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On the sharp constant for the weighted Trudinger–Moser type inequality of the scaling invariant form

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Abstract

In this article, we establish the weighted Trudinger–Moser inequality of the scaling invariant form including its best constant and prove the existence of a maximizer for the associated variational problem. The non-singular case was treated by Adachi and Tanaka (1999) [1] and the existence of a maximizer is a new result even for the non-singular case. We also discuss the relation between the best constants of the weighted Trudinger–Moser inequality and the Caffarelli–Kohn–Nirenberg inequality in the asymptotic sense.

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1. Introduction and main results

In this article, we shall establish the weighted Trudinger–Moser type inequality with its sharp constant and consider the existence of a maximizer associated with the (weighted) Trudinger–Moser type inequality. As is well-known, a function in $H^{1,N}(\mathbb{R}^N)$ $(N \geqslant 2)$ could have a local singularity and this causes the failure of the embedding $H^{1,N}(\mathbb{R}^N) \not\subset L^\infty(\mathbb{R}^N)$ although the continuous embedding $H^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ holds for all $N \leqslant q < \infty$.

As one of the characterizations of the critical embedding in $H_0^{1,N}(\Omega)$, Moser [17] and Trudinger [25] established the following: for any bounded domain $\Omega \subset \mathbb{R}^N$ with $N \geqslant 2$, there exists a positive constant $C = C(N)$ such that, for any $0 < \alpha \leqslant \alpha_N := N\omega_{N-1}^{\frac{1}{N-1}}$, where ω_{N-1} denotes the surface area of the unit ball in $\mathbb{R}^N$, there holds

$$\int_\Omega \exp\bigl(\alpha \bigl|u(x)\bigr|^{N'}\bigr)\,dx \leqslant C|\Omega| \tag{1.1}$$

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for all $u \in H_0^{1,N}(\Omega)$ with $\|\nabla u\|_{L^N(\Omega)} \leqslant 1$, where $N' := \frac{N}{N-1}$. Concerning the existence of a maximizer associated with (1.1), Carleson and Chang [6] showed the existence of a maximizer when Ω is a ball, $N \geqslant 2$ and $\alpha \leqslant \alpha_N$, although the problem suffers from a lack of the compactness when $\alpha = \alpha_N$, see also Struwe [24] for related works. After that, the existence of a maximizer was proved for any bounded domain by Flucker [8] when $N = 2$ and $\alpha = \alpha_2 = 4\pi$ and by Lin [16] when $N \geqslant 2$ and $\alpha = \alpha_N$.

There are several extensions of (1.1). As a scaling invariant form in $\mathbb{R}^N$, Adachi and Tanaka [1] proved the following: for $N \geqslant 2$ and $0 < \alpha < \alpha_N$, there exists a positive constant $C = C(N, \alpha)$ such that the inequality

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u(x)\right|^{N'}\right) dx \leqslant C\|u\|_{L^N(\mathbb{R}^N)}^N \tag{1.2}$$

holds for all $u \in H^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$, where

$$\Phi_N(t) := \sum_{j=N-1}^{\infty} \frac{t^j}{j!} \quad \text{for } t \geqslant 0. \tag{1.3}$$

In [1], it was also proved that (1.2) fails if $\alpha \geqslant \alpha_N$. The inequality (1.2) was originally obtained by Ogawa [19] with sufficiently small α when $N = 2$. Furthermore, (1.2) was extended to general critical Sobolev spaces in Ogawa and Ozawa [20] and Ozawa [22], see also Kozono, Sato and Wadade [14], Nagayasu and Wadade [18] and Ozawa [21] for related works. Moreover, another kind of the Trudinger–Moser type inequality is known. For $N \geqslant 2$, there exists a positive constant $C = C(N)$ such that for any $0 < \alpha \leqslant \alpha_N$, there holds

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u(x)\right|^{N'}\right) dx \leqslant C \tag{1.4}$$

for all $u \in H^{1,N}(\mathbb{R}^N)$ with $\|u\|_{H^{1,N}(\mathbb{R}^N)} \leqslant 1$. Li and Ruf [15] obtained (1.4) with the best constant α_N for $N \geqslant 3$, where the authors also proved the existence of a maximizer associated with (1.4) when $\alpha = \alpha_N$. For $N = 2$, (1.4) was proved by Cao [5], and Ruf [23] showed the sharpness of $\alpha = \alpha_2 (= 4\pi)$. In Ruf [23] and in Ishiwata [13], the existence of a maximizer associated with (1.4) was considered when $N = 2$ and it was also verified in [13] that the non-existence of a maximizer occurs when $N = 2$ and α is sufficiently small.

Keeping the historical remarks above in mind, we investigate (1.2) of the scaling invariant form into two directions. Our first aim is to extend (1.2) to the weighted inequality as follows: for $N \geqslant 2$ and $-\infty < s \leqslant t < N$,

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u(x)\right|^{N'}\right) \frac{dx}{|x|^t} \leqslant C\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{N-s}} \tag{1.5}$$

for all $u \in L^N(\mathbb{R}^N; |x|^{-s}\,dx) \cap \dot{H}^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$ with some positive constants α and C, where $L^N(\mathbb{R}^N; |x|^{-s}\,dx)$ denotes the weighted Lebesgue space endowed with the norm

$$\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} := \left(\int_{\mathbb{R}^N} \left|u(x)\right|^N |x|^{-s}\,dx\right)^{\frac{1}{N}},$$

see Theorem 1.1. This generalization from (1.2) to (1.5) is naturally motivated by the special case of the Caffarelli–Kohn–Nirenberg inequality obtained in [4], which states that, for $N \geqslant 2$, $-\infty < s \leqslant t < N$ and $N \leqslant q < \infty$, there exists a positive constant $C = C(N, s, t, q)$ such that the inequality

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}} \tag{1.6}$$

holds for all $u \in L^N(\mathbb{R}^N; |x|^{-s}\,dx) \cap \dot{H}^{1,N}(\mathbb{R}^N)$. We can regard (1.5) as a critical version of (1.6).

As another direction, we consider the existence of a maximizer associated with (1.5), which has not been discussed even for the non-singular inequality (1.2) as far as we know. In Theorem 1.3, we shall establish the existence of a maximizer to the above variational problem.

Next, we investigate the constant C in the Caffarelli–Kohn–Nirenberg inequality (1.6) including its asymptotic sharp constant, see Theorem 1.5. Among others, in Ozawa [21], the author gave the explicit relation between the constants appearing in (1.2) and the non-singular case $s = t = 0$ of (1.6). We shall revisit the strategy developed in [21] and apply it to the weighted inequalities obtained in Theorem 1.1.

Now we are in a position to give our main results. We often assume the condition for the exponents as follows:

$$N \geqslant 2, \quad -\infty < s \leqslant t < N \quad \text{and} \quad 0 < \alpha < \alpha_{N,t} := (N-t)\omega_{N-1}^{\frac{1}{N-1}}. \tag{1.7}$$

Theorem 1.1. (i) *Assume* (1.7). *Then there exists a positive constant* $C = C(N,s,t,\alpha)$ *such that the inequality*

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u(x)\right|^{N'}\right)\frac{dx}{|x|^t} \leqslant C\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{N-s}} \tag{1.8}$$

holds for all radially symmetric functions $u \in L^N(\mathbb{R}^N;|x|^{-s}\,dx) \cap \dot{H}^{1,N}(\mathbb{R}^N)$ *with* $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$, *where* Φ_N *is defined by* (1.3).

(ii) *Assume* (1.7). *The constant* $\alpha_{N,t}$ *is sharp for the weighted Trudinger–Moser type inequality* (1.8). *Indeed, the inequality* (1.8) *fails if* $\alpha \geqslant \alpha_{N,t}$.

We can remove the assumption of the radial symmetry on the functions for the special case $s = 0$ and $0 \leqslant t < N$ in Theorem 1.1 by virtue of the rearrangement argument, which may not work for the case $s \neq 0$, see Section 2 and Appendix A for the details. As a result, we obtain the following corollary of Theorem 1.1.

Corollary 1.2. (i) *Assume* (1.7) *with* $s = 0$. *Then there exists a positive constant* $C = C(N,t,\alpha)$ *such that the inequality*

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u(x)\right|^{N'}\right)\frac{dx}{|x|^t} \leqslant C\|u\|_{L^N(\mathbb{R}^N)}^{N-t} \tag{1.9}$$

holds for all $u \in H^{1,N}(\mathbb{R}^N)$ *with* $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$.

(ii) *Assume* (1.7) *with* $s = 0$. *If* $\alpha \geqslant \alpha_{N,t}$, *then the inequality* (1.9) *fails.*

Remark. The non-singular case $t = 0$ in Corollary 1.2 coincides with the result in Adachi and Tanaka [1, Theorems 0.1–0.2]. Moreover, the inequality (1.9) in Corollary 1.2 was obtained as a particular case of the result in Nagayasu and Wadade [18, Corollary 1.3] which does not include the consideration for its sharp constant with respect to α. However, both of Corollary 1.2(i) and (ii) are new results for the singular case $0 < t < N$.

Next, we shall discuss the existence of a maximizer associated with the Trudinger–Moser type inequality (1.8). We define the sharp constant $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$ for (1.8) by

$$\mu_{N,s,t,\alpha}\left(\mathbb{R}^N\right) := \sup_{\substack{u \in X_{s,rad}^{1,N},\\ \|\nabla u\|_{L^N(\mathbb{R}^N)}=1}} F_{N,s,t,\alpha}(u),$$

where

$$F_{N,s,t,\alpha}(u) := \frac{\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t}}{\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{N-s}}}, \tag{1.10}$$

and the function spaces $X_s^{1,N}$ and $X_{s,rad}^{1,N}$ denote the weighted Sobolev spaces defined by

$$\begin{cases} X_s^{1,N} := L^N\left(\mathbb{R}^N;|x|^{-s}\,dx\right) \cap \dot{H}^{1,N}\left(\mathbb{R}^N\right), \\ X_{s,rad}^{1,N} := \left\{u \in X_s^{1,N};\ u \text{ is radially symmetric}\right\}, \end{cases}$$

respectively. Though the variational problem associated with $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$ is a subcritical one from the viewpoint of the exponent α since $\alpha < \alpha_{N,t}$, the problem suffers from a lack of the compactness due to the scaling invariance of the inequality. To explain the non-compactness, let us assume there exists a strongly convergent maximizing sequence (w_n) for $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$ in $X^{1,N}_{s,rad}$ with $\|\nabla w_n\|_{L^N(\mathbb{R}^N)} = 1$ (otherwise we already have the non-compactness). Thus (w_n) satisfies $F_{N,s,t,\alpha}(w_n) \to \mu_{N,s,t,\alpha}(\mathbb{R}^N)$ and $w_n \to w$ as $n \to \infty$ strongly in $X^{1,N}_s$. It is easy to see that $w \neq 0$. Now let us define a sequence (u_n) by $u_n(x) := w_n(\lambda_n x)$ for $x \in \mathbb{R}^N$ with (λ_n) satisfying $\lambda_n \to \infty$ as $n \to \infty$. Then by the scaling invariance, we obtain $F_{N,s,t,\alpha}(u_n) = F_{N,s,t,\alpha}(w_n) \to \mu_{N,s,t,\alpha}(\mathbb{R}^N)$ and $\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = \|\nabla w_n\|_{L^N(\mathbb{R}^N)} = 1$. Furthermore, since $\|u_n\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} \to 0$ as $\lambda_n \to \infty$, (u_n) is bounded in $X^{1,N}_{s,rad}$ and up to a subsequence u_n converges 0 weakly in $X^{1,N}_s$. However, we see as $n \to \infty$,

$$\begin{aligned}\|u_n\|_{X^{1,N}_s} &:= \|u_n\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} + \|\nabla u_n\|_{L^N(\mathbb{R}^N)}\\ &\geqslant \|\nabla u_n\|_{L^N(\mathbb{R}^N)} = \|\nabla w_n\|_{L^N(\mathbb{R}^N)} = \|\nabla w\|_{L^N(\mathbb{R}^N)} + o(1) > 0,\end{aligned}$$

which implies that u_n cannot converge to 0 strongly in $X^{1,N}_s$, and then we have the non-compactness of $F_{N,s,t,\alpha}$ at the level $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$.

In spite of this difficulty, we can prove the following existence result by using a suitable renormalization argument.

Theorem 1.3. (i) *Assume* (1.7). *Then $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$ is attained.*

(ii) *Let $N \geqslant 2$ and $0 < \alpha < \alpha_N = \alpha_{N,0} = N\omega_{N-1}^{\frac{1}{N-1}}$, and define for any domain $D \subset \mathbb{R}^N$,*

$$\tilde{\mu}_{N,\alpha}(D) := \sup_{\substack{u \in H^{1,N}_0(D),\\ \|\nabla u\|_{L^N(D)}=1}} \frac{\int_D \Phi_N\big(\alpha|u(x)|^{N'}\big)\,dx}{\|u\|^N_{L^N(D)}},$$

where $H^{1,N}_0(D)$ denotes the completion of $C^\infty_c(D)$ over the norm $\|\cdot\|_{H^{1,N}(D)}$. Then $\tilde{\mu}_{N,\alpha}(D)$ is attained if and only if $D = \mathbb{R}^N$.

We can remove the assumption of the radial symmetry on functions in Theorem 1.3(i) when $s = 0$ and $0 \leqslant t < N$, see Lemma A.1 in Appendix A. As a consequence, we obtain the following corollary.

Corollary 1.4. *Assume* (1.7) *with $s = 0$. Then*

$$\mu_{N,0,t,\alpha}\big(\mathbb{R}^N\big) = \sup_{\substack{u \in H^{1,N}(\mathbb{R}^N),\\ \|\nabla u\|_{L^N(\mathbb{R}^N)}=1}} \frac{\int_{\mathbb{R}^N} \Phi_N\big(\alpha|u(x)|^{N'}\big)\frac{dx}{|x|^t}}{\|u\|^{N-t}_{L^N(\mathbb{R}^N)}}$$

is attained.

Remark. (i) As far as we know, Corollary 1.4 is a new result even for the non-singular case $t = 0$, which corresponds to (1.2).

(ii) Theorem 1.3(ii) shows that $\tilde{\mu}_{N,\alpha}(D)$ admits a maximizer only when $D = \mathbb{R}^N$. We here consider the case $s = t \neq 0$ and D is a radially symmetric domain with $0 \in D$. Let

$$\tilde{\mu}_{N,t,\alpha}(D) := \sup_{\substack{u \in X^{1,N}_{t,rad}(D),\\ \|\nabla u\|_{L^N(D)}=1}} \frac{\int_D \Phi_N\big(\alpha|u(x)|^{N'}\big)\frac{dx}{|x|^t}}{\|u\|^N_{L^N(D;|x|^{-t}\,dx)}},$$

where $X^{1,N}_{t,rad}(D)$ denotes the closure of the class of radially symmetric functions in $C^\infty_c(D)$ over the norm $\|\cdot\|_{L^N(D;|x|^{-t}\,dx)} + \|\nabla\cdot\|_{L^N(D)}$. Then Theorem 1.3(i) yields the attainability of $\tilde{\mu}_{N,t,\alpha}(\mathbb{R}^N)$ since $\tilde{\mu}_{N,t,\alpha}(\mathbb{R}^N) = \mu_{N,t,t,\alpha}(\mathbb{R}^N)$ by definition. On the other hand, $\tilde{\mu}_{N,t,\alpha}(D)$ is not attained if $D \neq \mathbb{R}^N$. Indeed, assume that $\tilde{\mu}_{N,t,\alpha}(D)$ is attained by $u_0 \in X^{1,N}_{t,rad}(D)$ for some radially symmetric domain $D \neq \mathbb{R}^N$ with $0 \in D$. By using the transformation $v(x) := \big(\frac{N-t}{N}\big)^{\frac{N-1}{N}}\tilde{u}\big(|x|^{\frac{N}{N-t}}\big)$ with $u(x) = \tilde{u}(|x|)$, see (2.3) and (2.4), we obtain $\tilde{\mu}_{N,t,\alpha}(D) = \tilde{\mu}_{N,0,\frac{N}{N-t}\alpha}(\tilde{D})$, where $\tilde{D} := \{x \in \mathbb{R}^N;\ |x|^{\frac{N}{N-t}} = |y| \text{ for some } y \in D\}$. On the other hand, since $0 \in \tilde{D}$, by the scaling and the rearrangement argument, it holds $(\tilde{\mu}_{N,t,\alpha}(D) =)\ \tilde{\mu}_{N,0,\frac{N}{N-t}\alpha}(\tilde{D}) = \tilde{\mu}_{N,\frac{N}{N-t}\alpha}(\tilde{D})$, which implies that $\tilde{\mu}_{N,\frac{N}{N-t}\alpha}(\tilde{D})$ is attained by

$v_0(x) := (\frac{N-t}{N})^{\frac{N-1}{N}} \tilde{u}_0(|x|^{\frac{N}{N-t}}) \in H_0^{1,N}(\tilde{D})$ with $u_0(x) = \tilde{u}_0(|x|)$. However, this is a contradiction to Theorem 1.3(ii) since $D \neq \mathbb{R}^N$ yields $\tilde{D} \neq \mathbb{R}^N$.

(iii) On the contrary, when $(s,t) \neq (0,0)$ and $0 \notin D$, we can guess that the corresponding assertion to Theorem 1.3(ii) is not true. For instance, the attainability for the best constant of the Caffarelli–Kohn–Nirenberg inequality (1.6) was extensively studied, see e.g. Chern and Lin [7], Ghoussoub and Kang [9], Ghoussoub and Robert [10,11] and Hsia, Lin and Wadade [12], where the authors proved the existence of a maximizer associated with the Caffarelli–Kohn–Nirenberg inequality for D with $0 \in \partial D$ (e.g. the half-space) when $(s,t) \neq (0,0)$.

Next, we investigate the asymptotic behavior of the constant for the Caffarelli–Kohn–Nirenberg inequality (1.6) with respect to the exponent q. Let us recall (1.6)

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}}$$

for all $u \in X_s^{1,N}$, where C depends only on N, s, t and q. Hereafter we fix N, s and t. It is known that $C = C_q$ behaves like $C_q \simeq \beta q^{\frac{1}{N'}}$ as $q \to \infty$ for some $\beta > 0$, see e.g., Nagayasu and Wadade [18]. Now we define

$$\beta_{N,t} := \limsup_{q\to\infty} \sup_{u \in X_s^{1,N}\setminus\{0\}} \frac{\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)}}{q^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}}},$$

which we call the asymptotic best constant of (1.6), see Section 4 for the precise definition of $\beta_{N,t}$. The next theorem gives $\beta_{N,t}$ explicitly and we obtain

$$\beta_{N,t} = \left(\frac{N-1}{eN(N-t)\omega_{N-1}^{\frac{1}{N-1}}}\right)^{\frac{N-1}{N}}.$$

More precisely, under the assumption

$$N \geqslant 2, \quad -\infty < s \leqslant t < N \quad \text{and} \quad \beta > \beta_{N,t}, \tag{1.11}$$

we obtain:

Theorem 1.5. *Assume (1.11). Then there exists a positive constant $q_0 = q_0(N,s,t,\beta) \geqslant N$ such that, for any $q \geqslant q_0$ the inequality*

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \beta q^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}} \tag{1.12}$$

holds for all $u \in X_{s,rad}^{1,N}$. Furthermore, the constant $\beta_{N,t}$ is sharp for (1.12). Indeed, (1.12) fails if $0 < \beta < \beta_{N,t}$ in the above asymptotic sense.

In particular, the case $s = 0$ and $0 \leqslant t < N$ in Theorem 1.5 yields the following corollary by virtue of the rearrangement argument.

Corollary 1.6. *Assume (1.11) with $s = 0$. Then there exists a positive constant $q_0 = q_0(N,t,\beta) \geqslant N$ such that, for any $q \geqslant q_0$ the inequality*

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \beta q^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N)}^{\frac{N-t}{q}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N-t}{q}} \tag{1.13}$$

holds for all $u \in H^{1,N}(\mathbb{R}^N)$. Furthermore, the constant $\beta_{N,t}$ is sharp for (1.13). Indeed, the inequality (1.13) fails if $0 < \beta < \beta_{N,t}$ in the above asymptotic sense.

Remark. In fact, we shall show the exact relation between the constants α in (1.8) and β in (1.12) explicitly given by $\beta = (\frac{1}{eN'\alpha})^{\frac{1}{N'}}$, which was established in Ozawa [21] for the non-singular case of the critical Sobolev space with the fractional derivatives. Then Theorem 1.5 will be proved by noting $\beta_{N,t} = (\frac{1}{eN'\alpha_{N,t}})^{\frac{1}{N'}}$.

Here, we describe the organization of this article. Section 2, Section 3 and Section 4 are devoted to prove Theorem 1.1, Theorem 1.3 and Theorem 1.5, respectively. Moreover, we shall collect several lemmas in Appendix A for the proof of the main theorems.

2. Proof of Theorem 1.1

First, we consider the case $s = t$ in Theorem 1.1(i), and the general case $s \leqslant t$ can be obtained by using the Caffarelli–Kohn–Nirenberg interpolation inequality (1.6). The case $s = t$ in Theorem 1.1(i) can be rewritten as follows. Throughout this paper, C is a positive constant independent of the function u and may vary from line to line.

Proposition 2.1. *Assume (1.7) with $s = t$. Then there exists a positive constant $C = C(N, t, \alpha)$ such that the inequality*

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t} \leqslant C\|u\|^N_{L^N(\mathbb{R}^N;|x|^{-t}\,dx)} \tag{2.1}$$

holds for all $u \in X^{1,N}_{t,rad}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$.

Once Proposition 2.1 has been established, Theorem 1.1(i) with $s < t$ will be its immediate consequence through the Caffarelli–Kohn–Nirenberg inequality (1.6) with $q = N$. Indeed, by combining the inequality

$$\|u\|_{L^N(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\|u\|^{\frac{N-t}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}\|\nabla u\|^{1-\frac{N-t}{N-s}}_{L^N(\mathbb{R}^N)} \tag{2.2}$$

with the Trudinger–Moser type inequality (2.1), we obtain Theorem 1.1(i) with $s < t$.

Proof of Proposition 2.1. Let $0 < \alpha < \alpha_{N,t}$ and let $u \in X^{1,N}_{t,rad}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$. We define the function $v \in H^{1,N}(\mathbb{R}^N)$ through the formula such as for $x \in \mathbb{R}^N$,

$$v(x) := \left(\frac{N-t}{N}\right)^{\frac{N-1}{N}} \tilde{u}\left(|x|^{\frac{N}{N-t}}\right), \tag{2.3}$$

where $u(x) = \tilde{u}(|x|)$ for $x \in \mathbb{R}^N$. Then direct computations show that

$$\begin{cases} \|\nabla u\|_{L^N(\mathbb{R}^N)} = \|\nabla v\|_{L^N(\mathbb{R}^N)}, \qquad \|u\|_{L^N(\mathbb{R}^N;|x|^{-t}\,dx)} = \left(\dfrac{N}{N-t}\right)^{\frac{1}{N}}\|v\|_{L^N(\mathbb{R}^N)}, \\ \displaystyle\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t} = \frac{N}{N-t}\int_{\mathbb{R}^N} \Phi_N\left(\frac{N}{N-t}\alpha|v(x)|^{N'}\right)dx. \end{cases} \tag{2.4}$$

Thus substituting (2.4) into (2.1), we see that (2.1) is transferred to the non-singular form equivalently in terms of the function v. Since $0 < \frac{N}{N-t}\alpha < \frac{N}{N-t}\alpha_{N,t} = N\omega_{N-1}^{\frac{1}{N-1}}$, by the Trudinger–Moser type inequality (1.2), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t} &= \frac{N}{N-t}\int_{\mathbb{R}^N} \Phi_N\left(\frac{N}{N-t}\alpha|v(x)|^{N'}\right)dx \\ &\leqslant C\|v\|^N_{L^N(\mathbb{R}^N)} = C\|u\|^N_{L^N(\mathbb{R}^N;|x|^{-t}\,dx)}, \end{aligned}$$

where we used $\|\nabla v\|_{L^N(\mathbb{R}^N)} \leqslant 1$, and we finish the proof of Proposition 2.1. □

Next, we proceed to the proof of the optimality for the constant $\alpha_{N,t}$ stated in Theorem 1.1(ii). In order to show this, we shall construct a sequence of functions in $X^{1,N}_{s,rad}$ so that the corresponding functional diverges.

Proof of Theorem 1.1(ii). For $k \in \mathbb{N}$, we define a sequence u_k of radially symmetric functions in $X^{1,N}_{s,rad}$ by

$$u_k(x) := \begin{cases} 0 & \text{if } |x| \geqslant 1, \\ \left(\frac{N-t}{\omega_{N-1}k}\right)^{\frac{1}{N}} \log\left(\frac{1}{|x|}\right) & \text{if } e^{-\frac{k}{N-t}} < |x| < 1, \\ \left(\frac{1}{\omega_{N-1}}\right)^{\frac{1}{N}} \left(\frac{k}{N-t}\right)^{\frac{1}{N'}} & \text{if } 0 \leqslant |x| \leqslant e^{-\frac{k}{N-t}}. \end{cases}$$

For the non-singular case $t = 0$, this sequence of functions $u_k \in H^{1,N}(\mathbb{R}^N)$ was used in Adachi and Tanaka [1, Theorem 0.2]. Then direct computations show that $\|\nabla u_k\|_{L^N(\mathbb{R}^N)} = 1$ for all $k \in \mathbb{N}$ and

$$\begin{cases} \|u_k\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} = o(1) \quad \text{as } k \to \infty, \\ \displaystyle\int_{\mathbb{R}^N} \Phi_N\left(\alpha_{N,t}|u_k(x)|^{N'}\right)\frac{dx}{|x|^t} \geqslant \frac{\omega_{N-1}}{N-t}\Phi_N(k)e^{-k} = \frac{\omega_{N-1}}{N-1}\left(1 - e^{-k}\sum_{j=0}^{N-2}\frac{k^j}{j!}\right). \end{cases}$$

Thus we have

$$\frac{\int_{\mathbb{R}^N} \Phi_N\left(\alpha_{N,t}|u_k(x)|^{N'}\right)\frac{dx}{|x|^t}}{\|u_k\|^{\frac{N(N-t)}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}} \geqslant \frac{1 - e^{-k}\sum_{j=0}^{N-2}\frac{k^j}{j!}}{o(1)} \to \infty$$

as $k \to \infty$, which implies that (1.8) fails when $\alpha = \alpha_{N,t}$, and we finish the proof of Theorem 1.1(ii). □

At the end of this section, we shall give a proof of Corollary 1.2 below. Since a particular case $s = 0$ and $0 \leqslant t < N$ in Theorem 1.1(ii) directly implies Corollary 1.2(ii), it remains to prove Corollary 1.2(i).

Proof of Corollary 1.2(i). By taking $s = 0$ and $0 \leqslant t < N$ in Theorem 1.1(i), we see that (1.9) holds for all radially symmetric functions $u \in H^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$. To remove the radially symmetric condition for the functions, we utilize the Schwarz symmetrization. Let $u^\# \in H^{1,N}(\mathbb{R}^N)$ be the Schwarz symmetrization of $u \in H^{1,N}(\mathbb{R}^N)$. Then we have, for any $q \geqslant N$,

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \left\|u^\#\right\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)}, \tag{2.5}$$

and

$$\left\|\nabla u^\#\right\|_{L^N(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^N(\mathbb{R}^N)}. \tag{2.6}$$

Among others, we refer to Almgren and Lieb [2, Theorem 2.7] for (2.6) and Bennett and Sharpley [3] for abundant information on the Schwarz symmetrization. We will prove (2.5) in Appendix A for the completeness of the paper, see Lemma A.1.

Since the integral on the left-hand side of (1.9) consists of a countable sum of the weighted Lebesgue norms $\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)}$ with $q \geqslant N$ through Taylor's expansion, by using (2.5), (2.6) and Theorem 1.1(i) with $s = 0$, we see for any $u \in H^{1,N}(\mathbb{R}^N)$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$,

$$\begin{aligned} \int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t} &= \sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\|u\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)} \\ &\leqslant \sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\left\|u^\#\right\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)} = \int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|u^\#(x)\right|^{N'}\right)\frac{dx}{|x|^t} \\ &\leqslant C\left\|u^\#\right\|^{N-t}_{L^N(\mathbb{R}^N)} = C\|u\|^{N-t}_{L^N(\mathbb{R}^N)}, \end{aligned}$$

which completes the proof of Corollary 1.2(i). □

3. Proof of Theorem 1.3

We distinguish two cases $s = t$ and $s \neq t$. The former case $s = t$ is more difficult to deal with compared to the case $s \neq t$ due to the non-compactness of the embedding $X^{1,N}_{s,rad} \hookrightarrow L^N(\mathbb{R}^N; |x|^{-s}\,dx)$ for $s < N$. Keeping this difficulty in mind, we first prepare the following lemma.

Lemma 3.1. *Assume (1.7) and let (u_n) be a bounded sequence of $X^{1,N}_{s,rad}$ with $\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1$. Moreover, assume*

$$u_n \rightharpoonup u \quad \text{weakly in } X^{1,N}_s$$

as $n \to \infty$. Then it holds as $n \to \infty$,

$$\int_{\mathbb{R}^N} \left(\Phi_N\left(\alpha |u_n(x)|^{N'}\right) - \frac{\alpha^{N-1}}{(N-1)!} |u_n(x)|^N \right) \frac{dx}{|x|^t} \to \int_{\mathbb{R}^N} \left(\Phi_N\left(\alpha |u(x)|^{N'}\right) - \frac{\alpha^{N-1}}{(N-1)!} |u(x)|^N \right) \frac{dx}{|x|^t}. \tag{3.1}$$

Proof. Let $\Psi_{N,k,\alpha}(\tau) := e^{\alpha \tau^{N'}} - \sum_{j=0}^{k} \frac{\alpha^j}{j!} \tau^{N'j}$ for $\tau \geqslant 0$, where $N \geqslant 2$, $k \in \mathbb{N} \cup \{0\}$ and $0 < \alpha < \alpha_{N,t}$. Then the desired convergence (3.1) can be rewritten as

$$\int_{\mathbb{R}^N} \Psi_{N,N-1,\alpha}\left(|u_n|\right) \frac{dx}{|x|^t} \to \int_{\mathbb{R}^N} \Psi_{N,N-1,\alpha}\left(|u|\right) \frac{dx}{|x|^t}$$

as $n \to \infty$. A direct computation shows for $\tau \geqslant 0$,

$$\Psi'_{N,N-1,\alpha}(\tau) = \frac{\alpha N}{N-1} \tau^{\frac{1}{N-1}} \Psi_{N,N-2,\alpha}(\tau).$$

Thus, by the mean value theorem and the convexity of the function $\Psi_{N,N-2,\alpha}$, we see for some $\theta \in [0, 1]$,

$$\begin{aligned}
&\left|\Psi_{N,N-1,\alpha}\left(|u_n|\right) - \Psi_{N,N-1,\alpha}\left(|u|\right)\right| \\
&\quad \leqslant \Psi'_{N,N-1,\alpha}\left(\theta|u_n| + (1-\theta)|u|\right)|u_n - u| \\
&\quad = \frac{\alpha N}{N-1} \left(\theta|u_n| + (1-\theta)|u|\right)^{\frac{1}{N-1}} \Psi_{N,N-2,\alpha}\left(\theta|u_n| + (1-\theta)|u|\right)|u_n - u| \\
&\quad \leqslant \frac{\alpha N}{N-1} \left(\theta|u_n| + (1-\theta)|u|\right)^{\frac{1}{N-1}} \left(\theta \Psi_{N,N-2,\alpha}\left(|u_n|\right) + (1-\theta)\Psi_{N,N-2,\alpha}\left(|u|\right)\right)|u_n - u| \\
&\quad \leqslant \frac{\alpha N}{N-1} \left(|u_n| + |u|\right)^{\frac{1}{N-1}} \left(\Psi_{N,N-2,\alpha}\left(|u_n|\right) + \Psi_{N,N-2,\alpha}\left(|u|\right)\right)|u_n - u|.
\end{aligned} \tag{3.2}$$

Take the numbers $a, b, c > 1$ satisfying $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, and then by (3.2) and the Hölder inequality, we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} \left(\Psi_{N,N-1,\alpha}\left(|u_n|\right) - \Psi_{N,N-1,\alpha}\left(|u|\right)\right) \frac{dx}{|x|^t} \right| \\
&\quad \leqslant \frac{\alpha N}{N-1} \int_{\mathbb{R}^N} \left(|u_n| + |u|\right)^{\frac{1}{N-1}} \left(\Psi_{N,N-2,\alpha}\left(|u_n|\right) + \Psi_{N,N-2,\alpha}\left(|u|\right)\right)|u_n - u| \frac{dx}{|x|^t} \\
&\quad \leqslant \frac{\alpha N}{N-1} \left\| |u_n| + |u| \right\|^{\frac{1}{N-1}}_{L^{\frac{a}{N-1}}(\mathbb{R}^N; |x|^{-t}\,dx)} \\
&\qquad \times \left\| \Psi_{N,N-2,\alpha}\left(|u_n|\right) + \Psi_{N,N-2,\alpha}\left(|u|\right) \right\|_{L^b(\mathbb{R}^N; |x|^{-t}\,dx)} \|u_n - u\|_{L^c(\mathbb{R}^N; |x|^{-t}\,dx)}.
\end{aligned} \tag{3.3}$$

We now use

$$\left(e^\tau - \sum_{j=0}^{N-2} \frac{\tau^j}{j!}\right)^b \leqslant e^{b\tau} - \sum_{j=0}^{N-2} \frac{(b\tau)^j}{j!} \tag{3.4}$$

for $\tau \geqslant 0$, see Lemma A.2 in Appendix A. Thus by using (1.8),

$$\begin{aligned} \left\| \Psi_{N,N-2,\alpha}\left(|u_n|\right) \right\|_{L^b(\mathbb{R}^N;|x|^{-t}\,dx)} &= \left(\int_{\mathbb{R}^N} \left(\Psi_{N,N-2,\alpha}\left(|u_n|\right)\right)^b \frac{dx}{|x|^t} \right)^{\frac{1}{b}} \\ &\leqslant \left(\int_{\mathbb{R}^N} \Psi_{N,N-2,b\alpha}\left(|u_n|\right) \frac{dx}{|x|^t} \right)^{\frac{1}{b}} \leqslant C \|u_n\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{b(N-s)}} \leqslant C, \end{aligned} \tag{3.5}$$

where we took $b > 1$ sufficiently close to 1 so that $b\alpha < \alpha_{N,t}$, which is possible since $\alpha < \alpha_{N,t}$. Similarly, we obtain

$$\left\| \Psi_{N,N-2,\alpha}\left(|u|\right) \right\|_{L^b(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C. \tag{3.6}$$

In the above arguments, we implicitly used

$$\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1 \quad \text{and} \quad \|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1.$$

Hence, by using (3.3), (3.5) and (3.6), we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \left(\Psi_{N,N-1,\alpha}\left(|u_n|\right) - \Psi_{N,N-1,\alpha}\left(|u|\right) \right) \frac{dx}{|x|^t} \right| \\ &\quad \leqslant C \left\| |u_n| + |u| \right\|_{L^{\frac{a}{N-1}}(\mathbb{R}^N;|x|^{-t}\,dx)}^{\frac{1}{N-1}} \|u_n - u\|_{L^c(\mathbb{R}^N;|x|^{-t}\,dx)}. \end{aligned} \tag{3.7}$$

Furthermore, by the compactness of $X^{1,N}_{s,rad} \hookrightarrow L^c(\mathbb{R}^N; |x|^{-t}\,dx)$, we have the convergence

$$\|u_n - u\|_{L^c(\mathbb{R}^N;|x|^{-t}\,dx)} \to 0 \tag{3.8}$$

as $n \to \infty$ up to a subsequence for all $c > N$, see Lemma A.3 in Appendix A. Then by the boundedness of $X^{1,N}_s \hookrightarrow L^{\frac{a}{N-1}}(\mathbb{R}^N; |x|^{-t}\,dx)$ for all $\frac{a}{N-1} \geqslant N$, which comes from the Caffarelli–Kohn–Nirenberg inequality (1.6), we have

$$\left\| |u_n| + |u| \right\|_{L^{\frac{a}{N-1}}(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\left(\|u_n\|_{X^{1,N}_s} + \|u\|_{X^{1,N}_s} \right) \leqslant C. \tag{3.9}$$

Summing-up (3.7), (3.8) and (3.9), we obtain the required convergence. □

We are now in a position to prove Theorem 1.3(i) by using Lemma 3.1.

Proof of Theorem 1.3(i). Let (u_n) be a maximizing sequence for $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$, that is, (u_n) is a sequence of functions in $X^{1,N}_{s,rad}$ with $\|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1$ and $F_{N,s,t,\alpha}(u_n) \to \mu_{N,s,t,\alpha}(\mathbb{R}^N)$ as $n \to \infty$. Let us define a new sequence (v_n) by $v_n(x) := u_n\left(\|u_n\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N}{N-s}} x\right)$ for $x \in \mathbb{R}^N$. Then we easily see that

$$\|\nabla v_n\|_{L^N(\mathbb{R}^N)} = \|\nabla u_n\|_{L^N(\mathbb{R}^N)} = 1, \qquad \|v_n\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} = 1,$$

and

$$F_{N,s,t,\alpha}(v_n) = F_{N,s,t,\alpha}(u_n) \to \mu_{N,s,t,\alpha}\left(\mathbb{R}^N\right)$$

as $n \to \infty$. Thus (v_n) is also a maximizing sequence for $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$, which is a bounded sequence of functions in $X^{1,N}_{s,rad}$. Therefore, up to a subsequence, v_n converges to some v weakly in $X^{1,N}_s$, and then v satisfies

$$\max\left\{ \|v\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}, \|\nabla v\|_{L^N(\mathbb{R}^N)} \right\} \leqslant 1. \tag{3.10}$$

In what follows, we distinguish two cases.

Case 1. We assume $s = t$. Let $\mu_{N,s,\alpha}(\mathbb{R}^N) := \mu_{N,s,s,\alpha}(\mathbb{R}^N)$ and $F_{N,s,\alpha}(u) := F_{N,s,s,\alpha}(u)$ for $u \in X^{1,N}_{s,rad}$. By applying Lemma 3.1, we see as $n \to \infty$,

$$\begin{aligned}\mu_{N,s,\alpha}(\mathbb{R}^N) &= F_{N,s,\alpha}(v_n) + o(1) = \int_{\mathbb{R}^N} \Phi_N\left(\alpha\left|v_n(x)\right|^{N'}\right)\frac{dx}{|x|^s} + o(1)\\ &= \frac{\alpha^{N-1}}{(N-1)!} + \int_{\mathbb{R}^N}\left(\Phi_N\left(\alpha\left|v_n(x)\right|^{N'}\right) - \frac{\alpha^{N-1}}{(N-1)!}\left|v_n(x)\right|^N\right)\frac{dx}{|x|^s} + o(1)\\ &= \frac{\alpha^{N-1}}{(N-1)!} + \int_{\mathbb{R}^N}\left(\Phi_N\left(\alpha\left|v(x)\right|^{N'}\right) - \frac{\alpha^{N-1}}{(N-1)!}\left|v(x)\right|^N\right)\frac{dx}{|x|^s}.\end{aligned} \tag{3.11}$$

Here, it is worth noting that $\mu_{N,s,\alpha}(\mathbb{R}^N) > \frac{\alpha^{N-1}}{(N-1)!}$. Indeed, pick up $u_0 \in X^{1,N}_{s,rad}$ satisfying $\|\nabla u_0\|_{L^N(\mathbb{R}^N)} = 1$ arbitrarily. Then we see

$$\begin{aligned}\mu_{N,s,\alpha}(\mathbb{R}^N) \geqslant F_{N,s,\alpha}(u_0) &= \frac{\int_{\mathbb{R}^N}\Phi_N\left(\alpha|u_0(x)|^{N'}\right)\frac{dx}{|x|^s}}{\|u_0\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\\ &= \frac{\sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\|u_0\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|u_0\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\\ &= \frac{\alpha^{N-1}}{(N-1)!} + \frac{\sum_{j=N}^{\infty}\frac{\alpha^j}{j!}\|u_0\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|u_0\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}} > \frac{\alpha^{N-1}}{(N-1)!}.\end{aligned}$$

Hence, (3.11) implies $v \neq 0$ in $X^{1,N}_{s,rad}$, and then from (3.10) and (3.11), we obtain

$$\begin{aligned}\mu_{N,s,\alpha}(\mathbb{R}^N) &\leqslant \frac{\alpha^{N-1}}{(N-1)!} + \frac{\int_{\mathbb{R}^N}\left(\Phi_N\left(\alpha|v(x)|^{N'}\right) - \frac{\alpha^{N-1}}{(N-1)!}|v(x)|^N\right)\frac{dx}{|x|^s}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\\ &= \frac{\int_{\mathbb{R}^N}\Phi_N\left(\alpha|v(x)|^{N'}\right)\frac{dx}{|x|^s}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}} = F_{N,s,\alpha}(v).\end{aligned} \tag{3.12}$$

Therefore, it remains to prove $\|\nabla v\|_{L^N(\mathbb{R}^N)} = 1$. Since $\|\nabla v\|_{L^N(\mathbb{R}^N)} \leqslant 1$ by (3.10), it suffices to show $\|\nabla v\|_{L^N(\mathbb{R}^N)} \geqslant 1$. By the definition of $\mu_{N,s,\alpha}(\mathbb{R}^N)$ and (3.10), we see

$$\begin{aligned}\mu_{N,s,\alpha}(\mathbb{R}^N) &\geqslant F_{N,s,\alpha}\left(\frac{v}{\|\nabla v\|_{L^N(\mathbb{R}^N)}}\right)\\ &= \frac{\|\nabla v\|^N_{L^N(\mathbb{R}^N)}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\int_{\mathbb{R}^N}\Phi_N\left(\alpha\left|\frac{v(x)}{\|\nabla v\|_{L^N(\mathbb{R}^N)}}\right|^{N'}\right)\frac{dx}{|x|^s}\\ &= \frac{\|\nabla v\|^N_{L^N(\mathbb{R}^N)}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\frac{\|v\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|\nabla v\|^{N'j}_{L^N(\mathbb{R}^N)}}\\ &= \sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\frac{\|v\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\|\nabla v\|^{N-N'j}_{L^N(\mathbb{R}^N)}\\ &\geqslant \frac{\alpha^{N-1}}{(N-1)!} + \frac{\alpha^N}{N!}\frac{\|v\|^{N'N}_{L^{N'N}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\|\nabla v\|^{-\frac{N}{N-1}}_{L^N(\mathbb{R}^N)} + \sum_{j=N+1}^{\infty}\frac{\alpha^j}{j!}\frac{\|v\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|v\|^N_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}\end{aligned}$$

$$= \frac{\alpha^{N-1}}{(N-1)!} + \sum_{j=N}^{\infty} \frac{\alpha^j}{j!} \frac{\|v\|_{L^{N'j}(\mathbb{R}^N;|x|^{-s}\,dx)}^{N'j}}{\|v\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^N} + \frac{\alpha^N}{N!}\left(\frac{1}{\|\nabla v\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-1}}} - 1\right)\frac{\|v\|_{L^{N'N}(\mathbb{R}^N;|x|^{-s}\,dx)}^{N'N}}{\|v\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^N}$$

$$= F_{N,s,\alpha}(v) + \frac{\alpha^N}{N!}\left(\frac{1}{\|\nabla v\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-1}}} - 1\right)\frac{\|v\|_{L^{N'N}(\mathbb{R}^N;|x|^{-s}\,dx)}^{N'N}}{\|v\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^N}. \tag{3.13}$$

Thus by (3.12) and (3.13), we have

$$\mu_{N,s,\alpha}(\mathbb{R}^N) \geqslant \mu_{N,s,\alpha}(\mathbb{R}^N) + \frac{\alpha^N}{N!}\left(\frac{1}{\|\nabla v\|_{L^N(\mathbb{R}^N)}^{\frac{N}{N-1}}} - 1\right)\frac{\|v\|_{L^{N'N}(\mathbb{R}^N;|x|^{-s}\,dx)}^{N'N}}{\|v\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^N},$$

which implies $\|\nabla v\|_{L^N(\mathbb{R}^N)} \geqslant 1$, and then it follows $\|\nabla v\|_{L^N(\mathbb{R}^N)} = 1$. As a consequence, (3.12) shows that v is a maximizer for $\mu_{N,s,\alpha}(\mathbb{R}^N)$.

Case 2. We assume $s < t$. By Lemma 3.1 and the compactness of the embedding $X^{1,N}_{s,rad} \hookrightarrow L^N(\mathbb{R}^N; |x|^{-t}\,dx)$, see Lemma A.3 in Appendix A, up to a subsequence, we obtain the convergence as $n \to \infty$,

$$\begin{aligned}\mu_{N,s,t,\alpha}(\mathbb{R}^N) &= F_{N,s,t,\alpha}(v_n) + o(1)\\ &= \int_{\mathbb{R}^N} \Phi_N\left(\alpha|v_n(x)|^{N'}\right)\frac{dx}{|x|^t} + o(1) = \int_{\mathbb{R}^N} \Phi_N\left(\alpha|v(x)|^{N'}\right)\frac{dx}{|x|^t},\end{aligned}$$

which implies $v \neq 0$ in $X^{1,N}_{s,rad}$. Then in a quite same way as in Case 1, we can prove $\|\nabla v\|_{L^N(\mathbb{R}^N)} = 1$ and v is a maximizer for $\mu_{N,s,t,\alpha}(\mathbb{R}^N)$. □

Proof of Corollary 1.4. Corollary 1.4 is an immediate consequence of the special case $s = 0$ in Theorem 1.3(i) and the rearrangement inequalities (2.5) and (2.6). □

We proceed to the proof of Theorem 1.3(ii). Since the attainability of $\tilde{\mu}_{N,\alpha}(\mathbb{R}^N) = \mu_{N,0,0,\alpha}(\mathbb{R}^N)$ has been already shown in Corollary 1.4 with $t = 0$, it remains to prove that $\tilde{\mu}_{N,\alpha}(D)$ admits a maximizer only when $D = \mathbb{R}^N$.

Proof of Theorem 1.3(ii). Now assume that $\tilde{\mu}_{N,\alpha}(D)$ with $D \neq \mathbb{R}^N$ is attained, and we derive a contradiction. Without loss of generality, we can assume $u \geqslant 0$ in D. Let $v(x) := \bar{u}(\|u\|_{L^N(D)}x)$ for $x \in \mathbb{R}^N$, where $\bar{u}$ is a zero-extension of u to $\mathbb{R}^N$. The scale-invariance of the problem yields $\tilde{\mu}_{N,\alpha}(D) = \tilde{\mu}_{N,\alpha}(\mathbb{R}^N)$. Thus we have

$$\|v\|_{L^N(\mathbb{R}^N)} = 1 \quad \text{and} \quad \sum_{j=N-1}^{\infty} \frac{\alpha^j}{j!}\|v\|_{L^{N'j}(\mathbb{R}^N)}^{N'j} = \tilde{F}_{N,\alpha}(v) = \tilde{\mu}_{N,\alpha}(\mathbb{R}^N), \tag{3.14}$$

where $\tilde{F}_{N,\alpha}(v) := F_{N,0,0,\alpha}(v)$ by (1.10). A direct computation yields

$$\begin{aligned}(d\tilde{F}_{N,\alpha})_w(\varphi) = \frac{1}{\|w\|_{L^N(\mathbb{R}^N)}^{2N}}\Bigg(&N'\|w\|_{L^N(\mathbb{R}^N)}^N \sum_{j=N-1}^{\infty}\frac{\alpha^j}{(j-1)!}\int_{\mathbb{R}^N}|w|^{N'j-2}w\varphi\,dx\\ &- N\int_{\mathbb{R}^N}|w|^{N-2}w\varphi\,dx \sum_{j=N-1}^{\infty}\frac{\alpha^j}{j!}\|w\|_{L^{N'j}(\mathbb{R}^N)}^{N'j}\Bigg)\end{aligned}$$

for all $w \in H^{1,N}(\mathbb{R}^N) \setminus \{0\}$ and $\varphi \in H^{1,N}(\mathbb{R}^N)$. The Lagrange multiplier rule together with (3.14) and the relation above show the existence of $\lambda \in \mathbb{R}$ satisfying

$$-\Delta_N v = \lambda\left(N' \sum_{j=N-1}^{\infty}\frac{\alpha^j}{(j-1)!}|v|^{N'j-2}v - N\tilde{\mu}_{N,\alpha}(\mathbb{R}^N)|v|^{N-2}v\right) \quad \text{in } \mathbb{R}^N, \tag{3.15}$$

where $\Delta_N v := \nabla \cdot (|\nabla v|^{N-2}\nabla v)$. Multiplying v by (3.15), integrating over $\mathbb{R}^N$ and using (3.14), we see that

$$\|\nabla v\|_{L^N(\mathbb{R}^N)}^N = \lambda\left(N' \sum_{j=N-1}^{\infty} \frac{\alpha^j}{(j-1)!}\|v\|_{L^{N'j}(\mathbb{R}^N)}^{N'j} - N\|v\|_{L^N(\mathbb{R}^N)}^N \sum_{j=N-1}^{\infty} \frac{\alpha^j}{j!}\|v\|_{L^{N'j}(\mathbb{R}^N)}^{N'j}\right)$$
$$= \lambda N \sum_{j=N-1}^{\infty} \frac{\alpha^j}{j!}\left(\frac{j}{N-1} - 1\right)\|v\|_{L^{N'j}(\mathbb{R}^N)}^{N'j},$$

whence $\lambda > 0$ follows. Consequently, (3.15) gives

$$-\Delta_N v + \lambda N \tilde{\mu}_{N,\alpha}(\mathbb{R}^N)|v|^{N-2}v = \lambda N' \sum_{j=N-1}^{\infty} \frac{\alpha^j}{(j-1)!}|v|^{N'j-2}v \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

This relation together with the strong maximum principle for the degenerate elliptic operator, see e.g., Vázquez [26, Theorem 5], yields $v > 0$ in $\mathbb{R}^N$, which is a contradiction since $v \equiv 0$ in $\{x \in \mathbb{R}^N;\ \|u\|_{L^N(D)}x \notin D\}$. This completes the proof of Theorem 1.3(ii). □

4. Proof of Theorem 1.5

Proof of Theorem 1.5. Let α_0 be the supremum of $\alpha > 0$ such that the inequality

$$\int_{\mathbb{R}^N} \Phi_N\left(\alpha|u(x)|^{N'}\right)\frac{dx}{|x|^t} \leqslant C\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{N-s}} \tag{4.1}$$

holds for all $u \in X_{s,rad}^{1,N}$ for some $C > 0$. Also let $\beta_{N,t}$ be the infimum of $\beta > 0$ satisfying the following: there exists $q_0 \geqslant N$ such that, for any $q \geqslant q_0$ the inequality

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \beta q^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}}\|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}} \tag{4.2}$$

holds for all $u \in X_{s,rad}^{1,N}$. Then Theorem 1.1 implies $\alpha_0 := \alpha_{N,t} = (N-t)\omega_{N-1}^{\frac{1}{N-1}}$, and hence, our goal is to prove the exact relation between α_0 and $\beta_{N,t}$ such as

$$\beta_{N,t} = \left(\frac{1}{eN'\alpha_0}\right)^{\frac{1}{N'}} = \left(\frac{N-1}{eN(N-t)\omega_{N-1}^{\frac{1}{N-1}}}\right)^{\frac{N-1}{N}}. \tag{4.3}$$

The definition of α_0 guarantees that for any $0 < \alpha < \alpha_0$, there exists a positive constant C such that (4.1) holds for all $u \in X_{s,rad}^{1,N}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$. By (4.1) without the normalization $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$, we have

$$C\left(\frac{\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}\right)^{\frac{N(N-t)}{N-s}} \geqslant \int_{\mathbb{R}^N} \Phi_N\left(\alpha\left(\frac{|u(x)|}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}\right)^{N'}\right)\frac{dx}{|x|^t}$$
$$= \sum_{k=N-1}^{\infty} \frac{\alpha^k}{k!}\left(\frac{\|u\|_{L^{N'k}(\mathbb{R}^N;|x|^{-t}\,dx)}}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}\right)^{N'k} \geqslant \frac{\alpha^j}{j!}\left(\frac{\|u\|_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)}}{\|\nabla u\|_{L^N(\mathbb{R}^N)}}\right)^{N'j}$$

for all $u \in X_{s,rad}^{1,N} \setminus \{0\}$ and all integers $j \geqslant N-1$. Thus we obtain

$$\|u\|_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \left(C\frac{j!}{\alpha^j}\right)^{\frac{1}{N'j}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{(N-s)N'j}}\|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{(N-s)N'j}} \tag{4.4}$$

for all $u \in X_{s,rad}^{1,N}$ and for all integers $j \geqslant N-1$. Moreover, for any $q \geqslant N$, there exists an integer $j \geqslant N-1$ satisfying $N'j \leqslant q < N'(j+1)$. Then by Hölder's inequality, it holds

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \|u\|_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)}^{\theta}\|u\|_{L^{N'(j+1)}(\mathbb{R}^N;|x|^{-t}\,dx)}^{1-\theta}, \tag{4.5}$$

where the interpolation index $\theta \in (0,1]$ enjoys $\frac{1}{q} = \frac{\theta}{N'j} + \frac{1-\theta}{N'(j+1)}$. Combining (4.4) with (4.5) yields

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C^{\frac{1}{q}} \alpha^{-\frac{1}{N'}} \big((j+1)!\big)^{\frac{1}{q}} \|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}} \tag{4.6}$$

for all $u \in X^{1,N}_{s,rad}$ and for all $q \geqslant N$. Now we investigate the positive constant which appears in the right-hand side of (4.6). Noting $\frac{q}{N'} \geqslant j$, we see

$$\big((j+1)!\big)^{\frac{1}{q}} = \Gamma(j+2)^{\frac{1}{q}} \leqslant \Gamma\left(\frac{q}{N'}+2\right)^{\frac{1}{q}},$$

where Γ denotes the Gamma function. Here let us recall Stirling's asymptotic formula

$$\lim_{t\to+\infty} \frac{\Gamma(t+1)}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t} = 1.$$

Thus we can compute as $q \to +\infty$,

$$\begin{aligned}\Gamma\left(\frac{q}{N'}+2\right)^{\frac{1}{q}} &= \left(\big(1+o(1)\big)\sqrt{2\pi\left(\frac{q}{N'}+1\right)}\left(\frac{\frac{q}{N'}+1}{e}\right)^{\frac{q}{N'}+1}\right)^{\frac{1}{q}} \\ &= \big(1+o(1)\big)\left(\frac{q}{eN'}\right)^{\frac{1}{N'}}.\end{aligned} \tag{4.7}$$

Hence, summing-up (4.6) and (4.7), we have as $q \to +\infty$,

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \big(1+o(1)\big)\left(\frac{q}{eN'\alpha}\right)^{\frac{1}{N'}} \|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q(N-s)}} \|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q(N-s)}},$$

which implies $\beta_{N,t} \leqslant \left(\frac{1}{eN'\alpha}\right)^{\frac{1}{N'}}$ for all $\alpha \in (0,\alpha_0)$, and then we obtain

$$\beta_{N,t} \leqslant \left(\frac{1}{eN'\alpha_0}\right)^{\frac{1}{N'}}. \tag{4.8}$$

Next, we shall prove the reverse inequality of (4.8). Take $\beta > \beta_{N,t}$ arbitrarily. From the definition of $\beta_{N,t}$, there exists $q_0 \geqslant N$ such that (4.2) holds for all $u \in X^{1,N}_{s,rad}$ and for all $q \geqslant q_0$. Then for positive α which will be chosen appropriately later, we see for any $u \in X^{1,N}_{s,rad}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$,

$$\begin{aligned}&\int_{\mathbb{R}^N} \Phi_N\big(\alpha|u(x)|^{N'}\big)\frac{dx}{|x|^t} \\ &\quad= \int_{\mathbb{R}^N}\Bigg(\sum_{\substack{N\leqslant N'j<q_0,\\ j\in\mathbb{N}}} \frac{\alpha^j}{j!}|u(x)|^{N'j}\Bigg)\frac{dx}{|x|^t} + \int_{\mathbb{R}^N}\Bigg(\sum_{\substack{N'j\geqslant q_0,\\ j\in\mathbb{N}}} \frac{\alpha^j}{j!}|u(x)|^{N'j}\Bigg)\frac{dx}{|x|^t} =: J_1 + J_2.\end{aligned}$$

As for the estimate of J_1, we have

$$J_1 = \sum_{\substack{N\leqslant N'j<q_0,\\ j\in\mathbb{N}}} \frac{\alpha^j}{j!}\|u\|_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)}^{N'j},$$

which consists of finite weighted Lebesgue norms. For each integer j with $N \leqslant N'j < q_0$, it holds by Hölder's inequality,

$$\|u\|_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \|u\|_{L^N(\mathbb{R}^N;|x|^{-t}\,dx)}^{\theta} \|u\|_{L^{q_0}(\mathbb{R}^N;|x|^{-t}\,dx)}^{1-\theta}, \tag{4.9}$$

where the interpolation index $\theta \in (0,1]$ enjoys $\frac{1}{N'j} = \frac{\theta}{N} + \frac{1-\theta}{q_0}$. Furthermore, by the assumption, (4.2) holds with $q = q_0$, that is, we have

$$\begin{aligned}\|u\|_{L^{q_0}(\mathbb{R}^N;|x|^{-t}\,dx)} &\leqslant \beta q_0^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q_0(N-s)}}\|\nabla u\|_{L^N(\mathbb{R}^N)}^{1-\frac{N(N-t)}{q_0(N-s)}}\\ &\leqslant \beta q_0^{\frac{1}{N'}}\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}^{\frac{N(N-t)}{q_0(N-s)}}\end{aligned}\tag{4.10}$$

for all $u \in X^{1,N}_{s,rad}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$. Thus by (2.2), (4.9) and (4.10), we obtain

$$\|u\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\|u\|^{\frac{N(N-t)}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}$$

for all $u \in X^{1,N}_{s,rad}$ with $\|\nabla u\|_{L^N(\mathbb{R}^N)} \leqslant 1$ and for all integers j with $N \leqslant N'j < q_0$. Thus we can estimate J_1 as

$$J_1 \leqslant C\Biggl(\sum_{\substack{N\leqslant N'j<q_0,\\ j\in\mathbb{N}}}\frac{\alpha^j}{j!}\Biggr)\|u\|^{\frac{N(N-t)}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} = C\|u\|^{\frac{N(N-t)}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}.\tag{4.11}$$

We proceed to the estimate of J_2. Applying (4.2), we see

$$\begin{aligned}J_2 &= \sum_{\substack{N'j\geqslant q_0,\\ j\in\mathbb{N}}}\frac{\alpha^j}{j!}\|u\|^{N'j}_{L^{N'j}(\mathbb{R}^N;|x|^{-t}\,dx)}\\ &\leqslant \sum_{\substack{N'j\geqslant q_0,\\ j\in\mathbb{N}}}\frac{\alpha^j}{j!}\Bigl(\beta(N'j)^{\frac{1}{N'}}\|u\|^{\frac{N(N-t)}{N'j(N-s)}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}\Bigr)^{N'j}\\ &= \Biggl(\sum_{\substack{N'j\geqslant q_0,\\ j\in\mathbb{N}}}\frac{j^j}{j!}\bigl(\alpha N'\beta^{N'}\bigr)^j\Biggr)\|u\|^{\frac{N(N-t)}{N-s}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}.\end{aligned}\tag{4.12}$$

We now take positive α satisfying $\alpha N'\beta^{N'} < \frac{1}{e}$ so that the power series in (4.12) converges. Therefore, by (4.11) and (4.12), we get $\alpha_0 \geqslant \frac{1}{eN'\beta^{N'}}$ for all $\beta > \beta_{N,t}$, which implies

$$\alpha_0 \geqslant \frac{1}{eN'\beta_{N,t}^{N'}} \quad \text{or equivalently} \quad \beta_{N,t} \geqslant \left(\frac{1}{eN'\alpha_0}\right)^{\frac{1}{N'}}.\tag{4.13}$$

Hence, by (4.8) and (4.13), the desired equality (4.3) can be obtained, and we finish the proof of Theorem 1.5. □

Proof of Corollary 1.6. Corollary 1.6 is an immediate consequence of the special case $s = 0$ in Theorem 1.5 and the rearrangement inequalities (2.5) and (2.6). □

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Appendix A

In this section, we collect several lemmas for the proof of the main theorems.

A.1. For Corollary 1.2, Corollary 1.4 and Corollary 1.6

First, we establish the rearrangement inequality (2.5). Precisely, the Schwarz symmetrization will be defined as follows. For a measurable function u on $\mathbb{R}^N$, $a_u : [0,\infty) \to [0,\infty]$ denotes the distribution function of u, that is, for $\lambda \geqslant 0$,

$$a_u(\lambda) := \bigl|\bigl\{x \in \mathbb{R}^N;\, \bigl|u(x)\bigr| > \lambda\bigr\}\bigr|,$$

where $|\Omega|$ means the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^N$. Then u^*: $[0,\infty) \to [0,\infty]$ and $u^\#$: $\mathbb{R}^N \to [0,\infty]$ are defined as

$$\begin{cases} u^*(t) := \inf\{\lambda > 0;\, a_u(\lambda) \leqslant t\} & \text{for } t \geqslant 0, \\ u^\#(x) := u^*\left(\dfrac{\omega_{N-1}}{N}|x|^N\right) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

We call u^* and $u^\#$ the rearrangement and the Schwarz symmetrization of u, respectively. We now prove (2.5) below.

Lemma A.1. *Let $N \geqslant 2$, $0 \leqslant t < N$ and $q \geqslant N$. Then it holds*

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant \|u^\#\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \tag{A.1}$$

for all functions u so that $u^\# \in L^q(\mathbb{R}^N;|x|^{-t}\,dx)$. Furthermore, the inequality (A.1) becomes the equality for the non-singular case $t = 0$.

Proof. Note that

$$\|u\|^q_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} = \int_{\mathbb{R}^N} |u(x)|^q \frac{dx}{|x|^t} = q\int_0^\infty \left(\int_{\{x\in\mathbb{R}^N;\,|u(x)|>\lambda\}} \frac{dx}{|x|^t}\right)\lambda^{q-1}\,d\lambda \tag{A.2}$$

holds. Therefore, in order to obtain (A.1), by recalling the fact that the functions u and $u^\#$ have the same distribution function since they are equi-measurable, it is enough to show the following inequality

$$\int_\Omega \frac{dx}{|x|^t} \leqslant \int_{\Omega^\#} \frac{dx}{|x|^t} \tag{A.3}$$

for all measurable sets $\Omega \subset \mathbb{R}^N$ with $|\Omega| < \infty$, where $\Omega^\#$ denotes the ball centered at the origin satisfying $|\Omega| = |\Omega^\#|$. Clearly, if $t = 0$, then the inequality (A.3) becomes the equality, and then by the representation (A.2), we obtain the equality (A.1).

Therefore, it remains to prove (A.3) for $0 \leqslant t < N$. By decomposing $\Omega = (\Omega \setminus \Omega^\#) \cup (\Omega \cap \Omega^\#)$ and $\Omega^\# = (\Omega^\# \setminus \Omega) \cup (\Omega^\# \cap \Omega)$, (A.3) is equivalent to

$$\int_{\Omega\setminus\Omega^\#} \frac{dx}{|x|^t} \leqslant \int_{\Omega^\#\setminus\Omega} \frac{dx}{|x|^t}. \tag{A.4}$$

Since $x \in \Omega \setminus \Omega^\#$ implies $|x| \geqslant \frac{\operatorname{diam}(\Omega^\#)}{2}$, we have

$$\int_{\Omega\setminus\Omega^\#} \frac{dx}{|x|^t} \leqslant \left(\frac{2}{\operatorname{diam}(\Omega^\#)}\right)^t |\Omega \setminus \Omega^\#|. \tag{A.5}$$

On the other hand, since $x \in \Omega^\# \setminus \Omega$ implies $|x| < \frac{\operatorname{diam}(\Omega^\#)}{2}$, we have

$$\int_{\Omega^\#\setminus\Omega} \frac{dx}{|x|^t} \geqslant \left(\frac{2}{\operatorname{diam}(\Omega^\#)}\right)^t |\Omega^\# \setminus \Omega|. \tag{A.6}$$

By noting $|\Omega \setminus \Omega^\#| = |\Omega^\# \setminus \Omega|$ and combining (A.5) with (A.6), we obtain (A.4), and then we have proved (A.1). □

A.2. For Theorem 1.3

Next, we shall prove an inequality (3.4), which will be needed to show Lemma 3.1.

Lemma A.2. *It holds*

$$f_N(\tau)^b \leqslant f_N(b\tau) \tag{A.7}$$

for all $N \geqslant 2$, $b \geqslant 1$ *and* $\tau \geqslant 0$, *where* $f_N(\tau) := e^\tau - \sum_{j=0}^{N-2} \frac{\tau^j}{j!}$.

Proof. For $N = 2$ and $N = 3$, it is easy to see that (A.7) holds. Now assume that (A.7) is true for $N_0 \geqslant 3$. For $\tau \geqslant 0$, let

$$g_{N_0+1}(\tau) := f_{N_0+1}(b\tau) - f_{N_0+1}(\tau)^b.$$

Since for any $\tau > 0$ and $N \geqslant 3$,

$$f_N'(\tau) = e^\tau - \sum_{j=0}^{N-3} \frac{\tau^j}{j!} = f_{N-1}(\tau),$$

we obtain for any $\tau > 0$,

$$\begin{aligned} g_{N_0+1}'(\tau) &= bf_{N_0+1}'(b\tau) - bf_{N_0+1}(\tau)^{b-1} f_{N_0+1}'(\tau) \\ &= b\big(f_{N_0}(b\tau) - f_{N_0+1}(\tau)^{b-1} f_{N_0}(\tau)\big) \geqslant b\big(f_{N_0}(\tau)^b - f_{N_0+1}(\tau)^{b-1} f_{N_0}(\tau)\big) \\ &= bf_{N_0}(\tau)\big(f_{N_0}(\tau)^{b-1} - f_{N_0+1}(\tau)^{b-1}\big) \geqslant 0, \end{aligned}$$

where we use $f_{N_0}(\tau) \geqslant f_{N_0+1}(\tau)$ for $\tau \geqslant 0$. Also, we have $g_{N_0+1}(0) = 0$, hence the inequality above yields $g_{N_0+1}(\tau) \geqslant 0$ for $\tau \geqslant 0$, that is,

$$f_{N_0+1}(\tau)^b \leqslant f_{N_0+1}(b\tau)$$

for all $\tau \geqslant 0$. By the induction argument with respect to $N \geqslant 3$, we have (A.7) for all $N \geqslant 3$. □

Finally, we shall prove the compactness of the embedding corresponding to the Caffarelli–Kohn–Nirenberg inequality, which will be used to prove Theorem 1.3.

Lemma A.3. *Let* $N \geqslant 2$ *and let* (s, t, q) *be exponents satisfying either*

$$-\infty < s < t < N \text{ and } N \leqslant q < \infty \quad \text{or} \quad -\infty < s = t < N \text{ and } N < q < \infty.$$

Then the embedding

$$X^{1,N}_{s,rad} \hookrightarrow L^q\big(\mathbb{R}^N; |x|^{-t}\,dx\big)$$

is compact.

Remark. By the Caffarelli–Kohn–Nirenberg inequality in [4], the continuous embedding $X^{1,N}_{s,rad} \hookrightarrow L^q(\mathbb{R}^N; |x|^{-t}\,dx)$ holds for all exponents (s, t, q) satisfying $-\infty < s \leqslant t < N$ and $N \leqslant q < \infty$. However, Lemma A.3 fails if $s = t$ and $q = N$, which includes the well-known case of the non-compact embedding $X^{1,N}_{0,rad} = \{u \in H^{1,N}(\mathbb{R}^N);\ u \text{ is radial}\} \hookrightarrow L^N(\mathbb{R}^N)$.

Proof of Lemma A.3. The Caffarelli–Kohn–Nirenberg inequality in [4] states that for any $N \geqslant 2$, $-\infty < s \leqslant t < N$ and $N \leqslant q < \infty$, there exists a positive constant C such that the inequality

$$\|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)} \leqslant C\|u\|^{\frac{N(N-t)}{q(N-s)}}_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)} \|\nabla u\|^{1-\frac{N(N-t)}{q(N-s)}}_{L^N(\mathbb{R}^N)} \tag{A.8}$$

holds for all $u \in X^{1,N}_s$. For any $u \in X^{1,N}_{s,rad}$, we define the radially symmetric function v by the formula

$$v(x) := \left(\frac{N-s}{N}\right)^{\frac{N-1}{N}} \tilde{u}\big(|x|^{\frac{N}{N-s}}\big),$$

where $u(x) = \tilde{u}(|x|)$ for $x \in \mathbb{R}^N$. Then direct computations show that $v \in H^{1,N}(\mathbb{R}^N)$ and

$$\begin{cases} \|\nabla v\|_{L^N(\mathbb{R}^N)} = \|\nabla u\|_{L^N(\mathbb{R}^N)}, \\ \|v\|_{L^N(\mathbb{R}^N)} = \left(\frac{N-s}{N}\right)\|u\|_{L^N(\mathbb{R}^N;|x|^{-s}\,dx)}, \\ \|v\|_{L^q(\mathbb{R}^N;|x|^{-\frac{N(t-s)}{N-s}}\,dx)} = \left(\frac{N-s}{N}\right)^{\frac{1}{q}+\frac{N-1}{N}} \|u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)}. \end{cases} \tag{A.9}$$

Thus by plugging (A.9) into (A.8) and by letting $\tilde{t} := \frac{N(t-s)}{N-s} \in [0, N)$, (A.8) can be transferred equivalently to

$$\|v\|_{L^q(\mathbb{R}^N;|x|^{-\tilde{t}}\,dx)} \leqslant C\|v\|_{L^N(\mathbb{R}^N)}^{\frac{N-\tilde{t}}{q}} \|\nabla v\|_{L^N(\mathbb{R}^N)}^{1-\frac{N-\tilde{t}}{q}}$$

for all radially symmetric function $u \in H^{1,N}(\mathbb{R}^N)$. Therefore, without loss of generality, we may assume $s = 0$ in order to prove Lemma A.3.

When $s = t = 0$, it is well-known that the embeddings

$$X^{1,N}_{0,rad} = \{u \in H^{1,N}(\mathbb{R}^N);\, u \text{ is radial}\} \hookrightarrow L^q(\mathbb{R}^N)$$

are compact for all $q > N$. Thus in what follows, we consider the case $0 < t < N$ and $N \leqslant q < \infty$. Let $(u_n) \subset H^{1,N}(\mathbb{R}^N)$ be a bounded sequence of radially symmetric functions, and take a number p arbitrarily so that $1 < p < \frac{N}{t}$. By the compactness, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H^{1,N}(\mathbb{R}^N), \\ u_n \to u & \text{strongly in } L^{qp'}(\mathbb{R}^N) \end{cases} \tag{A.10}$$

as $n \to \infty$, where we used $qp' > N$. For any $R > 0$, we decompose $\|u_n - u\|_{L^q(\mathbb{R}^N;|x|^{-t}\,dx)}$ as

$$\int_{\mathbb{R}^N} |u_n - u|^q \frac{dx}{|x|^t} = \int_{\{|x|<R\}} |u_n - u|^q \frac{dx}{|x|^t} + \int_{\{|x|\geqslant R\}} |u_n - u|^q \frac{dx}{|x|^t}.$$

Since $t > 0$, we see

$$\begin{aligned} \left(\int_{\{|x|\geqslant R\}} |u_n - u|^q \frac{dx}{|x|^t}\right)^{\frac{1}{q}} &\leqslant R^{-\frac{t}{q}}\|u_n - u\|_{L^q(\mathbb{R}^N)} \leqslant R^{-\frac{t}{q}}\left(\|u_n\|_{L^q(\mathbb{R}^N)} + \|u\|_{L^q(\mathbb{R}^N)}\right) \\ &\leqslant CR^{-\frac{t}{q}}\left(\|u_n\|_{H^{1,N}(\mathbb{R}^N)} + \|u\|_{H^{1,N}(\mathbb{R}^n)}\right) \leqslant CR^{-\frac{t}{q}} \end{aligned} \tag{A.11}$$

for all $n \in \mathbb{N}$ and all $R > 0$. On the other hand, by using the Hölder inequality and the latter convergence in (A.10), we see for any $R > 0$,

$$\begin{aligned} \int_{\{|x|<R\}} |u_n - u|^q \frac{dx}{|x|^t} &\leqslant \left(\int_{\{|x|<R\}} |u_n - u|^{qp'}\,dx\right)^{\frac{1}{p'}} \left(\int_{\{|x|<R\}} |x|^{-tp}\,dx\right)^{\frac{1}{p}} \\ &\leqslant CR^{\frac{N}{p}-t}\left(\int_{\mathbb{R}^N} |u_n - u|^{qp'}\,dx\right)^{\frac{1}{p'}} \to 0 \end{aligned} \tag{A.12}$$

as $n \to \infty$, where we used $1 < p < \frac{N}{t}$ for the local integrability. Thus combining (A.11) with (A.12), we have

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n - u|^q \frac{dx}{|x|^t} \leqslant CR^{-\frac{t}{q}}.$$

Then we have the desired strong convergence in $L^q(\mathbb{R}^N; |x|^{-t}\,dx)$ by letting $R \to \infty$. $\square$

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