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A generalization of Marstrand's theorem for projections of cartesian products ☆

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Abstract

We prove the following variant of Marstrand's theorem about projections of cartesian products of sets:

Let $K_1, \ldots, K_n$ be Borel subsets of $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_n}$ respectively, and $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a surjective linear map. We set

$$\mathfrak{m} := \min\left\{ \sum_{i\in I} \dim_H(K_i) + \dim \pi\left(\bigoplus_{i\in I^c} \mathbb{R}^{m_i} \right),\ I \subset \{1, \ldots, n\},\ I \neq \emptyset \right\}.$$

Consider the space $\Lambda_m = \{(t, O),\ t \in \mathbb{R},\ O \in SO(m)\}$ with the natural measure and set $\Lambda = \Lambda_{m_1} \times \ldots \times \Lambda_{m_n}$. For every $\lambda = (t_1, O_1, \ldots, t_n, O_n) \in \Lambda$ and every $x = (x^1, \ldots, x^n) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \ldots, t_n O_n x^n)$. Then we have

Theorem.

(i) *If* $\mathfrak{m} > k$, *then* $\pi_\lambda(K_1 \times \ldots \times K_n)$ *has positive* k*-dimensional Lebesgue measure for almost every* $\lambda \in \Lambda$.

(ii) *If* $\mathfrak{m} \leqslant k$ *and* $\dim_H(K_1 \times \ldots \times K_n) = \dim_H(K_1) + \ldots + \dim_H(K_n)$, *then* $\dim_H(\pi_\lambda(K_1 \times \ldots \times K_n)) = \mathfrak{m}$ *for almost every* $\lambda \in \Lambda$.

Keywords: Fractal geometry; Hausdorff dimensions; Potential theory; Fourier transform; Dynamical systems

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1. Introduction

Let us denote by $\dim_H(X)$ the Hausdorff dimension of the set X. For n and k integers with $0 < k < n$, $\Pi_{n.k}$ denotes the space of orthogonal projections from $\mathbb{R}^n$ to k-dimensional subspaces of $\mathbb{R}^n$, with natural measure. A fundamental result in dimensions of projections is the following theorem:

Theorem *(Marstrand–Kaufman–Mattila). Let $E \subset \mathbb{R}^n$ a Borel set. Then:*

(i) *If $\dim_H(E) > k$, then $\pi(E)$ has positive k-dimensional Lebesgue measure for almost every $\pi \in \Pi_{n.k}$.*
(ii) *If $\dim_H(E) \leqslant k$, then $\dim_H(\pi(E)) = \dim_H(E)$ for almost every $\pi \in \Pi_{n.k}$.*

This theorem was first proven for planar sets by Marstrand [3]. Marstrand's proof used geometric methods. Later, Kaufman [2] gave an alternative proof of the same result applying potential-theoretic methods. Finally, Mattila [4] generalized it to higher dimensions; his proof combines Marstrand and Kaufman methods.

There are other variants of Marstrand–Mattila's theorem that were unified in a more general result due to Peres and Schlag [7]. These authors studied general smooth families of projections, using some methods from harmonic analysis. The crucial characteristic that is common to all families of projections considered in Peres–Schlag's result is a transversality property (see [7, Definition 7.2]).

We are interested in Marstrand's projection result that actually is outside of Peres–Schlag's scheme (the families of projections considered here, in general, are not transversal). This result was motivated by the problem of understanding the behavior of projections of cartesian products of sets, by a fixed projection map.

Let $K_1, \dots, K_n$ be Borel subsets of $\mathbb{R}^{m_1}, \dots, \mathbb{R}^{m_n}$ respectively, and $\pi : \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a linear map. Then

$$\dim_H\big(\pi(K_1 \times \dots \times K_n)\big) \leqslant \min\left\{\sum_{i\in I} \dim_H(K_i) + \dim \pi\left(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}\right),\ I \subset \{1, \dots, n\}\right\}, \tag{1.1}$$

with the conventions $\sum_{i\in\emptyset} \dim_H(K_i) = 0$, $\dim\emptyset = 0$.

Consider the space $\Lambda_m = \{(t, O),\ t \in \mathbb{R},\ O \in SO(m)\}$ with the natural measure and set $\Lambda = \Lambda_{m_1} \times \dots \times \Lambda_{m_n}$. For every $x = (x^1, \dots, x^n) \in \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$ and every $\lambda = (t_1, O_1, \dots, t_n, O_n) \in \Lambda$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \dots, t_n O_n x^n)$. Suppose that π is surjective and set

$$\mathfrak{m} := \min\left\{\sum_{i\in I} \dim_H(K_i) + \dim \pi\left(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}\right),\ I \subset \{1, \dots, n\},\ I \neq \emptyset\right\}.$$

Then we have

Theorem 1.1.

(i) *If $\mathfrak{m} > k$, then $\pi_\lambda(K_1 \times \dots \times K_n)$ has positive k-dimensional Lebesgue measure for almost every $\lambda \in \Lambda$.*
(ii) *If $\mathfrak{m} \leqslant k$ and $\dim_H(K_1 \times \dots \times K_n) = \dim_H(K_1) + \dots + \dim_H(K_n)$, then $\dim_H(\pi_\lambda(K_1 \times \dots \times K_n)) = \mathfrak{m}$ for almost every $\lambda \in \Lambda$.*

We recover Marstrand–Mattila's theorem considering the cartesian product of only one set.

Theorem 2.3 is a fundamental tool in our forthcoming work which generalizes the result of Moreira and Yoccoz [6] about stable intersections of two regular Cantor sets for projections of cartesian products of several regular Cantor sets. We prove the following result: for any given surjective linear map $\pi : \mathbb{R}^n \to \mathbb{R}^k$, typically for regular Cantor sets on the real line $K_1, \dots, K_n$ with $\mathfrak{m} > k$, the set $\pi(K_1 \times \dots \times K_n)$ persistently contains non-empty open sets of $\mathbb{R}^k$. Such a result in particular implies an analogous result for simultaneous stable intersections of several regular Cantor sets on the real line.

In another forthcoming work, in collaboration with Pablo Shmerkin, we use the results of this paper combined with the techniques in [1] in order to obtain exact formulas for the Hausdorff dimensions of projections of cartesian products of (real or complex) regular Cantor sets under explicit irrationality conditions.

2. Statement the main results

Let μ be a finite Borel measure on $\mathbb{R}^m$. The *s-energy* of μ is

$$I_s(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^s}.$$

We know (see [5, Theorem 8.9(3)]) that for a Borel set $K \subset \mathbb{R}^m$

$$\dim_H(K) = \sup\{s \in \mathbb{R},\ \text{there is a compactly supported measure } \mu \text{ on } K \\ \text{with } 0 < \mu(\mathbb{R}^m) < \infty \text{ and } I_s(\mu) < \infty\}. \tag{2.1}$$

The *Fourier transform* of μ is denoted by $\hat{\mu}$ and defined as

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi\cdot x} d\mu(x).$$

It is well-known that if $\hat{\mu} \in L^2(\mathbb{R}^m)$, then μ is absolutely continuous with L^2-density. Energy and Fourier transform are related as follows (see [5, Lemma 12.12])

$$I_s(\mu) = (2\pi)^{-m} c(s,m) \int |\xi|^{s-m} |\hat{\mu}(\xi)|^2 d\xi,$$

for $0 < s < m$ and μ with compact support.

We summarize the above observations as the following result. Let $F \subset \mathbb{R}^k$ a Borel set supporting a probability measure ν with $\int |\xi|^{s-k} |\hat{\nu}(\xi)|^2 d\xi < \infty$. If $s \geqslant k$, then F has positive k-dimensional Lebesgue measure. Otherwise, if $0 < s < k$, then $\dim_H(F) \geqslant s$.

Let $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ be a linear map. For each $I \subset \{1, \ldots, n\}$, let $P_I : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ the orthogonal projection onto the subspace $\bigoplus_{i\in I} \mathbb{R}^{m_i}$, where $\mathbb{R}^{m_i}$ is as a canonical subspace of $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$. Then $\pi = \pi \circ P_I + \pi \circ P_{I^c}$ so, for $K_1, \ldots, K_n$ Borel subsets of $\mathbb{R}^{m_1}, \ldots, \mathbb{R}^{m_n}$ respectively we have

$$\begin{aligned}\dim_H(\pi(K_1 \times \ldots \times K_n)) &\leqslant \dim_H(\pi P_I(K_1 \times \ldots \times K_n) \times \pi P_{I^c}(K_1 \times \ldots \times K_n)) \\ &\leqslant \dim_H\Big(\pi P_I(K_1 \times \ldots \times K_n) \times \pi\Big(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}\Big)\Big) \\ &\leqslant \sum_{i\in I} \dim_H(K_i) + \dim \pi\Big(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}\Big).\end{aligned}$$

(In the last inequality, we assume that $\dim_H(K_1 \times \ldots \times K_n) = \dim_H(K_1) + \ldots + \dim_H(K_n)$.) This proves the inequality (1.1) and also motivates us to define:

Definition 2.1. For $\pi : \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ a surjective linear map and $d_1, \ldots, d_n$ nonnegative real numbers, we define $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n)$ as

$$\mathfrak{m} = \min\Big\{\sum_{i\in I} d_i + \dim \pi\Big(\bigoplus_{i\in I^c} \mathbb{R}^{m_i}\Big),\ I \subset \{1, \ldots, n\},\ I \neq \emptyset\Big\}.$$

Remark 2.2. If in addition $d_i \leqslant m_i$ (which holds for dimensions of subsets of $\mathbb{R}^{m_i}$), then, for the open and total measure family of linear maps π with the following transversality property:

$$\dim \pi\Big(\bigoplus_{i\in I} \mathbb{R}^{m_i}\Big) = \min\Big(k, \dim\Big(\bigoplus_{i\in I} \mathbb{R}^{m_i}\Big)\Big), \quad \text{for all } I \subset \{1, \ldots, n\},$$

the equivalence $\mathfrak{m}(\pi, d_1, \ldots, d_n) > k \Leftrightarrow d_1 + \ldots + d_n > k$ holds. However, in general we must check more than one of the $2^n - 1$ conditions appearing in the definition of $\mathfrak{m}$.

Consider the space $\Lambda_m = \{(t, O),\ t \in \mathbb{R},\ O \in SO(m)\}$, with the product measure $\mathcal{L}^1 \times \Theta^m$, where $\mathcal{L}^1$ denotes the one dimensional Lebesgue measure and Θ^m denotes the bi-invariant Haar probability measure on $SO(m)$. Notice that the set $C(m) = \{tO,\ t \in \mathbb{R},\ O \in SO(m)\}$ represents essentially the family of linear conformal maps on $\mathbb{R}^m$. $C(2) = \{\begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ a, b \in \mathbb{R}\}$, which can be viewed as the set of multiplications by a complex number.

We set $\Lambda = \Lambda_{m_1} \times \ldots \times \Lambda_{m_n}$. For every $x = (x^1, \ldots, x^n) \in \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$, and every $\lambda = (t_1, O_1, \ldots, t_n, O_n) \in \Lambda$ we define $\pi_\lambda(x) = \pi(t_1 O_1 x^1, \ldots, t_n O_n x^n)$. Also, given any finite measure μ on $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$, let $\nu_\lambda = (\pi_\lambda)_* \mu$. We also define

$$I_{d_1,\ldots,d_n}(\mu) = \iint \frac{d\mu(x)d\mu(y)}{|x^1 - y^1|^{d_1} \ldots |x^n - y^n|^{d_n}}.$$

Our main result is now the following:

Theorem 2.3. *Let π and $d_1, \ldots, d_n$ be as in Definition 2.1 with $\mathfrak{m} = \mathfrak{m}(\pi, d_1, \ldots, d_n) \neq 0, 1, \ldots, k-1$. Then, there exist $d'_1 \leqslant d_1, \ldots, d'_n \leqslant d_n$ such that for every Borel measure μ on $\mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n}$ we have*

$$\int_{\Lambda} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} \left|\widehat{\nu_\lambda}(\xi)\right|^2 \rho(\lambda) d\xi d\lambda \leqslant C_{\mathfrak{m}} I_{d'_1,\ldots,d'_n}(\mu),$$

where $\rho(\lambda) = |t_1|^{m_1-1} \ldots |t_n|^{m_n-1} e^{-\frac{1}{2}(|t_1|^2+\ldots+|t_n|^2)}$ and $C_{\mathfrak{m}} > 0$ is some constant depending only on $\pi, n, k, m_1, \ldots, m_n$ and $\mathfrak{m}$.

In the proof of Theorem 2.3 the key tool will be the following combinatorial lemma.

Lemma 2.4 *(Weights lemma). Let $s, d_1, \ldots, d_n \geqslant 0$ and $V_1, \ldots, V_n$ be vector subspaces of a fixed finite dimensional vector space satisfying the following 2^n conditions*

$$\sum_{i \in I} d_i + \dim\left(\sum_{i \in I^c} V_i\right) \geqslant s, \quad \textit{for every } I \subset \{1, \ldots, n\}$$

(with the conventions $\sum_{i \in \emptyset} d_i = 0$, $\dim \emptyset = 0$).

Fix a generating set $\{v^i_1, \ldots, v^i_{m_i}\}$ of V_i for each $i \in \{1, \ldots, n\}$. Consider the family $\mathbb{J}$ of all possible $J = (\mathrm{J}_1, \ldots, \mathrm{J}_n)$, $\mathrm{J}_i \subset \{v^i_1, \ldots, v^i_{m_i}\}$ such that $\mathrm{J}_1 \cup \ldots \cup \mathrm{J}_n$ is a linearly independent system with dimension greater than or equal to s. Define

$$\bar{\mathbb{J}} = \left\{(J, i) \in \mathbb{J} \times \{1, \ldots, n\},\ \widehat{J}(i) := (\#\mathrm{J}_1, \ldots, \#\,\mathrm{J}_n) + \left(s - (\#\mathrm{J}_1 + \ldots + \#\mathrm{J}_n)\right)e_i \geqslant 0\right\},$$

where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$ and $\geqslant$ means that the inequality is coordinate to coordinate.

Then, there exist non-negative real numbers $(\alpha_{(J,i)})_{(J,i) \in \bar{\mathbb{J}}}$ with sum equal to 1 such that

$$\sum_{(J,i) \in \bar{\mathbb{J}}} \alpha_{(J,i)} \widehat{J}(i) \leqslant d := (d_1, \ldots, d_n).$$

Proof of Theorem 1.1. The theorem follows immediately from Theorem 2.3 applied to $\mu = \mu_1 \times \ldots \times \mu_n$ for suitable measures μ_i compactly supported in K_i coming from Eq. (2.1). Indeed, this is so because in the part (i) the condition $\dim_H(K_1) > 0, \ldots, \dim_H(K_n) > 0$ follows from the hypotheses, and in the part (ii) we may assume the same condition by reduction to some cartesian product if necessary. □

Remark 2.5. We can derive the part (ii) of Theorem 1.1 from the part (i). Assume $\dim_H(K_i) > 0$. Let $k' < \mathfrak{m} \leqslant k' + 1 \leqslant k$ and consider any $k' < s < \mathfrak{m}$, and set $\Lambda^s = \{\lambda \in \Lambda, \dim_H(\pi_\lambda(K_1 \times \ldots \times K_n)) < s\}$. The idea is to add another factor to the cartesian product: Let $m_0 := k - k'$ and consider K_0 a sufficiently regular subset of $\mathbb{R}^{m_0}$ with $\dim_H(K_0) = k - s$, and $\tilde{\pi} : \mathbb{R}^{m_0} \times \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_n} \to \mathbb{R}^k$ with

$$\tilde{\pi} \circ P_{I^n} = \pi, \text{ where } I^n = \{1, \dots, n\},$$
$$\dim \tilde{\pi}\Big(\bigoplus_{i \in I \cup \{0\}} \mathbb{R}^{m_i}\Big) = \min\Big(k, m_0 + \dim \pi\Big(\bigoplus_{i\in I} \mathbb{R}^{m_i}\Big)\Big), \quad \text{for all } I \subset \{1, \dots, n\}.$$

In particular $\tilde{\pi}$ is surjective. Notice that

$$\sum_{i \in I} \dim_H(K_i) + \dim \tilde{\pi}\Big(\bigoplus_{i \in I^c} \mathbb{R}^{m_i}\Big) > k, \quad \text{for all } I \subset \{0, 1, \dots, n\},\ I \neq \emptyset,$$

and also that $\dim_H(\tilde{\pi}_{(\lambda_0, \lambda)}(K_0 \times K_1 \times \dots \times K_n)) < k$ for all $(\lambda_0, \lambda) \in \Lambda_{m_0} \times \Lambda^s$. Applying Theorem 1.1(i) in this new setting, we conclude that Λ^s is a zero measure subset of Λ.

Remark 2.6. Theorem 2.3, when combined with Proposition 7.5 of [7], also gives us a result on exceptional sets: In the setting of Theorem 1.1, part (i), we have

$$\dim_H\big(\big\{\lambda \in \Lambda,\ t_i \neq 0 \text{ if } m_i > 1,\ \mathcal{L}^k\big(\pi_\lambda(K_1 \times \dots \times K_n)\big) = 0\big\}\big) \leqslant l + k - \mathfrak{m},$$

where $l = \dim \Lambda_{m_1} \times \dots \times \Lambda_{m_n} = n + \sum_{i=1}^n m_i(m_i - 1)/2$.

3. Proof of the main results

Proof of Theorem 2.3 assuming Lemma 2.4. Notice that

$$\big|\widehat{\nu_\lambda}(\xi)\big|^2 = \iint e^{i\xi\cdot\pi_\lambda(y-x)} d\mu(x) d\mu(y),$$
$$= \iint e^{i\pi^T\xi\cdot(t_1 O_1(y^1 - x^1), \dots, t_n O_n(y^n - x^n))} d\mu(x) d\mu(y),$$

and that, for all $z \in \mathbb{R}^m$, $\eta \in \mathbb{R}^m$,

$$\int_{\mathbb{R}} \int_{SO(m)} e^{i\eta\cdot tOz} |t|^{m-1} e^{-\frac{1}{2}|t|^2} d\Theta^m dt = \int_{\mathbb{R}} \int_{S^{m-1}} e^{i|z|\eta\cdot t\theta} |t|^{m-1} e^{-\frac{1}{2}|t|^2} d\sigma^{m-1} dt$$
$$= 2\int_{\mathbb{R}^m} e^{i|z|\eta\cdot x} e^{-\frac{1}{2}|x|^2} dx$$
$$= 2\pi^{\frac{m}{2}} e^{-\frac{1}{2}(|z||\eta|)^2},$$

where σ^{m-1} denotes the normalized Lebesgue measure on S^{m-1}. Therefore by Fubini's theorem

$$\int_{\Lambda} \int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} \big|\widehat{\nu_\lambda}(\xi)\big|^2 \rho(\lambda) d\xi d\lambda = \lim_{a\to\infty} \int_{|\xi|\leqslant a} \int_{\Lambda} |\xi|^{\mathfrak{m}-k} \big|\widehat{\nu_\lambda}(\xi)\big|^2 \rho(\lambda) d\lambda d\xi$$
$$= c \lim_{a\to\infty} \iint \Big(\int_{|\xi|\leqslant a} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi\Big) d\mu(x) d\mu(y)$$
$$= c \iint \Big(\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi\Big) d\mu(x) d\mu(y),$$

where $D_{x,y} = (D^1(|y^1 - x^1|), \dots, D^n(|y^n - x^n|)) \circ \pi^T$, and $D^i(t) : \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$ is the diagonal transformation, $D^i(t) = t.Id$, for $t \in \mathbb{R}$.

We fix x, y assuming that $y^i - x^i \neq 0$ for all $i = 1, \dots, n$. We estimate $\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi$ separately, when $\mathfrak{m} \geqslant k$ and $\mathfrak{m} < k$. In both cases we apply Lemma 2.4 for $V_i = \pi(\mathbb{R}^{m_i})$, taking $v^i_j = \pi(e^i_j)$, where e^i_j, $j = 1, \dots, m_i$ is the canonical basis of $\mathbb{R}^{m_i}$ as subspace of $\mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$.

We use the notation $z^I = z_1^{i_1} \dots z_n^{i_n}$ if $z = (z_1, \dots, z_n) \in \mathbb{R}_+^n$ and $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, for $z = (|y^1 - x^1|, \dots, |y^n - x^n|)$.

Suppose $\mathfrak{m} \geqslant k$. Let i_0 be such that $z_{i_0} \leqslant z_i$ for all $i = 1, \dots, n$. Notice that $\mathfrak{m}(\pi, d - (\mathfrak{m} - k)e_{i_0}) \geqslant k$ and in particular $d - (\mathfrak{m} - k)e_{i_0} \geqslant 0$. We apply Lemma 2.4 to $d - (\mathfrak{m} - k)e_{i_0}$ and $s = k$. For each $J \in \mathbb{J}$, just looking for the sums in $\frac{1}{2}|D_{x,y}(\xi)|^2$ related to J and using the change of variables formula to an appropriate linear isomorphism of $\mathbb{R}^k$, we have

$$\int\limits_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leqslant c' z_{i_0}^{k-\mathfrak{m}} z^{-\widehat{J}} \int\limits_{\mathbb{R}^k} |\eta|^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta|^2} d\eta,$$

for some constant $c' > 0$ depending only on π, $\mathfrak{m}$ and k, where $\widehat{J} := (\#\mathrm{J}_1, \dots, \#\mathrm{J}_n)$. Therefore

$$\int\limits_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leqslant c'' z_{i_0}^{k-\mathfrak{m}} \prod_{J \in \mathbb{J}} z^{-\alpha_J \widehat{J}} = c'' z^{-(\sum_J \alpha_J \widehat{J} + (\mathfrak{m}-k)e_{i_0})} =: c'' z^{-d'}.$$

Suppose $\mathfrak{m} < k$. We apply Lemma 2.4 to d and $s = \mathfrak{m}$. Let $(J, i) \in \bar{\mathbb{J}}$. We define $k' := \#\mathrm{J}_1 + \dots + \#\mathrm{J}_n$ and $k_i' := \#\mathrm{J}_i$, then $\mathfrak{m} < k'$ and $k_i' > k' - \mathfrak{m}$. Similary to the previous case, notice that

$$\int\limits_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leqslant \tilde{c} z^{-\widehat{J}} \int\limits_{\mathbb{R}^{k_i'}} \int\limits_{\mathbb{R}^{k-k'}} \left(|\eta'|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'',$$

for some constant $\tilde{c} > 0$ depending only on $\pi, \mathfrak{m}, k, k', k_i'$. We affirm that

$$\int\limits_{\mathbb{R}^{k_i'}} \int\limits_{\mathbb{R}^{k-k'}} \left(|\eta'|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'' \leqslant \tilde{c}' z_i^{k'-\mathfrak{m}},$$

for some constant $\tilde{c}' > 0$ depending only on $\mathfrak{m}, k, k', k_i'$. If $k' = k$ the affirmation is true, since $\mathfrak{m} - k' > -k_i'$. If $k' < k$, applying polar coordinates in $\mathbb{R}^{k-k'}$ we have

$$\begin{aligned}\int\limits_{\mathbb{R}^{k_i'}} \int\limits_{\mathbb{R}^{k-k'}} \left(|\eta'|/z_i + |\eta''|\right)^{\mathfrak{m}-k} e^{-\frac{1}{2}|\eta'|^2} d\eta' d\eta'' &\leqslant C \int\limits_{\mathbb{R}^{k_i'}} \int\limits_{\mathbb{R}_+} \left(|\eta'|/z_i + r\right)^{\mathfrak{m}-k'-1} e^{-\frac{1}{2}|\eta'|^2} dr d\eta' \\ &= C\left(k' - m\right)^{-1} \int\limits_{\mathbb{R}^{k_i'}} \left(|\eta'|/z_i\right)^{\mathfrak{m}-k'} e^{-\frac{1}{2}|\eta'|^2} d\eta'.\end{aligned}$$

Then $\int_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leqslant \tilde{c}'' z^{-\widehat{J}(i)}$, and therefore

$$\int\limits_{\mathbb{R}^k} |\xi|^{\mathfrak{m}-k} e^{-\frac{1}{2}|D_{x,y}(\xi)|^2} d\xi \leqslant \tilde{c}'' \prod_{(J,i) \in \bar{\mathbb{J}}} z^{-\alpha_{(J,i)} \widehat{J}(i)} = \tilde{c}'' z^{-\sum_{(J,i)\in\bar{\mathbb{J}}} \alpha_{(J,i)} \widehat{J}(i)} =: \tilde{c}'' z^{-d'}. \qquad \square$$

Proof of Lemma 2.4.

Claim. *The vertices of the polyhedron*

$$P = \left\{(d_1, \dots, d_n) \in \mathbb{R}^n,\ d_1 \geqslant 0, \dots, d_n \geqslant 0,\ \sum_{i\in I} d_i + \dim\left(\sum_{i \in I^c} V_i\right) \geqslant s,\ \textit{for all } I \subset \{1, \dots, n\}\right\}$$

have all the form $\widehat{J}(i)$ *for some* $(J, i) \in \mathbb{J}$.

$P \subset \overline{\mathbb{R}}_+^n$, therefore P is a pointed polyhedron (i.e. it does not contain any non-trivial affine subspace). We proceed by induction on n. For $n = 1$ it is trivial. Let $x = (x_1, \dots, x_n)$ any vertex of the polyhedron. Then, there are n independent inequalities from the definition of P that become equality at x (see [8, p. 104]).

If $x_n = 0$, notice that $x' = (x_1, \dots, x_{n-1})$ is now a vertex of the polyhedron

$$P' = \left\{(d_1, \dots, d_{n-1}) \in \mathbb{R}^{n-1},\ d_1 \geqslant 0, \dots, d_{n-1} \geqslant 0,\ \sum_{i \in I} d_i + \dim\left(\sum_{i \in I^c} V_i\right) \geqslant s,\ \text{for all } I \subset \{1, \dots, n-1\}\right\}$$

(i.e. $x' \in P'$ and x' satisfies $n-1$ independent equalities). By induction hypothesis, there exist some $J' = (\mathtt{J}'_1, \dots, \mathtt{J}'_{n-1}) \in \mathbb{J}'$ and $i' \in \{1, \dots, n-1\}$ such that $x' = \widehat{J'}(i')$. Then, $J = (\mathtt{J}'_1, \dots, \mathtt{J}'_{n-1}, \emptyset) \in \mathbb{J}$ and $i = i'$ are such that $x = \widehat{J}(i)$.

Suppose $x_1 \neq 0, \dots, x_n \neq 0$. By simplicity, we denote $\sum_{i \in I} V_i$ by V_I. Consider

$$\mathcal{I} = \left\{I \subset \{1, \dots, n\},\ I \neq \emptyset,\ \sum_{i \in I} x_i + \dim V_{I^c} = s\right\}.$$

By the assumption on x, there are $I_1, \dots, I_n \in \mathcal{I}$ such that the associated $0, 1$ row vectors $\widetilde{I}_1, \dots, \widetilde{I}_n$ defining the equalities, are independent.

If $I, J \in \mathcal{I}$, then

$$\begin{aligned}
\dim V_{I^c} + \dim V_{J^c} &= 2s - \sum_{i \in I} x_i - \sum_{i \in J} x_i \\
&= 2s - \sum_{i \in I \cup J} x_i - \sum_{i \in I \cap J} x_i \\
&\leqslant \dim V_{I^c \cap J^c} + \dim V_{I^c \cup J^c} \\
&\leqslant \dim(V_{I^c} \cap V_{J^c}) + \dim(V_{I^c} + V_{J^c}) \\
&= \dim V_{I^c} + \dim V_{J^c},
\end{aligned}$$

therefore, $I \cup J \in \mathcal{I}$ and $I \cap J \in \mathcal{I}$. Let $I_0 \in \mathcal{I}$, $I_0 \neq \emptyset$ a minimal element by inclusion. Then, for any $J \in \mathcal{I}$, we have

$$I_0 \subset J \quad \text{or} \quad I_0 \cap J = \emptyset.$$

This means the invertible matrix of rows $\widetilde{I}_1, \dots, \widetilde{I}_n$ has $\#I_0$ identical columns, and therefore $\#I_0 = 1$, say $I_0 = \{n\}$, or, equivalently, $x_n = s - \dim(V_1 + \dots + V_{n-1})$.

Notice that now $\tilde{x} = (x_1, \dots, x_{n-1})$ is a vertex of the polyhedron

$$\begin{aligned}
\widetilde{P} = \Bigg\{&(d_1, \dots, d_{n-1}) \in \mathbb{R}^{n-1},\ d_1 \geqslant 0, \dots, d_{n-1} \geqslant 0, \\
&\sum_{i \in I} d_i + \dim\left(\sum_{i \in I^c} V_i\right) \geqslant \dim(V_1 + \dots + V_{n-1}),\ \text{for all } I \subset \{1, \dots, n-1\}\Bigg\}.
\end{aligned}$$

By induction hypothesis, there exist some appropriate $\widetilde{J} = (\tilde{\mathtt{J}}_1, \dots, \tilde{\mathtt{J}}_{n-1}) \in \widetilde{\mathbb{J}}$ such that $\tilde{x} = (\#\tilde{\mathtt{J}}_1, \dots, \#\tilde{\mathtt{J}}_{n-1})$. We can take $\mathtt{J}_n \subset \{v_1^n, \dots, v_{m_n}^n\}$ such that $V_1 + \dots + V_{n-1} + \langle \mathtt{J}_n \rangle = V_1 + \dots + V_n$ and $J = (\tilde{\mathtt{J}}_1, \dots, \tilde{\mathtt{J}}_{n-1}, \mathtt{J}_n) \in \mathbb{J}$. Notice that $x = \widehat{J}(n)$. This finishes the proof of the claim.

To finish the prove of the lemma, notice that for a pointed polyhedron P, we have

$$P = \text{conv.hull}\{x^1, \dots, x^r\} + \text{cone}\{y^1, \dots, y^t\}$$

where x^i are the vertices of P and y^i are its extremal rays (see [8, p. 107]); and we have necessarily $y^i \geqslant 0$ since $P \subset \overline{\mathbb{R}}^n_+$. □

Remark 3.1. Notice that $\widehat{J}(i) \in P$ for all $(J, i) \in \mathbb{J}$, hence we conclude from Lemma 2.4 that

$$P = \text{conv.hull}\{\widehat{J}(i),\ (J, i) \in \mathbb{J}\} + \text{cone}\{e_1, \dots, e_n\}.$$

Conflict of interest statement

None declared.

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