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The boundary value problem for the super-Liouville equation [☆]

Jürgen Jost [a], Guofang Wang [b], Chunqin Zhou [c], Miaomiao Zhu [d,*]

[a] Max Planck Institute for Mathematics in the Sciences, Inselstr. 22, D-04103 Leipzig, Germany
[b] Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, D-79104 Freiburg, Germany
[c] Department of Mathematics, Shanghai Jiaotong University, Shanghai, 200240, China
[d] Mathematics Institute, University of Warwick, CV4 7AL, Coventry, UK

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Abstract

We study the boundary value problem for the — conformally invariant — super-Liouville functional

$$E(u,\psi)=\int\limits_M\left\{\frac{1}{2}|\nabla u|^2+K_g u+\langle(\not{D}+e^u)\psi,\psi\rangle-e^{2u}\right\}dz$$

that couples a function u and a spinor ψ on a Riemann surface. The boundary condition that we identify (motivated by quantum field theory) couples a Neumann condition for u with a chirality condition for ψ. Associated to any solution of the super-Liouville system is a holomorphic quadratic differential $T(z)$, and when our boundary condition is satisfied, T becomes real on the boundary. We provide a complete regularity and blow-up analysis for solutions of this boundary value problem.

1. Introduction

In [12], we have introduced the *super-Liouville functional*, a conformally invariant functional that couples a real-valued function u and a spinor ψ on a Riemann surface M with conformal metric g and a spin structure,

$$E(u,\psi)=\int\limits_M\left\{\frac{1}{2}|\nabla u|^2+K_g u+\langle(\not{D}+e^u)\psi,\psi\rangle-e^{2u}\right\}dz. \tag{1}$$

Here K_g is the Gaussian curvature of M. The Dirac operator $\not{D}$ is defined by $\not{D}\psi:=\sum_{\alpha=1}^{2}e_\alpha\cdot\nabla_{e_\alpha}\psi$, where $\{e_1,e_2\}$ is an orthonormal basis on TM and ∇ is the connection on the spinor bundle ΣM of M, which is induced from the Levi-Civita connection on M with respect to g and $\cdot$ denotes the Clifford multiplication in the spinor bundle ΣM.

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* Corresponding author.
E-mail addresses: jost@mis.mpg.de (J. Jost), Guofang.Wang@math.uni-freiburg.de (G. Wang), cqzhou@sjtu.edu.cn (C. Zhou), Miaomiao.Zhu@warwick.ac.uk (M. Zhu).

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Finally, $\langle\cdot,\cdot\rangle$ is the natural Hermitian metric on ΣM induced by g. The system of Euler–Lagrange equations associated to (1) is called the super-Liouville equation. For the geometric background, see [15] or [10].

In this paper, we wish to address the boundary value problem for the super-Liouville functional. We therefore first of all need to identify the appropriate boundary condition for the spinor field ψ. Since the super-Liouville functional is inspired by quantum field theory, we likewise turn to the physics literature [2,6,16,17] to get some clue about a natural boundary condition. This boundary condition then will be of chirality type. The main point of the paper then is an analytical investigation of solutions of the boundary value problem. In particular, we shall show the regularity of solutions and identify the blow-up behavior for limits of sequences of solutions. In other words, we analytically understand the non-compactness of the solution space.

The key property of the functional is of course its conformal invariance. Therefore, the boundary conditions to be imposed likewise need to be conformally invariant. Conformal invariance on one hand makes the solution space non-compact, but on the other hand allows for a control of limits of solutions via a blow-up analysis. This is, of course, a well-known scheme, but the details are technically somewhat tricky and interesting.

Conformal invariance, like any invariance, by Noether's theorem leads to some conserved current. For two-dimensional conformally invariant variational problems, this conserved quantity can be identified with a holomorphic quadratic differential associated to a solution. From this perspective, our boundary condition is the natural one, because it renders that holomorphic quadratic differential real on the boundary. At a more technical level, this is important for the study of the asymptotic behavior of an entire solution on the upper half-plane with finite energy. Also, our boundary condition allows for the reflection of solutions across the boundary, which, at least heuristically, reduces boundary to interior regularity and which therefore, technically, is a useful device.

In [5], we have investigated the chirality boundary condition for Dirac-harmonic maps. Since in the present case, the coupling between the two fields is different, so then necessarily is the boundary analysis. Since the Liouville field u is scalar valued, in particular, here we can achieve a more precise blow-up analysis at the boundary. Since the chirality boundary condition is of physical interest, its general mathematical understanding should be useful.

2. The boundary value problem for the super-Liouville equation

In this section, we shall derive the boundary condition to be imposed on solutions of the super-Liouville equation. Thus, let M be a compact Riemann surface with smooth boundary ∂M and with a fixed spin structure. When $\partial M \neq \emptyset$, we know that the Laplacian operator Δ is in general not formally self-adjoint, and neither is the Dirac operator $\not{D}$. In fact, we have

$$\int\limits_M \langle\psi, \not{D}\varphi\rangle\, dv = \int\limits_M \langle\not{D}\psi, \varphi\rangle\, dv - \int\limits_{\partial M} \langle\vec{n}\cdot\psi, \varphi\rangle\, dv$$

for all $\psi, \varphi \in C^\infty(\Sigma M)$. Here $\vec{n}$ is the outward unit normal vector field on ∂M.

As is well known, the natural boundary condition for the function u is of Neumann type. This condition is clearly conformally invariant. We now shall derive a boundary conditions for the spinor field ψ that is likewise conformally invariant.

We recall the chirality boundary conditions for the Dirac operator $\not{D}$ first introduced in [7]. See also [9]. Let M be a compact Riemann surface with $\partial M \neq \emptyset$ and with a fixed spin structure, admitting a chirality operator G, which is an endomorphism of the spinor bundle ΣM satisfying:

$$G^2 = I, \qquad \langle G\psi, G\varphi\rangle = \langle\psi, \varphi\rangle,$$

and

$$\nabla_X(G\psi) = G\nabla_X\psi, \qquad X\cdot G\psi = -G(X\cdot\psi),$$

for any $X \in TM$, $\psi, \varphi \in \Gamma(\Sigma M)$. Here I denotes the identity endomorphism of ΣM.

We usually take $G = \gamma(\omega_2)$, the Clifford multiplication by the complex volume form $\omega_2 = ie_1e_2$, where e_1, e_2 is a local orthonormal frame on M.

Let

$$S := \Sigma M|_{\partial M}$$

denote the restricted spinor bundle with induced Hermitian product. The outward unit normal vector field $\vec{n}$ induces an operator $\vec{n}G : \Gamma(S) \to \Gamma(S)$, which is a self-adjoint endomorphism satisfying

$$(\vec{n}G)^2 = I, \qquad \langle \vec{n}G\psi, \varphi\rangle = \langle \psi, \vec{n}G\varphi\rangle.$$

Hence, we can decompose $S = V^+ \oplus V^-$, where $V^\pm$ is the eigensubbundle corresponding to the eigenvalue ± 1 of $\vec{n}G$. One can check that the orthogonal projection onto the eigensubbundle $V^\pm$:

$$\begin{aligned} B^\pm : L^2(S) &\to L^2(V^\pm) \\ \psi &\mapsto \frac{1}{2}(I \pm \vec{n}G)\psi, \end{aligned}$$

defines a local elliptic boundary condition for the Dirac operator $\not{D}$, see [9]. We say that a spinor $\psi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ satisfies the chirality boundary conditions $B^\pm$ if

$$B^\pm \psi|_{\partial M} = 0.$$

It is shown in [9] that if $\psi, \varphi \in W^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ satisfy the chirality boundary conditions, resp., then

$$\langle \vec{n}\cdot\psi, \varphi\rangle = 0, \quad \text{on } \partial M.$$

In particular,

$$\int_{\partial M} \langle \vec{n}\cdot\psi, \varphi\rangle = 0. \tag{2}$$

It follows that the Dirac operator $\not{D}$ is self-adjoint when we impose the chirality boundary conditions.

Let us note that on a surface the (usual) Dirac operator $\not{D}$ can be seen as the (doubled) Cauchy–Riemann operator. Consider $\mathbb{R}^2$ with the Euclidean metric $ds^2 + dt^2$. Let $e_1 = \frac{\partial}{\partial s}$ and $e_2 = \frac{\partial}{\partial t}$ be the standard orthonormal frame. A spinor field is simply a map $\Psi : \mathbb{R}^2 \to \Delta_2 = \mathbb{C}^2$, and the Cliford multiplication of e_1 and e_2 acting on spinor fields can be identified by the multiplication with matrices

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here, without loss of generality, we keep e_1 and e_2 consistence with that in [5]. If exchanging e_1 and e_2, then e_1 and e_2 are consistent with that in [12] and this case can be handled analogously. If $\Psi := \binom{f}{g} : \mathbb{R}^2 \to \mathbb{C}^2$ is a spinor field, then the Dirac operator is

$$\not{D}\Psi = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial g}{\partial s} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial g}{\partial t} \end{pmatrix} = 2i\begin{pmatrix} \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial \bar{z}} \end{pmatrix},$$

where

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial t}\right).$$

Therefore, the elliptic estimates developed for (anti-)holomorphic functions can be used to study the Dirac equation.

If M is the upper half Euclidean space $\mathbb{R}^2_+$, then the chirality operator is simply $G = ie_1e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that $\vec{n} = -e_2$, we get that

$$B^\pm = \frac{1}{2}(I \pm \vec{n}\cdot G) = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

By the standard chirality decomposition, we can write $\psi = \binom{\psi_+}{\psi_-}$, then the boundary condition becomes

$$\psi_+ = \mp\psi_- \quad \text{on } \partial\mathbb{R}^2_+.$$

In this paper, we will consider the functional

$$E_B(u,\psi)=\int_M\left\{\frac{1}{2}|\nabla u|^2+K_g u+\big\langle(\not{D}+e^u)\psi,\psi\big\rangle-e^{2u}\right\}dv+\int_{\partial M}\left\{h_g u-ce^u\right\}d\sigma, \tag{3}$$

where h_g is geodesic curvature of ∂M and c is a given constant.

Proposition 2.1. *The Euler–Lagrange system for $E_B(u,\psi)$ with Neumann/chirality boundary conditions is*

$$\begin{cases} -\Delta u=2e^{2u}-e^u\langle\psi,\psi\rangle-K_g, & \text{in } M^o,\\ \not{D}\psi=-e^u\psi, & \text{in } M^o,\\ \dfrac{\partial u}{\partial n}=ce^u-h_g, & \text{on } \partial M,\\ B^{\pm}\psi=0, & \text{on } \partial M. \end{cases} \tag{4}$$

Here Δ is the Laplacian with respect to g, and K_g is the Gaussian curvature in M, and h_g is the geodesic curvature of ∂M.

Proof. Let u_t be a family of function with $\frac{\partial u_t}{\partial t}|_{t=0}=\eta$, and let ψ_t be a family of spinor with $\frac{\partial\psi_t}{\partial t}|_{t=0}=\xi$. Since

$$\begin{aligned}\frac{dE_B(u,\psi_t)}{dt}\bigg|_{t=0}&=\int_M\langle\not{D}\xi,\psi\rangle+\langle\not{D}\psi,\xi\rangle+e^u\langle\xi,\psi\rangle+e^u\langle\psi,\xi\rangle\,dv\\ &=2\int_M\mathrm{Re}\langle\xi,\not{D}\psi\rangle+2e^u\,\mathrm{Re}\langle\xi,\psi\rangle\,dv-\int_{\partial M}\langle\xi,\vec{n}\cdot\psi\rangle\,d\sigma,\end{aligned}$$

and

$$\begin{aligned}&\frac{dE_B(u_t,\psi)}{dt}\bigg|_{t=0}\\ &=\int_M\nabla u\cdot\nabla\eta+K_g\eta+\eta e^u\langle\psi,\psi\rangle-2e^{2u}\eta\,dv+\int_{\partial M}h_g\eta-ce^u\eta\,d\sigma\\ &=-\int_M\eta\Delta u\,dv+\int_{\partial M}\eta\frac{\partial u}{\partial n}\,d\sigma+\int_M K_g\eta+\eta e^u\langle\psi,\psi\rangle-2e^{2u}\eta\,dv+\int_{\partial M}h_g\eta-ce^u\eta\,d\sigma,\end{aligned}$$

one can easily obtain (4). □

For simplicity, we shall call (4) the Neumann boundary problem in the sequel.

Now we come to an important property of the Neumann boundary problem (4).

Proposition 2.2. *Assume that (u,ψ) is a solution of (4). For any conformal diffeomorphism $\varphi: M\to M$, if we set*

$$\begin{aligned}\tilde{u}&=u\circ\varphi-\phi,\\ \tilde{\psi}&=e^{-\frac{\phi}{2}}\psi\circ\varphi\end{aligned} \tag{5}$$

where e^ϕ is the conformal factor of the conformal map φ, i.e., $\varphi^*(g)=e^{2\phi}g$, then $(\tilde{u},\tilde{\psi})$ is also a solution of (4). Moreover, the functional $E_B(u,\psi)$ is conformally invariant.

Proof. Let $\tilde{g}=\varphi^*g$, where g is the metric on M. Let $\widetilde{\not{D}}$, $\widetilde{B}$ be the Dirac operator and the chirality boundary operator with respect to the new metric $\tilde{g}$ respectively. We identify the new and old spin bundles as in [8]. Since the relation between the two Dirac operators $\not{D}$ and $\widetilde{\not{D}}$ is

$$\widetilde{\not{D}}\tilde{\psi}=\lambda^{-\frac{3}{2}}\not{D}\big(\lambda^{\frac{1}{2}}\tilde{\psi}\big)=\lambda^{-\frac{3}{2}}\not{D}\psi$$

for $\lambda = e^{\phi}$, and the relation between the two Gaussian curvatures and between the two geodesic curvatures are respectively

$$-\Delta_g \phi = K_{\tilde{g}} e^{2\phi} - K_g,$$
$$\frac{\partial \phi}{\partial n} = h_{\tilde{g}} e^{\phi} - h_g.$$

We can show by a direct computation that $(\tilde{u}, \tilde{\psi})$ satisfies

$$\begin{cases} -\Delta_{\tilde{g}} \tilde{u} = 2e^{2\tilde{u}} - e^{\tilde{u}} \langle \tilde{\psi}, \tilde{\psi} \rangle - K_{\tilde{g}}, & \text{in } M^o, \\ \widetilde{\not{D}} \tilde{\psi} = -e^{\tilde{u}} \tilde{\psi}, & \text{in } M^o, \\ \dfrac{\partial \tilde{u}}{\partial n} = c e^{\tilde{u}} - h_{\tilde{g}}, & \text{on } \partial M, \\ \widetilde{B}^{\pm} \tilde{\psi} = 0, & \text{on } \partial M. \end{cases}$$

Similarly, one can also show that the functional is conformally invariant. The proof of the proposition is complete. □

In the sequel, we will only consider the case of B^+ and omit the symbol "+". The case of B^- can, of course, be handled analogously.

Let us recall that a *Killing spinor* is a spinor ψ satisfying

$$\nabla_X \psi = \lambda X \cdot \psi, \qquad \text{for any vector field } X$$

for some constant λ. On the standard sphere, there are Killing spinors with the Killing constant $\lambda = \frac{1}{2}$, see for instance [3]. A Killing spinor is an eigenspinor, i.e.,

$$\not{D} \psi = -\psi, \tag{6}$$

with constant $|\psi|^2$. Choose a Killing spinor ψ with $|\psi|^2 = 1$. If we identify $\mathbb{S}^2 \backslash \{\text{north pole}\}$ by the stereographic projection with the Euclidean plane $\mathbb{R}^2$ with the metric

$$\frac{4}{(|1 + |x|^2)^2} |dx|^2,$$

then any Killing spinor has the form

$$\frac{v + x \cdot v}{\sqrt{1 + |x|^2}},$$

for a constant $v \in \mathbb{C}^2$, up to a translation or a dilation. See [3].

We can now construct some special solutions of (4).

Proposition 2.3. *Let $M = \mathbb{R}^2_+$. Then*

$$\left(\log \frac{\sqrt{2}}{1 + |x - x_0|^2}, 0 \right)$$

is a solution of (4), where $x_0 = (s_0, t_0)$ for $s_0 \in \mathbb{R}$ and $t_0 = -\frac{\sqrt{2}c}{2}$ for any constant c. Furthermore, if $c = 0$, then

$$\left(\log \frac{2}{1 + |x - x_1|^2}, \sqrt{2} \frac{v + (x - x_1) \cdot v}{1 + |x - x_1|^2} \right)$$

is also a solution of (4), where $x_1 = (s_1, 0)$ for $s_1 \in \mathbb{R}$, and $v = \binom{v_1}{v_2} \in \{v \in \mathbb{C}^2 \mid |v| = 1\}$ and $v_1 = -v_2$.

Proof. Set $\psi = \sqrt{2} \frac{v + (x - x_1) \cdot v}{1 + |x - x_1|^2}$. Since $\not{D} \psi = -\psi$, according to the argument in [12], it is sufficient to show that $B\psi|_{\partial \mathbb{R}^2_+} = 0$. Since

$$\begin{aligned} v + (x - x_1)\cdot v &= v + (s - s_1)e_1 \cdot v + te_2 \cdot v \\ &= \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + (s - s_1)\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 + i(s - s_1)v_2 + tv_2 \\ v_2 + i(s - s_1)v_1 - tv_1 \end{pmatrix}, \end{aligned}$$

we have

$$\psi = \frac{\sqrt{2}}{1 + |x - x_1|^2}\begin{pmatrix} v_1 + i(s - s_1)v_2 + tv_2 \\ v_2 + i(s - s_1)v_1 - tv_1 \end{pmatrix}.$$

Hence we have by using $v_1 = -v_2$ on $\partial\mathbb{R}^2_+$,

$$\psi = \frac{\sqrt{2}}{1 + |x - x_1|^2}\begin{pmatrix} v_1 - i(s - s_1)v_1 \\ -v_1 + i(s - s_1)v_1 \end{pmatrix} \quad \text{on } \partial\mathbb{R}^2_+.$$

This means that $B\psi|_{\partial\mathbb{R}^2_+} = 0$. □

3. Regularity of solutions for the Neumann boundary problem

In this section, we consider the regularity of solutions for the Neumann boundary problem (4) under the condition that

$$\int_M \left(e^{2u} + |\psi|^4 \, dv + \int_{\partial M} e^u \right) d\sigma < \infty.$$

First, we define weak solutions of (4). We say that (u, ψ) is a weak solution of (4), if $u \in W^{1,2}(M)$ and $\psi \in W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ satisfy

$$\int_M \nabla u \nabla \phi \, dv = \int_M \left(2e^{2u} - e^u|\psi|^2 - K_g\right)\phi \, dv + \int_{\partial M} \left(ce^u - h_g\right)\phi \, d\sigma,$$

$$\int_M \langle \psi, \not{D}\xi \rangle \, dv = -\int_M e^u \langle \psi, \xi \rangle \, dv$$

for $\phi \in C^\infty(M)$ and any smooth spinor $\xi \in C^\infty \cap W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$. Here

$$W_B^{1,\frac{4}{3}}\big(\Gamma(\Sigma M)\big) = \big\{\psi \bigm| \psi \in W^{1,\frac{4}{3}}\big(\Gamma(\Sigma M)\big),\ B\psi|_{\partial M} = 0\big\}.$$

It is clear that $(u, \psi) \in W^{1,2}(M) \times W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$ is a weak solution if and only if (u, ψ) is a critical point of $E_B(u, \psi)$ in $W^{1,2}(M) \times W_B^{1,\frac{4}{3}}(\Gamma(\Sigma M))$. A weak solution is a classical solution by the following.

Proposition 3.1. *Let (u, ψ) be a weak solution of (4) with $\int_M e^{2u} + |\psi|^4 \, dv + \int_{\partial M} e^u \, d\sigma < \infty$. Then $u \in C^{2,\alpha}(M^o) \cap C^{1,\alpha}(M)$ and $\psi \in C^{2,\alpha}(\Gamma(\Sigma M^o)) \cap C^{1,\alpha}(\Gamma(\Sigma M))$ for some $\alpha \in (0, 1)$.*

To prove this proposition, we need several lemmas.

Lemma 3.2. *(See [4].) Assume $\Omega \subset \mathbb{R}^2$ is a bounded domain and let u be a solution of*

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$. Then for every $\delta \in (0, 4\pi)$ we have

$$\int_\Omega \exp\left\{\frac{(4\pi-\delta)|u(x)|}{\|f\|_1}\right\}dx \leqslant \frac{4\pi^2}{\delta}(\operatorname{diam}\Omega)^2, \tag{7}$$

where $\|f\|_1 = \int_\Omega |f(x)|\,dx$.

Lemma 3.3. *(See [13].) Assume that u is a solution of*

$$\begin{cases} -\Delta u = 0, & \text{in } B_R^+, \\ \dfrac{\partial u}{\partial t} = f(x), & \text{on } \{t=0\}\cap \partial B_R^+, \\ u = 0, & \text{on } \partial B_R^+ \cap B_R^+, \end{cases}$$

with $f \in L^1(\{t=0\}\cap\partial B_R^+)$ *for any* $R>0$. *Then for every* $\delta_1 \in (0,4\pi)$ *we have*

$$\int_{B_R^+} \exp\left\{\frac{(4\pi-\delta_1)|u(x)|}{\|f\|_1}\right\}dx \leqslant \frac{16\pi^2R^2}{\delta_1}$$

and for every $\delta_2 \in (0,2\pi)$

$$\int_{\partial B_R^+\cap\{t=0\}} \exp\left\{\frac{(2\pi-\delta_2)|u(x)|}{\|f\|_1}\right\}ds \leqslant \frac{4\pi R}{\delta_2}$$

where $\|f\|_1 = \int_{\{t=0\}\cap\partial B_R^+} |f|\,ds$.

By Lemma 3.2 and Lemma 3.3, we obtain the following

Lemma 3.4. *If* (u,ψ) *is a weak solution to* (4) *with* $\int_M e^{2u} + |\psi|^4\,dv + \int_{\partial M} e^u\,d\sigma < \infty$, *then we have for* $0<\alpha<1$

$$u^+ \in L^\infty(M), \qquad \psi \in C^\alpha\big(\Gamma(\Sigma M)\big).$$

Proof. By the conformal invariance of (4) and by the interior regularity Lemma 4.3 in [12], it suffices to show that, for any $x_0 \in \partial M$, u is bounded from above in $B_r^M(x_0)\cap M$ and ψ is continuous in $\Gamma(\Sigma(B_r^M(x_0)\cap M))$, where $B_r^M(x_0)$ is a geodesic ball at x_0 of M. Without loss of generality, we assume that $x_0=0$ and $B_r^M(x_0)\cap M = \{x=(s,t)\mid s^2+t^2<r^2,\ t\geqslant 0\}\subset \overline{\mathbb{R}}^2_+$. Set $B_r^+ = \{x=(s,t)\mid s^2+t^2<r^2,\ t>0\}$, $B_r^- = \{(s,t)\mid s^2+t^2<r^2,\ t<0\}$ and $\Gamma_1 = \partial B_r^+\cap\partial\mathbb{R}^2_+$, $\Gamma_2 = \partial B_r^+\cap\mathbb{R}^2_+$. By using the conformality again, we may assume that

$$\int_M e^{2u}+|\psi|^4\,dv + \int_{\partial M} e^u\,d\sigma < \frac{1}{4}\pi.$$

First, we show the boundedness from above of u. Set

$$f = 2e^{2u} - e^u|\psi|^2 \quad \text{and} \quad g = ce^u.$$

Then we consider

$$\begin{cases} -\Delta u = f, & \text{in } B_r^+, \\ \dfrac{\partial u}{\partial n} = g, & \text{on } \Gamma_1. \end{cases}$$

It is clear that $g\in L^1(\Gamma_1)$. Set $g = g_1+g_2$ with $\|g_1\|_{L^1(\Gamma_1)} \leqslant \pi$ and $g_2 \in L^\infty(\Gamma_1)$. Define u_1, u_2 and u_3 by

$$\begin{cases} -\Delta u_1 = f, & \text{in } B_r^+, \\ \dfrac{\partial u_1}{\partial n} = 0, & \text{on } \Gamma_1, \\ u_1 = 0, & \text{on } \Gamma_2, \end{cases}$$

$$\begin{cases} -\Delta u_2 = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_2}{\partial n} = g_1, & \text{on } \Gamma_1, \\ u_2 = 0, & \text{on } \Gamma_2, \end{cases}$$

$$\begin{cases} -\Delta u_3 = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_3}{\partial n} = g_2, & \text{on } \Gamma_1, \\ u_3 = 0, & \text{on } \Gamma_2. \end{cases}$$

Extending u_1 and f evenly we have

$$\begin{cases} -\Delta u_1 = f, & \text{in } B_r, \\ u_1 = 0, & \text{on } \partial B_r. \end{cases}$$

Since $\int_{B_r^+} e^{2u} + |\psi|^4\,dx < \infty$, we know that $f \in L^1(B_r^+)$ with $\|f\|_{L^1} \leqslant \pi$. By applying Lemma 3.2 we have

$$e^{4|u_1|} \in L^1(B_r).$$

For u_2, by Lemma 3.3, we have

$$\int_{B_r^+} \exp\bigl(4|u_2|\bigr)\,dx \leqslant C, \qquad \int_{\Gamma_1} \exp\bigl(2|u_2|\bigr)\,ds \leqslant C.$$

For u_3, it is obvious that

$$\|u_3\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leqslant C.$$

Let $u_4 = u - u_1 - u_2 - u_3$. Then we have

$$\begin{cases} -\Delta u_4 = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_4}{\partial n} = 0, & \text{on } \Gamma_1. \end{cases}$$

Extending u_4 evenly, u_4 becomes a harmonic function in B_r. Then the mean value theorem for harmonic functions implies that

$$\bigl\|u_4^+\bigr\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leqslant C\bigl\|u_4^+\bigr\|_{L^1(B_r^+)}.$$

Notice that

$$u_4^+ \leqslant u^+ + |u_1| + |u_2| + |u_3|,$$

and

$$\int_{B_r^+} u^+\,dx \leqslant \frac{1}{2}\int_{B_r^+} e^{2u}\,dx < \infty.$$

We get

$$\bigl\|u_4^+\bigr\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leqslant C.$$

Altogether, we find that $f \in L^2(B_r^+)$ and $g \in L^2(\Gamma_1)$.

The standard elliptic estimates imply that

$$\bigl\|u^+\bigr\|_{L^\infty(\bar{B}^+_{\frac{r}{4}})} \leqslant C.$$

Next we show the continuity of the spinor field ψ. For this purpose, we extend (u, ψ) to the lower half disk B_r^-. Assume $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$, and define

$$u(\bar{x}) := u(x), \quad \bar{x} \in B_r^-,$$
$$\psi(\bar{x}) := ie_1 \cdot \psi(x), \quad \bar{x} \in B_r^-.$$

Since we have for a.e. $x \in \Gamma_1$

$$\psi(x) = -\vec{n}G\psi(x) = ie_1 \cdot \psi(x),$$

it is clear that the extension for ψ is well defined.

Now assume that (u, ψ) is a weak solution of (4) and ξ is in $W^{1,\frac{4}{3}}(\Gamma(\Sigma B_r))$ with compact support. Then we obtain

$$\begin{aligned}
\int_{B_r} \langle \psi, \not{D}\xi \rangle &= \int_{B_r^+} \langle \psi, \not{D}\xi \rangle + \int_{B_r^-} \langle \psi, \not{D}\xi \rangle \\
&= \int_{B_r^+} \langle \psi, \not{D}\xi \rangle + \int_{x \in B_r^+} \big\langle \psi(\bar{x}), \not{D}\xi(\bar{x}) \big\rangle \\
&= \int_{B_r^+} \langle \psi, \not{D}\xi \rangle + \int_{x \in B_r^+} \big\langle ie_1 \cdot \psi(x), \not{D}\xi(\bar{x}) \big\rangle \\
&= \int_{B_r^+} \langle \psi, \not{D}\xi \rangle + \int_{x \in B_r^+} \big\langle \psi(x), \not{D}\big(ie_1 \cdot \xi(\bar{x})\big) \big\rangle \\
&= \int_{B_r^+} \big\langle \psi(x), \not{D}\big(\xi(x) + ie_1 \cdot \xi(\bar{x})\big) \big\rangle.
\end{aligned}$$

By the definition of the chirality operator B, we have for a.e. $x \in \Gamma_1$

$$B\big(\xi(x) + ie_1 \cdot \xi(\bar{x})\big) = \frac{1}{2}(I - ie_1) \cdot \big(\xi(x) + ie_1 \cdot \xi(x)\big) = 0.$$

Then by the definition of a weak solution we obtain

$$\begin{aligned}
&\int_{B_r^+} \big\langle \psi(x), \not{D}\big(\xi(x) + ie_1 \cdot \xi(\bar{x})\big) \big\rangle \\
&\quad = \int_{B_r^+} \big\langle \not{D}\psi(x), \xi(x) + ie_1 \cdot \xi(\bar{x}) \big\rangle \\
&\quad = -\int_{B_r^+} e^u \big\langle \psi(x), \xi(x) + ie_1 \cdot \xi(\bar{x}) \big\rangle \\
&\quad = -\int_{B_r^+} e^u \big\langle \psi(x), \xi(x) \big\rangle - \int_{x \in B_r^+} e^u \big\langle \psi(x), ie_1 \cdot \xi(\bar{x}) \big\rangle \\
&\quad = -\int_{B_r^+} e^u \big\langle \psi(x), \xi(x) \big\rangle - \int_{x \in B_r^-} e^{u(\bar{x})} \big\langle \psi(\bar{x}), ie_1 \cdot \xi(x) \big\rangle \\
&\quad = -\int_{B_r^+} e^u \big\langle \psi(x), \xi(x) \big\rangle - \int_{x \in B_r^-} e^{u(\bar{x})} \big\langle \psi(x), \xi(x) \big\rangle.
\end{aligned}$$

Therefore we obtain that

$$\int_{B_r} \langle \psi, \not{D}\xi \rangle = -\int_{B_r^+} e^u \big\langle \psi(x), \xi(x) \big\rangle - \int_{x \in B_r^-} e^{u(\bar{x})} \big\langle \psi(x), \xi(x) \big\rangle.$$

Set

$$A(x)=\begin{cases} e^{u(x)}, & x\in B_r^+,\\ e^{u(\bar{x})}, & x\in B_r^-.\end{cases}$$

It follows that ψ satisfies

$$\not{D}\psi=-A(x)\psi, \quad \text{in } B_r.$$

Since $A(x)\in L^\infty(B_{\frac{r}{4}})$ and $\int_{B_r}|\psi|^4\,dx<\infty$, we have $\psi\in W^{1,4}(\Gamma(\Sigma B_{\frac{r}{8}}))$ and in particular $\psi\in C^\alpha(\Gamma(\Sigma\overline{B}^+_{\frac{r}{8}}))$ for some $0<\alpha<1$. $\square$

Proof of Proposition 3.1. Assume that (u,ψ) is a weak solution of (4). For any $q>2$, let $2>p=\frac{q}{q-1}>1$. Then we have

$$\|\nabla u\|_{L^q(M)}\leqslant \sup\left\{\left|\int_M \nabla u\nabla\varphi\,dv\right|\,\middle|\,\varphi\in W^{1,p}(M),\ \int_M\varphi\,dv=0,\ \|\varphi\|_{W^{1,p}(M)}=1\right\}.$$

Since from Lemma 3.4

$$\begin{aligned}\left|\int_M\nabla u\nabla\varphi\,dv\right|&=\left|\int_M-\Delta u\varphi\,dv+\int_{\partial M}\frac{\partial u}{\partial n}\varphi\,d\sigma\right|\\&=\left|\int_M\left(2e^{2u}-e^u|\psi|^2-K_g\right)\varphi\,dv+\int_{\partial M}\left(ce^u-h_g\right)\varphi\,d\sigma\right|\\&\leqslant C\int_M|\varphi|\,dv+\int_{\partial M}|\varphi|\,d\sigma\\&\leqslant C,\end{aligned}$$

we have $\|\nabla u\|_{L^q(M)}\leqslant C$ for any $q>2$. Therefore we have $u\in W^{1,q}(M)$ for any $q>2$. By $W^{2+k,q}$ estimates for the Neumann boundary problem (see [1], see also [14])

$$\|u\|_{W^{2+k,q}(M)}\leqslant C\left(\|\Delta u\|_{W^{k,q}(M)}+\left\|\frac{\partial u}{\partial n}\right\|_{W^{1+k,q}(\partial M)}+\|u\|_{W^{1+k,q}(M)}\right),$$

we have $u\in W^{2,q}(M)$ for any $q>2$. By the Sobolev embedding theorem we know $u\in C^{1,\alpha}(M)$ for some $\alpha\in(0,1)$. Similarly we obtain that $u\in C^{2,\alpha}(M^o)$ for some $\alpha\in(0,1)$.

For ψ, since (u,ψ) satisfies

$$\not{D}\psi=-A(x)\psi$$

in the neighborhood of $x_0\in\partial M$ after the reflection, see Lemma 3.4. By the well-know Lichnerowitz formula $\not{D}^2\psi=-\Delta\psi+\frac{1}{4}K_g\psi$ (see e.g. [10]), we know

$$-\Delta\psi=-dA(x)\cdot\psi+A^2(x)\psi-\frac{1}{4}K_g\psi.$$

It follows that $\psi\in W^{2,q}$ for any $q>1$ in the neighborhood of $x_0\in\partial M$. Hence we have $\psi\in C^{2,\alpha}(\Gamma(\Sigma M^o))\cap C^{1,\alpha}(\Gamma(\Sigma M))$ for some $\alpha\in(0,1)$. This concludes the proof. $\square$

We call (u,ψ) a *regular* solution to (4) if $u\in C^{2,\alpha}(M^o)\cap C^{1,\alpha}(M)$ and $\psi\in C^{2,\alpha}(\Gamma(\Sigma M^o))\cap C^{1,\alpha}(\Gamma(\Sigma M))$ for some $\alpha\in(0,1)$.

Next we discuss the convergence of a sequence of regular solutions to (4), under a smallness condition for the energy.

Lemma 3.5. *For $\varepsilon_1 < \pi$, and $\varepsilon_2 < \pi$. If a sequence of regular solutions (u_n, ψ_n) satisfy*

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n}\langle\psi_n, \psi_n\rangle, & \text{in } B_r^+, \\ \not{D}\psi_n = -e^{u_n}\psi_n, & \text{in } B_r^+, \\ \dfrac{\partial u_n}{\partial n} = ce^{u_n}, & \text{on } \partial B_r^+ \cap \{t=0\}, \\ B\psi_n = 0, & \text{on } \partial B_r^+ \cap \{t=0\} \end{cases}$$

and

$$\int_{B_r^+} e^{2u_n}\,dx < \varepsilon_1, \qquad |c| \int_{\partial B_r^+ \cap \{t=0\}} e^{u_n}\,d\sigma < \varepsilon_2, \qquad \int_{B_r^+} |\psi_n|^4\,dx < C.$$

Then $\|u_n^+\|_{L^\infty(\overline{B}^+_{\frac{r}{4}})}$ and $\|\psi_n\|_{L^\infty(\overline{B}^+_{\frac{r}{8}})}$ are uniformly bounded.

Proof. If $c = 0$, by extending (u_n, ψ_n) to the lower half disk B_r^- as Lemma 3.4, we have

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n}\langle\psi_n, \psi_n\rangle, & \text{in } B_r, \\ \not{D}\psi_n = -e^{u_n}\psi_n, & \text{in } B_r. \end{cases}$$

From Lemma 4.4 of [12], we obtain the conclusions.

Next we assume that $c \neq 0$. Let $\Gamma_1 = \partial B_r^+ \cap \{t = 0\}$ and $\Gamma_2 = \partial B_r^+ \cap \{t > 0\}$. Define $u_{1,n}, u_{2,n}$ by

$$\begin{cases} -\Delta u_{1,n} = 2e^{2u_n} - e^{u_n}|\psi_n|^2, & \text{in } B_r^+, \\ \dfrac{\partial u_{1,n}}{\partial n} = 0, & \text{on } \Gamma_1, \\ u_{1,n} = 0, & \text{on } \Gamma_2, \end{cases}$$

and

$$\begin{cases} -\Delta u_{2,n} = 0, & \text{in } B_r^+, \\ \dfrac{\partial u_{2,n}}{\partial n} = ce^{u_n}, & \text{on } \Gamma_1, \\ u_{2,n} = 0, & \text{on } \Gamma_2. \end{cases}$$

Extending $u_{1,n}$ evenly we have

$$\begin{cases} -\Delta u_{1,n} = 2e^{2u_n}, & \text{in } B_r, \\ u_{1,n} = 0, & \text{on } \partial B_r. \end{cases}$$

Since $\varepsilon_1 < \pi$, we can choose $\delta_1 > 0$ such that $4\pi - \delta_1 > (4\varepsilon_1 + 2\sqrt{C\varepsilon_1})(2 + \delta_1)$. By Lemma 3.2 we get

$$\int_{B_r} e^{(2+\delta_1)|u_{1,n}|} \leqslant C$$

for some constant C. In particular we have

$$\int_{B_r^+} e^{(2+\delta_1)|u_{1,n}|} \leqslant C.$$

For $u_{2,n}$, since $\varepsilon_2 < \pi$, by Lemma 3.3 we also can choose $\delta_2 > 0$, $\delta_3 > 0$ such that

$$\int_{B_r^+} e^{(2+\delta_2)|u_{2,n}|} \leqslant C, \qquad \int_{\Gamma_1} e^{(1+\delta_3)|u_{2,n}|} \leqslant C.$$

Now setting $w_n = u_n - u_{1,n} - u_{2,n}$, it follows

$$\begin{cases} \Delta w_n = e^{u_n}|\psi_n|^2 \geqslant 0, & \text{in } B_r^+, \\ \dfrac{\partial w_n}{\partial n} = 0, & \text{on } \Gamma_1. \end{cases}$$

Extending w_n evenly, we have w_n are subharmonic functions in B_r. Then the mean value theorem for subharmonic functions implies that

$$\|w_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leqslant C\|w_n^+\|_{L^1(B_r^+)}.$$

Notice that

$$\begin{aligned} \int_{B_r^+} w_n^+\,dx &\leqslant \int_{B_r^+} u_n^+ + |u_{1,n}| + |u_{2,n}|\,dx \\ &\leqslant C\int_{B_r^+} e^{2u_n} + e^{(2+\delta_1)|u_{1,n}|} + e^{(2+\delta_2)|u_{2,n}|}\,dx \\ &\leqslant C. \end{aligned}$$

Therefore we have

$$\|w_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{2}})} \leqslant C.$$

Finally, we write

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n}|\psi_n|^2 = f_n, & \text{in } B_r^+, \\ \dfrac{\partial u_n}{\partial n} = ce^{u_n} = g_n, & \text{on } \Gamma_1. \end{cases}$$

The standard elliptic estimates imply that

$$\|u_n^+\|_{L^\infty(\bar{B}^+_{\frac{r}{4}})} \leqslant C$$

since $\|f_n\|_{L^q(B^+_{\frac{r}{2}})} \leqslant C$ and $\|g_n\|_{L^q(\partial B^+_{\frac{r}{2}})\cap\{t=0\}} \leqslant C$ for some $q > 1$. Consequently, it follows that $\|\psi_n\|_{L^\infty(\bar{B}^+_{\frac{r}{8}})} \leqslant C$. □

4. Blow-up behavior

When the energy $\int_M e^{2u_n}\,dx$ and $\int_{\partial M} e^{u_n}\,dx$ are large, the blow-up phenomenon may occur as in the case of the Liouville equation. In this section we will analyze the asymptotic behavior of a sequence of regular solutions

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n}\langle\psi_n, \psi_n\rangle - K_g, & \text{in } M^o, \\ \not{D}\psi_n = -e^{u_n}\psi_n, & \text{in } M^o, \\ \dfrac{\partial u_n}{\partial n} = ce^{u_n} - h_g, & \text{on } \partial M, \\ B\psi_n = 0, & \text{on } \partial M, \end{cases} \tag{8}$$

with

$$\int_M e^{2u_n} + |\psi_n|^4\,dv + \int_{\partial M} e^{u_n}\,d\sigma \leqslant C. \tag{9}$$

The blow-up analysis was first introduced in [4] for the Liouville-type equation on an open bounded domain. Later, similar results for the Toda system and the super-Liouville equation, the natural generalization of the Liouville equation, were obtained in [11] and in [12] respectively. Here we will provide the blow-up analysis for the Neumann boundary problem (8) under condition (9). The key point is a Harnack inequality for the non-homogeneous Neumann-type boundary problem for second-order elliptic equations. See Lemma A.2 in Appendix A.

Theorem 4.1. *Let (u_n, ψ_n) be a sequence of regular solutions to (8) satisfying (9). Define*

$$\Sigma_1 = \{x \in M \mid \text{there is a sequence } y_n \to x \text{ such that } u_n(y_n) \to +\infty\},$$
$$\Sigma_2 = \{x \in M \mid \text{there is a sequence } y_n \to x \text{ such that } |\psi_n(y_n)| \to +\infty\}.$$

Then, we have $\Sigma_2 \subset \Sigma_1$. Moreover, (u_n, ψ_n) admits a subsequence, denoted still by (u_n, ψ_n), satisfying that:

a) *$|\psi_n|$ is bounded in $L^\infty_{loc}(M \setminus \Sigma_2)$.*
b) *For u_n, one of the following alternatives holds:*
   i) *u_n is bounded in $L^\infty(M)$.*
   ii) *$u_n \to -\infty$ uniformly on M.*
   iii) *Σ_1 is finite, nonempty and either*

$$u_n \text{ is bounded in } L^\infty_{loc}(M \setminus \Sigma_1) \tag{10}$$

or

$$u_n \to -\infty \text{ uniformly on compact subsets of } M \setminus \Sigma_1. \tag{11}$$

Proof. First of all, if $x \in M \setminus \Sigma_1$, then it follows from the equation $\not{D}\psi_n = -e^{u_n}\psi_n$ that $x \in M \setminus \Sigma_2$. Therefore we have $\Sigma_2 \subset \Sigma_1$. It is clear that $|\psi_n|$ are bounded in $L^\infty_{loc}(M \setminus \Sigma_2)$.

Since e^{2u_n} is bounded in $L^1(M)$ and e^{u_n} is bounded in $L^1(\partial M)$, we may extract a subsequence from u_n (still denoted u_n) such that

$$\int_M e^{2u_n}\varphi\, dv \to \int_M \varphi\, d\mu,$$
$$\int_{\partial M} e^{u_n}\phi\, d\sigma \to \int_{\partial M} \phi\, d\vartheta$$

for every $\varphi \in C(M)$ and $\phi \in C(\partial M)$. Here μ and ϑ are two nonnegative bounded measures. A point $x \in M$ is called an ε-regular point with respect to μ and ϑ if there is a function $\varphi \in C(M)$, $\operatorname{supp}\varphi \subset B^M_r(x) \subset M$ with $0 \leqslant \varphi \leqslant 1$ and $\varphi = 1$ in a neighborhood of x such that

$$\int_M \varphi\, d\mu < \varepsilon, \quad \text{if } x \in M^o,$$

or

$$\int_M \varphi\, d\mu < \varepsilon, \quad \text{and} \quad \int_{\partial M} \varphi\, d\vartheta < \varepsilon, \quad \text{if } x \in \partial M.$$

Here $B^M_r(x)$ is a geodesic ball at center x.

We define

$$\Omega(\varepsilon) = \{x \in M \mid x \text{ is not an } \varepsilon\text{-regular point with respect to } \mu \text{ and } \vartheta\}.$$

By $\int_M e^{2u_n} < C$ and $\int_{\partial M} e^{u_n} < C$, we have that $\Omega(\varepsilon)$ is finite. We divide the proof into three steps.

Step 1. $\Sigma_1 = \Omega(\varepsilon_0)$, where $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon_1, \varepsilon_2$ as in Lemma 3.5.

First we show that $\Omega(\varepsilon_0) \subset \Sigma_1$. Suppose that $x_0 \in \Omega(\varepsilon_0)$. If $x_0 \in M^o$, it is easy to show that $x_0 \in \Sigma_1$, see [12]. Next we assume that $x_0 \in \partial M$. We claim that for any $R > 0$, and $B^M_R(x_0) \subset M$, $\lim_{n\to+\infty} \|u_n^+\|_{L^\infty(B^M_R(x_0))} = +\infty$. We prove the claim by a contradiction. So we assume that there is some $R_0 > 0$ and $B^M_{R_0}(x_0) \subset M$ and a subsequence such that $\|u_n^+\|_{L^\infty(B_{R_0}^M(x_0))}$ is bounded. In particular we have $\|e^{2u_n}\|_{L^\infty(B_{R_0}^M(x_0))} \leqslant C$. Therefore $\int_{B_R^M(x_0)} e^{2u_n}\,dx \leqslant CR^\delta$ and $\int_{\partial B_R^M(x_0)\cap\partial M} e^{u_n}\,dx \leqslant CR^\delta$ for all $R < R_0$ and some $\delta > 0$. This implies

$$\int_M \varphi\,d\mu < \varepsilon_0, \quad \text{and} \quad \int_{\partial M} \varphi\,d\vartheta < \varepsilon_0 \quad \text{for some suitable } \varphi.$$

Therefore x_0 is regular, contradicting $x_0 \in \Omega(\varepsilon_0)$. The claim is proved. Now we choose $R > 0$ small enough so that $\overline{B}_R^M(x_0)$ does not contain any other point of $\Omega(\varepsilon_0)$. Let $x_n \in B_R^M(x_0)$ be such that

$$u_n^+(x_n) = \max_{\overline{B}_R^M(x_0)} u_n^+ \to +\infty.$$

We claim that $x_n \to x_0$, i.e., $x_0 \in \Sigma_1$. Otherwise there would be a subsequence

$$x_{n_k} \to \bar{x} \neq x_0 \quad \text{and} \quad \bar{x} \notin \Omega(\varepsilon_0)$$

that is, $\bar{x}$ is a regular point. This is a contradiction. Therefore we have proved that $\Omega(\varepsilon_0) \subset \Sigma_1$.

Next we show that $\Sigma_1 \subset \Omega(\varepsilon_0)$. Let $x_0 \in \Sigma_1$. There are two cases. Case 1. $x_0 \in M^o \Rightarrow x_0 \in \Omega(\varepsilon_0)$. Case 2. $x_0 \in \partial M \Rightarrow x_0 \in \Omega(\varepsilon_0)$.

Here we only show Case 2, since Case 1 easily follows from the argument in [12]. So next we assume that $x_0 \in \partial M$. We choose small $R > 0$ such that $\overline{B}_R^M(x_0) \cap \Sigma_1 = x_0$. We assume by contradiction that $x_0 \notin \Omega(\varepsilon_0)$. Thus we have

$$\int_{B_\delta^M(x_0)} e^{2u_n} < \varepsilon_1, \qquad \int_{\partial B_\delta^M(x_0)\cap\partial M} e^{u_n} < \varepsilon_2$$

for any small $\delta < R$. Since u_n satisfies that

$$\begin{cases} -\Delta u_n = 2e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g & \text{in } B_\delta^M(x_0) \cap M^o, \\ \dfrac{\partial u_n}{\partial n} = ce^{u_n} - h_g & \text{on } \partial B_\delta^M(x_0) \cap \partial M, \end{cases}$$

by Lemma 3.5, we also see that u_n^+ is uniformly bounded in $L^\infty(\overline{B_{\frac{\delta}{2}}^M(x_0)})$. Thus we have a contradiction with $x_0 \in \Sigma_1$. Therefore $x_0 \in \Omega(\varepsilon_0)$.

Step 2. $\Sigma_1 = \emptyset$ implies i) and ii) hold.

$\Sigma_1 = \emptyset$ means that u_n^+ is uniformly bounded in $L^\infty(M)$. Consequently ψ_n is bounded in $L^\infty(M)$. Thus, $f^n = 2e^{2u_n} - e^{u_n}|\psi_n|^2 - K_g$ is bounded in $L^p(M)$ for any $p > 1$ and $g^n = ce^{u_n} - h_g$ is bounded in $L^p(\partial M)$ for any $p > 1$. Applying the Harnack inequality in Lemma A.2 in Appendix A, we have i) or ii).

Step 3. $\Sigma_1 \neq \emptyset$ implies iii).

In this case, we know that u_n^+ is bounded in $L^\infty_{loc}(M\setminus\Sigma_1)$ and therefore f^n is bounded in $L^p_{loc}(M\setminus\Sigma_1)$ for any $p > 1$ and g^n is bounded in $L^p_{loc}(\partial M\setminus\Sigma_1)$ for any $p > 1$. Then as in Step 2 we know that either

$$u_n \text{ is bounded in } L^\infty_{loc}(M\setminus\Sigma_1),$$

or

$$u_n \to -\infty \text{ on any compact subset of } M\setminus\Sigma_1$$

Thus we complete the proof of the theorem. □

5. Asymptotic behavior of entire solutions

In the rest of the paper we will analyze the asymptotic behavior of an entire solution on the upper half-plane $\mathbb{R}^2_+$ with finite energy. Such an entire solution will be obtained after a suitable rescaling at a boundary blow-up point. We will show that an entire solution on $\mathbb{R}^2_+$ with finite energy can be extended to a spherical cap, i.e., the singularity at infinity is removable.

The considered equations are

$$\begin{cases} -\Delta u = 2e^{2u} - e^u\langle\psi,\psi\rangle, & \text{in } \mathbb{R}^2_+, \\ \not{D}\psi = -e^u\psi, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial n} = ce^u, & \text{on } \partial\mathbb{R}^2_+, \\ B\psi = 0, & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{12}$$

The energy condition is

$$I(u,\psi) = \int_{\mathbb{R}^2_+} \left(e^{2u} + |\psi|^4\right) dx + \int_{\partial\mathbb{R}^2_+} e^u\, ds < \infty. \tag{13}$$

First by a similar argument as Proposition 3.1 we have

Lemma 5.1. *Let (u,ψ) be a solution of (12) and (13) with $u \in H^{1,2}_{loc}(\mathbb{R}^2_+)$ and $\psi \in W^{1,\frac{4}{3}}_{loc}(\mathbb{R}^2_+)$. Then $u^+ \in L^\infty(\overline{\mathbb{R}}^2_+)$. Consequently it follows that $u \in C^{2,\alpha}_{loc}(\mathbb{R}^2_+) \cap C^{1,\alpha}_{loc}(\overline{\mathbb{R}}^2_+)$ and $\psi \in C^{2,\alpha}_{loc}(\Gamma(\Sigma\mathbb{R}^2_+)) \cap C^{1,\alpha}_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+))$.*

We call (u,ψ) a *regular* solution of (12) and (13) if $u \in C^{2,\alpha}_{loc}(\mathbb{R}^2_+) \cap C^{1,\alpha}_{loc}(\overline{\mathbb{R}}^2_+)$ and $\psi \in C^{2,\alpha}_{loc}(\Gamma(\Sigma\mathbb{R}^2_+)) \cap C^{1,\alpha}_{loc}(\Gamma(\Sigma\overline{\mathbb{R}}^2_+))$ for some $\alpha \in (0,1)$.

Proposition 5.2. *Let (u,ψ) be a regular solution of (12) and (13). Then the quadratic differential*

$$T(z)\,dz^2 = \left\{(\partial_z u)^2 - \partial_z^2 u + \frac{1}{4}\langle\psi, dz\cdot\partial_{\bar z}\psi\rangle + \frac{1}{4}\langle d\bar z\cdot\partial_z\psi,\psi\rangle\right\} dz^2$$

is holomorphic in $\mathbb{R}^2_+$ and $T(z)\,dz^2$ is real on $\partial\mathbb{R}^2_+$. Here $z = s + it \in \mathbb{R}^2$.

Proof. From Proposition 3.3 of [12], it is clear that $T(z)\,dz^2$ is holomorphic in $\mathbb{R}^2_+$. Next we show that $T(z)\,dz^2$ is real on $\partial\mathbb{R}^2_+$. Let

$$T_1(z) = (\partial_z u)^2 - \partial_z^2 u,$$
$$T_2(z) = \frac{1}{4}\langle\psi, dz\cdot\partial_{\bar z}\psi\rangle + \frac{1}{4}\langle d\bar z\cdot\partial_z\psi,\psi\rangle.$$

Then we have

$$\operatorname{Im}\big(T_1(z)\big) = \frac{1}{2}\left(\frac{\partial^2 u}{\partial s\partial t} - \frac{\partial u}{\partial s}\frac{\partial u}{\partial t}\right).$$

Since $\frac{\partial u}{\partial t} = -ce^u$ on $\partial\mathbb{R}^2_+$, we have

$$\operatorname{Im}\big(T_1(z)\big)\big|_{\partial\mathbb{R}^2_+} = \frac{1}{2}\left(-ce^u\frac{\partial u}{\partial s} + ce^u\frac{\partial u}{\partial s}\right) = 0$$

On the other hand, by a computation we have

$$\begin{aligned} T_2(z) &= \frac{1}{2}\Big(\big\langle\psi,(e_1+ie_2)\cdot(\nabla_{e_1}\psi + i\nabla_{e_2}\psi)\big\rangle + \big\langle(e_1-ie_2)\cdot(\nabla_{e_1}\psi - i\nabla_{e_2}\psi),\psi\big\rangle\Big) \\ &= \operatorname{Re}\langle\psi, e_1\cdot\nabla_{e_1}\psi\rangle - \operatorname{Re}\langle\psi, e_2\cdot\nabla_{e_2}\psi\rangle - 2i\operatorname{Re}\langle\psi, e_1\cdot\nabla_{e_2}\psi\rangle \\ &= \operatorname{Re}\langle\psi, e_1\cdot\nabla_{e_1}\psi\rangle - \operatorname{Re}\langle\psi, e_2\cdot\nabla_{e_2}\psi\rangle - 2i\operatorname{Re}\langle\psi, e_2\cdot\nabla_{e_1}\psi\rangle. \end{aligned}$$

Here e_1, e_2 constitute the standard orthonormal frame of $\mathbb{R}^2$. Notice that we can write $\psi = \binom{\psi_+}{\psi_-}$, then the chirality boundary condition becomes $\psi_+ = -\psi_-$ on $\partial\mathbb{R}^2_+$. Since $e_1 = \frac{\partial}{\partial s}$, it follows that $\nabla_{e_1}\psi_+ = -\nabla_{e_1}\psi_-$ on $\partial\mathbb{R}^2_+$. Therefore we obtain

$$\langle\psi, e_2\cdot\nabla_{e_1}\psi\rangle = \left\langle\begin{pmatrix}\psi_+\\ -\psi_+\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\nabla_{e_1}\psi_+\\ -\nabla_{e_1}\psi_+\end{pmatrix}\right\rangle = 0, \quad \text{on } \partial\mathbb{R}^2_+.$$

Consequently we have $\operatorname{Im} T_2(z)|_{\partial\mathbb{R}^2_+} = 0$. It follows that $\operatorname{Im} T(z)|_{\partial\mathbb{R}^2_+} = 0$. □

Next let (v, ϕ) be the Kelvin transformation of (u, ψ), i.e.,

$$v(x) = u\left(\frac{x}{|x|^2}\right) - 2\ln|x|,$$

$$\phi(x) = |x|^{-1}\psi\left(\frac{x}{|x|^2}\right).$$

Then (v, ϕ) satisfies

$$\begin{cases} -\Delta v = 2e^{2v} - e^v\langle\phi, \phi\rangle, & \text{in } \mathbb{R}^2_+,\\ \not{D}\phi = -e^v\phi, & \text{in } \mathbb{R}^2_+,\\ \dfrac{\partial v}{\partial n} = ce^v, & \text{on } \partial\mathbb{R}^2_+\backslash\{0\},\\ B\phi = 0, & \text{on } \partial\mathbb{R}^2_+\backslash\{0\}.\end{cases} \tag{14}$$

And, by change of variable,

$$\int_{|x|\leqslant r_0} e^{2v}\,dx = \int_{|x|\geqslant \frac{1}{r_0}} e^{2u}\,dx,$$

$$\int_{|x|\leqslant r_0} |\phi|^4\,dx = \int_{|x|\geqslant \frac{1}{r_0}} |\psi|^4\,dx,$$

$$\int_{|s|\leqslant r_0} e^v\,ds = \int_{|s|\geqslant \frac{1}{r_0}} e^u\,ds$$

can be made small if r_0 is small. Therefore, there is a small enough r_0 such that (v, ϕ) satisfies

$$\begin{cases} -\Delta v = 2e^{2v} - e^v\langle\phi, \phi\rangle, & \text{in } B^+_{r_0},\\ \not{D}\phi = -e^v\phi, & \text{in } B^+_{r_0},\\ \dfrac{\partial v}{\partial n} = ce^v, & \text{on } (\partial\mathbb{R}^2_+\cap\partial B^+_{r_0})\backslash\{0\},\\ B\phi = 0, & \text{on } (\partial\mathbb{R}^2_+\cap\partial B^+_{r_0})\backslash\{0\}\end{cases} \tag{15}$$

with energy condition

$$\int_{|x|\leqslant r_0} e^{2v}\,dx \leqslant \varepsilon_1 < 2\pi, \qquad \int_{|x|\leqslant r_0} |\phi|^4\,dx \leqslant C, \qquad |c|\int_{|s|\leqslant r_0} e^v\,ds \leqslant \varepsilon_2 < \pi. \tag{16}$$

Since (15) and (16) are conformally invariant, in the sequel we may assume $B^+_{r_0}$ to be the unit disk B^+_1. We have

Lemma 5.3. *There are $0 < \varepsilon_1 < \pi$ and $0 < \varepsilon_2 < \pi$ such that if (v, ϕ) is a regular solution to (15) with energy condition (16) (for $r_0 = 1$), then for any $x \in \overline{B}^+_{\frac{1}{2}}$ we have*

$$\big|\phi(x)\big||x|^{\frac{1}{2}} + \big|\nabla\phi(x)\big||x|^{\frac{3}{2}} \leqslant C\left(\int_{B^+_{2|x|}} |\phi|^4\,dx\right)^{\frac{1}{4}}. \tag{17}$$

Furthermore, if we assume that $e^{2v} = O(\frac{1}{|x|^{2-\varepsilon}})$, *then, for any* $x \in \overline{B}^+_{\frac{1}{2}}$, *we have*

$$\big|\phi(x)\big||x|^{\frac{1}{2}} + \big|\nabla\phi(x)\big||x|^{\frac{3}{2}} \leqslant C|x|^{\frac{1}{4C}} \Bigg(\int\limits_{B_1^+} |\phi|^4\, dx\Bigg)^{\frac{1}{4}}, \tag{18}$$

for some positive constant C. *Here* ε *is any sufficiently small positive number.*

Proof. Firstly by the chirality boundary condition of ϕ, we can extend (v,ϕ) to the lower half disk B_1^-. Assume $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$, and define

$$v(\bar{x}) := v(x), \quad \bar{x} \in B_1^-,$$
$$\phi(\bar{x}) := ie_1 \cdot \phi(x), \quad \bar{x} \in B_1^-.$$

Then from the argument in Lemma 3.4 we obtain that

$$\not{D}\psi = -A(x)\psi, \quad \text{in } B_1.$$

Here

$$A(x) = \begin{cases} e^{u(x)}, & x \in B_1^+, \\ e^{u(\bar{x})}, & x \in B_1^-. \end{cases}$$

The conclusions follow from applying similar arguments as in the proof of Lemma 6.2 of [12]. □

From Lemma 5.3 and the Kelvin transformation, we obtain the asymptotic estimate of the spinor $\psi(x)$

$$\big|\psi(x)\big| \leqslant C|x|^{-\frac{1}{2}-\delta_0} \quad \text{for } |x| \text{ near } \infty \tag{19}$$

for some positive number δ_0 provided that $e^{2v} = O(\frac{1}{|x|^{2-\varepsilon}})$.

Now let $\alpha = \int_{\mathbb{R}^2_+} 2e^{2u} - e^u|\psi|^2\, dx + \int_{\partial R^2_+} ce^u\, ds$ and define a constant spinor $\xi_0 = \int_{\mathbb{R}^2_+} e^u\psi\, dx$. It will turn out that the constant spinor ξ_0 is well defined. Then we have

Proposition 5.4. *Let* (u,ψ) *be a regular solution of* (12) *and* (13) *and let* c *be a nonnegative constant. Then we have*

$$u(x) = -\frac{\alpha}{\pi}\ln|x| + C + O\big(|x|^{-1}\big) \quad \textit{for } |x| \textit{ near } \infty, \tag{20}$$

$$\psi(x) = -\frac{1}{2\pi}\frac{x}{|x|^2}(I + ie_1)\cdot\xi_0 + o\big(|x|^{-1}\big) \quad \textit{for } |x| \textit{ near } \infty, \tag{21}$$

where $\cdot$ *is the Clifford multiplication,* C *is a positive universal constant, and* I *is the identity spinor. In particular we have*

$$\alpha = 2\pi.$$

Proof. We prove Proposition 5.4 in several steps.

Step 1. $\lim_{|x|\to\infty} \frac{u(x)}{\ln|x|} = -\frac{\alpha}{\pi}$.

Let

$$w(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2_+} \big(\log|x-y| + \log|\bar{x}-y| - 2\log|y|\big)\big(2e^{2u(y)} - e^{u(y)}\big|\psi(y)\big|^2\big)\, dy$$
$$+ \frac{1}{2\pi}\int\limits_{\partial\mathbb{R}^2_+} \big(\log|x-y| + \log|\bar{x}-y| - 2\log|y|\big)ce^{u(y)}\, dy,$$

where $\bar{x}$ is the reflection point of x about $\partial\mathbb{R}^2_+$. It is easy to check that $w(x)$ satisfies

$$\begin{cases} \Delta w = 2e^{2u} - e^{u}|\psi|^2, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial w}{\partial n} = -ce^{u}, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

and

$$\lim_{|x|\to\infty} \frac{w(x)}{\ln|x|} = \frac{\alpha}{\pi}.$$

Consider $v(x) = u + w$. Then $v(x)$ satisfies

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

We extend $v(x)$ to $\mathbb{R}^2$ by even reflection such that $v(x)$ is harmonic in $\mathbb{R}^2$. From Lemma 5.1 we know $v(x) \leqslant C(1 + \ln(|x| + 1))$ for some positive constant C. Thus $v(x)$ is a constant. This completes the proof of Step 1.

Step 2. $\alpha > \pi$.

Since $\int_{\mathbb{R}^2_+} e^{2u}\,dx < \infty$, we get that $\alpha \geqslant \pi$. Next we show that $\alpha > \pi$. Assume by contradiction that $\alpha = \pi$. Let (v, ϕ) be the Kelvin transformation of (u, ψ). Then (v, ϕ) satisfies

$$\begin{cases} -\Delta v = 2e^{2v} - e^{v}|\phi|^2, & \text{in } \mathbb{R}^2_+, \\ \not{D}\phi = -e^{v}\phi, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial n} = ce^{v}, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \\ B\phi = 0, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \end{cases}$$

with the energy conditions

$$\int_{\mathbb{R}^2_+} e^{2v} + |\phi|^4\,dx < \infty,$$

and

$$\int_{\partial\mathbb{R}^2_+} e^{v}\,ds < \infty.$$

Let D^+ be a small half disk centered at zero. Denote $f(x) := 2e^{2v} - e^{v}|\phi|^2$. From the asymptotic estimate (19) we know that $f(x) > 0$ in a small half disk D^+. Define $w(x)$ by

$$\begin{aligned} w(x) = {} & \frac{1}{2\pi} \int_{D^+} \big(\log|x-y| + \log|\bar{x}-y|\big) f(y)\,dy \\ & + \frac{1}{2\pi} \int_{\partial D^+ \cap \{t=0\}} \big(\log|x-y| + \log|\bar{x}-y|\big) ce^{v(y)}\,dy \end{aligned}$$

and define $g(x) = v(x) + w(x)$. It is clear that

$$\begin{cases} \Delta g = 0, & \text{in } D^+, \\ \dfrac{\partial g}{\partial n} = 0, & \text{on } \{\partial D^+ \cap \{t=0\}\} \setminus \{0\}. \end{cases}$$

Therefore by extending $g(x)$ to $D \setminus \{0\}$ evenly we obtain a harmonic $g(x)$ in $D \setminus \{0\}$.

On the other hand, we can check that

$$\lim_{|x|\to 0} \frac{w}{-\log|x|} = 0$$

by Step 1 which implies

$$\lim_{|x|\to 0}\frac{g(x)}{-\log|x|}=\lim_{|x|\to 0}\frac{v(x)+w(x)}{-\log|x|}=1.$$

Since $g(x)$ is harmonic in $D\backslash\{0\}$, we have $g(x)=-\log|x|+g_0(x)$ with a smooth harmonic function g_0 in D. By the definition, we have $w(x)<0$ since c is nonnegative and $f(x)>0$ in D^+. Thus, we have

$$\int_{D^+} e^{2v}\,dx=\int_{D^+} e^{2g-2w}\,dx\geqslant\int_{D^+}|x|^{-2}e^{2g_0}\,dx=\infty,$$

which is a contradiction with $\int_{\mathbb{R}^2_+} e^{2v}\,dx<\infty$. Hence we have shown that $\alpha>\pi$. Thus we finish the proof of Step 2.

Step 3. The proof of (20) and $\alpha=2\pi$.

From $\alpha>\pi$ we can improve the estimates for e^{2u} to

$$e^{2u}\leqslant C|x|^{-2-\varepsilon}\quad\text{for } |x| \text{ near } \infty.$$

Then by using the standard potential analysis we can obtain that

$$u(x)=-\frac{\alpha}{\pi}\ln|x|+C+O\big(|x|^{-1}\big)\quad\text{for } |x| \text{ near } \infty.$$

Furthermore, we can show that $\alpha=2\pi$. Since the quadratic differential $T(z)\,dz^2$ is holomorphic in $\mathbb{R}^2_+$ and is real on $\partial\mathbb{R}^2_+$, we can extend $T(z)$ to a holomorphic function in $\mathbb{R}^2$. Then by using (19) and (20), we have the following expansion of $T(z)$ near infinity

$$\frac{1}{4}\left(\frac{\alpha}{\pi}\right)^2\frac{1}{z^2}-\frac{1}{2}\frac{\alpha}{\pi}\frac{1}{z^2}+o\left(\frac{1}{z^2}\right)+\cdots=\frac{1}{2z^2}\left(\frac{1}{2}\left(\frac{\alpha}{\pi}\right)^2-\frac{\alpha}{\pi}\right)+o\left(\frac{1}{z^2}\right)+\cdots.$$

Hence, $T(z)$ is a constant and $\frac{1}{2}(\frac{\alpha}{\pi})^2-\frac{\alpha}{\pi}=0$, i.e., $\alpha=2\pi$.

Step 4. The proof of (21).

First from $\alpha=2\pi$, we can improve the estimate for e^{2u} to

$$e^{2u}\leqslant C|x|^{-4}\quad\text{for } |x| \text{ near } \infty. \tag{22}$$

This implies that the constant spinor ξ_0 is well defined.

Then by using the chirality boundary condition of spinor we have

$$\not{D}\psi=-A(x)\psi,\quad\text{in } \mathbb{R}^2.$$

Here $A(x)$ is defined as before. Define

$$\xi_1=\int_{\mathbb{R}^2}A(x)\psi\,dx.$$

The constant spinor ξ_1 is also well defined. From the asymptotic estimates (19) and (22) and a similar argument in [12] we obtain

$$\psi(x)=-\frac{1}{2\pi}\frac{x}{|x|^2}\cdot\xi_1+o\big(|x|^{-1}\big)\quad\text{for } |x| \text{ near } \infty. \tag{23}$$

Since

$$\begin{aligned}\xi_1&=\int_{\mathbb{R}^2_+}A(x)\psi\,dx+\int_{\mathbb{R}^2_-}A(x)\psi\,dx\\&=\int_{\mathbb{R}^2_+}e^u\psi\,dx+\int_{\mathbb{R}^2_-}e^{u(\bar{x})}ie_1\cdot\psi(\bar{x})\,dx\end{aligned}$$

$$= \int_{\mathbb{R}^2_+} e^u \psi \, dx + \int_{\mathbb{R}^2_-} e^{u(y)} i e_1 \cdot \psi(y) \, dy$$
$$= (I + i e_1) \cdot \int_{\mathbb{R}^2_+} e^u \psi \, dx$$
$$= (I + i e_1) \cdot \xi_0.$$

Hence we obtain from (23)

$$\psi(x) = -\frac{1}{2\pi} \frac{x}{|x|^2} (I + i e_1) \cdot \xi_0 + o\big(|x|^{-1}\big) \quad \text{for } |x| \text{ near } \infty.$$

Thus we finish the proof of Step 4 and we complete the proof of the proposition. □

Consequently, from Proposition 5.2, we shall show that an infinite singularity of regular solutions for (12) and (13) can be removed as in many other conformally invariant problems.

Theorem 5.5. *Let (u, ψ) be a regular solution of (12) and (13). Then (u, ψ) extends to a regular solution on a spherical cap $\mathbb{S}^2_{c'}$, where c' is the geodesic curvature of $\partial \mathbb{S}^2_{c'}$.*

Proof. Let (v, ϕ) be the Kelvin transformation of (u, ψ) as before. Then (v, ϕ) satisfies the system (14). To prove the theorem, by conformal invariance, it is sufficient to show that (v, ϕ) is regular on $\overline{\mathbb{R}}^2_+$. Applying Proposition 5.4, we get

$$v(x) = \left(\frac{\alpha}{\pi} - 2\right) \ln|x| + O(1) \quad \text{for } |x| \text{ near } 0. \tag{24}$$

Since $\alpha = 2\pi$, it follows that v is bounded near the singularity 0. Recall that ϕ is also bounded near 0, we can apply elliptic theory to obtain that (v, ϕ) is regular on $\overline{\mathbb{R}}^2_+$. □

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Appendix A

We present a Harnack inequality for a non-homogeneous Neumann-type boundary problem for second-order elliptic equations.

Lemma A.1. *Let $f \in L^p(B_r)$ for some $1 < p \leqslant +\infty$ and u satisfy*

$$\begin{cases} -\Delta u = f & \text{in } B_r, \\ u \leqslant 0 & \text{on } \partial B_r. \end{cases}$$

Then for any $0 < \theta < 1$, there exists a constant $\beta \in (0, 1)$ depending on r, θ only, and a constant $\gamma > 0$ depending on r, p only, such that

$$\sup_{B_{\theta r}} u \leqslant \beta \inf_{B_{\theta r}} u + (1 + \beta)\gamma \|f\|_{L^p(B_r)}.$$

Lemma A.2. *Let $f \in L^p(B_r^+)$ for some $1 < p \leqslant +\infty$, $g \in L^q(\partial B_r^+ \cap \{t = 0\})$ for some $1 < q \leqslant +\infty$ and u satisfy*

$$\begin{cases} -\Delta u = f & \text{in } B_r^+, \\ \dfrac{\partial u}{\partial t} = g & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ u \leqslant 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

Then for any $0 < \theta < 1$, there exist a constant $\beta \in (0,1)$ depending on r, θ only, and a constant $\gamma > 0$ depending on r, p, q only, such that

$$\sup_{\overline{B}^+_{\theta r}} u \leqslant \beta \inf_{\overline{B}^+_{\theta r}} u + (1+\beta)\gamma\big(\|f\|_{L^p(B^+_r)} + \|g\|_{L^q(\partial B^+_r \cap \{t=0\})}\big).$$

Proof. Let w satisfy

$$\begin{cases} -\Delta w = f & \text{in } B^+_r, \\ \dfrac{\partial w}{\partial t} = g & \text{on } \partial B^+_r \cap \{t=0\}, \\ w = 0 & \text{on } \partial B^+_r \cap \{t>0\} \end{cases}$$

and set $v = w - u$. Then v satisfies

$$\begin{cases} -\Delta v = 0 & \text{in } B^+_r, \\ \dfrac{\partial v}{\partial t} = 0 & \text{on } \partial B^+_r \cap \{t=0\}, \\ v \geqslant 0 & \text{on } \partial B^+_r \cap \{t>0\}. \end{cases}$$

By the maximum principle and Hopf lemma, $v \geqslant 0$ in B^+_r. By extending v evenly, v becomes a harmonic function in B_r with $v \geqslant 0$. Then from Harnack inequality of harmonic function we have

$$\sup_{\overline{B}^+_{\theta r}} v \leqslant \frac{1}{\beta} \inf_{\overline{B}^+_{\theta r}} v \tag{25}$$

for any $\theta \in (0,1)$ and for some $\beta \in (0,1)$ depending on r, θ only.

Next, assume that w_1 satisfies

$$\begin{cases} -\Delta w_1 = f & \text{in } B^+_r, \\ \dfrac{\partial w_1}{\partial t} = 0 & \text{on } \partial B^+_r \cap \{t=0\}, \\ w_1 = 0 & \text{on } \partial B^+_r \cap \{t>0\} \end{cases}$$

and that w_2 satisfies

$$\begin{cases} -\Delta w_2 = 0 & \text{in } B^+_r, \\ \dfrac{\partial w_2}{\partial t} = g & \text{on } \partial B^+_r \cap \{t=0\}, \\ w_2 = 0 & \text{on } \partial B^+_r \cap \{t>0\}. \end{cases}$$

It is clear from extending evenly that

$$\sup_{\overline{B}^+_r} |w_1| \leqslant \gamma \|f\|_{L^p(B^+_r)}.$$

For w_2, we define

$$\phi(x) = \frac{1}{2\pi} \int\limits_{\partial B^+_r \cap \{t=0\}} \left(\log \frac{2r}{|y-x|} + \log \frac{2r}{|y-\bar{x}|}\right) |g(y)|\, dy$$

where $\bar{x}$ is the reflection point of about $\{t=0\}$. Then ϕ satisfies

$$\begin{cases} -\Delta \phi = 0 & \text{in } B^+_r, \\ \dfrac{\partial w_2}{\partial t} = -|g| & \text{on } \partial B^+_r \cap \{t=0\}. \end{cases}$$

It is clear that $\phi \geqslant 0$ and

$$\sup_{\overline{B}_r^+} \phi \leqslant \gamma \|g\|_{L^q(B_r^+ \cap \{t=0\})}.$$

Since

$$\begin{cases} -\Delta(w_2 - \phi) = 0 & \text{in } B_r^+, \\ \dfrac{\partial(w_2 - \phi)}{\partial t} = g + |g| & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w_2 - \phi \leqslant 0 & \text{on } \partial B_r^+ \cap \{t > 0\}. \end{cases}$$

It follows from the maximum principle and Hopf lemma that $w_2 \leqslant \phi$ in $\overline{B}_r^+$. By a similar argument we also have

$$\begin{cases} -\Delta(w_2 + \phi) = 0 & \text{in } B_r^+, \\ \dfrac{\partial(w_2 + \phi)}{\partial t} = g - |g| & \text{on } \partial B_r^+ \cap \{t = 0\}, \\ w_2 + \phi \geqslant 0 & \text{on } \partial B_r^+ \cap \{t > 0\} \end{cases}$$

which implies that $w_2 \geqslant -\phi$ in $\overline{B}_r^+$. Thus we have $|w_2| \leqslant |\phi|$ in $\overline{B}_r^+$. Since $w = w_1 + w_2$, it follows that

$$\sup_{\overline{B}_r^+} |w| \leqslant \gamma \|g\|_{L^q(B_r^+ \cap \{t=0\})}. \tag{26}$$

By (25), (26) and $u = w - v$, it follows that

$$\sup_{\overline{B}_{\theta r}^+} u \leqslant \beta \inf_{\overline{B}_{\theta r}^+} u + (1 + \beta)\gamma \big(\|f\|_{L^p(B_r^+)} + \|g\|_{L^q(\partial B_r^+ \cap \{t=0\})}\big). \qquad \square$$

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