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Relative isoperimetric inequalities and sufficient conditions for finite perimeter on metric spaces [☆]

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Abstract

We study equivalence between the Poincaré inequality and several different relative isoperimetric inequalities on metric measure spaces. We then use these inequalities to establish sufficient conditions for the finite perimeter of sets.

Résumé

Nous étudions l'équivalence entre l'inégalité de Poincaré et plusieurs différentes inégalités isopérimétriques relatives sur les espaces métriques mesurés. Nous utilisons ensuite ces inégalités afin d'établir des conditions suffisantes sur le périmètre fini d'ensembles.

MSC: 28A12; 26A45; 30L99

1. Introduction

The Poincaré inequality is a standard assumption in analysis on metric spaces. This condition is associated with the connectedness properties of the space and the existence of a "thick" family of curves between any two points. There are several results concerning the characterization of Poincaré inequalities on metric spaces, see e.g. [6,20,23], and in particular the equivalence between a Poincaré inequality and an isoperimetric inequality, see [8,24]. In this paper, we consider several different relative isoperimetric inequalities and investigate equivalence between them and the Poincaré inequality.

For $1 \leqslant q, p < \infty$, we say that a metric space X supports a (q, p)-Poincaré inequality if for all locally integrable functions u on X and their p-weak upper gradients g, all balls $B = B(x, r)$, and some constants $c_P > 0$, $\lambda \geqslant 1$, we have

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$$\left(\fint_B |u-u_B|^q\,d\mu\right)^{1/q} \leqslant c_P r\left(\fint_{\lambda B} g^p\,d\mu\right)^{1/p}. \tag{1.1}$$

In this paper, we focus almost exclusively on the case $p=1$. The (ordinary) relative isoperimetric inequality holds if for some $q \geqslant 1$, every μ-measurable set $E \subset X$, every ball $B = B(x,r)$, and some constants $c_I > 0$, $\lambda \geqslant 1$, we have

$$\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_I r\frac{\|D\chi_E\|(\lambda B)}{\mu(\lambda B)}. \tag{1.2}$$

On the right-hand side, we have the perimeter of the set E in the ball λB. In addition, we consider four different formulations of a *strong relative isoperimetric inequality*:

$$\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r\frac{\mu^+(\lambda B\cap \partial E)}{\mu(\lambda B)}, \tag{1.3}$$

$$\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r\frac{\mu^+(\lambda B\cap \partial^* E)}{\mu(\lambda B)}, \tag{1.4}$$

$$\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r\frac{\mathcal{H}(\lambda B\cap \partial E)}{\mu(\lambda B)}, \tag{1.5}$$

$$\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r\frac{\mathcal{H}(\lambda B\cap \partial^* E)}{\mu(\lambda B)}, \tag{1.6}$$

where $q \geqslant 1$, $E \subset X$ is any μ-measurable set, $B = B(x,r)$ is any ball, and $c_S > 0$ and $\lambda \geqslant 1$ are constants that may vary from one inequality to the other. On the right-hand side, we have either the codimension one Minkowski content μ^+, or the codimension one Hausdorff measure $\mathcal{H}$, of either the topological boundary ∂E or the measure theoretic boundary $\partial^* E$. Later on we will often refer to the $(q,1)$-Poincaré inequality and to the relative isoperimetric inequality by their names, and to the strong relative isoperimetric inequalities by the numbers (1.3)–(1.6).

Our central result, which will be proved over the course of Section 3, is the following — the precise assumptions on the space are listed in Section 2.

Theorem 1.1. *Let X be a metric measure space equipped with a doubling outer measure μ. Then the Poincaré inequality* (1.1) *with $p = 1$, the relative isoperimetric inequality* (1.2)*, and the strong relative isoperimetric inequalities* (1.3), (1.4), *and* (1.5) *are all equivalent for any $q \geqslant 1$. The constants $c_P, c_I, c_S > 0$ and $\lambda \geqslant 1$ may vary but they depend quantitatively on each other, the doubling constant of the measure, and the number q.*

Especially the equivalence between (1.3) and (1.5) seems unexpected, since the codimension one Hausdorff measure and Minkowski content are not, in general, comparable.

The case $p > 1$ is briefly discussed in Section 4. Also in this case, an analogous characterization of the Poincaré inequality is available, but Hausdorff measures must be replaced by capacities.

In Section 5, we consider the fourth strong relative isoperimetric inequality (1.6), which implies all of the others. However, we are only able to establish a converse if we replace the measure theoretic boundary $\partial^* E$ with a slightly larger set $\partial_1^* E$. For the definition of the latter, see Definition 5.1.

In Section 6, we consider functions of bounded variation, abbreviated as *BV* functions. For the general theory and characterizations of *BV* functions on metric spaces, see e.g. [2,4,10,18,27,30,31] — see also [3,14,15,34] for the classical theory in the Euclidean case. In particular, *BV* functions can be characterized as functions that satisfy a Poincaré-type inequality with a (locally) finite Borel outer measure on the right-hand side [31]. On the other hand, the strong relative isoperimetric inequalities (1.5) and (1.6) with $q = 1$ are inequalities of precisely this type, written for the characteristic functions of sets. They can thus be used to establish sufficient conditions for the finite perimeter of a set, as was done in [26]. We simply need the finiteness of one of the quantities $\mathcal{H}(\partial E)$, $\mathcal{H}(\partial^* E)$, depending on which strong relative isoperimetric inequality we have at our disposal. The latter case is the more interesting, since for a set of finite perimeter, the perimeter measure and the codimension one Hausdorff measure restricted to the measure theoretic boundary are comparable (see [4]). However, we are only able to establish this type of condition in a slightly weaker form.

2. Notation and preliminaries

We assume throughout that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular, doubling outer measure μ. The doubling property means that there is a fixed constant $c_d > 0$, called the doubling constant of μ, such that

$$\mu(2B) \leqslant c_d \mu(B) \tag{2.1}$$

for every ball $B = B(x,r) := \{y \in X\colon d(y,x) < r\}$. Here $tB := B(x,tr)$, and furthermore we denote $\bar{B} = \bar{B}(x,r) := \{y \in X\colon d(y,x) \leqslant r\}$. We assume that the measure of every open set is positive and that the measure of every bounded set is finite. We also assume that X is complete; recall that a metric space with a doubling measure is complete if and only if the space is proper, that is, closed and bounded sets are compact. Since X is proper, for any open set $\Omega \subset X$ we define e.g. $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ as the space of functions that are Lipschitz in every open $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω.

The integral average of a function $u \in L^1(A)$ over a μ-measurable set A with finite and positive measure is $u_A := \fint_A u\,d\mu := \mu(A)^{-1} \int_A u\,d\mu$. The characteristic function of a set $E \subset X$ is denoted χ_E. In general, C will denote a positive constant whose value is not necessarily the same at each occurrence.

A curve is a rectifiable continuous mapping from a compact interval to X, and is usually denoted by the symbol γ. The image of a curve γ in X is denoted $|\gamma|$. For any given set $A \subset X$, the family of curves γ satisfying $|\gamma| \cap A \neq \emptyset$ is denoted Γ_A. The notation Γ_X then refers to the family of all curves in the space. The length of a curve γ is denoted $l(\gamma)$. We will assume every curve to be parameterized by arc-length, which can always be done (see e.g. [16] or [5]).

Definition 2.1. Let $\Gamma \subset \Gamma_X$ be a family of curves, and let $1 \leqslant p < \infty$. The *p-modulus* of Γ is defined as

$$\mathrm{Mod}_p(\Gamma) := \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all nonnegative Borel functions ρ such that $\int_\gamma \rho\, ds \geqslant 1$ for every $\gamma \in \Gamma$.

If a property fails only for a curve family with p-modulus zero, we say that it holds for p-almost every (p-a.e.) curve. For the properties of the modulus of curve families, see [21] or [32].

Definition 2.2. A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function u on X if for all curves $\gamma \in \Gamma_X$ with end points x and y, we have

$$\bigl|u(x) - u(y)\bigr| \leqslant \int_\gamma g\, ds \tag{2.2}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,ds = \infty$ otherwise. If g is a nonnegative μ-measurable function on X and (2.2) holds for p-almost every curve, then g is a *p-weak upper gradient* of u.

Natural upper gradients for locally Lipschitz functions $u \in \mathrm{Lip}_{\mathrm{loc}}(X)$ are the local Lipschitz constants

$$\mathrm{lip}\, u(x) := \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r} \tag{2.3}$$

and

$$\mathrm{Lip}\, u(x) := \limsup_{r \to 0} \sup_{y \in B(x,r)} \frac{|u(y) - u(x)|}{r} = \limsup_{y \to x} \frac{|u(y) - u(x)|}{d(y,x)}. \tag{2.4}$$

Definition 2.3. For $1 \leqslant p < \infty$ and $u \in L^p(X)$, let

$$\|u\|_{N^{1,p}(X)} := \biggl(\int_X |u|^p \, d\mu + \inf_g \int_X g^p \, d\mu \biggr)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The Newtonian space is the quotient space

$$N^{1,p}(X) := \{u\colon \ \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. Similarly, we can define $N^{1,p}(\Omega)$ for any open set $\Omega \subset X$.

It is known that for any $u \in N^{1,p}(\Omega)$, there exists a minimal p-weak upper gradient, denoted g_u, satisfying $g_u(x) \leqslant g(x)$ for μ-a.e. $x \in \Omega$, for any p-weak upper gradient $g \in L^p(\Omega)$ (see [16] or [6, p. 40]). Furthermore, if we consider any open $\Omega' \subset \Omega$, then $u \in N^{1,p}(\Omega')$ and the minimal p-weak upper gradient g_u is the same with respect to Ω' and Ω [6, p. 51].

Next we recall the definition of functions of bounded variation on metric spaces, given by Miranda in [31].

Definition 2.4. For $u \in L^1_{\mathrm{loc}}(X)$, we define

$$\|Du\|(X) := \inf\left\{\liminf_{i\to\infty} \int_X \operatorname{lip} u_i \, d\mu\colon \ u_i \in \operatorname{Lip}_{\mathrm{loc}}(X),\ u_i \to u \text{ in } L^1_{\mathrm{loc}}(X)\right\},$$

and we say that a function $u \in L^1(X)$ is of bounded variation, $u \in BV(X)$, if $\|Du\|(X) < \infty$. If the function u is the characteristic function of a set $E \subset X$ and $\|D\chi_E\|(X) < \infty$, we say that the set E has finite perimeter. Similarly, we can define $\|Du\|(\Omega)$ for any open set $\Omega \subset X$.

If $u \in BV(X)$, for an arbitrary set $A \subset X$ (not necessarily open), we define

$$\|Du\|(A) := \inf\{\|Du\|(\Omega)\colon \ A \subset \Omega,\ \Omega \text{ is open}\}.$$

According to Miranda, $\|Du\|(\cdot)$ is then a finite Borel outer measure. For a set E, we also define

$$P(E, A) := \|D\chi_E\|(A),$$

which we call the perimeter of E in A. Miranda also proves the following coarea formula: if $\Omega \subset X$ is open and $u \in L^1_{\mathrm{loc}}(\Omega)$, we have

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(\{u > t\}, \Omega)\, dt. \tag{2.5}$$

For any set $A \subset X$, the restricted spherical Hausdorff content of codimension one is defined as

$$\mathcal{H}_R(A) := \inf\left\{\sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i}\colon \ A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i \leqslant R\right\},$$

where $0 < R < \infty$. When $R = \infty$, the infimum is taken over coverings with finite radius. The Hausdorff measure of codimension one of a set $A \subset X$ is

$$\mathcal{H}(A) := \lim_{R\to 0} \mathcal{H}_R(A).$$

We define the codimension one Minkowski content of a set $A \subset X$ to be

$$\mu^+(A) := \liminf_{h\to 0} \frac{\mu(\bigcup_{x\in A} B(x, h))}{2h}.$$

The (topological) boundary ∂E of a set $E \subset X$ is defined as usual. The measure theoretic boundary $\partial^* E$ is defined as the set of points $x \in X$ in which both E and its complement have positive density, i.e.

$$\limsup_{r\to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

Furthermore, we define the measure theoretic interior of E to be

$$I := \left\{x \in X\colon \lim_{r\to 0} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} = 0\right\},$$

and the measure theoretic exterior to be

$$O := \left\{x \in X\colon \lim_{r\to 0} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} = 0\right\}.$$

3. Equivalence of isoperimetric inequalities

In this section, we prove Theorem 1.1, i.e. we show that the Poincaré inequality, the relative isoperimetric inequality, and the strong relative isoperimetric inequalities (1.3)–(1.5) are all equivalent.

A simple observation that we will need on several occasions is that for any $q \geqslant 1$, any μ-measurable set E, and any ball B, we have by an elementary calculation

$$\begin{aligned}\frac{1}{2}\left(\frac{\min\{\mu(B\cap E),\mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} &\leqslant \left(\fint_B |\chi_E-(\chi_E)_B|^q\,d\mu\right)^{1/q}\\ &\leqslant 2^{1/q}\left(\frac{\min\{\mu(B\cap E),\mu(B\setminus E)\}}{\mu(B)}\right)^{1/q}.\end{aligned} \tag{3.1}$$

For example, to obtain the first inequality, we simply note that if $(\chi_E)_B \geqslant \frac{1}{2}$, then

$$\int_B |\chi_E-(\chi_E)_B|^q\,d\mu \geqslant \left(\frac{1}{2}\right)^q \mu(B\setminus E).$$

If, on the other hand, $(\chi_E)_B < \frac{1}{2}$, then

$$\int_B |\chi_E-(\chi_E)_B|^q\,d\mu \geqslant \left(\frac{1}{2}\right)^q \mu(B\cap E).$$

Additionally we will need the following characterization of the (q,p)-Poincaré inequality, generalizing a result by Keith [23].

Theorem 3.1. *For $1 \leqslant q, p < \infty$, the space X supports a (q,p)-Poincaré inequality if and only if we have*

$$\left(\fint_B |u-u_B|^q\,d\mu\right)^{1/q} \leqslant c_L r\left(\fint_{\lambda B} (\operatorname{Lip} u)^p\,d\mu\right)^{1/p} \tag{3.2}$$

for every Lipschitz function with compact support $u \in \mathrm{Lip}_c(X)$, every ball $B = B(x,r)$, and some constants $c_L > 0$, $\lambda \geqslant 1$. Assuming (3.2), these constants remain the same for the Poincaré inequality, except that in the case $p = 1$ we may have $c_P = c_P(c_L, \lambda, c_d)$.

Proof. It is clear that if the space supports a (q,p)-Poincaré inequality, (3.2) holds with the same constants. Let us show the other direction. From (3.2) we immediately get by Hölder's inequality that

$$\fint_B |u-u_B|\,d\mu \leqslant c_L r\left(\fint_{\lambda B} (\operatorname{Lip} u)^p\,d\mu\right)^{1/p}$$

for every $u \in \mathrm{Lip}_c(X)$, and every ball $B = B(x,r)$. Keith has showed that this implies that the space X supports a $(1,p)$-Poincaré inequality [23]. Thus we know that locally Lipschitz functions are dense in $N^{1,p}(\Omega)$ for any open set $\Omega \subset X$ (see [7] or [6, p. 140]). Furthermore, we know that it is enough to check the (q,p)-Poincaré inequality for bounded measurable functions u on X and their upper gradients g [6, p. 88]. We pick any such pair, and any ball $B = B(x,r)$. Clearly we have $u \in L^p(\lambda B)$, and we can also assume that $g \in L^p(\lambda B)$, since otherwise the Poincaré inequality trivially holds. We conclude that $u \in N^{1,p}(\lambda B)$, so we can pick a sequence $(u_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\lambda B)$ such that

$u_i \to u$ in $N^{1,p}(\lambda B)$. We pick an arbitrary $a \in (0,1)$. Since the space is proper, we have $u_i \in \mathrm{Lip}(a\lambda B)$ for every $i \in \mathbb{N}$. Thus we can construct extensions $\tilde{u}_i \in \mathrm{Lip}_c(X)$ of these functions from $a\lambda B$ to the whole space. Inequality (3.2) now applies to the functions $\tilde{u}_i$. Since the space supports a $(1,p)$-Poincaré inequality, we know that $\mathrm{Lip}\,\tilde{u}_i(x) \leqslant C g_{\tilde{u}_i}(x)$ for μ-a.e. $x \in X$ and every $i \in \mathbb{N}$ [11, Proposition 4.26]. Here $g_{\tilde{u}_i}$ is the minimal p-weak upper gradient of $\tilde{u}_i$, and $C = C(c_d, c_L, \lambda)$. In the case $p > 1$, we have in fact $\mathrm{Lip}\,\tilde{u}_i(x) = g_{\tilde{u}_i}(x)$ [11, Proposition 6.1]. Since for every $i \in \mathbb{N}$, $\tilde{u}_i = u_i$ in $a\lambda B$, we have $\tilde{u}_i \to u$ in $N^{1,p}(a\lambda B)$. Thus we have

$$\limsup_{i\to\infty}\left(\fint_{a\lambda B}(\mathrm{Lip}\,\tilde{u}_i)^p\,d\mu\right)^{1/p} \leqslant C\limsup_{i\to\infty}\left(\fint_{a\lambda B} g_{\tilde{u}_i}^p\,d\mu\right)^{1/p} = C\left(\fint_{a\lambda B} g_u^p\,d\mu\right)^{1/p}.$$

Then the sequence $\{\tilde{u}_i - (\tilde{u}_i)_{aB}\}_{i=1}^{\infty}$ is, by inequality (3.2), bounded in $L^q(aB)$. By the uniform convexity of $L^q(aB)$ for $q > 1$ and Mazur's lemma — and by replacing the $\tilde{u}_i$'s with convex combinations, if necessary — we get

$$\lim_{i\to\infty}\left(\fint_{aB}|\tilde{u}_i - (\tilde{u}_i)_{aB}|^q\,d\mu\right)^{1/q} = \left(\fint_{aB}|u - u_{aB}|^q\,d\mu\right)^{1/q}.$$

Now we can combine the last two equations with the fact that inequality (3.2) applies to the functions $\tilde{u}_i$. By further noting that $g_u(x) \leqslant g(x)$ for μ-a.e. $x \in \lambda B$, and by letting $a \to 1$, we get the result by Lebesgue's dominated convergence theorem. □

Remark 3.2. We utilize the deep results by Cheeger [11] concerning the minimal p-weak upper gradients of Lipschitz functions only in the proof of Theorem 3.1. In the case $q = 1$, the theorem is true according to Keith [23], and no reference to [11] is necessary.

3.1. Equivalence of $(q,1)$-Poincaré and the relative isoperimetric inequality

In this section, we prove the following theorem.

Theorem 3.3. *For any $q \geqslant 1$, the $(q,1)$-Poincaré inequality and the relative isoperimetric inequality (1.2) are quantitatively equivalent.*

Proof. Let us assume that the space supports a $(q,1)$-Poincaré inequality, and take any μ-measurable set $E \subset X$, and any ball $B = B(x,r)$. By the $(q,1)$-Poincaré inequality, for any Lipschitz function $u \in \mathrm{Lip}(X)$ we have

$$\left(\fint_B |u - u_B|^q\,d\mu\right)^{1/q} \leqslant c_P r \fint_{\lambda B}\mathrm{lip}\,u\,d\mu, \tag{3.3}$$

since the local Lipschitz constant $\mathrm{lip}\,u$ is an upper gradient. We can assume that $\chi_E \in BV(\lambda B)$, since otherwise the right-hand side of the relative isoperimetric inequality (1.2) is infinite. By the definition of BV, we can pick a sequence of functions $(u_i) \subset \mathrm{Lip}_{\mathrm{loc}}(\lambda B)$ such that $u_i \to \chi_E$ in $L^1_{\mathrm{loc}}(\lambda B)$ and

$$\lim_{i\to\infty}\int_{\lambda B}\mathrm{lip}\,u_i\,d\mu = \|D\chi_E\|(\lambda B).$$

We fix any $a \in (0,1)$. Since we can assume that $0 \leqslant u_i \leqslant 2$ for every $i \in \mathbb{N}$, simply by truncating the functions if necessary, we also have (here we could even take $a = 1$)

$$\lim_{i\to\infty}\left(\fint_{aB}|u_i - (u_i)_{aB}|^q\,d\mu\right)^{1/q} = \left(\fint_{aB}|\chi_E - (\chi_E)_{aB}|^q\,d\mu\right)^{1/q}.$$

By the fact that $a < 1$, we have $u_i \in \mathrm{Lip}(a\lambda B)$ for every $i \in \mathbb{N}$, so we can construct extensions $\tilde{u}_i \in \mathrm{Lip}(X)$. If we now apply (3.3) to the functions $\tilde{u}_i$, which agree with the u_i's in $a\lambda B$, we get

$$\lim_{i\to\infty}\left(\fint_{aB}\left|u_i-(u_i)_{aB}\right|^q\,d\mu\right)^{1/q}\leqslant\limsup_{i\to\infty}c_P r\fint_{a\lambda B}\operatorname{lip}u_i\,d\mu\leqslant c_P r\frac{\|D\chi_E\|(\lambda B)}{\mu(a\lambda B)}.$$

By using the previous limit for the left-hand side, then letting $a\to 1$, and finally remembering (3.1), we get the relative isoperimetric inequality (1.2).

Let us then look at the converse. We assume that the relative isoperimetric inequality holds for some $q\geqslant 1$ and dilation factor $\lambda\geqslant 1$, and we want to establish the $(q,1)$-Poincaré inequality. We first note that according to Theorem 3.1, it is enough to show that

$$\left(\fint_B|u-u_B|^q\,d\mu\right)^{1/q}\leqslant Cr\fint_{\lambda B}\operatorname{lip}u\,d\mu \tag{3.4}$$

for every Lipschitz function with compact support $u\in\operatorname{Lip}_c(X)$, every ball $B=B(x,r)$, and some constant $C>0$. We take any function $u\in\operatorname{Lip}_c(X)$, and any ball $B=B(x,r)$. Since neither side of (3.4) changes if we add a constant to u, we can assume that $u\geqslant 0$. Now, since $u\in\operatorname{Lip}(\lambda B)\subset BV(\lambda B)$, we have by the definition of the variation measure

$$\|Du\|(\lambda B)\leqslant\int_{\lambda B}\operatorname{lip}u\,d\mu.$$

Define the level sets of the function u:

$$E_t:=\{x\in X:\ u(x)>t\}.$$

Now we use the last inequality, the coarea formula (2.5), the relative isoperimetric inequality, and (3.1) to obtain

$$\begin{aligned}r\fint_{\lambda B}\operatorname{lip}u\,d\mu&\geqslant r\frac{\|Du\|(\lambda B)}{\mu(\lambda B)}\\&=r\frac{\int_0^\infty\|D\chi_{E_t}\|(\lambda B)\,dt}{\mu(\lambda B)}\\&\geqslant 2^{-1/q}c_I^{-1}\int_0^\infty\left(\fint_B\left|\chi_{E_t}-(\chi_{E_t})_B\right|^q\,d\mu\right)^{1/q}dt.\end{aligned} \tag{3.5}$$

Now we essentially want to recover the function u by integrating the characteristic functions of the level sets χ_{E_t} with respect to t. By duality, for any $f\in L^q(B)$ we have

$$\left(\int_B|f|^q\,d\mu\right)^{1/q}=\sup\int_B fg\,d\mu, \tag{3.6}$$

where $1/q+1/q'=1$ and the supremum is taken over all $g\in L^{q'}(B)$ such that $\|g\|_{L^{q'}(B)}\leqslant 1$. If $q=1$, $q'=\infty$. We pick any $g\in L^{q'}(B)$, $\|g\|_{L^{q'}(B)}\leqslant 1$, and estimate the integral on the last line of (3.5):

$$\begin{aligned}&\int_0^\infty\left(\fint_B\left|\chi_{E_t}-(\chi_{E_t})_B\right|^q\,d\mu\right)^{1/q}dt\\&\quad\geqslant\frac{1}{\mu(B)^{1/q}}\int_0^\infty\left(\int_B\big(\chi_{E_t}(x)-(\chi_{E_t})_B\big)g(x)\,d\mu(x)\right)dt\\&\quad=\frac{1}{\mu(B)^{1/q}}\int_B\left(\int_0^\infty\big(\chi_{E_t}(x)-(\chi_{E_t})_B\big)g(x)\,dt\right)d\mu(x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mu(B)^{1/q}} \int_B \left(\int_0^\infty \chi_{E_t}(x) g(x)\,dt - \int_0^\infty (\chi_{E_t})_B\, g(x)\,dt \right) d\mu(x) \\
&= \frac{1}{\mu(B)^{1/q}} \left(\int_B ug\,d\mu - \int_B \int_0^\infty \fint_B \chi_{E_t}(z)\,d\mu(z)\,g(x)\,dt\,d\mu(x) \right) \\
&= \frac{1}{\mu(B)^{1/q}} \left(\int_B ug\,d\mu - \int_B \fint_B \int_0^\infty \chi_{E_t}(z)\,dt\,d\mu(z)\,g(x)\,d\mu(x) \right) \\
&= \frac{1}{\mu(B)^{1/q}} \left(\int_B ug\,d\mu - \int_B \fint_B u(z)\,d\mu(z)\,g(x)\,d\mu(x) \right) \\
&= \frac{1}{\mu(B)^{1/q}} \left(\int_B (u-u_B) g\,d\mu \right).
\end{aligned}
$$

Since $g \in L^{q'}(B)$, $\|g\|_{L^{q'}(B)} \leqslant 1$ was arbitrary, we obtain the desired $(q,1)$-Poincaré inequality by (3.6). □

3.2. *Inequality (1.3) implies $(q,1)$-Poincaré*

It is shown in [8] and [26] with slightly different formulations that inequality (1.3) implies the $(q,1)$-Poincaré inequality, but let us provide a complete proof here. The proof will contain essentially the same ingredients as the previous one.

Theorem 3.4. *For a given $q \geqslant 1$, assume that the strong relative isoperimetric inequality (1.3) holds, i.e.*

$$
\left(\frac{\min\{\mu(B\cap E), \mu(B\setminus E)\}}{\mu(B)} \right)^{1/q} \leqslant c_S r \frac{\mu^+(\lambda B \cap \partial E)}{\mu(\lambda B)}
$$

for every μ-measurable set $E \subset X$, every ball $B = B(x,r)$, and some constants $c_S > 0$ and $\lambda \geqslant 1$. Then the $(q,1)$-Poincaré inequality holds as well, with the constants depending on each other.

Again, according to Theorem 3.1, it is enough to show that

$$
\left(\fint_B |u-u_B|^q\,d\mu \right)^{1/q} \leqslant Cr \fint_{\lambda B} \operatorname{Lip} u\,d\mu \tag{3.7}
$$

for every Lipschitz function with compact support $u \in \operatorname{Lip}_c(X)$, every ball $B = B(x,r)$, and some constant $C > 0$. Additionally we need the following coarea inequality.

Proposition 3.5. *Assume that ν is a finite Borel outer measure. Then for every bounded Lipschitz function $u \in \operatorname{Lip}(X)$ we have the inequality*

$$
\int_X \operatorname{Lip} u\,d\nu \geqslant \int_{-\infty}^{\infty} \nu^+(\partial E_t)\,dt, \tag{3.8}
$$

where the level sets of u are again denoted E_t.

Proof. Clearly neither side of (3.8) changes if we add a constant to u. Thus we can assume that $u \geqslant 0$. For $h > 0$, define the Borel measurable functions

$$
u_h(x) := \sup_{d(y,x)<h} u(y), \qquad u_{-h}(x) := \inf_{d(y,x)<h} u(y).
$$

Also, define the level sets of these functions

$$E_t^h := \{x \in X\colon u_h(x) > t\}, \qquad E_t^{-h} := \{x \in X\colon u_{-h}(x) > t\}, \quad \text{for } t \in \mathbb{R}.$$

Now we can calculate using Cavalieri's principle

$$\begin{aligned}\int_X \frac{u_h - u_{-h}}{2h}\,d\nu &= \frac{1}{2h}\int_0^\infty \big(\nu(\{u_h > t\}) - \nu(\{u_{-h} > t\})\big)\,dt \\ &= \int_0^\infty \frac{\nu(E_t^h \setminus E_t^{-h})}{2h}\,dt. \end{aligned}\tag{3.9}$$

For any $x \in X$, we can estimate

$$\begin{aligned}\limsup_{h\to 0} \frac{u_h(x) - u_{-h}(x)}{2h} &\leqslant \frac{1}{2}\left[\limsup_{h\to 0}\frac{u_h(x) - u(x)}{h} + \limsup_{h\to 0}\frac{u(x) - u_{-h}(x)}{h}\right] \\ &\leqslant \frac{1}{2}\left[2\limsup_{y\to x}\frac{|u(y) - u(x)|}{d(y,x)}\right] \\ &= \operatorname{Lip} u(x). \end{aligned}\tag{3.10}$$

Next, we observe that for any $t \in \mathbb{R}$

$$E_t^h \setminus E_t^{-h} = \{y \in X\colon \forall \varepsilon > 0\ \exists z_1, z_2 \in B(y,h) \text{ s.t. } u(z_1) > t,\ u(z_2) < t + \varepsilon\}.$$

On the other hand,

$$\bigcup_{x\in\partial E_t} B(x,h) \subset \{y \in X\colon \exists z_1, z_2 \in B(y,h) \text{ s.t. } u(z_1) > t,\ u(z_2) \leqslant t\}.$$

Looking at these definitions, we see that $\bigcup_{x\in\partial E_t} B(x,h) \subset E_t^h \setminus E_t^{-h}$. Thus we get

$$\liminf_{h\to 0}\frac{\nu(E_t^h \setminus E_t^{-h})}{2h} \geqslant \liminf_{h\to 0}\frac{\nu(\bigcup_{x\in\partial E_t} B(x,h))}{2h} = \nu^+(\partial E_t).\tag{3.11}$$

Now we can use estimate (3.10), Fatou's lemma, (3.9), and (3.11) to calculate

$$\begin{aligned}\int_X \operatorname{Lip} u\,d\nu &\geqslant \int_X \limsup_{h\to 0}\frac{u_h - u_{-h}}{2h}\,d\nu \\ &\geqslant \limsup_{h\to 0}\int_X \frac{u_h - u_{-h}}{2h}\,d\nu = \limsup_{h\to 0}\int_0^\infty \frac{\nu(E_t^h \setminus E_t^{-h})}{2h}\,dt \\ &\geqslant \liminf_{h\to 0}\int_0^\infty \frac{\nu(E_t^h \setminus E_t^{-h})}{2h}\,dt \geqslant \int_0^\infty \liminf_{h\to 0}\frac{\nu(E_t^h \setminus E_t^{-h})}{2h}\,dt \\ &\geqslant \int_0^\infty \nu^+(\partial E_t)\,dt. \end{aligned}$$

Finally, we note that by the assumption $u \geqslant 0$, the lower limit in the last integral can as well be $-\infty$. This completes the proof. □

Now we can prove the main theorem of this section.

Proof of Theorem 3.4. We recall that it is enough to show that the $(q,1)$-Poincaré inequality holds for functions $u \in \operatorname{Lip}_c(X)$, and since we can again add any constant to u, we can assume that u is a nonnegative bounded Lipschitz function. We fix any $a \in (0,1)$, take any ball $B = B(x,r)$, and define $\nu := \mu|_{\lambda B}$. Note that ν is a finite Borel outer measure. Now we can use the coarea inequality, Proposition 3.5, to calculate

$$\begin{aligned}
\frac{r}{\mu(a\lambda B)} \int_{\lambda B} \operatorname{Lip} u \, d\mu &= \frac{r}{\mu(a\lambda B)} \int_X \operatorname{Lip} u \, d\nu \\
&\geqslant \frac{r}{\mu(a\lambda B)} \int_{-\infty}^{\infty} \nu^+(\partial E_t)\, dt \\
&\geqslant \frac{r}{\mu(a\lambda B)} \int_{-\infty}^{\infty} \mu^+(\partial E_t \cap a\lambda B)\, dt \\
&\geqslant c_S^{-1} \int_{-\infty}^{\infty} \left(\frac{\min\{\mu(aB \cap E_t), \mu(aB \setminus E_t)\}}{\mu(aB)} \right)^{1/q} dt \\
&\geqslant 2^{-1/q} c_S^{-1} \int_{-\infty}^{\infty} \left(\fint_{aB} |\chi_{E_t} - (\chi_{E_t})_{aB}|^q \, d\mu \right)^{1/q} dt \\
&\geqslant 2^{-1/q} c_S^{-1} \left(\fint_{aB} |u - u_{aB}|^q \, d\mu \right)^{1/q}.
\end{aligned}$$

The last inequality follows just as in Section 3.1. By letting $a \to 1$, we obtain the claim. □

3.3. Equivalence of inequalities (1.3) and (1.4)

Inequality (1.4) clearly implies (1.3), since the measure theoretic boundary is a subset of the topological boundary. Thus the following theorem implies that these two conditions are equivalent.

Theorem 3.6. *For any $q \geqslant 1$, the strong relative isoperimetric inequality* (1.3) *implies the strong relative isoperimetric inequality* (1.4)*, with the same constants.*

To show this, we will show that the measure theoretic boundary of a μ-measurable set E is dense in the topological boundary with a suitable choice of the L^1-representative of χ_E. We choose $E = I$ (measure theoretic interior); then for every $x \in \partial E$ and for arbitrarily small radii $r > 0$, we have

$$0 < \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))} < 1. \tag{3.12}$$

This is clear, for if the quotient were 1 for some $r > 0$, we would have $B(x,r) \subset I$, and if the quotient were 0, we would have $B(x,r) \subset O$ (measure theoretic exterior). In either case, x would not be on the boundary of E. Now we need to show that the ball $B(x,r)$, or a slightly enlarged ball $B(x,Cr)$, necessarily contains a point belonging to the measure theoretic boundary. To this end, we first show two simple lemmas. Recall that a space is *geodesic* if every two points $x, y \in X$ can be joined by a curve whose length is $d(x,y)$.

Lemma 3.7. *If X is a geodesic space, every sphere has zero measure, i.e. for every $x \in X$ and $r > 0$ we have*

$$\mu\big(\{y \in X\colon d(y,x) = r\}\big) = 0.$$

Proof. We know from [9] that μ (which we constantly assume to be doubling) has an *annular decay property*: there exist constants $K \geqslant 1$ and $0 < \delta \leqslant 1$, depending only on the doubling constant of μ, such that

$$\mu\big(B(x,r) \setminus B\big(x, r(1-\varepsilon)\big)\big) \leqslant K\varepsilon^\delta \mu\big(B(x,r)\big)$$

for every $x \in X$, $r > 0$ and $0 < \varepsilon < 1$. With a change of variables we get

$$\mu\big(B\big(x, r/(1-\varepsilon)\big) \setminus B(x,r)\big) \leqslant K\varepsilon^{\delta}\mu\big(B\big(x, r/(1-\varepsilon)\big)\big) \leqslant K\varepsilon^{\delta}\mu\big(B(x, 2r)\big)$$

for every $x \in X$, $r > 0$ and $0 < \varepsilon < 1/2$. By noting that

$$\big\{y \in X\colon\ d(y,x) = r\big\} \subset B\big(x, r/(1-\varepsilon)\big) \setminus B(x,r)$$

for arbitrarily small $\varepsilon > 0$, we get the desired result. □

With the help of the previous lemma, we can prove a second lemma.

Lemma 3.8. *Let X be a geodesic space, and let $E \subset X$ be any μ-measurable set. Then the functions*

$$X \ni x \mapsto \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}$$

for any constant $r > 0$, and

$$\mathbb{R}_+ \ni r \mapsto \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r))}$$

for any constant $x \in X$, are continuous.

Proof. The proofs are almost identical, so we only give the proof for the first function. Take any point $x \in X$, and any sequence $x_i \to x$. We see that $\chi_{B(x_i,r)\cap E} \to \chi_{B(x,r)\cap E}$ pointwise outside the sphere $\{y \in X\colon\ d(y,x) = r\}$. However, by the previous lemma, the sphere has zero measure, so $\chi_{B(x_i,r)\cap E} \to \chi_{B(x,r)\cap E}$ pointwise for μ-a.e. $y \in X$. Now we get by Lebesgue's dominated convergence theorem that

$$\big|\mu\big(B(x_i,r) \cap E\big) - \mu\big(B(x,r) \cap E\big)\big| \leqslant \int_X |\chi_{B(x_i,r)\cap E} - \chi_{B(x,r)\cap E}|\, d\mu \to 0,$$

as $i \to \infty$. Similarly we obtain the continuity of the denominator, giving us the result. □

Now we proceed to prove the desired existence result for the measure theoretic boundary. We first prove it in the case of a geodesic space.

Proposition 3.9. *Let X be a geodesic space, and assume that for a μ-measurable set $E \subset X$, a point $x \in X$, and $R > 0$, we have*

$$0 < \frac{\mu(B(x,R) \cap E)}{\mu(B(x,R))} < 1. \tag{3.13}$$

Then there is a point $y \in B(x, CR) \cap \partial^* E$, where $C \geqslant 1$ is a universal constant.

Proof. We know that

$$\frac{\mu(B(x,R) \cap E)}{\mu(B(x,R))} = a \in (0,1).$$

Our strategy is to find a smaller ball that contains the "same proportion" of the set E. Let us consider the balls $B(y, R/2)$, $y \in B(x,R)$. If one of these balls satisfies

$$\frac{\mu(B(y,R/2) \cap E)}{\mu(B(y,R/2))} = a,$$

we pick this ball. Suppose next that we do not find such a ball, but that we do find balls $B(y_1, R/2)$, $B(y_2, R/2)$, with $y_1, y_2 \in B(x,R)$, satisfying

$$\frac{\mu(B(y_1,R/2) \cap E)}{\mu(B(y_1,R/2))} > a \quad \text{and} \quad \frac{\mu(B(y_2,R/2) \cap E)}{\mu(B(y_2,R/2))} < a.$$

By the fact that the space is geodesic, we can join the points y_1 and y_2 by a curve γ satisfying $l(\gamma) = d(y_1, y_2) < 2R$. This tells us that $|\gamma| \subset B(x, 2R)$. By Lemma 3.8 we know that the function

$$y \mapsto \frac{\mu(B(y, R/2) \cap E)}{\mu(B(y, R/2))}$$

is continuous. This means that some point $y \in |\gamma|$ must satisfy

$$\frac{\mu(B(y, R/2) \cap E)}{\mu(B(y, R/2))} = a,$$

in which case we pick the ball $B(y, R/2)$. Finally, let us assume that for every point $y \in B(x, R)$ we have

$$\frac{\mu(B(y, R/2) \cap E)}{\mu(B(y, R/2))} > a$$

(the case "$<$" is similar). In this case, we simply pick a point $y \in O \cap B(x, R)$ — which exists by the assumption (3.13) — and note that by definition

$$\lim_{r \to 0} \frac{\mu(B(y, r) \cap E)}{\mu(B(y, r))} = 0.$$

By the continuity of the function

$$r \mapsto \frac{\mu(B(y, r) \cap E)}{\mu(B(y, r))},$$

given again by Lemma 3.8, we can conclude that

$$\frac{\mu(B(y, r) \cap E)}{\mu(B(y, r))} = a$$

for some $r \in (0, R/2)$. So in each case, we find a point $y_1 \in B(x, 2R)$ and a radius $0 < r_1 \leqslant R/2$ satisfying

$$\frac{\mu(B(y_1, r_1) \cap E)}{\mu(B(y_1, r_1))} = a.$$

Now we can repeat this process: we next find a point $y_2 \in B(y_1, 2r_1)$ and a radius $0 < r_2 \leqslant r_1/2$ satisfying

$$\frac{\mu(B(y_2, r_2) \cap E)}{\mu(B(y_2, r_2))} = a.$$

Continuing like this, we get a sequence of ball centers y_i and a sequence of radii $r_i \to 0$. The sequence (y_i) is, by its construction, clearly Cauchy, and by the completeness of the space we have $y_i \to y \in X$. Moreover,

$$d(x, y) \leqslant d(x, y_1) + \sum_{i=1}^{\infty} d(y_i, y_{i+1}) \leqslant \sum_{i=0}^{\infty} 2^{-i}(2R) = 4R.$$

Now we wish to show that $y \in \partial^* E$. For any $i \in \mathbb{N}$, we have

$$d(y_i, y) \leqslant \sum_{j=i}^{\infty} d(y_j, y_{j+1}) \leqslant \sum_{j=0}^{\infty} (2r_i)2^{-j} = 4r_i.$$

So, we have $B(y, 5r_i) \supset B(y_i, r_i)$, from which we get by the doubling property of μ

$$\frac{\mu(B(y, 5r_i) \cap E)}{\mu(B(y, 5r_i))} \geqslant \frac{\mu(B(y_i, r_i) \cap E)}{\mu(B(y, 5r_i))} \geqslant \frac{\mu(B(y_i, r_i) \cap E)}{c_d^4 \mu(B(y_i, r_i))} = \frac{a}{c_d^4} > 0$$

for every $i \in \mathbb{N}$. By a similar calculation, we get

$$\frac{\mu(B(y, 5r_i) \setminus E)}{\mu(B(y, 5r_i))} \geqslant \frac{1-a}{c_d^4} > 0$$

for every $i \in \mathbb{N}$. Since we had $r_i \to 0$, this shows that

$$\limsup_{r\to 0} \frac{\mu(B(y,r)\cap E)}{\mu(B(y,r))} \geqslant Ca > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(y,r)\setminus E)}{\mu(B(y,r))} \geqslant C(1-a) > 0,$$

so by definition $y \in \partial^* E$. $\square$

Now we want to remove the assumption that X is geodesic. Recall that an identity mapping $\Gamma : (X, d, \mu) \to (X, d', \mu)$, where d' is another metric in the space X, is *bi-Lipschitz* if for some constant $D \geqslant 1$ and for all points $x, y \in X$, we have

$$\frac{1}{D} d(x,y) \leqslant d'(x,y) \leqslant Dd(x,y). \tag{3.14}$$

Now we prove one more lemma.

Lemma 3.10. *For any set $E \subset X$, the sets I, O, and $\partial^* E$ (measure theoretic interior, exterior and boundary) are invariant under a bi-Lipschitz identity mapping $\Gamma : (X, d, \mu) \to (X, d', \mu)$.*

Proof. We only prove the claim for $\partial^* E$; the other cases are similar. Take any $x \in \partial^*_d E$ — the subscript means that the measure theoretic boundary is determined with respect to the metric d. By definition,

$$\limsup_{r\to 0} \frac{\mu(B_d(x,r)\cap E)}{\mu(B_d(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B_d(x,r)\setminus E)}{\mu(B_d(x,r))} > 0,$$

where again the subscript means that the balls are taken with respect to the metric d. Now we can compute

$$\begin{aligned}
\limsup_{r\to 0} \frac{\mu(B_{d'}(x,r)\cap E)}{\mu(B_{d'}(x,r))} &\geqslant \limsup_{r\to 0} \frac{\mu(B_d(x,r/D)\cap E)}{\mu(B_d(x,Dr))} \\
&\geqslant \limsup_{r\to 0} C \frac{\mu(B_d(x,r/D)\cap E)}{\mu(B_d(x,r/D))} \\
&= C \limsup_{r\to 0} \frac{\mu(B_d(x,r)\cap E)}{\mu(B_d(x,r))} > 0.
\end{aligned}$$

We see that the constant $C > 0$ only depends on D and the doubling constant of the measure μ. For the set $B_{d'}(x,r)\setminus E$ we can perform a similar calculation, so we get $x \in \partial^*_{d'} E$. By exactly the same reasoning, $x \in \partial^*_{d'} E$ implies $x \in \partial^*_d E$, so in conclusion $\partial^*_d E = \partial^*_{d'} E$. $\square$

Now we can easily prove the desired existence result for the measure theoretic boundary.

Theorem 3.11. *Let X support a Poincaré inequality, and assume that for a μ-measurable set $E \subset X$, a point $x \in X$, and $R > 0$, we have*

$$0 < \frac{\mu(B(x,R)\cap E)}{\mu(B(x,R))} < 1.$$

Then there is a point $y \in B(x, CR) \cap \partial^* E$, where the constant $C \geqslant 1$ only depends on the doubling constant of the measure and the constant c_P in the Poincaré inequality.

Proof. We know that

$$\frac{\mu(B_d(x,R)\cap E)}{\mu(B_d(x,R))} = a \in (0,1), \tag{3.15}$$

where again the subscripts emphasize the fact that the balls are taken with respect to the metric d. Now, let us define the *length metric*

$$\rho(x,y) := \inf l(\gamma_{xy}),$$

where the infimum is taken over curves connecting x and y. Since X supports a Poincaré inequality, it is quasiconvex (see [17] or [6, p. 100]). This means that for some constant $D \geqslant 1$, which only depends on the doubling constant of the measure and the constant c_P in the Poincaré inequality, and for all points $x, y \in X$, we have

$$d(x,y) \leqslant \rho(x,y) \leqslant Dd(x,y).$$

This means that the identity mapping $\Gamma : (X, d, \mu) \to (X, \rho, \mu)$ is bi-Lipschitz. The reason for wanting to work with the space (X, ρ, μ) is that it is geodesic (see e.g. [5, p. 62]). By (3.15) and the doubling property of the measure, we get

$$\frac{\mu(B_\rho(x, DR) \cap E)}{\mu(B_\rho(x, DR))} = \tilde{a} \in (0,1).$$

Now, by Proposition 3.9, we find a point $y \in B_\rho(x, CR) \cap \partial_\rho^* E$. By Lemma 3.10, we have in fact $y \in B_d(x, CR) \cap \partial_d^* E$, where the constant $C \geqslant 1$ only depends on the doubling constant of the measure and the constant c_P in the Poincaré inequality. □

Now we can show that the strong relative isoperimetric inequality (1.3) implies the strong relative isoperimetric inequality (1.4).

Proof of Theorem 3.6. Take any μ-measurable set E. Since inequality (1.4) does not depend on the choice of the L^1-representative of χ_E, it is enough to prove the result for some representative. We pick $E = I$ (measure theoretic interior), as discussed in the beginning of this section. Since (1.3) implies some Poincaré inequality according to Theorem 3.4, we can use Eq. (3.12) and Theorem 3.11 to conclude that the measure theoretic boundary $\partial^* E$ is dense in the topological boundary ∂E. By the definition of the codimension one Minkowski content, we get

$$\mu^+\bigl(\lambda B \cap \partial^* E\bigr) = \mu^+(\lambda B \cap \partial E)$$

for every ball B, and every $\lambda > 0$. This gives the result. □

3.4. Inequality (1.5) implies inequality (1.3)

It follows immediately from the following proposition that (1.5) implies (1.3).

Proposition 3.12. *For any set $A \subset X$, we have $\mathcal{H}(A) \leqslant C\mu^+(A)$, where the constant C only depends on the doubling constant of the measure μ.*

Proof. Choose any $h > 0$ and cover A with the balls $\{B(x,h)\}_{x\in A}$. The 5-covering lemma gives us a countable collection of pairwise disjoint balls $\{B(x_i,h)\}_{i=1}^\infty$ such that $A \subset \bigcup_{i=1}^\infty B(x_i, 5h)$. Now we can calculate

$$\begin{aligned}\mathcal{H}_{5h}(A) &\leqslant \sum_{i=1}^\infty \frac{\mu(B(x_i,5h))}{5h} \\ &\leqslant c_d^3 \sum_{i=1}^\infty \frac{\mu(B(x_i,h))}{2h} \\ &\leqslant c_d^3 \frac{\mu(\bigcup_{x\in A} B(x,h))}{2h}.\end{aligned}$$

By taking $\liminf$ as $h \to 0$ on both sides, we get $\mathcal{H}(A) \leqslant C\mu^+(A)$, as desired. □

Given the simplicity of this proof and the fact that an opposite inequality is not true at all, the equivalence between (1.3) and (1.5) seems quite surprising. To obtain this equivalence, we need one more implication, which is established in the following.

3.5. The $(q,1)$-Poincaré inequality implies inequality (1.5)

In this section, we prove the following theorem.

Theorem 3.13. *Assume that the space X supports a $(q,1)$-Poincaré inequality for a given $q \geqslant 1$. Then we have*

$$\left(\frac{\min\{\mu(B\cap E),\mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r \frac{\mathcal{H}(\tilde{\lambda}B\cap\partial E)}{\mu(\tilde{\lambda}B)} \tag{3.16}$$

for every μ-measurable set $E\subset X$, every ball $B=B(x,r)$, and some constants $c_S>0$ and $\tilde{\lambda}\geqslant 1$, with $c_S = c_S(c_P,c_d,q)$ and $\tilde{\lambda}=\tilde{\lambda}(\lambda,c_P,c_d)$.

Proof. By changing the metric as in Theorem 3.11, we may assume that the space is geodesic. For the effect this has on constants, see e.g. [6, p. 89]. Now, we fix an arbitrary μ-measurable set $E\subset X$, and a ball $B=B(x,r)$. If $\mathcal{H}(\lambda\bar{B}\cap\partial E)=\infty$, inequality (3.16) holds with the closed ball $\lambda\bar{B}$ on the right-hand side. Let us therefore assume that $\mathcal{H}(\lambda\bar{B}\cap\partial E)<\infty$. The idea is to approximate the characteristic function χ_E in the ball B by suitable functions whose upper gradients' L^1-norms converge to $\mathcal{H}(\lambda\bar{B}\cap\partial E)$. For any $\delta>0$, we can cover $\lambda\bar{B}\cap\partial E$ with balls $\{B_i\}_{i=1}^\infty$ such that $r_i<\delta$ for every $i\in\mathbb{N}$, and

$$\sum_{i=1}^\infty \frac{\mu(B_i)}{r_i} < \mathcal{H}(\lambda\bar{B}\cap\partial E)+1.$$

Define the Borel measurable function

$$g := \sum_{i=1}^\infty \frac{\chi_{2B_i}}{r_i} + \infty\chi_{X\setminus\lambda\bar{B}}. \tag{3.17}$$

Then define

$$u(x) := \min\left\{1, \inf_\gamma \int_\gamma g\,ds\right\}, \tag{3.18}$$

where the infimum is taken over curves connecting x to $(B\setminus E)\setminus\bigcup_{i=1}^\infty 2B_i$. With this definition, the function g is an upper gradient of u (for a proof, see [7] or [6, p. 128]). We also know that u is μ-measurable in λB (see [22, Theorem 1.11] and [19, Corollary 3.2]). The function u is also measurable in $X\setminus\lambda\bar{B}$, where it takes the value one. By the assumption that the space is geodesic and by Lemma 3.7, we know that spheres have zero measure, ensuring the μ-measurability of u in X. Now we wish to show that u approximates χ_E. Clearly $u=0$ in $(B\setminus E)\setminus\bigcup_{i=1}^\infty 2B_i$ (take constant curves). Next, take a point $x\in E\cap B$ and any curve γ connecting x to $(B\setminus E)\setminus\bigcup_{i=1}^\infty 2B_i$. Clearly γ must pass through a point $y\in\partial E$. If y belongs to the open set $X\setminus\lambda\bar{B}$, we have

$$\int_\gamma g\,ds = \infty.$$

If $y\in\lambda\bar{B}$, and thus $y\in\lambda\bar{B}\cap\partial E$, then $y\in B_i$ for some $i\in\mathbb{N}$. This means that γ travels at least the distance r_i in the dilated ball $2B_i$ before reaching the point y. Thus we get

$$\int_\gamma g\,ds \geqslant \int_\gamma \frac{\chi_{2B_i}}{r_i}\,ds \geqslant \frac{r_i}{r_i} = 1.$$

Altogether we conclude that $u=1$ in $E\cap B$. Now we want to show that $u\to\chi_E$ in $L^q(B)$, when $\delta\to 0$. We note that both u and χ_E take values between 0 and 1, and that in B they differ only in the balls $2B_i$, $i\in\mathbb{N}$. Now a straightforward calculation gives

$$\begin{aligned}\int_B |u-\chi_E|^q\,d\mu &\leqslant \int_{\bigcup_{i=1}^\infty 2B_i} |u-\chi_E|^q\,d\mu\\ &\leqslant \sum_{i=1}^\infty \mu(2B_i)\\ &\leqslant c_d\sum_{i=1}^\infty \mu(B_i)\\ &\leqslant c_d\delta\sum_{i=1}^\infty \frac{\mu(B_i)}{r_i}\\ &\leqslant c_d\delta\big(\mathcal{H}(\lambda\bar B\cap\partial E)+1\big)\xrightarrow{\delta\to 0} 0.\end{aligned}$$

Thus we get

$$\left(\fint_B |u-u_B|^q\,d\mu\right)^{1/q}\xrightarrow{\delta\to 0}\left(\fint_B \big|\chi_E-(\chi_E)_B\big|^q\,d\mu\right)^{1/q}.$$

Furthermore, with a suitable choice of the coverings $\{B_i\}_{i=1}^\infty$, we have

$$\begin{aligned}\int_{\lambda B} g\,d\mu &\leqslant c_d\sum_{i=1}^\infty \frac{\mu(B_i)}{r_i}\\ &\xrightarrow{\delta\to 0} c_d\mathcal{H}(\lambda\bar B\cap\partial E).\end{aligned}$$

Applying the $(q,1)$-Poincaré inequality to u, g, and remembering also (3.1), we obtain the result. $\square$

Let us look at two simple implications of the strong relative isoperimetric inequality proved above.

Corollary 3.14. *Let X support a $(1,1)$-Poincaré inequality. Then for any μ-measurable set $E\subset X$ that satisfies* $\operatorname{diam}(E)\leqslant a\operatorname{diam}(X)$ *and* $\mu(X\setminus E)>0$*, we have*

$$\mu(E)\leqslant C\operatorname{diam}(E)\mathcal{H}(\partial E),\tag{3.19}$$

where the constants $0<a<1$ and $C>0$ satisfy $a=a(c_d)$ and $C=C(c_P,c_d)$.

Proof. Let us first assume that E is bounded. Since the space supports a $(1,1)$-Poincaré inequality, it is connected, and thus there exist constants $b>0$ and $\alpha>0$, depending only on the doubling constant of the measure, such that for all $x\in X$ and all $0<r\leqslant R\leqslant\operatorname{diam}(X)/2$, we have [6, p. 67]

$$\frac{\mu(B(x,r))}{\mu(B(x,R))}\leqslant b\left(\frac{r}{R}\right)^\alpha.$$

This easily gives another constant $H>1$ such that

$$\mu\big(B(x,r)\big)\leqslant\frac{1}{2}\mu\big(B(x,Hr)\big)\tag{3.20}$$

for every ball $B=B(x,r)$ with $r\leqslant\operatorname{diam}(X)/2H$. If $\operatorname{diam}(E)<\operatorname{diam}(X)/2H$, we can take any $x\in E$ and $r>\operatorname{diam}(E)$, so that $E\subset B(x,r)$, and of course $\mu(E)\leqslant\mu(B(x,r))$. Using (3.20) and the strong relative isoperimetric inequality proved in this section, we can calculate

$$\begin{aligned}\mu(E)&=\min\big\{\mu\big(B(x,Hr)\cap E\big),\mu\big(B(x,Hr)\setminus E\big)\big\}\\ &\leqslant c_S Hr\mathcal{H}\big(B(x,\tilde\lambda Hr)\cap\partial E\big)\\ &=c_S Hr\mathcal{H}(\partial E).\end{aligned}$$

Since this holds for any $r > \operatorname{diam}(E)$, and since $c_S = c_S(c_P, c_d)$ and $H = H(c_d)$, we get the desired result. Let us then assume that E is unbounded. If $\mu(E) = 0$, the claim is true, so assume $\mu(E) > 0$. By the assumption $\mu(X \setminus E) > 0$ and inequality (3.16), we have

$$\mathcal{H}(\partial E) \geqslant \mathcal{H}\big(B(x,r) \cap \partial E\big) > 0$$

for some ball $B(x,r)$. Thus the right-hand side of (3.19) is infinity, giving us the result. □

When E is itself a ball, its boundary will of course be a subset of a sphere. Somewhat similar results concerning the size of spheres can be found in [28]. Our second corollary is similar to the *boxing inequality*, with the difference that on the right-hand side the perimeter is replaced by the Hausdorff measure of the boundary.

Corollary 3.15. *If X supports a $(1,1)$-Poincaré inequality, for any set $E \subset X$ we have the upper bound*

$$\mathcal{H}_\infty(E) \leqslant c \inf\big\{\mathcal{H}(\partial U):\ E \subset U,\ U \text{ is open},\ \mu(U) < \infty\big\},$$

where the constant $c > 0$ only depends on the doubling constant and the constants in the $(1,1)$-Poincaré inequality.

Proof. See [25, Theorem 3.1, Remark 3.3]. □

4. The case $p > 1$

The main focus of this paper is on the case $p = 1$. As we have seen, geometric quantities such as Hausdorff measures and Minkowski contents naturally arise in this context. When $p > 1$, the codimension p Hausdorff measure or Minkowski content of the boundary of any set with positive measure typically blows up, rendering inequalities of type (1.3)–(1.6) essentially meaningless. In this case, more meaningful quantities are the variously defined *capacities* of sets.

For a bounded open set $\Omega \subset X$, let $E \subset \Omega$ be any closed set and let $G \subset \Omega$ be any open set containing E. The p-*conductivity* of the *conductor* $G \setminus E$ is defined as

$$\operatorname{con}_p(E, G, \Omega) := \inf_{u \in \mathcal{B}(E,G,\Omega)} \int_\Omega g_u^p \, d\mu,$$

where

$$\mathcal{B}(E, G, \Omega) := \big\{u \in N^{1,p}(\Omega) \cap C(\Omega):\ u \geqslant 1 \text{ in } E \text{ and } u \leqslant 0 \text{ in } \Omega \setminus G\big\}.$$

In [24, Theorem 4.3], there is the following characterization of a (q,p)-Poincaré inequality.

Theorem 4.1. *Let $1 \leqslant p \leqslant q < \infty$, let ν be a Borel regular outer measure, and let $\Omega \subset X$ be a bounded open set.*

(i) *If there is a constant γ such that*

$$\nu(E)^{p/q} \leqslant \gamma \operatorname{con}_p(E, G, \Omega) \tag{4.1}$$

for every conductor $G \setminus E$ with $\nu(G) \leqslant \nu(\Omega)/2$, then

$$\inf_{a \in \mathbb{R}} \Big(\int_\Omega |u - a|^q \, d\nu\Big)^{1/q} \leqslant c\Big(\int_\Omega g_u^p \, d\mu\Big)^{1/p} \tag{4.2}$$

for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$, with $c = C(p)\gamma^{1/p}$.

(ii) *If (4.2) holds for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$, then (4.1) holds for every conductor $G \setminus E$ with $\nu(G) \leqslant \nu(\Omega)/2$.*

By slightly modifying Theorem 4.1, we obtain the following formulation for the equivalence of the (q,p)-Poincaré inequality and an *isocapacitary inequality*. For the sake of completeness, we include the proof here.

Theorem 4.2. *Let $1 \leqslant p \leqslant q < \infty$. Then the (q,p)-Poincaré inequality for continuous Newtonian functions, with dilation factor $\lambda \geqslant 1$, is quantitatively equivalent with the relative isocapacitary inequality*

$$\left(\frac{\min\{\mu(B \cap E), \mu(B \setminus G)\}}{\mu(B)}\right)^{1/q} \leqslant c_I r \left(\frac{\operatorname{con}_p(E, G, \lambda B)}{\mu(\lambda B)}\right)^{1/p}, \tag{4.3}$$

where $c_I > 0$ is a constant, $B = B(x,r)$ is any ball, $E \subset \lambda B$ is any closed set, and $G \subset \lambda B$ is any open set containing E.

Proof. Suppose first that the space supports a (q,p)-Poincaré inequality for continuous Newtonian functions. Fix $\varepsilon > 0$ and take $u \in \mathcal{B}(E, G, \lambda B)$ such that

$$\int_{\lambda B} g_u^p \, d\mu \leqslant \operatorname{con}_p(E, G, \lambda B) + \varepsilon.$$

It follows from the (q,p)-Poincaré inequality that

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leqslant c_P r \left(\fint_{\lambda B} g_u^p \, d\mu\right)^{1/p} \leqslant c_P r \left(\frac{\operatorname{con}_p(E, G, \lambda B) + \varepsilon}{\mu(\lambda B)}\right)^{1/p}.$$

Just as in (3.1), we can easily derive the inequality

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \geqslant \frac{1}{2}\left(\frac{\min\{\mu(B \cap E), \mu(B \setminus G)\}}{\mu(B)}\right)^{1/q}.$$

By combining the above estimates, we conclude that X satisfies the isocapacitary inequality (4.3) with the constant $c_I = 2c_P$.

Now suppose that X satisfies the isocapacitary inequality (4.3). This direction follows directly from Theorem 4.1. For a given ball $B = B(x,r)$, we choose $\nu = \mu|_B$, $\Omega = \lambda B$, and $\gamma = c_I^p r^p \mu(B)^{p/q} \mu(\lambda B)^{-1}$. Now the condition (4.1) is clearly satisfied since, when $\nu(G) \leqslant \nu(\lambda B)/2$, we have $\mu(B \cap E) = \nu(E) \leqslant \nu(B)/2 \leqslant \nu(B \setminus G) = \mu(B \setminus G)$ and thus $\min\{\mu(B \cap E), \mu(B \setminus G)\} = \mu(B \cap E) = \nu(E)$.

Notice also that

$$\left(\int_B |u - u_B|^q \, d\mu\right)^{1/q} \leqslant 2 \inf_{a \in \mathbb{R}} \left(\int_B |u - a|^q \, d\mu\right)^{1/q},$$

see for example [24, p. 1103]. □

5. The strong relative isoperimetric inequality (1.6)

It is clear that inequality (1.6) implies inequality (1.5), and thus in fact all of the isoperimetric inequalities we have considered, as well as the $(q,1)$-Poincaré inequality. In this section, we consider the converse. More precisely, we show that the $(q,1)$-Poincaré inequality implies at least a slightly weakened formulation of (1.6).

We begin with the following definition.

Definition 5.1. For any $p > 0$ and $E \subset X$, let

$$\partial_p E := \left\{\liminf_{r \to 0} \frac{\mu(B(x,r) \cap E)}{\mu(B(x,r)) r^p} > 0 \text{ and } \liminf_{r \to 0} \frac{\mu(B(x,r) \setminus E)}{\mu(B(x,r)) r^p} > 0\right\},$$

and furthermore, let $\partial_p^* E := \partial_p E \cup \partial^* E$. We call $\partial_p^* E$ the *p-extended measure theoretic boundary* of E.

We note that the p-extended measure theoretic boundary contains the measure theoretic boundary by definition, but it also contains points that satisfy a slightly different density condition found in the definition of $\partial_p E$. This density condition is weaker in the sense that there is an extra factor r^p in the denominator, but on the other hand it is more restrictive in the sense that there is "lim inf" instead of "lim sup". This means that we do not necessarily have $\partial^* E \subset \partial_p E$. However, we clearly do always have $\partial^* E \subset \partial_p^* E \subset \partial E$. Thus the p-extended measure theoretic boundary is a subset of the topological boundary, and often strictly smaller regardless of the choice of the L^1-representative of E.

With these definitions, we will be able to show the following result.

Theorem 5.2. *For any $q \geqslant 1$, the space X supports a $(q,1)$-Poincaré inequality if and only if we have*

$$\left(\frac{\min\{\mu(B\cap E),\mu(B\setminus E)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r \frac{\mathcal{H}(\tilde{\lambda}B\cap \partial_1^* E)}{\mu(\tilde{\lambda}B)} \tag{5.1}$$

for every μ-measurable set $E \subset X$, every ball $B = B(x,r)$, and some constants $c_S > 0$ and $\tilde{\lambda} \geqslant 1$. Assuming the $(q,1)$-Poincaré inequality, we have $c_S = c_S(c_P, c_d, q)$ and $\tilde{\lambda} = \tilde{\lambda}(\lambda, c_P, c_d)$.

Inequality (5.1) differs from inequality (1.6) in the fact that on the right-hand side we have $\partial_1^* E$ instead of $\partial^* E$. Of course, this makes no difference if the set E satisfies $\mathcal{H}(\partial_1^* E \setminus \partial^* E) = 0$. It is clear that (5.1) implies the $(q,1)$-Poincaré inequality, since by Theorem 1.1 we know that (1.5) implies the $(q,1)$-Poincaré, and $\partial_1^* E \subset \partial E$. To show that the $(q,1)$-Poincaré inequality implies (5.1), we use the same strategy as when proving that Poincaré implies (1.5). The only difference is that now we are dealing with the 1-extended measure theoretic boundary instead of the whole boundary. This does not cause problems, unless there are too many curves that pass through the boundary of E but not through the 1-extended measure theoretic boundary. Thus we start by showing the following result.

Theorem 5.3. *Let E be any μ-measurable set, let $1 \leqslant p < \infty$, and define the family of curves*

$$\Gamma := \bigl\{\gamma \in \Gamma_X\colon\ |\gamma| \cap \partial_p^* E = \varnothing,\ \text{and either}\ \gamma(0) \in O,\ \gamma(l_\gamma) \in I\ \text{or}\ \gamma(0) \in I,\ \gamma(l_\gamma) \in O\bigr\}.$$

Then $\mathrm{Mod}_p(\Gamma) = 0$.

Remark 5.4. Of course, it is enough to consider the case $\gamma(0) \in O$, $\gamma(l_\gamma) \in I$. Essentially we are claiming that very few curves can travel from the measure theoretic exterior of a set to the measure theoretic interior without passing through the p-extended measure theoretic boundary.

Proof of Theorem 5.3. In accordance with the above remark, we assume throughout the proof that $\gamma(0) \in O$ and $\gamma(l_\gamma) \in I$ for each $\gamma \in \Gamma$. For every $m \in \mathbb{N}$, we define the curve family

$$\Gamma_m := \left\{\gamma \in \Gamma\colon \text{ for some } t \in [0,l_\gamma],\ \gamma(t) \in I \text{ and } \frac{\mathcal{L}^1(B(t,s)\cap \gamma^{-1}(O))}{s} > \frac{1}{9} \text{ for all } s \in (0,1/m)\right\}.$$

We note that in the domain of a curve, a ball is simply a subinterval of $[0,l_\gamma]$. Roughly speaking, Γ_m consists of curves that pass through a point belonging to the measure theoretic interior, but immediately after it (or before) travel in the measure theoretic exterior. Similarly, we define $\tilde{\Gamma}_m$, with the roles of I and O reversed. Later in the proof we will show that

$$\Gamma = \bigcup_{m=1}^{\infty} \Gamma_m \cup \tilde{\Gamma}_m,$$

so that it is enough to show that $\mathrm{Mod}_p(\Gamma_m) = 0$ for every $m \in \mathbb{N}$. Let us thus consider the family Γ_m for a fixed $m \in \mathbb{N}$. We take a cover of the set $I \setminus \partial_p^* E$: $\{B(x,r(x))\}_{x\in I\setminus\partial_p^* E}$, $r(x) < 1/(5m)$. By the 5-covering lemma, we can pick a countable collection of pairwise disjoint balls $\{B_i\}_{i=1}^{\infty}$ such that $\{5B_i\}_{i=1}^{\infty}$ is a cover of $I \setminus \partial_p^* E$. Let us consider the balls $10B_i$, $i \in \mathbb{N}$. Clearly each $\gamma \in \Gamma_m$ travels at least the length $5r_i$ in some $10B_i$ right after (and before) passing through a point in $I \setminus \partial_p^* E$ (unless the curve isn't defined for such a length). Now we can define a test function for the family of curves Γ_m:

$$\rho := 18 \sup_{i\in\mathbb{N}} \frac{\chi_{10B_i\cap O}}{10r_i}.$$

By the above discussion and by the definition of Γ_m, for each $\gamma \in \Gamma_m$ we have for some $i \in \mathbb{N}$

$$\int_\gamma \rho\,ds \geqslant 18\int_\gamma \frac{\chi_{10B_i\cap O}}{10r_i}\,ds \geqslant 18\frac{1}{9}\frac{5r_i}{10r_i} = 1.$$

Thus ρ can indeed be used as a test function for the modulus of Γ_m. Let $\varepsilon > 0$. By definition of $\partial_p^* E$, we can assume that the cover $\{B(x, r(x))\}_{x\in I\setminus\partial_p^* E}$ was chosen in such a way that

$$\frac{\mu(10B_i \cap O)}{\mu(10B_i)(10r_i)^p} < \varepsilon$$

for every $i \in \mathbb{N}$. We can now calculate for any ball $B(x, R)$ (here the sums are taken such that $10B_i \cap B(x, R) \neq \varnothing$ for every i)

$$\begin{aligned}
\|\rho\|^p_{L^p(B(x,R))} &= 18^p \int_{B(x,R)} \left(\sup_{i\in\mathbb{N}} \frac{\chi_{10B_i\cap O}}{10r_i}\right)^p d\mu \\
&\leqslant C\int_X \sum_{i=1}^\infty \left(\frac{\chi_{10B_i\cap O}}{10r_i}\right)^p d\mu \\
&= C\int_X \sum_{i=1}^\infty \frac{\chi_{10B_i\cap O}}{(10r_i)^p}\,d\mu \\
&= C\sum_{i=1}^\infty \frac{\mu(10B_i\cap O)}{(10r_i)^p} \\
&\leqslant C\varepsilon \sum_{i=1}^\infty \mu(10B_i) \\
&\leqslant C\varepsilon \sum_{i=1}^\infty \mu(B_i) \\
&\leqslant C\varepsilon\mu\big(B(x, R+3)\big).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $\operatorname{Mod}_p(\Gamma_m \setminus \Gamma_{X\setminus B(x,R)}) = 0$ for any $R > 0$. Since R can be made arbitrarily large, we get $\operatorname{Mod}_p(\Gamma_m) = 0$. As pointed out earlier, this proves that $\operatorname{Mod}_p(\Gamma) = 0$, which was the claim. We only need to verify that there cannot exist a curve

$$\gamma \in \Gamma \setminus \left(\bigcup_{m=1}^\infty \Gamma_m \cup \tilde{\Gamma}_m\right).$$

Assume that there is such a curve γ. We see that if $t \in [0, l_\gamma]$ and $\gamma(t) \in I$, then

$$\limsup_{\delta\to 0} \frac{\mathcal{L}^1(B(t,\delta)\cap\gamma^{-1}(I))}{\mathcal{L}^1(B(t,\delta))} \geqslant \frac{8}{9}. \tag{5.2}$$

Similarly, if $t \in [0, l_\gamma]$ and $\gamma(t) \in O$, then

$$\limsup_{\delta\to 0} \frac{\mathcal{L}^1(B(t,\delta)\cap\gamma^{-1}(O))}{\mathcal{L}^1(B(t,\delta))} \geqslant \frac{8}{9}. \tag{5.3}$$

Now it seems intuitively clear that we should have $\gamma(u) \notin I \cup O$ for some $u \in [0, l_\gamma]$. Precisely speaking, we use Lemma 5.5 (proved below) so that we take the set $\gamma^{-1}(I)$ to be the set "A". Since I is a Borel set in X, its preimage by the continuous function γ is a Borel set in $\mathbb{R}$. Due to (5.2) and (5.3) applied to the end points 0 and l_γ, we see that $0 < \mathcal{L}^1(\gamma^{-1}(I)) < \mathcal{L}^1([0, l_\gamma])$, as required. Now, according to the lemma, there is some $u \in [0, l_\gamma]$ for which

$$\limsup_{\delta\to 0} \frac{\mathcal{L}^1(B(u,\delta)\cap\gamma^{-1}(I))}{\mathcal{L}^1(B(u,\delta))} \leqslant \frac{7}{8}$$

and

$$\limsup_{\delta\to 0} \frac{\mathcal{L}^1(B(u,\delta)\cap \gamma^{-1}(X\setminus I))}{\mathcal{L}^1(B(u,\delta))} \leqslant \frac{7}{8}.$$

We now see from Eqs. (5.2) and (5.3) that $\gamma(u) \notin I \cup O$, so necessarily $\gamma(u) \in \partial^* E \subset \partial_p^* E$. This is a contradiction, since $\gamma \in \Gamma$, completing the proof. □

Lemma 5.5. *Take the space $X = [a,b] \subset \mathbb{R}$, and an $\mathcal{L}^1$-measurable set $A \subset [a,b]$ satisfying $0 < \mathcal{L}^1(A) < \mathcal{L}^1([a,b])$. Then there is a point $y \in [a,b]$ that satisfies*

$$\frac{1}{8} \leqslant \liminf_{r\to 0} \frac{\mathcal{L}^1(\bar{B}(y,r)\cap A)}{\mathcal{L}^1(\bar{B}(y,r))} \leqslant \limsup_{r\to 0} \frac{\mathcal{L}^1(\bar{B}(y,r)\cap A)}{\mathcal{L}^1(\bar{B}(y,r))} \leqslant \frac{7}{8}.$$

Remark 5.6. Both the claim and the proof we present for this lemma are similar to those of Proposition 3.9. In fact, the problem of the type presented in the lemma has been closely studied in [33,12], and [29], with the latter providing the best possible constant that could be used in place of the number $1/8$.

Proof of Lemma 5.5. We prove the result for the case $a = 0$, $b = 1$; the general case is proved similarly. By assumption, we have

$$\frac{\mathcal{L}^1([0,1]\cap A)}{\mathcal{L}^1([0,1])} = c \in (0,1).$$

First we show that we can in fact assume $c = 1/2$. If $c < 1/2$, we pick any Lebesgue point $x \in A$ and observe that the continuous function

$$r \mapsto \frac{\mathcal{L}^1(\bar{B}(x,r)\cap A)}{\mathcal{L}^1(\bar{B}(x,r))}$$

gets values close to one for small $r > 0$, and a value smaller than $1/2$ for $r = 1$. Some $r \in (0,1)$ thus gives an interval $\bar{B}(x,r) \subset [0,1]$ for which the function attains the value $1/2$, and we can then consider this interval. The case $c > 1/2$ is similar. Now, we wish to find a shorter interval that contains the "same proportion" of the set A. To this end, we consider the two subintervals $[0,1/2]$ and $[1/2,1]$. If we have

$$\frac{\mathcal{L}^1([0,1/2]\cap A)}{\mathcal{L}^1([0,1/2])} = \frac{\mathcal{L}^1([1/2,1]\cap A)}{\mathcal{L}^1([1/2,1])} = \frac{1}{2},$$

then both of these subintervals are of the desired kind. If not, we necessarily have

$$\frac{\mathcal{L}^1([0,1/2]\cap A)}{\mathcal{L}^1([0,1/2])} > \frac{1}{2} \quad \text{and} \quad \frac{\mathcal{L}^1([1/2,1]\cap A)}{\mathcal{L}^1([1/2,1])} < \frac{1}{2},$$

or the other way around. By continuity of the function

$$y \mapsto \frac{\mathcal{L}^1(\bar{B}(y,r)\cap A)}{\mathcal{L}^1(\bar{B}(y,r))}$$

for constant $r > 0$, there must be a point $y_1 \in [1/4,3/4]$ for which

$$\frac{\mathcal{L}^1(\bar{B}(y_1,1/4)\cap A)}{\mathcal{L}^1(\bar{B}(y_1,1/4))} = \frac{1}{2}.$$

Now we repeat this step, always picking a ball (interval) from within the previous ball, containing the same proportion of the set A. This means that we get a sequence of radii $r_i = 2^{-i-1}$ and a sequence of ball centers $(y_i) \subset [0,1]$. The latter is, by construction, clearly also Cauchy, so it converges, $y_i \to y \in [0,1]$. The sequence of balls $\{\bar{B}(y_i,r_i)\}_{i=1}^{\infty}$ satisfies

$$\frac{\mathcal{L}^1(\bar{B}(y_i,r_i)\cap A)}{\mathcal{L}^1(\bar{B}(y_i,r_i))} = \frac{1}{2}$$

for every $i \in \mathbb{N}$. Clearly $y \in \bar{B}(y_i, r_i)$ for every $i \in \mathbb{N}$. Thus we have $\bar{B}(y, r_i) \supset \bar{B}(y_{i+1}, r_{i+1})$ for every $i \in \mathbb{N}$. We can now calculate

$$\frac{\mathcal{L}^1(\bar{B}(y, r_i) \cap A)}{\mathcal{L}^1(\bar{B}(y, r_i))} \geqslant \frac{\mathcal{L}^1(\bar{B}(y_{i+1}, r_{i+1}) \cap A)}{2\mathcal{L}^1(\bar{B}(y_{i+1}, r_{i+1}))} \geqslant \frac{1}{4}$$

for every $i \in \mathbb{N}$. Furthermore, for every $i \in \mathbb{N}$ and any $r \in [r_{i+1}, r_i]$,

$$\frac{\mathcal{L}^1(\bar{B}(y, r) \cap A)}{\mathcal{L}^1(\bar{B}(y, r))} \geqslant \frac{\mathcal{L}^1(\bar{B}(y, r_{i+1}) \cap A)}{\mathcal{L}^1(\bar{B}(y, r_i))} \geqslant \frac{\mathcal{L}^1(\bar{B}(y, r_{i+1}) \cap A)}{2\mathcal{L}^1(\bar{B}(y, r_{i+1}))} \geqslant \frac{1}{8}.$$

Altogether, for any $r \in (0, 1/4)$ we have

$$\frac{\mathcal{L}^1(\bar{B}(y, r) \cap A)}{\mathcal{L}^1(\bar{B}(y, r))} \geqslant \frac{1}{8}.$$

In the same fashion, we get for the complement of the set A:

$$\frac{\mathcal{L}^1(\bar{B}(y, r) \setminus A)}{\mathcal{L}^1(\bar{B}(y, r))} \geqslant \frac{1}{8}$$

for any $r \in (0, 1/4)$. This shows that the point y fulfills the desired requirements. □

Proof of Theorem 5.2. As mentioned immediately after Theorem 5.2, it is clear that (5.1) implies the $(q, 1)$-Poincaré inequality. To show that the $(q, 1)$-Poincaré inequality implies (5.1), we essentially use the same method as in the proof of Theorem 3.13. Instead of the whole boundary ∂E we just need to work with the 1-extended measure theoretic boundary $\partial_1^* E$. Similar to Lemma 3.10, we can show that $\partial_1^* E$ is also invariant under bi-Lipschitz identity mappings. The function g is defined as in (3.17), except that we take the family $\{B_i\}_{i=1}^{\infty}$ to cover $\lambda \bar{B} \cap \partial_1^* E$. The function u is now defined as

$$u(x) := \min\left\{1, \inf_{\gamma} \int_{\gamma} g\, ds\right\},$$

where the infimum is taken over curves $\gamma \in \Gamma_X \setminus \Gamma$ connecting x to $(B \setminus I) \setminus \bigcup_{i=1}^{\infty} 2B_i$. Here Γ is the family of curves defined in Theorem 5.3, satisfying $\mathrm{Mod}_1(\Gamma) = 0$, and I is again the measure theoretic interior of the set E. If there are no such admissible curves, we let the infimum be infinity. Just as in the proof of Theorem 3.13 we obtained $u = 1$ in $E \cap B$, we now get $u = 1$ in $I \cap B$ (due to the restriction on the curves in the definition of u). Since the sets I and E are equivalent in the L^1-sense, as are the sets O and $X \setminus E$, the rest of the proof goes similarly. The only thing that needs to be checked is that the function g is in fact a 1-weak upper gradient of u — the proof will again be similar to that given in [7] or [6, p. 128], but we need to make sure that the family of curves Γ with 1-modulus zero does not cause problems. We check this in the following, completing the proof. □

Let us define the "gluing" together of two curves in a natural way: for any two curves $\gamma_1, \gamma_2 \in \Gamma_X$ satisfying $\gamma_1(l_{\gamma_1}) = \gamma_2(0)$, we define the "glued" curve $\gamma_1 + \gamma_2 \in \Gamma_X$ as

$$(\gamma_1 + \gamma_2)(t) := \begin{cases} \gamma_1(t) & \text{if } 0 \leqslant t \leqslant l_{\gamma_1}, \\ \gamma_2(t - l_{\gamma_1}) & \text{if } l_{\gamma_1} \leqslant t \leqslant l_{\gamma_1} + l_{\gamma_2}. \end{cases}$$

Now, with the curve family Γ defined as in Theorem 5.3, it is easy to check that the "good" curves $\gamma \in \Gamma_X \setminus \Gamma$ have the following property: if $\gamma_1, \gamma_2 \in \Gamma_X \setminus \Gamma$, then $\gamma_1 + \gamma_2 \in \Gamma_X \setminus \Gamma$ (if defined). Thus we need to prove the following proposition.

Proposition 5.7. *Let $A \subset X$ and let $g : X \mapsto [0, \infty]$ be a Borel measurable function. Let $1 \leqslant p < \infty$ and let Γ be a family of curves satisfying $\mathrm{Mod}_p(\Gamma) = 0$ and the following condition:*

$$\text{If } \gamma_1, \gamma_2 \in \Gamma_X \setminus \Gamma, \text{ then } \gamma_1 + \gamma_2 \in \Gamma_X \setminus \Gamma \text{ (if defined).} \tag{5.4}$$

Finally, let

$$u(x) := \min\left\{1, \inf_{\gamma} \int_{\gamma} g\,ds\right\},$$

where the infimum is taken over curves $\gamma \in \Gamma_X \setminus \Gamma$ connecting x to A. If there are no such curves, we let the infimum be infinity. Then the function g is a p-weak upper gradient of the function u.

Proof. Let $x, y \in X$ be arbitrary and let γ_{xy} be any curve belonging to $\Gamma_X \setminus \Gamma$ and connecting y to x. We can assume that $0 \leqslant u(x) < u(y) \leqslant 1$. If $\gamma \in \Gamma_X \setminus \Gamma$ is a curve connecting x to A, then by (5.4) the curve $\gamma + \gamma_{xy}$ belongs to $\Gamma_X \setminus \Gamma$ and connects y to A. Thus we get

$$u(y) \leqslant \int_{\gamma+\gamma_{xy}} g\,ds = \int_{\gamma} g\,ds + \int_{\gamma_{xy}} g\,ds.$$

Taking infimum over all curves $\gamma \in \Gamma_X \setminus \Gamma$ connecting x to A gives us

$$u(y) \leqslant \int_{\gamma_{xy}} g\,ds + u(x).$$

Thus g is a p-weak upper gradient of u. □

Remark 5.8. Naturally Corollaries 3.14 and 3.15 still hold, with the topological boundary replaced by the 1-extended measure theoretic boundary.

6. Sufficient conditions for finite perimeter and finite variation

We know from [4, Theorems 4.4, 4.6] that if X supports a $(1,1)$-Poincaré inequality and E is a set of finite perimeter, the measure $P(E,\cdot)$ is concentrated on the measure theoretic boundary $\partial^* E$, and

$$\frac{1}{C} P(E,A) \leqslant \mathcal{H}\big(\partial^* E \cap A\big) \leqslant C P(E,A) \tag{6.1}$$

for any Borel set A and some constant C, which only depends on the doubling constant and the constants in the Poincaré inequality. In the Euclidean case (with the Euclidean metric and Lebesgue measure), we also have a converse: if a Lebesgue measurable set E satisfies $\mathcal{H}(\partial^* E) < \infty$, it is a set of finite perimeter; see [14] or [13]. We would like to have a similar result in the case of a general metric space. As was mentioned in the introduction, the following characterization by Miranda holds [31, Theorem 3.8].

Theorem 6.1. *Let X support a $(1,1)$-Poincaré inequality, and let $u \in L^1_{\mathrm{loc}}(X)$. Then $\|Du\|(X) < \infty$ if and only if there exist a constant $\lambda > 0$ and a finite Borel outer measure ν such that*

$$\int_B |u - u_B|\,d\mu \leqslant r\nu(\lambda B) \tag{6.2}$$

for each ball $B = B(x,r)$. Moreover, $\|Du\| \leqslant C\nu$, where $C = C(c_d, \lambda)$.

Remembering Eq. (3.1), we see that the strong relative isoperimetric inequalities (1.5) and (1.6) with $q = 1$ are inequalities of the type (6.2), written for the characteristic functions of sets. The only requirement is that the measure on the right-hand side is finite.

Following precisely this idea, it was shown in [26, Theorem 4.6] that in a metric space X supporting the strong relative isoperimetric inequality (1.6) with $q = 1$, any μ-measurable set satisfying $\mathcal{H}(\partial^* E) < \infty$ is of finite perimeter. On the other hand, we know that the $(1,1)$-Poincaré inequality implies at least the weaker inequality (5.1) with $q = 1$. By taking the measure ν in Theorem 6.1 to be $\mathcal{H}|_{\partial_1^* E}$, we conclude that the finiteness of $\mathcal{H}(\partial_1^* E)$ is a sufficient condition for the finite perimeter of E. By contrast, since the codimension one Minkowski content μ^+ is not an outer measure, we cannot apply Theorem 6.1 to the two strong relative isoperimetric inequalities that have the Minkowski content on the right-hand side.

The proof of Theorem 6.1, and likewise the proof of [26, Theorem 4.6], is based on covering the space with balls of uniform radii. We can obtain the same results in an arbitrary open set $\Omega \subset X$ simply by using Whitney-type coverings instead (for the construction of such a covering, see e.g. [1, Lemma 4.1]). Thus we have the following theorem.

Theorem 6.2. *Let X support a $(1,1)$-Poincaré inequality, let $\Omega \subset X$ be any open set, and let $E \subset X$ be any μ-measurable set. Then the condition $\mathcal{H}(\partial_1^* E \cap \Omega) < \infty$ implies $P(E,\Omega) < \infty$. If X further supports the strong relative isoperimetric inequality* (1.6) *with $q = 1$, then $\mathcal{H}(\partial^* E \cap \Omega) < \infty$ implies $P(E,\Omega) < \infty$.*

The coarea formula enables us to apply the results obtained for sets of finite perimeter to general BV functions. For an open set $\Omega \subset X$, take any function $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfying

$$\int_{-\infty}^{\infty} \mathcal{H}\big(\partial_1^* E_t \cap \Omega\big)\, dt < \infty, \tag{6.3}$$

where the sets $E_t = \{u > t\}$ are again the level sets of the function u. Then we know that for a.e. $t \in \mathbb{R}$, $\mathcal{H}(\partial_1^* E_t \cap \Omega) < \infty$ and thus $P(E_t,\Omega) < \infty$. Furthermore, by (6.1) we have for a.e. $t \in \mathbb{R}$

$$P(E_t,\Omega) \leqslant C\mathcal{H}\big(\partial^* E_t \cap \Omega\big) \leqslant C\mathcal{H}\big(\partial_1^* E_t \cap \Omega\big),$$

whence

$$\int_{-\infty}^{\infty} P(E_t,\Omega)\, dt \leqslant C \int_{-\infty}^{\infty} \mathcal{H}\big(\partial_1^* E_t \cap \Omega\big)\, dt < \infty.$$

By the coarea formula (2.5), this implies $\|Du\|(\Omega) < \infty$. Thus (6.3) is a sufficient condition for the bounded variation of a function.

Again, of course, we would like to work with the (ordinary) measure theoretic boundary $\partial^* E$, and not the extended one. Let us take a brief look at another way of defining BV functions.

Definition 6.3. For any open set $\Omega \subset X$ and any function $u \in L^1_{\mathrm{loc}}(\Omega)$, define the alternative variation measure

$$\|Du\|^*(\Omega) := \int_{-\infty}^{\infty} \mathcal{H}\big(\partial^* E_t \cap \Omega\big)\, dt.$$

Then define the class $BV^*(\Omega)$ so that $u \in BV^*(\Omega)$ if and only if $u \in L^1(\Omega)$ and $\|Du\|^*(\Omega) < \infty$.

It is natural to ask whether this definition is equivalent with the ordinary definition of BV functions. Clearly, if $u \in BV(\Omega)$, we have $u \in L^1(\Omega)$, and by (6.1),

$$\begin{aligned} \infty > \|Du\|(\Omega) &= \int_{-\infty}^{\infty} P(E_t,\Omega)\, dt \\ &\geqslant C^{-1} \int_{-\infty}^{\infty} \mathcal{H}\big(\partial^* E_t \cap \Omega\big)\, dt. \end{aligned}$$

By definition, $\|Du\|^*(\Omega) < \infty$ and thus $u \in BV^*(\Omega)$. On the other hand, if $u \in BV^*(\Omega)$, we again have $u \in L^1(\Omega)$, but the question of bounded variation is unclear. If we assume that for any μ-measurable set E the condition $\mathcal{H}(\partial^* E \cap \Omega) < \infty$ implies $P(E,\Omega) < \infty$, which is again true if the strong relative isoperimetric inequality (1.6) holds with $q = 1$, we can conclude that

$$\|Du\|(\Omega) = \int_{-\infty}^{\infty} P(E_t,\Omega)\, dt \leqslant C \int_{-\infty}^{\infty} \mathcal{H}\big(\partial^* E_t \cap \Omega\big)\, dt < \infty.$$

Thus the seminorms $\|Du\|$ and $\|Du\|^*$ will be comparable.

However, Definition 6.3 would probably not be very useful as a definition of BV functions, since it does not give lower semicontinuity. If $\|Du\|$ and $\|Du\|^*$ are comparable and $u_i \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, by the lower semicontinuity of the (ordinary) variation measure we get

$$\|Du\|^*(\Omega) \leqslant C\|Du\|(\Omega) \leqslant C \liminf_{i\to\infty} \|Du_i\|(\Omega) \leqslant C \liminf_{i\to\infty} \|Du_i\|^*(\Omega).$$

So, the alternative variation measure is, up to a constant, lower semicontinuous with respect to convergence in L^1. However, a simple example shows that this constant is impossible to get rid of in general.

Example 6.4. In $\mathbb{R}^2$, define a weight function such that $w = 1$ on the unit ball $B(0, 1)$, and $w = 2$ in $\mathbb{R}^2 \setminus B(0, 1)$. Then define the doubling measure $d\mu = w\, d\mathcal{L}^2$, where $\mathcal{L}^2$ is the 2-dimensional Lebesgue measure. Define the sequence of sets $E_i := B(0, 1 - 1/i)$, $i \in \mathbb{N}$. Clearly $\chi_{E_i} \to \chi_{B(0,1)}$ in $L^1(\mathbb{R}^2, \mu)$ as $i \to \infty$. We also have $\mathcal{H}(\partial^* E_i) = \pi^2(1 - 1/i)$ (note that our definition of the Hausdorff measure gives slightly different constants than the usual definition).

On the other hand, it is fairly easy to see that the quantity $\mathcal{H}(\partial^* B(0, 1))$ is strictly larger than π^2 (which is the value it would take if the weight function was 1 everywhere). The reason is that in order to eliminate the impact of the larger value of the weight function outside the unit ball, one would have to choose coverings in which the balls are almost completely inside the unit ball, but then one would need a much larger number of balls. More precisely, consider a ball $B(x, r)$, with $r < 1/100$, in the covering of $\partial^* B(0, 1)$. If $|x| \geqslant 1 - r/2$, then $\mu(B(x, r)) \geqslant \mathcal{L}^2(B(x, r))(1 + \frac{1}{16})$. And if $|x| \leqslant 1 - r/2$, then $\mathcal{H}^1(B(x, r) \cap \partial^* B(0, 1)) \leqslant \frac{9}{5}r$, where $\mathcal{H}^1$ is the usual 1-dimensional Hausdorff measure (length measure). From these estimates, it is easy to see that $\mathcal{H}(\partial^* B(0, 1)) \geqslant (1 + \frac{1}{16})\pi^2$. Thus the lower semicontinuity fails to hold.

Let us finally note that from every strong relative isoperimetric inequality, we can derive a Poincaré-type inequality that applies to every function $u \in L^1_{\mathrm{loc}}(X)$. For example, the inequality (5.1) written for the level sets of u reads

$$\left(\frac{\min\{\mu(B \cap E_t), \mu(B \setminus E_t)\}}{\mu(B)}\right)^{1/q} \leqslant c_S r \frac{\mathcal{H}(\tilde{\lambda} B \cap \partial_1^* E_t)}{\mu(\tilde{\lambda} B)}. \tag{6.4}$$

By integrating both sides of this inequality with respect to t (analyzing separately the positive and negative parts of u and truncating, if necessary), we get just as in the proof of Theorem 3.3 that

$$\left(\fint_B |u - u_B|^q \, d\mu\right)^{1/q} \leqslant \frac{Cr}{\mu(\tilde{\lambda} B)} \int_{-\infty}^{\infty} \mathcal{H}\bigl(\tilde{\lambda} B \cap \partial_1^* E_t\bigr)\, dt. \tag{6.5}$$

This holds for every function $u \in L^1_{\mathrm{loc}}(X)$, and is equivalent with the $(q, 1)$-Poincaré inequality. This is immediate, since inequality (5.1) was earlier found to be equivalent with the $(q, 1)$-Poincaré inequality, and (6.5) implies (5.1) simply by choosing $u = \chi_E$. Inequality (6.5) with $q = 1$ is also easily seen to be of the form (6.2). Thus it could also be used to show that (6.3) implies $\|Du\|(\Omega) < \infty$ for any $u \in L^1_{\mathrm{loc}}(\Omega)$.

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