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Rigidity of pairs of quasiregular mappings whose symmetric part of gradient are close

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Abstract

For $A \in M^{2\times 2}$ let $S(A) = \sqrt{A^T A}$, i.e. the symmetric part of the polar decomposition of A. We consider the relation between two quasiregular mappings whose symmetric part of gradient are close. Our main result is the following. Suppose $v, u \in W^{1,2}(B_1(0) : \mathbb{R}^2)$ are Q-quasiregular mappings with $\int_{B_1(0)} \det(Du)^{-p} dz \le C_p$ for some $p \in (0, 1)$ and $\int_{B_1(0)} |Du|^2 dz \le \pi$. There exists constant $M > 1$ such that if $\int_{B_1(0)} |S(Dv) - S(Du)|^2 dz = \epsilon$ then

$$\int_{B_{\frac{1}{2}}(0)} |Dv - RDu| dz \le c C_p^{\frac{2}{p}} \epsilon^{\frac{p^2}{MQ^5 \log(10 C_p Q)}} \quad \text{for some } R \in SO(2).$$

Taking $u = Id$ we obtain a special case of the quantitative rigidity result of Friesecke, James and Müller [13]. Our main result can be considered as a first step in a new line of generalization of Theorem 1 of [13] in which *Id* is replaced by a mapping of non-trivial degree.

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Rigidity and stability of differential inclusions is a classical subject. Reshetnyak's monograph [23] is devoted to proving a quantitative stability result generalizing Liouville's classic theorem [18] that solutions of the differential inclusion $Du \in CO_+(n) := \{\lambda R : \lambda > 0, R \in SO(n)\}$, $n \ge 3$ are affine or Mobius. Korn's inequality is an optimal quantitative stability result for the fact that the differential inclusion $Du \in \text{Skew}(n \times n) := \{M \in M^{n\times n} : M^T = -M\}$ is satisfied only by an affine map.

This subject has received considerable impetus from the work of Friesecke, James and Müller [13] who proved an optimal quantitative stability result for the corollary to Liouville's theorem that states solutions to the differential inclusion $Du \in SO(n)$ are affine.

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Theorem 1. *(See Friesecke, James and Müller, 2002.) For every bounded open connected Lipschitz domain $U \subset \mathbb{R}^n$, $n \geq 2$, and every $q > 1$, there exists a constant $C = C(U, q, n)$ such that writing $K := SO(n)$,*

$$\inf_{R \in K} \|Dv - R\|_{L^q(U)} \leq C \left\| \operatorname{dist}(Dv, K) \right\|_{L^q(U)} \quad \text{for every } v \in W^{1,q}\left(U : \mathbb{R}^n\right).$$

Previously strong partial results controlling the function (rather than the gradient) have been established by John [16], Kohn [17].

The simplicity of the statement of Theorem 1 can lead to the strength of the advance that is represented by this theorem being overlooked. It is rare in contemporary research in analysis to prove a new and deep result about elementary mathematical objects; Theorem 1 is exactly such a result. It has had wide application in applied analysis and is one of the main tools used to make a rigorous and complete analysis of the multiple thin shell theories in classical elasticity [13–15]. Beyond this it has the merit of being a statement whose significance would be clear to mathematicians of two hundred years ago.

A number of works have extended Theorem 1 to cover various larger classes of matrices than $SO(n)$. Faraco and Zhong proved the corresponding result with $K = \Pi SO(n)$ where $\Pi \subset \mathbb{R}_+ \backslash \{0\}$ is a compact set, [12]. Chaudhuri and Müller [5] and later DeLellis and Szekelyhidi [9] considered a set of the form $K = SO(n)A \cup SO(n)B$ where A and B are *strongly incompatible* in the sense of Matos [22].

In this paper, following an approach started by Ciarlet and Mardare [6] and also suggested by Müller, we start a different line of generalization of Theorem 1. The initial observation is that Theorem 1 is a special case of the following question. Recall we defined $S(M) = \sqrt{M^T M}$ to be the symmetric part of a matrix.

Question 1. If $\Omega \subset \mathbb{R}^n$ is a connected domain and $u, v \in W^{1,2}(\Omega : \mathbb{R}^2)$, $\det(Du) > 0$, $\det(Dv) > 0$ and $\int_\Omega |S(Dv) - S(Du)|^2 dx = \epsilon$ does this imply there exists $R \in SO(n)$ such that $\int_\Omega |Du - RDv|^2 dx \leq \delta$ where δ is some small quantity depending on ϵ.

It turns out that the answer to Question 1 is no, even in the "absolute" version of this question where $\epsilon = 0$, see Example 1 [19] or see the example in Section 4, [6]. For a positive result for the case where $\epsilon = 0$ it suffices to consider the class of functions of integrable dilatation as shown in Theorem 1 [19] (or see Theorem 1 of [20] for a more general result). Theorem 1 of [20] and the 2d version of Theorem 1 of [19] are sharp in the sense that no result of this kind is possible outside the space of mappings of integrable dilatation.

In this paper we will provide a positive answer to Question 1 for pairs of Quasiregular mappings in two dimensions. Note in Theorem 2 and throughout the paper a ball of radius r centred on zero will be denoted B_r.

Theorem 2. *Suppose $v, u \in W^{1,2}(B_1 : \mathbb{R}^2)$ are Q-quasiregular mappings with $\int_{B_1} \det(Du)^{-p} dz \leq C_p$ for some $p \in (0, 1)$ and $\int_{B_1} |Du|^2 dz \leq \pi$. If*

$$\int_{B_1} \left|S(Dv) - S(Du)\right|^2 dz = \epsilon \tag{1}$$

then there exists $R \in SO(2)$ such that

$$\int_{B_{\frac{1}{2}}} |Dv - RDu| dz \leq c C_p^{\frac{2}{p}} \epsilon^{\frac{p^2}{10^{10} Q^5 \log(10 C_p Q)}}. \tag{2}$$

Theorem 2 to a certain extent shares the property that Theorem 1 has of being a new and interesting statement about the classical objects of mathematical analysis. The credit for this however is largely due to Theorem 1 as the methods of proof of this theorem are used in an essential way in the proof of Theorem 2. In this author's opinion there are a number of results in the area of classical Quasiconformal analysis that can be harvested by use of the ideas in the proof of Theorem 1, Theorem 2 is just one of them. Note if we take $u = Id$ hypothesis (1) is exactly $\int_{B_1} d^2(Dv, SO(2)) dz = \epsilon$ and the conclusion is $\int_{B_{\frac{1}{2}}} |Dv - R| dz \leq c \epsilon^{\frac{p^2}{10^{10} \log(10)}}$ for some $R \in SO(2)$. While this is much weaker than Theorem 1 it is still a result that was not known prior to the publication of [13]. In some sense the line of generalization that this paper contributes to is the desire to replace *Id* by a mapping of non-trivial degree.

Ciarlet and Mardare were motivated to study Question 1 as part of a program to develop a theory of elasticity based on study the "Cauchy Green" tensor $Du^T Du$ of a deformation u, [6–8]. They proved a version of Theorem 2 for C^1 mappings with the property that $\det(Du) > 0$ everywhere in the domain and the constant c in (2) depends on u. Their method was again to apply Theorem 1, this will be sketched in the next section.

Theorem 2 is clearly suboptimal however we believe the power of ϵ in inequality (2) is of the right form in the sense that the power decreases as the degree of the mapping u increases or as Q increases. As the dependence on the degree is a key issue an example showing the dependence will be presented in [21]. We give a sketch of the construction of the example in Section 5.

1. Proof sketch

1.1. Absolute case with global invertibility

First suppose we have C^1 functions u, v where u is globally invertible and $S(Du) = S(Dv)$ everywhere. By polar decomposition we have $A = R(A)S(A)$ for some $R(A) \in SO(n)$. Form $w(z) = v(u^{-1}(z))$ and note that

$$\begin{aligned} Dw(x) &= Dv\big(u^{-1}(x)\big)\big(Du\big(u^{-1}(x)\big)\big)^{-1} \\ &= R\big(Dv\big(u^{-1}(x)\big)\big)\big(R\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1} \in SO(n) \end{aligned}$$

by Liouville's theorem it is clear there exists $R \in SO(n)$ such that $Dw(z) = R$ for all $z \in B_1$. Thus

$$Dv = RDu \quad \text{on } B_1 \tag{3}$$

and result is established.

1.2. Quantitative case with global invertibility

Now assume u, v are C^1 and u is globally invertible and $\int_{B_1} |S(Dv) - S(Du)|^2 dz = \epsilon$ and $\inf\{\det(Du(z)) : z \in B_1\} > 0$. Apart from where $|Du| \sim 0$ and $|Dv| \sim 0$ we know $|(S(Du(z)))^{-1} - (S(Dv(z)))^{-1}| \approx |S(Du(z)) - S(Dv(z))|$ and hence letting

$$E(z) = \big(S\big(Du(z)\big)\big)^{-1} - \big(S\big(Dv(z)\big)\big)^{-1}$$

we have

$$\begin{aligned} Dw(x) &= R\big(Dv\big(u^{-1}(x)\big)\big)S\big(Dv\big(u^{-1}(x)\big)\big)\big(S\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1}\big(R\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1} \\ &= R\big(Dv\big(u^{-1}(x)\big)\big)S\big(Dv\big(u^{-1}(x)\big)\big)\big(\big(S\big(Dv\big(u^{-1}(x)\big)\big)\big)^{-1} + E\big(u^{-1}(x)\big)\big)\big(R\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1} \\ &= R\big(Dv\big(u^{-1}(x)\big)\big)\big(R\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1} \\ &\quad + R\big(Dv\big(u^{-1}(x)\big)\big)S\big(Dv\big(u^{-1}(x)\big)\big)E\big(u^{-1}(x)\big)\big(R\big(Du\big(u^{-1}(x)\big)\big)\big)^{-1}. \end{aligned}$$

So for any compact subset $\Pi \Subset B_1$ using the upper bound on $\det(Du)$ on Π we get from the fact u is C^1

$$\begin{aligned} \int_{u(\Pi)} \operatorname{dist}\big(Dw(x), SO(2)\big)dx &\leq \int_{\Pi} \big|S\big(Dv(z)\big)\big|\big|E(z)\big| \det\big(Du(z)\big)dz \\ &\leq c\Big(\int_{B_1} \big|Dv(z)\big|^2 dz\Big)^{\frac{1}{2}}\Big(\int_{B_1} \big|E(z)\big|^2 dz\Big)^{\frac{1}{2}} \\ &\leq c\sqrt{\epsilon}. \end{aligned} \tag{4}$$

So applying the L^1 version of Theorem 1 we have that there is constant $\mathcal{C} = \mathcal{C}(u)$ such that

$$\int_{u(\Pi)} \left|Dw(z) - R_0\right| dz \leq \mathcal{C} \log\left(\epsilon^{-1}\right)\sqrt{\epsilon}.$$

and unwrapping gives the estimate we seek, however with a constant depending on u.

1.3. Sketch of the general case

Our problem is that we do not have global invertibility and we would like an estimate that depends on u in a more explicit way. Under the hypothesis that the mappings u, v are Q-quasiregular we know that u is locally invertible at all but countably many points, but we have no estimates of the size of the of neighbourhoods of invertibility. If we wanted to prove an estimate of the form (2) where the constant c depended on u we could patch together neighbourhoods of invertibility so long as we knew the "size" of the neighbourhoods were bounded below on all compact subdomains. Under the hypothesis $\det(Du) > 0$ everywhere for a C^1 function u this is true and this is how Ciarlet and Mardare established their estimate [6].

For quasiregular mappings there is no way to patch together the argument shown in Subsection 1.2. The key to making progress is to use the *Stoilow decomposition* to translate the information we have from the hypotheses into information about the analytic functions of the Stoilow decomposition. Let us recall the basics of the Stoilow decomposition, any Q-quasiregular mapping $u : \Omega \to \mathbb{R}^2$ can be written as the composition of a Q-quasiconformal homeomorphism $w_u : \Omega \to \mathbb{R}^2$ and an analytic function $\phi_u : w_u(\Omega) \to \mathbb{R}^2$ so that

$$u(z) = \phi_u\left(w_u(z)\right). \tag{5}$$

A good reference are the monographs of Astala, Iwaniec and Martin [3] Section 5.5. and Ahlfors [2].

The heart of the Stoilow decomposition is the fact that it is possible to solve Beltrami's equation. This allows us to find a Q-quasiconformal mapping w_u that has the same *Beltrami Coefficient* as Du. The *Beltrami Coefficient* of a matrix M is a 2×2 conformal matrix μ_M (or more typically a complex number) that encodes the *geometry* of the deformation of the unit ball by M, but *not* the orientation or the size (formally $[M]_a\mathcal{I} = \mu_M[M]_c$ where $[M]_c, [M]_a$ are the conformal and anti-conformal parts of M and $\mathcal{I}$ is a reflection across the y-axis, see Subsection 2.1 for more details). By solving Beltrami's equation we can find a homeomorphism w_u with the property that

$$\mu_{Dw_u(z)} = \mu_{Du(z)} \quad \text{for a.e. } z \in B_1 \tag{6}$$

and $w_u(z) - z = O(1/z)$, $w_v(z) - z = O(1/z)$. So for any $z \in B_1$ the shape of the image of the unit ball under $Du(z)$ is *similar* to the shape of the image of the unit ball under $Dw_u(z)$. Hence the factorization represented by (5) is entirely natural.

Now the symmetric part of a gradient encodes both the geometry and the size. So a key result that starts the proof is a bound of the difference between Beltrami coefficients of two Q-quasiconformal matrices A, B by $|S(A) - S(B)|$

$$|\mu_A - \mu_B| \leq 4\sqrt{Q}\min\left\{\det(A)^{-\frac{1}{2}}, \det(B)^{-\frac{1}{2}}\right\}\left|S(A) - S(B)\right|. \tag{7}$$

This is the contents of Lemma 2. Note as the determinants of Q-quasiconformal matrices A, B get very small their norm gets very small so $|S(A) - S(B)| \leq |S(A)| + |S(B)| \leq 4Q(\det(A) + \det(B)) \approx 0$ but the *geometry* of the deformation of the unit ball by A, B could be very different hence the factor of $\min\{\det(A)^{-\frac{1}{2}}, \det(B)^{-\frac{1}{2}}\}$ in the right hand side (7) is to be expected.

Now the solutions of the Beltrami equation w_u, w_v are essentially given by solving $\mathcal{C}(1 - \mu_{Du}\mathcal{S})^{-1}$ and $\mathcal{C}(1 - \mu_{Dv}\mathcal{S})^{-1}$ where $\mathcal{C}$ is the Cauchy transform and $\mathcal{S}$ is the Beurling–Ahlfors transform. Hence it should seem reasonable that we can prove an estimate showing Dw_u, Dw_v are close in L^q norm. As a consequence we establish

$$\|w_u - w_v\|_{L^\infty(B_{\frac{1}{2}})} \leq c\epsilon^{\frac{p}{180Q}}. \tag{8}$$

This is part of the contents of Lemma 8 and Lemma 9.

Having established a quantitative relation between w_u, w_v in order to prove the estimate on Du, Dv we need to establish the relation $\phi_v' - \zeta\phi_u' \approx 0$ for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. We will establish this relation by applying Theorem 1 but first we have to set up some preliminary estimates. Since w_u is a solution of the Beltrami equation we have explicit estimates on its L^q norm and the L^q norm of its inverse in terms of Q. Hence we are able to establish the existence of constants $\xi = \xi(Q) < \gamma = \gamma(Q) < \mu = \mu(Q)$ such that

$$B_{2\mu}\big(w_u(0)\big) \subset w_u(B_{\frac{1}{2}}) \quad \text{and} \quad B_{2\mu}\big(w_v(0)\big) \subset w_v(B_{\frac{1}{2}}). \tag{9}$$

And

$$B_{\xi}\big(w_u(0)\big) \subset w_u(B_\gamma) \subset B_{\frac{\mu}{8}}\big(w_u(0)\big), \qquad B_{\xi}\big(w_v(0)\big) \subset w_v(B_\gamma) \subset B_{\frac{\mu}{8}}\big(w_v(0)\big). \tag{10}$$

This is the contents of part of Lemma 6 and Lemma 7.

Now by (10) we know $w_u(B_\gamma) \subset B_{\frac{\mu}{8}}(w_u(0))$ and $w_v(B_\gamma) \subset B_{\frac{\mu}{8}}(w_v(0))$, so ϕ_u and ϕ_v are defined on both of these sets. Since the hypotheses are that the symmetric part of gradient are close we also know the size of the gradients Du and Dv are close. We can use this and the estimates for ϕ_v', ϕ_v'' on $B_\mu(w_v(0))$ to show that

$$\int\limits_{w_u(B_\gamma)} \big||\phi_u'|^2 - |\phi_v'|^2\big| dz \le c_1 \epsilon^{\frac{p}{180Q}}, \tag{11}$$

this is the content of Lemma 10. We would like to apply Theorem 1 so a natural thing to do would be to use Cauchy's Theorem to find an analytic function ψ such that $\psi' = \frac{\phi_v'}{\phi_u'}$ then establish appropriate lower bounds on $|\phi_u'|$ on some ball $B_{h_0}(z_0)$ to conclude

$$\int\limits_{B_{h_0}(z_0)} \big|1 - |\psi'(z)|\big|^2 dz \le c_2 \epsilon^{\frac{p}{180Q}}. \tag{12}$$

The non-degeneracy condition $\int_{B_1} \det(Du(z))^{-p} dz \le C_p$ allows to find such a ball centred somewhere in $B_{\frac{\xi}{2}}(w_u(0))$, this is the contents of Lemma 11. Specifically we find some $h_0 = h_0(Q, C_p) > 0$ and some $\varpi = \varpi(Q, C_p) > 0$ such that for some $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ (since recall, $B_\xi(w_u(0)) \subset w_u(B_\gamma)$ by (10))

$$\inf\big\{\big|\phi_u'(y)\big| : y \in B_{h_0}(z_0)\big\} \ge \varpi. \tag{13}$$

Let $\tilde{\psi}(x, y) = (\mathrm{Re}(\psi(x + iy)), \mathrm{Im}(\psi(x + iy)))$. Reformulating (12) in matrix notation gives $\int_{B_{h_0}(z_0)} \mathrm{dist}^2(D\tilde{\psi}, SO(2)) dz \le c_2 \epsilon^{\frac{p}{180Q}}$. So we can apply Theorem 1, however for reasons we will explain later we will instead use a more restricted version of it given by Proposition 2 proved in Appendix A. So we can conclude there exists some rotation R such that

$$\int\limits_{B_{h_0}(z_0)} |D\tilde{\psi} - R| dz \le c_3 \epsilon^{\frac{p}{720Q}}. \tag{14}$$

Returning this into complex notation and unwrapping it using the definition of ψ we have

$$\int\limits_{B_{h_0}(z_0)} \big|\phi_v'(z) - \zeta \phi_u'(z)\big| dz \le c_4 \epsilon^{\frac{p}{720Q}}. \tag{15}$$

We need to extend control on $\phi_v' - \zeta \phi_u'$ to include an explicit neighbourhood of $w_u(0)$. We are able to do this by the fact that we are dealing with an analytic function $\phi_v - \zeta \phi_u$ and so have Taylor's Theorem. Since we already know $B_{2\mu}(w_u(0)) \overset{(9)}{\subset} w_u(B_{\frac{1}{2}})$ and $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ so we can use Taylor's Theorem to extend control to $B_{\frac{\mu}{4}}(z_0)$ which contains $B_{\frac{\mu}{5}}(w_u(0))$.

So let $w(z) = \phi_v'(z) - \zeta_1 \phi_u'(z)$. By the local Taylor Theorem we have $w(z) = \sum_{k=0}^m \frac{w^{(k)}(z_0)}{k!}(z - z_0)^k + (z - z_0)^{m+1} w_m(z)$ where $w_m(z) = \frac{1}{2\pi i} \int_{\partial B_{\frac{\mu}{2}}(z_0)} \frac{w(\zeta)}{(\zeta - z_0)^{m+1}(\zeta - z)} d\zeta$.

By the Coarea formula we can find $q \in (\frac{h_0}{8}, h_0)$ such that $\int_{\partial B_q(z_0)} |w(z)| dH^1 z \le 8 c_4 \epsilon^{\frac{p}{720Q}}$. So by Cauchy's integral formula

$$\left|w^{(k)}(z_0)\right| = \frac{k!}{2\pi} \int\limits_{\partial B_q(z_0)} \left|\frac{w(\zeta)}{(\zeta - z_0)^{k+1}}\right| d\zeta \leq c_4 k! \frac{\epsilon^{\frac{p}{720Q}}}{q^k}.$$

We can also use the upper bound $\|Du\|_{L^2(B_1)} \leq \pi$ and the upperbounds on w_u, w_v to get upper bounds on ϕ_u and ϕ_v on $B_\mu(w_u(0))$ (this is part of the contents of Lemma 7) so can estimate the remainder term $\|w_m\|_{L^\infty(B_{\frac{\mu}{4}}(z_0))} \leq 32\pi\mu^{-2}(\frac{\mu}{2})^{-m}$. Thus we have

$$\left|w(z)\right| \leq \sum_{k=0}^{m} c_5 \epsilon^{\frac{p}{720Q}} \left(\frac{\mu}{4q}\right)^k + \frac{8\pi}{\mu}\left(\frac{1}{2}\right)^m \quad \text{for any } z \in B_{\frac{\mu}{4}}(z_0). \tag{16}$$

The key is to make the right choice of m. If we choose m too large then $\sum_{k=0}^{m} c_5(\frac{\mu}{4q})^k$ will dominate $\epsilon^{\frac{p}{720Q}}$ and the upperbound will be weak. If m is too small then $\frac{8\pi}{\mu}(\frac{1}{2})^m$ will not be small enough. The answer is to find m that roughly equalizes these two quantities. An *essential* point is that finding this m requires knowing what the constants h_0, c_5, μ are. To estimate these constants we need to know c_1, c_2, c_3, c_4 and ϖ in (11), (12), (14), (15) and (13). For this reason much effort will be made to track all constants in the estimates in this paper, since the methods are not close to being sharp we do not attempt to consistently calculate the best possible constants, but we do make efforts to prevent the constants blowing up too much throughout the paper. The reason we need the simplified version of Theorem 1 that is given by Proposition 2 is that we need to know explicitly the constant in this inequality. This requires us to rewrite the proof of an estimate from [13] while tracking the constants. The fact we are able to do this with the methods of [13] is one of the reasons that Theorem 2 was not in practical terms accessible before the ideas introduced in [13]. So making these estimates (recalling the fact $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$) we have

$$\left\|\phi_v' - \zeta\phi_u'\right\|_{L^\infty(B_{\frac{\mu}{5}}(w_u(0)))} \leq c_5 C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}} \tag{17}$$

This is the contents of Lemma 12. By using the estimates on the closeness of Dw_u and Dw_v in L^q we can then conclude that for some constant $\gamma = \gamma(Q)$ that

$$\|Dv - RDu\|_{L^1(B_\gamma)} \leq c C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}. \tag{18}$$

This is the contents of Proposition 1 below. Theorem 2 follows by a straightforward covering argument that gives estimate (2).

Proposition 1. *Suppose $v, u \in W^{1,2}(B_1 : \mathbb{R}^2)$ are a Q-quasiregular mappings with $\int_{B_1} \det(Du)^{-p} dz \leq C_p$ for some $p \in (0, 1)$ and $\int_{B_1} |Du|^2 dz \leq \pi$. If*

$$\int\limits_{B_1} \left|S(Dv) - S(Du)\right|^2 dz = \epsilon \tag{19}$$

then there exists $R \in SO(2)$ and constant $\gamma = \gamma(Q) > 0$ such that

$$\int\limits_{B_\gamma} |Dv - RDu| dz \leq c C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}. \tag{20}$$

Remark. We can assume without loss of generality

$$u(0) = 0, \tag{21}$$

since if not the quasiregular mapping defined by $\tilde{u}(x) = u(x) - u(0)$ has this property.

2. Preliminaries

2.1. Conformal, anti-conformal decomposition of 2×2 matrices

Given $A \in M^{2\times 2}$ we can decompose A into conformal and anti-conformal parts as follows

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -(a_{21}-a_{12}) \\ a_{21}-a_{12} & a_{11}+a_{22} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a_{11}-a_{22} & a_{21}+a_{12} \\ a_{21}+a_{12} & -(a_{11}-a_{22}) \end{pmatrix}. \tag{22}$$

So for arbitrary matrix A let

$$[A]_c := \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -(a_{21}-a_{12}) \\ a_{21}-a_{12} & a_{11}+a_{22} \end{pmatrix} \quad \text{and} \quad [A]_a := \frac{1}{2}\begin{pmatrix} a_{11}-a_{22} & a_{21}+a_{12} \\ a_{21}+a_{12} & -(a_{11}-a_{22}) \end{pmatrix}. \tag{23}$$

It will often be convenient to write this decomposition as $A = \alpha R_\theta + \beta N_\psi$ for $\alpha > 0$, $\beta > 0$ where

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad N_\psi := \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}.$$

Let $\mathcal{I} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $\|\cdot\|$ denote the operator norm and $|\cdot|$ be the Hilbert Schmidt norm. Note

$$\|x\| \leq |x| \leq 2\|x\| \quad \text{for any } x \in \mathbb{R}^n. \tag{24}$$

The **Beltrami Coefficient** of a matrix A that relates the conformal and anti-conformal parts of A is the conformal matrix μ_A defined by

$$[A]_a\mathcal{I} = \mu_A[A]_c. \tag{25}$$

Now

$$\|A\|^2 \leq Q \det A \quad \Rightarrow \quad \frac{(\alpha+\beta)^2}{\alpha^2-\beta^2} \leq Q \quad \Rightarrow \quad \frac{\beta}{\alpha} \leq \frac{Q-1}{Q+1}. \tag{26}$$

And

$$\big|[A]_a\big| = \frac{1}{2}\left|\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} -a_{22} & a_{21} \\ a_{12} & -a_{11} \end{pmatrix}\right| \leq |A|. \tag{27}$$

As $\beta N_\psi \mathcal{I} = \mu_A \alpha R_\theta$, so

$$|\mu_A| = \sqrt{2}\frac{\beta}{\alpha} \tag{28}$$

2.2. The Beltrami equation

The Beltrami equation is a linear complex PDE the relates the conformal part of the gradient to the anti-conformal, we briefly describe the connection between the classical complex formulation and the matrix formulation we will be using in this paper.

Take function from the complex plane to itself, $f(x+iy) = u(x,y) + iv(x,y)$. As is standard, $\frac{\partial}{\partial \bar{z}} f(x,y) = \frac{1}{2}(\partial_x + i\partial_y)f$ and $\frac{\partial}{\partial z} f(x,y) = \frac{1}{2}(\partial_x - i\partial_y)f$.

If we take a $\Omega \subset \mathbb{C}$ and a function $f : \Omega \to \mathbb{C}$ then define the $\mathbb{R}^2$ valued function $\tilde{f}(x,y) = (\mathrm{Re}(f(x+iy)), \mathrm{Im}(f(x+iy)))$. Let $CO_+(2)$ denote the set of conformal 2×2 matrices. And let $[\cdot]_M$ denote the homomorphism between $\mathbb{C}$ and $CO_+(2)$, so $[a+ib]_M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

So note

$$\sqrt{2}|a+ib| = \big|[a+ib]_M\big|. \tag{29}$$

It is straight forward to see that

$$\left[\frac{\partial f}{\partial z}\right]_M = [D\tilde{f}]_c \quad \text{and} \quad \left[\frac{\partial f}{\partial \bar{z}}\right]_M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = [D\tilde{f}]_a,$$

(recall the decomposition into conformal and anticonformal parts given by (22), (23)).

Now as in 2.9.1. [3] letting $Df(z): \mathbb{C} \to \mathbb{C}$ denote the linear map that is the derivative of f at z, then we have $Df(z)h = \frac{\partial f}{\partial z}(z)h + \frac{\partial f}{\partial \bar{z}}(z)\bar{h}$. Let $[\cdot]_{\mathbb{C}}$ be the identification of $\mathbb{R}^2$ with $\mathbb{C}$, i.e. $\left[\binom{a}{b}\right]_{\mathbb{C}} = a + ib$. Let $f = u + iv$ so we have

$$\begin{aligned} Df(z)h &= \frac{1}{2}\big((u_x + v_y) + i(v_x - u_y)\big)(h_1 + ih_2) + \frac{1}{2}\big(u_x - v_y + i(v_x + u_y)\big)(h_1 - ih_2) \\ &= \left[\frac{1}{2}\begin{pmatrix} u_x + v_y & -(v_x - u_y) \\ v_x - u_y & u_x + v_y \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right]_{\mathbb{C}} + \left[\frac{1}{2}\begin{pmatrix} u_x - v_y & v_x + u_y \\ v_x + u_y & -(u_x - v_y) \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right]_{\mathbb{C}} \\ &= \left[\left(\frac{1}{2}\big[D\tilde{f}(x,y)\big]_c + \frac{1}{2}\big[D\tilde{f}(x,y)\big]_a\right)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right]_{\mathbb{C}}. \end{aligned} \tag{30}$$

Given $f : \Omega \to \mathbb{C}$ one of the basic equations of Quasiregular analysis is the Beltrami equation

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z). \tag{31}$$

As above define $\tilde{f} = (\mathrm{Re}(f), \mathrm{Im}(f))$ then $\tilde{f}$ satisfies

$$\big[D\tilde{f}(x,y)\big]_a \mathcal{I} = \big[\mu(x+iy)\big]_M \big[D\tilde{f}(x,y)\big]_c. \tag{32}$$

By uniqueness this implies that

$$\big[\mu(x+iy)\big]_M = \mu_{D\tilde{f}(x,y)}. \tag{33}$$

The basic theorem about the solvability of the Beltrami equation (sometimes known as the measurable Riemann mapping theorem) is the following

Theorem *(Bojarski–Morrey). Suppose that* $0 \le k < 1$ *and that* $|\mu(z)| \le \kappa \mathbb{1}_{B_r}(z)$, $z \in \mathbb{C}$. *Then there is a unique* $f \in W^{1,q}_{loc}(\mathbb{C})$ *(for every* $q \in [2, 1 + \frac{1}{\kappa})$*) such that*

$$\frac{\partial f}{\partial \bar{z}} = \mu(z)\frac{\partial f}{\partial z} \quad \textit{for almost every } z \in \mathbb{C}, \tag{34}$$

$$f(z) = z + O\left(\frac{1}{z}\right) \quad \textit{as } z \to \infty. \tag{35}$$

We say a Quasiconformal mapping that satisfies (34) is the *principle solution* of the Beltrami equation if it in addition satisfies (35).

Definition 1. Given a Q-quasiregular mapping u we say the pair $w_u : B_1 \to \mathbb{R}^2$, $\phi_u : w_u(B_1) \to u(B_1)$ are the Stoilow decomposition of u iff

$$u(z) = \phi_u\big(w_u(z)\big) \quad \text{for all } z \in B_1 \tag{36}$$

where function w_u is the principle solution of the Beltrami equation

$$\big[Dw_u(z)\big]_a \mathcal{I} = \mu(z)\big[Dw_u(z)\big]_c \tag{37}$$

for

$$\mu(z) := \begin{cases} [Du(z)]_a \mathcal{I} [Du(z)]_c^{-1} & \text{for } z \in B_1 \\ 0 & \text{for } z \notin B_1. \end{cases} \tag{38}$$

Note that (37), (38) are just the reformulation of the standard Beltrami equation and Beltrami coefficient in matrix notation as explained in Subsection 2.1 (25) and Eqs. (32), (33) of this subsection.

As explained in the introduction, a consequence of (36), (37) we have that $D\phi_u \in CO_+(2) = \{\lambda R : \lambda > 0, R \in SO(2)\}$. So considered as a complex valued function of a complex variable, function ϕ_u is holomorphic. We will often consider ϕ_u as a holomorphic function of a complex variable without relabelling it.

2.3. The Beltrami coefficient of gradient whose symmetric parts agree

We require Lemma 1 from [20]. It is stated below

Lemma 1. *Let $A \in M^{2\times 2}$, $\det(A) > 0$. Let the Beltrami coefficient of A be defined by (25). The Beltrami coefficient of A and A^{-1} are related in the following way*

$$\mu_A [A]_c \mathcal{I} = -\mu_{A^{-1}} \mathcal{I} [A]_c. \tag{39}$$

Notice as a consequence of (39) we have

$$|\mu_{A^{-1}}| = |\mu_A|. \tag{40}$$

3. Lemmas for Theorem 2

Lemma 2. *Suppose $A, B \in M^{2\times 2}$ with $\det(A) > 0$ and $\det(B) > 0$ and $\|A\|^2 \leq Q \det(A)$, $\|B\|^2 \leq Q \det(B)$ then*

$$|\mu_A - \mu_B| \leq \frac{4\sqrt{Q}}{\max\{\sqrt{\det(A)}, \sqrt{\det(B)}\}} \left|S(A) - S(B)\right|. \tag{41}$$

Proof of Lemma 2. Note by Cauchy Schwartz inequality the Hilbert Schmidt norm is submultiplicative, i.e. $|AB| \leq |A||B|$. Recall also since $|A| = \text{Trace}(A^T A)$ we can easily see that the Hilbert Schmidt norm is invariant under compositions with rotations.

Now note also $R(A)S(A)S(B)^{-1}R(B)^{-1} = AB^{-1} = [AB^{-1}]_a + [AB^{-1}]_c$. Thus

$$S(A)S(B)^{-1} = R(A)^{-1}\left[AB^{-1}\right]_a R(B) + R(A)^{-1}\left[AB^{-1}\right]_c R(B) \tag{42}$$

as the decomposition into conformal and anti-conformal parts are unique, so

$$\left[S(A)S(B)^{-1}\right]_a \overset{(42)}{=} R(A)^{-1}\left[AB^{-1}\right]_a R(B). \tag{43}$$

Note $|ADJ(B)| = |B| \overset{(24)}{\leq} 2\|B\| \leq 2\sqrt{Q}\sqrt{\det(B)}$. Let $\delta = |S(A) - S(B)|$, so

$$\begin{aligned}\left|S(A)S(B)^{-1} - Id\right| &\leq \left|S(A) - S(B)\right| \frac{|ADJ(B)|}{\det(B)} \\ &\leq \frac{2|S(A) - S(B)|\sqrt{Q}}{\sqrt{\det(B)}} \\ &\leq \frac{2\delta\sqrt{Q}}{\sqrt{\det(B)}}.\end{aligned} \tag{44}$$

Now

$$\begin{aligned}\left|\left[AB^{-1}\right]_a\right| &\overset{(43)}{\leq} \left|\left[S(A)S(B)^{-1}\right]_a\right| \\ &= \left|\left[S(A)S(B)^{-1} - Id\right]_a\right| \\ &\overset{(27)}{\leq} \left|S(A)S(B)^{-1} - Id\right| \\ &\overset{(44)}{\leq} \frac{2\delta\sqrt{Q}}{\sqrt{\det(B)}}.\end{aligned} \tag{45}$$

Thus as we know from (39) Lemma 1 applied to B^{-1} that

$$\mu_{B^{-1}}\left[B^{-1}\right]_c \mathcal{I} = -\mu_B \mathcal{I}\left[B^{-1}\right]_c, \tag{46}$$

so

$$\begin{aligned}\left[AB^{-1}\right]_a &:= \left[\left([A]_c+[A]_a\right)\left(\left[B^{-1}\right]_c+\left[B^{-1}\right]_a\right)\right]_a\\ &= [A]_a\left[B^{-1}\right]_c+[A]_c\left[B^{-1}\right]_a\\ &\overset{(25)}{=} \mu_A[A]_c\mathcal{I}\left[B^{-1}\right]_c+[A]_c\mu_{B^{-1}}\left[B^{-1}\right]_c\mathcal{I}\\ &\overset{(46)}{=} \mu_A[A]_c\mathcal{I}\left[B^{-1}\right]_c-[A]_c\mu_B\mathcal{I}\left[B^{-1}\right]_c\\ &= (\mu_A-\mu_B)[A]_c\mathcal{I}\left[B^{-1}\right]_c. \end{aligned} \tag{47}$$

For any matrix A let $\Pi(A):=\inf\{|Av|:|v|=1\}$. Note that $\Pi(AB)\geq\Pi(A)\Pi(B)$. Thus

$$\Pi(\mu_A-\mu_B)\Pi\left([A]_c\right)\Pi\left(\left[B^{-1}\right]_c\right)\overset{(47)}{\leq}\Pi\left(\left[AB^{-1}\right]_a\right)\leq\left|\left[AB^{-1}\right]_a\right|\overset{(45)}{\leq}\frac{2\delta\sqrt{Q}}{\sqrt{\det(B)}}. \tag{48}$$

Now $\Pi([A]_c)=\sqrt{(\det([A]_c))}\geq\sqrt{\det(A)}$. And $\Pi([B^{-1}]_c)\geq\sqrt{\det(B^{-1})}=\frac{1}{\sqrt{\det(B)}}$. So putting these things together we have that

$$\frac{\sqrt{\det(A)}}{\sqrt{\det(B)}}\Pi(\mu_A-\mu_B)\overset{(48)}{\leq}\frac{2\delta\sqrt{Q}}{\sqrt{\det(B)}}.$$

So $\Pi(\mu_A-\mu_B)\leq\frac{2\delta\sqrt{Q}}{\sqrt{\det(A)}}$. By definition of Π for any $\epsilon>0$ we can find $w\in S^1$ such that $|(\mu_A-\mu_B)w|\leq\frac{2\delta\sqrt{Q}}{\sqrt{\det(A)}}+\epsilon$. Since $\mu_A-\mu_B$ is conformal so $|(\mu_A-\mu_B)e_1|\leq\frac{2\delta\sqrt{Q}}{\sqrt{\det(A)}}+\epsilon$ and $|(\mu_A-\mu_B)e_2|\leq\frac{2\delta\sqrt{Q}}{\sqrt{\det(A)}}+\epsilon$ and thus $|\mu_A-\mu_B|\leq\frac{4\delta\sqrt{Q}}{\sqrt{\det(A)}}$. Now since the hypotheses on A, B are the same this implies $|\mu_A-\mu_B|\leq\frac{4\delta\sqrt{Q}}{\sqrt{\det(B)}}$ and hence we have established (41). □

3.1. Estimates on Beltrami equations

3.1.1. Estimates of the Holder norm of solutions of the Beltrami equation

We need bounds on the Holder norm of solutions of the Beltrami equation.

The first is a well known lemma whose constant we explicitly estimate.

Lemma 3. *Suppose $q>2$ and $u\in W^{1,q}(B_w(\zeta))$, then for any x, y with*

$$|x-y|<\frac{1}{2}\min\left\{\operatorname{dist}\left(x,\partial B_w(\zeta)\right),\operatorname{dist}\left(y,\partial B_w(\zeta)\right)\right\} \tag{49}$$

$$\left|u(x)-u(y)\right|\leq 8\left(\frac{q-1}{q-2}\right)|x-y|^{\frac{q-2}{q}}\left(\int_{B_{2|x-y|}(x)}|Du|^q dx\right)^{\frac{1}{q}}. \tag{50}$$

Proof of Lemma 3. We will use the following Poincare type inequality (see page 267 [11])

$$\int_{B_r(x)}\left|u(x)-u(z)\right|dz\leq\frac{r^2}{2}\int_{B_r(x)}\frac{|Du(z)|}{|z-x|}dz \tag{51}$$

Let $W=B_r(x)\cap B_r(y)$ with $r=|x-y|$. Note by (49), $B_{2r}(x)\subset B_w(\zeta)$. Let q' denote the Holder conjugate of q. So

$$\begin{aligned}\left|u(x)-u(y)\right| &\leq \fint_W\left|u(x)-u(z)\right|dz+\fint_W\left|u(y)-u(z)\right|dz\\ &\leq\left(\pi\left(\frac{r}{2}\right)^2\right)^{-1}\left(\int_{B_r(x)}\left|u(x)-u(z)\right|dz+\int_{B_r(y)}\left|u(y)-u(z)\right|dz\right)\\ &\overset{(51)}{\leq}\frac{2}{\pi}\int_{B_r(x)}\frac{|Du(z)|}{|x-z|}dz+\frac{2}{\pi}\int_{B_r(y)}\frac{|Du(z)|}{|y-z|}dz\end{aligned}$$

$$\leq \frac{2}{\pi}\Big(\int_{B_r(x)} |Du|^q\Big)^{\frac{1}{q}}\Big(\int_{B_r(x)} |x-z|^{-q'}dz\Big)^{\frac{1}{q'}}$$
$$+\frac{2}{\pi}\Big(\int_{B_r(y)} |Du|^q\Big)^{\frac{1}{q}}\Big(\int_{B_r(y)} |y-z|^{-q'}dz\Big)^{\frac{1}{q'}}. \tag{52}$$

Now

$$\Big(\int_{B_r(y)} |y-z|^{-q'}\Big)^{\frac{1}{q'}} \leq 2\pi\Big(\frac{r^{2-q'}}{2-q'}\Big)^{\frac{1}{q'}} = 2\pi\Big(\frac{q-1}{q-2}\Big)^{\frac{q-1}{q}} r^{\frac{q-2}{q}}.$$

Putting this together with (52) we have

$$\big|u(x)-u(y)\big| \leq 4\Big(\frac{q-1}{q-2}\Big)^{\frac{q-1}{q}} r^{\frac{q-2}{q}}\Big(\int_{B_r(y)} |Du|^q\Big)^{\frac{1}{q}} + 4\Big(\frac{q-1}{q-2}\Big)^{\frac{q-1}{q}} r^{\frac{q-2}{q}}\Big(\int_{B_r(x)} |Du|^q\Big)^{\frac{1}{q}}$$
$$\leq 8\Big(\frac{q-1}{q-2}\Big)^{\frac{q-1}{q}} r^{\frac{q-2}{q}}\Big(\int_{B_{2r}(x)} |Du|^q\Big)^{\frac{1}{q}}$$

and hence we have established (50). □

Lemma 4. *Suppose $0 \leq \kappa < 1$ and $\mu : \mathbb{R}^2 \to \mathbb{C}$ is measurable and for some $x_0 \in \mathbb{R}^2$, $|\mu(z)| \leq \kappa \mathbb{1}_{B_\tau(x_0)}(z)$ for all $z \in \mathbb{R}^2$ and f is a principle solution of the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z).$$

Let $q \in [2, 2+\frac{1-\kappa}{3\kappa})$. For any $x \in \mathbb{R}^2$, $r > 0$ we have

$$\Big(\int_{B_r(x)} |Df|^q dz\Big)^{\frac{1}{q}} \leq \pi^{\frac{1}{q}} r^{\frac{2}{q}} + \frac{4(1+3(q-2))}{1-\kappa(1+2(q-2))}\tau^{\frac{2}{q}}. \tag{53}$$

In addition letting $\mathcal{S}$ denote the Beurling transform

$$\big\|(Id-\mu\mathcal{S})^{-1}\big\|_{L^q(\mathbb{C})\to L^q(\mathbb{C})} \leq \frac{1}{1-\kappa(1+3(q-2))}. \tag{54}$$

Proof of Lemma 4. Let S_q denote the L_q norm of the Beurling transform $\mathcal{S}$. Consider the operator

$$(Id-\mu\mathcal{S})^{-1} = Id + \mu\mathcal{S} + \mu\mathcal{S}\mu\mathcal{S} + \mu\mathcal{S}\mu\mathcal{S}\mu\mathcal{S}\ldots.$$

Note that if $\phi \in L^q$ then

$$\|\mu\mathcal{S}\mu\mathcal{S}\ldots\mu\mathcal{S}\phi\|_{L^q(\mathbb{C})} \leq (\kappa S_q)^n \|\phi\|_{L^q(\mathbb{C})}. \tag{55}$$

So we require $\kappa S_q < 1$ in order for $(Id-\mu\mathcal{S})^{-1}$ to be well defined. By inequality (4.89) Section 4.5.2 [3] we have

$$S_q < 1 + 3(q-2). \tag{56}$$

Thus it is sufficient for $\kappa(1+3(q-2)) < 1$ which is equivalent to $q < \frac{1-\kappa}{3\kappa} + 2$. If this inequality is satisfied then

$$\big\|(Id-\mu\mathcal{S})^{-1}\phi\big\|_{L^q(\mathbb{C})} \overset{(55),(56)}{\leq} \sum_{m=0}^{\infty}\big(\kappa\big(1+3(q-2)\big)\big)^m \|\phi\|_{L^q(\mathbb{C})} \leq \frac{1}{1-\kappa(1+3(q-2))}\|\phi\|_{L^q(\mathbb{C})}. \tag{57}$$

Thus (54) is established. So defining $\sigma = \mathcal{C}((Id - \mu\mathcal{S})^{-1}\mu)$ where $\mathcal{C}$ is the Cauchy transform. As in the proof of Theorem 5.1.1. [3] we know that

$$\frac{\partial \sigma}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\big(\mathcal{C}\big((Id - \mu\mathcal{S})^{-1}\mu\big)\big) = (Id - \mu\mathcal{S})^{-1}\mu \tag{58}$$

and

$$\frac{\partial \sigma}{\partial z} = \frac{\partial}{\partial z}\big(\mathcal{C}\big((Id - \mu\mathcal{S})^{-1}\mu\big)\big) = \mathcal{S}\big((Id - \mu\mathcal{S})^{-1}\mu\big). \tag{59}$$

Thus

$$\left\|\frac{\partial \sigma}{\partial \bar{z}}\right\|_{L^q(\mathbb{C})} \overset{(57),(58)}{\leq} \frac{\pi^{\frac{1}{q}}\tau^{\frac{2}{q}}}{1-\kappa(1+3(q-2))} \tag{60}$$

and

$$\left\|\frac{\partial \sigma}{\partial z}\right\|_{L^q(\mathbb{C})} \overset{(59),(57)}{\leq} S_q \frac{\pi^{\frac{1}{q}}\tau^{\frac{2}{q}}}{1-\kappa(1+3(q-2))} \overset{(56)}{\leq} \frac{(1+3(q-2))\pi^{\frac{1}{q}}\tau^{\frac{2}{q}}}{1-\kappa(1+3(q-2))}. \tag{61}$$

Hence

$$\|D\sigma\|_{L^q(\mathbb{C})} \overset{(60),(61)}{\leq} \frac{4(1+3(q-2))\tau^{\frac{2}{q}}}{1-\kappa(1+3(q-2))}. \tag{62}$$

Now as in the proof of Theorem 5.1.2 [3] we see that $f(z) = z + \sigma(z)$ is the principle solution of the Beltrami equation, i.e. the function that satisfies

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z) \quad \text{for a.e. } z$$

and $f(z) = z + \mathcal{O}(\frac{1}{z})$ as $z \to \infty$. So note that for any $x \in \mathbb{R}^2$ we have that

$$\left(\int_{B_r(x)} |Df|^q dz\right)^{\frac{1}{q}} \leq \pi^{\frac{1}{q}} r^{\frac{2}{q}} + \left(\int_{B_r(x)} |D\sigma|^q dz\right)^{\frac{1}{q}}.$$

Putting this together with (62) we have (53). □

Lemma 5. *Suppose* $0 \leq \kappa < 1$ *and* $\mu, \nu : \mathbb{R}^2 \to \mathbb{C}$ *is measurable and* $|\mu(z)| \leq \kappa \mathbb{1}_{B_1}(z)$, $|\nu(z)| \leq \kappa \mathbb{1}_{B_1}(z)$ *for all* $z \in \mathbb{R}^2$. *Suppose* f, g *are the a principle solutions of the Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z), \qquad \frac{\partial g}{\partial \bar{z}}(z) = \nu(z)\frac{\partial g}{\partial z}(z) \quad \textit{for a.e. } z. \tag{63}$$

Then for $q \in [2, 2 + \frac{1-\kappa}{3\kappa})$ *and* $s > 1$ *such that* $sq < 2 + \frac{1-\kappa}{3\kappa}$ *we have that*

$$\left\|\frac{\partial f}{\partial \bar{z}} - \frac{\partial g}{\partial \bar{z}}\right\|_{L^q(\mathbb{C})} \leq 6(qs)^2\big(1-\kappa\big(1+3(qs-2)\big)\big)^{-2}\|\mu-\nu\|_{L^{\frac{qs}{s-1}}(\mathbb{C})} \tag{64}$$

and

$$\left\|\frac{\partial f}{\partial z} - \frac{\partial g}{\partial z}\right\|_{L^q(\mathbb{C})} \leq 18(qs)^2\big(1-\kappa\big(1+3(qs-2)\big)\big)^{-2}\|\mu-\nu\|_{L^{\frac{qs}{s-1}}(\mathbb{C})}. \tag{65}$$

Proof of Lemma 5. This is essentially a very minor strengthening of Lemma 5.3.1 [3], the main difference is that we estimate the constants and control the difference of the conformal part of the gradient.

Now from (5.8) p. 165 [3] we know that $\frac{\partial f}{\partial \bar{z}} = \mu + \mu\mathcal{S}(\frac{\partial f}{\partial \bar{z}})$ and so

$$\frac{\partial f}{\partial \bar{z}} = (Id - \mu\mathcal{S})^{-1}\mu. \tag{66}$$

Now from the first inequality of the proof of Lemma 5.3.1 page 168 [3] and from Lemma 4 we know

$$\begin{aligned}\left\|\frac{\partial f}{\partial \bar z}-\frac{\partial g}{\partial \bar z}\right\|^q_{L^q(\mathbb{C})} &\leq \left\|(Id-\nu\mathcal{S})^{-1}\right\|^q_{L^q(\mathbb{C})\to L^q(\mathbb{C})}\left\|(\mu-\nu)\left(\mathbb{1}_{B_1}+\mathcal{S}\left(\frac{\partial f}{\partial \bar z}\right)\right)\right\|^q_{L^q(\mathbb{C})}\\ &\overset{(54),(66)}{\leq} \left(1-\kappa\left(1+3(q-2)\right)\right)^{-q}\|\mu-\nu\|^q_{L^{qs'}(\mathbb{C})}\left\|\mathbb{1}_{B_1}+\mathcal{S}\left((Id-\mu\mathcal{S})^{-1}\mu\right)\right\|^q_{L^{qs}(\mathbb{C})}. \end{aligned} \tag{67}$$

Now by Lemma 4 (recalling that $sq<2+\frac{1-\kappa}{3\kappa}$)

$$\begin{aligned}\left\|(Id-\mu\mathcal{S})^{-1}\mu\right\|_{L^{qs}(\mathbb{C})} &\overset{(54)}{\leq} \left(1-\kappa\left(1+3(qs-2)\right)\right)^{-1}\|\mu\|_{L^{qs}(\mathbb{C})}\\ &\leq 2\left(1-\kappa\left(1+3(qs-2)\right)\right)^{-1}.\end{aligned} \tag{68}$$

So

$$\begin{aligned}\left\|\frac{\partial f}{\partial \bar z}-\frac{\partial g}{\partial \bar z}\right\|_{L^q(\mathbb{C})} &\overset{(67),(56)}{\leq} \left(1-\kappa\left(1+3(q-2)\right)\right)^{-1}\|\mu-\nu\|_{L^{qs'}(\mathbb{C})}\\ &\qquad\times\left(\pi^{\frac{1}{qs}}+\left(1+3(qs-2)\right)\left\|(Id-\mu\mathcal{S})^{-1}\mu\right\|_{L^{qs}(\mathbb{C})}\right)\\ &\overset{(68)}{\leq} \left(1-\kappa\left(1+3(q-2)\right)\right)^{-1}\|\mu-\nu\|_{L^{qs'}}\\ &\qquad\times\left(2+2\left(1+3(qs-2)\right)\left(1-\kappa\left(1+3(qs-2)\right)\right)^{-1}\right)\\ &\leq 6qs\left(1-\kappa\left(1+3(qs-2)\right)\right)^{-2}\|\mu-\nu\|_{L^{qs'}(\mathbb{C})}.\end{aligned} \tag{69}$$

Now the Beurling transform $\mathcal{S}$ of the anti-conformal part of the gradient of the L^2 function gives the conformal part of the gradient, see (4.18) Chapter 4 [3]. So

$$\mathcal{S}\left(\frac{\partial}{\partial \bar z}(f-g)\right)=\frac{\partial}{\partial z}(f-g).$$

Using the fact that f and g are holomorphic outside B_1 (see (38)) for the last inequality

$$\begin{aligned}\left\|\frac{\partial f}{\partial z}-\frac{\partial g}{\partial z}\right\|_{L^q(\mathbb{C})} &= \left\|\mathcal{S}\left(\frac{\partial f}{\partial \bar z}\right)-\mathcal{S}\left(\frac{\partial g}{\partial \bar z}\right)\right\|_{L^q(\mathbb{C})}\\ &\overset{(56)}{\leq} \left(1+3(q-2)\right)\left\|\frac{\partial f}{\partial \bar z}-\frac{\partial g}{\partial \bar z}\right\|_{L^2(\mathbb{C})}\\ &\overset{(69)}{\leq} 6\left(1+3(q-2)\right)qs\left(1-\kappa\left(1+3(qs-2)\right)\right)^{-2}\|\mu-\nu\|_{L^{qs'}(\mathbb{C})}\\ &\leq 18(qs)^2\left(1-\kappa\left(1+3(qs-2)\right)\right)^{-2}\|\mu-\nu\|_{L^{qs'}(\mathbb{C})}.\end{aligned} \tag{70}$$

So (69) gives (64) and (70) gives (65). □

Lemma 6. *Suppose $0\leq\kappa<1$ and $\mu:\mathbb{R}^2\to\mathbb{C}$ is measurable and $|\mu(z)|\leq\kappa\mathbb{1}_{B_1}(z)$ for all $z\in\mathbb{R}^2$. Let $f:\mathbb{R}^2\to\mathbb{C}$ be the principle solution of the Beltrami equation*

$$\frac{\partial f}{\partial \bar z}(z)=\mu(z)\frac{\partial f}{\partial z}(z)\quad \text{for a.e. } z$$

and let $h:\mathbb{C}\to\mathbb{C}$ be the global inverse of f. Let $q\in(2,2+\frac{1-\kappa}{3\kappa})$. Then

$$\left|f(z_1)-f(z_2)\right|\leq 48|z_1-z_2|^{\frac{q-2}{q}}\left(\frac{q-1}{q-2}\right)\left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)\quad \text{for any } z_1,z_2\in B_1. \tag{71}$$

And

$$\left|h(z_1)-h(z_2)\right|\leq 2400|z_1-z_2|^{\frac{q-2}{q}}\left(\frac{q-1}{q-2}\right)^2\left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)^2\quad \text{for any } z_1,z_2\in f(B_1). \tag{72}$$

As a consequence for any $B_r(z_0) \subset B_1$

$$B_{(\frac{(1-\kappa)^6 r}{4.3\times 10^{10}})^{\frac{12}{1-\kappa}}}\big(f(z_0)\big) \subset f\big(B_r(z_0)\big). \tag{73}$$

In addition for any $\alpha > 0$ *such that* $B_{(\frac{\alpha(1-\kappa)^2}{3456})^{\frac{12\kappa}{1-\kappa}}}(z_0) \subset B_1$

$$f\big(B_{(\frac{\alpha(1-\kappa)^2}{3456})^{\frac{12}{1-\kappa}}}(z_0)\big) \subset B_\alpha\big(f(z_0)\big). \tag{74}$$

Proof of Lemma 6. *Step 1*. We will establish (71).

Proof of Step 1. By Lemma 4 we have that

$$\Big(\int_{B_4(z_0)} |Df|^q dz\Big)^{\frac{1}{q}} \overset{(53)}{\le} 8 + \frac{4(1+3(q-2))}{1-\kappa(1+2(q-2))} \quad \text{for any } z_0 \in \mathbb{R}^2. \tag{75}$$

So by Lemma 3 we know that (using the fact $0 < 1-\kappa(1+2(q-2)) \le \frac{1}{3}$ for the last inequality)

$$\begin{aligned}\big|f(z_1)-f(z_2)\big| &\overset{(50)}{\le} 8\Big(\frac{q-1}{q-2}\Big)|z_1-z_2|^{\frac{q-2}{q}}\Big(\int_{B_{2|z_1-z_2|}(x)} |Df|^q dz\Big)^{\frac{1}{q}} \\ &\overset{(75)}{\le} 8|z_1-z_2|^{\frac{q-2}{q}}\Big(\frac{q-1}{q-2}\Big)\Big(8+\frac{4(1+3(q-2))}{1-\kappa(1+2(q-2))}\Big) \\ &\le 48|z_1-z_2|^{\frac{q-2}{q}}\Big(\frac{q-1}{q-2}\Big)\Big(\frac{(1+3(q-2))}{1-\kappa(1+2(q-2))}\Big)\end{aligned} \tag{76}$$

so estimate (71) holds true.

Step 2. We will establish (72).

Proof of Step 2. Now if we consider the Beltrami equation of f we have that $(Df(x))_a\mathcal{I} = \mu_{Df(x)}(Df(x))_c$, so if $z = f(x)$ then $Dh(z) = (Df(h(z)))^{-1} = (Df(x))^{-1}$. Now the Beltrami equation for h is $(Dh(z))_a\mathcal{I} = \mu_{Dh(z)}(Dh(z))_c$. By (40) we have that

$$|\mu_{Dh(z)}| = |\mu_{(Df(x))^{-1}}| = |\mu_{Df(x)}|. \tag{77}$$

Now if $z \notin f(B_1)$, since $Dh(z) = (Df(h(z)))^{-1}$ and since $h(z) \notin B_1$, $Df(h(z)) \in CO_+(2)$ so $Dh(z) \in CO_+(2)$ thus $\mu_{Dh(z)} = 0$.

Let

$$\Lambda_q^\kappa = 48\Big(\frac{q-1}{q-2}\Big)\Big(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\Big). \tag{78}$$

Note that for any $z \in B_1$, $|f(z)-f(0)| \overset{(71)}{\le} \Lambda_q^\kappa$ so

$$f(B_1) \subset B_{\Lambda_q^\kappa}\big(f(0)\big). \tag{79}$$

So returning to complex notation we have $\frac{\partial h}{\partial \bar{w}}(w) = \gamma(w)\frac{\partial h}{\partial w}(w)$ where

$$\big|\gamma(w)\big| \overset{(33),(77)}{\le} \kappa \mathbb{1}_{B_{\Lambda_q^\kappa}(f(0))}.$$

By Lemma 4 (53) we know

$$\begin{aligned}\Big(\int_{B_{4\Lambda_q^\kappa}(y_1)} |Dh|^q dz\Big)^{\frac{1}{q}} &\le \pi^{\frac{1}{q}}\big(4\Lambda_q^\kappa\big)^{\frac{2}{q}} + \Big(\frac{4(1+3(q-2))}{1-\kappa(1+2(q-2))}\Big)\big(\Lambda_q^\kappa\big)^{\frac{2}{q}} \\ &\overset{(78)}{\le} 300\Big(\frac{q-1}{q-2}\Big)\Big(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\Big)^2.\end{aligned} \tag{80}$$

Now for any $z_1, z_2 \in f(B_1)$ by (79) we know $z_2 \in B_{2\Lambda_q^\kappa}(y_1)$ so by Lemma 3

$$\left|h(z_1) - h(z_2)\right| \overset{(80),(50)}{\leq} 2400\left(\frac{q-1}{q-2}\right)^2 |z_1 - z_2|^{\frac{q-2}{q}} \left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)^2 \tag{81}$$

and hence (72) is established.

Step 3. Let

$$q = \min\left\{2 + \frac{1-\kappa}{6\kappa}, 3\right\}. \tag{82}$$

We will establish

$$\Lambda_q^\kappa \leq \frac{3456}{(1-\kappa)^2} \tag{83}$$

and

$$0 \leq \frac{q-2}{q} - \frac{1-\kappa}{12} < 1. \tag{84}$$

Proof of Step 3. If $q < 3$ then

$$\begin{aligned}\left(\frac{q-1}{q-2}\right)^2 \left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)^2 &\overset{(82)}{\leq} \left(\frac{12}{1-\kappa}\right)^2 \left(\frac{4}{1-\kappa(1+2(\frac{1-\kappa}{6\kappa}))}\right)^2 \\ &= \frac{(72)^2}{(1-\kappa)^4}.\end{aligned} \tag{85}$$

If $q = 3$ then $2 + \frac{1-\kappa}{6\kappa} \geq 3$, $1-\kappa \geq 6\kappa$ so $0 < \kappa \leq \frac{1}{7}$. So

$$\left(\frac{q-1}{q-2}\right)^2 \left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)^2 \leq \frac{200}{(1-\kappa)^4}.$$

So for any q we have that

$$\left(\frac{q-1}{q-2}\right)^2 \left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right)^2 \leq \frac{(72)^2}{(1-\kappa)^4} \tag{86}$$

and so $\frac{\Lambda_q^\kappa}{48} \overset{(78)}{=} (\frac{q-1}{q-2})(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}) \leq \frac{72}{(1-\kappa)^2}$. Thus (83) is established.

Now if $q \in (2,3)$ then by (82) we know $2 + \frac{1-\kappa}{6\kappa} < 3$ and so $\frac{1}{7} < \kappa < 1$. Thus

$$\frac{q-2}{q} \overset{(82)}{=} \frac{1-\kappa}{6q\kappa} \overset{(82)}{=} \frac{1-\kappa}{6(2+\frac{1-\kappa}{6\kappa})\kappa} = \frac{1-\kappa}{11\kappa+1} \geq \frac{1-\kappa}{12} \tag{87}$$

and

$$\frac{q-2}{q} - \frac{1-\kappa}{12} \leq \frac{q-2}{q} \leq \frac{1}{2}. \tag{88}$$

So (87), (88) together establishes (84) for the case $q \in (2,3)$. And if $q = 3$ since $\kappa \leq \frac{1}{7}$ we have $\frac{q-2}{q} = \frac{1}{3} \geq \frac{1-\kappa}{12}$ and $0 \leq \frac{q-2}{q} - \frac{1-\kappa}{12} \leq \frac{1}{3} < 1$. Thus in all cases we have established (84).

Proof of Lemma completed. Let $\varpi \in f(\partial B_r(z_0))$ be such that

$$\left|\varpi - f(z_0)\right| = \inf\left\{\left|f(z) - f(z_0)\right| : z \in \partial B_r(z_0)\right\}.$$

So

$$\left|\varpi - f(z_0)\right| \overset{(71)}{\leq} 48 r^{\frac{q-2}{q}} \left(\frac{q-1}{q-2}\right)\left(\frac{1+3(q-2)}{1-\kappa(1+2(q-2))}\right) \overset{(78)}{=} \Lambda_q^\kappa r^{\frac{q-2}{q}}. \tag{89}$$

Hence as

$$\left|\varpi - f(z_0)\right|^{\frac{q-2}{2}} = \left|\varpi - f(z_0)\right|^{\frac{1-\kappa}{12\kappa}}\left|\varpi - f(z_0)\right|^{\frac{q-2}{q}-\frac{1-\kappa}{12\kappa}} \overset{(84),(89)}{\leq} \left|\varpi - f(z_0)\right|^{\frac{1-\kappa}{12\kappa}}\Lambda_q^{\kappa}, \tag{90}$$

thus

$$\begin{aligned} r &= \left|h(\varpi) - h\big(f(z_0)\big)\right| \\ &\overset{(81),(86),(90)}{\leq} \frac{2400\times(72)^2\Lambda_q^{\kappa}}{(1-\kappa)^4}\left|\varpi - f(z_0)\right|^{\frac{1-\kappa}{12}} \\ &\overset{(83)}{\leq} \frac{4.3\times 10^{10}}{(1-\kappa)^6}\left|\varpi - f(z_0)\right|^{\frac{1-\kappa}{12}}. \end{aligned}$$

Thus $\frac{(1-\kappa)^6 r}{4.3\times 10^{10}} \leq |\varpi - f(z_0)|^{\frac{1-\kappa}{12}}$ so $(\frac{(1-\kappa)^6 r}{4.3\times 10^{10}})^{\frac{12}{1-\kappa}} \leq |\varpi - f(z_0)|$ which implies (73). Now finally

$$\begin{aligned} \left|f(z_0) - f(z_1)\right| &\overset{(71),(78)}{\leq} \Lambda_q^{\kappa}|z_0 - z_1|^{\frac{q-2}{q}} \\ &\overset{(83),(84)}{\leq} \frac{3456}{(1-\kappa)^2}|z_0 - z_1|^{\frac{1-\kappa}{12}} \quad \text{for any } |z_0 - z_1| < 1. \end{aligned}$$

Thus for any $z_1 \in B_{(\frac{\alpha(1-\kappa)^2}{3456})^{\frac{12}{1-\kappa}}}(z_0)$ we have $|f(z_0) - f(z_1)| < \alpha$ which implies (74). □

3.2. *Estimates on Stoilow decompositions of Quasiregular mappings u, v satisfying the hypotheses of Theorem 2*

In this section we will prove several lemmas that provide quantitative estimates on the functions that make up the Stoilow decompositions of the Quasiregular mappings u, v that satisfy the hypotheses of Theorem 2. Specifically our hypotheses are

$$u, v \in W^{1,2}\big(B_1 : \mathbb{R}^2\big) \text{ are Q-quasiregular,} \tag{91}$$

$$\int_{B_1} |Du|^2 dz \leq \pi, \tag{92}$$

$$\int_{B_1} \det(Du)^{-p} dz \leq C_p, \tag{93}$$

and

$$\left\|S(Du) - S(Dv)\right\|_{L^2(B_1)} = \epsilon^2. \tag{94}$$

As described in Definition 1 the functions ϕ_u, w_u provide the Stoilow decomposition of u and ϕ_v, w_v provide the Stoilow decomposition of v. Formally

$$u = \phi_u \circ w_u \quad \text{and} \quad v = \phi_v \circ w_v \quad \text{on } B_1 \tag{95}$$

and the Beltrami coefficients of w_u, u and w_v, v agree, i.e.

$$[Dw_u]_a\mathcal{I}[Dw_u]_c = [Du]_a\mathcal{I}[Du]_c \quad \text{and} \quad [Dw_v]_a\mathcal{I}[Dw_v]_c = [Dv]_a\mathcal{I}[Dv]_c \quad \text{a.e. on } B_1. \tag{96}$$

Lemma 7. *Let u, v satisfy (91), (92) and (94) and let ϕ_u, w_u and ϕ_v, w_v be defined by (95), (96). We will show that for constants*

$$\mu = \big(2\times 10^{10}(Q+1)^6\big)^{-6(Q+1)}, \qquad \gamma = \left(\frac{\mu}{8000(Q+1)^2}\right)^{6(Q+1)} \quad \textit{and} \quad \xi = \left(\frac{\gamma}{10^{10}(Q+1)^6}\right)^{6(Q+1)} \tag{97}$$

we have

$$B_{2\mu}\big(w_u(0)\big) \subset w_u(B_{\frac{1}{2}}), \qquad B_{2\mu}\big(w_v(0)\big) \subset w_v(B_{\frac{1}{2}}) \tag{98}$$

and

$$B_{\xi}\big(w_u(0)\big) \subset w_u(B_\gamma) \subset B_{\frac{\mu}{8}}\big(w_u(0)\big), \qquad B_{\xi}\big(w_v(0)\big) \subset w_v(B_\gamma) \subset B_{\frac{\mu}{8}}\big(w_v(0)\big). \tag{99}$$

In addition

$$\|\phi_u\|_{L^\infty(B_{2\mu}(w_u(0)))} \le 2\pi, \qquad \|\phi_v\|_{L^\infty(B_{2\mu}(w_v(0)))} \le 2\pi \tag{100}$$

$$\|\phi_u'\|_{L^\infty(B_{\mu}(w_u(0)))} \le \frac{4\pi}{\mu}, \qquad \|\phi_v'\|_{L^\infty(B_{\mu}(w_v(0)))} \le \frac{4\pi}{\mu} \tag{101}$$

and

$$\|\phi_u''\|_{L^\infty(B_{\mu}(w_u(0)))} \le \frac{8\pi}{\mu^2}, \qquad \|\phi_v''\|_{L^\infty(B_{\mu}(w_v(0)))} \le \frac{8\pi}{\mu^2}. \tag{102}$$

Proof of Lemma 7. We will argue the estimate for u, ϕ_u. The estimates for v, ϕ_v follow by exactly the same arguments.

Now recall from (26), (28) we can take $\kappa = \frac{Q-1}{Q+1}$. Now

$$1-\kappa = \frac{Q+1-(Q-1)}{Q+1} = \frac{2}{Q+1}. \tag{103}$$

Thus

$$\begin{aligned}\left(\frac{r}{10^{10}(Q+1)^6}\right)^{6(Q+1)} &\le \left(\frac{64}{10}\times\frac{r}{10^{10}(Q+1)^6}\right)^{6(Q+1)} \\ &\overset{(103)}{=} \left(\frac{r(1-\kappa)^6}{10^{11}}\right)^{\frac{12}{1-\kappa}} \\ &\le \left(\frac{r}{2}\frac{(1-\kappa)^6}{4.3\times 10^{10}}\right)^{\frac{12}{1-\kappa}}.\end{aligned}$$

So by Lemma 6

$$B_{(\frac{r}{10^{10}(Q+1)^6})^{6(Q+1)}}\big(w_u(z)\big) \overset{(73)}{\subset} w_u\big(B_{\frac{r}{2}}(z)\big) \tag{104}$$

so defining $\mu = (2\times 10^{10}(Q+1)^6)^{-6(Q+1)}$, since $2\mu \le (10^{10}(Q+1)^6)^{-6(Q+1)}$ we do indeed have $B_{2\mu}(w_u(0)) \overset{(104)}{\subset} w_u(B_{\frac{1}{2}})$ and so (98) is established.

Note we have

$$B_{(\frac{\mu}{8000(Q+1)^2})^{6(Q+1)}}(0) \subset B_{(\frac{4\mu}{8\times 3456(Q+1)^2})^{6(Q+1)}}(0) \overset{(103)}{=} B_{(\frac{(1-\kappa)^2\mu}{8\times 3456})^{\frac{12}{1-\kappa}}}(0).$$

So by (74) we have $w_u(B_{(\frac{\mu}{8000(Q+1)^2})^{6(Q+1)}}(0)) \subset B_{\frac{\mu}{8}}(w_u(0))$ which establishes half of (99). Finally by (104) we have that $B_{(\frac{\gamma}{10^{10}(Q+1)^6})^{6(Q+1)}}(w_u(0)) \subset w_u(B_{\frac{\gamma}{2}})$ which establishes the other half of (99).

Recall from the hypothesis (92). As $\int_{B_1}|Du|dz \le \pi$ we can find $h \in (\frac{1}{2},1)$ such that $\int_{\partial B_h}|Du|dH^1 z \le 2\pi$. Since u is open $\partial u(B_h) \subset u(\partial B_h)$ so $H^1(\partial u(B_h)) \le 2\pi$.

So

$$u(B_{\frac{1}{2}}) \subset u(B_h) \subset B_{2\pi}\big(u(0)\big). \tag{105}$$

Now by (98) since w_u is a homeomorphism $w_u^{-1}(B_{2\mu}(w_u(0))) \subset B_{\frac{1}{2}}$ so as $\phi_u = u\circ w_u^{-1}$ we have

$$\phi_u\big(B_{2\mu}\big(w_u(0)\big)\big) \overset{(105)}{\subset} B_{2\pi}\big(u(0)\big) \overset{(21)}{=} B_{2\pi}(0).$$

So

$$\|\phi_u\|_{L^\infty(B_{2\mu}(w_u(0)))} \le 2\pi. \tag{106}$$

Thus for any $z \in B_\mu(w_u(0))$

$$\begin{aligned}\left|\phi_u'(z)\right| &\le \frac{1}{2\pi} \int_{\partial B_{2\mu}(w_u(0))} \frac{|\phi_u(\zeta)|}{|\zeta - z|^2} |d\zeta| \\ &\le (2\pi)^{-1} \sup_{B_{2\mu}(w_u(0))} |\phi_u| \int_{\partial B_{2\mu}(w_u(0))} \frac{1}{|\zeta - z|^2}|d\zeta| \overset{(106)}{\le} \frac{4\pi}{\mu}.\end{aligned}$$

In the same way for any $z \in B_\mu(w_u(0))$

$$\left|\phi_u''(z)\right| \le \frac{1}{\pi} \sup_{B_{2\mu}(w_u(0))} |\phi_u| \int_{\partial B_{2\mu}(w_u(0))} \frac{1}{|\zeta - z|^3}|d\zeta| \overset{(106)}{\le} \frac{8\pi}{\mu^2}. \qquad \square$$

Lemma 8. *Let u, v satisfy (91), (92) and (94) and let ϕ_u, w_u and ϕ_v, w_v be defined by (95), (96). We will show*

$$\|Dw_u - Dw_v\|_{L^2(B_1)} \le 127\,280 Q^4 C_p \epsilon^{\frac{p}{72Q}}. \tag{107}$$

Proof of Lemma 8. Recall from (26), (28) we can take $\kappa = \frac{Q-1}{Q+1}$. Note

$$\frac{1}{2}\left(4 + \frac{1-\kappa}{3\kappa}\right) = 2 + \frac{1-\kappa}{6\kappa} = 2 + \frac{1}{3(Q-1)}.$$

So define

$$\mathcal{P}_Q = \begin{cases} 2 + \frac{1}{3(Q-1)} & \text{for } Q \ge \frac{4}{3} \\ 3 & \text{for } Q < \frac{4}{3}. \end{cases} \tag{108}$$

Now we require s such that $s(2 + \frac{1-\kappa}{6\kappa}) < 2 + \frac{1-\kappa}{3\kappa}$. So

$$s < \frac{2 + 10\kappa}{1 + 11\kappa} = \frac{2 + 10(\frac{Q-1}{Q+1})}{1 + 11(\frac{Q-1}{Q+1})} = \frac{12Q - 8}{12Q - 10}.$$

Now define

$$s_Q = \begin{cases} \frac{12Q-9}{12Q-10} & \text{for } Q \ge \frac{4}{3} \\ \frac{7}{6} & \text{for } Q < \frac{4}{3}. \end{cases} \tag{109}$$

Now for $\mathcal{P}_Q$ and s_Q defined in this way we need to estimate various exponents and constants in Lemma 5. To begin with note since $\frac{d}{dQ}(\frac{3(Q-1)}{6Q-5}) = \frac{3}{(5-6Q)^2}$ so we know $\frac{3(Q-1)}{6Q-5} \ge \frac{1}{3}$ for all $Q \ge \frac{4}{3}$. Thus

$$\frac{s_Q - 1}{\mathcal{P}_Q s_Q} = \begin{cases} \frac{3(Q-1)}{6Q-5}\frac{1}{12Q-9} \ge \frac{1}{36Q} & \text{for } Q \ge \frac{4}{3} \\ \frac{1}{3} \times \frac{1}{7} = \frac{1}{21} & \text{for } Q < \frac{4}{3}. \end{cases} \tag{110}$$

Step 1. We will establish

$$(s_Q\mathcal{P}_Q)^2\big(1 - \kappa\big(1 + 3(\mathcal{P}_Q s_Q - 2)\big)\big)^{-2} \le 784 Q^3. \tag{111}$$

Proof of Step 1. If $Q \ge \frac{4}{3}$ define

$$\alpha = \frac{18Q - 15}{24Q - 20}. \tag{112}$$

It is a calculation to see that

$$\frac{6Q-5}{3Q-3}\frac{12Q-9}{12Q-10}-2-\frac{2\alpha}{3(Q-1)}=0. \tag{113}$$

So note as $\frac{1-\kappa}{\kappa}=\frac{2}{Q+1}\frac{Q+1}{Q-1}=\frac{2}{Q-1}$

$$\mathcal{P}_Q s_Q \overset{(108),(109)}{=} \frac{6Q-5}{3Q-3}\frac{12Q-9}{12Q-10} \overset{(113)}{=} 2+\frac{2\alpha}{3(Q-1)}=2+\frac{\alpha(1-\kappa)}{3\kappa}. \tag{114}$$

Thus

$$\left(1-\kappa\left(1+3(\mathcal{P}_Q s_Q-2)\right)\right)^2 \overset{(114)}{=} \left((1-\alpha)(1-\kappa)\right)^2 \tag{115}$$

and

$$1-\alpha \overset{(112)}{=} \frac{6Q-5}{24Q-20} \geq \frac{3}{24Q-20} \geq \frac{1}{8Q}. \tag{116}$$

Thus finally as $1-\kappa=\frac{2}{Q+1}$

$$\left(1-\kappa\left(1+3(\mathcal{P}_Q s_Q-2)\right)\right)^2 \overset{(115),(116)}{\geq} \left(\frac{2}{Q+1}\right)^2 \frac{1}{(8Q)^2} \geq \frac{1}{64Q^3}. \tag{117}$$

If $Q\in[1,\frac{4}{3})$ noting in this case $\mathcal{P}_Q \overset{(108)}{=} 3$, $s_Q \overset{(109)}{=} \frac{7}{6}$ we have

$$\begin{aligned}
1-\kappa\left(1+3(\mathcal{P}_Q s_Q-2)\right) &= \frac{1}{Q+1}\left(Q+1-\frac{11}{2}(Q-1)\right)\\
&= \frac{1}{2(Q+1)}(13-9Q)\\
&\geq \frac{1}{4Q}.
\end{aligned} \tag{118}$$

Now for any value of Q since $\frac{d}{dQ}(\frac{12Q-9}{12Q-10})=\frac{-3}{(5-6Q)^2}$ from (109) we see $s_Q\leq\frac{7}{6}$ for all Q. And from (108) we have $\mathcal{P}_Q\leq 3$ so

$$(s_Q\mathcal{P}_Q)^2\left(1-\kappa\left(1+3(\mathcal{P}_Q s_Q-2)\right)\right)^{-2} \overset{(117),(118)}{\leq} \left(\frac{7}{2}\right)^2 64Q^3 \tag{119}$$

which establishes (111).

Proof of Lemma 8 completed. Let $\mathcal{B}=\{z\in B_1:\det(Du(z))\leq\sqrt{\epsilon}\}$. So

$$C_p \geq \int_{B_1} \det\left(Du(z)\right)^{-p} dz \geq \epsilon^{-\frac{p}{2}}|\mathcal{B}|.$$

Thus

$$|\mathcal{B}|\leq C_p \epsilon^{\frac{p}{2}}. \tag{120}$$

Now for any $z\in B_1\backslash\mathcal{B}$ by Lemma 2 we have

$$|\mu_{Dw_u(z)}-\mu_{Dw_v(z)}|=|\mu_{Du(z)}-\mu_{Dv(z)}| \overset{(41)}{\leq} \frac{4\sqrt{Q}}{\sqrt{\epsilon}}\left|S\left(Du(z)\right)-S\left(Dv(z)\right)\right|. \tag{121}$$

And note

$$|\mu_{Dv(z)}-\mu_{Du(z)}|\leq 3 \quad \text{for any } z\in B_1. \tag{122}$$

Hence

$$\int_{B_1} |\mu_{Dw_u(z)} - \mu_{Dw_v(z)}|dz \overset{(122),(121)}{\leq} 3|\mathcal{B}| + \frac{4\sqrt{Q}}{\sqrt{\epsilon}} \int_{B_1} \left|S\big(Du(z)\big) - S\big(Dv(z)\big)\right|dz$$
$$\overset{(94),(120)}{\leq} 3C_p \epsilon^{\frac{p}{2}} + 4\sqrt{Q}\sqrt{\pi}\epsilon$$
$$\leq 7C_p\sqrt{\pi}\sqrt{Q}\epsilon^{\frac{p}{2}}. \tag{123}$$

Now by Lemma 5 and using interpolation of L^p norms (see Section B2, (h) of the Appendix of [10])

$$\|Dw_u - Dw_v\|_{L^p(\mathbb{C})} \leq \left\|[Dw_u]_c - [Dw_v]_c\right\|_{L^p(\mathbb{C})} + \left\|[Dw_u]_a - [Dw_v]_a\right\|_{L^p(\mathbb{C})}$$
$$\overset{(64),(65),(111)}{\leq} 2 \times (6 \times 784 + 18 \times 784) Q^3 \|\mu_{Dw_u} - \mu_{Dw_v}\|_{L^{\frac{\mathcal{P}_Q s_Q}{s_Q - 1}}}$$
$$\overset{(122)}{\leq} 3 \times 37\,632 Q^3 \big(\|\mu_{Dw_u} - \mu_{Dw_v}\|_{L^1(B_1)}\big)^{\frac{s_Q-1}{\mathcal{P}_Q s_Q}}$$
$$\overset{(123)}{\leq} 112\,896 Q^3 \big(7C_p\sqrt{\pi}\sqrt{Q}\epsilon^{\frac{p}{2}}\big)^{\frac{s_Q-1}{\mathcal{P}_Q s_Q}}$$
$$\overset{(110)}{\leq} 121\,077 Q^4 \big(C_p\epsilon^{\frac{p}{2}}\big)^{\frac{1}{36Q}}$$
$$\leq 121\,077 Q^4 C_p \epsilon^{\frac{p}{72Q}}. \tag{124}$$

Now in the case $Q < \frac{4}{3}$ in the same way as before, using Lemma 5.3.1 [3] and interpolation of L^p norms, from the third line of (124) we have

$$\|Dw_u - Dw_v\|_{L^p(B_1)} \overset{(64),(65),(111),(123),(110)}{\leq} 112\,896 Q^3 \big(7C_p\sqrt{\pi}\sqrt{Q}\epsilon^{\frac{p}{2}}\big)^{\frac{1}{21}}$$
$$\leq 127\,280 Q^4 C_p \epsilon^{\frac{p}{42}}. \tag{125}$$

Putting (125) and (124) together we have (107). □

Lemma 9. *Let u, v satisfy (91), (92) and (94) and let ϕ_u, w_u and ϕ_v, w_v be defined by (95), (96). We will show*

$$\|Dw_u\|_{L^2(B_1)} \leq 26\pi Q \quad \textit{and} \quad \|Dw_v\|_{L^2(B_1)} \leq 26\pi Q. \tag{126}$$

Let $\varpi = \min\{3, 2 + \frac{1}{3(Q-1)}\}$,

$$\|Dw_u\|_{L^{\varpi}(B_1)} \leq 26Q, \qquad \|Dw_v\|_{L^{\varpi}(B_1)} \leq 26Q. \tag{127}$$

And

$$\|w_u - w_v\|_{L^\infty(B_{\frac{1}{2}})} \leq 8.6 \times 10^8 Q^5 C_p \epsilon^{\frac{p}{180Q}}. \tag{128}$$

Proof of Lemma 9. As before we will take $\kappa = \frac{Q-1}{Q+1}$, so

$$\varpi = \min\left\{3, 2 + \frac{1-\kappa}{6\kappa}\right\} \quad \text{so } \varpi - 2 = \min\left\{1, \frac{1-\kappa}{6\kappa}\right\}. \tag{129}$$

Note that

$$1 - \kappa\big(1 + 2(\varpi - 2)\big) \overset{(129)}{\geq} 1 - \kappa\left(1 + \frac{1-\kappa}{3\kappa}\right) = \frac{2}{3}(1-\kappa) \overset{(103)}{=} \frac{4}{3(Q+1)}. \tag{130}$$

And note

$$\frac{1-\kappa}{6\kappa} = \frac{1}{6}\left(\frac{Q+1}{Q-1}\right)\left(\frac{2}{Q+1}\right) = \frac{1}{3(Q-1)} \tag{131}$$

and thus

$$1 \overset{(129)}{\geq} \varpi - 2 \overset{(131),(129)}{=} \min\left\{1, \frac{1}{3(Q-1)}\right\} \geq \frac{1}{3Q}. \tag{132}$$

So from Lemma 4 (53) (and recalling (129))

$$\left(\int_{B_1} |Dw_u|^{\varpi} dz\right)^{\frac{1}{\varpi}} \overset{(53),(130),(132)}{\leq} \pi^{\frac{1}{q}} + 12(Q+1) \leq 26Q. \tag{133}$$

In the same way $\|Dw_v\|_{L^{\varpi}(B_1)} \leq 26Q$, thus (127) is established. By Holder if we let $r = \frac{\varpi}{2}$ and $r' > 0$ be such that $\frac{1}{r} + \frac{1}{r'} = 1$ and so

$$\int_{B_1} |Dw_u|^2 dz \leq \pi\left(\int_{B_1} |Dw_u|^{\varpi} dz\right)^{\frac{1}{r}} \overset{(127)}{\leq} \pi(26Q)^{\frac{\varpi}{r}} \leq 26^2 \pi Q^2.$$

So $\|Dw_u\|_{L^2(B_1)} \leq 26\pi Q$ and in the same way $\|Dw_v\|_{L^2(B_1)} \leq 26\pi Q$. So (126) is established.

Let

$$r = 1 + \frac{\varpi}{2} = \frac{2+\varpi}{2}. \tag{134}$$

Since by (129), $2 < \varpi \leq 3$, so $r \in (2, \varpi)$ and thus $\frac{1}{\varpi} < \frac{1}{r} < \frac{1}{2}$ and thus there exists $\theta \in (0,1)$ such that

$$\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{\varpi}. \tag{135}$$

By interpolation of L^p norms we know

$$\|Dw_u - Dw_v\|_{L^r(B_1)} \leq \|Dw_u - Dw_v\|^{\theta}_{L^2(B_1)} \|Dw_u - Dw_v\|^{1-\theta}_{L^{\varpi}(B_1)} \overset{(107),(127)}{\leq} \left(127\,280 C_p Q^4 \epsilon^{\frac{p}{72Q}}\right)^{\theta} (52Q)^{1-\theta}. \tag{136}$$

Now since $r = \frac{2+\varpi}{2}$ so $\frac{2}{2+\varpi} - \frac{1}{\varpi} \overset{(135)}{=} \theta(\frac{1}{2} - \frac{1}{\varpi})$. So

$$\theta\left(\frac{\varpi-2}{2\varpi}\right) = \theta\left(\frac{1}{2} - \frac{1}{\varpi}\right) = \frac{2}{2+\varpi} - \frac{1}{\varpi} = \frac{\varpi-2}{\varpi(2+\varpi)}. \tag{137}$$

So again since by (129) $2 < \varpi \leq 3$, thus $\frac{2}{5} \leq \theta \overset{(137)}{=} \frac{2}{(2+\varpi)} < \frac{1}{2}$. Thus

$$\|Dw_u - Dw_v\|_{L^r(B_1)} \overset{(136)}{\leq} 127\,280^{\theta}(52)^{1-\theta} C_p Q^4 \epsilon^{\frac{p}{180Q}} \leq 2 \times 10^7 Q^4 C_p \epsilon^{\frac{p}{180Q}}. \tag{138}$$

Now from the proof of Lemma 4.28 of [1] letting $Q_r(\zeta)$ denote the square of side length r centred on ζ, we have that for any $z_0 \in Q_{\frac{1}{\sqrt{2}}}(0)$,

$$\left|(w_u - w_v)(z_0) - 2\int_{Q_{\frac{1}{\sqrt{2}}}(0)} (w_u - w_v)(z) dz\right| \leq K\left(\frac{1}{\sqrt{2}}\right)^{1-\frac{2}{r}} \|Dw_u - Dw_v\|_{L^r(B_1)} \tag{139}$$

where

$$K = \sqrt{2}\int_0^1 t^{-\frac{2}{r}} dt \overset{(134)}{=} \sqrt{2}\int_0^1 t^{-\frac{4}{2+\varpi}} dt = \sqrt{2}\left(\frac{2+\varpi}{\varpi-2}\right)\int_0^1 \frac{d}{dt}\left(t^{\frac{\varpi-2}{2+\varpi}}\right) dt = \sqrt{2}\left(\frac{2+\varpi}{\varpi-2}\right) \overset{(132),(129)}{\leq} 15\sqrt{2}Q. \tag{140}$$

So by (138), (139), (140) we have that

$$\left|(w_u - w_v)(z_0) - 2\int_{Q_{\frac{1}{\sqrt{2}}}(0)} (w_u - w_v)(z)dz\right| \leq 15\sqrt{2}Q \times 2 \times 10^7 Q^4 C_p \epsilon^{\frac{p}{180Q}}$$

$$\leq 4.3 \times 10^8 Q^5 C_p \epsilon^{\frac{p}{180Q}}.$$

Thus

$$\left|(w_u - w_v)(z_0) - (w_u - w_v)(z_1)\right| \leq 8.6 \times 10^8 Q^5 C_p \epsilon^{\frac{p}{180Q}} \quad \text{for any } z_1, z_2 \in Q_{\frac{1}{\sqrt{2}}}(0).$$

This establishes (128). □

Lemma 10. *Let u, v satisfy (91), (92) and (94) and let ϕ_u, w_u and ϕ_v, w_v be defined by (95), (96). We will show that for all small enough $\epsilon > 0$*

$$\int_{w_u(B_\gamma)} \left||\phi_u'|^2 - |\phi_v'|^2\right| dz \leq 2.25 \times 10^{16} Q^7 \mu^{-3} C_p \epsilon^{\frac{p}{180Q}}. \tag{141}$$

Proof of Lemma 10. Note $Du(z) = D\phi_u(w_u(z))Dw_u(z)$, so

$$Du(z)^T Du(z) = Dw_u(z)^T D\phi_u\big(w_u(z)\big)^T D\phi_u\big(w_u(z)\big) Dw_u(z) = \left|D\phi_u\big(w_u(z)\big)\right|^2 Dw_u(z)^T Dw_u(z). \tag{142}$$

We know

$$\phi_v'(z) = \mathrm{Re}\big(\phi_v(z)\big)_x + i\,\mathrm{Im}\big(\phi_v(z)\big)_x = \mathrm{Im}\big(\phi_v(z)\big)_y - i\,\mathrm{Re}\big(\phi_v(z)\big)_y. \tag{143}$$

So to simplify notation let

$$\lambda(z) = \left|D\phi_u(z)\right|^2 = 2\left|\phi_u'(z)\right|^2 \quad \text{and} \quad \varrho(z) \overset{(143)}{=} \left|D\phi_v(z)\right|^2 = 2\left|\phi_v'(z)\right|^2. \tag{144}$$

Thus from (142), (144)

$$Du(z)^T Du(z) = \lambda\big(w_u(z)\big) Dw_u(z)^T Dw_u(z) \quad \text{and} \quad Dv(z)^T Dv(z) = \varrho\big(w_v(z)\big) Dw_v(z)^T Dw_v(z). \tag{145}$$

Note since the Hilbert Schmidt norm is invariant under rotation $|S(Du)| \leq |Du|$, so

$$\left\|S(Du)\right\|_{L^2(B_1)} \leq \pi \quad \text{and} \quad \left\|S(Dv)\right\|_{L^2(B_1)} \overset{(19)}{\leq} 2\pi. \tag{146}$$

Thus

$$\begin{aligned}
&\int_{B_1} \left|S(Du)^2 - S(Dv)^2\right| dz \\
&\leq \int_{B_1} \left|S(Du)\big(S(Du) - S(Dv)\big)\right| + \left|\big(S(Du) - S(Dv)\big)S(Dv)\right| dz \\
&\leq \left\|S(Du)\right\|_{L^2(B_1)} \left\|S(Du) - S(Dv)\right\|_{L^2(B_1)} + \left\|S(Dv)\right\|_{L^2(B_1)} \left\|S(Du) - S(Dv)\right\|_{L^2(B_1)} \\
&\overset{(146)}{\leq} 3\pi\sqrt{\epsilon}.
\end{aligned} \tag{147}$$

Recall constant $\mu = (2 \times 10^{10}(Q+1)^6)^{-6(Q+1)}$ and $\gamma = (\frac{\mu}{8000(Q+1)^2})^{6(Q+1)}$. Note $w_u(B_\gamma) \overset{(99)}{\subset} B_\mu(w_u(0)) \overset{(128)}{\subset} B_{2\mu}(w_v(0)) \overset{(98)}{\subset} w_v(B_{\frac{1}{2}})$ for all small enough ϵ. Thus $\rho \circ w_u$ is defined on B_γ. Now since $S(Du)^2 = Du^T Du$ and $S(Dv)^2 = Dv^T Dv$

$$20\sqrt{\epsilon} \overset{(147)}{\geq} \int_{B_\gamma} \left|Tr\left(Du^T Du\right) - Tr\left(Dv^T Dv\right)\right| dz$$
$$\overset{(145)}{=} \int_{B_\gamma} \left|\lambda(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_v) Tr\left(Dw_v^T Dw_v\right)\right| dz$$
$$\geq \int_{B_\gamma} \left|\lambda(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_u) Tr\left(Dw_u^T Dw_u\right)\right| dz$$
$$- \int_{B_\gamma} \left|\varrho(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_v) Tr\left(Dw_v^T Dw_v\right)\right| dz. \tag{148}$$

Now note

$$\sup\left\{\left|\varrho\left(w_u(z)\right)\right| : x \in B_\gamma\right\} \overset{(99)}{\leq} \sup\left\{\left|\varrho(z)\right| : z \in B_\mu\left(w_u(0)\right)\right\} \overset{(144),(101)}{\leq} 2 \times \left(\frac{4\pi}{\mu}\right)^2 = \frac{32\pi^2}{\mu^2}. \tag{149}$$

Now from (143) as

$$\phi_v'' = \mathrm{Re}(\phi_v)_{xx} + i\mathrm{Im}(\phi_v)_{xx} = \mathrm{Im}(\phi_v)_{xy} - i\mathrm{Re}(\phi_v)_{xy} = -\mathrm{Re}(\phi_v)_{yy} - i\mathrm{Im}(\phi_v)_{yy}.$$

Thus we have

$$\left|D^2\phi_v(z)\right| \leq 4\left|\phi_v''(z)\right|. \tag{150}$$

And we can estimate the first and second partial derivatives of ϱ by

$$\left|\varrho_{,k}(z)\right| \overset{(144)}{\leq} 2\left|D^2\phi_v(z)\right|\left|D\phi_v(z)\right|$$
$$\overset{(150),(144)}{\leq} 16\left|\phi_v''(z)\right|\left|\phi_v'(z)\right|$$
$$\overset{(101),(102)}{\leq} \frac{512\pi^2}{\mu^3} \quad \text{for any } z \in B_\mu\left(w_u(0)\right) \text{ for } k = 1, 2.$$

Thus

$$\left|D\varrho(z)\right| \leq \frac{1024\pi^2}{\mu^3} \quad \text{for any } z \in B_\mu\left(w_u(0)\right). \tag{151}$$

Hence

$$\sup\left\{\left|\varrho\left(w_u(z)\right) - \varrho\left(w_v(z)\right)\right| : z \in B_\gamma\right\} \overset{(99),(128),(151)}{\leq} \frac{1024\pi^2}{\mu^3} \sup\left\{\left|w_u(z) - w_v(z)\right| : z \in B_\gamma\right\}$$
$$\overset{(128)}{\leq} 8.9 \times 10^{11} Q^5 C_p \frac{\pi^2}{\mu^3} \epsilon^{\frac{p}{180Q}}. \tag{152}$$

Thus

$$\int_{B_\gamma} \left|\varrho(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_v) Tr\left(Dw_v^T Dw_v\right)\right| dz$$
$$\leq \int_{B_\gamma} \left|\varrho(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_u) Tr\left(Dw_v^T Dw_v\right)\right| dz$$
$$+ \left|\left(\varrho(w_u) - \varrho(w_v)\right) Tr\left(Dw_v^T Dw_v\right)\right| dz$$
$$\overset{(149),(152)}{\leq} \frac{64\pi^2}{\mu^2} \int_{B_\gamma} \left|Dw_u^T Dw_u - Dw_v^T Dw_v\right| dz + 8.9 \times 10^{11} C_p Q^5 \frac{\pi^2}{\mu^3} \epsilon^{\frac{p}{180Q}} \int_{B_{\frac{1}{2}}} |Dw_v|^2 dz. \tag{153}$$

Now

$$\int_{B_1} \left|Dw_u^T Dw_u - Dw_v^T Dw_v\right| dz \le \int_{B_1} \left|Dw_u^T (Dw_u - Dw_v)\right| dz + \int_{B_1} \left|\left(Dw_u^T - Dw_v^T\right) Dw_v\right| dz$$

$$\overset{(126)}{\le} 52\pi Q \|Dw_u - Dw_v\|_{L^2(B_1)}$$

$$\overset{(107)}{\le} 6\,618\,560\pi C_p Q^5 \epsilon^{\frac{p}{72Q}}. \tag{154}$$

So applying (154) and (126) to (153)

$$\int_{B_\gamma} \left|\varrho(w_u) Tr\left(Dw_u^T Dw_u\right) - \varrho(w_v) Tr\left(Dw_v^T Dw_v\right)\right| dz$$

$$\overset{(126),(153),(154)}{\le} \frac{64\pi^2}{\mu^2} \times 6\,618\,560\pi C_p Q^5 \epsilon^{\frac{p}{72Q}} + 8.9 \times 10^{11} C_p Q^5 \frac{\pi^2}{\mu^3} \epsilon^{\frac{p}{180Q}} \times 25^2 \pi^2 Q^2$$

$$\le 5.42 \times 10^{16} \mu^{-3} Q^7 C_p \epsilon^{\frac{p}{180Q}}.$$

Putting this together with (148) we have that

$$5.5 \times 10^{16} \mu^{-3} Q^7 C_p \epsilon^{\frac{p}{180Q}} \ge \int_{B_\gamma} \left|\left(\lambda(w_u) - \varrho(w_u)\right) Tr\left(Dw_u^T Dw_u\right)\right| dz$$

$$\ge \int_{B_\gamma} \left|\lambda(w_u) - \varrho(w_u)\right| \det(Dw_u) dz$$

$$\overset{(144)}{=} 2 \int_{w_u(B_\gamma)} \left|\left|\phi_u'\right|^2 - \left|\phi_v'\right|^2\right| dz \qquad \square$$

Lemma 11. *Let μ, γ, ξ be as defined by (97). Recall that $B_\xi(w_u(0)) \subset w_u(B_\gamma)$. Fix constant*

$$h_0 = \frac{\mu^2}{16\pi} \left(\frac{\xi^{38Q}}{3C_p (5408)^{18Q} Q^{40Q}}\right)^{\frac{1}{2p}} \tag{155}$$

We can find $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ such that

$$\inf\left\{\left|\phi_u'(z)\right| : z \in B_{h_0}(z_0)\right\} \ge \frac{1}{2} \left(\frac{\xi^{38Q}}{3C_p (5408)^{18Q} Q^{40Q}}\right)^{\frac{1}{2p}}. \tag{156}$$

Proof of Lemma 11. Note

$$C_p \ge \int_{B_1} \det\left(Du(z)\right)^{-p} dz$$

$$= \int_{B_1} \det\left(D\phi_u\left(w_u(z)\right)\right)^{-p} \det\left(Dw_u(z)\right)^{-p} dz$$

$$\overset{(98)}{\ge} \int_{w_u^{-1}(B_\mu(w_u(0)))} \det\left(D\phi_u\left(w_u(z)\right)\right)^{-p} \det\left(Dw_u(z)\right) \det\left(Dw_u(z)\right)^{-p-1} dz$$

$$= \int_{w_u^{-1}(B_\mu(w_u(0)))} \det\left(D\phi_u\left(w_u(z)\right)\right)^{-p} \det\left(Dw_u\left(w_u^{-1}\left(w_u(z)\right)\right)\right)^{-p-1} \det\left(Dw_u(z)\right) dz$$

$$= \int_{B_\mu(w_u(0))} \det\big(D\phi_u(z)\big)^{-p} \det\big(Dw_u\big(w_u^{-1}(z)\big)\big)^{-p-1} dz. \tag{157}$$

Let $\varsigma > 4Q$ be some constant we decide on later $D_\varsigma = \{z \in B_1 : \det(Dw_u(z)) > \varsigma\}$. Thus by Theorem 13.1.4 [3]

$$Q\pi\left(\frac{|D_\varsigma|}{\pi}\right)^{\frac{1}{Q}} \geq \int_{D_\varsigma} \det\big(Dw_u(z)\big)dz \geq \varsigma|D_\varsigma|.$$

So $\frac{|D_\varsigma|}{\pi} \geq (\frac{\varsigma}{Q\pi})^Q |D_\varsigma|^Q$ and thus $|D_\varsigma|^{Q-1} \leq \frac{Q^Q\pi^{Q-1}}{\varsigma^Q}$. Hence as $\varsigma > 4Q$

$$|D_\varsigma| \leq \pi\left(\frac{Q}{\varsigma}\right)^{\frac{Q}{Q-1}} \leq \pi\left(\frac{Q}{\varsigma}\right). \tag{158}$$

In particular

$$|D_\varsigma| < 1. \tag{159}$$

Now let $\varphi = \min\{\frac{3}{2}, 1 + \frac{1}{6(Q-1)}\}$. Note $\varphi \geq 1 + \frac{1}{6Q}$.

$$\frac{\varphi - 1}{\varphi} \geq \frac{1}{6Q\varphi} \geq \frac{1}{9Q}. \tag{160}$$

Now note $2\varphi = \varpi$ where ϖ is the constant from the statement of Lemma 9. Note

$$\begin{aligned}
\int_{D_\varsigma} \det(Dw_u)dz &\leq \int_{D_\varsigma} |Dw_u|^2 dz \\
&\leq \left(\int_{D_\varsigma} |Dw_u|^{2\varphi} dz\right)^{\frac{1}{\varphi}} |D_\varsigma|^{\frac{1}{\varphi'}} \\
&= \left(\left(\int_{D_\varsigma} |Dw_u|^{\varpi} dz\right)^{\frac{1}{\varpi}}\right)^2 |D_\varsigma|^{\frac{1}{\varphi'}} \\
&\overset{(127)}{\leq} (26Q)^2 |D_\varsigma|^{\frac{\varphi-1}{\varphi}} \\
&\overset{(159),(160)}{\leq} (26Q)^2 |D_\varsigma|^{\frac{1}{9Q}} \\
&\overset{(158)}{\leq} (26Q)^2 \pi\left(\frac{Q}{\varsigma}\right)^{\frac{1}{9Q}}.
\end{aligned} \tag{161}$$

Now let

$$\varsigma = (5408)^{9Q} Q^{(18Q+1)} \xi^{-18Q}, \tag{162}$$

so $\varsigma^{\frac{1}{9Q}} = 5408 Q^{\frac{1}{9Q}+2} \xi^{-2}$ and $\xi^2 = 5408 Q^2 (\frac{Q}{\varsigma})^{\frac{1}{9Q}}$ hence

$$\pi\xi^2 8^{-1} = (26Q)^2 \pi\left(\frac{Q}{\varsigma}\right)^{\frac{1}{9Q}}. \tag{163}$$

So note by (161) we have that

$$\begin{aligned}
\left|B_{\frac{\xi}{2}}\big(w_u(0)\big)\backslash w_u(D_\varsigma)\right| &\overset{(161)}{\geq} \pi\frac{\xi^2}{4} - (26Q)^2\pi\left(\frac{Q}{\varsigma}\right)^{\frac{1}{9Q}} \\
&\overset{(163)}{\geq} \pi\frac{\xi^2}{8}.
\end{aligned} \tag{164}$$

Thus

$$C_p \overset{(157)}{\geq} \int_{B_{\frac{\xi}{2}}(w_u(0))\backslash w_u(D_\varsigma)} \det\big(D\phi_u(z)\big)^{-p} \det\big(Dw_u\big(w_u^{-1}(z)\big)\big)^{-p-1} dz$$

$$\geq \varsigma^{-p-1} \int_{B_{\frac{\xi}{2}}(w_u(0))\backslash w_u(D_\varsigma)} \det\big(D\phi_u(z)\big)^{-p} dz \tag{165}$$

$$\overset{(164)}{\geq} \inf\big\{\det\big(D\phi_u(z)\big)^{-p} : z \in B_{\frac{\xi}{2}}\big(w_u(0)\big)\backslash w_u(D_\varsigma)\big\}\pi\varsigma^{-p-1}\frac{\xi^2}{8}. \tag{166}$$

So there must exist $z_0 \in B_{\frac{\xi}{2}}(w_u(0))\backslash w_u(D_\varsigma)$ such that $\det(D\phi_u(z_0))^{-p} \leq \frac{3C_p\varsigma^{p+1}}{\xi^2}$. Note

$$\varsigma^2 \overset{(162)}{\leq} (5408)^{18Q} Q^{40Q} \xi^{-36Q}, \tag{167}$$

thus (recalling $p \in (0, 1)$)

$$\det\big(D\phi_u(z_0)\big) \geq \frac{\xi^{\frac{2}{p}}}{3^{\frac{1}{p}} C_p^{\frac{1}{p}} \varsigma^{\frac{p+1}{p}}} \geq \frac{\xi^{\frac{2}{p}}}{3^{\frac{1}{p}} C_p^{\frac{1}{p}} \varsigma^{\frac{2}{p}}} \overset{(167)}{\geq} \left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{\frac{1}{p}}. \tag{168}$$

So if $z_1 \in B_{h_0}(z_0)$ then

$$\big|\phi_u'(z_1) - \phi_u'(z_0)\big| \leq \int_{[z_0,z_1]} \big|\phi_u''(z)\big| dH^1 z$$

$$\overset{(102)}{\leq} h_0 \frac{8\pi}{\mu^2}$$

$$\overset{(155)}{\leq} \frac{1}{2}\left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{\frac{1}{2p}}. \tag{169}$$

Hence

$$\big|\phi_u'(z)\big| \overset{(169),(168)}{\geq} \frac{1}{2}\left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{\frac{1}{2p}} \quad \text{for any } z \in B_{h_0}(z_0). \qquad \square$$

Lemma 12. *We will show there exists $\zeta \in \mathbb{C}$ such that*

$$\sup\big\{\big|\phi_u'(z) - \zeta\phi_v'(z)\big| : z \in B_{\frac{\mu}{5}}\big(w_u(0)\big)\big\} \leq C_p^2 \mathcal{C}_1 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10C_pQ)}} \quad \textit{for all small enough } \epsilon > 0. \tag{170}$$

Proof of Lemma 12. Let h_0 be the constant defined by (155) of Lemma 11 and let $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ be the point from Lemma 11 that satisfies (156). Note since $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ for all small enough $\epsilon > 0$ and $h_0 \overset{(155)}{\leq} \frac{\xi}{2} \leq \frac{\mu}{2}$ thus $B_{h_0}(z_0) \subset B_\mu(w_u(0)) \overset{(98),(128)}{\subset} w_v(B_{\frac{1}{2}})$ and so ϕ_v is defined on this set.

Now

$$2.25 \times 10^{16} Q^7 \mu^{-3} C_p \epsilon^{\frac{p}{180Q}} \overset{(141)}{\geq} \int_{B_{h_0}(z_0)} \big|\big|\phi_u'(y)\big|^2 - \big|\phi_v'(y)\big|^2\big| dy$$

$$= \int_{B_{h_0}(z_0)} \big|\big|\phi_u'(y)\big| - \big|\phi_v'(y)\big|\big|\big|\big|\phi_u'(y)\big| + \big|\phi_v'(y)\big|\big| dy$$

$$\overset{(156)}{\geq} \frac{1}{2}\left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{\frac{1}{2p}} \int_{B_{h_0}(z_0)} \left||\phi_u'(y)| - |\phi_v'(y)|\right| dy.$$

Thus

$$\int_{B_{h_0}(z_0)} \left||\phi_u'(y)| - |\phi_v'(y)|\right| dy \leq \left(\frac{3C_p(5408)^{18Q}Q^{40Q}}{\xi^{38Q}}\right)^{\frac{1}{2p}} 4.5\times 10^{16} Q^7 \mu^{-3} C_p \epsilon^{\frac{p}{180Q}}. \tag{171}$$

By Cauchy's theorem we can find an analytic function ψ such that

$$\psi'(z) = \frac{\phi_v'(z)}{\phi_u'(z)} \quad \text{for } z \in B_{h_0}(z_0). \tag{172}$$

So

$$\begin{aligned}
\int_{B_{h_0}(z_0)} \left|1 - |\psi'(z)|\right|^2 dz &= \int_{B_{h_0}(z_0)} |\phi_u'(z)|^{-2} \left||\phi_u'(z)| - |\phi_v'(z)|\right|^2 dz \\
&\overset{(156),(101)}{\leq} \frac{32\pi}{\mu}\left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{-\frac{1}{p}} \int_{B_{h_0}(z_0)} \left||\phi_u'(z)| - |\phi_v'(z)|\right| dz \\
&\overset{(171)}{\leq} 144\times 10^{16}\left(\frac{3C_p(5408)^{18Q}Q^{40Q}}{\xi^{38Q}}\right)^{\frac{3}{2p}} \pi Q^7 \mu^{-4} C_p \epsilon^{\frac{p}{180Q}}.
\end{aligned} \tag{173}$$

Now since $[\psi'(z)]_M \in CO_+(2)$, $\sqrt{2}|1 - |\psi'(z)|| \overset{(29)}{=} \text{dist}(D\psi(z), SO(2))$. So

$$\begin{aligned}
\fint_{B_{h_0}(z_0)} \text{dist}^2\big(D\psi(z), SO(2)\big) dz &\leq \frac{288\times 10^{16}}{h_0^2}\left(\frac{3C_p(5408)^{18Q}Q^{40Q}}{\xi^{38Q}}\right)^{\frac{3}{2p}} Q^7\mu^{-4}C_p\epsilon^{\frac{p}{180Q}} \\
&\overset{(155)}{\leq} (16\pi)^2\left(\frac{3C_p(5408)^{18Q}Q^{40Q}}{\xi^{38Q}}\right)^{\frac{5}{2p}} 288\times 10^{16} Q^7\mu^{-8}C_p\epsilon^{\frac{p}{180Q}} \\
&\leq \left((16\pi)^{\frac{4}{5}}\times 3\times (5408)^{18}\times (288)^{\frac{2}{5}}\times 10^{\frac{32}{5}}\right)^{\frac{5Q}{2p}}\left(\frac{Q}{\xi}\right)^{\frac{107Q}{p}} C_p^{\frac{4}{p}}\epsilon^{\frac{p}{180Q}} \\
&\leq \left(2.7\times 10^{76}\right)^{\frac{3Q}{p}}\left(\frac{Q}{\xi}\right)^{\frac{107Q}{p}} C_p^{\frac{4}{p}}\epsilon^{\frac{p}{180Q}}.
\end{aligned}$$

Let $\zeta(z) = \psi(z_0 + h_0 z)h_0^{-1}$. Thus

$$\int_{B_1} \text{dist}^2\big(D\zeta(z), SO(2)\big) dz \leq \pi\left(2.7\times 10^{76}\right)^{\frac{3Q}{p}}\left(\frac{Q}{\xi}\right)^{\frac{107Q}{p}} C_p^{\frac{4}{p}}\epsilon^{\frac{p}{180Q}}.$$

Thus by applying Proposition 2 we have that there exists $R \in SO(2)$ such that

$$\int_{B_{\frac{1}{4}}} \left|D\zeta(z) - R\right| dz \leq 15\pi\times\left(2.7\times 10^{76}\right)^{\frac{Q}{p}}\left(\frac{Q}{\xi}\right)^{\frac{27Q}{p}} C_p^{\frac{1}{p}}\epsilon^{\frac{p}{720Q}}.$$

By rescaling we obtain that there exists R such that

$$\fint_{B_{\frac{h_0}{4}}(z_0)} \left|D\psi(z) - R\right| dz \leq 240\times\left(2.7\times 10^{76}\right)^{\frac{Q}{p}}\left(\frac{Q}{\xi}\right)^{\frac{27Q}{p}} C_p^{\frac{1}{p}}\epsilon^{\frac{p}{720Q}}.$$

Returning to complex notation for some $\zeta_1 \in \mathbb{C} \cap \{z : |z| = 1\}$ we have

$$\int_{B_{\frac{h_0}{4}}(z_0)} \left|\psi'(z) - \zeta_1\right| dz \leq 15\pi h_0^2 \times \left(2.7 \times 10^{76}\right)^{\frac{Q}{p}} \left(\frac{Q}{\xi}\right)^{\frac{27Q}{p}} C_p^{\frac{1}{p}} \epsilon^{\frac{p}{720Q}}. \tag{174}$$

Now by the Co-area formula we know

$$\int_{\frac{h_0}{8}}^{\frac{h_0}{4}} \int_{\partial B_s(z_0)} \left|\psi'(z) - \zeta_1\right| dH^1 x\, ds \leq \int_{B_{\frac{h_0}{4}}(z_0)} \left|\psi'(z) - \zeta_1\right| dz \tag{175}$$

So we must be able to find

$$q \in \left(\frac{h_0}{8}, \frac{h_0}{4}\right) \tag{176}$$

such that

$$\begin{aligned} \int_{\partial B_q(z_0)} \left|\psi'(z) - \zeta_1\right| dH^1 z \;&\leq\; \frac{8}{h_0} \int_{B_{\frac{h_0}{4}}(z_0)} \left|\psi'(z) - \zeta_1\right| dz \\ &\overset{(174)}{\leq} 120\pi h_0 \times \left(2.7 \times 10^{76}\right)^{\frac{Q}{p}} \left(\frac{Q}{\xi}\right)^{\frac{27Q}{p}} C_p^{\frac{1}{p}} \epsilon^{\frac{p}{720Q}}. \end{aligned} \tag{177}$$

So

$$\begin{aligned} \int_{\partial B_q(z_0)} \left|\phi_v'(z) - \zeta_1 \phi_u'(z)\right| dH^1 z &\overset{(172)}{=} \int_{\partial B_q(z_0)} \left|\left(\psi'(z) - \zeta_1\right)\phi_u'(z)\right| dH^1 z \\ &\overset{(101)}{\leq} \frac{4\pi}{\mu} \int_{\partial B_q(z_0)} \left|\psi'(z) - \zeta_1\right| dH^1 z \\ &\overset{(177)}{\leq} 480\pi^2 h_0 \times \left(2.7 \times 10^{76}\right)^{\frac{Q}{p}} \left(\frac{Q}{\xi}\right)^{\frac{28Q}{p}} C_p^{\frac{1}{p}} \epsilon^{\frac{p}{720Q}} \\ &\leq\; h_0 10^{\frac{81Q}{p}} \left(\frac{Q}{\xi}\right)^{\frac{28Q}{p}} C_p^{\frac{1}{p}} \epsilon^{\frac{p}{720Q}}. \end{aligned} \tag{178}$$

Let

$$\varpi = 10^{\frac{81Q}{p}} \left(\frac{Q}{\xi}\right)^{\frac{28Q}{p}} C_p^{\frac{1}{p}} \quad \text{and} \quad \beta = \frac{p}{720Q}. \tag{179}$$

Note

$$h_0 \overset{(155)}{\geq} \frac{\xi^{\frac{21Q}{p}} Q^{-\frac{20Q}{p}} C_p^{-\frac{1}{2p}}}{16\pi\left(3(5408)^{18Q}\right)^{\frac{1}{2p}}}. \tag{180}$$

Now let

$$w(z) = \phi_u'(z) - \zeta_1 \phi_v'(z). \tag{181}$$

Hence by Cauchy's integral formula we have that

$$\begin{aligned}\left|w^{(k)}(z_0)\right| &= \frac{k!}{2\pi}\int_{\partial B_q(z_0)}\left|\frac{w(\zeta)}{(\zeta-z_0)^{k+1}}\right|d\zeta\\ &\leq \frac{k!}{2\pi q^{k+1}}\int_{\partial B_q(z_0)}\left|w(\zeta)\right|d\zeta\\ &\overset{(179),(178)}{\leq} \frac{k!}{2\pi q^{k+1}}\varpi h_0\epsilon^{\beta}\\ &\overset{(176)}{\leq} \frac{2k!\varpi\epsilon^{\beta}}{q^k}.\end{aligned} \tag{182}$$

By the local Taylor Theorem we have

$$w(z)=\sum_{k=0}^{m}\frac{w^{(k)}(z_0)}{k!}(z-z_0)^k+(z-z_0)^{m+1}w_m(z) \tag{183}$$

where $w_m(z)=\frac{1}{2\pi i}\int_{\partial B_{\frac{\mu}{2}}(z_0)}\frac{w(\zeta)}{(\zeta-z_0)^{m+1}(\zeta-z)}d\zeta$ for any $z\in B_{\frac{\mu}{2}}(z_0)$. Hence for $z\in B_{\frac{\mu}{4}}(z_0)$

$$\begin{aligned}\left|w_m(z)\right| &\leq \frac{1}{2\pi}\int_{\partial B_{\frac{\mu}{2}}(z_0)}\frac{|w(\zeta)|}{|\zeta-z_0|^{m+1}|\zeta-z|}d\zeta\\ &\overset{(181),(101)}{\leq} \frac{4}{\mu}\int_{\partial B_{\frac{\mu}{2}}(z_0)}\frac{1}{(\frac{\mu}{2})^{m+1}\frac{\mu}{4}}\\ &\leq 32\pi\mu^{-2}\left(\frac{\mu}{2}\right)^{-m}.\end{aligned} \tag{184}$$

So for any $z\in B_{\frac{\mu}{4}}(z_0)$ we have

$$\begin{aligned}\left|w(z)\right| &\overset{(183),(184)}{\leq} \sum_{k=0}^{m}\frac{|w^{(k)}(z_0)|}{k!}|z-z_0|^k+|z-z_0|^{m+1}32\pi\mu^{-2}\left(\frac{\mu}{2}\right)^{-m}\\ &\overset{(182)}{\leq} 2\sum_{k=0}^{m}\varpi\epsilon^{\beta}\left(\frac{\mu}{4q}\right)^k+\frac{8\pi}{\mu}\left(\frac{1}{2}\right)^m.\end{aligned} \tag{185}$$

Let

$$\begin{aligned}\alpha=\frac{h_0}{\mu} &\overset{(155)}{=} \frac{\mu}{16\pi}\left(\frac{\xi^{38Q}}{3C_p(5408)^{18Q}Q^{40Q}}\right)^{\frac{1}{2p}}\\ &\geq \frac{\xi^{\frac{20Q}{p}}}{(16\pi\times 3\times(5408)^9)^{\frac{Q}{p}}C_pQ^{\frac{20Q}{p}}}\geq\frac{\xi^{\frac{20Q}{p}}}{C_pQ^{\frac{20Q}{p}}10^{\frac{36Q}{p}}}.\end{aligned} \tag{186}$$

Now note that $q\in(\frac{\alpha\mu}{2},\alpha\mu)$, $(\frac{\mu}{4q})^k\leq(\frac{1}{2\alpha})^k$. So note as

$$2\alpha\overset{(186)}{<}1. \tag{187}$$

Hence

$$\sum_{k=0}^{m}\left(\frac{1}{2\alpha}\right)^k\leq\frac{(\frac{1}{2\alpha})^{m+1}}{\frac{1}{2\alpha}-1}\leq\left(\frac{1}{2\alpha}\right)^{m+1}. \tag{188}$$

Thus

$$\sum_{k=0}^{m} \varpi\epsilon^{\beta}\left(\frac{\mu}{4q}\right)^{k} \leq \varpi\epsilon^{\beta}\sum_{k=0}^{m}\left(\frac{1}{2\alpha}\right)^{k} \overset{(188)}{\leq} \varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m+1}. \tag{189}$$

So

$$\left|w(z)\right| \overset{(185),(189)}{\leq} 2\varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m+1} + \frac{8\pi}{\mu}\left(\frac{1}{2}\right)^{m} \quad \text{for } z \in B_{\frac{\mu}{4}}(z_0). \tag{190}$$

Let m be the smallest integer such that

$$\varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m} \geq \pi\left(\frac{1}{2}\right)^{m}. \tag{191}$$

So

$$\epsilon^{\beta} \geq \frac{\pi}{\varpi}(2\alpha)^{m}\left(\frac{1}{2}\right)^{m} = \left(\left(\frac{\pi}{\varpi}\right)^{\frac{1}{m}}\alpha\right)^{m}. \tag{192}$$

Thus as $\varpi \overset{(179)}{>} \pi$ and $\alpha \overset{(187)}{<} 1$. So $|\log(\epsilon^{\beta})| \overset{(192)}{\leq} |\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)^{m}| = m|\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)|$. Hence

$$\frac{\log(\epsilon^{\beta})}{\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)} \leq m. \tag{193}$$

So

$$\frac{8\pi}{\mu}\left(\frac{1}{2}\right)^{m} \overset{(193)}{\leq} \frac{8\pi}{\mu}\left(\frac{1}{2}\right)^{\frac{\log(\epsilon^{\beta})}{\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)}} = \frac{8\pi}{\mu}\left(e^{\log(\frac{1}{2})}\right)^{\frac{\log(\epsilon^{\beta})}{\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)}} = \frac{8\pi}{\mu}\left(\epsilon^{\beta}\right)^{\frac{\log(\frac{1}{2})}{\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)}}. \tag{194}$$

Using the fact that $\varpi \overset{(179)}{>} \pi$ so $1 > \frac{\pi}{\varpi}$ and thus $1 > (\frac{\pi}{\varpi})^{\frac{1}{m}} > \frac{\pi}{\varpi}$. Thus we have $1 \overset{(187)}{\geq} (\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha \geq \frac{\pi\alpha}{\varpi}$, so

$$\left|\log\left(\left(\frac{\pi}{\varpi}\right)^{\frac{1}{m}}\alpha\right)\right| \leq \left|\log\left(\frac{\pi\alpha}{\varpi}\right)\right|. \tag{195}$$

So we have

$$\frac{8\pi}{\mu}\left(\frac{1}{2}\right)^{m} \overset{(194)}{\leq} \frac{8\pi}{\mu}\epsilon^{\frac{\beta\log(\frac{1}{2})}{\log((\frac{\pi}{\varpi})^{\frac{1}{m}}\alpha)}} \overset{(195)}{\leq} \frac{8\pi}{\mu}\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}}. \tag{196}$$

Since m is the smallest integer such that (191) holds true we have

$$2\varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m-1} \overset{(191)}{<} 2\pi\left(\frac{1}{2}\right)^{m-1} = 4\pi\left(\frac{1}{2}\right)^{m} \overset{(196)}{\leq} 4\pi\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}}. \tag{197}$$

Thus as $\alpha^2 \overset{(186)}{\geq} \frac{\xi^{\frac{40Q}{p}}}{C_p^2 Q^{\frac{40Q}{p}} 10^{\frac{72Q}{p}}}$, so

$$\begin{aligned}
\varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m+1} &= \frac{1}{4\alpha^2}\varpi\epsilon^{\beta}\left(\frac{1}{2\alpha}\right)^{m-1} \\
&\overset{(197)}{\leq} \frac{1}{8\alpha^2} \times 4\pi\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}} \\
&\leq \frac{4\pi \times 10^{\frac{72Q}{p}} C_p^2 Q^{\frac{40Q}{p}}}{\xi^{\frac{40Q}{p}}}\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}} \\
&\leq C_p^2\mathcal{C}_0\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}},
\end{aligned} \tag{198}$$

where $\mathcal{C}_0 = \mathcal{C}_0(p, Q)$.

So putting (198) and (196) together with (190) we have

$$\begin{aligned}|w(z)| &\leq 2C_p^2\mathcal{C}_0\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}} + \frac{8\pi}{\mu}\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}}. \\ &\leq C_p^2\mathcal{C}_1\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}} \quad \text{for all } z \in B_{\frac{\mu}{4}}(z_0).\end{aligned} \tag{199}$$

Hence

$$\left\|\phi_u' - \zeta_1\phi_v'\right\|_{L^\infty(B_{\frac{\mu}{4}}(z_0))} \overset{(199),(181)}{\leq} C_p^2\mathcal{C}_1\epsilon^{\frac{-\beta}{2\log(\frac{\pi\alpha}{\varpi})}} = C_p^2\mathcal{C}_1\epsilon^{\frac{\beta}{2\log(\frac{\varpi}{\pi\alpha})}}. \tag{200}$$

Now

$$\frac{\pi\alpha}{\varpi} \overset{(186)}{\geq} \frac{\xi^{\frac{20Q}{p}}}{\varpi C_p Q^{\frac{20Q}{p}} 10^{\frac{36Q}{p}}}. \tag{201}$$

And note

$$\begin{aligned}\gamma &\overset{(97)}{\geq} \left(\frac{\mu}{2\times 10^{10}(Q+1)^6}\right)^{6(Q+1)} \\ &\overset{(97)}{\geq} \frac{1}{(2\times 10^{10}(Q+1)^6)^{6(Q+1)}}\frac{1}{(2\times 10^{10}(Q+1)^6)^{36(Q+1)^2}}.\end{aligned} \tag{202}$$

And so

$$\begin{aligned}\xi &\overset{(97)}{\geq} \frac{\gamma^{6(Q+1)}}{(2\times 10^{10}(Q+1)^6)^{6(Q+1)}} \\ &\overset{(202)}{\geq} \frac{1}{(2\times 10^{10}(Q+1)^6)^{6(Q+1)+36(Q+1)^2+216(Q+1)^3}} \\ &\geq \frac{1}{(2^7\times 10^{10}Q)^{1884Q^3}}.\end{aligned} \tag{203}$$

Now

$$\begin{aligned}\frac{\xi^{\frac{20Q}{p}}}{\varpi C_p Q^{\frac{20}{p}} 10^{\frac{36Q}{p}}} &\overset{(179)}{\geq} \frac{\xi^{\frac{48Q}{p}}}{10^{\frac{117Q}{p}} Q^{\frac{48Q}{p}} C_p^{\frac{2}{p}}} \\ &\overset{(203)}{\geq} \frac{1}{10^{\frac{117Q}{p}} Q^{\frac{48Q}{p}} C_p^{\frac{2}{p}} (10^{13}Q)^{90\,432\frac{Q^4}{p}}} \\ &\geq \frac{1}{(10C_pQ)^{1\,175\,733\frac{Q^4}{p}}}\end{aligned} \tag{204}$$

So

$$\frac{\beta}{2\log(\frac{\varpi}{\pi\alpha})} \overset{(201),(179)}{\geq} \frac{p}{1440Q\log(\frac{\varpi C_p Q^{\frac{20}{p}} 10^{\frac{36Q}{p}}}{\xi^{\frac{20Q}{p}}})} \overset{(204)}{\geq} \frac{p^2}{2\times 10^9 Q^5 \log(10C_pQ)}. \tag{205}$$

Thus

$$\left\|\phi_u' - \zeta_1\phi_v'\right\|_{L^\infty(B_{\frac{\mu}{4}}(z_0))} \overset{(200),(205)}{\leq} C_p^2\mathcal{C}_1\epsilon^{\frac{p^2}{2\times 10^9 Q^5\log(10C_pQ)}}.$$

Since $z_0 \in B_{\frac{\xi}{2}}(w_u(0))$ by (97) this implies (170). □

3.3. Proof of Proposition 1 completed

Now

$$Du(z) = D\phi_u\big(w_u(z)\big)Dw_u(z) \quad \text{and} \quad Dv(z) = D\phi_v\big(w_v(z)\big)Dw_v(z).$$

So

$$\begin{aligned}\int_{B_\gamma} \big|Du(z) - RDv(z)\big|dz &= \int_{B_\gamma} \big|D\phi_u\big(w_u(z)\big)Dw_u(z) - RD\phi_v\big(w_v(z)\big)Dw_v(z)\big|dz \\ &\leq \int_{B_\gamma} \big|\big(D\phi_u\big(w_u(z)\big) - RD\phi_v\big(w_u(z)\big)\big)Dw_u(z)\big|dz \\ &\quad + \int_{B_\gamma} \big|D\phi_v\big(w_u(z)\big)\big(Dw_u(z) - Dw_v(z)\big)\big|dz \\ &\quad + \int_{B_\gamma} \big|\big(D\phi_v\big(w_u(z)\big) - D\phi_v\big(w_v(z)\big)\big)Dw_v(z)\big|dz.\end{aligned}$$

So to deal with the last term

$$\begin{aligned}&\int_{B_\gamma} \big|\big(D\phi_v\big(w_u(z)\big) - D\phi_v\big(w_v(z)\big)\big)Dw_v(z)\big|dz \\ &\overset{(102),(99),(150)}{\leq} \frac{32\pi}{\mu^2}\int_{B_\gamma} \big|w_u(z) - w_v(z)\big|\big|Dw_v(z)\big|dz \\ &\overset{(128)}{\leq} \frac{32\pi}{\mu^2} \times 8.6 \times 10^8 Q^5 C_p \epsilon^{\frac{p}{180Q}} \sqrt{\pi}\bigg(\int_{B_\gamma} \big|Dw_u(z)\big|^2 dz\bigg)^{\frac{1}{2}} \\ &\overset{(126)}{\leq} \mathcal{C}_2 C_p \epsilon^{\frac{p}{180Q}}.\end{aligned}$$

And

$$\begin{aligned}\int_{B_\gamma} \big|D\phi_v\big(w_u(z)\big)\big(Dw_u(z) - Dw_v(z)\big)\big|dz &\overset{(101),(107)}{\leq} \frac{8\pi}{\mu} \times 127\,280 Q^4 C_p \epsilon^{\frac{p}{72Q}} \\ &\leq \mathcal{C}_3 C_p Q^4 \epsilon^{\frac{p}{72Q}}.\end{aligned} \tag{206}$$

So

$$\begin{aligned}\int_{B_\gamma} |Du - RDv|dz &= \sqrt{\pi}\bigg(\int_{B_\gamma} \big|D\phi_u\big(w_u(z)\big) - RD\phi_v\big(w_u(z)\big)\big|^2 \big|Dw_u(z)\big|^2 dz\bigg)^{\frac{1}{2}} + \mathcal{C}_2 C_p \epsilon^{\frac{p}{180Q}} + \mathcal{C}_3 C_p \epsilon^{\frac{p}{72Q}} \\ &\leq c\sqrt{Q}\bigg(\int_{B_\gamma} \big|D\phi_u\big(w_u(z)\big) - RD\phi_v\big(w_u(z)\big)\big|^2 \det\big(Dw_u(z)\big)\, dz\bigg)^{\frac{1}{2}} + (\mathcal{C}_2 + \mathcal{C}_3) C_p \epsilon^{\frac{p}{180Q}} \\ &\overset{(99)}{\leq} \sqrt{Q\pi}\bigg(\int_{B_{\frac{\mu}{8}}(w_u(0))} \big|D\phi_u(z) - RD\phi_v(z)\big|^2 dz\bigg)^{\frac{1}{2}} + (\mathcal{C}_2 + \mathcal{C}_3) C_p \epsilon^{\frac{p}{180Q}} \\ &\overset{(170)}{\leq} C_p^2 \mathcal{C}_4 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}. \qquad \square\end{aligned} \tag{207}$$

4. Proof of Theorem 2

Let $\tilde{u}(z) = \frac{u(z)}{4}$ and $\tilde{v}(z) = \frac{v(z)}{4}$. So

$$\int_{B_1} |D\tilde{u}| dz \leq \frac{\pi}{4} \tag{208}$$

and

$$\int_{B_1} \det(D\tilde{u})^{-p} dz = 16^p \int_{B_1} \det(Du)^{-p} dz = 16^p C_p. \tag{209}$$

Note also from (1)

$$\int_{B_1} \left| S\big(D\tilde{u}(z)\big) - S\big(D\tilde{v}(z)\big)\right|^2 dz \leq \epsilon. \tag{210}$$

Step 1. For any set $S \subset B_1$ with $|S| > 0$ we will show

$$\int_S \det\big(D\tilde{u}(z)\big) dz \geq 16^{-1} C_p^{-\frac{1}{p}} |S|^{1+\frac{1}{p}}. \tag{211}$$

Proof of Step 1. Note

$$\begin{aligned} |S| &= \int_S \det\big(D\tilde{u}(z)\big)^{\frac{p}{2}} \det\big(D\tilde{u}(z)\big)^{-\frac{p}{2}} dz \\ &\overset{(209)}{\leq} \Big(\int_S \det\big(D\tilde{u}(z)\big)^p dz\Big)^{\frac{1}{2}} 16^{\frac{p}{2}} \sqrt{C_p}. \end{aligned} \tag{212}$$

Let $q = \frac{1}{p}$, $q' = \frac{q}{q-1} = \frac{p^{-1}}{p^{-1}-1} = \frac{1}{1-p}$. So using Holder's inequality for the second inequality

$$\begin{aligned} \left(\frac{|S|}{16^{\frac{p}{2}}\sqrt{C_p}}\right)^2 &\overset{(212)}{\leq} \int_S \det\big(Du(z)\big)^p dz \\ &= \Big(\int_S \det\big(Du(z)\big)^{pq} dz\Big)^{\frac{1}{q}} |S|^{\frac{1}{q'}} \\ &= \Big(\int_S \det\big(Du(z)\big) dz\Big)^p |S|^{1-p}. \end{aligned} \tag{213}$$

So $16^{-p} C_p^{-1} |S|^{1+p} \overset{(213)}{\leq} (\int_S \det(Du(z))dz)^p$. Thus so we have established (211).

Step 2. Let $\{B_{\frac{\gamma}{4}}(x_k) : k = 1, 2, \ldots N\}$ be collection such that

$$\sum_{k=1}^N \mathbb{1}_{B_{\frac{\gamma}{4}}(x_k)} \leq 5 \tag{214}$$

and

$$B_{\frac{1}{2}} \subset \bigcup_{k=1}^N B_{\frac{\gamma}{4}}(x_k). \tag{215}$$

We will order these balls so that $B_{\frac{\gamma}{4}}(x_k) \cap B_{\frac{\gamma}{4}}(x_{k+1}) \neq \emptyset$ for $k = 1, 2, \ldots N-1$.

Let $u_k(z) = 2\tilde{u}(x_k + \frac{z}{2})$ and $v_k(z) = 2\tilde{v}(x_k + \frac{z}{2})$. Note

$$\int_{B_1} \left|Du_k(z)\right| dz = \int_{B_1} \left|D\tilde{u}\left(x_k + \frac{z}{2}\right)\right| dz \leq 4 \int_{B_{\frac{1}{2}}(x_k)} \left|D\tilde{u}(z)\right| dz \overset{(208)}{\leq} \pi.$$

And

$$\int_{B_1} \det\left(D\tilde{u}\left(x_k + \frac{z}{2}\right)\right)^{-p} dz \leq 4 \int_{B_1} \det\left(D\tilde{u}(z)\right)^{-p} dz \overset{(209)}{\leq} 64 C_p. \tag{216}$$

Note also

$$\begin{aligned} \int_{B_1} \left|S(Du_k) - S(Dv_k)\right|^2 dz &\leq \int_{B_1} \left|S\left(D\tilde{u}\left(x_k + \frac{z}{2}\right)\right) - S\left(D\tilde{u}\left(x_k + \frac{z}{2}\right)\right)\right|^2 dz \\ &\leq 4 \int_{B_1} \left|S\left(D\tilde{u}(z)\right) - S\left(D\tilde{v}(z)\right)\right|^2 dz \\ &\overset{(210)}{\leq} 4\epsilon. \end{aligned} \tag{217}$$

So we can apply Proposition 1 and for some $R_k \in SO(2)$ we have

$$\int_{B_\gamma} \left|Dv_k(z) - R_k Du_k(z)\right| dz \leq \mathcal{C}_4 C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}.$$

We will show that

$$|R_1 - R_k| \leq c\gamma^{-4-\frac{1}{p}} C_p^{\frac{2}{p}} \epsilon^{\frac{p^2}{10^{10} Q^5 \log(10 C_p Q)}} \quad \text{for } k = 1, 2, \ldots N-1. \tag{218}$$

Proof of Step 2. The existence of a collection $\{B_{\frac{\gamma}{4}}(x_1), B_{\frac{\gamma}{4}}(x_2), \ldots B_{\frac{\gamma}{4}}(x_N)\}$ satisfying (214), (215) follows by the $5r$ covering theorem. Rescaling v_k and u_k we have

$$\int_{B_{\frac{\gamma}{2}}(x_k)} \left|D\tilde{v}(z) - R_k D\tilde{u}(z)\right| dz \leq \mathcal{C}_4 C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}. \tag{219}$$

So

$$\int_{B_{\frac{\gamma}{2}}(x_k) \cap B_{\frac{\gamma}{2}}(x_{k+1})} \left|(R_k - R_{k+1}) D\tilde{u}(z)\right| dz \leq \mathcal{C}_4 C_p^2 \epsilon^{\frac{p^2}{2\times 10^9 Q^5 \log(10 C_p Q)}}. \tag{220}$$

Let

$$\mathcal{B}_1 = \left\{z : \left|D\tilde{u}(z)\right| > 8\gamma^{-2}\right\}. \tag{221}$$

So $|\mathcal{B}_1| \overset{(208)}{\leq} \frac{\pi\gamma^2}{32}$. Since $B_{\frac{\gamma}{4}}(x_k) \cap B_{\frac{\gamma}{4}}(x_{k+1}) \neq \emptyset$ for $k = 1, 2, \ldots N-1$. So $|B_{\frac{\gamma}{2}}(x_k) \cap B_{\frac{\gamma}{2}}(x_{k+1})| \geq \frac{\pi\gamma^2}{16}$ for $k = 1, 2, \ldots N-1$. Thus

$$\left|B_{\frac{\gamma}{2}}(x_k) \cap B_{\frac{\gamma}{2}}(x_{k+1}) \backslash \mathcal{B}_1\right| \geq \frac{\pi\gamma^2}{32} \quad \text{for } k = 1, 2, \ldots N-1. \tag{222}$$

Now

$$
\begin{aligned}
&\int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \det(R_k - R_{k+1})\det\big(D\tilde{u}(z)\big)dz \\
&\quad= \int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \det\big((R_k - R_{k+1})D\tilde{u}(z)\big)dz \\
&\quad\le \int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \big|(R_k - R_{k+1})D\tilde{u}(z)\big|^2 dz \\
&\quad\le 2\int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \big|D\tilde{u}(z)\big|\big|(R_k - R_{k+1})D\tilde{u}(z)\big|dz \\
&\quad\overset{(221)}{\le} 16\gamma^{-2}\int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \big|(R_k - R_{k+1})D\tilde{u}(z)\big|dz \\
&\quad\overset{(220)}{\le} c\gamma^{-2}C_p^2\epsilon^{\frac{p^2}{2\times10^9 Q^5\log(10C_pQ)}}.
\end{aligned} \tag{223}
$$

Hence

$$
\begin{aligned}
\det(R_k - R_{k+1})C_p^{-\frac{1}{p}}\gamma^{2+\frac{2}{p}} &\overset{(222)}{\le} c\det(R_k - R_{k+1})C_p^{-\frac{1}{p}}\big|B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1\big|^{1+\frac{1}{p}} \\
&\overset{(211)}{\le} \int_{B_{\frac{\gamma}{2}}(x_k)\cap B_{\frac{\gamma}{2}}(x_{k+1})\backslash\mathcal{B}_1} \det(R_k - R_{k+1})\det\big(D\tilde{u}(z)\big)dz \\
&\overset{(223)}{\le} c\gamma^{-2}C_p^2\epsilon^{\frac{p^2}{2\times10^9 Q^5\log(10C_pQ)}}.
\end{aligned}
$$

Thus

$$
\det(R_k - R_{k+1}) \le c\gamma^{-4-\frac{2}{p}}C_p^{\frac{3}{p}}\epsilon^{\frac{p^2}{2\times10^9 Q^5\log(10C_pQ)}}. \tag{224}
$$

Note that if $R_\alpha = \begin{pmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}$, $R_\beta = \begin{pmatrix}\cos(\beta) & -\sin(\beta)\\ \sin(\beta) & \cos(\beta)\end{pmatrix}$ then $\det(R_\alpha - R_\beta) = 2(1-\cos(\alpha-\beta))$ thus from (224) we have

$$
|R_k - R_{k+1}| \le c\gamma^{-2-\frac{1}{p}}C_p^{\frac{2}{p}}\epsilon^{\frac{p^2}{10^{10}Q^5\log(10C_pQ)}} \quad \text{for } k = 1,2,\dots N-1. \tag{225}
$$

So we have established (218).

Proof of Theorem 2 completed.

$$
\begin{aligned}
\int_{B_{\frac{\gamma}{2}}(x_k)} \big|D\tilde{v}(z) - R_1 D\tilde{u}(z)\big|dz &\le \int_{B_{\frac{\gamma}{2}}(x_k)} \big|D\tilde{v}(z) - R_k D\tilde{u}(z)\big|dz + \big|(R_k - R_1)D\tilde{u}(z)\big|dz \\
&\overset{(219),(218)}{\le} \mathcal{C}_4 C_p^2\epsilon^{\frac{p^2}{2\times10^9 Q^5\log(10C_pQ)}} + \mathcal{C}_5 C_p^{\frac{2}{p}}\epsilon^{\frac{p^2}{10^{10}Q^5\log(10C_pQ)}} \int_{B_{\frac{\gamma}{2}}(x_k)} |D\tilde{u}|dz.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{B_{\frac{1}{2}}} |D\tilde{v} - R_1 D\tilde{u}|dz &\overset{(214),(215),}{\le} c\sum_{k=1}^{N} C_p^{\frac{2}{p}}\epsilon^{\frac{p^2}{10^{10}Q^5\log(10C_pQ)}} \int_{B_{\frac{\gamma}{2}}(x_k)} |D\tilde{u}|dz + cC_p^2\epsilon^{\frac{p^2}{2\times10^9 Q^5\log(10C_pQ)}} \\
&\overset{(214),(208)}{\le} cC_p^{\frac{2}{p}}\epsilon^{\frac{p^2}{10^{10}Q^5\log(10C_pQ)}}.
\end{aligned} \tag{226}
$$

Rescaling gives (2) and this completes the proof of Theorem 2. □

5. Examples

We can show that any estimate has to lose at least a root power.

Example 1. Let $f(z)=\frac{z^{k+1}}{k+1}$, $g(z)=\frac{z^{k+2}}{k+2}$. So rewriting these functions as vector valued functions of two variables we have

$$D\tilde{f}(x,y)=\left[z^k\right]_M \quad \text{and} \quad D\tilde{g}(x,y)=\left[z^{k+1}\right]_M. \tag{227}$$

Now

$$\left[z^k\right]_M=|z|^k\begin{pmatrix}\cos(k\text{Arg}(z)) & -\sin(k\text{Arg}(z))\\ \sin(k\text{Arg}(z)) & \cos(k\text{Arg}(z))\end{pmatrix} \tag{228}$$

and

$$\left[z^{k+1}\right]_M=|z|^{k+1}\begin{pmatrix}\cos((k+1)\text{Arg}(z)) & -\sin((k+1)\text{Arg}(z))\\ \sin((k+1)\text{Arg}(z)) & \cos((k+1)\text{Arg}(z))\end{pmatrix} \tag{229}$$

Thus

$$\text{Sym}\big(D\tilde{f}(x,y)\big)=\big(x^2+y^2\big)^{\frac{k}{2}}Id \quad \text{and} \quad \text{Sym}\big(D\tilde{g}(x,y)\big)=\big(x^2+y^2\big)^{\frac{k+1}{2}}Id.$$

So note

$$\begin{aligned}\int_{B_1}\big|\text{Sym}(D\tilde{f})-\text{Sym}(D\tilde{g})\big|dz&=\int_0^1\int_{\partial B_r}\big|r^k-r^{k+1}\big|dH^1zdr\\ &=\frac{2\pi}{(k+1)(k+2)}.\end{aligned}$$

A slightly longer calculation shows that

$$\int_{B_1}|D\tilde{f}-R_\theta D\tilde{g}|dz\geq\frac{c}{k} \quad \text{for any } \theta\in(0,2\pi].$$

Conjecture 1. *There exists a sequence of positive numbers $\epsilon_k\to 0$ and a sequence of pairs of Q-Quasiregular maps $u_k:B_1\to\mathbb{R}^2$, $v_k:B_1\to\mathbb{R}^2$ with $\int_{B_1}|Du_k|^2dz\leq 1$ such that*

$$\int_{B_1}\big|S(Du_k)-S(Dv_k)\big|^2dz=\epsilon_k$$

and

$$\int_{B_{\frac{1}{2}}}|Du_k-R_\theta Dv_k|dz\geq 1 \quad \textit{for all } R_\theta\in SO(2).$$

Sketch of proof of Conjecture 1. Let k be a large integer. Let $w_m=e^{\frac{2\pi im}{k}}$. A natural approach is to define function

$$\varrho(z):=\prod_{m=1}^{k}\left(\rho\big(|z-w_m|\big)\frac{z-w_m}{|z-w_m|}\right)^k. \tag{230}$$

If $\rho(x)=x$ this is just a holomorphic function with order k zero at $\{w_1,w_2,\dots w_k\}$. The idea is to create a function whose gradient close to an annulus of radius 1 is very small. And whose gradient in the inside of the annulus and the outside of the annulus is large.

Specifically we want estimates of the form

$$\int_{B_{1-h}} |D\varrho|dz = O(1) \quad \text{and} \quad \int_{B_2(0)\backslash B_{1+h}} |D\varrho|dz = O(1). \tag{231}$$

And for $\epsilon \ll h$

$$\int_{B_{1+h}\backslash B_{1-h}} |D\varrho|dz \leq \epsilon. \tag{232}$$

Let R be a rotation and l_R the affine map with $l_R(0) = 0$, $Dl_R = R$. Now defining

$$w(z) := \begin{cases} \varrho(z) - \fint_{\partial B_{1-h}} \varrho dH^1 x & \text{for } z \in B_{1-h} \\ l_R \circ \varrho(z) - \fint_{\partial B_{1+h}} l_R \circ \varrho dH^1 x & \text{for } z \in B_2 \backslash B_{1+h} \end{cases} \tag{233}$$

We can interpolate across $B_{1+h}\backslash B_{1-h}$ to create a function $\tilde{w}$ with the property that

$$D\tilde{w}(z) := \begin{cases} D\varrho(z) & \text{for } z \in B_{1-h} \\ RD\varrho(z) & \text{for } z \in B_2 \end{cases} \tag{234}$$

and $\|D\tilde{w}\|_{L^\infty(B_{1+h}\backslash B_{1-h})} \leq c\epsilon$. If h could be showed to be Quasiregular then we can use the method of [4] "project" $\tilde{w}$ onto the space of Quasiregular mappings to obtain a Quasiregular mappings with the properties required. So the main obstacle is to obtain a Quasiregular mapping that has properties (231), (232).

Let

$$G(z) := \prod_{m=1}^{k} \left(\rho\left(|z - w_m|\right)\right)^k = e^{\frac{k^2}{2\pi}\left(\sum_{m=1}^{k} \frac{2\pi}{k} \log(\rho(|z-w_m|))\right)}. \tag{235}$$

Take $z = 1$. Then

$$\begin{aligned} |z - w_m| &= \sqrt{\left(\left(1 - \cos\left(\frac{2\pi m}{k}\right)\right)^2 + \left(\sin\left(\frac{2\pi m}{k}\right)\right)^2\right)} \\ &= \sqrt{2\left(1 - \cos\left(\frac{2\pi m}{k}\right)\right)}. \end{aligned} \tag{236}$$

So

$$\begin{aligned} \sum_{m=1}^{k} \frac{2\pi}{k} \log\left(\rho\left(|1 - w_m|\right)\right) &= \sum_{m=1}^{k} \frac{2\pi}{k} \log\left(\rho\left(\sqrt{2\left(1 - \cos\left(\frac{2\pi m}{k}\right)\right)}\right)\right) \\ &\to \int_0^{2\pi} \log\left(\rho\left(\sqrt{2\left(1 - \cos(x)\right)}\right)\right) dx \\ &= \int_0^2 \log\left(\rho(r)\right) \frac{4}{\sqrt{4 - r^2}} dr \\ &=: A_\rho. \end{aligned} \tag{237}$$

Since 1 is a typical point on ∂B_1 by symmetry of $z_1, z_2, \ldots z_m$ so we have

$$\inf_{z \in \partial B_1(0)} G(z) \leq c e^{\frac{k^2}{2\pi} A_\rho}. \tag{238}$$

Let $\varpi(x) = \sum_{m=1}^{k} k \log(\rho(|z - w_m|))$. So

$$\int_{B_1} \left|G(z)\right| dz = \int_{B_1} e^{\log(|G(z)|)} dz$$
$$\stackrel{(235)}{=} \int_{B_1} e \circ \varpi(z) dz \tag{239}$$

Since e^x is convex by Jensen's inequality we know

$$e^{(\int_{B_1} \varpi(z)dz)} \leq \int_{B_1} e \circ \varpi(z) dz. \tag{240}$$

Let

$$D_\rho := \int_0^2 2r \cos^{-1}\left(\frac{r}{2}\right) \log\bigl(\rho(r)\bigr) dr. \tag{241}$$

And note

$$\int_{B_1} \varpi(z) dz = \sum_{m=1}^{k} k \int_{B_1} \log\bigl(\rho\bigl(|z - w_m|\bigr)\bigr) dz$$
$$= k^2 \int_{B_1} \log\bigl(\rho\bigl(\left|z - (-1, 0)\right|\bigr)\bigr) dz$$
$$= k^2 \int_0^2 2r \cos^{-1}\left(\frac{r}{2}\right) \log\bigl(\rho(r)\bigr) dr$$
$$\stackrel{(241)}{=} k^2 D_\rho. \tag{242}$$

$$\int_{B_1} \left|G(z)\right| dz \stackrel{(241),(240),(239)}{\geq} e^{k^2 D_\rho}$$

Thus a counter example can be constructed by finding an increasing function ρ that satisfies the following two inequalities

$$D_\rho = \int_0^2 2r \cos^{-1}\left(\frac{r}{2}\right) \log\bigl(\rho(r)\bigr) dr > 0 \quad \text{and} \quad A_\rho = \int_0^2 \log\bigl(\rho(r)\bigr) \frac{4}{\sqrt{4 - r^2}} dr < 0 \tag{243}$$

and for which function G defined (235) forms a quasiregular mapping. These things will be addressed in forthcoming preprint [21].

Conflict of interest statement

The author declares there is no conflict of interest.

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Appendix A

We will prove an estimate from [13] where we track the constants explicitly. All the arguments are from [13].

Proposition 2. *Suppose* $u \in W^{1,2}(B_1 : \mathbb{R}^2)$ *with* $\int_{B_1} \mathrm{dist}^2(Du, SO(2))dz \leq 1$ *then there exists* $R \in SO(2)$ *such that*

$$\int_{B_{\frac{1}{4}}} |Du - R|dz \leq 15\Big(\int_{B_1} \mathrm{dist}^2\big(Du, SO(2)\big)dz\Big)^{\frac{1}{4}}. \tag{244}$$

Step 1. We will show

$$\big|\mathrm{cof}(M) - M\big| \leq 2\mathrm{dist}\big(M, SO(2)\big) \quad \text{for any } M \in M^{2\times 2}. \tag{245}$$

Proof of Step 1. Let $R_M \in SO(2)$ be such that $|M - R_M| = \mathrm{dist}(M, SO(2))$. Note $|\mathrm{cof}(M) - R_M| = \mathrm{dist}(M, SO(2))$. So $|\mathrm{cof}(M) - M| \leq |\mathrm{cof}(M) - R_M| + |R_M - M| = 2\mathrm{dist}(M, SO(2))$. Which establishes (245).

Step 2. For any $\psi \in W^{1,2}(B_1 : \mathbb{R}^2)$ we will show

$$\int_{B_1} \big||D\psi|^2 - 2\big|dx \leq \Big(\int_{B_1} \mathrm{dist}^2\big(D\psi, SO(2)\big)dx\Big)^{\frac{1}{2}}\big(\|D\psi\|_{L^2(B_1)} + \sqrt{2\pi}\big). \tag{246}$$

And in particular function u that satisfies the hypotheses of Proposition 2 has the property that

$$\|Du\|_{L^2(B_1)} \leq 2\pi. \tag{247}$$

Proof of Step 2. For any $x \in B_1$ let $R_x \in SO(2)$ be such that $|D\psi(x) - R_x| = \mathrm{dist}(D\psi(x), SO(2))$. So

$$\begin{aligned}\int_{B_1} \big|\big|D\psi(x)\big|^2 - 2\big|dx &= \int_{B_1} \big|\big(\big|D\psi(x)\big| - |R_x|\big)\big(\big|D\psi(x)\big| + \sqrt{2}\big)\big|dx \\ &\leq \Big(\int_{B_1} \big|D\psi(x) - R_x\big|^2 dx\Big)^{\frac{1}{2}}\Big(\int_{B_1} \big(\big|D\psi(x)\big| + \sqrt{2}\big)^2 dx\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_{B_1} \mathrm{dist}^2\big(D\psi(x), SO(2)\big)dx\Big)^{\frac{1}{2}}\big(\|D\psi\|_{L^2(B_1)} + \sqrt{2\pi}\big)\end{aligned} \tag{248}$$

which establishes (246).

We will now establish (247) for function u. Suppose $\|Du\|_{L^2(B_1)} \geq 2\pi$, then

$$\begin{aligned}\int_{B_1} \big||Du|^2 - 2\big|dx &\overset{(246)}{\leq} \Big(\int_{B_1} \mathrm{dist}^2\big(Du(x), SO(2)\big)dx\Big)^{\frac{1}{2}}\big(\|Du\|_{L^2(B_1)} + \sqrt{2\pi}\big) \\ &\leq 2\|Du\|_{L^2(B_1)}.\end{aligned}$$

So $\int_{B_1} |Du|^2 dx \leq 2\|Du\|_{L^2(B_1)} + 2\pi \leq 3\|Du\|_{L^2(B_1)}$. Thus $\|Du\|_{L^2(B_1)} \leq 3$ which contradicts the assumption that $\|Du\|_{L^2(B_1)} \geq 2\pi$. Thus (247) is established.

Proof of Proposition completed. Let $\zeta : B_1 \to \mathbb{R}^2$ be the solution of

$$\Delta\zeta = -\mathrm{div}\big(\mathrm{cof}(Du) - Du\big), \quad \zeta = 0 \text{ on } \partial B_1.$$

So testing the equation with z itself we have

$$\int_{B_1} |D\zeta|^2 dx = \int_{B_1} \big(\mathrm{cof}(Du) - Du\big) : D\zeta\, dx$$
$$\leq \Big(\int_{B_1} \big|\mathrm{cof}(Du) - Du\big|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{B_1} |D\zeta|^2 dx\Big)^{\frac{1}{2}}$$
$$\overset{(245)}{\leq} 2\Big(\int_{B_1} \mathrm{dist}^2\big(Du, SO(2)\big) dx\Big)^{\frac{1}{2}} \|D\zeta\|_{L^2(B_1)}.$$

So

$$\int_{B_1} |D\zeta|^2 dx \leq 4 \int_{B_1} \mathrm{dist}^2\big(Du, SO(2)\big) dx. \tag{249}$$

Let

$$w = u - \zeta. \tag{250}$$

Now using the identity

$$\frac{1}{2}\Delta\big(|Df|^2\big) = Df \cdot \Delta Df + \big|D^2 f\big|^2 \quad \text{for any scalar valued function } f \in C^2.$$

As w is a vector valued function both of whose co-ordinates are harmonic we have

$$\frac{1}{2}\Delta\big(|Dw|^2 - 2\big) = \big|D^2 w\big|^2. \tag{251}$$

Let $\eta \in C_0(B_1)$ be such that $\eta = 1$ on $B_{\frac{1}{2}}$ and $\|D^2\eta\|_{L^\infty(B_1)} \leq 8$. So

$$\int_{B_1} \big|D^2 w\big|^2 \eta\, dz \overset{(251)}{=} \int_{B_1} \frac{1}{2}\Delta\big(|Dw|^2 - 2\big)\eta\, dx$$
$$= \int_{B_1} \frac{1}{2}\big(|Dw|^2 - 2\big)\Delta\eta\, dx$$
$$\leq \frac{1}{2}\sup_{B_1}|\Delta\eta| \int_{B_1} \big||Dw|^2 - 2\big| dx$$
$$\overset{(250)}{\leq} 4\int_{B_1} \big||Du|^2 - 2Du : D\zeta + |D\zeta|^2 - 2\big| dx$$
$$\leq 4\Big(\int_{B_1} \big||Du|^2 - 2\big| dx + \int_{B_1} |D\zeta|^2 dx + 2\Big(\int_{B_1} |D\zeta|^2 dx\Big)^{\frac{1}{2}} \Big(\int_{B_1} |Du|^2 dx\Big)^{\frac{1}{2}}\Big)$$
$$\overset{(246),(249)}{\leq} 4\Big(\int_{B_1} \mathrm{dist}^2\big(Du(x), SO(2)\big) dx\Big)^{\frac{1}{2}} \big(\|Du\|_{L^2(B_1)} + \sqrt{2\pi}\big)$$
$$+ 16\int_{B_1} \mathrm{dist}^2\big(Du(x), SO(2)\big) dx + 16\Big(\int_{B_1} \mathrm{dist}^2\big(Du(x), SO(2)\big) dx\Big)^{\frac{1}{2}} \|Du\|_{L^2(B_1)}$$
$$\overset{(247)}{\leq} (40\pi + 4\sqrt{2\pi} + 16)\Big(\int_{B_1} \mathrm{dist}^2\big(Du(x), SO(2)\big) dx\Big)^{\frac{1}{2}}. \tag{252}$$

So

$$\left(\int_{B_{\frac{1}{2}}} \left|D^2 w\right|^2 dx\right)^{\frac{1}{2}} \leq 13\left(\int_{B_1} \operatorname{dist}^2\big(Du(x), SO(2)\big)dx\right)^{\frac{1}{4}}. \tag{253}$$

Note $\int_{B_{\frac{1}{2}}} |D^2 w| dx \leq (\int_{B_{\frac{1}{2}}} |D^2 w|^2 dx)^{\frac{1}{2}} \sqrt{\frac{\pi}{4}} \overset{(253)}{\leq} 13(\int_{B_1} \operatorname{dist}^2(Du(x), SO(2))dx)^{\frac{1}{4}}$. Let $y \in B_{\frac{1}{4}}$, by the Mean Value Theorem

$$D^2 w(y) = \fint_{B_{\frac{1}{4}}(y)} D^2 w(x) dx.$$

So

$$\begin{aligned}
\left|D^2 w(y)\right| &\leq \left(\frac{\pi}{16}\right)^{-1} \int_{B_{\frac{1}{4}}(y)} \left|D^2 w(x)\right| dx \\
&\overset{(253)}{\leq} \frac{16}{\pi} \times 13\left(\int_{B_1} \operatorname{dist}^2\big(Du(x), SO(2)\big)dx\right)^{\frac{1}{4}} \\
&\leq 67\left(\int_{B_1} \operatorname{dist}^2\big(Du(x), SO(2)\big)dx\right)^{\frac{1}{4}}.
\end{aligned} \tag{254}$$

So

$$\left\|D^2 w(y)\right\|_{L^\infty(B_{\frac{1}{4}})} \leq 67\left(\int_{B_1} \operatorname{dist}^2\big(Du(x), SO(2)\big)dx\right)^{\frac{1}{4}}. \tag{255}$$

Let $z_0 \in B_{\frac{1}{4}}$. Thus

$$\sup\left\{\left|Dw(x) - Dw(z_0)\right| : x \in B_{\frac{1}{4}}\right\} \overset{(255)}{\leq} 34\left(\int_{B_1} \operatorname{dist}^2\big(Du(x), SO(2)\big)dx\right)^{\frac{1}{4}}. \tag{256}$$

Now

$$\int_{B_{\frac{1}{4}}} |D\zeta| dx \leq \left(\int_{B_{\frac{1}{4}}} |D\zeta|^2 dx\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{4} \overset{(249)}{\leq} \frac{\sqrt{\pi}}{2}\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{2}}. \tag{257}$$

And

$$\begin{aligned}
\int_{B_{\frac{1}{4}}} \left|Du(x) - Dw(z_0)\right| dx &\leq \int_{B_{\frac{1}{4}}} \left|Du(x) - Dw(x)\right| dx + \int_{B_{\frac{1}{4}}} \left|Dw(x) - Dw(z_0)\right| dx \\
&\overset{(250)}{\leq} \int_{B_{\frac{1}{4}}} \left|Dz(x)\right| dx + \frac{\pi}{16}\left\|Dw - Dw(z_0)\right\|_{L^\infty(B_{\frac{1}{4}})} \\
&\overset{(256)}{\leq} \int_{B_{\frac{1}{4}}} \left|Dz(x)\right| dx + \frac{34\pi}{16}\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{4}}
\end{aligned}$$

$$\overset{(257)}{\leq} \left(\frac{34\pi}{16} + \frac{\sqrt{\pi}}{2}\right)\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{4}}. \tag{258}$$

Recall $w = u - z$. So

$$\begin{aligned}\left(\int_{B_1} \operatorname{dist}^2\big(Dw, SO(2)\big)dx\right)^{\frac{1}{2}} &\leq \left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{2}} + \|D\zeta\|_{L^2(B_1)} \\ &\overset{(249)}{\leq} 3\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{2}}. \end{aligned} \tag{259}$$

Hence $\int_{B_{\frac{1}{4}}} \operatorname{dist}(Dw, SO(2))dx \leq \sqrt{\frac{\pi}{16}}(\int_{B_1} \operatorname{dist}^2(Dw, SO(2))dx)^{\frac{1}{2}}$. Thus

$$\int_{B_{\frac{1}{4}}} \operatorname{dist}\big(Dw, SO(2)\big)dx \overset{(259)}{\leq} \frac{3\sqrt{\pi}}{4}\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{2}}.$$

So there must exist $z_0 \in B_{\frac{1}{4}}$ such that

$$\operatorname{dist}\big(Dw(z_0), SO(2)\big) \leq \frac{12}{\sqrt{\pi}}\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{2}}. \tag{260}$$

Let $R \in SO(2)$ be such that $|Dw(z_0) - R| = \operatorname{dist}(Dw(z_0), SO(2))$. By (258), (260) we have that

$$\int_{B_{\frac{1}{4}}} \big|Du(x) - R\big|dx \leq 15\left(\int_{B_1} \operatorname{dist}^2\big(Du, SO(2)\big)dx\right)^{\frac{1}{4}}. \tag{261}$$

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