 0, z) = 0. \tag{2.3}$$

It turns out that it is much easier to get a good estimate for $\frac{u_r}{r}$ than $\partial_z u_z$, which is strong enough to show that the blowup of $\partial_z \Pi$. From these explanations, one can also understand why it is difficult to show that the singularity occurs outside the axis. In the following, we will do some a priori estimates to get a bound for $\int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty} ds$ by distinguishing the viscous and inviscid cases. In the next section, we will use these estimates to show that $\partial_z \Pi$ will blow up in finite time.

2.1. A priori estimates: Viscous case $\nu = 1$

First we give some a priori estimates for solutions to (1.3). The following lemma shows that the maximum principle for Π . The proof is easy, we omit the details.

Lemma 2.1. *For any smooth solution (u_r, u_z, b_θ, p) to (1.3) with initial data $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z \in C_c^\infty(\mathbb{R}^3)$, $B_0(x) = b_{0\theta}(r, z)e_\theta \in C_c^\infty(\mathbb{R}^3)$ satisfying $\Pi_0(r, z) \in L^\infty(\mathbb{R}^3)$, then we have*

$$\|\Pi(t, r, z)\|_{L^\infty} \leq \|\Pi_0(r, z)\|_{L^\infty}.$$

If $\Pi_0 \in L^2(\mathbb{R}^3)$, then

$$\|\Pi(t, \cdot)\|_{L^2(\mathbb{R}^3)} = \|\Pi_0\|_{L^2(\mathbb{R}^3)}.$$

Lemma 2.2 (L^2 estimate of Ω). *Assume that the initial data (u_0, B_0) satisfy $u_0 \in H^2(\mathbb{R}^3)$, $B_0 \in H^1(\mathbb{R}^3)$ and $\Pi_0 \in L^\infty(\mathbb{R}^3)$. Then we have the following estimate for Ω*

$$\begin{aligned} & \|\Omega(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla \Omega(s, \cdot)\|_{L^2}^2 ds + 2\pi \int_0^t |\Omega(s, 0, z)|^2 dz \\ & \leq C_1(\|u_0\|_{H^2(\mathbb{R}^3)}, \|B_0\|_{H^1(\mathbb{R}^3)}, \|\Pi_0\|_{L^\infty(\mathbb{R}^3)})(1+t). \end{aligned}$$

Proof. By (1.4), one can easily obtain the L^2 estimate for Ω

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 + 2\pi \int_{\mathbb{R}} |\Omega(t, 0, z)|^2 dz \\ & = - \int \Omega \partial_z \Pi^2 dx = \int \Pi^2 \partial_z \Omega dx \\ & \leq \|\Pi\|_{L^\infty} \|\Pi\|_{L^2} \|\partial_z \Omega\|_{L^2} \leq 4\|\Pi\|_{L^\infty}^2 \|\Pi\|_{L^2}^2 + \frac{1}{2} \|\partial_z \Omega\|_{L^2}^2. \end{aligned}$$

Hence we obtain

$$\begin{aligned} & \frac{d}{dt} \|\Omega\|_{L^2}^2 + \|\nabla \Omega\|_{L^2}^2 + 2\pi \int_{\mathbb{R}} |\Omega(t, 0, z)|^2 dz \\ & \leq 4\|\Pi\|_{L^\infty}^2 \|\Pi\|_{L^2}^2 \leq 4\|\Pi_0\|_{L^\infty}^2 \|\Pi_0\|_{L^2}^2. \end{aligned}$$

This will imply the following estimate for Ω

$$\begin{aligned} & \|\Omega(t, \cdot)\|_{L^2}^2 + \int_0^t \|\nabla \Omega(s, \cdot)\|_{L^2}^2 ds + 2\pi \int_0^t |\Omega(s, 0, z)|^2 dz ds \\ & \leq \|\Omega_0\|_{L^2}^2 + 4\|\Pi_0\|_{L^\infty}^2 \|\Pi_0\|_{L^2}^2 t \\ & \leq \|u_0\|_{H^2}^2 + 4\|\Pi_0\|_{L^\infty}^2 \|B_0\|_{H^1}^2 t \leq C_1(\|u_0\|_{H^2}, \|\Pi_0\|_{L^\infty}, \|B_0\|_{H^1})(1+t). \quad \square \end{aligned}$$

Remark 2.1. If one consider Eqs. (1.1) with only partial viscosity in the z -direction, i.e. replace Δu by $\partial_z^2 u$, then we still have the following estimate

$$\begin{aligned} & \|\Omega(t, \cdot)\|_{L^2}^2 + \int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2}^2 ds \\ & \leq C_1 (\|u_0\|_{H^2(\mathbb{R}^3)}, \|B_0\|_{H^1(\mathbb{R}^3)}, \|\Pi_0\|_{L^\infty(\mathbb{R}^3)}) (1+t). \end{aligned} \tag{2.4}$$

We also need the following estimate for $\frac{u_r}{r}$. A similar estimate has appeared in Lemma 3.1 in [17].

Lemma 2.3. *The following estimate holds for $\frac{u_r}{r}$:*

$$\int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds \leq \sup_{0 \leq s \leq t} \|\Omega(s, \cdot)\|_{L^2} \int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2} ds \leq C_*(1+t)t^{1/2}, \tag{2.5}$$

where C_* depends only on $\|u_0\|_{H^2(\mathbb{R}^3)}, \|B_0\|_{H^1(\mathbb{R}^3)}, \|\Pi_0\|_{L^\infty(\mathbb{R}^3)}$.

Proof. For the convenience of the reader we give a sketch of proof. For more details of the proof, one may refer to [17]. By the divergence free condition, $\partial_r(ru_r) + \partial_z(ru_z) = 0$, one can introduce a stream function ψ^θ such that

$$u_r = -\partial_z \psi_\theta, \quad u_z = \frac{1}{r} \partial_r(r\psi_\theta).$$

Since $\omega_\theta = \partial_z u_r - \partial_r u_z$, we have

$$-(\partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2}) \psi_\theta = \omega_\theta.$$

Setting $\varphi = \frac{\psi_\theta}{r}$, then it is easy to see that

$$-(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2) \varphi = \Omega.$$

As in [17], the second order operator $(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2)$ can be interpreted as the Laplace operator in 5-dimensional space. We introduce

$$y = (y_1, y_2, y_3, y_4, z), \quad r = \sqrt{y_1^2 + y_2^2 + y_3^2 + y_4^2}, \quad \Delta_y = (\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2).$$

Hence we have $\varphi = (-\Delta_y)^{-1} \Omega$. To get an estimate of $\|\frac{u_r}{r}\|_{L^\infty}$, by a simple interpolation inequality $\|f\|_{L^\infty}^2 \leq C_2 \|\nabla f\|_{L^2} \|\nabla^2 f\|_{L^2}$, we have

$$\begin{aligned} \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds &= \int_0^t \|\partial_z \varphi(s, \cdot)\|_{L^\infty}^2 ds \\ &\leq C_2 \int_0^t \|\nabla \partial_z \varphi(s, \cdot)\|_{L^2} \|\nabla^2 \partial_z \varphi(s, \cdot)\|_{L^2} ds. \end{aligned}$$

By simple calculations, one has

$$|\nabla_y^2 \varphi|^2 \simeq |\partial_r^2 \varphi|^2 + |\frac{1}{r} \partial_r \varphi|^2 + |\partial_z^2 \varphi|^2 + |\partial_{rz}^2 \varphi|^2$$

and

$$\begin{aligned}
 \int |\nabla^2 \varphi|^2 dx &\leq C_3 \int_{-\infty}^{\infty} \int_0^{\infty} \left(|\partial_r^2 \varphi|^2 + \left| \frac{1}{r} \partial_r \varphi \right|^2 + |\partial_z^2 \varphi|^2 + |\partial_{rz}^2 \varphi|^2 \right) r dr dz \\
 &= C_3 \int_{-\infty}^{\infty} \int_0^{\infty} \left(|\partial_r^2 \varphi|^2 + \left| \frac{1}{r} \partial_r \varphi \right|^2 + |\partial_z^2 \varphi|^2 + |\partial_{rz}^2 \varphi|^2 \right) w(r) r^3 dr dz \\
 &\leq C_4 \int_{-\infty}^{\infty} \int_0^{\infty} |\nabla_y^2 \varphi|^2 w(r) r^3 dr dz = C_4 \int_{-\infty}^{\infty} \int_0^{\infty} |\nabla_y^2 (-\Delta_y)^{-1} \Omega|^2 w(r) r^3 dr dz \\
 &= C_4 \int |\nabla_y^2 (-\Delta_y)^{-1} \Omega|^2 w(r) dy \\
 &\leq C_5 \int |\Omega|^2 w(r) dy = C_5 \int |\Omega|^2 dx,
 \end{aligned}$$

where $w(r) = r^{-2}$ and in the last step we have used the boundedness of Riesz operators in weighted Sobolev spaces (Lemma 2 in [16]). See also Corollary 2 in [3] for a similar weighted estimate for a singular integral operator.

Similarly, we also have

$$\int |\nabla^2 \partial_z \varphi|^2 dx \leq C_6 \int |\partial_z \Omega|^2 dx.$$

Hence

$$\begin{aligned}
 \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds &\leq C_2 \int_0^t \|\nabla \partial_z \varphi(s, \cdot)\|_{L^2} \|\nabla^2 \partial_z \varphi(s, \cdot)\|_{L^2} ds \\
 &\leq C_7 \int_0^t \|\Omega(s, \cdot)\|_{L^2} \|\partial_z \Omega(s, \cdot)\|_{L^2} ds \\
 &\leq C_7 \sup_{0 \leq s \leq t} \|\Omega(s, \cdot)\|_{L^2} \int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2} ds \\
 &\leq C_7 C_1^{\frac{1}{2}} (1+t)^{\frac{1}{2}} \left(\int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2}^2 ds \right)^{\frac{1}{2}} t^{\frac{1}{2}} \\
 &\leq C_* (1+t) t^{\frac{1}{2}},
 \end{aligned}$$

where C_* depends only on $\|u_0\|_{H^2(\mathbb{R}^3)}$, $\|B_0\|_{H^1(\mathbb{R}^3)}$, $\|\Pi_0\|_{L^\infty(\mathbb{R}^3)}$. \square

Remark 2.2. If one consider Eqs. (1.1) with only partial viscosity in the z -direction, i.e. replace Δu by $\partial_z^2 u$, then we still have the following estimate

$$\int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds \leq \sup_{0 \leq s \leq t} \|\Omega(s, \cdot)\|_{L^2} \int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2} ds \leq C_* (1+t) t^{\frac{1}{2}}. \tag{2.6}$$

2.2. A priori estimates: Inviscid case $\nu = 0$

In this case, then the equations satisfied by Ω and Π will reduce to

$$\begin{cases} \partial_t \Omega + (u_r \partial_r + u_z \partial_z) \Omega + 2\Pi \partial_z \Pi = 0, \\ \partial_t \Pi + (u_r \partial_r + u_z \partial_z) \Pi - 2\Pi \partial_z \Pi = 0, \\ (\Omega, \Pi)(t=0, r, z) = (\Omega_0, \Pi_0)(r, z). \end{cases} \tag{2.7}$$

Putting $\Gamma = \Omega + \Pi$, it is easy to see that

$$\begin{cases} \partial_t \Gamma + (u_r \partial_r + u_z \partial_z) \Gamma = 0, \\ \Gamma(t = 0, r, z) = \Omega_0(r, z) + \Pi_0(r, z) := \Gamma_0(r, z). \end{cases} \tag{2.8}$$

This simple, but important observation plays a key role in our following argument. Note that (2.8) indeed comes from (1.1) with $v = 0$ by observing that $R = \text{curl } u + B$ satisfies the following equation

$$\partial_t R + u \cdot \nabla R - R \cdot \nabla u = 0. \tag{2.9}$$

Lemma 2.4. *For any smooth solution (u_r, u_z, b_θ, p) to (1.3) with initial data $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z \in C_c^\infty(\mathbb{R}^3)$, $B_0(x) = b_{0\theta}(r, z)e_\theta \in C_c^\infty(\mathbb{R}^3)$ satisfying $(\Omega_0, \Pi_0)(r, z) \in L^1 \cap L^\infty$, then we have*

$$\|\Pi(t, r, z)\|_{L^1 \cap L^\infty} \leq \|\Pi_0(r, z)\|_{L^1 \cap L^\infty}, \tag{2.10}$$

$$\|\Omega(t, r, z)\|_{L^1 \cap L^\infty} \leq \|\Omega_0(r, z)\|_{L^1 \cap L^\infty} + \|\Pi_0(r, z)\|_{L^1 \cap L^\infty}. \tag{2.11}$$

Next we need the following inequality, which comes from the Biot–Savart law (note that $\text{curl}(u_r(t, r, z)e_r + u_z(t, r, z)e_z) = \omega_\theta(t, r, z)e_\theta$) and has been proved in [21] long time ago. One can refer to Lemma 2 in [7] for more details.

Lemma 2.5. *There exists a universal constant C_8 such that*

$$|u_r(t, x)| \leq C_8 \int_{\mathbb{R}^3} \min\left(1, \frac{r}{|x' - x|}\right) \frac{|\omega_\theta(t, x')|}{|x - x'|^2} dx', \tag{2.12}$$

which yields

$$\frac{|u_r(t, x)|}{r} \leq 2C_8 \int_{\mathbb{R}^3} \frac{1}{|x - x'|^2} \frac{|\omega_\theta(t, x')|}{r'} dx'. \tag{2.13}$$

Note that here we use the notation

$$u_r(t, x) := u_r(t, \sqrt{x_1^2 + x_2^2}, x_3), \quad \omega_\theta(t, x) := \omega_\theta(t, \sqrt{x_1^2 + x_2^2}, x_3).$$

From (2.13), we have for any $t > 0$

$$\begin{aligned} \left\| \frac{u_r}{r}(t, \cdot) \right\|_{L^\infty(\mathbb{R}^3)} &\leq 2C_9 \int_{\mathbb{R}^3} \frac{1}{|x - x'|^2} |\Omega(t, x')| dx' \\ &\leq 2C_9 \left(\int_{|x-x'| \leq 1} + \int_{|x-x'| > 1} \right) \frac{1}{|x - x'|^2} |\Omega(t, x')| dx' \\ &\leq 2C_9 \left(\|\Omega(t)\|_{L^\infty} \int_{|x-x'| \leq 1} \frac{1}{|x - x'|^2} dx' + \|\Omega(t)\|_{L^1(\mathbb{R}^3)} \right) \\ &\leq C_{10} \|\Omega(t)\|_{L^1 \cap L^\infty} \\ &\leq C_{10} \left(\|\Omega_0(r, z)\|_{L^1 \cap L^\infty} + \|\Pi_0(r, z)\|_{L^1 \cap L^\infty} \right) := C_\sharp, \end{aligned} \tag{2.14}$$

where C_{10} is also a universal constant.

3. Singularity formation

3.1. Viscous case $\nu = 1$

Proof of Theorem 1.1. Suppose the incompressible viscous Hall-MHD system without resistivity (1.1) is globally well-posed in some Sobolev space $H^m(\mathbb{R}^3)$, where $m > \frac{7}{2}$. That is to say, for any initial data $(u_0, B_0) \in H^m(\mathbb{R}^3)$, there exists a unique smooth solution $(u, B) \in C([0, \infty); H^m(\mathbb{R}^3))$ to the Hall-MHD (1.1). We will derive a contradiction to this.

In the following, we will choose a special class of smooth axisymmetric initial data with the form $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z \in (C_c^\infty(\mathbb{R}^3))^3$ and $B_0(x) = b_{0\theta}(r, z)e_\theta \in (C_c^\infty(\mathbb{R}^3))^3$, such that the corresponding solution $(u, B)(t, x)$ to (1.1) will develop a singularity for the magnetic field in finite time, which will also induce a singularity in the velocity field. Hence we can conclude that the Hall-MHD system (1.1) is not global well-posedness in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$.

For Hall-MHD system with initial data $u_0(x) = u_{0r}(r, z)e_r + u_{0z}(r, z)e_z$ and $B_0(x) = b_{0\theta}(r, z)e_\theta$, by uniqueness, we can show that the corresponding solution $(u, B)(t, x)$ should be axisymmetric and has the form

$$u(t, x) = u_r(t, r, z)e_r + u_z(t, r, z)e_z, \quad B(t, x) = b_\theta(t, r, z)e_\theta,$$

where $(u_r, u_z, b_\theta)(t, r, z)$ should solve the system (1.3) with initial data $(u_{0r}, u_{0z}, b_{0\theta})$. Indeed, for any $\alpha \in [0, 2\pi)$, we define the following change of coordinate

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Setting

$$\tilde{u}(t, y) = Au(t, A^{-1}y), \quad \tilde{B}(t, y) = AB(t, A^{-1}y), \quad \tilde{p}(t, y) = p(t, A^{-1}y),$$

then it is easy to verify that $(\tilde{u}(t, y), \tilde{B}(t, y), \tilde{p}(t, y))$ solves (1.1) with initial data

$$\tilde{u}(t = 0, y) = Au_0(A^{-1}y), \quad \tilde{B}(t = 0, y) = AB_0(A^{-1}y).$$

By the axisymmetric property of $(u_0(x), B_0(x))$, we have $Au_0(A^{-1}y) = u_0(y)$, $AB_0(A^{-1}y) = B_0(y)$. Hence by uniqueness of (1.1), we have

$$\tilde{u}(t, y) \equiv u(t, y), \quad \tilde{B}(t, y) \equiv B(t, y), \quad \tilde{p}(t, y) \equiv p(t, y).$$

Since $\alpha \in [0, 2\pi)$ is arbitrary, we find that $(u, B, p)(t, x)$ must be axisymmetric and is of the form

$$u(t, x) = u_r(t, r, z)e_r + u_z(t, r, z)e_z, \quad B(t, x) = b_\theta(t, r, z)e_\theta, \quad p(t, x) = p(t, r, z)$$

where $(u_r, u_z, b_\theta, p)(t, r, z)$ solve the problem (1.3) (see lines below (1.2)).

Hence the a priori estimates established in Section 2 hold for $(u_r, u_z, b_\theta)(t, r, z)$. In particular, we have the following estimate

$$\int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds \leq \sup_{0 \leq s \leq t} \|\Omega(s, \cdot)\|_{L^2} \int_0^t \|\partial_z \Omega(s, \cdot)\|_{L^2} ds \leq C_*(1+t)t^{\frac{1}{2}},$$

where C_* depends only on $\|u_0\|_{H^2(\mathbb{R}^3)}$, $\|B_0\|_{H^1(\mathbb{R}^3)}$, $\|\Pi_0\|_{L^\infty(\mathbb{R}^3)}$.

As we explained in the beginning of the second section, we know that $\partial_z \Pi(t, 0, z)$ will satisfy the following equation

$$\partial_t \partial_z \Pi(t, 0, z) + (u_z - 2\Pi) \partial_z \partial_z \Pi(t, 0, z) - 2 \left(\partial_z \Pi(t, 0, z) + \lim_{r \rightarrow 0} \frac{u_r(t, r, z)}{r} \right) \partial_z \Pi(t, 0, z) = 0. \tag{3.1}$$

Define the particle trajectory on the axis of symmetry $\phi(t, z)$ as follows

$$\begin{cases} \frac{d}{dt}\phi(t, z) = (u_z - 2\Pi)(t, 0, \phi(t, z)), \\ \phi(0, z) = z. \end{cases}$$

Then by setting $f(t, z) = \partial_z \Pi(t, 0, \phi(t, z))$ and $g(t, z) = \partial_r u_r(t, 0, \phi(t, z)) = \lim_{r \rightarrow 0} \frac{u_r}{r}(t, 0, \phi(t, z))$, we know that

$$\begin{aligned} \frac{d}{dt} f(t, z) &= 2f^2(t, z) - 2g(t, z)f(t, z) \\ &\geq f^2(t, z) - g^2(t, z). \end{aligned}$$

Integrating over $[0, t]$, we have

$$\begin{aligned} f(t, z) - f(0, z) &\geq \int_0^t f^2(s, z) ds - \int_0^t g^2(s, z) ds \\ &\geq \int_0^t f^2(s, z) ds - \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds. \end{aligned} \tag{3.2}$$

Fix $z = 0$ and set $y_0 = f(0, 0) = \partial_z \Pi_0(0, 0)$, then by employing the estimate (2.5) in Lemma 2.3, we obtain

$$f(t, 0) \geq \int_0^t f^2(s, 0) ds + y_0 - \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds \tag{3.3}$$

$$\geq \int_0^t f^2(s, 0) ds + y_0 - C_*(1+t)t^{\frac{1}{2}}. \tag{3.4}$$

Now take $y_0 \geq 10^4 C_*^2$ and $T_* = \frac{4}{y_0}$, for $t \in [0, T_*]$, we have

$$\begin{aligned} f(t, 0) &\geq \int_0^t f^2(s, 0) ds + y_0 - 4C_* \times \frac{1}{100C_*} \\ &\geq \int_0^t f^2(s, 0) ds + \frac{1}{2}y_0. \end{aligned}$$

Define a new function $F(t) = \int_0^t f^2(s, 0) ds + \frac{1}{2}y_0$, then $F(t)$ satisfies the following inequality

$$\begin{aligned} F'(t) &\geq F^2(t), \quad t \in [0, T_*], \\ F(0) &= \frac{1}{2}y_0. \end{aligned}$$

Hence we have

$$F(t) \geq \frac{y_0}{2 - ty_0}.$$

This simply implies that

$$\limsup_{t \rightarrow t_0} F(t) = \infty, \quad \limsup_{t \rightarrow t_0} f(t, 0) = \limsup_{t \rightarrow t_0} \partial_z \Pi(t, 0, \phi(t, 0)) = \infty,$$

where $t_0 = \frac{2}{y_0} < T_*$.

Note that on the axis $r = 0$, the equation for Π can be reduced to

$$\partial_t \Pi(t, 0, z) + (u_z - 2\Pi)(t, 0, z) \partial_z \Pi(t, 0, z) = 0.$$

By the definition of $\phi(t, z)$, we have $\frac{d}{dt} \Pi(t, 0, \phi(t, z)) \equiv 0$. This enables us to get

$$\Pi(t, 0, \phi(t, 0)) = \Pi(0, 0, \phi(0, 0)) = \Pi_0(0, 0).$$

Hence, if we choose $\Pi_0(0, 0) = J_0 > 0$, then

$$\limsup_{t \rightarrow t_0} (\Pi \partial_z \Pi)(t, 0, \phi(t, 0)) = \infty.$$

From Eq. (1.4) for Ω , we get

$$2\Pi \partial_z \Pi = -\partial_t \Omega - (u_r \partial_r + u_z \partial_z) \Omega + \left(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2 \right) \Omega. \tag{3.5}$$

Therefore we see that at least one of the terms on the right side in (3.5) blows up

$$\limsup_{t \rightarrow t_0} \left(\partial_t \Omega + (u_r \partial_r + u_z \partial_z) \Omega - \left(\partial_r^2 + \frac{3}{r} \partial_r + \partial_z^2 \right) \Omega \right) (t, 0, \phi(t, 0)) = \infty.$$

This contradicts to our assumption that (1.1) is globally well-posedness in some Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. Hence the incompressible viscous Hall-MHD system without resistivity is not globally well-posedness in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. This yields two possibility: either the system is locally ill-posed in $H^m(\mathbb{R}^3)$, or it is locally well-posed, but there exists an initial data in $H^m(\mathbb{R}^3)$, for which the $H^m(\mathbb{R}^3)$ norm of solution blows-up in finite time if $m > \frac{7}{2}$. \square

3.2. Inviscid case $\nu = 0$

Proof of Theorem 1.2. As in the proof of Theorem 1.1, we will argue by contradiction. Same argument as before shows that (3.2) also holds in the inviscid case, so

$$f(t, z) - f(0, z) \geq \int_0^t f^2(s, z) ds - \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds. \tag{3.6}$$

Fix $z = 0$ and set $y_0 = f(0, 0) = \partial_z \Pi_0(0, 0)$, then by employing the estimate (2.14), we obtain

$$f(t, 0) \geq \int_0^t f^2(s, 0) ds + y_0 - \int_0^t \left\| \frac{u_r}{r}(s, \cdot) \right\|_{L^\infty}^2 ds \tag{3.7}$$

$$\geq \int_0^t f^2(s, 0) ds + y_0 - C_{\sharp} t. \tag{3.8}$$

Now take $y_0 \geq 4C_{\sharp}^{\frac{1}{2}}$ and $T_{\sharp} = \frac{4}{y_0} \leq C_{\sharp}^{-1/2}$, for $t \in [0, T_{\sharp}]$, we have

$$\begin{aligned} f(t, 0) &\geq \int_0^t f^2(s, 0) ds + y_0 - C_{\sharp} C_{\sharp}^{-1/2} \\ &\geq \int_0^t f^2(s, 0) ds + \frac{1}{2} y_0. \end{aligned}$$

Then $F(t) = \int_0^t f^2(s, 0)ds + \frac{1}{2}y_0$ satisfies the following inequality

$$F'(t) \geq F^2(t), \quad t \in [0, T_{\sharp}^*],$$

$$F(0) = \frac{1}{2}y_0.$$

Hence we have

$$F(t) \geq \frac{y_0}{2 - ty_0}.$$

This simply implies that

$$\limsup_{t \rightarrow t_{\sharp}^*} F(t) = \infty, \quad \limsup_{t \rightarrow t_{\sharp}^*} f(t, 0) = \limsup_{t \rightarrow t_{\sharp}^*} \partial_z \Pi(t, 0, \phi(t, 0)) = \infty,$$

where $t_{\sharp}^* = \frac{2}{y_0} < T_{\sharp}^*$.

As before, if we choose $\Pi_0(0, 0) = J_0 > 0$, then

$$\limsup_{t \rightarrow t_{\sharp}^*} (\Pi \partial_z \Pi)(t, 0, \phi(t, 0)) = \infty,$$

and also by (2.7), the velocity field will also blow up

$$\limsup_{t \rightarrow t_{\sharp}^*} \left(\partial_t \Omega + (u_r \partial_r + u_z \partial_z) \Omega \right)(t, 0, \phi(t, 0)) = \infty.$$

This contradicts to our assumption that (1.1) is globally well-posedness in some Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. Hence the incompressible viscous Hall-MHD system without resistivity is not globally well-posedness in any Sobolev space $H^m(\mathbb{R}^3)$ for $m > \frac{7}{2}$. This also yields two possibilities: either the system is locally ill-posed in $H^m(\mathbb{R}^3)$, or it is locally well-posed, but there exists an initial data in $H^m(\mathbb{R}^3)$, for which the $H^m(\mathbb{R}^3)$ norm of solution blows-up in finite time if $m > \frac{7}{2}$. \square

Remark 3.1. As one can see from the above proof, the convective term $(u_r \partial_r + u_z \partial_z) \Pi$ may prevent the shock formation. For the incompressible viscous Hall-MHD by restricting on the axis, we have good control on the gradient of u_r and u_z , showing that the smoothing effect of the convective term is not strong enough and cannot prevent the formation of shock-type singularity in the magnetic field.

Conflict of interest statement

There does not exist any kind of potential conflicts.

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