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Critical travelling waves for general heterogeneous one-dimensional reaction–diffusion equations

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Abstract

This paper investigates time-global wave-like solutions of heterogeneous reaction–diffusion equations: $\partial_t u - a(x)\partial_{xx}u - b(x)\partial_x u = f(x,u)$ in $\mathbb{R}\times\mathbb{R}$, where the coefficients a, a_x, a_{xx}, $1/a$, b, b_x and f are only assumed to be measurable and bounded in $x \in \mathbb{R}$ and the nonlinearity f is Lipschitz-continuous in $u \in [0,1]$, with $f(x,0) = f(x,1) = 0$ for all $x \in \mathbb{R}$. In this general framework, the notion of spatial transition wave has been introduced by Berestycki and Hamel [4]. Such waves always exist for one-dimensional ignition-type equations [22,27], but not for monostable ones [26]. We introduce in the present paper a new notion of wave-like solutions, called *critical travelling waves* since their definition relies on a geometrical comparison in the class of time-global solutions trapped between 0 and 1. Critical travelling waves always exist, whatever the nonlinearity of the equation is, are monotonic in time and unique up to normalization. They are spatial transition waves if such waves exist. Moreover, if the equation is of monostable type, for example if $b \equiv 0$ and $f(x,u) = c(x)u(1-u)$, with $\inf_{\mathbb{R}} c > 0$, then critical travelling waves have minimum least mean speed. If the coefficients are homogeneous/periodic, then we recover the classical notion of planar/pulsating travelling wave. If the heterogeneity of the coefficients is compactly supported, then critical transition waves are either a spatial transition wave with minimal global mean speed or bump-like solutions if spatial transition does not exist. In the bistable framework, the nature of the critical travelling waves depends on the existence of non-trivial steady states. Hence, the notion of critical travelling wave provides a unifying framework to earlier scattered existence results for wave-like solutions. We conclude by proving that in the monostable framework, critical travelling waves attract, in a sense and under additional assumptions, the solution of the Cauchy problem associated with a Heaviside initial datum.

MSC: 35B08; 35B40; 35B50; 35C07; 35K57

Keywords: Travelling waves; Heterogeneous reaction–diffusion equations; Monostable equations; Bistable equations; Ignition-type equations; Zero set of solutions of parabolic equations

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1. Previous notions of waves for reaction–diffusion equations

1.1. General framework

This paper investigates time-global wave-like solutions of general heterogeneous reaction–diffusion equations

$$\partial_t u - a(x)\partial_{xx}u - b(x)\partial_x u = f(x,u) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{E}$$

We shall make the following assumptions on the coefficients throughout the paper:

$$\begin{aligned}
&a, a_x, a_{xx}, 1/a, b, b_x \in L^\infty(\mathbb{R}), \qquad f \in L^\infty\big(\mathbb{R}\times[0,1]\big),\\
&\exists C>0 \quad \text{s.t.} \quad \big|f(x,u)-f(x,v)\big| \leqslant C|u-v| \quad \text{a.e. } (x,u,v)\in\mathbb{R}\times[0,1]\times[0,1],\\
&f(x,0)=f(x,1)=0 \quad \text{a.e. } x\in\mathbb{R}.
\end{aligned} \tag{H}$$

This equation arises in many scientific fields such as genetics, population dynamics, chemistry or combustion. The underlying models rely on a simple mechanism: $u\in[0,1]$ is the proportion of a population or of a product, which diffuses and reacts in the environment. Hence, $a(x)$ is sometimes called the diffusion coefficient, $b(x)$ the advection coefficient and $f(x,u)$ the reaction term. The last hypothesis $f(x,0)=f(x,1)=0$ means that Eq. (E) admits two uniform steady states $u=0$ and $u=1$. The key question in all these models is to understand how the steady state 1 invades the steady state 0. One way to address this question is to investigate the existence of wave-like solutions.

Note that one can consider the more general framework where the two steady states $u_-=u_-(x)$ and $u_+=u_+(x)$ depend on space, under the conditions that $u_\pm$ are measurable, essentially bounded and $u_+>u_-$, just by performing the change of variables $v(t,x):=(u(t,x)-u_-(x))/(u_+(x)-u_-(x))$. Hence, there is no loss of generality in assuming that $u_-\equiv 0$ and $u_+\equiv 1$.

We underline that our results will be stated without making any other structural hypothesis on the dependence in x of the coefficients, such as periodicity, almost periodicity or ergodicity for examples. We do not even need the coefficients to be continuous in general. Considering such a general heterogeneity is natural in many applications, for example in population dynamics models. Before stating our results in this general framework, we will first review earlier existence results for wave-like solutions in homogeneous and periodic one-dimensional reaction–diffusion equations.

1.2. Planar travelling waves for homogeneous equations

Eq. (E) has first been investigated by Kolmogorov, Petrovsky, Piskunov [17] and Fisher [12] in the 30's when $a\equiv 1$, $b\equiv 0$, and f does not depend on x:

$$\partial_t u - \partial_{xx}u = f(u) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{1.1}$$

Kolmogorov, Petrovsky and Piskunov proved that, if f is derivable, $f(u)>0$ and $f'(u)\leqslant f'(0)$ for all $u\in(0,1)$, then (1.1) admits a *planar travelling wave* of speed c for all $c\geqslant c^*=2\sqrt{f'(0)}$. That is, for all $c\geqslant c^*$, there exists a function $u=u(t,x)$ which satisfies (1.1) and which can be written

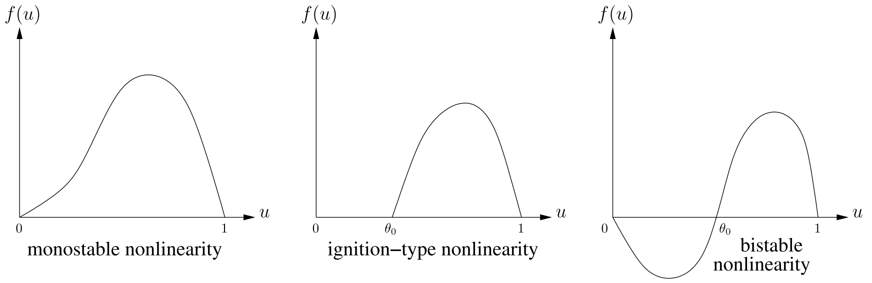
$$u(t,x)=U(x-ct), \quad \text{with } U\in\mathcal{C}^2(\mathbb{R}),\ 0<U<1, \qquad U(-\infty)=1, \qquad U(+\infty)=0. \tag{1.2}$$

The quantity c is called the *speed* of the planar travelling wave u and U is called its *profile*. Moreover, if v is a solution of

$$\begin{cases} \partial_t v - \partial_{xx}v = f(v) & \text{in } (0,\infty)\times\mathbb{R},\\ v(0,x)=1 \quad \text{if } x\leqslant 0, & v(0,x)=0 \quad \text{if } x>0, \end{cases} \tag{1.3}$$

then for all $\theta\in(0,1)$, there exists a unique function $X\in\mathcal{C}^0(\mathbb{R})$ such that $v(t,X(t))=\theta$ for all $t>0$ and one has $\lim_{t\to+\infty} v(t,x+X(t))=u^*(0,x)$ uniformly in $x\in\mathbb{R}$, where u^* is a planar travelling wave with speed $c^*=2\sqrt{f'(0)}$ such that $u^*(0,0)=\theta$ [17]. Hence, the travelling wave with minimal speed c^* is attractive in a sense.

More general types of nonlinearities, still independent of x, have been considered by Aronson and Weinberger [2]. These authors distinguished three classes of equations. When $f(u)>0$ for all $u\in(0,1)$, then (1.1) is called a *monostable* equation because the steady state 0 is unstable while 1 is globally attractive. In combustion models, it may be relevant to assume that the reaction only starts when the temperature is large enough. These models gave rise to *ignition-type* equations, for which there exists $\theta_0 \in (0,1)$ such that $f(u) = 0$ for all $u \in [0, \theta_0]$ and $f(u) > 0$ for all $u \in (\theta_0, 1)$. The quantity θ_0 can be viewed as an ignition temperature. Lastly, if there exists $\theta_0 \in (0,1)$ such that $f(u) < 0$ for all $u \in (0, \theta_0)$ and $f(u) > 0$ for all $u \in (\theta_0, 1)$, then the equation is called *bistable* since the two steady states 0 and 1 are both stable, while θ_0 is an unstable steady state.



The classical result proved by Aronson and Weinberger [2] (see also [11,33]) is the following:

- if Eq. (1.1) is monostable, then there exists a speed $c^* > 0$ such that (1.1) admits a planar travelling wave with speed c if and only if $c \geqslant c^*$,
- if Eq. (1.1) is bistable or of ignition-type, then there exists a speed c^* such that (1.1) admits a planar travelling wave with speed c if and only if $c = c^*$. Moreover, c^* has the same sign as $\int_0^1 f(s)ds$.

For bistable and ignition-type equations, planar travelling waves (of speed c^*) attract, in a sense, the solutions of the Cauchy problem (1.3) [11]. For monostable equations, the solutions of the Cauchy problem (1.3) are attracted, in the same meaning as in [17], by a travelling wave with minimal speed $c = c^*$ [33]. Hence, the travelling wave with minimal speed is the most important one in order to understand the dynamics of the Cauchy problem (1.3) in the monostable framework.

1.3. Pulsating travelling waves for periodic equations

A first heterogeneous generalization of the Fisher-KPP reaction–diffusion equation (1.1) investigated in the last decades was the periodic reaction–diffusion equation. Assume that the coefficients are L-periodic in x, with $L > 0$, that is,

$$a(x+L) = a(x), \qquad b(x+L) = b(x), \qquad f(x+L,u) = f(x,u) \quad \text{for all } (x,u) \in \mathbb{R} \times [0,1].$$

In this case the notion of *pulsating travelling wave* has been introduced in parallel ways by Shigesada, Kawazaki and Teramoto [32] and Xin [34]. A solution u of Eq. (E) is called a pulsating travelling wave with speed $c > 0$ if for all $(t,x) \in \mathbb{R} \times \mathbb{R}$:

$$u(t + L/c, x) = u(t, x - L), \quad 0 < u < 1, \qquad \lim_{x \to -\infty} u(t,x) = 1, \qquad \lim_{x \to +\infty} u(t,x) = 0 \tag{1.4}$$

where the limits hold locally in $t \in \mathbb{R}$.

The existence of pulsating travelling waves has been proved by Xin [34] when only the diffusion a is heterogeneous in the ignition-type framework, and by Berestycki and Hamel [3] for general monostable and ignition-type equations. For periodic bistable equations, the existence of pulsating travelling waves is due to Xin when the equation is a uniform perturbation of a homogeneous equation [35], to Heinze if the equation is close to some homogenization limit [16], and to Ducrot, Giletti and Matano [9] or Fang and Zhao [10] when the only stationary solutions between 0 and 1 are assumed to be unstable.

The literature on the properties of these pulsating travelling waves is very dense and we will not describe it here since this is not the main topic of the present paper. Let us just mention that the attractivity of pulsating travelling waves has been proved by Xin [35] in the ignition-type framework and by Ducrot, Giletti and Matano [9], in a less accurate meaning, in ignition-type, monostable and bistable frameworks for Heaviside initial data and by Giletti [13] for more general initial data in the monostable framework.

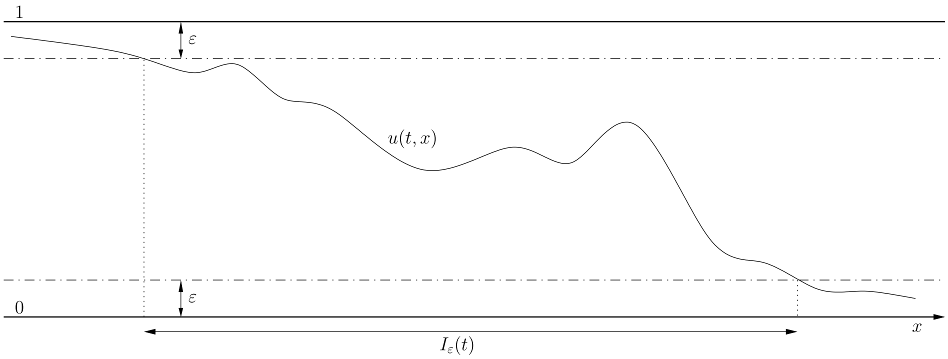


Fig. 1. The interface $I_\varepsilon(t)$ of the spatial transition wave u.

1.4. Spatial transition waves for general heterogeneous equations

1.4.1. Definition of the spatial transition waves

A generalization of the notions of planar and pulsating travelling waves to heterogeneous equations like (E) has been given by Berestycki and Hamel in [4,5].

Definition 1.1. (See [4,5].) A *spatial transition wave* (to the right) of Eq. (E) is a time-global (weak) solution $u \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R})$, with $0 < u < 1$, such that there exists a function $X : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x \to -\infty} u\big(t, x + X(t)\big) = 1 \quad \text{and} \quad \lim_{x \to +\infty} u\big(t, x + X(t)\big) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \tag{1.5}$$

Heuristically, this definition means that the solution $x \mapsto u(t, x)$ connects 0 to 1 for all t, and that the widths of the spatial interfaces $I_\varepsilon(t) = \{x \in \mathbb{R},\ \varepsilon < u(t, x) < 1 - \varepsilon\}$ are bounded with respect to $t \in \mathbb{R}$ for all $\varepsilon \in (0, 1/2)$ (see Fig. 1).

Such solutions were called generalized transition waves in [4,5]. We add the term "spatial" here because $\lim_{x \to -\infty} u(t, x) = 1$ and $\lim_{x \to +\infty} u(t, x) = 0$ for all $t \in \mathbb{R}$. This will be useful later in order to emphasize a difference with other notions of waves, see Section 3.4 below. Note that Definition 1.1 holds in a general framework: one could consider time-dependent coefficients and multidimensional equations. Several properties of spatial transition waves have been proved in [4,5]. Berestycki and Hamel showed in particular that the notion of spatial transition waves includes all the previous notions of fronts in homogeneous or periodic media, including non-trivial ones such as fronts with a speed which changes with respect to time (which have been proved to exist in [15]). Thus Definition 1.1 could be a good generalization of the notion of waves to heterogeneous equations. But such a generalization is meaningful only if one could prove that such spatial transition waves exist.

1.4.2. Existence results for spatial transition waves

The first existence result of spatial transition waves in a heterogeneous framework is due to Shen, for bistable and time-dependent equations [30]. Next, spatial transition waves for the space-heterogeneous equation (E) has been proved to exist for ignition-type equations in parallel by Nolen and Ryzhik [27] and Mellet, Roquejoffre and Sire [22] (see Section 3.2 for the definition of such equations). Then, Mellet, Nolen, Roquejoffre and Ryzhik [21] proved that these spatial transition waves are unique (up to translation in time) and stable, still for ignition-type equations. These results have been extended by Zlatos [36] to multidimensional ignition-type equations with periodic shear heterogeneities.

The existence of spatial transition waves for monostable time-dependent equations has been proved when the coefficients are assumed to be uniquely ergodic by Shen [31] and in the general framework by Rossi and the author [24]. In these two papers, the nonlinearity is assumed to be KPP, that is, it is $\mathcal{C}^1$ in $u = 0$ and $f(t, u) \leqslant f_u'(t, 0)u$ for all $(t, u) \in \mathbb{R} \times [0, 1]$. Due to this property, one can construct $X(t)$ in Definition 1.1 explicitly with respect to the coefficients and thus the spatial transition waves satisfy the same types of properties as the coefficients. This existence result has been extended to KPP equations with a time-heterogeneous reaction term and space-periodic diffusion and advection terms by Rossi and Ryzhik [29], and to general space-periodic and time-heterogeneous equations by Rossi and the author [25]. Note that in this last paper some conditions ensuring the existence of spatial transition waves for space–time general heterogeneous Fisher-KPP equations were derived, involving a global Harnack-type property for the solution of the linearized equation near the unstable equilibrium.

The existence of waves for the spatially heterogeneous equation (E) is still poorly understood when the nonlinearity is bistable, even in the periodic case. Spatial transition waves solutions of the homogeneous multidimensional equation with a convex obstacle have been constructed by Berestycki, Hamel and Matano [6]. However, in other frameworks the bistability could produce new steady states which could block the propagation between 0 and 1 [6,8,19,28]. Such non-trivial steady states are spatial transition waves with null speed, i.e. $X \equiv 0$.

1.4.3. Nonexistence results for spatial transition waves

A remaining gap was the existence of spatial transition waves for the spatially heterogeneous monostable equation (E). In this framework, a counter-example to the existence of spatial transition waves has been constructed by Nolen, Roquejoffre, Ryzhik and Zlatos [26]. These authors proved that if $a \equiv 1$ and $b \equiv 0$, if there exists $R > 0$ such that $f(x,u) = f_{\min}(u)$ for all $|x| > R$ and $u \in [0,1]$, if $f'_{\min}(0) = 1$, if $f(x,s) \leqslant f'_u(x,0)u$ for all $(x,u) \in \mathbb{R} \times [0,1]$ and if there exists a principal eigenvalue λ, defined by the existence of a positive function $\psi \in L^2(\mathbb{R})$ such that $\psi'' + f'_u(x,0)\psi = \lambda\psi$, which satisfies $\lambda > 2$, then for all time-global solution u of (E) such that $0 < u < 1$, and for all $c < \lambda/\sqrt{\lambda - 1}$, there exists $C_c > 0$ such that

$$u(t,x) \leqslant C_c e^{-|x|+ct} \quad \text{for all } (t,x) \in \mathbb{R}^- \times \mathbb{R}. \tag{1.6}$$

In particular, spatial transition waves do not exist in this framework, since any time-global solution converges to 0 as $|x| \to +\infty$ locally in time.

On the other hand, if $\lambda < 2$, then spatial transition waves exist with various global mean speeds [26]. This existence result has been extended by Zlatos [37] to Eq. (E), still under a hypothesis which is the analogous of $\lambda < 2$. Namely, assume that $a \equiv 1$, $b \equiv 0$ and $f(x,u) = c(x)u(1-u)$ in order to simplify the presentation. Let λ be the supremum of the spectrum of the operator $\frac{d^2}{dx^2} + c(x)$ and assume that $\lambda < 2\inf_{x\in\mathbb{R}} c(x)$. Then, for all $\gamma \in (\lambda, 2\inf_{\mathbb{R}} c)$, there exists a unique solution $\varphi_\gamma > 0$ of equation $\varphi''_\gamma + c(x)\varphi_\gamma = \gamma\varphi_\gamma$ in $\mathbb{R}$ such that $\varphi_\gamma(0) = 1$ and $\lim_{x\to+\infty}\varphi_\gamma(x) = 0$, and there exists a spatial transition wave u_γ of Eq. (E) which is increasing in time and such that $u_\gamma(t,x) \sim e^{\gamma t}\varphi_\gamma(x)$ when $u_\gamma(t,x) \to 0$. Zlatos' result [37] is indeed more general: it holds for heterogeneous diffusion and advection terms a and b.

However, we repeat that spatial transition waves do not exist in general because of the counter-example in [26] and that the identification of optimal conditions on the coefficients which ensure the existence of spatial transition waves is still an open problem.

Lastly, in multi-dimensional media, Zlatos [38] has recently provided a counter-example showing that spatial transition waves do not exist in general even for ignition-type nonlinearities.

1.4.4. Matano's alternative definition

Let us mention to conclude another notion of wave introduced by Matano [20]. This definition relies on a different point of view involving a translation property of the wave with respect to the environment (a,b,f). Namely, assume that the coefficients are uniformly continuous with respect to $x \in \mathbb{R}$ and define the *hull* $\mathcal{H}$ of the coefficients by

$$\mathcal{H} := cl\{(\pi_y a, \pi_y b, \pi_y f),\ y \in \mathbb{R}\}$$
$$\text{where } \pi_y a(x) := a(x+y),\ \pi_y b(x) := b(x+y) \text{ and } \pi_y f(x,u) := f(x+y,u) \tag{1.7}$$

and the closure is associated with the topology of the local convergence. The uniform continuity of the coefficients in x ensures that this set is relatively compact.

Definition 1.2. (See [20,30].) A *generalized travelling wave* (in the sense of Matano) is a continuous function $u : \mathbb{R} \times \mathbb{R} \times \mathcal{H} \to [0,1]$ such that

- for all $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$, $(t,x) \mapsto u(t,x;(\tilde{a},\tilde{b},\tilde{f}))$ is a solution of Eq. (E) with coefficients $(\tilde{a},\tilde{b},\tilde{f})$,

- there exists a function $X : \mathbb{R} \to \mathbb{R}$ such that $\lim_{x\to-\infty} u(t, x + X(t); (a, b, f)) = 1$ and $\lim_{x\to+\infty} u(t, x + X(t); (a, b, f)) = 0$ uniformly in $t \in \mathbb{R}$,
- (translation property) $u(t, x + X(t); (a, b, f)) = u(0, x; \pi_{X(t)}(a, b, f))$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$.

This last property implies in particular that if the coefficients are homogeneous/periodic, generalized travelling waves are necessarily planar/pulsating travelling waves.

The second property in Definition 1.2 means that $(t, x) \mapsto u(t, x; (a, b, f))$ is a spatial transition wave of Eq. (E). The reciprocal assertion is not true since, when the coefficients do not depend on x, then Matano's waves are necessarily planar travelling waves, while Hamel and Nadirashvili [15] proved that there exist spatial transition waves which are not planar.

The two additional properties required on u are the continuity with respect to the environment $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$ and the translation property. Shen proved in a general framework that if there exists a family of spatial transition waves $v(\cdot, \cdot; (\tilde{a}, \tilde{b}, \tilde{f}))$ for all $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$ such that the convergences in Definition 1.1 are uniform with respect to $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$, then there exists a function $u : \mathbb{R} \times \mathbb{R} \times \mathcal{H} \to [0, 1]$ which satisfies all the above properties *except the continuity* with respect to $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$ [30]. Shen noticed that u was continuous on a residual subset of $\mathcal{H}$.

It is easily checked that the translation property in Definition 1.2 is immediately satisfied if the spatial transition wave is unique (up to translation in time). Hence, in all the cases where spatial transition waves are known to exist and to be unique [21,24,30], these solutions are indeed generalized travelling waves in the sense of Matano.

As generalized travelling waves are necessarily spatial transition waves, such waves do not exist for the monostable equations with compactly supported heterogeneities considered in [26]. Hence, this alternative notion does not help to fill the non-existence gap exhibited in [26].

1.5. Scope of the paper

The aim of the present paper is to find a new generalization of the notion of wave for Eq. (E). In order to find a meaningful generalization, we want to

- prove the existence of this new notion of wave in a general setting,
- recover the earlier notions of planar/pulsating waves with minimal speed when the coefficients are homogeneous/periodic and of spatial transition waves for ignition-type equations with general heterogeneities.

Ideally, we would also like this new type of waves to give some information about the large-time behaviour of the solution of the Cauchy problem associated with Eq. (E). For example, we would like such waves to attract the solutions of the Cauchy problem associated with Heaviside-type initial data, as in homogeneous media.

We introduce the notion of critical travelling waves and prove their existence for general spatially-heterogeneous equations in Section 2. We then compare this notion and the notion of spatial transition waves in Section 3. In Section 4, we prove a translation property of critical travelling waves close to [20], from which we derive that this new notion of wave fits with earlier notions in homogenous, periodic or compactly supported heterogeneities. We also derive new results in the random stationary ergodic monostable framework, for which no notion of wave-like solution was known to exist before. We discuss the attractivity of these waves and state some open problems in this direction in Section 5. A particular example of bistable equations admitting non-trivial steady states is investigated in Section 6. Sections 7 to 11 are devoted to the proof of the results. Lastly, we give a brief summary of the results in Section 12.

2. Statement of the main results

2.1. Definition of critical travelling waves

Our generalization of the notion of wave to general spatially heterogeneous reaction–diffusion equations (E) is stated in the next definition.

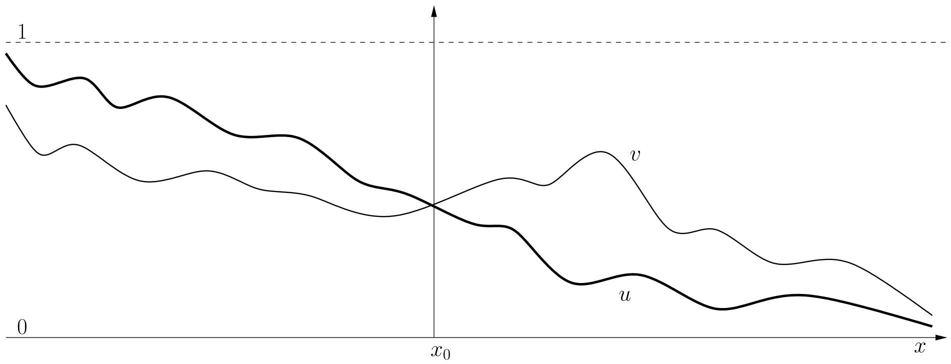


Fig. 2. The critical travelling wave u compared with a time-global solution v at $t = t_0$.

Definition 2.1. We say that a time-global (weak) solution $u \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R})$ of (E), with $0 < u < 1$, is a *critical travelling wave* (to the right) if for all $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$, if $v \in \mathcal{C}^0(\mathbb{R} \times \mathbb{R})$ is a time-global (weak) solution of (E) such that $v(t_0, x_0) = u(t_0, x_0)$ and $0 < v < 1$, then either $u \equiv v$ or

$$u(t_0, x) > v(t_0, x) \quad \text{if } x < x_0 \quad \text{and} \quad u(t_0, x) < v(t_0, x) \quad \text{if } x > x_0. \tag{2.1}$$

In Fig. 2, the critical travelling wave u converges to 1 as $x \to -\infty$ and to 0 as $x \to +\infty$. This is just one possible behaviour: it may happen that u converges to 0 as $|x| \to +\infty$ for example, as in Section 4.4 below.

Critical travelling wave to the left could be defined similarly, only by reversing the inequalities in (2.1). In the sequel, only critical travelling waves to the right will be considered.

The notion of criticality is not new. Shen introduced a similar notion in [30] but as a property of travelling waves (in the sense of Matano). She proved the existence of such waves under the assumption that there exists a particular family of spatial transition waves (see the discussion after Theorem 3.1 below). The contribution of the present paper is to use the criticality property as a definition for a new notion of wave. As we require a comparison in the class of time-global solutions instead of that of travelling waves like in [30], our definition is meaningful independently of the existence of travelling waves. Many properties, in particular existence, will be derived from this simple definition. We will prove in Section 4 that if the coefficients are homogeneous/periodic, then the critical travelling wave is a planar/pulsating travelling wave with minimal speed. Hence, the notion of criticality is, somehow, the generalization to heterogeneous equations (E) of the minimality of the speed.

A similar notion, called "steepness", has also been introduced recently by Ducrot, Giletti and Matano [9] in order to investigate large time behaviour of the solution of Cauchy problem for periodic reaction–diffusion equations with Heaviside initial data. These authors used the fact that such solutions are always "steeper" than any entire solution in order to prove that they converge to particular solutions called propagating terraces in [9], which could be identified as pulsating fronts in several particular cases. Our technical approach is close, but we are focused in the present paper on entire solutions more than initial value problems (even if this question will be addressed in Section 5 below), and we aim at handling general heterogeneous reaction–diffusion equations, without any periodicity assumption.

2.2. Existence, uniqueness and monotonicity of critical travelling waves in a general setting

We are now in position to state our main result.

Theorem 2.2. *Assume that (H) is satisfied.*

1. *(Existence and uniqueness) For all $\theta \in (0, 1)$ and $x_0 \in \mathbb{R}$, Eq. (E) admits a unique critical transition wave u such that $u(0, x_0) = \theta$.*
2. *(Monotonicity in time) $t \mapsto u(t, x)$ is either decreasing for all $x \in \mathbb{R}$, increasing for all $x \in \mathbb{R}$ or constant for all $x \in \mathbb{R}$.*
3. *(Monotonicity in space) If f does not depend on x, then $x \mapsto u(t, x)$ is nonincreasing for all $t \in \mathbb{R}$.*

One can check that Definition 2.1 and Theorem 2.2 still hold if one considers space–time heterogeneous reaction–diffusion equations. However, as all the applications we provide in the present paper involve time-independent coefficients, we chose to present all the results in the spatially heterogeneous framework.

We underline that critical travelling waves exist for all reaction–diffusion equations, even when spatial transition waves do not exist as in [26], whatever the stabilities of the steady states 0 and 1 are and for general heterogeneous coefficients.

Definition 2.1 only fits to dimension 1. If one considers multidimensional versions of Eq. (E), then it is not clear at all whether there exist good generalizations of Definition 2.1 and Theorem 2.2 or not. This might help to understand why most of the known existence results of spatial transition waves hold in dimension 1.

The monotonicity in time of the critical travelling waves yields that the limits $u_\pm(x) := \lim_{t\to\pm\infty} u(t,x)$ always exist and are steady states of Eq. (E). Of course these limits are not necessarily $u \equiv 0$ or $u \equiv 1$ and may depend on the normalization $u(0,x_0) = \theta$. This may be the case in particular if non-trivial steady states exist as illustrated in Section 6 below. Hence, critical travelling waves can always be viewed as temporal heteroclinic connections between steady states.

3. Comparison with spatial transition waves

3.1. Comparison in the general framework

Let now consider equations which admit a spatial transition wave and investigate the properties of critical travelling waves in this framework.

Theorem 3.1. *Assume that Eq. (E) admits a spatial transition wave v (in the sense of Definition 1.1) such that $\lim_{t\to-\infty} v(t,x) = 0$ and $\lim_{t\to+\infty} v(t,x) = 1$ for all $x \in \mathbb{R}$. Then any critical transition wave u of Eq. (E) is a spatial transition wave and one has $\lim_{t\to-\infty} u(t,x) = 0$ and $\lim_{t\to+\infty} u(t,x) = 1$ for all $x \in \mathbb{R}$.*

Hence, if there exists a spatial transition wave which connects 0 to 1 in time, then critical travelling waves are spatial transition waves. However, the two notions are not necessarily equivalent. First, critical travelling waves always exist, unlike spatial transition waves. Second, some spatial transition waves are not critical travelling waves (see the discussion in Section 4.2 below).

This result is close to Theorem A in Shen's paper [30]. She proved that if there exists a so called "wave-like solution" (in the sense of Definition 2.3 in [30]) then there exists a travelling wave solution which is critical (in the sense of Definition 2.2 in [30]). The notion of criticality we use here is very close from Shen's one, except that in Shen's paper the criticality is only related to comparison with respect to other travelling waves, unlike in the present paper where all the time-global solutions are involved. This enables us to use the criticality notion as a definition of critical travelling waves, unlike Shen who considered criticality as a property of particular spatial transition waves, which are thus required to exist in [30]. The other difference with Shen's result lies in her definition of "wave-like solutions". Such solutions are spatial transition wave such that the convergences in Definition 1.1 are uniform with respect to translations of the coefficients. We do not require such a uniformity in Theorem 3.1: we consider the classical notion of spatial transition wave (in the sense of Berestycki–Hamel).

Let now consider particular classes of nonlinearities for which further links between the two notions can be proved.

3.2. The case of ignition-type equations: equivalence between the two notions

Consider ignition-type equations in the sense of [21]:

$$a \equiv 1, \qquad b \equiv 0, \qquad f(x,u) = g(x) f_0(u),$$

g is uniformly bounded and Lipschitz-continuous over $\mathbb{R}$ and $\inf_{\mathbb{R}} g > 0$,

f_0 is Lipschitz-continuous and there exists $\theta_0 \in (0,1)$ such that

$$f_0(s) = 0 \quad \text{for } s \in [0,\theta_0], \qquad f_0(1) = 0, \qquad f_0(s) > 0 \quad \text{for } s \in (\theta_0, 1), \qquad f_0'(1) < 0. \tag{3.1}$$

Under these assumptions, spatial transition waves are known to exist and to be unique up to translation in time.

Theorem 3.2. *(See [21,22,27].) Assume that (3.1) is satisfied. Then (E) admits a spatial transition wave v. Moreover, v is increasing in time and if $\tilde{v}$ is another spatial transition wave, then there exists $\tau > 0$ such that $v(t,x) = \tilde{v}(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$.*

We just mentioned here the main properties of these spatial transition waves and we refer to [21,22,27] for further results.

In this framework, Theorems 3.1 and 3.2 yield that the notions of spatial transition waves and critical travelling waves are equivalent.

Corollary 3.3. *Assume that (3.1) is satisfied. Then a solution u of (E), with $0 < u < 1$, is a spatial transition wave if and only if it is a critical travelling wave. Consequently, critical travelling waves are increasing with respect to t and if u and $\tilde{u}$ are two critical travelling waves of Eq. (E), then there exists $\tau \in \mathbb{R}$ such that $u(t,x) = \tilde{u}(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$.*

We underline that this result does not trivially follow from the uniqueness of spatial transition waves proved in [21]. A new property of the spatial transition wave is proved: it is critical. Hence, Corollary 3.3 provides a new characterization of spatial transition waves in this framework.

Note that the difference between the uniqueness results of Theorem 2.2 and that of Corollary 3.3 is that in the corollary we do not assume that $\tilde{u}(0,x_0) = \theta$. This is why we get a uniqueness result in Corollary 3.3 up to translation in time.

3.3. The case of monostable equations: minimization of the least mean speed

The notion of monostability we will use in this paper is the following.

Definition 3.4. We say that Eq. (E) is *monostable* if for all continuous function $u_0 \not\equiv 0$, with $0 \leqslant u_0 \leqslant 1$, if $u \in \mathcal{C}^0([0,\infty)\times\mathbb{R})$ is the (weak) solution of

$$\begin{cases} \partial_t u - a(x)\partial_{xx}u - b(x)\partial_x u = f(x,u) & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{for all } x \in \mathbb{R}, \end{cases} \tag{3.2}$$

then $\lim_{t\to+\infty} u(t,x) = 1$ locally uniformly in $x \in \mathbb{R}$.

In other words, any perturbation of 0 converges to 1, meaning that 0 is unstable while 1 is globally attractive. This implies in particular that there exists no non-trivial steady state between 0 and 1.

It immediately follows from the parabolic maximum principle that Eq. (E) is monostable in the case where $a \equiv 1$, $b \equiv 0$ and there exists a Lipschitz-continuous function $f_{\min} : [0,1] \to \mathbb{R}$ such that

$$\begin{aligned} &f_{\min}(0) = f_{\min}(1) = 0, \qquad f_{\min}(s) > 0 \quad \text{for all } s \in (0,1), \qquad f'_{\min}(0) > 0, \\ &f(x,u) \geqslant f_{\min}(u) \quad \text{a.e. } (x,u) \in \mathbb{R}\times[0,1]. \end{aligned} \tag{3.3}$$

We refer to [7] for more general conditions on the coefficients guaranteeing the monostability of the equation. In particular, the results of [7] yield that the equation associated with $b \equiv 0$, f as in (3.3) and an arbitrary a (satisfying (H)) is monostable.

First, in this framework, critical travelling waves are time-increasing and unique up to translation in time, as in the ignition-type setting.

Proposition 3.5. *Assume that (H) is satisfied and that Eq. (E) is monostable in the sense of Definition 3.4. Let $\theta \in (0,1)$ and u be a critical travelling wave of Eq. (E). Then u is increasing in time and there exists a unique continuous function $T : \mathbb{R} \to \mathbb{R}$ such that $u(T(x),x) = \theta$ for all $x \in \mathbb{R}$. Moreover, if $\tilde{u}$ is another critical travelling waves of Eq. (E), then there exists $\tau \in \mathbb{R}$ such that $\tilde{u}(t,x) = u(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$.*

Note that this result holds in the general monostable framework: it does not depend on the existence of spatial transition waves.

Such a uniqueness up to translation does not hold in general. The shape of critical travelling waves may depend on their normalizations (see Section 6 below).

Next, assume that Eq. (E) is monostable and admits a spatial transition wave (which is not always true, see [26]). Together with their definition of spatial transition waves, Berestycki and Hamel [4,5] gave a generalization of the notion of speed. If v is a spatial transition wave of Eq. (E), let $\theta \in (0,1)$ and define for all $t \in \mathbb{R}$, $Y(t) := \sup\{x \in \mathbb{R}, v(t,x) \geqslant \theta\}$. Then

$$\text{if} \quad \lim_{t \to +\infty} \frac{Y(s+t) - Y(s)}{t} = c \quad \text{exists uniformly in } s \in \mathbb{R},$$

we call c the *global mean speed* of the spatial transition wave v. It is easily checked that this quantity does not depend on θ. Of course spatial transition waves do not always admit global mean speeds. This is why we are led to introduce another quantity here. Namely, with the same notations as above, let

$$\underline{c} := \liminf_{t \to +\infty} \inf_{s \in \mathbb{R}} \frac{Y(s+t) - Y(s)}{t}.$$

We call $\underline{c}$ the *least mean speed* of the spatial transition wave v. This quantity is always well-defined in $[-\infty, \infty)$ and it follows from the definition of spatial transition waves that it does not depend on θ. Of course if v admits a global mean speed c, then c is also the least mean speed of v.

A similar quantity has been introduced by Rossi and the author in [24]. They proved that, in the framework of time-dependent monostable equations, there exists an explicit threshold c^* such that spatial transition waves with least mean speed $\underline{c}$ exist for all $\underline{c} > c^*$ and do not exist if $\underline{c} < c^*$. This result generalized the classical existence results in monostable homogeneous/periodic equations, where planar/pulsating travelling waves with speed c exist if and only if $c \geqslant c^*$. Hence, least mean speed seems to be an appropriate quantity in order to compute existence threshold in monostable heterogeneous equations.

Let now turn back to the spatially heterogeneous equation (E).

Theorem 3.6. *Assume that Eq. (E) is monostable in the sense of Definition 3.4 and admits a spatial transition wave solution v. Assume in addition that $s \mapsto f(x,s)/s$ is nonincreasing for all $x \in \mathbb{R}$. Let $\theta \in (0,1)$, u be a critical travelling wave of Eq. (E) and*

$$X(t) := \sup\{x \in \mathbb{R},\ u(t,x) > \theta\} \quad \text{and} \quad Y(t) := \sup\{x \in \mathbb{R},\ v(t,x) > \theta\}.$$

Then there exists $L > 0$ such that

$$\forall t > 0, \quad \inf_{s \in \mathbb{R}} \big(X(s+t) - X(s)\big) \leqslant \inf_{s' \in \mathbb{R}} \big(Y(s'+t) - Y(s')\big) + L.$$

Therefore, the least mean speed of the critical travelling wave u is smaller than the least mean speed of any spatial transition wave v.

We remind to the reader that $X(t)$ is well-defined since the critical travelling wave u is a spatial transition wave by Theorem 3.1.

Theorem 3.6 means that, in the monostable framework, critical travelling waves generalize the notion of waves with minimal speed. The monotonicity hypothesis on $s \mapsto f(x,s)/s$ is indeed quite strong and we do not know if this result holds in a more general setting. If we assume the coefficients to be homogeneous or periodic, then we will check in Propositions 4.2 and 4.3 that this result still holds without any monotonicity hypothesis on $s \mapsto f(x,s)/s$, under the additional assumption that the principal eigenvalue associated with the linearization near $u = 1$ is negative. This assumption is not milder or stronger than the monotonicity hypothesis on $s \mapsto f(x,s)/s$, it is just different.

3.4. Spatial/temporal connections between steady states

Let conclude this section with some comments on the definition of spatial transition waves in the sense of Berestycki and Hamel [4,5]. The convergences in Berestycki–Hamel's Definition 1.1 are convergences as $x \to \pm\infty$. Hence, spatial transition waves are connections in x between two steady states. Definition 1.1 does not involve any convergences as $t \to \pm\infty$. This has very important consequences.

For example, assume that there exists a steady state $w = w(x)$ of Eq. (E) such that $0 < w < 1$, $w(-\infty) = 1$ and $w(+\infty) = 0$. Such a situation typically arises in bistable equations, even simple ones, see [19,28,35] and Section 6 for example. Then $w = w(x)$ is a spatial transition wave with global mean speed 0 (with $X \equiv 0$), but it does not converge to 0 or 1 as $t \pm \infty$ since it does not depend on time. Moreover, it could block the convergence to 1 as $t \to +\infty$ of the solution of the Cauchy problem associated with front-like initial data. This is why in the literature some authors [19,35] consider that the existence of such non-trivial steady states proves that travelling waves do not exist, although such steady states are spatial transition waves according to Berestycki–Hamel's definition.

On the other hand, some monostable equations have been constructed by Nolen, Roquejoffre, Ryzhik and Zlatos [26], for which any time-global solution u satisfies $\lim_{|x|\to+\infty} u(t,x) = 0$ for all $t \in \mathbb{R}$, $\lim_{t\to-\infty} u(t,x) = 0$ and $\lim_{t\to+\infty} u(t,x) = 1$ for all $x \in \mathbb{R}$. Hence, spatial transition waves do not exist, but any time-global solution, in particular critical travelling waves, is a temporal connection between 0 and 1.

These two examples show that when one investigates the existence and properties of wave-like solutions of (E), that is, heteroclinic connections between two steady states, one could be led to different conclusions depending upon the nature of the connection: is this a connection in space or in time? This is why we use the name "spatial transition wave" in the present paper instead of "generalized transition waves" as in the original articles [4,5].

Note that our definition of critical travelling waves does not involve any convergence to the steady states, which enables us to go beyond the difficulties described above. Indeed, Definition 2.1 yields that if there exists a spatial connection between the steady states, then any critical travelling wave converges to these steady states as $x \to \pm\infty$. On the other hand, as critical travelling waves are monotonic in time, they are always temporal connections between steady states (which may be constant in time).

4. Identification and properties of critical travelling waves for particular classes of heterogeneities

The aim of this section is to identify critical travelling waves when particular structural dependences of the coefficients of Eq. (E) in x are prescribed. We will first prove that if the coefficients are homogeneous/periodic, then the critical travelling waves are planar/pulsating travelling waves (with minimal speed in the monostable framework). If the heterogeneity is compactly supported and if the equation is monostable, then the critical travelling wave is either a spatial transition wave with minimal speed, or a bump-like solution in the sense of [21] if spatial transition waves do not exist. Lastly, if the coefficients are random stationary ergodic variables and the equation is almost surely monostable, then we prove that the critical travelling wave depends in a random stationary ergodic way, in a sense, on the environment. Most of these results rely on a translation property of critical travelling waves.

4.1. The translation property

We are interested here in proving some analogous of the translation property introduced by Matano (see Definition 1.2).

Proposition 4.1. *Assume that (H) is satisfied and let $\theta \in (0,1)$. For all $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$, let $u(\cdot,\cdot;(\tilde{a}, \tilde{b}, \tilde{f})) = u(t,x;(\tilde{a}, \tilde{b}, \tilde{f}))$ the solution of (E) associated with the coefficients $(\tilde{a}, \tilde{b}, \tilde{f})$ constructed in Theorem 2.2 and normalized by $u(0,0;(\tilde{a}, \tilde{b}, \tilde{f})) = \theta$. Assume that $\lim_{t\to-\infty} u(t,x;(a,b,f)) = 0$ and $\lim_{t\to+\infty} u(t,x;(a,b,f)) = 1$ for all $x \in \mathbb{R}$.*

Then there exists a unique point $T(y) \in \mathbb{R}$ such that $u(T(y), y; (a,b,f)) = \theta$ for all $y \in \mathbb{R}$ and one has for all $(t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$:

$$u\big(t + T(y), x + y; (a,b,f)\big) = u\big(t, x; (\pi_y a, \pi_y b, \pi_y f)\big). \tag{4.1}$$

Note that the hypothesis $\lim_{t\to-\infty} u(t,x;(a,b,f)) = 0$ and $\lim_{t\to+\infty} u(t,x;(a,b,f)) = 1$ is at least checked for monostable and ignition-type equations and implies that u is time-increasing. If the limits are reversed, that is, if $\lim_{t\to-\infty} u(t,x;(a,b,f)) = 1$ and $\lim_{t\to+\infty} u(t,x;(a,b,f)) = 0$, then the results still hold (just let $v := 1 - u$). If the critical travelling wave is time-independent or does not connect 0 to 1, then Proposition 4.1 does not hold anymore, as emphasized by the example investigated in Section 6.

The difference with Matano's translation property of Definition 1.2 is that here we translate in time instead of space, which is natural since, as already underlined, critical travelling waves are somehow temporal transitions.

4.2. Homogeneous equations

Let first consider coefficients which do not depend on x. Up to some well-chosen change of variables, one can always assume that $a \equiv 1$ and $b \equiv 0$.

Proposition 4.2. *Assume that $a \equiv 1$, $b \equiv 0$ and $f = f(u)$ is a Lipschitz-continuous function which does not depend on x such that $f(0) = f(1) = 0$. Then*

- *if there exists $\theta_0 \in [0, 1)$ such that $f(u) \leqslant 0$ when $u \in [0, \theta_0]$, $f(u) > 0$ when $u \in (\theta_0, 1]$ and $\int_0^1 f(u)du > 0$, then the critical travelling waves are unique up to translation in time and are planar travelling waves (in the sense of (1.2)),*
- *moreover, if $\theta_0 = 0$, $f'(1) < 0$ and if we write the critical travelling wave $u(t, x) = U(x - ct)$, then there exists no planar travelling wave with speed $c' < c$.*

Heuristically, this result means that if 0 is less stable than 1 (that is, when $\int_0^1 f(u)du > 0$), then we recover the classical notion of planar travelling waves. If this condition is not met, for example if $f(u) = u(1 - u)(u - 1/2)$, then such a result still holds except that one would get a uniqueness up to translation in space instead of time in general. The second part of the result means that, in the monostable framework ($\theta_0 = 0$), if $f'(1) < 0$, critical travelling waves are a planar travelling wave with minimal speed.

Note that in the monostable framework, there exist many other wave-like solutions: planar travelling waves with speed $c' > c$ and even non-planar travelling waves with a changing speed (see [15]). Such waves are spatial transition waves in the sense of Definition 1.1 but not critical travelling waves. Hence, the notion of critical travelling wave is not more general than the notion of spatial transition wave. One can just claim that critical travelling waves always exist, while spatial transition waves do not for heterogeneous equations (see [26]), but on the other hand some wave-like solutions are spatial transition waves but not critical travelling waves. This is why critical travelling wave is only a good generalization of the notion of waves *with minimal speed* in the monostable framework.

Lastly, Proposition 4.2 gives a new geometrical characterization of the waves with minimal speed in the homogeneous framework: theses waves are necessarily critical travelling waves in the sense of Definition 2.1, which is a new result of independent interest.

4.3. Periodic heterogeneity

Assume now that the coefficients are periodic. That is, there exists $L > 0$ such that

$$a(x + L) = a(x), \qquad b(x + L) = b(x) \quad \text{and} \quad f(x + L, u) = f(x, u) \quad \text{for all } (x, u) \in \mathbb{R} \times [0, 1].$$

Assume that $f = f(x, u)$ is differentiable at $u = 1$. The statement of the results will involve the elliptic operator $\mathcal{L}_1$ associated with the linearization near the stable steady state $u = 1$, defined for all $\varphi \in \mathcal{C}^2(\mathbb{R})$ by

$$\mathcal{L}_1\varphi = a(x)\varphi'' + b(x)\varphi' + f_u'(x, 1)\varphi.$$

As its coefficients are periodic and bounded, this operator admits a unique periodic principal eigenvalue, that is, a unique μ associated with a periodic function $\varphi \in W^{2,\infty}(\mathbb{R}) > 0$ such that $\mathcal{L}_1\varphi = \mu\varphi$. Note that if $f_u'(x, 1)$ does not depend on x, then μ is just the constant function $f'(1)$.

Proposition 4.3. *Assume that (H) is satisfied, that the coefficients are periodic and that f is differentiable at $u = 1$.*

- *If Eq. (E) is monostable in the sense of Definition 3.4, then a critical travelling wave of Eq. (E) is a pulsating travelling wave with speed c. Moreover, if $(x, u) \mapsto (a(x), b(x), f(x, u))$ is of class $\mathcal{C}^{1,\gamma}(\mathbb{R} \times [0, 1])$ for some $\gamma \in (0, 1)$ and if $\mu < 0$, then for all $c' < c$, there exists no pulsating travelling wave with speed c'.*
- *If $(x, u) \mapsto (a(x), b(x), f(x, u))$ is of class $\mathcal{C}^{2,\gamma}(\mathbb{R} \times [0, 1])$ for some $\gamma \in (0, 1)$, $b \equiv \partial_x a$, there exists $\beta \in (0, 1)$ such that $u \in (1 - \beta, 1] \mapsto f(x, u)$ is nonincreasing for all $x \in \mathbb{R}$ and there exists $\theta_0 \in (0, 1)$ such that*

$$\forall (x, u) \in \mathbb{R} \times [0, \theta_0], \quad f(x, u) = 0 \quad \text{and} \quad \forall u \in (\theta_0, 1), \quad \max_{x \in \mathbb{R}} f(x, u) > 0,$$

then critical travelling waves are unique up to translation in time and are pulsating travelling waves of speed c^*. Moreover, there exists no pulsating travelling wave of speed $c \neq c^*$.

It is easy to check that the monostability hypothesis ensures that $\mu \leqslant 0$. Even if it is not involved in the existence of pulsating travelling waves, hypothesis $\mu < 0$ ensures the uniqueness, monotonicity or exponential decay of these waves (see [3] for example). If $\mu = 0$, there are still many open questions stated in the literature. We do not know if the critical travelling wave is the pulsating travelling wave with minimal speed if $\mu = 0$.

We will prove in Section 5 below that, if the coefficients are periodic and if the equation is monostable, then the critical travelling wave attracts the solution of the Cauchy problem associated with Heaviside-type initial data. This result does not involve any hypothesis like $\mu < 0$. In other words, in the case $\mu = 0$, if the critical transition wave was not the pulsating travelling wave with minimal speed, it would still be attractive. Hence, from the point of view of attractivity, criticality is a more relevant notion than being of minimal speed.

The hypotheses of the second part of Proposition 4.3 mean that Eq. (E) is of ignition-type and in the divergence form. Under these hypotheses, the existence and uniqueness of pulsating travelling waves have been proved in [3]. Our contribution is the identification of critical travelling waves in this framework.

Lastly, we leave possible extensions to bistable equations using [9,10,16,34] to the reader.

4.4. Compactly supported heterogeneity

We consider in this section the same type of equation as in [26]. In other words, we assume that $a \equiv 1$, $b \equiv 0$, f is a uniformly Lipschitz-continuous function over $\mathbb{R} \times [0, 1]$ such that $f(x, 0) = f(x, 1) = 0$ for all $x \in \mathbb{R}$ and

$$\begin{aligned}
&f'_u(x,0) \text{ exists} \quad \text{and} \quad f(x,u) \leqslant f'_u(x,0)u \quad \text{for all } (x,u) \in \mathbb{R}\times[0,1],\\
&\exists C,\delta>0 \quad \text{such that} \quad f(x,u) \geqslant f'_u(x,0)u - Cu^{1+\delta} \quad \text{for all } (x,u)\in\mathbb{R}\times[0,1],\\
&f'_u(\cdot,0) \text{ is continuous} \quad \text{and} \quad \inf_{x\in\mathbb{R}} f'_u(x,0) > 0,\\
&\exists R>0,\ \forall |x|>R, \quad f'_u(x,0)=1.
\end{aligned} \tag{4.2}$$

We also assume that the supremum λ of the spectrum of the operator $\partial_{xx} + f'_u(x, 0)$ is strictly larger than 1. Due to (4.2), it is equivalent to assume that

$$\exists \lambda \in (1,\infty),\ \exists \psi \in L^2(\mathbb{R}),\ \psi > 0 \quad | \quad \psi'' + f'_u(x,0)\psi = \lambda\psi \quad \text{in } \mathbb{R}. \tag{4.3}$$

Under these hypotheses, Nolen, Roquejoffre, Ryzhik and Zlatos [26] proved that such spatial transition waves exist for a given range of speeds if $\lambda > 2$ and do not exist if $\lambda < 2$. Moreover, they proved that another class of time-global solutions, that they called *bump-like solutions*, is always non-empty.

Theorem 4.4. *(See [26].) Assume that (4.2) and (4.3) hold.*

1. *If $\lambda > 2$, then any time-global solution v of (E) such that $0 < v < 1$ satisfies (with $C_c > 0$) $v(t, x) \leqslant C_c e^{-|x|+ct}$ for any $c < \lambda/\sqrt{\lambda - 1}$ and $(t, x) \in (-\infty, 0) \times \mathbb{R}$. In particular, no spatial transition wave exists.*
2. *If $\lambda \in (1, 2)$, then for all $c \in (2, \lambda/\sqrt{\lambda - 1})$, Eq. (E) admits a spatial transition wave with global mean speed c. If in addition $x \mapsto f'_u(x, 0)$ is even, then there exists no spatial transition wave with global mean speed $c > \lambda/\sqrt{\lambda - 1}$.*
3. *For all $\lambda > 1$, if there exists $\theta_0 \in (0, 1)$ such that $f(x, u) = f'_u(x, 0)u$ for all $(x, u) \in \mathbb{R} \times [0, \theta_0]$, then there exists a solution v of Eq. (E) such that $0 < v < 1$ and $v(t, \cdot) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$. If in addition $\lambda > 2$, then there exists a unique (up to translation in time) time-global solution v such that $0 < v < 1$.*

Let now identify the critical transition waves in this framework.

Proposition 4.5. *Assume (4.2)–(4.3) and $f'_u(x, 0) \geqslant 1$ for all $x \in \mathbb{R}$. Let u be a critical travelling wave.*

1. *If $\lambda > 2$, then $u(t, \cdot) \in L^1(\mathbb{R})$ for all $t \in \mathbb{R}$.*

2. *If $\lambda \in (1,2)$ and if $s \mapsto f(x,s)/s$ is nonincreasing for all $x \in \mathbb{R}$, then u is a spatial transition wave with global mean speed $c = 2$.*

In other words, if $\lambda > 2$, then the critical travelling waves are bump-like solutions, while if $\lambda \in (1,2)$, critical travelling waves are spatial transition waves with minimal speed. We underline that the existence of spatial transition waves with global mean speed $c = 2$ is a new result: only the existence of spatial transition waves with global mean speed $c > 2$ was proved in [26].

4.5. Random stationary ergodic heterogeneities

We consider here reaction–diffusion equations with random coefficients

$$\partial_t u - a(x,\omega)\partial_{xx} u - b(x,\omega)\partial_x u = f(x,\omega,u). \tag{4.4}$$

The functions $a : \mathbb{R} \times \Omega \to (0,\infty)$, $b : \mathbb{R} \times \Omega \to \mathbb{R}$ and $f : \mathbb{R} \times \Omega \times [0,1] \to \mathbb{R}$ are random variables defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$. We assume that the coefficients are random stationary ergodic functions with respect to x. Namely, there exists a group $(\pi_x)_{x\in\mathbb{R}}$ of measure-preserving transformations of Ω such that for all $(x,y,\omega,u) \in \mathbb{R} \times \mathbb{R} \times \Omega \times [0,1]$:

$$a(x+y,\omega) = a(x,\pi_y\omega), \qquad b(x+y,\omega) = b(x,\pi_y\omega), \qquad f(x+y,\omega,u) = f(x,\pi_y\omega,u),$$

and for all $A \in \mathcal{F}$, if $\pi_x A = A$ for all $x \in \mathbb{R}$, then $\mathbb{P}(A) = 0$ or 1.

The case of ignition-type equations with random stationary ergodic equations has been addressed by Nolen and Ryzhik [27].

Theorem 4.6. *(See [27].) Assume that $a \equiv 1$, $b \equiv 0$ and $f(x,\omega,u) = g(x,\omega) f_0(u)$, where f_0 is of ignition-type and $x \mapsto g(x,\omega)$ is a uniformly Lipschitz-continuous and bounded function, with $\inf_{x\in\mathbb{R}} g(x,\omega) > 0$, for almost every $\omega \in \Omega$. Then there exists a measurable function $u : \mathbb{R} \times \mathbb{R} \times \Omega \to [0,1]$ such that*

- *for almost every $\omega \in \Omega$, $(t,x) \mapsto u(t,x,\omega)$ is a spatial transition wave of Eq. (4.4) which is increasing in t,*
- *if $X(t,\omega)$ is defined by $u(t, X(t,\omega), \omega) = \theta_0$, then X is increasing in t, measurable in ω and*

$$\text{the limit } c_{\theta_0} = \lim_{t\to+\infty} \frac{X(t,\omega)}{t} \text{ exists almost surely and is deterministic,}$$

- *$u(t,x,\omega) = u(0, x - X(t,\omega), \pi_{X(t,\omega)}\omega)$ for almost every $(t,x,\omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$.*

Such a family of solutions is called a *random travelling wave* in [27,30]. Moreover, the speed c_{θ_0} is the spreading speed associated with compactly supported initial data (see [27]).

When the equation is monostable almost surely in $\omega \in \Omega$, the existence of random travelling waves is still a fully open problem. Nevertheless, we know that for almost every $\omega \in \Omega$, there exists a critical travelling wave $(t,x) \mapsto u(t,x,\omega)$ and this solution satisfies properties related to the stationary ergodicity of the equation, as stated in the next result.

Proposition 4.7. *Assume that for almost every $\omega \in \Omega$, Eq. (4.4) (where ω is fixed) is monostable in the sense of Definition 3.4 and satisfies (H). Take $\theta \in (0,1)$. For almost every $\omega \in \Omega$, let $u : (t,x) \mapsto u(t,x,\omega)$ the critical travelling wave of Eq. (4.4) normalized by $u(0,0,\omega) = \theta$ and define $T : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $u(T(x,\omega), x, \omega) = \theta$ for all $(x,\omega) \in \mathbb{R} \times \Omega$. Then*

- *$\omega \mapsto u(t,x,\omega)$ and $\omega \mapsto T(x,\omega)$ are measurable for all $(t,x) \in \mathbb{R} \times \mathbb{R}$,*
- *$u(t + T(y,\omega), x + y, \omega) = u(t,x,\pi_y\omega)$ and $T(x+y,\omega) = T(y,\omega) + T(x,\pi_y\omega)$ for all $(t,x,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and almost every $\omega \in \Omega$,*
- *the limit*

$$c^* = \lim_{x\to+\infty} \frac{x}{T(x,\omega)} \text{ exists almost surely and is deterministic.}$$

As $T(x,\omega)$ is the localization of the temporal interface of the wave u between 0 and 1, the quantity c^* can be viewed as the propagation speed of u.

This result does not solve the problem of the existence of random travelling waves in the monostable framework. However, it shows that there exists a family of wave-like solutions satisfying some random stationarity property, which may be a random travelling wave.

5. Attractivity of critical travelling waves along a subsequence for recurrent at infinity coefficients

This section investigates the attractivity of critical travelling waves. Recall that in homogeneous media, the solution of the Cauchy problem associated with a Heaviside initial datum is attracted, in a sense (see the Introduction), by the planar travelling wave with minimal speed [17], which is indeed the critical travelling wave in this framework.

We were not able to fully extend this result to general heterogeneous framework. However, we proved that such an attractivity holds along a subsequence, if the coefficients are recurrent at infinity.

Definition 5.1. We say that a uniformly continuous function $g:\mathbb{R}\to\mathbb{R}$ is *recurrent at infinity* if there exists a sequence $(y_n)_n$ such that $\lim_{n\to+\infty} y_n=+\infty$ and $g(x)=\lim_{n\to+\infty} g(x+y_n)$ locally uniformly in $x\in\mathbb{R}$.

Heuristically, this notion means that the structure of g is repeated along a sequence of translations which diverge to $+\infty$. It is easy to check that periodic and almost periodicity functions are recurrent at infinity. Typical function that is not recurrent at infinity is compactly supported ones: if $g(x)=0$ when x is large enough and $g\not\equiv 0$, then for all sequence $(y_n)_n$ such that $\lim_{n\to+\infty} y_n=+\infty$, one has $\lim_{n\to+\infty} g(x+x_n)\equiv 0$ locally in x, which contradicts the recurrence at infinity by taking $x_n=0$ for all n.

Theorem 5.2. *Assume that a, b and $f(\cdot,u)$ are uniformly continuous and recurrent at infinity for all $u\in[0,1]$, that (H) is satisfied and that Eq. (E) is monostable. Let v be the solution of the Cauchy problem*

$$\begin{cases}\partial_t v-a(x)\partial_{xx}v-b(x)\partial_x v=f(x,v) & \text{in } (0,\infty)\times\mathbb{R},\\ v(0,x)=\begin{cases}1 & \text{if } x\leqslant 0,\\ 0 & \text{if } x>0\end{cases} & \text{for all } x\in\mathbb{R}.\end{cases} \tag{5.1}$$

Let $\theta\in(0,1)$ and

$$S(y):=\sup\{t>0,\ v(t,y)\leqslant\theta\}\quad\text{for all } y>0.$$

Then there exists a sequence $(y_n)_n$ such that $\lim_{n\to+\infty} y_n=+\infty$ and

$$v\big(S(y_n),x+y_n\big)-u\big(T(y_n),x+y_n\big)\to 0\quad\text{as } n\to+\infty \text{ locally in } x\in\mathbb{R}, \tag{5.2}$$

where u is the unique critical travelling wave normalized by $u(0,0)=\theta$ and T is uniquely defined by $u(T(y),y)=\theta$ for all $y\in\mathbb{R}$.

We do not know if this attractivity holds along any sequence $(y_n)_n$ converging to $+\infty$ and if more general heterogeneities could be handled. Some partial results in this direction are stated in Section 10. In particular, we prove that there is a strong link between the attractivity and the continuity of critical travelling waves with respect to the coefficients.

If the nonlinearity is periodic, then we can improve this result and get the full convergence instead of the convergence along a subsequence.

Proposition 5.3. *Assume that a, b and f are L-periodic in x, with $L>0$, that (H) is satisfied and that Eq. (E) is monostable in the sense of Definition 3.4. Then, with the same notations as in Theorem 5.2, one has*

$$\lim_{y\to+\infty}\Big(v\big(S(y),x+y\big)-u\big(T(y),x+y\big)\Big)=0\quad\text{locally in } x\in\mathbb{R}. \tag{5.3}$$

We recall that this result has been proved in a parallel work by Ducrot, Giletti and Matano [9]. Note that we do not need any hypothesis involving the linearization of Eq. (E) near the steady state $u = 1$. Hence, Proposition 4.3 above yields that the critical travelling wave is a pulsating travelling wave but we do not know if its speed is the minimal speed of such waves. However, Proposition 5.3 ensures that the critical travelling wave is always attractive.

A natural open problem is the extension of Theorem 5.2 to more general wave-like initial data. We recall here that Giletti [13] proved that such an extension holds for periodic monostable equations.

6. Wave-blocking phenomena and critical travelling waves

Several papers [6,8,19,28] observed in various framework that heterogeneous bistable equations might admit non-trivial stationary solutions. In this case the monotonicity and the convergences as $t \to \pm\infty$ of the critical travelling waves will strongly depend on the normalization of the wave. Typically, these non-trivial steady states could be critical travelling waves (see Proposition 11.1 below for a result in this direction in a general bistable framework).

We will now focus on an example investigated in [19] in order to illustrate this phenomenon. Consider

$$\partial_t u - \partial_{xx} u = f(x,u) = \begin{cases} f_0(u) & \text{if } x < 0 \text{ or } x > L, \\ 0 & \text{if } 0 \leqslant x \leqslant L \end{cases} \tag{6.1}$$

where $L > 0$ and f_0 satisfies $f_0(0) = f_0(\theta_0) = f_0(1) = 0$, f_0 is convex and negative in $(0, \theta_0)$, f_0 is concave and positive in $(\theta_0, 1)$, $f_0'(0) \neq 0 \neq f_0'(1)$ and $\int_0^1 f_0(s)ds > 0$. An example of nonlinearity satisfying this set of hypotheses is $f_0(u) = u(1-u)(u-\theta_0)$, with $\theta_0 \in (0, 1/2)$.

In this case the existence of stationary solutions has been investigated by Lewis and Keener [19]. They proved that there exists $L^* > 0$ such that for all $L > L^*$, there exist two (and only two) $\mathcal{C}^1(\mathbb{R})$ solutions $w_- < w_+$ of

$$-w'' = f(x,w) \quad \text{in } \mathbb{R}, \qquad w(-\infty) = 1, \qquad w(+\infty) = 0, \quad 0 < w < 1. \tag{6.2}$$

Moreover, $w_\pm$ are decreasing, $w_-(0) < \theta_0 < w_+(0)$, w_- is stable and w_+ is unstable (see [19] for a precise statement on the stability).

Proposition 6.1. *Consider $L > L^*$ and let $x_\pm$ be the unique points such that $w_\pm(x_\pm) = \theta_0$. Let u be the critical travelling wave normalized by $u(0, x_0) = \theta_0$,*

- *if $x_0 < x_-$, then u is time-increasing, $u(-\infty, x) = 0$ and $u(+\infty, x) = w_-(x)$,*
- *if $x_0 = x_-$, then u does not depend on time and $u \equiv w_-$,*
- *if $x_- < x_0 < x_+$, then u is time-decreasing, $u(-\infty, x) = w_+(x)$ and $u(+\infty, x) = w_-(x)$,*
- *if $x_0 = x_+$, then u does not depend on time and $u \equiv w_+$,*
- *if $x_0 > x_+$, then u is time-increasing, $u(-\infty, x) = w_+(x)$ and $u(+\infty, x) = 1$,*

where all these convergences are locally uniform in $x \in \mathbb{R}$ (see Fig. 3).

This example shows that in the multistable setting, the shape of the critical travelling wave is not unique up to translation in time unlike in monostable or ignition-type framework. Indeed, different normalizations of the critical travelling wave could give very different behaviours.

7. Construction and properties of critical travelling waves

7.1. Preliminaries: zero set of the solution of a parabolic equation

Our main tool in the sequel will be Proposition 7.1, which is an extension of Angenent's classical result [1]. It basically states that if the solution u of a linear parabolic equation admits only one zero at $t = 0$, then $u(t, \cdot)$ will admit at most one zero for all $t > 0$. Angenent's result [1] states that the number of zeros of $u(t, \cdot)$ is nonincreasing with respect to $t > 0$, but it does not include the time $t = 0$. Hence, we need to start the proof through direct arguments, before being able to use Angenent's result. Moreover, we will need in the sequel a slightly more general assumption on the initial datum.

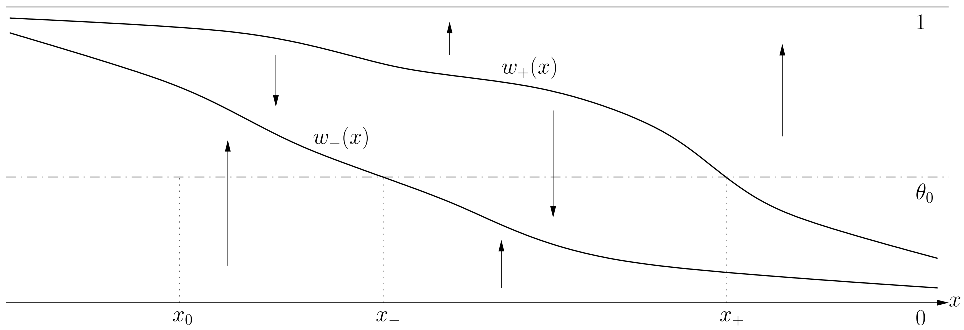


Fig. 3. A representation of Proposition 6.1. The arrows indicate the evolution with respect to time of the critical travelling waves lying in the area.

Proposition 7.1. *Assume that a, $1/a$, a_t, a_x, a_{xx}, b, b_t, b_x and c are measurable and essentially bounded functions over $(0,\infty)\times\mathbb{R}$. Consider a bounded weak solution $u\in\mathcal{C}^0((0,\infty)\times\mathbb{R})\cap L^\infty((0,\infty)\times\mathbb{R})$ of*

$$\begin{cases} \partial_t u = a(t,x)\partial_{xx}u + b(t,x)\partial_x u + c(t,x)u & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,x)=u_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{7.1}$$

where $u_0\not\equiv 0$ is continuous by parts and bounded over $\mathbb{R}$ and there exists $x_0\in\mathbb{R}$ such that

$$u_0(x)\geqslant 0 \quad \text{if } x<x_0, \qquad u_0(x)\leqslant 0 \quad \text{if } x>x_0.$$

Then, for all $t>0$, there exists a unique $\xi(t)\in[-\infty,\infty]$ such that

$$u(t,x)>0 \quad \text{if } x<\xi(t), \qquad u(t,x)<0 \quad \text{if } x>\xi(t).$$

Proof. 1. Take $T>0$ and let $\Omega:=\{(t,x)\in(0,T)\times\mathbb{R},\ u(t,x)>0\}$. Assume that Ω is non-empty and write $\Omega=\bigcup_{i\in I}\Omega_i$, where Ω_i are disjoint non-empty connected open sets of $(0,T)\times\mathbb{R}$ for all $i\in I$. Assume that there exists $i_0\in I$ such that $\{(0,x),\ x\in\mathbb{R}\}\cap\overline{\Omega_{i_0}}=\emptyset$. Define

$$\mathcal{V}=\big((0,T)\times\mathbb{R}\big)\backslash\bigcup_{i\neq i_0}\overline{\Omega}_i.$$

It is easy readily checked that

$$\partial\mathcal{V}\backslash\big\{(T,x),\ x\in\mathbb{R}\big\}\subset\{u\leqslant 0\} \quad \text{since } \big\{(0,x),\ x\in\mathbb{R}\big\}\cap\overline{\Omega_{i_0}}=\emptyset.$$

The parabolic weak maximum principle yields that $u\leqslant 0$ in $\mathcal{V}$, which is a contradiction since $\Omega_{i_0}\subset\mathcal{V}$ and $u>0$ in Ω_{i_0}. Hence, $\{(0,x),\ x\in\mathbb{R}\}\cap\overline{\Omega_{i_0}}\neq\emptyset$ for all $i\in I$.

2. Define $\Omega(t):=\{x\in\mathbb{R},\ u(t,x)>0\}$ for all $t>0$. Assume that $y_1\in\Omega(T)$ and $y_2\in\Omega(T)$, with $T>0$ and $y_1<y_2$. We will prove that $[y_1,y_2]\in\Omega(T)$. Define Ω as in the first step. As $\overline{\Omega}=\bigcup_{i\in I}\overline{\Omega_i}$, there exist $i_1,i_2\in I$ such that $(T,y_1)\in\overline{\Omega_{i_1}}$ and $(T,y_2)\in\overline{\Omega_{i_2}}$. For all i, as Ω_i is connected, $\overline{\Omega_i}$ is connected and, as the first step yields that $\{(0,x),\ x\in\mathbb{R}\}\cap\overline{\Omega_{i_0}}\neq\emptyset$ for $i=1,2$, there exist two continuous paths $\phi_1:[0,1]\to\overline{\Omega_1}$ and $\phi_2:[0,1]\to\overline{\Omega_2}$ such that $\phi_k(1)\in\{(0,x),\ x\in\mathbb{R}\}$ for $k=1,2$, $\phi_1(0)=(T,y_1)$ and $\phi_2(0)=(T,y_2)$. We can assume that these paths are non-self-intersecting and that $\phi_k(s)\notin\{(T,x),\ x\in\mathbb{R}\}$ for $k=1,2$.

3. Consider first the case where there exist $s_1,s_2\in(0,1)$ such that $\phi_1(s_1)=\phi_2(s_2)$. We can assume that s_1 is the smallest s such that $\phi_1(s)$ intersects the curve associated with ϕ_2. Define the path: for all $s\in[0,s_1+s_2+1]$,

$$\psi(s):=\begin{cases} s(T,y_1)+(1-s)(T,y_2) & \text{if } s\in[0,1], \\ \phi_1(s-1) & \text{if } s\in[1,1+s_1], \\ \phi_2(s_2+s_1+1-s) & \text{if } s\in[1+s_1,1+s_1+s_2]. \end{cases}$$

This path is a Jordan curve since it is continuous, non-self-intersecting and $\psi(0)=\psi(s_1+s_2+1)=(T,y_2)$. Let K be its (compact) interior region. Then

$$\partial K\backslash\big\{(T,x),\ x\in\mathbb{R}\big\}\subset\big\{\phi_1(s),\ s\in[0,s_1]\big\}\cup\big\{\phi_2(s),\ s\in[0,s_2]\big\}\subset\{u\geqslant 0\},$$

and the weak parabolic maximum principle gives $u \geqslant 0$ in K. If there exists $x \in (y_1, y_2)$ such that $u(T,x) = 0$, then for all $\delta > 0$ such that $[T-\delta, T] \times [x-\delta, x+\delta] \subset K$, the Krylov–Safonov–Harnack inequality (see [18]) would give $u(T,\cdot) \equiv 0$ on $[x-\delta, x+\delta]$. Iterating, one would eventually get by continuity $u(T, y_1) = 0$, which would be a contradiction. Hence, $u(T,x) > 0$ for all $x \in [y_1, y_2]$.

If the paths ϕ_1 and ϕ_2 do not intersect, constructing a Jordan curve by connecting the points where ϕ_1 and ϕ_2 touch $\{(0,x),\ x \in \mathbb{R}\}$ through a segment, one concludes similarly.

This proves that $\Omega(t)$ is connected for all $t > 0$. In other words, it is an interval. Similarly, $\{x \in \mathbb{R},\ u(t,x) < 0\}$ is an interval for all $t > 0$ if it is not empty. Hence, one can define $\xi^-(t) := \sup\{x,\ u(t,x) > 0\}$ and $\xi^+(t) := \inf\{x,\ u(t,x) < 0\}$ for all $t > 0$. If $\xi^-(t) < \xi^+(t)$ for some $t > 0$, then $u(t,x) = 0$ for all $\xi^-(t) \leqslant x \leqslant \xi^+(t)$ and Theorem A in [1] would give $u \equiv 0$, which is a contradiction. Letting $\xi(t) := \xi^-(t) = \xi^+(t)$ concludes the proof. □

7.2. Construction of the function u

The construction of the wave is similar to the construction of random travelling waves in earlier works of Nolen and Ryzhik [27] and Shen [30]. However, we will diverge from these two papers in the next subsections since the properties of the wave we seek to prove are different.

Define for all $s < 0$, $y \in \mathbb{R}$, the solution $u_s^y = u_s^y(t,x)$ of Eq. (E) with initial condition at $t = s$:

$$u_s^y(s,x) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 0 & \text{if } x > y. \end{cases}$$

Take $\theta \in (0,1)$ and $x_0 \in \mathbb{R}$ as in the statement of Theorem 2.2.

Lemma 7.2. *For all $s < 0$, there exists a unique $x_s \in \mathbb{R}$ such that $u_s^{x_s}(0, x_0) = \theta$.*

Proof. The parabolic maximum principle and the parabolic regularity estimates respectively yield that $y \mapsto u_s^y(0,x_0)$ is increasing and continuous. Let $m := \lim_{y \to -\infty} u_s^y(0,x_0)$. Take a sequence $(y_n)_n$ such that $\lim_{n\to+\infty} y_n = -\infty$. The parabolic regularity estimates yield that one can assume, up to extraction, that the sequence $(u_s^{y_n})_n$ converges locally uniformly to the solution v of (E) associated with the initial datum $v(s,x) = 0$ for all $x \in \mathbb{R}$. Hence $v \equiv 0$ and $m = \lim_{n\to+\infty} u_s^{y_n}(0,x_0) = v(0,x_0) = 0$. Similarly, one can prove that $\lim_{y\to+\infty} u_s^y(0,x_0) = 1$. The existence and uniqueness of $x_s \in \mathbb{R}$ follow from the intermediate value theorem and the monotonicity of $y \mapsto u_s^y(0,x_0)$. □

In the sequel, we will denote $u_s := u_s^{x_s}$ in order to enlight the notations.

Lemma 7.3. *The limit*

$$u(t,x) := \lim_{s\to-\infty} u_s(t,x) \tag{7.2}$$

exists locally uniformly in $(t,x) \in \mathbb{R} \times \mathbb{R}$ and it is a solution of (E).

Proof. Take $s_1 < s_2$ and define $w(t,x) := u_{s_2}(t,x) - u_{s_1}(t,x)$ for all $t \geqslant s_2$ and $x \in \mathbb{R}$. The function w satisfies the parabolic equation

$$\partial_t w - a(x)\partial_{xx} w - b(x)\partial_x w = c(t,x)w \quad \text{in } (s,\infty) \times \mathbb{R}$$

where

$$c(t,x) = \begin{cases} \frac{f(x,u_{s_2}(t,x)) - f(x,u_{s_1}(t,x))}{u_{s_2}(t,x) - u_{s_1}(t,x)} & \text{if } u_{s_2}(t,x) \neq u_{s_1}(t,x), \\ 0 & \text{if } u_{s_2}(t,x) = u_{s_1}(t,x). \end{cases}$$

As f is Lipschitz-continuous, the function c is bounded and measurable. On the other hand, we know from the parabolic maximum principle that $0 < u_{s_1}(t,x) < 1$ for all $t > s_1$ and $x \in \mathbb{R}$. Hence, $w(s_2,x) > 0$ if $x < x_{s_2}$ and $w(s_2,x) < 0$ if $x > x_{s_2}$. It follows from Proposition 7.1 that for all $t > s_2$, $\{w(t,\cdot) > 0\}$ and $\{w(t,\cdot) > 0\}$ are intervals. But we also know that $w(0,x_0) = \theta - \theta = 0$. Hence, we eventually get

$$u_{s_2}(0,x) \geqslant u_{s_1}(0,x) \quad \text{if } x < x_0 \quad \text{and} \quad u_{s_2}(0,x) \leqslant u_{s_1}(0,x) \quad \text{if } x > x_0.$$

In other words, $s \mapsto u_s(0,x)$ is nondecreasing if $x < x_0$ and nonincreasing if $x > x_0$. Thus, the limit $u_0(x) := \lim_{s\to-\infty} u_s(0,x)$ is well-defined for all $x \in \mathbb{R}$ and as the solution $u = u(t,x)$ of the Cauchy problem associated with Eq. (E) and the initial datum $u_0 = u_0(x)$ is unique, parabolic regularity estimates give the conclusion. □

7.3. Criticality of the wave

Lemma 7.4. *Assume that $v \in \mathcal{C}^0(\mathbb{R}\times\mathbb{R})$ is a time-global solution of (E) such that $v(t_0,x_0) = u(t_0,x_0)$ for some $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}$ and $0 < v < 1$, then either $u \equiv v$ or*

$$u(t_0,x) > v(t_0,x) \quad \text{if } x < x_0 \quad \text{and} \quad u(t_0,x) < v(t_0,x) \quad \text{if } x > x_0,$$

where u is defined by (7.2).

Proof. Assume that $u \not\equiv v$. Let $w(t,x) := u_s(t,x) - v(t,x)$, where $s < 0$. The definition of u_s yields that $w(s,x) > 0$ if $x < x_s$ and $w(s,x) < 0$ if $x > x_s$. As in the proof of Lemma 7.3, it follows from Proposition 7.1 that for all $t > s$, there exists $\xi_s(t) \in \mathbb{R}$ such that $u_s(t,\xi_s(t)) = v(t,\xi_s(t))$ and

$$u_s(t,x) > v(t,x) \quad \text{if } x < \xi_s(t) \quad \text{and} \quad u_s(t,x) < v(t,x) \quad \text{if } x > \xi_s(t). \tag{7.3}$$

Assume that there exists a sequence $(s_n)_n$ such that $s_n \to -\infty$ and $\xi_{s_n}(t_0-1) \to +\infty$. Then letting $s = s_n$, $t = t_0 - 1$ and $n \to +\infty$ in (7.3) gives $u(t_0-1,x) \geqslant v(t_0-1,x)$ for all $x \in \mathbb{R}$. It follows from the parabolic strong maximum principle that $u(t_0,x) > v(t_0,x)$ for all $x \in \mathbb{R}$ since $u \not\equiv v$, which is a contradiction at $x = x_0$. Similarly, one can prove that $\xi_{s_n}(t_0-1) \to -\infty$ would lead to a contradiction. Hence, $s \mapsto \xi_s(t_0-1)$ is bounded and there exists a sequence $(s_n)_n$ such that $s_n \to -\infty$ and $(\xi_{s_n}(t_0-1))_n$ converges to a limit $\xi_\infty \in \mathbb{R}$. One gets from (7.3)

$$u(t_0-1,x) \geqslant v(t_0-1,x) \quad \text{if } x \leqslant \xi_\infty \quad \text{and} \quad u(t_0-1,x) \leqslant v(t_0-1,x) \quad \text{if } x \geqslant \xi_\infty. \tag{7.4}$$

It follows from Proposition 7.1 that the function $x \mapsto u(t_0,x) - v(t_0,x)$ admits a unique zero, which is necessarily x_0, and that

$$u(t_0,x) > v(t_0,x) \quad \text{if } x < x_0 \quad \text{and} \quad u(t_0,x) < v(t_0,x) \quad \text{if } x > x_0. \qquad \square$$

7.4. Monotonicity of the wave in time

Lemma 7.5. *Let $\tau > 0$.*

- *If there exists a sequence $(s_n)_n$ such that $\lim_{n\to+\infty} s_n = -\infty$ and $x_{s_n+\tau} \leqslant x_{s_n}$ for all $n \in \mathbb{N}$, then $t \mapsto u(t,x)$ is nonincreasing for all $x \in \mathbb{R}$.*
- *If there exists a sequence $(s_n)_n$ such that $\lim_{n\to+\infty} s_n = -\infty$ and $x_{s_n+\tau} \geqslant x_{s_n}$ for all $n \in \mathbb{N}$, then $t \mapsto u(t,x)$ is nondecreasing for all $x \in \mathbb{R}$.*

Hence, $t \mapsto u(t,x)$ is either nondecreasing for all $x \in \mathbb{R}$ or nonincreasing for all $x \in \mathbb{R}$.

Proof. Assume first that there exists a sequence $(s_n)_n$ such that $\lim_{n\to+\infty} s_n = -\infty$ and $x_{s_n+\tau} \leqslant x_{s_n}$ for all $n \in \mathbb{N}$. Define for all $n \in \mathbb{N}$, $(t,x) \in (s_n,\infty)\times\mathbb{R}$:

$$v_n(t,x) := u_{s_n+\tau}(t+\tau,x).$$

This function satisfies (E) in $(s_n,\infty)\times\mathbb{R}$ together with the initial condition

$$v_n(s_n,x) = \begin{cases} 1 & \text{if } x \leqslant x_{s_n+\tau}, \\ 0 & \text{if } x > x_{s_n+\tau}. \end{cases}$$

The function u_{s_n} satisfies the same Cauchy problem but with x_{s_n} instead of $x_{s_n+\tau}$ in the definition of the initial condition. Hence, as $x_{s_n+\tau} \leqslant x_{s_n}$, the parabolic maximum principle gives

$$u_{s_n}(t,x) \geqslant v_n(t,x) = u_{s_n+\tau}(t+\tau,x) \quad \text{for all } (t,x) \in (s_n,\infty)\times\mathbb{R}.$$

Letting $n \to +\infty$ in this inequality, as $u(t,x) = \lim_{s\to+\infty} u_s(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$, one gets $u(t,x) \geqslant u(t+\tau,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}$. This proves the first part of Lemma 7.5.

If there exists a sequence $(s_n)_n$ such that $\lim_{n\to+\infty} s_n = -\infty$ and $x_{s_n+\tau} \geqslant x_{s_n}$ for all $n \in \mathbb{N}$, then the monotonicity of $t \mapsto u(t,x)$ follows through similar arguments.

If there exists no sequence $(s_n)_n$ such that $\lim_{n\to+\infty} s_n = -\infty$ and $x_{s_n+\tau} \leqslant x_{s_n}$ for all $n \in \mathbb{N}$, then there exists $S < 0$ such that $x_{s+\tau} > x_s$ for all $s < S$ and thus the second part of the lemma yields that $t \mapsto u(t,x)$ is nondecreasing for all $x \in \mathbb{R}$. We conclude that $t \mapsto u(t,x)$ is either nondecreasing for all $x \in \mathbb{R}$ or nonincreasing for all $x \in \mathbb{R}$. □

7.5. Monotonicity of the wave in space for homogeneous f

Lemma 7.6. *Assume that f does not depend on x. Then $x \mapsto u(t,x)$ is nonincreasing for all $t \in \mathbb{R}$.*

Proof. Define u_s^y as above for all $s < 0$ and $y \in \mathbb{R}$. It is clear from the previous proof that we only need to prove that $x \mapsto u_s^y(t,x)$ is nonincreasing for all $t > s$ in order to conclude.

Assume first that the coefficients a, b and f are $\mathcal{C}^\infty$ functions and let $v := \partial_x u_s^y$. This function satisfies the smooth parabolic equation

$$\partial_t v - a(x)\partial_{xx} v - \big(a'(x) + b(x)\big)\partial_x v = \big(f'\big(u(t,x)\big) + b'(x)\big)v \quad \text{in } (s,\infty)\times\mathbb{R}.$$

Moreover, $v(t,\cdot) \rightharpoonup -\delta_y$ as $t \to s^+$ in the sense of measures, where δ_y is the Dirac measure localized at y. Hence, the weak parabolic maximum principle yields that $v(t,x) \leqslant 0$ for all $(t,x) \in (s,\infty)\times\mathbb{R}$, meaning that $x \mapsto u_s^y(t,x)$ is nonincreasing for all $t > s$.

If $a = a(x)$, $b = b(x)$ and $f = f(u)$ satisfy (H), then the result follows from the previous step by approximation. □

7.6. End of the proof of Theorem 2.2

Proof of Theorem 2.2. 1. The existence of the critical travelling wave immediately follows from Lemmas 7.3 and 7.4. Assume that $\tilde{u}$ is a critical travelling wave of (E) such that $\tilde{u}(0,x_0) = \theta$. As u is a critical travelling wave and $\tilde{u}$ is a time-global solution, one has $u(0,x) \geqslant \tilde{u}(0,x)$ if $x < x_0$ and $u(0,x) \leqslant \tilde{u}(0,x)$ if $x > x_0$. But as $\tilde{u}$ is a critical travelling wave, we also have $u(0,x) \leqslant \tilde{u}(0,x)$ if $x < x_0$ and $u(0,x) \geqslant \tilde{u}(0,x)$ if $x > x_0$. Hence $\tilde{u}(0,x) = u(0,x)$ for all $x \in \mathbb{R}$. As u and $\tilde{u}$ both satisfy the parabolic equation (E), the uniqueness follows.

2. Next, we know from Lemmas 7.4 and 7.5 that u is either nonincreasing or nondecreasing in time. Let $v := \partial_t u$. This function satisfies

$$\partial_t v - a(x)\partial_{xx} v - b(x)\partial_x v = f_u'\big(x, u(t,x)\big)v \quad \text{in } \mathbb{R}\times\mathbb{R}.$$

It follows from parabolic regularity estimates that $\partial_t v \in L^p_{loc}(\mathbb{R}\times\mathbb{R})$ and $\partial_{xx} v \in L^p_{loc}(\mathbb{R}\times\mathbb{R})$ for all $p \in (1,\infty)$. In particular, v is a continuous function on $\mathbb{R}\times\mathbb{R}$.

Assume that u is nonincreasing in t. Then $v := \partial_t u \geqslant 0$ on $\mathbb{R}\times\mathbb{R}$. If there exists $(t_0,x_0) \in \mathbb{R}\times\mathbb{R}$ such that $v(t_0,x_0) = 0$, then the Harnack–Krylov–Safonov inequality for parabolic equations [18] implies that $v(t,x) = 0$ for all $t < t_0$, $x \in \mathbb{R}$ and thus $v \equiv 0$. Hence u would be constant with respect to time. If v is positive, this means that $\partial_t u > 0$ and thus u is increasing in time. Similarly, if u is nonincreasing in time, then one can prove that either u is either constant or decreasing in time.

3. Lastly, when f does not depend on x, the monotonicity in x immediately follows from Lemma 7.6. □

8. Proof of the comparison results with spatial transition waves

8.1. Proof of the results in the general framework

Proof of Theorem 3.1. Consider a critical transition wave u. As $\lim_{t\to-\infty} v(t,0) = 0$ and $\lim_{t\to+\infty} v(t,0) = 1$, there exists $\tau \in \mathbb{R}$ such that $v(\tau,0) = u(0,0)$. The criticality of u yields

$$u(0,x) \geqslant v(\tau,x) \quad \text{if } x \leqslant 0 \quad \text{and} \quad u(0,x) \leqslant v(\tau,x) \quad \text{if } x \geqslant 0.$$

Hence, $\lim_{x\to-\infty} u(0,x) = 1$ and $\lim_{x\to+\infty} u(0,x) = 0$ and it easily follows from parabolic regularity estimates and the parabolic strong maximum principle that

$$\lim_{x\to-\infty} u(t,x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(t,x) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Next, let $\varepsilon \in (0, 1/2)$ and take $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}$ such that $\varepsilon < u(t_0, x_0) < 1 - \varepsilon$. There exists $\tau' \in \mathbb{R}$ such that $v(t_0 + \tau', x_0) = u(t_0, x_0)$. Define for all $t \in \mathbb{R}$, $I_\varepsilon(t) := \{x \in \mathbb{R},\ \varepsilon < u(t,x) < 1 - \varepsilon\}$ and $J_\varepsilon(t) := \{x \in \mathbb{R},\ \varepsilon < v(t,x) < 1 - \varepsilon\}$. As v is a spatial transition wave, we know that there exists $L > 0$ such that the *diam* $J_\varepsilon(t) \leqslant L$ for all $t \in \mathbb{R}$.

On the other hand, one has

$$u(t_0, x) \geqslant v\big(t_0 + \tau', x\big) \quad \text{if } x \leqslant x_0 \quad \text{and} \quad u(t_0, x) \leqslant v\big(t_0 + \tau', x\big) \quad \text{if } x \geqslant x_0.$$

Also, $x_0 \in J_\varepsilon(t_0 + \tau') \cap I_\varepsilon(t_0)$. Take $x < \inf J_\varepsilon(t_0 + \tau')$. One has $x < x_0$ and $v(t_0 + \tau', x) \geqslant 1 - \varepsilon$, which gives $u(t_0, x) \geqslant 1 - \varepsilon$. Hence $x < \inf I_\varepsilon(t_0)$. Similarly, one can prove that $x > \sup J_\varepsilon(t_0 + \tau')$ implies $x > \sup I_\varepsilon(t_0)$. We conclude that *diam* $I_\varepsilon(t_0) \leqslant$ *diam* $J_\varepsilon(t_0 + \tau') \leqslant L$. As L does not depend on t_0, we conclude that u is a spatial transition wave. □

8.2. Proof of the results in the ignition-type framework

Proof of Corollary 3.3. Consider a critical travelling wave u. We know from [22,27] that there exists a spatial transition wave v such that $\lim_{t\to+\infty} v(t,x) = 1$ and $\lim_{t\to-\infty} v(t,x) = 0$ for all $x \in \mathbb{R}$. Hence, Theorem 3.1 yields that u is a spatial transition wave.

On the other hand, if v is an arbitrary spatial transition wave, then as u is a spatial transition wave, it follows from [21] that there exists $\tau \in \mathbb{R}$ such that $u(t,x) = v(t+\tau, x)$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$. Thus, v is a critical travelling wave.

The monotonicity and the uniqueness immediately follow from Theorem 3.2. □

8.3. Proof of the results in the monostable framework

Proof of Proposition 3.5. We know from Theorem 2.2 that u is either increasing, decreasing or constant in time from Theorem 2.2. Moreover, the monostability of Eq. (E) ensures that $\lim_{t\to+\infty} u(t,x) = 1$ since $u(0,\cdot) \not\equiv 0$. Hence, as $0 < u < 1$, u is necessarily increasing in time. Let $u_\infty(x) = \lim_{t\to-\infty} u(t,x)$ for all $x \in \mathbb{R}$. The parabolic regularity estimates yield that $u_\infty \in W^{2,p}(\mathbb{R})$ for all $p \in (1,\infty)$ and that it is a weak solution of $a(x)u_\infty'' + b(x)u_\infty' + f(x, u_\infty) = 0$ over $\mathbb{R}$. Taking u_∞ as an initial datum in the Cauchy problem (3.2), we get from the monostability hypothesis that $u_\infty \equiv 1$ if $u_\infty \not\equiv 0$. Hence, as $u_\infty(x) = \lim_{t\to-\infty} u(t,x) \leqslant u(0,x)$ for all $x \in \mathbb{R}$, one gets $u_\infty \equiv 0$.

The existence and the uniqueness of $T(x)$ follow from the monotonicity of u with respect to t and the convergences as $t \to \pm\infty$. The continuity of T in x immediately follows from the continuity of u and the uniqueness of T.

Assume that $\tilde{u}$ is another critical travelling wave of (E). As $\lim_{t\to-\infty} \tilde{u}(t,0) = 0$ and $\lim_{t\to+\infty} \tilde{u}(t,0) = 1$, there exists $\tau \in \mathbb{R}$ such that $u(\tau, 0) = \tilde{u}(0,0)$. Theorem 2.2 ensures that $\tilde{u}(\tau, x) = u(0,x)$ for all $x \in \mathbb{R}$. As u and $\tilde{u}$ both satisfy the parabolic equation (E), the conclusion follows. □

Lemma 8.1. *Assume that v is a spatial transition wave of Eq. (E). Let $\theta \in (0,1)$ and $Y(t) := \sup\{x \in \mathbb{R},\ v(t,x) > \theta\}$. Then,* $\inf_{t\in\mathbb{R}} \inf_{x\leqslant Y(t)} v(t,x) > 0$.

Proof. Assume by contradiction that $\inf_{t\in\mathbb{R}} \inf_{x\leqslant Y(t)} v(t,x) = 0$. Consider a sequence $(t_n, x_n)_n$ in $\mathbb{R} \times \mathbb{R}^-$ such that $x_n \leqslant 0$ for all n and $\lim_{n\to+\infty} v(t_n, x_n + Y(t_n)) = 0$. The definition of spatial transition waves yields that, as $v(t_n, Y(t_n)) = \theta$ for all n, there exists $L > 0$ such that $v(t_n, x + Y(t_n)) \geqslant \theta$ for all $x \leqslant -L$. Hence the sequence $(x_n)_n$ is bounded and one can assume that it converges to a limit $x_\infty \in [-L, 0]$.

Let $c_n(t,x) := f(x + Y(t_n), v(t+t_n, x + Y(t_n)))/v(t+t_n, x+Y(t_n))$. We know from (H) that this function is bounded over $\mathbb{R} \times \mathbb{R}$. One can assume, up to extraction, that the sequences $(a(\cdot + Y(t_n)))_n$, $(b(\cdot + Y(t_n)))_n$ and $(c_n)_n$ converge respectively in the $W^{2,\infty}(\mathbb{R})$, $W^{1,\infty}(\mathbb{R})$ and $L^\infty(\mathbb{R} \times \mathbb{R})$ weak-* topologies. Let $a_\infty, b_\infty, c_\infty$ be their respective limits. The parabolic regularity estimates yield that the sequence $(v(\cdot + t_n, \cdot + Y(t_n)))_n$ converges (up to extraction) locally uniformly to a function v_∞ which is a weak solution of

$$\partial_t v_\infty - a_\infty(x)\partial_{xx} v_\infty + b_\infty(x)\partial_x v_\infty = c_\infty(t,x)v_\infty \quad \text{over } \mathbb{R}\times\mathbb{R}.$$

Moreover, $0 \leqslant v_\infty \leqslant 1$ and $v_\infty(0, x_\infty) = 0$. The Krylov–Safonov–Harnack inequality [18] thus gives $v_\infty \equiv 0$, which is a contradiction since $v_\infty(0,x) \geqslant \theta$ for all $x \leqslant -L$. □

Proof of Theorem 3.6. Consider a critical travelling wave u and a spatial transition wave v as in the statement of Theorem 3.6. One can define through Lemma 8.1 the quantity

$$\kappa := 1 / \inf_{\substack{s\in\mathbb{R},\\ x<Y(s)}} v(s,x) > 1.$$

As u and v are both spatial transition waves, the width of their interfaces is bounded and thus there is no loss of generality in assuming that $\kappa\theta < 1/2$.

Take $s_0 \in \mathbb{R}$. As Eq. (E) is monostable, one has $\lim_{t\to+\infty} v(t,x) = 1$ for all $x \in \mathbb{R}$. It can easily be proved that $\lim_{t\to-\infty} v(t,x) = 0$ for all $x\in\mathbb{R}$ since v is a time-global solution. Hence, there exists $s_1 \in \mathbb{R}$ such that $v(s_1, X(s_0)) = u(s_0, X(s_0))$ $(= \theta)$. In particular, $Y(s_1) \geqslant X(s_0)$. Moreover, as u is a critical travelling wave, one has $u(s_0,x) \leqslant v(s_1,x)$ if $x \geqslant X(s_0)$. The definition of κ gives

$$u(s_0,x) \leqslant \kappa v(s_1,x) \quad \text{for all } x\in\mathbb{R}.$$

Next, as $s \mapsto f(x,s)/s$ is nonincreasing for all $x\in\mathbb{R}$ and $\kappa > 1$, one has

$$\kappa\partial_t v - \kappa a(x)\partial_{xx} v - \kappa b(x)\partial_x v = \kappa f(x,v) \geqslant f(x,\kappa v) \quad \text{in } \mathbb{R}\times\mathbb{R},$$

which means that κv is a supersolution of (E). It thus follows from the parabolic maximum principle that

$$u(s_0+t,x) \leqslant \kappa v(s_1+t,x) \quad \text{for all } (t,x)\in\mathbb{R}^+\times\mathbb{R}.$$

Taking $t > 0$ and $x = Y(s_1+t)$ in this inequality gives $u(s_0+t, Y(s_1+t)) \leqslant \kappa\theta$. Hence,

$$Y(s_1+t) \geqslant \xi \quad \text{where } \xi := \inf\{x\in\mathbb{R},\ u(s_0+t,x) \leqslant \kappa\theta\}.$$

Take $\varepsilon < \min\{\theta, 1-\theta\}$ and define $I_\varepsilon(t) = \{x\in\mathbb{R},\ \varepsilon < u(t,x) < 1-\varepsilon\}$, $J_\varepsilon(t) = \{x\in\mathbb{R},\ \varepsilon < v(t,x) < 1-\varepsilon\}$ and L such that $diam\, I_\varepsilon(t) \leqslant L$ and $diam\, J_\varepsilon(t) \leqslant L$ for all $t\in\mathbb{R}$. As $u(s_0+t,\xi) = \kappa\theta \in (\theta, 1/2)$, one has $\xi \in I_\varepsilon(s_0+t)$ and thus

$$\left|\xi - X(s_0+t)\right| \leqslant L.$$

Similarly, as $v(s_1, Y(s_1)) = v(s_1, X(s_0)) = \theta$, one has $|X(s_0) - Y(s_1)| \leqslant L$. We eventually get

$$\begin{aligned} X(s_0+t) - X(s_0) &\leqslant L + \xi - X(s_0)\\ &\leqslant 2L + \xi - Y(s_1)\\ &\leqslant Y(s_1+t) - Y(s_1) + 2L, \end{aligned}$$

which gives the conclusion since L does not depend on s_0 and s_1. □

9. Proof of the results for particular classes of heterogeneities

9.1. The translation property

Proof of Proposition 4.1. As $\lim_{t\to-\infty} u(t,x;(a,b,f)) = 0$ and $\lim_{t\to+\infty} u(t,x;(a,b,f)) = 1$ for all $x\in\mathbb{R}$, we know from Theorem 2.2 that u is increasing in time. Hence, $T(y)$ is uniquely defined for all $y\in\mathbb{R}$.

Let $y\in\mathbb{R}$ and define

$$\tilde{u}(t,x) := u\big(t - T(y), x - y; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{for all } (t,x)\in\mathbb{R}\times\mathbb{R}.$$

We need to prove that $u \equiv \tilde{u}$. We know that

$$\tilde{u}\big(T(y), y\big) = u\big(0, 0, (\pi_y a, \pi_y b, \pi_y f)\big) = \theta = u\big(T(y), y; (a, b, f)\big).$$

Moreover, $\tilde{u}$ is a time-global solution of (E) and $0 < \tilde{u} < 1$. Hence

$$u\big(T(y), y + x; (a, b, f)\big) \geqslant \tilde{u}\big(T(y), x + y\big) = u\big(0, x; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{if } x < 0 \quad \text{and}$$
$$u\big(T(y), y + x; (a, b, f)\big) \leqslant \tilde{u}\big(T(y), x + y\big) = u\big(0, x; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{if } x > 0.$$

On the other hand, $(t, x) \mapsto u(t, x + y; (a, b, f))$ is a time-global solution of Eq. (E) with coefficients $(\pi_y a, \pi_y b, \pi_y f)$ instead of (a, b, f). As $u(T(y), y; (a, b, f)) = \theta = u(0, 0, \pi_y(a, b, f))$, we know that

$$u\big(T(y), y + x; (a, b, f)\big) \leqslant u\big(0, x; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{if } x < 0 \quad \text{and}$$
$$u\big(T(y), y + x; (a, b, f)\big) \geqslant u\big(0, x; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{if } x > 0.$$

Hence, $u(T(y), y + x; (a, b, f)) = u(0, x; (\pi_y a, \pi_y b, \pi_y f))$ for all $x \in \mathbb{R}$. As $u(\cdot + T(y), \cdot + y; (a, b, f))$ and $u(\cdot,\cdot; (\pi_y a, \pi_y b, \pi_y f))$ both satisfy (E) with coefficients $(\pi_y a, \pi_y b, \pi_y f)$ instead if (a, b, f), it follows from the well-posedness of the Cauchy problem that

$$u\big(t + T(y), y + x; (a, b, f)\big) = u\big(t, x; (\pi_y a, \pi_y b, \pi_y f)\big) \quad \text{for all } (t, x) \in \mathbb{R} \times \mathbb{R}. \qquad \square$$

9.2. Comparison with planar and pulsating travelling waves

We first prove Proposition 4.3, from which we will derive partially Proposition 4.2. The next lemma, which proves that monostable equations (in the sense of Definition 3.4) always admit pulsating travelling waves with positive speeds, is crucial since we will need to compare critical travelling waves with these pulsating travelling waves in order to obtain estimates on the speed of the critical travelling wave. Of course the existence of pulsating travelling waves has already been studied in earlier papers, but always under more restrictive notions of monostability. Hence, Lemma 9.1 cannot be trivially derived from earlier results.

Lemma 9.1. *Assume that Eq. (E) is monostable and let $\theta \in (0, 1)$ and $x_0 \in \mathbb{R}$. Then there exists $\underline{c} > 0$ such that for all $c \geqslant \underline{c}$, Eq. (E) admits a pulsating travelling wave v of speed c such that $v(0, x_0) = \theta_0$.*

Note that we do not know if $\underline{c}$ is a minimal speed. In other words, we do not know if there could exist some pulsating travelling waves with speed $c < \underline{c}$ under our mild monostability hypothesis.

Proof of Lemma 9.1. Let λ be the periodic principal eigenvalue associated with the linearized operator near $u = 0$, defined for all $\varphi \in \mathcal{C}^2(\mathbb{R})$ by

$$M\varphi = a(x)\varphi'' + b(x)\varphi' + f_u'(x, 0)\varphi.$$

Assume first that $\lambda > 0$. As Eq. (E) is monostable, we know that if v is a solution of $-a(x)v'' - b(x)v' = f(x, v)$ over $\mathbb{R}$ such that $0 \leqslant v \leqslant 1$ and v is L-periodic, then $v \equiv 0$ or $v \equiv 1$. Hence, all the hypotheses of Theorem 2.3 of [23] are satisfied and the conclusion follows: there exists a speed $\underline{c}$ such that for all $c \geqslant \underline{c}$, Eq. (E) admits a pulsating travelling wave v of speed c such that $v(0, x_0) = \theta$. Moreover, one can take $\underline{c} = 2\sqrt{\|a\|_\infty \|f\|_{Lip}} + \|b\|_\infty$ (this immediately follows from basic estimates on the speed $c_e^*(A, q, f)$ using the same notations as in [23]) and the pulsating travelling wave we obtain is increasing in time. Note that we refer to [23], which investigates space–time periodic media, instead of classical papers such as [3,34]. This is because this paper gives the only proof of existence of pulsating travelling waves under the very mild hypothesis $\lambda > 0$, as far as we know.

Next, assume that $\lambda < 0$, then let φ be the periodic principal eigenfunction associated with λ and normalized by $\|\varphi\|_\infty = 1$. That is, ϕ is a positive periodic solution of $M\varphi = \lambda\varphi$. As $\lambda < 0$, it is easy to check that $-\kappa a(x)\varphi'' - \kappa b(x)\varphi' \geqslant f(x, \kappa\varphi)$ over $\mathbb{R}$ if $\kappa > 0$ is small enough. Hence, the solution v of $\partial_t v - a(x)\partial_{xx} v - b(x)\partial_x v = f(x, v)$ in $(0, \infty) \times \mathbb{R}$ with initial condition $v(0, x) = \kappa\varphi(x)$ for all $x \in \mathbb{R}$ is nonincreasing in time. Thus it cannot converges to 1 as $t \to +\infty$, which contradicts the fact that Eq. (E) is monostable. Hence the monostability implies $\lambda \geqslant 0$.

Lastly, assume that $\lambda = 0$. For all $\varepsilon > 0$ and $(x, u) \in \mathbb{R} \times [0, 1]$, define $f_\varepsilon(x, u) := f(x, u) + \varepsilon u(1-u)$. As $f_\varepsilon \geqslant f$, it is easy to see that Eq. (E) with nonlinearity f_ε instead of f is still monostable. Moreover, if λ_ε is the periodic principal eigenvalue associated with the linearization at 0, then one has $\lambda_\varepsilon = \lambda + \varepsilon = \varepsilon > 0$ since $(f_\varepsilon)'_u(x, 0) = f'_u(x, 0) + \varepsilon$ for all $x \in \mathbb{R}$. Hence, it follows from our first case that for all $c \geqslant 2\sqrt{\|a\|_\infty(\|f\|_{Lip} + \varepsilon)} + \|b\|_\infty$, there exists a pulsating travelling wave with speed c associated with the nonlinearity f_ε, which is increasing in time.

Define $\underline{c} = 2\sqrt{\|a\|_\infty(\|f\|_{Lip} + 1)} + \|b\|_\infty$ and take $c \geqslant \underline{c}$. Consider for all $\varepsilon \in (0, 1)$ a pulsating travelling wave v_ε with speed c, normalized (up to translation in time) by $v_\varepsilon(0, x_0) = \theta$, which is increasing in time. It follows from parabolic regularity estimates that there exists a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n \to 0$ and $(v_{\varepsilon_n})_n$ converges locally uniformly. Let v_0 be its limit. Then v_0 satisfies Eq. (E), $v_0(0, x_0) = \theta$, v_0 is nondecreasing in time and one has $v_0(t + L/c, x + L) = v_0(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. Moreover, it follows from the monostability of Eq. (E) that $\lim_{t\to+\infty} v_0(t, x) = 1$ locally in $x \in \mathbb{R}$ and one also gets $\lim_{t\to-\infty} v_0(t, x) = 0$ since v_0 is nondecreasing in time. Hence, v_0 is a pulsating travelling wave with speed c, which concludes the proof. □

Proof of Proposition 4.3. Let u be the critical travelling wave of Eq. (E) normalized by $u(0, x_0) = \theta$ and $T(y)$ the unique solution of $u(T(y), y) = \theta$ for all $y \in \mathbb{R}$. As a, b and f are L-periodic, one has $\pi_L a = a$, $\pi_L b = b$ and $\pi_L f$. Proposition 4.1 implies that $u(t + T(L), x + L) = u(t, x)$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}$.

Next, we know from Lemma 9.1 that there exists a pulsating travelling wave $v(t, x)$ with speed $c' > 0$ such that $v(0, x_0) = \theta$. The criticality of u gives $u(0, x) \leqslant v(0, x)$ for all $x > x_0$. Taking $x = x_0 + nL$ with $n \in \mathbb{N}$, one gets $u(-nT(L), x_0) = u(0, x_0 + nL) \leqslant v(0, x_0 + nL) = v(-nL/c', x_0)$ for all $n \in \mathbb{N}$ and the right hand-side converges to 0 as $n \to +\infty$ since $c' > 0$. Hence, $T(L) > 0$ and one can define $c := L/T(L)$. One has $u(t + L/c, x + L) = u(t, x)$ for all (t, x) and thus u is a pulsating travelling wave.

Assume that there exists a pulsating travelling wave solution v of Eq. (E) with speed $c' < c$. Define $\phi(z, x) := u((z + x)/c, x)$ and $\psi(z, x) := v((z + x)/c', x)$ for all $(z, x) \in \mathbb{R} \times \mathbb{R}$. It follows from the definition of pulsating travelling waves that ϕ and ψ are L-periodic in x. Assume furthermore that the coefficients are of class $\mathcal{C}^{1,\gamma}(\mathbb{R} \times [0, 1])$ for some $\gamma \in (0, 1)$ and that the periodic principal eigenvalue associated with the linearization of Eq. (E) near $u = 1$ is positive: $\mu > 0$. For all $\alpha > 0$, let L_α be the elliptic operator defined for all $\varphi \in \mathcal{C}^2(\mathbb{R})$ by

$$L_\alpha \varphi = a(x)\varphi'' + \big(b(x) + 2\alpha a(x)\big)\varphi' + \big(f'_u(x, 1) + \alpha b(x) + \alpha^2 a(x)\big)\varphi.$$

As the coefficients of this operator are L-periodic, it admits a periodic principal eigenvalue $k(\alpha)$. When $\alpha = 0$, we recover the linearization of the equation near $u = 1$ and thus $k(0) = \mu < 0$. It has been proved by Hamel in [14] (see Eq. (1.29) in [14]) that

$$\lim_{z\to+\infty} \frac{1}{z} \ln\big(1 - \phi(z, x)\big) = -\alpha_c \quad \text{and} \quad \lim_{z\to+\infty} \frac{1}{z} \ln\big(1 - \psi(z, x)\big) = -\alpha_{c'} \quad \text{uniformly in } x \in \mathbb{R}, \tag{9.1}$$

where for all $\tilde{c} > 0$ we define $\alpha_{\tilde{c}}$ the unique positive solution of equation $k(\alpha) = -\alpha\tilde{c}$. As $\alpha \mapsto k(\alpha)$ is convex (see [14]), $\alpha > 0$ and $c' < c$, simple graphical considerations yield that $\alpha_c < \alpha_{c'}$.

On the other hand, we know from the criticality of the wave u that $u(0, x) \geqslant v(0, x)$ for all $x < x_0$, which reads

$$\phi(-x, x) \geqslant \psi(-x, x) \quad \text{for all } x < x_0.$$

It follows from (9.1) that $-\alpha_c \leqslant -\alpha_{c'}$, which is a contradiction.

Let now prove the second part of Proposition 4.3. It has been proved in [3] that under the hypotheses of the second part of Proposition 4.3, there exists a pulsating travelling wave v of Eq. (E) with speed c^*, which is increasing in time, such that $\lim_{t\to-\infty} v(t, x) = 0$, $\lim_{t\to+\infty} v(t, x) = 1$, and that this pulsating travelling wave is unique up to translation in time. Let $\theta \in (0, 1)$, $x_0 \in \mathbb{R}$ and u be the critical travelling wave normalized by $u(0, x_0) = \theta$. We can assume that $v(0, x_0) = \theta$ by translating in time and thus it follows from the criticality of u that $u(0, x) \geqslant v(0, x)$ for all $x < x_0$. In particular, $\lim_{x\to-\infty} u(0, x) = 1$, which yields that $u(t, x) \to 1$ as $t \to +\infty$ locally in $x \in \mathbb{R}$. Hence, Theorem 2.2 implies that u is increasing in time.

Next, as $\lim_{t\to-\infty} u(t, x) \leqslant u(0, x) \leqslant v(0, x)$ for all $x > x_0$ and as $\lim_{x\to+\infty} v(0, x) = 0$, one gets $\lim_{t\to-\infty} u(t, x) = 0$. It follows from Proposition 4.1 that u satisfies the translation property. Thus, the same arguments as above yield that u is a pulsating travelling waves, meaning that $u \equiv v$ by the uniqueness proved in [3]. The uniqueness up to translation in time follows from the same arguments as in the proof of Proposition 3.5 since u is time-increasing and connect 0 to 1. □

Proof of Proposition 4.2. If $\theta_0 = 0$, then Eq. (E) is monostable and we know from Proposition 4.3 that a critical travelling wave is a pulsating travelling wave of period L for all $L > 0$. This means that it is a planar travelling wave and the other results are proved similarly.

If $\theta_0 > 0$, then Theorem 2.4 in [11] asserts that there exists a planar travelling wave v with speed $c > 0$, which is unique up to translation and time-increasing, and that there exists no planar travelling wave with speed $c' \neq c$. The conclusion follows from the same arguments as in the proof of Proposition 4.3. □

9.3. The case of compactly supported heterogeneities

Proof of Proposition 4.5. 1. Let $c < \lambda/\sqrt{\lambda - 1}$. For all $t \in \mathbb{R}$, as $u(\cdot + t, \cdot)$ is a time-global solution, it follows from Theorem 4.4.1 that there exists $C_c = C_c(t) > 0$ such that $u(t+s, x) \leqslant C_c e^{-|x|+cs}$ for all $(s, x) \in (-\infty, 0) \times \mathbb{R}$. Letting $s \to 0^+$, this gives $u(t, x) \leqslant C_c e^{-|x|}$ for all $x \in \mathbb{R}$. Thus $u(t, \cdot) \in L^1(\mathbb{R})$.

2. We know from the parabolic regularity estimates that $\partial_x u$ is bounded over $\mathbb{R} \times \mathbb{R}$. Let $C > 0$ such that $|\partial_x u(t, x)| \leqslant C$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. For all $s \in \mathbb{R}$, as $u(s, X(s)) = \theta$, one has $u(s, x + X(s)) \geqslant (\theta - C|x|)_+$ for all $x \in \mathbb{R}$. Define $\underline{u}$ the solution of the Cauchy problem

$$\begin{cases} \partial_t \underline{u} - \partial_{xx} \underline{u} = \underline{u}\big(1 - C\underline{u}^\delta\big) & \text{in } (0, \infty) \times \mathbb{R}, \\ \underline{u}(0, x) = \big(\theta - C|x|\big)_+ & \text{for all } x \in \mathbb{R}. \end{cases}$$

It follows from $f_u'(x, 0) \geqslant 1$, (4.2) and the parabolic maximum principle that

$$u\big(t + s, x + X(s)\big) \geqslant \underline{u}(t, x) \quad \text{for all } s \in \mathbb{R} \text{ and } (t, x) \in (0, \infty) \times \mathbb{R}.$$

On the other hand, as $x \mapsto \underline{u}(0, x)$ is compactly supported and $u \mapsto u - Cu^{1+\delta}$ is of KPP type, it follows from [2] that $\lim_{t \to +\infty} \underline{u}(t, ct) = 1$ for all $c < 2$. Hence, $u(t + s, ct + X(s)) \to 1$ as $t \to +\infty$ uniformly in $s \in \mathbb{R}$, for all $c < 2$. Take $c < 2$ and let $T > 0$ such that $u(t + s, ct + X(s)) > \theta$ for all $t \geqslant T$ and $s \in \mathbb{R}$.

As u is also a spatial transition wave, there exists $L > 0$ such that $u(t, x + X(t)) < \theta$ for all $x > L$ and for all $t \in \mathbb{R}$. It follows that $ct + X(s) \leqslant X(t + s) + L$ for all $t > T$ and $s \in \mathbb{R}$. We eventually get

$$\liminf_{t \to +\infty} \inf_{s \in \mathbb{R}} \frac{X(t+s) - X(s)}{t} \geqslant c \quad \text{for all } c < 2.$$

On the other hand, as $\lambda \in (1, 2)$, we know from Theorem 4.4 that there exists a spatial transition wave v with global mean speed c for all $c \in (2, \lambda/\sqrt{\lambda - 1})$. Let $Y(t)$ be such that $v(t, Y(t)) = \theta$. It follows from Theorem 3.6 that

$$\limsup_{t \to +\infty} \sup_{s \in \mathbb{R}} \frac{1}{t}\big(X(s+t) - X(s)\big) \leqslant \lim_{t \to +\infty} \sup_{s \in \mathbb{R}} \frac{1}{t}\big(Y(s+t) - Y(s)\big) = c.$$

We conclude by letting $c \to 2$ in the two inequalities. □

9.4. The case of random stationary ergodic coefficients

Proof of Proposition 4.7. We first use the same types of arguments as in [27,30] in order to prove the measurability of u and T. For all $s < 0$, let $u_s^y = u_s^y(t, x, \omega)$ the unique solution of

$$\begin{cases} \partial_t u_s^y - a(x, \omega)\partial_{xx} u_s^y - b(x, \omega)\partial_x u_s^y = f\big(x, \omega, u_s^y\big) & \text{in } (s, \infty) \times \mathbb{R}, \\ u_s^y(s, x, \omega) = 1 \quad \text{if } x \leqslant y \quad \text{and} \quad u_s^y(s, x, \omega) = 0 \quad \text{if } x > y, \end{cases}$$

and $x_s(\omega)$ the unique real number such that $u_s^{x_s(\omega)}(0, 0) = \theta$. It can be proved exactly as in [27] that $\omega \in \Omega \mapsto x_s(\omega)$ and $\omega \mapsto u_s(t, x, \omega)$ are measurable functions for all $(t, x) \in \mathbb{R} \times \mathbb{R}$. As we know from the proof of Theorem 2.2 that $u(t, x, \omega) = \lim_{s \to +\infty} u_s^{x_s(\omega)}(t, x, \omega)$ for all $(t, x, \omega) \in \mathbb{R} \times \mathbb{R} \times \Omega$, the measurability of u in ω follows.

Let now prove the measurability of $\omega \mapsto T(x, \omega)$ for all $x \in \mathbb{R}$. Take $x \in \mathbb{R}$ and define for all $j, m \in \mathbb{N}$:

$$A_j^m = \big\{\omega \in \Omega,\ u(t, x, \omega) \geqslant \theta \text{ for all } t \leqslant j2^{-m}\big\}.$$

The measurability of A_j^m follows from the measurability of u. Define the measurable function:

$$\tau^m(\omega) := \max_{j\in\mathbb{N}} 2^{-m} j \chi_{A_j^m}(\omega).$$

It is easy to check that for all $\omega \in \Omega$, $\tau^m(\omega) = j2^{-m}$, where j is the unique integer such that $j2^{-m} \leqslant T(x,\omega) < (j+1)2^{-m}$. It follows that for all $\omega \in \Omega$,

$$\tau^m(\omega) \leqslant T(x,\omega) \leqslant \tau^m(\omega) + 2^{-m}.$$

Hence, $\lim_{m\to+\infty} \tau^m(\omega) = T(x,\omega)$ and thus $\omega \in \Omega \mapsto T(x,\omega)$ is measurable.

Next, let $\omega \in \Omega$ and $y \in \mathbb{R}$. As, for all $(x, y, u, \omega) \in \mathbb{R} \times \mathbb{R} \times [0, 1] \times \Omega$,

$$a(x+y,\omega) = a(x,\pi_y\omega), \qquad b(x+y,\omega) = b(x,\pi_y\omega) \quad \text{and} \quad f(x+y,\omega,u) = f(x,\pi_y\omega,u),$$

the functions $(t,x) \mapsto u(t+T(y,\omega), x+y, \omega)$ and $(t,x) \mapsto u(t,x,\pi_y\omega)$ are both critical travelling waves of Eq. (4.4) with $\pi_y\omega$ instead of ω. As $u(T(y,\omega), y, \omega) = \theta$ by definition of $T(y,\omega)$, $u(0,0,\pi_y\omega) = \theta$ by definition of u and the critical travelling wave is unique up to normalization, we eventually get

$$u\big(t+T(y,\omega), x+y, \omega\big) = u(t,x,\pi_y\omega) \quad \text{for all } (t,x,y,\omega) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega. \tag{9.2}$$

Considering this inequality at $t = T(x,\pi_y\omega)$, we obtain $u(T(x,\pi_y\omega) + T(y,\omega), x+y, \omega) = \theta$ and thus, as u is increasing in t, the definition of T implies that

$$T(x,\pi_y\omega) + T(y,\omega) = T(x+y,\omega) \quad \text{for all } (x,y,\omega) \in \mathbb{R}\times\mathbb{R}\times\Omega. \tag{9.3}$$

It follows from the Birkhoff ergodic theorem that the limit

$$\frac{1}{c^*} := \lim_{x\to+\infty} \frac{T(x,\omega)}{x} \text{ exists almost surely and does not depend on } \omega \in \Omega. \qquad \square$$

10. Proof of the results on attractivity and continuity

Theorem 5.2 will follow from several intermediate results, which are more general but require hypotheses involving the continuity of critical travelling waves with respect to the environment. It is not clear if such a continuity holds in general. Assume that a, b and f are uniformly continuous in $(x,u) \in \mathbb{R} \times [0,1]$, uniformly with respect to $u \in [0,1]$. As in Section 4, take $\in (0,1)$ and let

$$\mathcal{H} = cl\big\{\pi_y C = (\pi_y a, \pi_y b, \pi_y f),\ y \in \mathbb{R}\big\},$$

where the closure is associated with topology of the local convergence. This set is a complete metric space. Let $u(\cdot,\cdot;\tilde{C})$ be the unique critical travelling wave associated with the coefficients $\tilde{C} \in \mathcal{H}$ and normalized by $u(0,0;\tilde{C}) = \theta \in (0,1)$ for all $\tilde{C} \in \mathcal{H}$.

We say that $\tilde{C} \in \mathcal{H} \mapsto u(0,\cdot;\tilde{C}) \in \mathcal{C}^0(\mathbb{R})$ is *continuous* at $C_* \in \mathcal{H}$, or that the critical travelling wave is continuous at C_* if there is no ambiguity, if for all sequence $(C_n)_n$ in $\mathcal{H}$ which converges locally uniformly in $\mathbb{R} \times [0,1]$ to $C_* \in \mathcal{H}$, one has $\lim_{n\to+\infty} u(0,x;C_n) = u(0,x;C_*)$ locally uniformly in $x \in \mathbb{R}$. Then if the critical travelling wave is continuous at C_*, it attracts the solutions of the Cauchy problem associated with the Heaviside initial datum along a subsequence, as stated in the next proposition.

Proposition 10.1. *Assume that (H) is satisfied and that Eq. (E) is monostable. Let $\theta \in (0,1)$. Assume that $C \in \mathcal{H} \mapsto u(\cdot,\cdot;C)$ is continuous at $C_* = (a_*, b_*, f_*) \in \mathcal{H}$ and that there exists a sequence $(y_n)_n$ such that $\lim_{n\to+\infty} y_n = +\infty$, $\lim_{n\to+\infty} a(x+y_n) = a_*(x)$, $\lim_{n\to+\infty} b(x+y_n) = b_*(x)$ and $\lim_{n\to+\infty} f(x+y_n,u) = f_*(x,u)$ locally in $(x,u) \in \mathbb{R}\times[0,1]$. Let v be the solution of (5.1). Then*

$$v\big(S(y_n), x+y_n\big) - u\big(T(y_n), x+y_n; (a,b,f)\big) \to 0 \quad \textit{as } n \to +\infty \textit{ locally uniformly in } x \in \mathbb{R}.$$

This result shows that the continuity of the wave with respect to the coefficients of the equation is another relevant open question. As already mentioned in Definition 1.2, Matano requires the continuity of the wave with respect to $(\tilde{a}, \tilde{b}, \tilde{f}) \in \mathcal{H}$. Shen [30] proved the existence of such waves for general heterogeneous equations under the assumption that there exists a spatial transition wave which converges uniformly with respect to the translations of the coefficients, except that she did not prove the continuity of the wave with respect to $\tilde{C} \in \mathcal{H}$. She only proved such a continuity when the nonlinearity is bistable and depends on time. The full continuity remains an open problem. As already mentioned above, another open problem is the extension of these results to more general wave-like initial data.

However, Shen also proved in a general framework that the critical travelling wave is always continuous at least on a residual subset of $\mathcal{H}$. We recall that a residual subset is the intersection of countably many open dense subsets and that as $\mathcal{H}$ is a complete metric space, the Baire theorem yields that such a residual subset is dense in $\mathcal{H}$. Hence, the critical travelling waves are always continuous with respect to the environment at least at one point $C_* \in \mathcal{H}$. We will prove a similar property in Proposition 10.2 below, but through direct arguments, which provide an alternative approach to the topological one of [30].

Many other properties can be derived from continuity on the whole set $\mathcal{H}$, as observed in the framework of spatial transition waves by Shen [30]. Basically, the continuity of the wave ensures that any property of the coefficients, such as almost periodicity, almost automorphy or recurrence, is also satisfied, in a sense, by the wave. We will not go any further on this topic since we do not know if the wave is continuous over the whole set $\mathcal{H}$ or not.

Open Problem. Is it true that the critical travelling wave $u(\cdot,\cdot;g)$ is continuous with respect to $g \in \mathcal{H}$?

Proof of Proposition 10.1. First, as $\lim_{t\to+\infty} v(t,x) = 1$ for all $x \in \mathbb{R}$ and $v(0,x) = 0$ for all $x > 0$, the quantity $S(y) := \sup\{t > 0,\ v(t,x) \leqslant \theta\}$ is well-defined for all $y > 0$.

Assume by contradiction that there exists a sequence $(z_n)_n$ such that $\lim_{n\to+\infty} z_n = +\infty$ and $(S(z_n))_n$ is bounded. Then there exists $M > 0$ such that $S(z_n) \leqslant M$ for all n. In other words, one has $v(M, z_n) \geqslant \theta$ for all n. On the other hand, it easily follows from the strong maximum principle and the fact that $v(0,x) = 0$ for all $x > 0$ that $\lim_{x\to+\infty} v(t,x) = 0$ for all $t > 0$, which gives a contradiction. Hence $(S(z_n))_n$ is unbounded and as this is true for all sequence $(z_n)_n$ converging to $+\infty$, we get

$$\lim_{y\to+\infty} S(y) = +\infty.$$

Next, consider the critical travelling wave $u = u(t,x;C)$, where $C = (a,b,f)$. For all $n \in \mathbb{N}$, as

$$v(0,x) = 1 > u\big(T(y_n) - S(y_n), x; C\big) \quad \text{if } x < 0 \quad \text{and}$$
$$v(0,x) = 0 < u\big(T(y_n) - S(y_n), x; C\big) \quad \text{if } x > 0,$$

we know from Proposition 7.1 that the sets $\{v(t,\cdot) > u(t + T(y_n) - S(y_n), \cdot; C)\}$ and $\{v(t,\cdot) < u(t + T(y_n) - S(y_n), \cdot; C)\}$ are intervals for all $t > 0$. Taking $t = S(y_n)$, as $v(S(y_n), y_n) = \theta = u(T(y_n), y_n; C)$, one gets

$$v\big(S(y_n), x + y_n\big) > u\big(T(y_n), x + y_n; C\big) = u(0, x; \pi_{y_n} C) \quad \text{if } x < 0 \quad \text{and}$$
$$v\big(S(y_n), x + y_n\big) < u\big(T(y_n), x + y_n; C\big) = u(0, x; \pi_{y_n} C) \quad \text{if } x > 0, \tag{10.1}$$

where we have used the translation property (see Proposition 4.1).

On the other hand, we know that, as u is continuous at $C_* = (a_*, b_*, f_*) \in \mathcal{H}$ by assumption and as

$$\pi_{y_n} C = (\pi_{y_n} a, \pi_{y_n} b, \pi_{y_n} f) \to (a_*, b_*, f_*) = C_* \quad \text{as } n \to +\infty,$$

locally uniformly in $(x,u) \in \mathbb{R} \times [0,1]$, one has

$$\lim_{n\to+\infty} u(0, x; \pi_{y_n} C) = u(0, x; C_*) \quad \text{locally in } x \in \mathbb{R}.$$

Let $w_n(t,x) := v(t + S(y_n), x + y_n)$. This function satisfies

$$\partial_t w_n - a(x + y_n)\partial_{xx} w_n - b(x + y_n)\partial_x w_n = f(x + y_n, w_n) \quad \text{in } \big(-S(y_n), \infty\big) \times \mathbb{R}.$$

Hence, one can assume from parabolic regularity estimates that the sequence $(w_n)_n$ converges to a function w_∞ locally uniformly in $\mathbb{R}\times\mathbb{R}$. As $\lim_{n\to+\infty} S(y_n)=+\infty$, the function w_∞ is a time-global solution of

$$\partial_t w_\infty - a_*(x)\partial_{xx} w_\infty - b_*(x)\partial_x w_\infty = f_*(x, w_\infty) \quad \text{in } \mathbb{R}\times\mathbb{R}. \tag{10.2}$$

Moreover, letting $n\to+\infty$ in (10.1), one gets

$$w_\infty(0,x) \geqslant u(0,x;C_*) \quad \text{if } x<0 \quad \text{and} \quad w_\infty(0,x)\leqslant u(0,x;C_*) \quad \text{if } x>0.$$

But as u is a critical travelling wave and w_∞ is a time-global solution of (10.2), the reverse inequalities hold. Hence $w_\infty(0,x)=u(0,x;C_*)$ for all $x\in\mathbb{R}$. In other words, up to extraction, one has for all $x\in\mathbb{R}$:

$$\begin{aligned}\lim_{n\to+\infty}\big(v\big(S(y_n),x+y_n\big)-u\big(T(y_n),x+y_n;C\big)\big) &= \lim_{n\to+\infty}\big(w_n(0,x)-u(0,x;\pi_{y_n}C)\big)\\ &= w_\infty(0,x)-u(0,x;C_*)=0.\end{aligned}$$

As this convergence does not depend on the extraction, the convergence along the full sequence $(y_n)_n$ follows from standard arguments. □

Proposition 10.2. *Assume that (H) is satisfied and let $\theta\in(0,1)$. Define*

$$F:\mathcal{H}\to\mathcal{C}^0(\mathbb{R}),$$
$$\tilde{C}\mapsto u(0,\cdot;\tilde{C})$$

where u is the unique critical travelling wave associated with the coefficients $\tilde{C}$ and normalized by $u(0,0;\tilde{C})=\theta$. Then there exists at least one set of coefficients $C_*\in\mathcal{H}$ such that F is continuous at C_*.

Proof. Take $\tilde{C}=(\tilde{a},\tilde{b},\tilde{f})\in\mathcal{H}$ and $(C_n)_n$ a sequence of $\mathcal{H}$ such that $C_n\to\tilde{C}$ as $n\to+\infty$ in $\mathcal{H}$. For all $\tilde{C}\in\mathcal{H}$, define $u_s=u_s(t,x;\tilde{C})$ as in the proof of Theorem 2.2, with coefficients $\tilde{C}$ instead of (a,b,f), and $x_s(\tilde{C})$ such that $u_s(s,x;\tilde{C})=1$ if $x<x_s(\tilde{C})$, $u_s(s,x;\tilde{C})=0$ if $x>x_s(\tilde{C})$ and $u_s(0,0;\tilde{C})=\theta$.

We know from the parabolic regularity estimates that one can assume, up to extraction, that the sequence $(u_s(\cdot,\cdot;C_n))_n$ converges to a function v locally uniformly in $(s,\infty)\times\mathbb{R}$ as $n\to+\infty$. This function satisfies

$$\partial_t v-\tilde{a}(x)\partial_{xx}v-\tilde{b}(x)\partial_x v=\tilde{f}(x,v) \quad \text{in } (s,\infty)\times\mathbb{R} \text{ and } v(0,0)=\theta.$$

If $(x_s(C_n))_n$ converges to $+\infty$, then $u_s(s,x;C_n)\to1$ as $n\to+\infty$ locally in x and thus $v\equiv1$, which is a contradiction since $v(0,0)=\theta$. One gets a similar contradiction if $(x_s(C_n))_n$ converges to $-\infty$. As this is true along any subsequence, we conclude that $(x_s(C_n))_n$ is bounded. Extracting one more time, we can assume that this sequence converges, let $X:=\lim_{n\to+\infty}x_s(g_n)$. Then $v(s,x)=1$ if $x<X$, $v(s,x)=0$ if $x>X$. Hence, it follows from Lemma 7.2 that $X=x_s(\tilde{C})$ since $v(0,0)=\theta$ and thus $v(t,x)=u_s(t,x;\tilde{C})$. As this eventual limit does not depend on the previous extractions, we conclude that the full sequence $(x_s(C_n))_n$ converges to $(x_s(\tilde{C}))$ as $n\to+\infty$. Hence, the sequence $(u(0,\cdot;C_n))_n$ converges to $u_s(0,\cdot;\tilde{C})$ as $n\to+\infty$, which means that

$$\begin{aligned}&F_s:\mathcal{H}\to\mathcal{C}^0(\mathbb{R}),\\ &\tilde{C}\mapsto u_s(0,\cdot;\tilde{C})\end{aligned} \quad \text{is continuous for all } s<0.$$

But we also know that $u(0,\cdot;\tilde{C})=\lim_{s\to-\infty}u_s(0,\cdot;\tilde{C})$ for all $\tilde{C}\in\mathcal{H}$ locally uniformly over $\mathbb{R}\times\mathbb{R}$. In other words,

$$F_s(\tilde{C})\to F(\tilde{C}) \quad \text{as } s\to-\infty \text{ pointwise in } \tilde{C}\in\mathcal{H}.$$

It is a classical application of Baire theorem that the set of continuity points of the pointwise limit of continuous functions is a residual set. Hence, F is continuous on a non-empty (residual) subset of $\mathcal{H}$. □

Proof of Theorem 5.2. First, we know from Proposition 10.2 that there exists a set of coefficients $C_*\in\mathcal{H}$ such that the critical travelling wave is continuous at C_*. Let $(y_n)_n$ be such that $\pi_{y_n}C\to C_*$ as $n\to+\infty$ locally in $(x,u)\in\mathbb{R}\times[0,1]$. If $\lim_{n\to+\infty}y_n=+\infty$ Proposition 10.1 applies and gives the conclusion. If $(y_n)_n$ is bounded, then the recurrence at infinity assumption yields that we can change $(y_n)_n$ so that this sequence converges to $+\infty$ and conclude as in the previous case. □

Proof of Corollary 5.3. We can assume that the period L is the minimal periodic of f. That is: for all $\ell \in (0, L)$, there exists $(x, u) \in \mathbb{R} \times [0, 1]$ such that $a(x + \ell) \neq a(x)$, $b(x + \ell) \neq b(x)$ or $f(x + \ell, u) \neq f(x, u)$. Next, as the coefficients a, b and f are periodic in x, one has $\pi_L C = C$ and thus $\mathcal{H} = \{\pi_y C;\ y \in [0, L]\}$. Consider $\tilde{C} \in \mathcal{H}$ and a sequence $(C_n)_n$ in $\mathcal{H}$ such that $C_n \to \tilde{C}$ locally uniformly. Then there exist $(y_n)_n$ and $\tilde{y}$ in $[0, L)$ such that $C_n = \pi_{y_n} C$ for all n and $\tilde{C} = \pi_{\tilde{y}} C$. Consider an extraction $(y_{n'})_{n'}$ such that this sequence converges to a limit $y_\infty \in [0, L]$. One has $\pi_{y_\infty} C = \pi_{\tilde{y}} C$. As $-L < y_\infty - y < L$ and as L is the minimal period of the coefficients, one gets $y_\infty = \tilde{y}$. Hence, the full sequence $(y_n)_n$ converges to $\tilde{y}$. We conclude that $\tilde{C} \in \mathcal{H} \mapsto u(0, \cdot; \tilde{C}) \in \mathcal{C}^0(\mathbb{R})$ is continuous. The conclusion follows from Proposition 10.1. □

11. Proof of the results in the bistable framework

We start with a general result in the bistable framework. We do not consider the particular equation (6.1) yet.

Proposition 11.1. *Assume that (H) is satisfied and that there exists $\theta_0 \in (0, 1)$ such that for all $x \in \mathbb{R}$, $f(x, \theta_0) = 0$ and*

$$u \in (0, \theta_0) \mapsto f(x, u) \text{ is convex}, \qquad u \in (\theta_0, 1) \mapsto f(x, u) \text{ is concave}. \tag{11.1}$$

Let $x_0 \in \mathbb{R}$ and assume that there exists a stationary solution w of Eq. (E) such that $w(x_0) = \theta_0$, $w(x) > \theta_0$ for all $x < x_0$, $w(x) < \theta_0$ for all $x > x_0$, $\liminf_{x \to -\infty} w(x) > \theta_0$ and $\limsup_{x \to +\infty} w(x) < \theta_0$. Let u be the critical travelling wave normalized by $u(0, x_0) = \theta_0$. Then u is constant with respect to time and $u \equiv w$.

In other words, if there exists a non-trivial steady state w and if one considers a critical travelling waves u which crosses w, then, under mild bistability hypotheses, u does not depend on time and $u \equiv w$. Note f is not assumed to be positive (resp. negative) over $(\theta_0, 1)$ (resp. $(0, \theta_0)$).

This result will be derived from the following comparison result.

Lemma 11.2. *Assume that (H) and (11.1) are satisfied. Consider two $\mathcal{C}^1(\mathbb{R})$ weak solutions w_1 and w_2 of*

$$\begin{cases} -a(x)w_1'' - b(x)w_1' \geqslant f(x, w_1) & \text{in } (-\infty, x_1), \\ -a(x)w_2'' - b(x)w_2' \leqslant f(x, w_2) & \text{in } (-\infty, x_1), \\ \liminf\limits_{x \to -\infty} w_1(x) > \theta_0, \\ w_1(x) > \theta_0 \quad \text{for all } x \leqslant x_1 \quad \text{and} \quad w_1(x_1) \geqslant w_2(x_1), \end{cases}$$

for some $x_1 \in \mathbb{R}$. Then $w_1 \geqslant w_2$ in $(-\infty, x_1)$.

Proof of Lemma 11.2. Let

$$\kappa^* = \inf\bigl\{\kappa > 0,\ (1 + \kappa)\bigl(w_1(x) - \theta_0\bigr) \geqslant w_2(x) - \theta_0 \text{ for all } x \in (-\infty, x_1)\bigr\}.$$

As $w_1(x) > \theta_0$ for all $x \leqslant x_1$, $\liminf_{x \to -\infty} w_1(x) > \theta_0$ and w_2 is bounded, this quantity is well-defined. If $\kappa^* = 0$, then $w_1(x) \geqslant w_2(x)$ for all $x \in (-\infty, x_1)$, which ends the proof. Assume by contradiction that $\kappa^* > 0$. Then $(1 + \kappa^*)(w_1(x) - \theta_0) \geqslant w_2(x) - \theta_0$ for all $x \in (-\infty, x_1]$ and there exists $x_* \in (-\infty, x_1]$ such that $(1 + \kappa^*)(w_1(x_*) - \theta_0) = w_2(x_*) - \theta_0$. As $w_1(x_1) \geqslant w_2(x_1)$ and $\kappa^* > 0$, one gets $x_* \neq x_1$.

Define $z(x) := (1 + \kappa^*)(w_1(x) - \theta_0) - w_2(x) + \theta_0$ for all $x \in \mathbb{R}$. This function is nonnegative, vanishes at $x = x_*$ and it follows from the Lipschitz-continuity and the concavity of f on $[\theta_0, 1]$ that

$$\begin{aligned} -a(x)z'' - b(x)z' &\geqslant \bigl(1 + \kappa^*\bigr) f(x, w_1 - \theta_0 + \theta_0) - f(x, w_2) \\ &\geqslant f\bigl(x, \bigl(1 + \kappa^*\bigr)(w_1 - \theta_0) + \theta_0\bigr) - f(x, w_2) \\ &\geqslant -C\bigl|\bigl(1 + \kappa^*\bigr)(w_1 - \theta_0) + \theta_0 - w_2\bigr| = -Cz \end{aligned}$$

in $(-\infty, x_1)$. As z reaches its minimum and vanishes at the interior point x_*, the strong maximum principle yields that $z \equiv 0$, which is a contradiction since

$$z(x_1) = \big(1+\kappa^*\big)\big(w_1(x_1)-\theta_0\big) - w_2(x_1) - \theta_0 \geqslant \kappa^*\big(w_1(x_1)-\theta_0\big) > 0.$$

Hence $\kappa^* = 0$ and $w_1 \geqslant w_2$ in $(-\infty, x_1)$. □

Proof of Proposition 11.1. We know from Theorem 2.2 that u is either increasing, decreasing or constant with respect to time. Assume first that u is time-increasing. We know from the proof of Theorem 2.2 above that $\partial_t u > 0$ in $\mathbb{R}\times\mathbb{R}$. As $u(0,x_0) = w(x_0) = \theta_0$ and $u(0,\cdot) \not\equiv w$ since u is increasing in time, the criticality of u yields $u(0,x) > w(x)$ for all $x < x_0$ and $u(0,x) < w(x)$ for all $x > x_0$.

Next, we know from Lemma 5.4 in [1] that there exist $\tau > 0$ and a continuous function $\xi : [-\tau, 0] \to \mathbb{R}$ such that $\xi(0) = x_0$ and $u(t,\xi(t)) = w(\xi(t))$ for all $t \in [-\tau, 0]$. Take $t \in [-\tau, 0)$ and let $w_2(x) := u(t,x)$ for all $x \in \mathbb{R}$. As $\partial_t u > 0$, w_2 satisfies

$$-a(x)w_2'' - b(x)w_2' < f(x, w_2) \quad \text{in } \mathbb{R}. \tag{11.2}$$

As u is time-increasing and $t \in [-\tau, 0)$, one has $u(t,\xi(t)) = w(\xi(t)) < u(0,\xi(t))$ and thus $\xi(t) < x_0$ since $u(0,x) \leqslant w(x)$ for all $x \geqslant x_0$. Hence, $w(\xi(t)) > \theta_0$ by hypothesis. Thus, Lemma 11.2 applies and gives $w_2(x) = u(t,x) \leqslant w(x) = w_1(x)$ for all $x < \xi(t)$ and $t \in [-\tau, 0)$. Letting $t \to 0^-$, as ξ is continuous, one gets $u(0,x) \leqslant w(x)$ for all $x < \xi(0) = x_0$, a contradiction.

A contradiction is reached similarly if u is time-decreasing, by applying the first step to the auxiliary function $v = 1 - u$. Hence, $u = u(x)$ does not depend on time and thus $u \equiv w$. □

Proof of Proposition 6.1. If $x_0 = x_-$ or $x_0 = x_+$, then the conclusion immediately follows from Proposition 11.1.

Take now $x_0 \in \mathbb{R}\backslash\{x_+, x_-\}$ and u the critical travelling wave normalized by $u(0,x_0) = \theta_0$. First, assume by contradiction that $u = u(x)$ is a stationary solution of Eq. (E). We know that $u \not\equiv w_+$ and $u \not\equiv w_-$ since $w_\pm(x_0) \neq u(x_0) = \theta_0$. The criticality of u ensures that $u(x) > \theta_0$ for all $x < x_0$ and $u(x) < \theta_0$ for all $x > x_0$. Otherwise, one would have $u \equiv \theta_0$ and it would follow from Proposition 11.1 that $u \equiv w_+$ since $\theta_0 = u(x_+) = w_+(x_+)$, which is excluded. Together with the hypotheses on f, this implies in particular that $-u''(x) = f(x, u(x)) \geqslant 0$ (resp. $\leqslant 0$) for all $x \leqslant x_0$ (resp. $x \geqslant x_0$), from which it is easy to derive that u is nonincreasing in x and that $u(-\infty) = 1$ and $u(+\infty) = 0$. But then, the uniqueness of the solutions of (6.2), proved in [28], yields that $u \equiv w_+$ or w_-, which gives the final contradiction.

Hence, u is not a stationary solution and Theorem 2.2 ensures that u is either increasing or decreasing with respect to time. One has $u(t,x_-) \neq w_-(x_-) = \theta_0$ (resp. $u(t,x_+) \neq w_+(x_+)$) for all $t \in \mathbb{R}$, otherwise Proposition 11.1 would yield that $u(t_0,\cdot) \equiv w_-$ (resp. w_+) for some $t_0 \in \mathbb{R}$ and thus u would be constant in time.

Next, assume that $x_0 < x_-$. As u is a critical travelling wave and as $u \not\equiv \theta_0$, we know that

$$u(0,x) > \theta_0 \quad \text{if } x < x_0 \quad \text{and} \quad u(0,x) < \theta_0 \quad \text{if } x > x_0.$$

It follows that $u(0,x_-) < u(0,x_0) = \theta_0 = w_-(x_-)$ and the previous remark yields that $u(t,x_-) < w_-(x_-)$ for all $t \in \mathbb{R}$. Take $t \in \mathbb{R}$, if there exists $x_t \in \mathbb{R}$ such that $u(t,x_t) = w_-(x_t)$, then as u is critical, one has $u(t,x) < w_-(x)$ if $x > x_t$ and $u(t,x) > w_-(x)$ if $x < x_t$. Hence, $x_- > x_t$ and one has

$$u(t,x) < w_-(x) \quad \text{for all } t \in \mathbb{R},\ x \geqslant x_-. \tag{11.3}$$

Assume by contradiction that u is decreasing with respect to time. Let $u_\infty(x) := \lim_{t\to-\infty} u(t,x)$. This function satisfies $\theta_0 < u(0,x) < u_\infty(x)$ for all $x \leqslant x_0$ and $-u_\infty'' = f(x, u_\infty)$ over $\mathbb{R}$. It is easy to derive from this, together with the hypotheses on f, that $x \in (-\infty, x_0) \mapsto u_\infty(x)$ is nonincreasing, from which we get $\lim_{x\to-\infty} u_\infty(x) = 1$. Moreover, (11.3) implies $u_\infty(x) \leqslant w_-(x)$ for all $x \geqslant x_-$ and thus $\lim_{x\to+\infty} u_\infty(x) = 0$. We conclude from the uniqueness result proved in [19] that $u_\infty \equiv w_-$.

Next, we know from [19] that there exists a subsolution $0 < \underline{w} < w_-$ of Eq. (6.2) with $\underline{w}(-\infty) = 1$ and $\underline{w}(+\infty) = 0$. Consider $X \in \mathbb{R}$ such that $\max\{\underline{w}(X), u(0,X)\} < \theta_0 < w_-(X)$. As $\lim_{t\to-\infty} u(t,X) = w_-(X)$, there exists $T < 0$ such that $u(T,X) = \theta_0$. As u is decreasing with respect to time, one has $-\partial_{xx}u(T,x) \geqslant f(x, u(T,x))$ over $\mathbb{R}$. As u is critical, $u(T,x) > \theta_0$ for all $x < X$ and $u(T,x) < \theta_0$ for all $x > X$. Hence, Lemma 11.2 applies with $w_1 = u(T,\cdot)$,

$w_2 = \underline{w}$ and $x_1 < X$ small, leading to $u(T, x) \geqslant \underline{w}(x)$ for all $x < X$. Hence, $u^\infty(x) := \lim_{t\to+\infty} u(t, x)$ is a steady state such that $\underline{w} \leqslant u^\infty$ in $(-\infty, X)$ and $u_\infty < w_-$, which contradicts the uniqueness of $w_\pm$.

We have thus proved that

$$u \text{ is increasing with respect to time.}$$

Define $u^\infty(x) = \lim_{t\to+\infty} u(t, x)$. This function is a steady state such that $u^\infty(x) \geqslant u(0, x)$ and $u^\infty(x) \leqslant w_-(x)$ for all $x \geqslant x_-$. This gives $\lim_{x\to-\infty} u^\infty(x) = 1$ and $\lim_{x\to+\infty} u^\infty(x) = 0$, and thus

$$\lim_{t\to+\infty} u(t, x) = u^\infty(x) = w_-(x) \quad \text{for all } x \in \mathbb{R}.$$

Next, let $u_\infty(x) = \lim_{t\to-\infty} u(t, x) \leqslant u^\infty(x) = w_-(x)$ for all $x \in \mathbb{R}$. We know from Lemma 5.5 in [1] that there exists a continuous function $\xi : (-\infty, 0]$ such that $u(t, \xi(t)) = \theta_0$ for all $t \leqslant 0$, with $\xi(0) = x_0$. The criticality of u implies

$$u(t, x) > \theta_0 \quad \text{for all } x < \xi(t) \quad \text{and} \quad u(t, x) < \theta_0 \quad \text{for all } x > \xi(t).$$

Clearly, $\xi(t) \leqslant x_-$ for all $t \leqslant 0$ since $u(t, x) < w_-(x)$ for all $x \in \mathbb{R}$. Assume that there exists a sequence $(t_n)_n$ such that $\lim_{n\to+\infty} t_n = -\infty$ and the sequence $(\xi(t_n))_n$ converges to a limit $\xi_\infty \in \mathbb{R}$ as $n \to +\infty$. Then $u_\infty(x) \geqslant \theta_0$ for all $x < \xi_\infty$ and $u_\infty(x) \leqslant \theta_0$ for all $x > \xi_\infty$. The hypotheses on f ensure that $u''_\infty(x) = -f(x, u_\infty(x)) \leqslant 0$ for all $x \leqslant \xi_\infty$. As $u'_\infty(\xi_\infty) \leqslant 0$, it follows that $x \in (-\infty, \xi_\infty) \mapsto u_\infty(x)$ is nonincreasing and thus $\ell := \lim_{x\to-\infty} u_\infty(x)$ is well-defined. This limit satisfies $f_0(\ell) = 0$, implying that $\ell = \theta_0$ or $\ell = 1$. If $\ell = \theta_0$, then $u_\infty(x) = \theta_0$ for all $x < \xi_\infty$ and thus $u_\infty \equiv \theta_0$, which contradicts $u_\infty < w_-$. Thus $\ell = 1$, that is, $\lim_{x\to-\infty} u_\infty(x) = 1$. The uniqueness of $w_\pm$ would thus give $u_\infty \equiv w_-$, a contradiction. Hence, $\lim_{t\to-\infty} \xi(t) = -\infty$, which eventually gives $u_\infty \leqslant \theta_0$ over $\mathbb{R}$. Hence, $u''_\infty = -f(x, u_\infty) \geqslant 0$, and as u_∞ is bounded, it is constant. As $u_\infty < w_-$, one has $u_\infty \equiv 0$. In other words,

$$\lim_{t\to-\infty} u(t, x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

The identification of u when $x_0 \in (x_-, x_+)$ or $x > x_+$ is proved through similar arguments. □

12. Summary of the results

We have introduced a new notion of critical travelling waves for reaction–diffusion equations with arbitrary nonlinearity and general heterogeneous coefficients (Definition 2.1). These waves always exist, are monotonic in time and unique up to normalization (Theorem 2.2). If there exists a spatial transition wave, then critical travelling waves are necessarily spatial transition waves (Theorem 3.1). Hence, for ignition-type equations, the two notions are equivalent (Corollary 3.3). For monostable equations, critical travelling waves always exist, unlike spatial transition waves, and if there exists a spatial transition wave, then critical travelling waves have minimal least mean speed (Theorem 3.6).

In the cases where the critical travelling waves are unique up to translation in time, such as ignition-type or monostable equations, these waves satisfy a property which is close to the translation property introduced in [20] (Proposition 4.1). We derive from this result that if the coefficients are homogeneous/periodic, then the critical transition waves are planar/pulsating travelling waves in the ignition-type and monostable frameworks (Propositions 4.2 and 4.3) as well as in the bistable setting for homogeneous coefficients. If the heterogeneity of the coefficients is compactly supported, as in [26], then critical travelling waves are spatial transition waves with minimal speed if such waves exist, and bump-like solutions otherwise (Proposition 4.5). If the equation is monostable and the coefficients are random stationary ergodic in space, then the wave and its interface also satisfy such a dependence in space and admit a propagation speed, in a sense (Proposition 4.7). In the monostable setting, if the coefficients are "recurrent at infinity", then critical travelling waves attract the solution of the Cauchy problem with a Heaviside initial datum (Theorem 5.2) along a subsequence. Lastly, if the equation is bistable, then there might exist non-trivial steady states which block the propagation and in this case the identification of critical transition waves depends on the normalization of these waves (Proposition 6.1).

Conflict of interest statement

The author certifies that there is no conflict of interest regarding the results presented in this article.

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References

[1] S. Angenent, The zero set of a solution of a parabolic equation, J. Reine Angew. Math. 390 (1988) 79–96.
[2] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, Adv. Math. 30 (1978) 33–76.
[3] H. Berestycki, F. Hamel, Front propagation in periodic excitable media, Commun. Pure Appl. Math. 55 (2002) 949–1032.
[4] H. Berestycki, F. Hamel, Generalized travelling waves for reaction–diffusion equations, in: Perspectives in Nonlinear Partial Differential Equations. In Honor of H. Brezis, in: Contemp. Math., vol. 446, Amer. Math. Soc., 2007, pp. 101–123.
[5] H. Berestycki, F. Hamel, Generalized transition waves and their properties, Commun. Pure Appl. Math. 65 (2012) 592–648.
[6] H. Berestycki, F. Hamel, H. Matano, Bistable travelling waves around an obstacle, Commun. Pure Appl. Math. 62 (2009) 729–788.
[7] H. Berestycki, F. Hamel, L. Rossi, Liouville-type results for semilinear elliptic equations in unbounded domains, Ann. Mat. Pura Appl. 186 (2007) 469–507.
[8] G. Chapuisat, E. Grenier, Existence and nonexistence of traveling wave solutions for a bistable reaction–diffusion equation in an infinite cylinder whose diameter is suddenly increased, Commun. Partial Differ. Equ. 30 (2005) 1805–1816.
[9] A. Ducrot, T. Giletti, H. Matano, Existence and convergence to a propagating terrace in one-dimensional reaction–diffusion equations, Trans. Am. Math. Soc. (2014), http://dx.doi.org/10.1090/S0002-9947-2014-06105-9, in press.
[10] J. Fang, X.-Q. Zhao, Bistable traveling waves for monotone semiflows with applications, J. Eur. Math. Soc. (2014), in press.
[11] P.C. Fife, J.B. McLeod, The approach of solutions of nonlinear diffusion equations by travelling front solutions, Arch. Ration. Mech. Anal. 65 (1977) 335–361.
[12] R.A. Fisher, The advance of advantageous genes, Annu. Eugen. 7 (1937) 335–369.
[13] T. Giletti, Convergence to pulsating traveling waves with minimal speed in some KPP heterogeneous problems, Calc. Var. Partial Differ. Equ. (2014), http://dx.doi.org/10.1007/s00526-013-0674-9, in press.
[14] F. Hamel, Qualitative properties of monostable pulsating fronts: exponential decay and monotonicity, J. Math. Pures Appl. 89 (2008) 355–399.
[15] F. Hamel, N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$, Arch. Ration. Mech. Anal. 157 (2001) 91–163.
[16] S. Heinze, Wave solutions for reaction–diffusion systems in perforated domains, Z. Anal. Anwend. 20 (2001) 661–670.
[17] A.N. Kolmogorov, I.G. Petrovsky, N.S. Piskunov, Etude de l équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, Bull. Univ. Etat. Moscou (Bjul. Moskowskogo Gos. Univ.) (1937) 1–26.
[18] N.V. Krylov, M.V. Safonov, A certain property of solutions of parabolic equations with measurable coefficients, Math. USSR, Izv. 16 (1981) 151–164.
[19] J. Lewis, J.P. Keener, Wave-block in excitable media due to regions of depressed excitability, SIAM J. Appl. Math. 61 (2000) 293–316.
[20] H. Matano, Traveling waves in spatially inhomogeneous diffusive media, Oral communications.
[21] A. Mellet, J. Nolen, J.-M. Roquejoffre, L. Ryzhik, Stability of generalized transition fronts, Commun. Partial Differ. Equ. 34 (2009) 521–552.
[22] A. Mellet, J.-M. Roquejoffre, Y. Sire, Existence of generalized transition fronts in reaction–diffusion equations, Discrete Contin. Dyn. Syst., Ser. A 26 (2010) 303–312.
[23] G. Nadin, Traveling fronts in space–time periodic media, J. Math. Pures Appl. 92 (2009) 232–262.
[24] G. Nadin, L. Rossi, Propagation phenomenas for time heterogeneous KPP reaction–diffusion equations, J. Math. Pures Appl. 98 (2012) 633–653.
[25] G. Nadin, L. Rossi, Transition waves for Fisher-KPP equations with general time-heterogeneous and space-periodic coefficients, preprint.
[26] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, A. Zlatos, Existence and non-existence of Fisher-KPP transition fronts, Arch. Ration. Mech. Anal. 203 (2012) 217–246.
[27] J. Nolen, L. Ryzhik, Traveling waves in a one-dimensional random medium, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 26 (2009) 1021–1047.
[28] J.P. Pauwelussen, Nerve impulse propagation in a branching nerve system: a simple model, Phys. D 4 (1981) 67–88.
[29] L. Rossi, L. Ryzhik, Transition waves for a class of space–time dependent monostable equations, Commun. Math. Sci. (2014), in press.
[30] W. Shen, Traveling waves in diffusive random media, J. Dyn. Differ. Equ. 16 (2004) 1011–1060.
[31] W. Shen, Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations, J. Dyn. Differ. Equ. 23 (2011) 1–44.
[32] N. Shigesada, K. Kawasaki, E. Teramoto, Traveling periodic waves in heterogeneous environments, Theor. Popul. Biol. 30 (1986) 143–160.
[33] K. Uchiyama, The behavior of solutions of some nonlinear diffusion equations for large time, J. Math. Kyoto Univ. 18 (1978) 453–508.
[34] J. Xin, Existence of planar flame fronts in convective–diffusive periodic media, Arch. Ration. Mech. Anal. 121 (1992) 205–233.

[35] J. Xin, Existence and nonexistence of traveling waves and reaction–diffusion front propagation in periodic media, J. Stat. Phys. 73 (1993) 893–926.
[36] A. Zlatos, Generalized travelling waves in disordered media: existence, uniqueness, and stability, Arch. Ration. Mech. Anal. 208 (2013) 447–480.
[37] A. Zlatos, Transition fronts in inhomogeneous Fisher-KPP reaction–diffusion equations, J. Math. Pures Appl. 98 (2012) 89–102.
[38] A. Zlatos, Propagation of reaction in inhomogeneous media, preprint.