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On Clark's theorem and its applications to partially sublinear problems

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Abstract

In critical point theory, Clark's theorem asserts the existence of a sequence of negative critical values tending to 0 for even coercive functionals. We improve Clark's theorem, showing that such a functional has a sequence of critical points tending to 0. Our result also gives more detailed structure of the set of critical points near the origin. An extension of Clark's theorem is also given. Our abstract results are powerful in applications, and thus lead to much stronger results than those in the literature on existence of infinitely many solutions for partially sublinear problems such as elliptic equations and Hamiltonian systems.

Résumé

En théorie des points critiques, le théorème de Clark assure l'existence d'une suite de valeurs critiques négatives tendant vers 0 pour des fonctionnelles paires et coercitives. Nous étendons le théorème de Clark en montrant qu'une telle fonctionnelle possède une suite de points critiques tendant vers 0. Notre résultat permet aussi une description plus précise de l'ensemble des points critiques autour de l'origine. Une extension du théorème de Clark est aussi donnée. Nos résultats abstraits s'avèrent puissants dans les applications et conduisent à des résultats nouveaux concernant l'existence d'une infinité de solutions pour des problèmes partiellement sous linéaires comme des équations elliptiques ou des systèmes hamiltoniens.

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1. Introduction

The Clark theorem [6] is an important tool in critical point theory and is constantly and effectively applied to sublinear differential equations with symmetry. The purpose of this paper is to investigate the structure of the set of critical points given in critical point theorems of the Clark type and to prove existence of infinitely many solutions to partially sublinear problems including partially sublinear elliptic equations and partially sublinear Hamiltonian systems. A variant of the Clark Theorem was given by Heinz in [11].

Theorem A. *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then Φ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.*

Theorem A asserts the existence of a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$, without giving any information on the structure of the set of critical points. In applications to nonlinear boundary value problems, if one knows in addition that there exist u_k with $\Phi(u_k) = c_k$, $\Phi'(u_k) = 0$ such that $\|u_k\| \to 0$ as $k \to \infty$ then the behavior of the nonlinearity for $|u|$ large may not be needed to assert the existence of infinitely many critical points. This idea was first explored in [19] for a variety of nonlinear boundary value problems. For example, consider the classical Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, and $f(x, u)$ and $F(x, u) = \int_0^u f(x, t)dt$ satisfy

(a1) $f \in C(\bar{\Omega} \times (-\delta, \delta), \mathbb{R})$ for some $\delta > 0$, $f(x, u)$ is odd in u, and uniformly in $x \in \bar{\Omega}$

$$\lim_{|u| \to 0} \frac{F(x, u)}{u^2} = +\infty,$$

(a2) $2F(x, u) - f(x, u)u > 0$ for $x \in \bar{\Omega}$ and $u \in (-\delta, \delta)$, $u \neq 0$.

Then it was shown in [19] that the problem has a sequence of solutions $u_k \neq 0$ with $\|u_k\| \to 0$ as $k \to \infty$. In applying Theorem A, the nonlinear function f is extended to $\bar{\Omega} \times \mathbb{R}$ so that the resulting variational functional satisfies the conditions of Theorem A. In fact only (a1) is needed in verifying the assumptions of Theorem A. Then a sequence of critical values $c_k < 0$, $c_k \to 0$ as $k \to \infty$, is obtained. (a2) is used to guarantee that the associated functional Φ has $u = 0$ as the only critical point with the critical value 0. Therefore, if (u_k) are critical points corresponding to c_k then (u_k) is a (PS) sequence and it has a convergent subsequence which must converge to 0 with respect to the Sobolev norm $\|\cdot\|$. Finally a regularity argument shows $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$ so for k large, (u_k) are solutions of the original equation. This unified and generalized several results from [10,1,3] under various additional technical conditions (see Section 3.1 for details).

A very interesting question arising from applications of Theorem A is whether there is a sequence of critical points u_k such that $\Phi(u_k) \to 0$ and $\|u_k\| \to 0$ as $k \to \infty$ under the assumptions of Theorem A. If this question has a positive answer then, when for example applying the abstract result to (1.1), (a2) is completely unnecessary and (a1) can be much weakened. We shall give this question a positive answer and our result gives the structure of the set of critical points near the original in the abstract setting of Clark's theorem. Our first abstract result is the following.

Theorem 1.1. *Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.*
(ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\Phi(u) = 0$.*

As a corollary we obtain that in the setting of Theorem A there exist a sequence of critical points $u_k \neq 0$ such that $\Phi(u_k) \le 0$, $\Phi(u_k) \to 0$, and $\|u_k\| \to 0$ as $k \to \infty$. This is exactly what is needed in the above application without (a2). In fact we have much more general results concerning (1.1) (see Theorems 1.3 and 3.1).

Theorem A has some variants which are applicable to indefinite problems such as problems on periodic solutions of first order Hamiltonian systems [3, Theorem 2], [19, Theorem 3.1]. We can also give information on the set of critical points near the origin in the setting of indefinite problems, and our next abstract result is for this case.

Theorem 1.2. *Let X be a Banach space, $\{X_n\}_0^\infty$ be a sequence of infinitely dimensional closed subspaces of X such that $X_0 \subset X_1 \subset X_2 \subset \cdots$, the codimension d_n of X_0 in X_n is finite, and $X = \overline{\bigcup_0^\infty X_n}$, and $\Phi \in C^1(X, \mathbb{R}^1)$. Assume that Φ is even and satisfies the $(PS)^*$ condition with respect to $\{X_n\}_0^\infty$, $\Phi|_{X_0}$ is bounded below and satisfies the (PS) condition, and $\Phi(0) = 0$. If there exists $n_0 > 0$ such that for any $k \in \mathbb{N}$, there exist $\epsilon_k > 0$, $\rho_k > 0$ with $\rho_k \to 0$, and a symmetric set $A_k \subset \{u \in X \mid \|u\| = \rho_k\}$ such that $\gamma(X_n \cap A_k) = d_n + k$ and $\sup_{X_n \cap A_k} \Phi < -\epsilon_k$ for all $n \ge n_0$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points $\{u_k\}$ satisfying $\Phi(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.*
(ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $\Phi(u) = 0$.*

Recall that Φ is said to satisfy the $(PS)^*$ condition with respect to $\{X_n\}_0^\infty$ if for any subsequence $\{n_j\}$ of $\{n\}$ any sequence $\{u_{n_j}\}$ such that $u_{n_j} \in X_{n_j}$, $\Phi(u_{n_j})$ is bounded, and $(\Phi|_{X_{n_j}})'(u_{n_j}) \to 0$ as $j \to \infty$ contains a subsequence converging to a critical point of Φ. Denote by Σ the family of closed symmetric subsets of X which do not contain 0. For $A \in \Sigma$, the genus $\gamma(A)$ of A is by definition the smallest integer n for which there exists an odd and continuous mapping $h : A \to \mathbb{R}^n \backslash \{0\}$, $\gamma(A) = +\infty$ if no such mapping exists, and $\gamma(\emptyset) = 0$.

From the proofs of Theorems 1.1 and 1.2 below, we will see that the conclusion (ii) in these two theorems can be strengthened as: There exist $r > 0$ and a continuum component $\mathcal{C}$ of $\{u \in X \mid \Phi'(u) = 0,\ \Phi(u) = 0\}$ which joins the sphere $S_r = \{u \in X \mid \|u\| = r\}$ and the origin 0, that is $\mathcal{C} \cap S_r \neq \emptyset$ and $0 \in \mathcal{C}$.

The additional information on the norms of the critical points we obtain in these abstract theorems is very important in applications. For example, as a consequence of Theorem 1.1 in its application to (1.1), (a1) itself guarantees existence of infinitely many solutions of (1.1). In fact, our result gives infinitely many solutions under only the partially sublinear condition (a3), a condition which is much weaker than (a1).

Theorem 1.3. *Assume*

(a3) *$f \in C(\bar{\Omega} \times (-\delta, \delta), \mathbb{R})$ for some $\delta > 0$, $f(x, u)$ is odd in u, and there exists a ball $B_r(x_0) \subset \Omega$ such that uniformly in $x \in B_r(x_0)$*

$$\lim_{|u| \to 0} \frac{F(x, u)}{u^2} = +\infty.$$

Then Eq. (1.1) has infinitely many solutions u_k with $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$.

As an application of Theorem 1.2, we study existence of periodic solutions of the first order Hamiltonian system

$$-Jz' = H_z(t, z), \quad z(0) = z(2\pi), \tag{1.2}$$

where J is the standard symplectic matrix in $\mathbb{R}^{2N}$, and $H \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ with $B_\delta = \{z \in \mathbb{R}^{2N} \mid |z| \le \delta\}$ for some $\delta > 0$. We assume that

(a4) H is 2π-periodic in t and is even in z, and $H(t, 0) = 0$;
(a5) $\lim_{|z| \to 0} H(t, z)/|z|^2 = +\infty$ uniformly in $t \in [0, 2\pi]$.

Theorem 1.4. *Under the assumptions* (a4)–(a5), *(1.2) has infinitely many 2π-periodic solutions z_k such that $\|z_k\|_{L^\infty} \to 0$ as $k \to \infty$.*

Theorems 1.3 and 1.4 generalize many results in the literature. For example, Theorem 1.3 generalizes [1, Theorem 2.5.1], [3, Theorem 1(b)] and [19, Theorems 1.1 and 2.5], while Theorem 1.4 generalizes [3, Theorem 3(b)] and [19, Theorem 3.1].

The paper is organized as follows. Theorems 1.1 and 1.2 are proved in Section 2. In Section 3 we will use the abstract results to study three types of elliptic partial differential equations or systems, and we will prove results which are much more general than Theorem 1.3. We prove Theorem 1.4 in Section 4, and periodic solutions of second order Hamiltonian systems are also considered in this section.

In the following C and C_i will stand for positive constants which may be variant from place to place.

After this paper was submitted, we learned that similar results to Theorems 1.1 and 1.3 were obtained in [12]. Nevertheless, our proofs of these two theorems are different here.

2. Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorems 1.1 and 1.2. We first recall some definitions and known facts which will be used. Let X be a Banach space and $\Phi \in C^1(X,\mathbb{R})$. Denote $K=\{u\in X \mid \Phi'(u)=0\}$. A pseudogradient vector field of Φ is by definition a Lipschitz continuous mapping $W: X\backslash K\to X$ which satisfies

$$\|W(u)\|\le 2\|\Phi'(u)\| \quad \text{and} \quad \langle \Phi'(u), W(u)\rangle \ge \|\Phi'(u)\|^2 \quad \text{for } u\in X\backslash K.$$

According to [16] (see also [18, Lemma A.2]), a pseudogradient vector field of a C^1 functional always exists, and an even functional has an odd pseudogradient vector field. The properties of genus in the following lemma are well known. See [18] for more detailed discussion about genus.

Lemma 2.1. *Let $A, B\in \Sigma$.*

1° *If there exists an odd and continuous mapping $f: A\to B$, then $\gamma(A)\le\gamma(B)$.*
2° *$\gamma(A\cup B)\le \gamma(A)+\gamma(B)$.*
3° *If A is compact, then $\gamma(A)<\infty$ and there is a $\delta>0$ such that $N_\delta(A):=\{u\in X \mid \operatorname{dist}(u,A)\le\delta\}\in\Sigma$ and $\gamma(N_\delta(A))=\gamma(A)$.*

Now we give the idea of the proof of Theorem 1.1. Assuming both (i) and (ii) in Theorem 1.1 are false, we will construct a set $\mathcal{O}\subset\Phi^0=\{u\in X \mid \Phi(u)\le 0\}$ which is invariant with respect to the descending flow of Φ defined by integrating $-W$ and which is away from the origin. In addition, $\mathcal{O}$ is symmetric with respect to the origin and contains subsets of $\operatorname{int}(\Phi^0)$ having arbitrarily preassigned large genus. We will define an increasing sequence of minimax values on $\overline{\mathcal{O}}$. Then an argument based on genus will lead to a contradiction.

Proof of Theorem 1.1. Assume to the contrary that neither (i) nor (ii) is true. Since (i) is false there exists $r_0>0$ such that if $\|u\|\le r_0$ and $\Phi(u)<0$ then $\Phi'(u)\ne 0$. Since (ii) is false there exists r_1 with $0<r_1<r_0$ such that if $\|u\|=r_1$ and $\Phi(u)=0$ then $\Phi'(u)\neq 0$. Then the (PS) condition implies that there exist a and b with $0<a<r_1<b\le r_0$ and $\nu>0$ such that

$$\|\Phi'(u)\|\ge\nu, \quad \text{if } a\le\|u\|\le b \text{ and } \Phi(u)\le 0. \tag{2.1}$$

Let $W: X\backslash K\to X$ be a pseudogradient vector field of Φ such that $W(-u)=-W(u)$, where $K=\{u\in X \mid \Phi'(u)=0\}$. For $u\in X\backslash K$ and $t\ge 0$ define $\phi^t(u)$ to be the unique solution of the initial value problem considered on $X\backslash K$

$$\frac{d}{dt}\phi^t(u)=-W\big(\phi^t(u)\big), \qquad \phi^0(u)=u, \tag{2.2}$$

with $[0,T(u))$ the maximum interval of existence for $t\ge 0$. We also define $\phi^t(u)=u$ for $t\ge 0$ and $T(u)=+\infty$ if $u\in K$. Then $\phi^t(u)$ is odd in u, $T(u)$ is even in u, and

$$\frac{d}{dt}\Phi\big(\phi^t(u)\big)\le 0.$$

The mapping $(t,u)\mapsto\phi^t(u)$ from $\{(t,u)|\, u\in X,\ 0\le t<T(u)\}$ to X is called a descending flow of Φ. For $c\in\mathbb{R}$ and $A\subset X$, let $\Phi_A^c=\{u\in A|\ \Phi(u)\le c\}$. Denote $B_r=\{u\in X|\ \|u\|<r\}$ for $r>0$. Fix b^* such that $a<b^*<b$ and set

$$\mathcal{O}=\left\{\phi^t(u)\,\middle|\, u\in\Phi^0_{X\setminus B_{b^*}},\ 0\le t<T(u)\right\}.$$

Then $\mathcal{O}$ is invariant with respect to the descending flow, that is, $\phi^t(u)\in\mathcal{O}$ for all $0\le t<T(u)$ if $u\in\mathcal{O}$. It is easy to see that $\overline{\mathcal{O}}$ is also invariant respect to the descending flow. In addition, $\mathcal{O}$, and thus $\overline{\mathcal{O}}$, are symmetric with respect to the origin.

We show that

$$\Phi(u)\le-\frac{(b^*-a)\nu}{2},\quad \text{for } u\in\overline{\mathcal{O}}\cap\overline{B}_a. \tag{2.3}$$

Let $u\in\mathcal{O}\cap\overline{B}_a$. Then there exist $u_1\in\Phi^0_{X\setminus B_{b^*}}$ and $0\le t_1<t_2\le t_3<T(u_1)$ such that $\|\phi^{t_1}(u_1)\|=b^*$, $\|\phi^{t_2}(u_1)\|=a$, $a<\|\phi^t(u_1)\|<b^*$ for $t_1<t<t_2$, and $\phi^{t_3}(u_1)=u$. Using (2.1), (2.2) and the properties of pseudogradient vector field we have

$$\Phi\big(\phi^{t_1}(u_1)\big)-\Phi\big(\phi^{t_2}(u_1)\big)=\int_{t_1}^{t_2}\big\langle\Phi'\big(\phi^s(u_1)\big),W\big(\phi^s(u_1)\big)\big\rangle ds\ge\nu\int_{t_1}^{t_2}\big\|\Phi'\big(\phi^s(u_1)\big)\big\|ds \tag{2.4}$$

and

$$b^*-a\le\big\|\phi^{t_2}(u_1)-\phi^{t_1}(u_1)\big\|\le\left\|\int_{t_1}^{t_2}W\big(\phi^s(u_1)\big)ds\right\|\le2\int_{t_1}^{t_2}\big\|\Phi'\big(\phi^s(u_1)\big)\big\|ds. \tag{2.5}$$

Thus for $u\in\mathcal{O}\cap\overline{B}_a$, since $\Phi(\phi^{t_1}(u_1))\le0$,

$$\Phi(u)=\Phi\big(\phi^{t_3}(u_1)\big)\le\Phi\big(\phi^{t_2}(u_1)\big)\le\Phi\big(\phi^{t_1}(u_1)\big)-\frac{(b^*-a)\nu}{2}\le-\frac{(b^*-a)\nu}{2},$$

and (2.3) follows.

Let $u\in X$ and assume $\|u\|<b^*$ and $\Phi(u)<0$. We prove that there exists $T_1(u)\in(0,T(u))$ such that $\|\phi^{T_1(u)}(u)\|>b^*$. Since $\|u\|<b^*<b\le r_0$ and $\Phi(u)<0$, the choice of r_0 at the beginning of the proof implies that $\Phi'(u)\ne0$ and thus $u\in X\setminus K$. For $0<t_1<t_2<T(u)$, as (2.4) and (2.5), we have

$$\Phi\big(\phi^{t_1}(u)\big)-\Phi\big(\phi^{t_2}(u)\big)\ge\int_{t_1}^{t_2}\big\|\Phi'\big(\phi^s(u)\big)\big\|^2ds$$

and

$$\big\|\phi^{t_2}(u)-\phi^{t_1}(u)\big\|\le2\left(\int_{t_1}^{t_2}\big\|\Phi'\big(\phi^s(u)\big)\big\|^2ds\right)^{1/2}(t_2-t_1)^{1/2}.$$

Thus

$$\int_0^{T(u)}\big\|\Phi'\big(\phi^s(u)\big)\big\|^2ds\le-m \tag{2.6}$$

and for $0<t_1<t_2<T(u)$,

$$\big\|\phi^{t_2}(u)-\phi^{t_1}(u)\big\|\le2(-m)^{1/2}(t_2-t_1)^{1/2}, \tag{2.7}$$

where $m:=\inf_X\Phi\in(-\infty,0)$. If $T(u)<\infty$ then (2.7) implies that the limit $u^*:=\lim_{t\to T(u)-}\phi^t(u)$ exists, and then the definition of the initial value problem (2.2) implies $u^*\in K$. Again the choice of r_0 implies that $\|u^*\|>r_0$. Therefore $\|u^*\|>b>b^*$ and there exists $T_1(u)\in(0,T(u))$ such that $\|\phi^{T_1(u)}(u)\|>b^*$. If $T(u)=\infty$ then (2.6) implies existence of an increasing sequence $\{s_n\}$ of positive numbers such that $s_n \to \infty$ and $\Phi'(\phi^{s_n}(u)) \to 0$. Since $m \le \Phi(\phi^{s_n}(u)) < 0$, using the (PS) condition we may assume that the limit $u^* := \lim_{n\to\infty} \phi^{s_n}(u)$ exists and $u^* \in K$. Since $\|u^*\| > b$, if we let n be large enough and choose $T_1(u) = s_n$ then $\|\phi^{T_1(u)}(u)\| > b^*$.

We now show that for any k there exists $A \in \Sigma$ such that $A \subset \mathcal{O}$, $\sup_A \Phi < 0$, and $\gamma(A) \ge k$. Let X^k and ρ_k be as in the theorem. If $\rho_k \ge b^*$ then we just choose $A = X^k \cap S_{\rho_k}$. Thus we may assume $\rho_k < b^*$ in what follows. Let $u \in X^k \cap S_{\rho_k}$. According to the assertion in the last paragraph, there exists $T_1(u) \in (0, T(u))$ such that $\|\phi^{T_1(u)}(u)\| > b^*$. Then continuity of the solution of (2.2) with respect to the initial data and the definition of $\mathcal{O}$ imply existence of a neighborhood $N(u)$ of u in $X^k \cap S_{\rho_k}$ such that $T(v) > T_1(u)$ and $\phi^t(v) \in \mathcal{O}$ for all $t \in [T_1(u), T(v))$ and all $v \in N(u)$. Since $X^k \cap S_{\rho_k}$ is compact, we may choose a finite number of points $u_1, \cdots, u_n \in X^k \cap S_{\rho_k}$ such that

$$\bigcup_{i=1}^{n} N(u_i) = X^k \cap S_{\rho_k}.$$

Let $\lambda_1(u), \cdots, \lambda_n(u)$ be a C^0 partition of unity subordinated to $\{N(u_i)\}_{i=1}^n$. Define

$$T^*(u) = \frac{1}{2}\sum_{i=1}^{n}\big(\lambda_i(u) + \lambda_i(-u)\big)T_1(u_i), \quad u \in X^k \cap S_{\rho_k}.$$

Then $T^* : X^k \cap S_{\rho_k} \to \mathbb{R}^+$ is even and continuous. For any $u \in X^k \cap S_{\rho_k}$, let $N(u_{i_1}), \cdots, N(u_{i_m})$ be all the neighborhoods from $\{N(u_i)\}_1^n$ such that

$$u \in \bigcap_{j=1}^{m}\big(N(u_{i_j}) \cup \big(-N(u_{i_j})\big)\big).$$

We may assume that $N(u_{i_1}), \cdots, N(u_{i_m})$ are ordered in such a way that

$$T_1(u_{i_1}) \le T_1(u_{i_2}) \le \cdots \le T_1(u_{i_m}).$$

Then

$$T_1(u_{i_1}) \le T^*(u) = \frac{1}{2}\sum_{j=1}^{m}\big(\lambda_{i_j}(u) + \lambda_{i_j}(-u)\big)T_1(u_{i_j}) \le T_1(u_{i_m}) < T(u),$$

where the fact that T is even and $u \in N(u_{i_m}) \cup (-N(u_{i_m}))$ is used to obtain the last inequality. This implies

$$T_1(u_{i_1}) \le T^*(-u) < T(-u).$$

Therefore $\phi^{T^*(u)}(u) \in \mathcal{O}$, since $u \in N(u_{i_1}) \cup (-N(u_{i_1}))$. Define

$$h(u) = \phi^{T^*(u)}(u), \quad u \in X^k \cap S_{\rho_k}.$$

Then $h : X^k \cap S_{\rho_k} \to \mathcal{O}$ is odd and continuous. Now if we choose $A = h(X^k \cap S_{\rho_k})$, then $A \in \Sigma$, $A \subset \mathcal{O}$, $\sup_A \Phi < 0$, and $\gamma(A) \ge k$.

For any $k \in \mathbb{N}$, define

$$c_k = \inf_{A\in\Sigma,\ A\subset\overline{\mathcal{O}},\ \gamma(A)\ge k}\ \sup_{u\in A} \Phi(u).$$

Since $m = \inf_X \Phi$, the discussion above implies that

$$m \le c_1 \le c_2 \le \cdots \le c_k \le \cdots < 0.$$

Denote $c_\infty = \lim_{k\to\infty} c_k$. Then

$$m \le c_\infty \le 0.$$

We will divide the rest of the proof into two cases: $c_\infty < 0$ and $c_\infty = 0$, and in either case we will arrive at a contradiction.

First we assume $c_\infty < 0$. Then $K_{c_\infty} \in \Sigma$, where $K_c = \{u \in K \mid \Phi(u) = c\}$ for a real number c. Since K_{c_∞} is compact, there is $\delta_1 > 0$ such that the closed neighborhood $\mathcal{N}_{\delta_1} = \{u \in X \mid \operatorname{dist}(u, K_{c_\infty}) \le \delta_1\}$ of K_{c_∞} satisfies $\mathcal{N}_{\delta_1} \in \Sigma$ and $l := \gamma(\mathcal{N}_{\delta_1}) = \gamma(K_{c_\infty}) < \infty$. The (PS) condition implies existence of $\epsilon_1 > 0$ and $\nu_1 > 0$ such that

$$\left\|\Phi'(u)\right\| \ge \nu_1, \quad \text{for } u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \left(\mathcal{N}_{\delta_1/2} \cup \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\right). \tag{2.8}$$

Without loss of generality, we assume that $\epsilon_1 \le \frac{\delta_1\nu_1}{16}$. Then the same argument as in the proof of (2.3) shows that if $u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \operatorname{int}(\mathcal{N}_{\delta_1})$ and $0 \le t_1 < t_2 < T(u)$ are such that

$$\operatorname{dist}\left(\phi^{t_1}(u), K_{c_\infty}\right) = \delta_1, \quad \operatorname{dist}\left(\phi^{t_2}(u), K_{c_\infty}\right) = \delta_1/2$$

and, for $t_1 < t < t_2$,

$$\phi^t(u) \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \left(\mathcal{N}_{\delta_1/2} \cup \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\right),$$

then

$$\Phi\left(\phi^{t_2}(u)\right) \le c_\infty + \epsilon_1 - \frac{\delta_1\nu_1}{4}.$$

This implies

$$c_\infty - \epsilon_1 \le c_\infty + \epsilon_1 - \frac{\delta_1\nu_1}{4},$$

which contradicts $\epsilon_1 \le \frac{\delta_1\nu_1}{16}$. Therefore, for any $u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \operatorname{int}(\mathcal{N}_{\delta_1})$, the curve $\phi^t(u)$ must enter $\{u \in \overline{\mathcal{O}} \mid \Phi(u) < c_\infty - \epsilon_1\}$ before it reaches possibly $\mathcal{N}_{\delta_1/2}$, that is, there exists $t_0 > 0$ such that $\Phi(\phi^{t_0}(u)) < c_\infty - \epsilon_1$ and $\operatorname{dist}(\phi^t(u), K_{c_\infty}) > \delta_1/2$ for $0 \le t \le t_0$. Now a similar argument as used above for defining the mapping h yields an odd and continuous mapping $\eta : \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \operatorname{int}(\mathcal{N}_{\delta_1}) \to \operatorname{int}(\Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1})$, observing that $\overline{\mathcal{O}}$ is invariant with respect to the descending flow defined by (2.2). Choose c_k such that $c_k > c_\infty - \epsilon_1$. Then the definition of c_k implies that

$$\gamma\left(\Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\right) < k.$$

By the definition of c_{k+l} there exists a set $A \in \Sigma$ with $A \subset \overline{\mathcal{O}}$ and $\gamma(A) \ge k + l$ such that $\sup_{u\in A} \Phi(u) < c_\infty + \epsilon_1$. Since $A \subset \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1}$, one has $\eta(A \setminus \operatorname{int}(\mathcal{N}_{\delta_1})) \subset \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}$. Now properties of the genus function γ imply

$$\gamma(A) \le \gamma\left(A \setminus \operatorname{int}(\mathcal{N}_{\delta_1})\right) + \gamma(\mathcal{N}_{\delta_1}) \le \gamma\left(\Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\right) + \gamma(\mathcal{N}_{\delta_1}) < k + l,$$

which contradicts the fact that $\gamma(A) \ge k + l$.

Now assume $c_\infty = 0$. Set $K_{0,b^*} := \{u \in K_0 \mid \|u\| \ge b^*\}$. It is easy to see that $K_{0,b^*} \in \Sigma$ and K_{0,b^*} is compact. Thus there exists $\delta_2 > 0$ such that the closed neighborhood $\mathcal{N}_{b^*,\delta_2} := \{u \in X \mid \operatorname{dist}(u, K_{0,b^*}) \le \delta_2\}$ of K_{0,b^*} satisfies $\mathcal{N}_{b^*,\delta_2} \in \Sigma$ and $l := \gamma(\mathcal{N}_{b^*,\delta_2}) = \gamma(K_{0,b^*}) < \infty$. We claim that there exist $\epsilon_2 > 0$ and $\nu_2 > 0$ such that

$$\left\|\Phi'(u)\right\| \ge \nu_2, \quad \text{for } u \in \Phi_{\overline{\mathcal{O}}}^{0} \setminus \left(\Phi_{\overline{\mathcal{O}}}^{-\epsilon_2} \cup \mathcal{N}_{b^*,\delta_2/2}\right). \tag{2.9}$$

If not then there exists a sequence $\{u_n\} \subset \overline{\mathcal{O}}$ such that

$$\Phi(u_n) \to 0, \quad \Phi'(u_n) \to 0, \quad u_n \notin \mathcal{N}_{b^*,\delta_2/2}.$$

By the (PS) condition, one may assume that $u_n \to u^*$. Then

$$u^* \in (K_0 \cap \overline{\mathcal{O}}) \setminus \operatorname{int}(\mathcal{N}_{b^*,\delta_2/2}).$$

From $u^* \in K_0 \cap \overline{\mathcal{O}}$ and using (2.1) and (2.3), one sees that $\|u^*\| > b^*$. Therefore $u^* \in K_{0,b^*} \subset \operatorname{int}(\mathcal{N}_{b^*,\delta_2/2})$, which is a contradiction. Hence (2.9) holds. We may assume that $\epsilon_2 \le \frac{\delta_2\nu_2}{8}$. Starting from (2.9) in stead of (2.8), one comes to a contradiction in the same way as in the last paragraph. □

Instead of using a pseudogradient vector field to define a descending flow, one may use an equivariant variational system in the sense of [4] to define a descending flow on the whole space.

The proof of Theorem 1.2 needs a generalized version of the genus which we recall now (see [2] for more detailed information). Denote by Σ' the family of symmetric subsets of X. For $A \in \Sigma'$, as in the introduction, the genus of A, still denoted by $\gamma(A)$, is the smallest integer n for which there exists an odd and continuous mapping $h : A \to \mathbb{R}^n \backslash \{0\}$, $\gamma(A) = +\infty$ if no such mapping exists, and $\gamma(\emptyset) = 0$. Obviously, $\Sigma \subset \Sigma'$ and this new genus generalizes the original version. For $B \subset A \subset X$, B is said to be a locally closed subset of A if $B = O \cap C$ with O open in A and C closed in A.

Lemma 2.2. *(See [2].) Let $A, B \in \Sigma'$.*

1° *If there exists an odd and continuous mapping $f : A \to B$, then $\gamma(A) \le \gamma(B)$.*
2° *If A and B are locally closed subsets of $A \cup B$, then $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.*
3° *If A is a closed subset of X, then there exists an open neighborhood U of A in X with $\gamma(U) = \gamma(A)$.*

As a consequence of Lemma 2.2, if $A, A_1 \in \Sigma'$ and A_1 is open or closed in A then $\gamma(A) = \gamma(A_1) + \gamma(A \backslash A_1)$.

Proof of Theorem 1.2. Assume to the contrary that neither (i) nor (ii) is true. Then as in the proof of Theorem 1.1 there exist $0 < r_1 < r_0$ such that if $\|u\| \le r_0$ and $\Phi(u) < 0$ or if $\|u\| = r_1$ and $\Phi(u) = 0$ then $\Phi'(u) \ne 0$. Since $\Phi(0) = 0$, decreasing r_0 if necessary, we may assume that

$$\Phi(u) > -1, \quad \text{if } \|u\| \le r_0. \tag{2.10}$$

The $(PS)^*$ condition implies that there exist a and b with $0 < a < r_1 < b \le r_0$, $\nu_1 > 0$, and $n_1 \ge n_0$ such that for $n \ge n_1$

$$\big\|(\Phi|_{X_n})'(u)\big\| \ge \nu_1, \quad \text{if } u \in \Phi^0_{X_n} \text{ and } a \le \|u\| \le b. \tag{2.11}$$

Let $n \ge n_1$ and W_n be an odd pseudogradient vector field of $\Phi|_{X_n}$. Denote $K^n = \{u \in X_n |\ (\Phi|_{X_n})'(u) = 0\}$. For $u \in X_n \backslash K^n$ and $t \ge 0$, consider the initial value problem on $X_n \backslash K^n$

$$\frac{d}{dt}\phi^t(u) = -W_n\big(\phi^t(u)\big), \qquad \phi^0(u) = u. \tag{2.12}$$

Let $\phi_n^t(u)$, with maximal interval $[0, T_n(u))$ of existence, be the solution of (2.12). We also define $\phi_n^t(u) = u$ for $t \ge 0$ and $T_n(u) = +\infty$ if $u \in K^n$. Using (2.11), the same reasoning leading to (2.3) shows that if $u \in \Phi^0_{X_n \backslash K^n}$ and $0 \le t_1 < t_2 < T_n(u)$ are such that $\|\phi_n^{t_1}(u)\| = a$, $\|\phi_n^{t_2}(u)\| = b$, and $a < \|\phi_n^t(u)\| < b$ for $t_1 < t < t_2$, then

$$\Phi\big(\phi_n^{t_2}(u)\big) \le -\mu_0 := -\frac{(b-a)\nu_1}{2}. \tag{2.13}$$

Since $\Phi|_{X_0}$ is even, bounded below, satisfies the (PS) condition, and $\Phi(0) = 0$, a standard argument shows that

$$k_0 := \gamma\big(\Phi_{X_0}^{-\mu_0}\big) < \infty. \tag{2.14}$$

Here we include a sketch of the proof for completeness. If $\gamma(\Phi_{X_0}^{-\mu_0}) = \infty$, then the sequence $\{c_j\}$ defined by

$$c_j = \inf_{A \subset \Phi_{X_0}^{-\mu_0},\ \gamma(A) \ge j} \sup_{u \in A} \Phi(u)$$

satisfies

$$-\infty < c_1 \le c_2 \le \cdots \le c_j \le \cdots \le -\mu_0.$$

Then a contradiction can be reached in the same way as the reasoning in the case $c_\infty < 0$ in the proof of Theorem 1.1.

For any $n \ge n_1$, since $\Phi_{X_0}^{-\mu_0}$ is closed in $\Phi_{X_n}^{-\mu_0}$ and since the codimension of X_0 in X_n is d_n, using Lemma 2.2 and (2.14) we see that

$$\gamma\big(\Phi_{X_n}^{-\mu_0}\big) \le \gamma\big(\Phi_{X_n}^{-\mu_0} \backslash \Phi_{X_0}^{-\mu_0}\big) + \gamma\big(\Phi_{X_0}^{-\mu_0}\big) \le \gamma(X_n \backslash X_0) + \gamma\big(\Phi_{X_0}^{-\mu_0}\big) = d_n + k_0. \tag{2.15}$$

Let $k_1 \geq k_0$ be such that $\rho_k < a$ for $k \geq k_1$. Fix $k \geq k_1$. Then the choice of r_0 together with the $(PS)^*$ condition implies that there exist $\nu_2 = \nu_2(k) > 0$ and $n_2 = n_2(k) \geq n_1$ such that for $n \geq n_2$

$$\left\|(\Phi|_{X_n})'(u)\right\| \geq \nu_2, \quad \text{if } u \in \Phi_{X_n}^{-\epsilon_k} \text{ and } \|u\| \leq r_0. \tag{2.16}$$

For any $n \geq n_2$, we will construct an odd and continuous map from $X_n \cap A_k$ to $\Phi_{X_n}^{-\mu_0}$. We first show that for any $u \in X_n \cap A_k$ there exists $s_n(u) \in (0, T_n(u))$ such that $\|\phi_n^{s_n(u)}(u)\| > b$. Otherwise, there exists $u \in X_n \cap A_k$ such that $\|\phi_n^t(u)\| \leq b \leq r_0$ for $0 < t < T_n(u)$. From (2.10) we see that $\Phi(\phi_n^t(u)) > -1$ for $0 < t < T_n(u)$. Since $A_k \subset \{u \in X \mid \|u\| = \rho_k\} \subset B_a$ and $\sup_{X_n \cap A_k} \Phi < -\epsilon_k$, we have $X_n \cap A_k \subset \Phi_{X_n \cap B_a}^{-\epsilon_k}$. As a consequence,

$$-1 < \Phi\big(\phi_n^t(u)\big) \leq \Phi(u) \leq -\epsilon_k$$

for $0 < t < T_n(u)$. Using (2.16) we infer that

$$1 \geq 1 - \epsilon_k \geq \Phi(u) - \Phi\big(\phi_n^t(u)\big) \geq \int_0^t \left\|(\Phi|_{X_n})'\big(\phi_n^s(u)\big)\right\|^2 ds \geq \nu_2^2 t$$

for $0 < t < T_n(u)$, which implies $T_n(u) \leq \nu_2^{-2}$. Moreover, since for $0 < t_1 < t_2 < T_n(u)$

$$\left\|\phi_n^{t_1}(u) - \phi_n^{t_2}(u)\right\| \leq 2\left(\int_{t_1}^{t_2} \left\|(\Phi|_{X_n})'\big(\phi_n^s(u)\big)\right\|^2 ds\right)^{1/2} (t_2 - t_1)^{1/2} \leq 2(t_2 - t_1)^{1/2},$$

the limit $u^* := \lim_{t \to T_n(u)-} \phi_n^t(u)$ exists. Clearly $u^* \in \Phi_{X_n \cap B_b}^{-\epsilon_k}$ since $\phi_n^t(u) \in \Phi_{X_n \cap B_b}^{-\epsilon_k}$ for $0 < t < T_n(u)$. By (2.16) again, we see that $u^* \in X_n \backslash K^n$ and $\phi_n^t(u)$ as a solution of (2.12) can be extended beyond $T_n(u)$, a contradiction to the maximality of the interval $[0, T_n(u))$. Thus the assertion concerning existence of $s_n(u)$ holds.

Let $u \in X_n \cap A_k$. Since $\|u\| < a$ and $\|\phi_n^{s_n(u)}(u)\| > b$, there exists $0 < t_1 < t_2 < s_n(u)$ such that $\|\phi_n^{t_1}(u)\| = a$, $\|\phi_n^{t_2}(u)\| = b$, and $a < \|\phi_n^t(u)\| < b$ for $t_1 < t < t_2$. Thus (2.13) implies

$$\Phi\big(\phi_n^{s_n(u)}(u)\big) < \Phi\big(\phi_n^{t_2}(u)\big) \leq -\mu_0.$$

By continuity, u has a neighborhood $N(u)$ in $X_n \cap A_k$ such that for any $v \in N(u)$, $T_n(v) > s_n(u)$ and $\Phi(\phi_n^{s_n(u)}(v)) < -\mu_0$. This implies $\Phi(\phi_n^t(v)) < -\mu_0$ for $v \in N(u)$ and $s_n(u) \leq t < T_n(v)$. Let $\{\lambda_\alpha \mid \alpha \in \Lambda\}$ be a locally finite partition of unity with $\operatorname{supp} \lambda_\alpha \subset N(u_\alpha)$ for some $u_\alpha \in X_n \cap A_k$ and $\sum_{\alpha \in \Lambda} \lambda_\alpha \equiv 1$ on $X_n \cap A_k$. Define $T_n^* : X_n \cap A_k \to \mathbb{R}$ as

$$T_n^*(u) = \frac{1}{2} \sum_{\alpha \in \Lambda} \big(\lambda_\alpha(u) + \lambda_\alpha(-u)\big) s_n(u_\alpha).$$

Then T_n^* is even and continuous. The same argument as in the proof of Theorem 1.1 shows that for any $u \in X_n \cap A_k$, $0 < T_n^*(u) < T_n(u)$ and $\phi_n^{T_n^*(u)}(u) \in \Phi_{X_n}^{-\mu_0}$. Therefore the odd and continuous mapping h_n defined by $h_n(u) = \phi_n^{T_n^*(u)}(u)$ maps $X_n \cap A_k$ into $\Phi_{X_n}^{-\mu_0}$.

Since $\gamma(X_n \cap A_k) = d_n + k$, using (2.15) we have

$$d_n + k = \gamma(X_n \cap A_k) \leq \gamma\big(h_n(X_n \cap A_k)\big) \leq \gamma\big(\Phi_{X_n}^{-\mu_0}\big) \leq d_n + k_0,$$

which contradicts the fact that $k > k_0$. □

3. Applications to elliptic equations

The abstract Theorems 1.1 and 1.2 can be applied to many types of elliptic partial differential equations and systems. We will give three applications here, one to an elliptic system on a bounded domain, one to a p-Laplacian equation on $\mathbb{R}^N$, and one to another quasilinear elliptic equation on $\mathbb{R}^N$, refraining from showing more applications to other type of elliptic equations. For the application to the quasilinear elliptic equation on $\mathbb{R}^N$, we will have to relax the C^1 assumption on Φ in Theorem 1.1, since the functional corresponding to the quasilinear elliptic equation is not in C^1.

3.1. An elliptic system on a bounded domain

Consider the elliptic system

$$\begin{cases} -\Delta_p u = F_u(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, the unknown function $u = (u_1, u_2, \cdots, u_n)$ is an n-dimensional vector function, $\Delta_p u = (\Delta_p u_1, \Delta_p u_2, \cdots, \Delta_p u_n)$, $\Delta_p u_k = \operatorname{div}(|\nabla u_k|^{p-2}\nabla u_k)$, and $p > 1$. Let $\delta > 0$ and denote by $B_\delta(0)$ the ball in $\mathbb{R}^n$ with center 0 and radius δ. Assume

(a6) $F \in C^1(\bar{\Omega} \times B_\delta(0), \mathbb{R})$ for some $\delta > 0$, F is even in u, $F(x,0) = 0$, and there exists a ball $B_r(x_0) \subset \Omega$ such that uniformly in $x \in B_r(x_0)$

$$\lim_{|u|\to 0} \frac{F(x,u)}{|u|^p} = +\infty. \tag{3.2}$$

Theorem 1.3 is a special case of the following theorem.

Theorem 3.1. *Under* (a6) *system* (3.1) *has infinitely many solutions u_k such that $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$.*

Proof. Choose $\hat{F} \in C^1(\bar{\Omega} \times \mathbb{R}^n, \mathbb{R})$ so that $\hat{F}$ is even in u, $\hat{F}(x,u) = F(x,u)$ for $x \in \bar{\Omega}$ and $|u| < \delta/2$, and $\hat{F}(x,u) = 0$ for $x \in \bar{\Omega}$ and $|u| > \delta$. We then consider

$$\begin{cases} -\Delta_p u = \hat{F}_u(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

and its associated functional

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega \hat{F}(x,u), \quad u \in X \triangleq W_0^{1,p}(\Omega, \mathbb{R}^n).$$

It is standard to see that $\Phi \in C^1(X)$, is even, coercive, bounded below, and satisfies the (PS) condition. For any $k \in \mathbb{N}$, if X^k is a k-dimensional subspace of $C_0^\infty(B_r(x_0), \mathbb{R}^n)$ and $\rho_k > 0$ is sufficiently small, then (3.2) implies that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X : \|u\| = \rho\}$. According to Theorem 1.1, Φ has a sequence of nontrivial critical points $\{u_k\}$ satisfying $\Phi(u_k) \le 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$. A standard regularity argument then shows that $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$, and therefore for k large u_k are solutions of (3.1). □

Remark 3.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. For the problem of single equation

$$\begin{cases} -\Delta_p u = \lambda|u|^{q-2}u + |u|^{p^*-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

with $1 < q < p < N$ and $p^* = pN/(N-p)$, Garcia Azorero and Peral Alonso in [10] proved that there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ there exist infinitely many solutions. For the problem of single equation

$$\begin{cases} -\Delta u = \lambda|u|^{q-2}u + \mu|u|^{r-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

with $1 < q < 2 < r \le 2^*$ and $\mu = 1$, Ambrosetti, Brezis and Cerami in [1] proved that there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$ there exist infinitely many solutions. In the subcritical case, the latter result was strengthened by Bartsch and Willem in [3] and they showed that if $1 < q < 2 < r < 2^*$ then for any $\lambda > 0$ and $\mu \in \mathbb{R}$ the problem has infinitely many solutions. The second author in [19] first explored the idea of cut-off modification and L^∞ estimate and unified and generalized all the above results. In fact under conditions (a_1) and (a_2) infinitely many solutions converging to zero was established in [19]. Here our abstract results allow us to obtain Theorem 3.1 generalizing all these results by removing (a_2).

3.2. A p-Laplacian equation on $\mathbb{R}^N$

Consider

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = Q(x)f(x,u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{3.4}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ and $p > 1$. Set $F(x,u) = \int_0^u f(x,s)ds$. Assume

(a7) there exist $\delta > 0$, $1 \le \gamma < p$, $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta,\delta], \mathbb{R})$, f is odd in u, $|f(x,u)| \le C|u|^{\gamma-1}$, and $\lim_{u\to 0} F(x,u)/|u|^p = +\infty$ uniformly in some ball $B_r(x_0) \subset \mathbb{R}^N$,

(a8) $V, Q \in C(\mathbb{R}^N, \mathbb{R}^1)$, $V(x) \ge \alpha_0$ and $0 < Q(x) \le \beta_0$ for some $\alpha_0 > 0$, $\beta_0 > 0$, and $M := Q^{\frac{p}{p-\gamma}} V^{-\frac{\gamma}{p-\gamma}} \in L^1(\mathbb{R}^N)$.

Theorem 3.3. *If* (a7) *and* (a8) *are satisfied then* (3.4) *has infinitely many solutions* $\{u_k\}$ *such that* $\|u_k\|_{L^\infty} \to 0$ *as* $k \to \infty$.

Proof. Choose $\hat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ so that $\hat{f}$ is odd in $u \in \mathbb{R}$, $\hat{f}(x,u) = f(x,u)$ for $x \in \mathbb{R}^N$ and $|u| < \delta/2$, and $\hat{f}(x,u) = 0$ for $x \in \mathbb{R}^N$ and $|u| > \delta$. In order to obtain solutions of (3.4) we study

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = Q(x)\hat{f}(x,u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{3.5}$$

which is the Euler equation of the functional

$$\Phi(u) = \frac{1}{p}\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right) - \int_{\mathbb{R}^N} Q(x)\hat{F}(x,u), \quad u \in X,$$

where X is the Banach space

$$X = \left\{u \in W^{1,p}(\mathbb{R}^N) \,\Big|\, \int_{\mathbb{R}^N} V(x)|u|^p < \infty\right\}$$

endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^p + V(x)|u|^p\right)\right)^{\frac{1}{p}},$$

and $\hat{F}(x,u) = \int_0^u \hat{f}(x,s)ds$. It is standard to check that $\Phi \in C^1(X,\mathbb{R})$, Φ is even, and $\Phi(0) = 0$. For $u \in X$,

$$\int_{\mathbb{R}^N} Q(x)\left|\hat{F}(x,u)\right| \le C_1 \int_{\mathbb{R}^N} Q(x)|u|^\gamma \le C_1 \|M\|_{L^1(\mathbb{R}^N)}^{\frac{p-\gamma}{p}} \left\|V|u|^p\right\|_{L^1(\mathbb{R}^N)}^{\frac{\gamma}{p}} \le C_2\|u\|^\gamma.$$

Therefore,

$$\Phi(u) \ge \frac{1}{p}\|u\|^p - C_2\|u\|^\gamma, \quad u \in X,$$

and then Φ is coercive and bounded below. Now let $\{u_n\}$ be a (PS) sequence, that is, $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$. Then $\{u_n\}$ is bounded. Assume without loss of generality that u_n converges to u weakly in X and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Then $\Phi'(u) = 0$. Wright

$$\left\langle \Phi'(u_n) - \Phi'(u), u_n - u\right\rangle = \left(\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\right.$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n-|u|^{p-2}u\big)(u_n-u)\Big)\\
&+\int_{\mathbb{R}^N} Q(x)\big(\hat f(x,u_n)-\hat f(x,u)\big)(u_n-u)\\
&\triangleq I_1+I_2.
\end{aligned}
$$

Using the elementary inequalities

$$
\big(|\xi|^{p-2}\xi-|\eta|^{p-2}\eta\big)\cdot(\xi-\eta)\ge\begin{cases} c(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2 & \text{for } \xi,\eta\in\mathbb{R}^N,\\ c|\xi-\eta|^p & \text{for } \xi,\eta\in\mathbb{R}^N \text{ if } p\ge 2,\end{cases}
$$

where $c>0$ is a constant, it is easy to see that

$$
I_1\ge\begin{cases} C_3\|u_n-u\|^p, & \text{if } p\ge 2,\\ C_3\|u_n-u\|^2, & \text{if } 1<p<2.\end{cases}
$$

Here, in the case $1<p<2$, we have used the fact that $\{u_n\}$ is bounded together with the inequality

$$
\int_{\mathbb{R}^N}|u|^p\le\Big(\int_{\mathbb{R}^N}|u|^2|v|^{p-2}\Big)^{\frac{p}{2}}\Big(\int_{\mathbb{R}^N}|v|^p\Big)^{\frac{2-p}{2}},\quad \text{for } u,v\in L^p\big(\mathbb{R}^N\big),
$$

which can be deduced from the Hölder inequality. We estimate I_2 as, for any $R>0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} Q(x)\big|\hat f(x,u_n)-\hat f(x,u)\big||u_n-u|\\
&\quad\le C_4\int_{\mathbb{R}^N\setminus B_R(0)} Q(x)\big(|u_n|^\gamma+|u|^\gamma\big)+C_4\int_{B_R(0)}\big(|u_n|^{\gamma-1}+|u|^{\gamma-1}\big)(u_n-u)\\
&\quad\le C_4\big(\big\|V|u_n|^p\big\|^{\frac{\gamma}{p}}_{L^1(\mathbb{R}^N\setminus B_R(0))}+\big\|V|u|^p\big\|^{\frac{\gamma}{p}}_{L^1(\mathbb{R}^N\setminus B_R(0))}\big)\|M\|^{\frac{p-\gamma}{p}}_{L^1(\mathbb{R}^N\setminus B_R(0))}\\
&\qquad+C_4\big(\|u_n\|^{\gamma-1}_{L^\gamma(B_R(0))}+\|u\|^{\gamma-1}_{L^\gamma(B_R(0))}\big)\|u_n-u\|_{L^\gamma(B_R(0))}\\
&\quad\le C_5\|M\|^{\frac{p-\gamma}{p}}_{L^1(\mathbb{R}^N\setminus B_R(0))}+C_5\|u_n-u\|_{L^\gamma(B_R(0))},
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N} Q(x)\big|\hat f(x,u_n)-\hat f(x,u)\big||u_n-u|=0.
$$

Therefore, $\{u_n\}$ converges strongly in X and the (PS) condition holds for Φ. For any $K>0$, there exists $\delta=\delta(K)>0$ such that if $u\in C_0^\infty(B_r(x_0))$ and $|u|_\infty<\delta$ then $Q(x)\hat F(x,u(x))\ge K|u(x)|^p$, and thus

$$
\Phi(u)\le\frac{1}{p}\|u\|^p-K\|u\|^p_{L^p(\mathbb{R}^N)}.
$$

This implies, for any $k\in\mathbb{N}$, if X^k is a k-dimensional subspace of $C_0^\infty(B_r(x_0))$ and $\rho_k>0$ is sufficiently small then $\sup_{X^k\cap S_{\rho_k}}\Phi<0$, where $S_\rho=\{u\in X|\ \|u\|=\rho\}$. Now we appeal to Theorem 1.1 to obtain infinitely many solutions $\{u_k\}$ for (3.5) such that $\|u_k\|\to 0$ as $k\to\infty$.

Finally we show that $\|u_k\|_{L^\infty}\to 0$ as $k\to\infty$. We assume $1<p<N$ and denote $p^*=Np/(N-p)$. For $p\ge N$ the argument is similar and even simpler. Let u be a solution of (3.5) and $\alpha>0$. Let $M>0$ and set $u^M(x)=\max\{-M,\min\{u(x),M\}\}$. Multiplying both sides of (3.5) with $|u^M|^\alpha u^M$ implies

$$
\frac{p^p}{(\alpha+p)^p}\int_{\mathbb{R}^N}\big|\nabla|u^M|^{\frac{\alpha}{p}+1}\big|^p\le C\int_{\mathbb{R}^N}|u^M|^{\alpha+1}
$$

which together with the Sobolev inequality yields

$$\|u^M\|_{L^{(\alpha+p)N/(N-p)}(\mathbb{R}^N)} \le \big(C_1(\alpha+p)\big)^{p/(\alpha+p)} \|u^M\|^{(\alpha+1)/(\alpha+p)}_{L^{\alpha+1}(\mathbb{R}^N)},$$

for some $C_1 \ge 1$ independent of u and α. Set $\alpha_0 = p^* - 1$ and $\alpha_k = \frac{(\alpha_{k-1}+p)N}{N-p} - 1$, that is $\alpha_k = \frac{(p^*/p)^{k+1}-1}{(p^*/p)-1}\alpha_0$, for $k = 1, 2, \cdots$. From the last inequality, an iterating process leads to

$$\|u^M\|_{L^{\alpha_{k+1}+1}(\mathbb{R}^N)} \le \exp\left(\sum_{i=0}^{k} \frac{p\ln(C_1(\alpha_i+p))}{\alpha_i+p}\right) \|u^M\|^{\nu_k}_{L^{p^*}(\mathbb{R}^N)},$$

where $\nu_k = \prod_{i=0}^{k}(\alpha_i+1)/(\alpha_i+p)$. Sending M to infinity and then k to infinity, as a consequence, we have

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le \exp\left(\sum_{i=0}^{\infty} \frac{p\ln(C_1(\alpha_i+p))}{\alpha_i+p}\right) \|u\|^{\nu}_{L^{p^*}(\mathbb{R}^N)},$$

where $\nu = \prod_{i=0}^{\infty}(\alpha_i+1)/(\alpha_i+p)$ is a number in (0, 1) and $\exp(\sum_{i=0}^{\infty} \frac{p\ln(C_1(\alpha_i+p))}{\alpha_i+p})$ is a positive number. Therefore, $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$, and u_k with k sufficiently large are solutions of (3.4). □

Remark 3.4. The case of $p = 2$ was first studied by the authors in [14] and our Theorem 3.3 generalizes the result even for the case of $p = 2$ in [14].

3.3. A quasilinear elliptic equation on $\mathbb{R}^N$

Consider

$$\begin{cases} -\partial_i\big(a_{ij}(u)\partial_j u\big) + \frac{1}{2}a'_{ij}(u)\partial_i u\partial_j u + V(x)u = Q(x)f(x,u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,2}\big(\mathbb{R}^N\big). \end{cases} \tag{3.6}$$

Set $F(x,u) = \int_0^u f(x,s)ds$. Assume

- (a9) there exist $\delta > 0$, $1 \le \gamma < 2$, $C > 0$ such that $f \in C(\mathbb{R}^N \times [-\delta,\delta], \mathbb{R})$, f is odd in u, $|f(x,u)| \le C|u|^{\gamma-1}$, and $\lim_{u\to 0} F(x,u)/|u|^2 = +\infty$ uniformly in some ball $B_r(x_0) \subset \mathbb{R}^N$,
- (a10) $V, Q \in C(\mathbb{R}^N, \mathbb{R}^1)$, $V(x) \ge \alpha_0$ and $0 < Q(x) \le \beta_0$ for some $\alpha_0 > 0$ and $\beta_0 > 0$, and $M := Q^{\frac{2}{2-\gamma}} V^{-\frac{\gamma}{2-\gamma}} \in L^1(\mathbb{R}^N)$,
- (a11) $a_{ij} \in C^1([-\delta,\delta], \mathbb{R})$, a_{ij} is even, $a_{ij} = a_{ji}$, and there exists $c_0 > 0$ such that $a_{ij}\xi_i\xi_j \ge c_0|\xi|^2$.

Theorem 3.5. *If* (a9), (a10) *and* (a11) *are satisfied then* (3.6) *has infinitely many solutions* $\{u_k\}$ *such that* $\|u_k\|_{L^\infty} \to 0$ *as* $k \to \infty$.

We first choose $\hat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ so that $\hat{f}$ is odd in $u \in \mathbb{R}$, $\hat{f}(x,u) = f(x,u)$ for $x \in \mathbb{R}^N$ and $|u| < \delta/2$, and $\hat{f}(x,u) = 0$ for $x \in \mathbb{R}^N$ and $|u| > \delta$. Then we extend a_{ij} to be defined on $\mathbb{R}$ by the following form

$$\hat{a}_{ij}(u) = \eta\big(|u|\big)a_{ij}(u) + \big(1 - \eta\big(|u|\big)\big)c^*\delta_{ij},$$

where $c^* > 0$ is a constant such that $a_{ij}\xi_i\xi_j \le c^*|\xi|^2$, η is a cut-off function such that $\eta(t) = 1$ for $0 \le t \le \delta/2$, $\eta(t) = 0$ for $t \ge \delta$, and $-\frac{4}{\delta} \le \eta'(t) \le 0$ for $\delta/2 \le t \le \delta$. Then it is easy to check that there exists $C > 0$ such that $|\hat{a}_{ij}(u)| + |\hat{a}'_{ij}(u)| \le C$ for $u \in \mathbb{R}$. In order to obtain solutions of (3.6) we study

$$\begin{cases} -\partial_i\big(\hat{a}_{ij}(u)\partial_j u\big) + \frac{1}{2}\hat{a}'_{ij}(u)\partial_i u\partial_j u + V(x)u = Q(x)\hat{f}(x,u), \quad x \in \mathbb{R}^N, \\ u \in W^{1,2}\big(\mathbb{R}^N\big), \end{cases} \tag{3.7}$$

which is formally the Euler equation of the functional

$$\Phi(u) = \frac{1}{2}\int\limits_{\mathbb{R}^N} \big(\hat{a}_{ij}(u)\partial_i u \partial_j u + V(x)u^2\big) - \int\limits_{\mathbb{R}^N} Q(x)\hat{F}(x,u), \quad u \in X,$$

where X is the Banach space

$$X = \Big\{u \in W^{1,2}(\mathbb{R}^N)\Big| \int\limits_{\mathbb{R}^N} V(x)u^2 < \infty\Big\}$$

endowed with the norm

$$\|u\| = \Big(\int\limits_{\mathbb{R}^N} \big(|\nabla u|^2 + V(x)|u|^2\big)\Big)^{\frac{1}{2}},$$

and $\hat{F}(x,u) = \int_0^u \hat{f}(x,s)ds$.

Due to the quasilinear nature of the problem the variational functional Φ is continuous but continuously differentiable, and accordingly Theorem 1.1 cannot be applied directly to (3.7). It is therefore necessary to extend Theorem 1.1 to include the present case. There is an abstract critical point theory for nonsmooth functionals (e.g., [5,7–9]) developed over the years which may apply to our situation. Nevertheless the functional Φ is E-differentiable in the directions $h \in E := X \cap L^\infty(\mathbb{R}^N)$, a case which was well studied in [13]. In a more direct approach for functionals with similar property like Φ here a critical point theory was developed in [13] and we will use the techniques in [13] in extending Theorem 1.1 to a more general theorem. Let us recall a few definitions first.

Definition 3.6. Let X be a Banach space and Φ be a continuous functional defined on X. Let E be a dense subspace of X. We say that Φ is E-differentiable if

(1) for all $u \in X$ and all $h \in E$, the derivative of Φ in the direction h at u exists, denoted by $\langle D\Phi(u), h\rangle$,

$$\big\langle D\Phi(u), h\big\rangle = \lim_{t\to 0+} \frac{1}{t}\big(\Phi(u+th) - \Phi(u)\big)$$

(2) the mapping $(u,h) \mapsto \langle D\Phi(u), h\rangle$ satisfies
  (i) $\langle D\Phi(u), h\rangle$ is linear in h,
  (ii) $\langle D\Phi(u), h\rangle$ is continuous in u, that is $\langle D\Phi(u_n), h\rangle \to \langle D\Phi(u), h\rangle$ as $u_n \to u$ in X.

Definition 3.7. The slope of an E-differentiable functional Φ at u denoted by $|D\Phi(u)|$ is an extended number in $[0,\infty]$:

$$\big|D\Phi(u)\big| = \sup\big\{\big\langle D\Phi(u), h\big\rangle\big|\ h \in E, \|h\| = 1\big\},$$

where $\|\cdot\|$ denotes the norm in X. A point $u \in X$ is said to be a critical point of Φ at the level c if $|D\Phi(u)| = 0$ and $\Phi(u) = c$.

Definition 3.8. An E-differentiable functional f satisfies the (PS) condition if any sequence $\{u_n\} \subset X$, satisfying $|D\Phi(u_n)| \to 0$ and $\Phi(u_n) \to c$, possesses a convergent subsequence.

The proof of Theorem 3.5 relies on the following abstract theorem, which extend Theorem 1.1 by relaxing the C^1 assumption of Φ.

Theorem 3.9. *Let X be a Banach space and E be a dense subspace of X. Let Φ be a continuous functional defined on X which is E-differentiable. Assume Φ satisfies the (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k of X and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X|\ \|u\| = \rho\}$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points* $\{u_k\}$ *satisfying* $\Phi(u_k) < 0$ *for all* k *and* $\|u_k\| \to 0$ *as* $k \to \infty$.
(ii) *There exists* $r > 0$ *such that for any* $0 < a < r$ *there exists a critical point* u *such that* $\|u\| = a$ *and* $\Phi(u) = 0$.

Proof. Assume to the contrary that the conclusions are false. As in the proof of Theorem 1.1, there exist $0 < a < b$ and $\nu > 0$ such that

$$\left|D\Phi(u)\right| \neq 0, \quad \text{if } \|u\| \le a \text{ and } \Phi(u) < 0, \tag{3.8}$$

and

$$\left|D\Phi(u)\right| \ge \nu, \quad \text{if } a \le \|u\| \le b \text{ and } \Phi(u) \le 0. \tag{3.9}$$

We construct a pseudogradient vector field as in [13]. Define $\Phi^0 = \{u \in X | \ \Phi(u) \le 0\}$, $K^* = \{u \in X | \ u \in \Phi^0, |D\Phi(u)| = 0\}$, and

$$\alpha_n = \frac{1}{2}\inf\left\{\left|D\Phi(u)\right| \middle| \ u \in \Phi^0, \ \operatorname{dist}\left(u, K^*\right) \ge \frac{1}{n}\right\}, \quad n = 2, 3, \cdots.$$

Then $\alpha_n \in (0, +\infty]$. We may assume all α_n are finite. The argument is similar if some of α_n or even all α_n are $+\infty$. Define an increasing function on $(0, \infty)$ by

$$g(t) = \begin{cases} \alpha_2, & t \ge 1, \\ \alpha_{n+1}, & t = \frac{1}{n}, \ n = 2, 3, \cdots, \\ (n+1)(\alpha_{n+1} - \alpha_{n+2})(nt - 1) + \alpha_{n+1}, & t \in [\frac{1}{n+1}, \frac{1}{n}], \ n = 1, 2, \cdots. \end{cases}$$

For $u \in \Phi^0 \backslash K^*$, we have $|D\Phi(u)| \ge 2g(\operatorname{dist}(u, K^*))$ and there exist $h = h(u) \in E$ and an open neighborhood $U(u)$ such that $\|h\| = 1$ and

$$\left\langle D\Phi(v), h\right\rangle \ge g\left(\operatorname{dist}\left(v, K^*\right)\right) \quad \text{for } v \in U(u).$$

Let $\{\eta_i | \ i \in \Lambda\}$ be a locally finite partition of unity with $\operatorname{supp} \eta_i \subset U(u_i)$ for some $u_i \in \Phi^0 \backslash K^*$ and $\sum_{i \in \Lambda} \eta_i \equiv 1$ on $\Phi^0 \backslash K^*$. Define

$$W(u) = \sum_{i \in \Lambda}\left(\eta_i(u) - \eta_i(-u)\right)h(u_i)/2g\left(\operatorname{dist}\left(u, K^*\right)\right).$$

Then for $u \in \Phi^0 \backslash K^*$,

$$\left\|W(u)\right\| \le \frac{1}{g(\operatorname{dist}(u, K^*))}, \qquad \left\langle D\Phi(u), W(u)\right\rangle \ge 1. \tag{3.10}$$

For $u \in \Phi^0 \backslash K^*$ and $t \ge 0$, define $\phi^t(u)$ to be the unique solution of the initial value problem considered on $\Phi^0 \backslash K^*$:

$$\frac{d}{dt}\phi^t(u) = -W\left(\phi^t(u)\right), \qquad \phi^0(u) = u, \tag{3.11}$$

with $[0, T(u))$ the maximum interval of existence for $t \ge 0$. We also define $\phi^t(u) = u$ for $t \ge 0$ and $T(u) = +\infty$ if $u \in K^*$. Then $\phi^t(u)$ is odd in u and $T(u)$ is even in u. Using the second inequality in (3.10) and [13, Lemma 2.1], we see that

$$\frac{d}{dt}\Phi\left(\phi^t(u)\right) \le -1 \quad \text{if } u \in \Phi^0 \backslash K^* \text{ and } 0 \le t < T(u). \tag{3.12}$$

Let $b^* := \frac{a+2b}{3}$. As in the proof of Theorem 1.1, we set

$$\mathcal{O} = \left\{\phi^t(u) \middle| \ u \in \Phi^0_{X \backslash B_{b^*}}, \ 0 \le t < T(u)\right\}.$$

Then $\overline{\mathcal{O}}$ is invariant respect to the descending flow and symmetric with respect to the origin.

Denote $\xi = \frac{b-a}{3}$. We show that

$$\Phi(u) \le -\xi g(\xi), \quad \text{for } u \in \overline{\mathcal{O}} \cap \overline{B}_a. \tag{3.13}$$

Let $u \in \mathcal{O} \cap \overline{B}_a$ and $a^* := \frac{2a+b}{3}$. Then there exist $u_1 \in \Phi^0_{X \backslash B_{b^*}}$ and $0 \le t_1 < t_2 < t_3 < T(u_1)$ such that $\|\phi^{t_1}(u_1)\| = b^*$, $\|\phi^{t_2}(u_1)\| = a^*$, $a^* < \|\phi^t(u_1)\| < b^*$ for $t_1 < t < t_2$, and $\phi^{t_3}(u_1) = u$. In view of (3.9) we see that $\operatorname{dist}(\phi^t(u_1), K^*) \ge \xi$

and then the first inequality in (3.10) implies $\|W(\phi^t(u_1))\| \le 1/g(\xi)$ for $t_1 < t < t_2$. Using (3.11) and (3.12) we have

$$-\Phi(u) \ge \Phi\big(\phi^{t_1}(u_1)\big) - \Phi\big(\phi^{t_2}(u_1)\big) = -\int_{t_1}^{t_2} \frac{d}{ds}\Phi\big(\phi^t(u_1)\big)ds \ge t_2 - t_1 \tag{3.14}$$

and

$$\xi \le \big\|\phi^{t_2}(u_1) - \phi^{t_1}(u_1)\big\| \le \int_{t_1}^{t_2} \big\|W\big(\phi^s(u_1)\big)\big\|ds \le (t_2 - t_1)/g(\xi), \tag{3.15}$$

and then (3.13) follows.

Let $u \in X$ and assume $\|u\| < b^*$ and $\Phi(u) < 0$. We prove that there exists $T_1(u) \in (0, T(u))$ such that $\|\phi^{T_1(u)}(u)\| > b^*$. Suppose this were false. Then $\|\phi^t(u)\| \le b^*$ for all $t \in (0, T(u))$. By (3.8) and (3.9) we see that $\kappa := \inf_{0\le t<T(u)} \operatorname{dist}(\phi^t(u), K^*) > 0$. For $0 < t_1 < t_2 < T(u)$, as (3.14) and (3.15), we have

$$-m \ge \Phi\big(\phi^{t_1}(u)\big) - \Phi\big(\phi^{t_2}(u)\big) \ge t_2 - t_1$$

and

$$\big\|\phi^{t_2}(u) - \phi^{t_1}(u)\big\| \le (t_2 - t_1)/g(\kappa),$$

where $m = \inf_X \Phi$. Therefore $T(u) < \infty$ and the limit $u^* := \lim_{t\to T(u)-} \phi^t(u)$ exists and lies outside of K^*. Then the solution $\phi^t(u)$ can be extended beyond $T(u)$, which is a contradiction.

Once $T_1(u) \in (0, T(u))$ satisfying $\|\phi^{T_1(u)}(u)\| > b^*$ exists, then, as in the proof of Theorem 1.1, for any k there exists $A \in \Sigma$ such that $A \subset \mathcal{O}$, $\sup_A \Phi < 0$, and $\gamma(A) \ge k$. Define

$$c_k = \inf_{A\in\Sigma,\ A\subset\overline{\mathcal{O}},\ \gamma(A)\ge k}\ \sup_{u\in A} \Phi(u).$$

Then $\{c_k\}$ is an increasing and bounded sequence:

$$m \le c_1 \le c_2 \le \cdots \le c_k \le \cdots < 0,$$

and its limit $c_\infty = \lim_{k\to\infty} c_k$ satisfies

$$m \le c_\infty \le 0.$$

First we assume $c_\infty < 0$. Then $K_{c_\infty} \in \Sigma$ and there is $\delta_1 > 0$ such that $\mathcal{N}_{\delta_1} := \{u \in X \mid \operatorname{dist}(u, K_{c_\infty}) \le \delta_1\} \in \Sigma$ and $l := \gamma(\mathcal{N}_{\delta_1}) = \gamma(K_{c_\infty}) < \infty$. The (PS) condition implies existence of $\epsilon_1 > 0$ such that

$$\inf\big\{\big|D\Phi(u)\big|\ \big|\ u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1}\backslash\big(\operatorname{int}(\mathcal{N}_{\delta_1/2}) \cup \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\big)\big\} > 0,$$

from which it is easy to see that

$$\kappa_1 := \inf\big\{\operatorname{dist}\big(u, K^*\big)\ \big|\ u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1}\backslash\big(\operatorname{int}(\mathcal{N}_{\delta_1/2}) \cup \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\big)\big\} > 0. \tag{3.16}$$

Without loss of generality, we assume that $\epsilon_1 \le \frac{\delta_1 g(\kappa_1)}{8}$. If $u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1}\backslash \operatorname{int}(\mathcal{N}_{\delta_1})$ and $0 \le t_1 < t_2 < T(u)$ are such that

$$\operatorname{dist}\big(\phi^{t_1}(u), K_{c_\infty}\big) = \delta_1, \qquad \operatorname{dist}\big(\phi^{t_2}(u), K_{c_\infty}\big) = \delta_1/2$$

and, for $t_1 \le t \le t_2$,

$$\phi^t(u) \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1}\backslash\big(\operatorname{int}(\mathcal{N}_{\delta_1/2}) \cup \Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1}\big),$$

then, using the first inequality in (3.10), (3.12), and (3.16), we have

$$\frac{\delta_1}{2} \le \big\|\phi^{t_2}(u) - \phi^{t_1}(u)\big\| \le \int_{t_1}^{t_2} \big\|W\big(\phi^t(u)\big)\big\|dt \le \frac{t_2 - t_1}{g(\kappa_1)} \le \frac{2\epsilon_1}{g(\kappa_1)},$$

which contradicts $\epsilon_1 \le \frac{\delta_1 g(\kappa_1)}{8}$. Therefore, for any $u \in \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \mathrm{int}(\mathcal{N}_{\delta_1})$, the curve $\phi^t(u)$ enters $\{u \in \overline{\mathcal{O}} | \ \Phi(u) < c_\infty - \epsilon_1\}$ before it reaches possibly $\mathcal{N}_{\delta_1/2}$, and this fact implies existence of an odd and continuous mapping $\eta : \Phi_{\overline{\mathcal{O}}}^{c_\infty+\epsilon_1} \setminus \mathrm{int}(\mathcal{N}_{\delta_1}) \to \mathrm{int}(\Phi_{\overline{\mathcal{O}}}^{c_\infty-\epsilon_1})$. Once we have such a mapping η, we then come to a contradiction as in the proof of Theorem 1.1.

Finally we consider $c_\infty = 0$. Set $K_{0,b^*} := \{u \in K_0 | \ \|u\| \ge b^*\}$. Then $K_{0,b^*} \in \Sigma$ and there exists $\delta_2 > 0$ such that $\mathcal{N}_{b^*,\delta_2} := \{u \in X | \ \mathrm{dist}(u, K_{0,b^*}) \le \delta_2\} \in \Sigma$ and $l := \gamma(\mathcal{N}_{b^*,\delta_2}) = \gamma(K_{0,b^*}) < \infty$. We claim that there exists $\epsilon_2 > 0$ such that

$$\inf\big\{\big|D\Phi(u)\big| \ \big| \ u \in \Phi_{\overline{\mathcal{O}}}^{0} \setminus \big(\Phi_{\overline{\mathcal{O}}}^{-\epsilon_2} \cup \mathrm{int}(\mathcal{N}_{b^*,\delta_2/2})\big)\big\} > 0. \tag{3.17}$$

If not then there exists a sequence $\{u_n\} \subset \overline{\mathcal{O}}$ such that

$$\Phi(u_n) \to 0, \qquad \big|D\Phi(u_n)\big| \to 0, \quad u_n \notin \mathrm{int}(\mathcal{N}_{b^*,\delta_2/2}).$$

By the (PS) condition, one may assume that $u_n \to u^*$. Then

$$u^* \in (K_0 \cap \overline{\mathcal{O}}) \setminus \mathrm{int}(\mathcal{N}_{b^*,\delta_2/2}).$$

From $u^* \in K_0 \cap \overline{\mathcal{O}}$ and using (3.9) and (3.13), one sees that $\|u^*\| > b^*$. Therefore $u^* \in K_{0,b^*} \subset \mathrm{int}(\mathcal{N}_{b^*,\delta_2/2})$, which is a contradiction. Hence (3.17) holds and then

$$\kappa_2 := \inf\big\{\mathrm{dist}\big(u, K^*\big) \big| \ u \in \Phi_{\overline{\mathcal{O}}}^{0} \setminus \big(\Phi_{\overline{\mathcal{O}}}^{-\epsilon_2} \cup \mathrm{int}(\mathcal{N}_{b^*,\delta_2/2})\big)\big\} > 0. \tag{3.18}$$

We may assume that $\epsilon_2 \le \frac{\delta_2 g(\kappa_2)}{8}$. Starting from (3.18) in stead of (3.16), one comes to a contradiction in the same way as in the last paragraph. □

Now we are ready to prove Theorem 3.5. We need to verify all the assumptions of Theorem 3.9.

Proof of Theorem 3.5. Obviously, Φ is continuous on X, E-differentiable with respect to $E = X \cap L^\infty(\mathbb{R}^N)$ and even, and $\Phi(0) = 0$. Moreover, for any $u \in X$,

$$\big\langle D\Phi(u), u\big\rangle := \lim_{t\to 0+} \frac{1}{t}\big(\Phi(u+tu) - \Phi(u)\big)$$

exists and takes the expression

$$\big\langle D\Phi(u), u\big\rangle = \int_{\mathbb{R}^N} \left(\hat{a}_{ij}(u)\partial_i u \partial_j u + \frac{1}{2}\hat{a}'_{ij}(u)u\partial_i u\partial_j u + V(x)u^2\right) - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u)u.$$

For $u \in X$,

$$\int_{\mathbb{R}^N} Q(x)\big|\hat{F}(x,u)\big| \le C \int_{\mathbb{R}^N} Q(x)|u|^\gamma \le C\|M\|_{L^1(\mathbb{R}^N)}^{\frac{2-\gamma}{2}} \big\|V|u|^2\big\|_{L^1(\mathbb{R}^N)}^{\frac{\gamma}{2}} \le C_1\|u\|^\gamma.$$

Therefore,

$$\Phi(u) \ge \frac{1}{2}\|u\|^2 - C_1\|u\|^\gamma, \quad u \in X,$$

and then Φ is coercive and bounded below. Now let $\{u_n\}$ be a (PS) sequence, that is, $\Phi(u_n)$ is bounded and $|D\Phi(u_n)| \to 0$. Then $\{u_n\}$ is bounded. Assume without loss of generality that u_n converges to u weakly in X and a.e. on $\mathbb{R}^N$. Then similar to the above we have $Q\hat{F}(x,u_n) \to Q\hat{F}(x,u)$ strongly in $L^1(\mathbb{R}^N)$. Now we show that u is a critical point of Φ. Take $K > \delta$ and define $u^K(x) = u(x)$ if $|u(x)| \le K$ and $u^K(x) = \pm K$ if $\pm u(x) \ge K$. Choose $M > 0$ large so that $M\hat{a}_{ij} - \frac{1}{2}\hat{a}'_{ij}$ is positive definite. Let $\psi \in X \cap L^\infty(\mathbb{R}^N)$ and $\psi \ge 0$. Take test function $\varphi_n = \psi \exp(-Mu_n^K)$ in $\langle D\Phi(u_n), \varphi_n\rangle = o(1)$. Then we have

$$\int_{\mathbb{R}^N} \hat{a}_{ij}(u_n)\partial_i u_n\big(\partial_j\psi - M\psi\partial_j u_n^K\big)\exp\big(-Mu_n^K\big) + \frac{1}{2}\hat{a}'_{ij}(u_n)\partial_i u_n\partial_j u_n\psi\exp\big(-Mu_n^K\big)dx$$
$$+\int_{\mathbb{R}^N} V(x)u_n\psi\exp\big(-Mu_n^K\big)dx - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u_n)\psi\exp\big(-Mu_n^K\big)dx = o(1).$$

Passing to limit as $n \to \infty$ and invoking [17, Theorem 1.6] we obtain

$$\int_{\mathbb{R}^N} \hat{a}_{ij}(u)\partial_i u\big(\partial_j\psi - M\psi\partial_j u^K\big)\exp\big(-Mu^K\big) + \frac{1}{2}\hat{a}'_{ij}(u)\partial_i u\partial_j u\psi\exp\big(-Mu^K\big)dx$$
$$+\int_{\mathbb{R}^N} V(x)u\psi\exp\big(-Mu^K\big)dx - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u)\psi\exp\big(-Mu^K\big)dx \ge 0,$$

which can be written back in

$$\int_{\mathbb{R}^N} \hat{a}_{ij}(u)\partial_i u\partial_j\big(\psi\exp\big(-Mu^K\big)\big) + \frac{1}{2}\hat{a}'_{ij}(u)\partial_i u\partial_j u\psi\exp\big(-Mu^K\big)dx$$
$$+\int_{\mathbb{R}^N} V(x)u\psi\exp\big(-Mu^K\big)dx - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u)\psi\exp\big(-Mu^K\big)dx \ge 0,$$

for all $0 \le \psi \in X \cap L^\infty(\mathbb{R}^N)$. By taking $\psi = \phi\exp(Mu^K)$ for $\phi \in X \cap L^\infty(\mathbb{R}^N)$ and $\phi \ge 0$ we obtain

$$\int_{\mathbb{R}^N} \hat{a}_{ij}(u)\partial_i u\partial_j\phi + \frac{1}{2}\hat{a}'_{ij}(u)\partial_i u\partial_j u\phi dx + \int_{\mathbb{R}^N} V(x)u\phi dx - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u)\phi dx \ge 0.$$

Repeating this with test function $\varphi_n = \psi\exp(Mu_n^K)$ we obtain for all $\phi \ge 0$, $\phi \in X \cap L^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \hat{a}_{ij}(u)\partial_i u\partial_j\phi + \frac{1}{2}\hat{a}'_{ij}(u)\partial_j u\partial_j u\phi dx + \int_{\mathbb{R}^N} V(x)u\phi dx - \int_{\mathbb{R}^N} Q(x)\hat{f}(x,u)\phi dx \le 0.$$

Thus u is a critical point of Φ. Since $\|u_n^K\| \le C$ with C independent of K and n, we have $|\langle D\Phi(u_n), u_n^K\rangle| \le C|D\Phi(u_n)|$. Letting $K \to +\infty$, we see that $|\langle D\Phi(u_n), u_n\rangle| \le C|D\Phi(u_n)|$. Therefore,

$$\big\langle D\Phi(u_n), u_n\big\rangle = o(1).$$

In the same way, $\langle D\Phi(u), u\rangle = 0$. The choice of c^* and η in the definition of $\hat{a}_{ij}$ implies $\hat{a}'_{ij}(u)u\xi_i\xi_j \ge \eta(|u|)a'_{ij}(u)u\xi_i\xi_j$. Decreasing δ if necessary, we may assume

$$\left(\hat{a}_{ij}(u) + \frac{1}{2}\eta\big(|u|\big)a'_{ij}(u)u\right)\xi_i\xi_j \ge \frac{c_0}{2}|\xi|^2.$$

Then $(\hat{a}_{ij}(u) + \frac{1}{2}\hat{a}'_{ij}(u)u)\xi_i\xi_j \ge \frac{c_0}{2}|\xi|^2$. Now we have

$$c_1\|u_n - u\|^2 \le \int_{\mathbb{R}^N} \left(\hat{a}_{ij}(u_n) + \frac{1}{2}\hat{a}'_{ij}(u_n)u_n\right)\partial_i(u_n - u)\partial_j(u_n - u) + V(u_n - u)^2 dx$$
$$\le \big\langle D\Phi(u_n), u_n\big\rangle - \big\langle D\Phi(u), u\big\rangle + o(1) = o(1),$$

where $c_1 = \min\{c_0/2, 1\}$, i.e., $\|u_n - u\|^2 \to 0$.

For any $K > 0$, there exists $\hat{\delta} = \hat{\delta}(K) > 0$ such that if $u \in C_0^\infty(B_r(x_0))$, $|u|_\infty < \hat{\delta}$, and $u(x) \ne 0$ then $Q(x)\hat{F}(x,u(x)) > K|u(x)|^2$, and thus

$$\Phi(u) \le \frac{1}{2}\|u\|^2 - K\|u\|^2_{L^2(\mathbb{R}^N)}.$$

This implies, for any $k \in \mathbb{N}$, if X^k is a k-dimensional subspace of $C_0^\infty(B_r(x_0))$ and $\rho_k > 0$ is sufficiently small then $\sup_{X^k \cap S_{\rho_k}} \Phi < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$. Now we apply Theorem 3.9 to the functional Φ and obtain infinitely many solutions $\{u_k\}$ for (3.7) such that $\|u_k\| \to 0$ as $k \to \infty$.

Finally we show that for k large the L^∞ norm of these solutions are less than $\delta/2$. Therefore, u_k with k sufficiently large are solutions of (3.6). Note that $Q|\hat{f}(x,u)| \le C|u|^{\gamma-1}$ for some $C > 0$. Let $\varphi = |u^M|^\alpha u^M$ with $M \ge \delta$ and $\alpha > 0$. Using the equation we have

$$\int_{|u|\le M} \hat{a}_{ij}(u)\partial_i u \partial_j u(\alpha+1)\big|u^M\big|^\alpha + \frac{1}{2}\hat{a}'_{ij}(u)\partial_i u\partial_j u\big|u^M\big|^\alpha u^M + \int_{\mathbb{R}^N} Vu\big|u^M\big|^\alpha u^M dx = \int_{\mathbb{R}^N} Q\hat{f}(x,u)\big|u^M\big|^\alpha u^M dx,$$

which implies

$$\frac{2c_0}{(\alpha+2)^2}\int_{\mathbb{R}^N} \big|\nabla\big|u^M\big|^{\frac{\alpha}{2}+1}\big|^2 \le \int_{\mathbb{R}^N} C|u|^{\alpha+\gamma}.$$

By Sobolev inequality we have

$$\frac{2c_0 S}{(\alpha+2)^2}\big\|u^M\big\|^{\alpha+2}_{L^{(\alpha+2)N/(N-2)}(\mathbb{R}^N)} \le C\|u\|^{\alpha+\gamma}_{L^{\alpha+\gamma}(\mathbb{R}^N)},$$

where $S = \inf\{\int_{\mathbb{R}^N} |\nabla u|^2 dx \mid \int_{\mathbb{R}^N} |u|^{2N/(N-2)} dx = 1\}$. Here we just consider the case $N \ge 3$. The case $N = 1, 2$ is similar. Sending $M \to \infty$, we have

$$\|u\|_{L^{(\alpha+2)N/(N-2)}(\mathbb{R}^N)} \le \big(C_1(\alpha+2)\big)^{\frac{2}{\alpha+2}}\|u\|^{(\alpha+\gamma)/(\alpha+2)}_{L^{\alpha+\gamma}(\mathbb{R}^N)}.$$

Set $\alpha_0 = 2^* - \gamma$ and $\alpha_k = \frac{(\alpha_{k-1}+2)N}{N-2} - \gamma$, that is $\alpha_k = \frac{(2^*/2)^{k+1}-1}{(2^*/2)-1}\alpha_0$, for $k = 1, 2, \cdots$. We may assume $C_1 \ge 1$. Since

$$\big(C_1(\alpha_i+2)\big)^{(\alpha_j+\gamma)/(\alpha_j+2)} \le C_1(\alpha_i+2) \quad \text{for } i < j,$$

doing iteration, we obtain

$$\|u\|_{L^{\alpha_{k+1}+\gamma}(\mathbb{R}^N)} \le \exp\left(\sum_{i=0}^{k}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)\|u\|^{\nu_k}_{L^{2^*}(\mathbb{R}^N)},$$

where $\nu_k = \prod_{i=0}^{k}(\alpha_i+\gamma)/(\alpha_i+2)$. Letting $k \to \infty$, we see that

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le \exp\left(\sum_{i=0}^{\infty}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2}\right)\|u\|^{\nu}_{L^{2^*}(\mathbb{R}^N)},$$

where $\nu = \prod_{i=0}^{\infty}(\alpha_i+\gamma)/(\alpha_i+2)$ is a number in $(0,1)$ and $\exp(\sum_{i=0}^{\infty}\frac{2\ln(C_1(\alpha_i+2))}{\alpha_i+2})$ is a positive number. We then see that for k large enough $\|u_k\|_{L^\infty(\mathbb{R}^N)} \le \frac{\delta}{2}$ and u_k are solutions of the original equation (3.6). □

Remark 3.10. The abstract theorems from Section 1 can also be applied to other types of elliptic equations and elliptic systems. For example, Theorem 1.2 can be applied to the systems

$$\begin{cases} \Delta u + F_v(x,u,v) = \Delta v + F_u(x,u,v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta u + F_u(x,u,v) = \Delta v - F_v(x,u,v) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

We leave the details to the interested readers.

4. Applications to Hamiltonian systems

In this section, we apply Theorems 1.1 and 1.2 to existence of multiple period solutions to Hamiltonian systems of first and second order.

4.1. Second order Hamiltonian systems

We first consider the second order Hamiltonian system

$$-u'' = V_u(t,u), \tag{4.1}$$

where $V \in C^1(\mathbb{R} \times B_\delta, \mathbb{R})$ with $B_\delta = \{u \in \mathbb{R}^N \mid |u| \le \delta\}$ for some $\delta > 0$. We assume that

(a12) V is 2π-periodic in t and even in u, and $V(t,0) = 0$;
(a13) there exist $0 < \alpha < \beta < 2\pi$ such that $\lim_{|u|\to 0} V(t,u)/|u|^2 = +\infty$ uniformly in $t \in [\alpha, \beta]$.

Theorem 4.1. *Under* (a12) *and* (a13)*,* (4.1) *has infinitely many* 2π*-periodic solutions* u_k *such that* $\|u_k\|_{L^\infty} \to 0$ *as* $k \to \infty$.

Proof. Choose $\hat{V} \in C(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ such that $\hat{V}$ is 2π-periodic in t and even in u, $\hat{V}(t,u) = V(t,u)$ for $(t,u) \in \mathbb{R} \times B_\delta$, and $\hat{V}(t,u) = 0$ for $(t,u) \in \mathbb{R} \times (\mathbb{R}^N \setminus B_{2\delta})$. Instead of (4.1), we consider the equation

$$-u'' = \hat{V}_u(t,u) \tag{4.2}$$

and its associated functional

$$\Phi(u) = \frac{1}{2}\int_0^{2\pi} |u'|^2 dt - \int_0^{2\pi} \hat{V}(t,u)dt, \quad u \in H^1(S^1, \mathbb{R}^N).$$

It is readily checked that all the assumptions of Theorem 1.1 are satisfied, and thus (4.2) has infinitely many 2π-periodic solutions u_k such that $\|u_k\|_{H^1} \to 0$ as $k \to \infty$. Then it is easy to see that $\|u_k\|_{L^\infty} \to 0$ as $k \to \infty$ and u_k with k sufficiently large are 2π-periodic solutions of (4.1). □

Remark 4.2. There have been many papers concerning multiple solutions for second order Hamiltonian systems with various additional conditions [15]. Our Theorem 4.1 unifies and generalizes the results for cases with sublinear nonlinearity near zero.

4.2. First order Hamiltonian systems

Proof of Theorem 1.4. Choose $\hat{H} \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ such that $\hat{H}$ is 2π-periodic in t and even in z, $\hat{H}(t,z) = H(t,z)$ for $(t,z) \in \mathbb{R} \times B_\delta$, and $\hat{H}(t,z) = 0$ for $(t,z) \in \mathbb{R} \times (\mathbb{R}^{2N} \setminus B_{2\delta})$. Instead of (1.2), consider

$$-J\dot{z} = \hat{H}_z(t,z). \tag{4.3}$$

We verify the assumptions of Theorem 1.2 for system (4.3). Let $S_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$, and let $X = W^{\frac{1}{2},2}(S_{2\pi}, \mathbb{R}^{2N})$ be equipped with the usual norm. Denote

$$E_j = \exp(jtJ)\mathbb{R}^{2N} = \big((\cos jt)I + (\sin jt)J\big)\mathbb{R}^{2N}, \quad j \in \mathbb{Z}$$

and define

$$X_n = \bigoplus_{j=-n+1}^{\infty} E_j, \quad n = 0, 1, 2, \cdots.$$

Then $\{X_n\}_0^\infty$ is a sequence of infinitely dimensional closed subspaces of X such that $X_0 \subset X_1 \subset X_2 \subset \cdots$, the codimension of X_0 in X_n is $d_n = 2nN$, and $X = \overline{\bigcup_0^\infty X_n}$. Set

$$X^+ = \bigoplus_{j=1}^{\infty} E_j, \qquad X^0 = E_0, \qquad X^- = \bigoplus_{j=1}^{\infty} E_{-j}.$$

Then any $z \in X$ has a unique orthogonal decomposition $z = z^+ + z^0 + z^-$, $z^\pm \in X^\pm$, $z^0 \in X^0$. It is known that 2π-periodic solutions of (4.3) correspond to critical points of the functional

$$\Phi(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \int_0^{2\pi} \hat{H}\big(t, z(t)\big)dt, \quad z \in X.$$

It is easy to see that $\Phi \in C^1(X, \mathbb{R}^N)$, Φ is even and satisfies the $(PS)^*$ condition with respect to $\{X_n\}_0^\infty$, $\Phi|_{X_0}$ is bounded below and satisfies the (PS) condition, and $\Phi(0) = 0$.

For any $T > 0$, use (a5) and the definition $\hat{H}$ to find $C = C(T) > 0$ such that

$$\hat{H}(t, z) \geq T|z|^2 - C(T)|z|^4, \quad t \in [0, 2\pi], \ z \in \mathbb{R}^{2N}.$$

Let $m \geq 1$ and $z \in \bigoplus_{j=-\infty}^m E_j$. Decompose z as $z = z^+ + z^0 + z^-$. Then we have

$$\begin{aligned}\Phi(z) &\leq \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - T\int_0^{2\pi} |z|^2 + C(T)\int_\alpha^\beta |z|^4 \\ &= \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - T\int_0^{2\pi} |z^+|^2 - T\int_0^{2\pi}|z^-|^2 - 2\pi T|z^0|^2 + C(T)\int_\alpha^\beta |z|^4.\end{aligned}$$

Since $\bigoplus_{j=1}^m E_j$ is of finite dimension, there exists $T_m > 0$ large enough such that, for $z^+ \in \bigoplus_{j=1}^m E_j$,

$$T_m \int_0^{2\pi} |z^+|^2 \geq \|z^+\|^2.$$

As a consequence, for $z \in \bigoplus_{j=-\infty}^m E_j$,

$$\Phi(z) \leq -\frac{1}{2}\|z\|^2 + C(T_m)\int_0^{2\pi} |z|^4 \leq -\frac{1}{2}\|z\|^2 + C_1(m)\|z\|^4,$$

where $C_1(m)$ is a constant depending on m. If $z \in \bigoplus_{j=-\infty}^m E_j$ and $\|z\| = (4C_1(m))^{-1/2}$ then $\Phi(z) \leq -(16C_1(m))^{-1}$. We may assume $C_1(m) \geq m$. For any $k \in \mathbb{N}$, let $m = m_k$ be the unique integer such that $k \in \{2N(m-1)+1, 2N(m-1)+2, \cdots, 2Nm\}$. For $j \geq 1$, write $E_j = \bigoplus_{i=1}^{2N} E_{j,i}$ with $\dim E_{j,i} = 1$ for each i. Define

$$\rho_k = \big(4C_1(m_k)\big)^{-1/2}, \qquad \epsilon_k = \big(16C_1(m_k)\big)^{-1},$$

and

$$A_k = \left\{ z \in \left(\bigoplus_{j=-\infty}^{m-1} E_j\right) \oplus \left(\bigoplus_{i=1}^{k-2N(m-1)} E_{m,i}\right) \Big| \ \|z\| = \rho_k \right\}.$$

Then $\rho_k \to 0$ as $k \to \infty$, $\gamma(X_n \cap A_k) = d_n + k$ and $\sup_{X_n \cap A_k} \Phi \leq -\epsilon_k$ for all $n, k \in \mathbb{N}$. Now all the assumptions of Theorem 1.2 has been verified and (4.3) has a sequence of 2π-periodic solutions $\{z_k\}$ such that $\|z_k\| \to 0$ as $k \to \infty$. Then it is easy to see that $\|z_k\|_{L^\infty} \to 0$ as $k \to \infty$ and z_k are 2π-periodic solutions of (1.2) for k large. □

Remark 4.3. Exchanging the positions of p and q in $z = (p, q)$, we easily see that Theorem 1.4 also holds if (a5) is replaced with the following assumption.

(a5′) $\lim_{|z| \to 0} H(t, z)/|z|^2 = -\infty$ uniformly in $t \in [0, 2\pi]$.

Remark 4.4. Theorem 1.4 was first given in [3] under some global conditions on the nonlinearity and was extended in [19]. We are able to remove a more technical condition like (a_2). Exchanging the positions of p and q in $z = (p, q)$, we easily see that Theorem 1.4 also holds if (a5) is replaced with the following assumption.

Remark 4.5. We conjecture that (a5) in Theorem 1.4 can be weakened as:

(a5″) $\lim_{|z|\to 0} H(t, z)/|z|^2 = \infty$ uniformly in $t \in I$ for some subinterval I of $[0, 2\pi]$.

A positive answer to this conjecture would imply the following: if (a4) is satisfied and there exist two subintervals I_1, I_2 of $[0, 2\pi]$ such that $\lim_{|z|\to 0} H(t, z)/|z|^2 = +\infty$ uniformly in I_1 and $\lim_{|z|\to 0} H(t, z)/|z|^2 = -\infty$ uniformly in I_2, then either

(i) Eq. (1.2) has two sequences $\{z_k\}$, $\{z_k'\}$ of 2π-periodic solutions such that $\Phi(z_k) < 0 < \Phi(z_k')$, and $\|z_k\|_{L^\infty}$, $\|z_k'\|_{L^\infty} \to 0$,

or

(ii) there exists $r > 0$ such that, for any $0 < a < r$, (1.2) has a 2π-periodic solution z such that $\|z\| = a$ and $\Phi(z) = 0$, and thus (1.2) has uncountably infinitely many 2π-periodic solutions.

Remark 4.6. The abstract theorems can also be applied to existence of multiple homoclinic solutions of first order and second order Hamiltonian systems.

Conflict of interest statement

The authors declare there is no conflict of interest.

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