# Analysis of a Ginzburg-Landau type energy model for smectic C* liquid crystals with defects 

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#### Abstract

This work investigates properties of a smectic C* liquid crystal film containing defects that cause distinctive spiral patterns in the film's texture. The phenomena are described by a Ginzburg-Landau type model and the investigation provides a detailed analysis of minimal energy configurations for the film's director field. The study demonstrates the existence of a limiting location for the defects (vortices) so as to minimize a renormalized energy. It is shown that if the degree of the boundary data is positive then the vortices each have degree +1 and that they are located away from the boundary. It is proved that the limit of the energies for a sequence of minimizers minus the sum of the energies around their vortices, as the G-L parameter $\varepsilon$ tends to zero, is equal to the renormalized energy for the limiting state.


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## Résumé

Dans ce travail on étudie les propriétés d'un film de cristaux liquides smectiques $\mathrm{C}^{*}$ contenant des défauts qui produisent des motifs en spirale dans la texture du film. Les phénomènes sont décrits par un modèle de type Ginzburg-Landau dans un domaine borné du plan, et cet article fournit une analyse détaillée des configurations d'énergie minimale du champ de directions du film. On montre l'existence d'une configuration limite pour les défauts (tourbillons) qui minimise une énergie renormalisée. On démontre que si le degré du champ sur le bord du domaine est positif, alors les tourbillons sont dans l'intérieur du domaine et sont chacun de degré +1 . On prouve que quand le paramètre $\varepsilon$ de Ginzburg-Landau tend vers zéro, pour une suite de minimiseurs, la limite de l'énergie moins la somme des énergies autour des tourbillons est égale à l'énergie renormalisée de l'état limite.
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## 1. Introduction

We study the occurrence of point defects in a thin ferroelectric smectic $\mathrm{C}^{*}\left(\mathrm{SmC}^{*}\right)$ liquid crystal by using a director field description based on the Ginzburg-Landau theory. The unknown function $u$ is a vector field in $\mathbb{R}^{2}$. When convenient, for ease of notation, we view it as a $\mathbb{C}$-valued function such that $u=u^{1}\left(x_{1}, x_{2}\right)+i u^{2}\left(x_{1}, x_{2}\right)$ for $\left(x_{1}, x_{2}\right)$ in a bounded domain $\Omega$ in $\mathbb{R}^{2}$. We assume $\Omega$ has a smooth $\left(C^{3}\right)$ boundary in the plane and that $\Omega$ represents the reference configuration of a very thin liquid crystal material.

[^0]We analyze minimizers for a Ginzburg-Landau type energy,

$$
\begin{equation*}
J_{\varepsilon}(u)=\frac{1}{2} \int_{\Omega}\left(k_{s}(\operatorname{div} u)^{2}+k_{b}(\operatorname{curl} u)^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x=\int_{\Omega} j_{\varepsilon}(u, \nabla u) d x, \tag{1.1}
\end{equation*}
$$

where $k_{s}$ and $k_{b}$ represent the two-dimensional splay and bend moduli for the film respectively, with $k_{s}, k_{b}>0$. Here $J_{\varepsilon}(\cdot)$ is defined for $u \in H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$, consisting of fields $u \in H^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ with Dirichlet boundary conditions, $\left.u\right|_{\partial \Omega}=g \in C^{3}\left(\partial \Omega ; \mathbb{S}^{1}\right)$ such that $\operatorname{deg}(g, \partial \Omega)=d>0$. The variable $\varepsilon>0$ represents the radius of the defect cores. Previous work has considered the case where $k_{s}=k_{b}$, reducing to the classical Ginzburg-Landau functional [1-5]. Our work focuses on the cases where $k_{s} \neq k_{b}$. The elastic energy term of (1.1) is used to model thin film liquid crystals with chirality, such as an $\mathrm{SmC}^{*}$ material. The resulting pattern consists of a family of point defects in the film forming vortices in the molecular texture that spiral in a fashion determined by the relative values for $k_{s}$ and $k_{b}$; see [6].

### 1.1. Main results

By denoting $\underline{k}=\min \left\{k_{s}, k_{b}\right\}$, we can express (1.1) as

$$
\begin{equation*}
J_{\varepsilon}(u)=\bar{J}_{\varepsilon}(u)+\underline{k} \int_{\Omega} \operatorname{det} \nabla u d x=\bar{J}_{\varepsilon}(u)+\underline{k} \pi d \tag{1.2}
\end{equation*}
$$

for $u \in H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ where

$$
\bar{J}_{\varepsilon}(u)=\int_{\Omega} \bar{j}_{\varepsilon}(u, \nabla u) d x= \begin{cases}\frac{1}{2} \int_{\Omega}\left(k_{s}|\nabla u|^{2}+\left(k_{b}-k_{s}\right)(\operatorname{curl} u)^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x & \text { if } \underline{k}=k_{s},  \tag{1.3}\\ \frac{1}{2} \int_{\Omega}\left(k_{b}|\nabla u|^{2}+\left(k_{s}-k_{b}\right)(\operatorname{div} u)^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x & \text { if } \underline{k}=k_{b} .\end{cases}
$$

Then $u \in H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ is a minimizer of $J_{\varepsilon}(u)$ if and only if $u$ is a minimizer for $\bar{J}_{\varepsilon}(u)$. Hence, it suffices to consider the minimizers of Eq. (1.3) and analyze this functional. In this way, by the strict convexity of the integral in (1.3), we have the existence of a minimizer $u_{\varepsilon}$ for each $\varepsilon$; see [7].

We need a detailed description of $\Omega$. Let $D \subset \mathbb{R}^{2}$ be a bounded, simply connected domain with a $C^{3}$ boundary $\Gamma_{0}$. For $\ell=1, \ldots, k$ let $\bar{\Lambda}_{\ell} \subset D$ be pair-wise disjoint, simply connected sets with $C^{3}$ boundaries $\Gamma_{\ell}$. Consider the domain $\Omega=D \backslash \bigcup_{\ell=1}^{k} \bar{\Lambda}_{\ell}$ where we take the natural orientation for $\partial \Omega=\bigcup_{\ell=0}^{k} \partial \Gamma_{\ell}$, such that $\Gamma_{0}$ is oriented counterclockwise and $\Gamma_{\ell}$ are oriented clockwise for $1 \leqslant \ell \leqslant k$. For each $g \in C^{3}\left(\partial \Omega ; \mathbb{S}^{1}\right)$ set $d_{\ell}:=$ winding number of $\left.g\right|_{\Gamma_{\ell}}$ with respect to the curve's orientation, and denote the degree $d(g, \partial \Omega):=d=\sum_{\ell=0}^{k} d_{\ell}$. We fix $k$ points, $y_{\ell} \in \Lambda_{\ell}$, and set $w(x)=\prod_{\ell=1}^{k}\left(\frac{x-y_{\ell}}{\left|x-y_{\ell}\right|}\right)^{-d_{\ell}}=e^{i \zeta(x)}$ for $x \in \bar{\Omega}$. Thus $\zeta$ is a multi-valued, harmonic expression such that $\nabla \zeta(x)$ is point-wise well defined. We use $w(x)$ to fix specific representations of functions having boundary values with winding numbers $d_{\ell}$ with respect to $\Gamma_{\ell}$ for $1 \leqslant \ell \leqslant k$. The minimizers of the energy functional over $H_{g}^{1}$ have a number of structural properties that lead to the first main result of the paper.

Theorem A. Let $\left\{u_{\varepsilon}\right\}$ be a sequence of minimizers for $J_{\varepsilon}(u)$ over $H_{g}^{1}$ such that $\varepsilon \rightarrow 0$. Then there is a subsequence $\left\{u_{\varepsilon \ell}\right\}$, a function $h \in H^{1}(\Omega)$ and $d$ points $\left\{a_{1}, \ldots, a_{d}\right\} \in \Omega$ such that
$\left|u_{\varepsilon_{\ell}}\right| \rightarrow 1 \quad$ uniformly on compact subsets of $\bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}$,
and more generally $u_{\varepsilon_{\ell}}(x) \rightarrow u_{*}(x)=e^{i\left(h(x)+\zeta(x)+\sum_{n=1}^{d} \theta_{a_{n}}(x)\right)}$
in $C_{l o c}^{\alpha}\left(\bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}\right)$ and in $C_{l o c}^{m}\left(\Omega \backslash\left\{a_{1}, \ldots, a_{d}\right\}\right)$ for every $0<\alpha<1$ and integer $m \geqslant 0$, in which $\theta_{a_{n}}=\theta_{a_{n}}(x)$ denotes the polar angle of $x$ with respect to the center $a_{n}$.

The d-tuple $\mathbf{a}=\left(a_{1}, \ldots, a_{d}\right) \in \Omega^{d}$ represents the point defects within the domain $\Omega$. The energy functional, just as in [1], has a renormalized form,

$$
\begin{equation*}
\underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right) \quad \text { for } \mathbf{b} \in \Omega^{d} \tag{1.5}
\end{equation*}
$$

here

$$
\begin{align*}
W(\mathbf{b})= & \frac{1}{2} \int_{\partial \Omega}\left(2 G_{\mathbf{b}}\left(g \times \partial_{\tau} g\right)-\left(\partial_{\nu} G_{\mathbf{b}}\right) G_{\mathbf{b}}\right) d \sigma+\pi d \\
& -\sum_{m \neq n} \pi \ln \left(\left|b_{n}-b_{m}\right|\right)+\sum_{n=1}^{d} \sum_{\ell=1}^{k} \pi d_{\ell} \ln \left(\left|b_{n}-y_{\ell}\right|\right) \tag{1.6}
\end{align*}
$$

and

$$
H\left(\mathbf{b}, k_{s}, k_{b}\right)=\inf _{\phi} \mathscr{H}\left(\mathbf{b}, \phi, k_{s}, k_{b}\right)= \begin{cases}\inf _{\phi} \frac{1}{2} \int_{\Omega}\left(k_{s}|\nabla \phi|^{2}+\left(k_{b}-k_{s}\right)(\operatorname{curl} v)^{2}\right) d x & \text { if } \underline{k}=k_{s}  \tag{1.7}\\ \inf _{\phi} \frac{1}{2} \int_{\Omega}\left(k_{b}|\nabla \phi|^{2}+\left(k_{s}-k_{b}\right)(\operatorname{div} v)^{2}\right) d x & \text { if } \underline{k}=k_{b}\end{cases}
$$

where $G_{\mathbf{b}}(x)=\sum_{n=1}^{d} \ln \left(\left|x-b_{n}\right|\right)-\sum_{\ell=1}^{k} d_{\ell} \ln \left(\left|x-y_{\ell}\right|\right), v(x)=\prod_{n=1}^{d} \frac{x-b_{n}}{\left|x-b_{n}\right|} e^{i(\phi(x)+\zeta(x))}$, and $\mathscr{H}\left(\mathbf{b}, \phi, k_{s}, k_{b}\right)$ is minimized over the class of functions $\phi \in H^{1}(\Omega)$ such that $v=g$ on $\partial \Omega$. The expression (1.5) is a variant of the renormalized energy from [1]. Moreover the two agree for the case where $k_{s}=k_{b}$ and $\Omega$ is simply connected. By the definition of (1.5), we have that in order for the renormalized energy to be finite, $b_{n} \neq b_{m}$ for $n \neq m$ and $b_{n} \notin \partial \Omega$ for every $n$. For each such set of configurations $\mathbf{b}$, there exists a function $h_{\mathbf{b}}$, in a particular class of functions, such that $H\left(\mathbf{b}, k_{s}, k_{b}\right)=\mathscr{H}\left(\mathbf{b}, h_{\mathbf{b}}, k_{s}, k_{b}\right)$, which leads to the second main theorem of this work.

Theorem B. Let $\left\{u_{\ell}\right\}$ be a sequence of minimizers for $J_{\varepsilon_{\ell}}$, for which $\mathbf{a}=\left(a_{1}, \ldots, a_{d}\right)$ is a limiting configuration of distinct defects as $\varepsilon_{\ell} \rightarrow 0$ as described in Theorem A and $h \in H^{1}(\Omega)$ is as in Theorem A. Then it holds that $H\left(\mathbf{a}, k_{s}, k_{b}\right)=\mathscr{H}\left(\mathbf{a}, h, k_{s}, k_{b}\right)$ and

$$
\lim _{\ell \rightarrow \infty}\left(J_{\varepsilon_{\ell}}\left(u_{\ell}\right)-\underline{k} \pi d \ln \left(\frac{1}{\varepsilon_{\ell}}\right)\right)=\underline{k} W(\mathbf{a})+H\left(\mathbf{a}, k_{s}, k_{b}\right)+d \gamma
$$

where $\gamma$ is a fixed constant associated with each of the defect core's energy. Moreover, the renormalized energy attains its minimum among $\mathbf{b} \in \Omega^{d}$ with distinct components at $\mathbf{b}=\mathbf{a}$.

The term $k \pi d \ln \left(\frac{1}{\varepsilon}\right)$ represents the energy around the vortices to leading order. The limit as $\varepsilon$ tends to zero of the difference between this term and (1.1) gives the remaining energy over the domain $\Omega$ minus the vortices, with their location a, minimizing (1.5). For the case $k_{s}=k_{b}$ the proofs for Theorems A and B follow from the results in [1] and [3] if $\Omega$ is simply connected. Moreover their proofs can be extended if the domain is multiply connected. Theorem B allows us to characterize the limiting pattern, $u_{*}(x)$, near each $a_{m}$. This follows from the fact that $h$ minimizes (1.7). For the case $k_{s}=k_{b}$ this implies that $h$ is a harmonic function such that $v=g$ on $\partial \Omega$. Thus

$$
u_{*}\left(\rho y+a_{m}\right) \rightarrow \beta_{m} y
$$

as $\rho \rightarrow 0$ for each $y \in \partial B_{1}(0)$ where $\beta_{m}=e^{i\left(h\left(a_{m}\right)+\zeta\left(a_{m}\right)+\sum_{n \neq m} \theta_{a_{n}}\left(a_{m}\right)\right)}$. For $k_{s} \neq k_{b}$ we find a much different structure. In this case the integral in (1.7) involving either the term curl $v$ or $\operatorname{div} v$ must be finite, and as a result pins the values of $h$ near each $a_{m}$ so that

$$
u_{*}\left(\rho y+a_{m}\right) \rightarrow \begin{cases} \pm y & \text { if } k_{s}<k_{b} \\ \pm i y & \text { if } k_{b}<k_{s}\end{cases}
$$

in $L^{2}\left(\partial B_{1}(0) ; \mathbb{C}\right)$ as $\rho \rightarrow 0$. Thus if $k_{s}<k_{b}$ the limiting texture $u_{*}$ has a pure splay pattern near each defect and if $k_{b}<k_{s}$ then $u_{*}$ asymptotically has a pure bend pattern near each $a_{m}$.

### 1.2. Applications

Smectic C materials are made of layers of liquid crystal molecules that pack so that their long axes form a fixed angle $0<\theta_{0}<\pi / 2$ with the layer normal. The pattern is described using the layer structure and a director field $n(x)$ for the liquid crystal. The director is a unit vector field that lies parallel to the local average of the molecular long axes at $x$. One can then express $n(x)=\cos \left(\theta_{0}\right) \nu(x)+\sin \left(\theta_{0}\right) c(x)$ where $v(x)$ and $c(x)$ each are unit vector fields that are respectively parallel and perpendicular to the layer normal at $x$. These two fields are the fundamental unknowns that are used to characterize the material's configuration [8]. For the case of a thin film the layers are planer, given by the
domain $\Omega \subset \mathbb{R}^{2}$ such that $v(x)=\langle 0,0,1\rangle$ and $c(x)=\left\langle c^{1}(x), c^{2}(x), 0\right\rangle$ with $x=\left(x_{1}, x_{2}\right)$. The film can be just several layers thick and the elastic energy for the liquid crystal pattern is given by the Oseen-Frank energy. Each layer in a smectic $\mathrm{C}(\mathrm{SmC})$ liquid crystal can be represented as a two-dimensional liquid [9] and the integral is taken over the film,

$$
\begin{equation*}
\frac{1}{2} \int_{\Omega}\left(k_{s}(\operatorname{div} c)^{2}+k_{b}(\operatorname{curl} c)^{2}\right) d x \tag{1.8}
\end{equation*}
$$

An $\mathrm{SmC}^{*}$ liquid crystal additionally forms a spontaneous polarization field that produces elastic and electro-static contributions to the energy. The polar field generates an elastic stress on the film whose effect is modeled by introducing boundary values for $c(x)=g(x)$ on $\partial \Omega$; see [6]. The field induces an electro-static contribution that appears in our energy by increasing the splay constant $k_{b}$ above its bare elastic value [10] and this is a motivation for studying the case $k_{s} \neq k_{b}$.

If a smoke or dust particle lands on a free-standing film a defect forms in the film's texture. The particle induces a singularity in the spontaneous polarization field that in turn causes an island, several layers thicker than the film, to form around the defect. The island's shape eventually stabilizes and the island migrates within the film so as to reduce the total energy. Various experiments have been conducted and models put forward to investigate this phenomenon. See [6,10-13]. In these papers the notion of the $c$-director is generalized to allow for defects. The experiments reported in [6,10-12] indicate that a stable island is disk-like and that the $c$-director is tangential at the island's edge, so that the winding number of $c$ on the edge of the disk is +1 . In [6], Lee et al. model the island-defect ensemble by setting $\Omega=B_{R}(0) \backslash B_{\varepsilon}(0)$ with the defect represented by the $\varepsilon$ void at the origin and investigate numerically the stability of rotationally invariant equilibria for (1.8). Their simulations for the case $k_{s}<k_{b}$ and $\varepsilon$ sufficiently small, indicate that minimizers for $(1.8)$ over $H^{1}\left(\Omega ; \mathbb{S}^{1}\right)$ subject to tangential boundary conditions on $\partial B_{R}(0)$ form a simple spiral, turning from the tangential pattern at the edge of the disk to radial near the defect at the center.

In this paper we follow an order parameter approach as in [13] for the energy (1.1). The unknown field $u(x)$ is taken to be a generalization of the $c$ director. In this case $u$ need not have unit length and vanishes at a defect where the smectic order is allowed to melt. This description, in contrast to the one above, does not presuppose the nature or location of individual defects. We can apply Theorem A to obtain information on minimizers for the problem of an island, $\Omega=B_{R}(0)$ with the tangential boundary values $g(x)= \pm \frac{x^{\perp}}{R}$ (where $\left.x^{\perp}=\left(-x_{2}, x_{1}\right) \equiv i x\right)$. If $k_{s}<k_{b}$ and $\varepsilon$ is taken sufficiently small it follows that a minimizer has one defect with degree +1 in $B_{R}(0)$, moreover the minimizer's pattern is near radial in a neighborhood of the defect. This is consistent with what was observed in the experiments and simulations from [6].

In [13], Silvestre et al. investigate a different aspect of the problem. They consider a free-standing $\mathrm{SmC}^{*}$ film occupying a simply connected region $D$ containing $d$ disjoint circular islands $\left\{\bar{B}_{R_{j}}\left(x_{j}\right) ; 1 \leqslant j \leqslant d\right\}$ and they investigate the texture in the background film $\Omega \equiv D \backslash \bigcup_{j=1}^{d} \bar{B}_{R_{j}}\left(x_{j}\right)$. In this case $\partial \Omega$ has $d+1$ components, $\partial D$ and $\partial B_{R_{j}}\left(x_{j}\right)$ for $1 \leqslant j \leqslant d$. On $\partial D$ they take $g=$ const. and $g= \pm \frac{\left(x-x_{j}\right)^{\perp}}{R_{j}}$ on $\partial B_{R_{j}}\left(x_{j}\right)$ for $1 \leqslant j \leqslant d$. It follows that $\operatorname{deg}(g, \partial \Omega)=-d$. The simulations and experiments in [13] exhibit $d$ topological defects in $\Omega$, each with degree -1 , as companions to the $d$ chiral islands. Our results do not directly apply to this setting. For the case of the classic Ginzburg-Landau energy (2.7), structure proved for the case $\operatorname{deg}(g, \partial \Omega)>0$ directly translates to the same information for the case $\operatorname{deg}(g, \partial \Omega)<0$. This is not obvious for the energy (1.1). Nevertheless the present analysis should be useful for investigating the case of boundary data with negative degree.

Our paper is organized as follows. In Section 2, we prove Theorem A, developing a number of qualitative features for the minimizers of $J_{\varepsilon}$. We express the integral in the form of (1.2)-(1.3). Written in this way one sees that minimizers form a family of low energy states for the Ginzburg-Landau functional (2.7), that is a family $\left\{u_{\varepsilon}\right\} \subset H_{g}^{1}(\Omega)$ satisfying (2.9) for a fixed constant $K$. It is proved in [4,5,14] that such a family has a number of structure and compactness properties, in particular it is shown that for a sequence $\varepsilon_{k} \rightarrow 0$, there exists a subsequence $\left\{u_{\varepsilon_{k(\ell)}}\right\}$ and a function $u_{*}$ such that $u_{\varepsilon_{k(\ell)}} \rightarrow u_{*}$. The analysis in [14] due to Fanghua Lin, contains the detailed description of $u_{*}$ that is needed here and we expand on this work in Proposition 2.3. We then use the fact that the $u_{\varepsilon}$ are minimizers to refine the notion of convergence away from the defects of $u_{*}$. Our work here builds on the investigation of minimizers for the Ginzburg-Landau energy (2.7) carried out by Brezis, Bethuel and Hélein in [1]. Their work however relies on a priori estimates for $\sup \left|u_{\varepsilon}\right|$, which they obtain by applying a maximum principle that is not available in the case at
hand. Here we proceed as in [15], obtaining bounds on the bulk term of the energy, giving us a priori $L^{4}$ estimates for the sequence of minimizers. A uniform bound on sup $\left|u_{\varepsilon}\right|$ follows from this and elliptic estimates. In Section 3 we analyze a class of polar functions for a particular set of points $\mathbf{b}=\left(b_{1}, \ldots, b_{d}\right)$ and show that the spherical average of these functions around $b_{n}$ tend to either 0 or $\frac{\pi}{2}, \bmod \pi$, depending on the relative values of $k_{s}$ and $k_{b}$. In Section 4 , we construct the renormalized energy in (1.5). We show that for each a, there exists a polar function $h_{\mathbf{a}}$ that minimizes the renormalized energy for a. In Section 5, we prove Theorem B.

## 2. Qualitative properties of minimizers

We begin by developing a number of structural properties for minimizers of (1.1) for each $\varepsilon$. As stated before, the minimizers in $H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ of (1.1) are also minimizers in $H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ of (1.3). In translating the variational problem into minimizing the energy (1.3) in $H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$, we have a strictly convex integrand (in the gradient), giving us the existence of a minimizer to our problem [7]. Then, taking the first variation of (1.3), we get that the minimizer $u_{\varepsilon}$ satisfies

$$
\begin{align*}
& \int_{\Omega}\left(\left(k_{s} u_{x_{1}}^{1}\right) v_{x_{1}}^{1}+\left(k_{s} u_{x_{2}}^{1}+\left(k_{b}-k_{s}\right)\left(u_{x_{2}}^{1}-u_{x_{1}}^{2}\right)\right) v_{x_{2}}^{1}+\left(k_{s} u_{x_{1}}^{2}+\left(k_{b}-k_{s}\right)\left(u_{x_{1}}^{2}-u_{x_{2}}^{1}\right)\right) v_{x_{1}}^{2}\right. \\
& \left.\quad+\left(k_{s} u_{x_{2}}^{2}\right) v_{x_{2}}^{2}+\frac{1}{\varepsilon^{2}}\left(u\left(1-|u|^{2}\right)\right) \cdot v\right) d x=0 \quad \text { if } \underline{k}=k_{s}, \\
& \int_{\Omega}\left(\left(k_{b} u_{x_{1}}^{1}+\left(k_{s}-k_{b}\right)\left(u_{x_{1}}^{1}+u_{x_{2}}^{2}\right)\right) v_{x_{1}}^{1}+\left(k_{b} u_{x_{2}}^{1}\right) v_{x_{2}}^{1}+\left(k_{b} u_{x_{1}}^{2}\right) v_{x_{1}}^{2}\right. \\
& \left.\quad+\left(k_{b} u_{x_{2}}^{2}+\left(k_{s}-k_{b}\right)\left(u_{x_{1}}^{1}+u_{x_{2}}^{2}\right)\right) v_{x_{2}}^{2}+\frac{1}{\varepsilon^{2}}\left(u\left(1-|u|^{2}\right)\right) \cdot v\right) d x=0 \quad \text { if } \underline{k}=k_{b} \tag{2.1}
\end{align*}
$$

for $v \in H_{0}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$. From regularity theory [16,17], we have $u_{\varepsilon} \in C^{\infty}(\Omega) \cap C^{2, \alpha}(\bar{\Omega})$ for each $0<\alpha<1$ and $\varepsilon>0$, since $\partial \Omega$ is $C^{3}, g \in C^{3}$, and the coefficients of the elliptic elastic energy term are constant. Thus we conclude that $\left|u_{\varepsilon}\right|_{2, \alpha ; \bar{\Omega}} \leqslant C(\varepsilon)$. Using integration by parts, we have that the Euler-Lagrange equation the minimizer satisfies is

$$
\begin{align*}
& -k_{s} \Delta u^{1}-\left(k_{b}-k_{s}\right)\left(u_{x_{2} x_{2}}^{1}-u_{x_{1} x_{2}}^{2}\right)=\frac{1}{\varepsilon^{2}} u^{1}\left(1-|u|^{2}\right), \\
& -k_{s} \Delta u^{2}-\left(k_{b}-k_{s}\right)\left(u_{x_{1} x_{1}}^{2}-u_{x_{2} x_{1}}^{1}\right)=\frac{1}{\varepsilon^{2}} u^{2}\left(1-|u|^{2}\right) \tag{2.2}
\end{align*}
$$

if $\underline{k}=k_{s}$, and

$$
\begin{align*}
& -k_{b} \Delta u^{1}-\left(k_{s}-k_{b}\right)\left(u_{x_{1} x_{1}}^{1}+u_{x_{2} x_{1}}^{2}\right)=\frac{1}{\varepsilon^{2}} u^{1}\left(1-|u|^{2}\right), \\
& -k_{b} \Delta u^{2}-\left(k_{s}-k_{b}\right)\left(u_{x_{2} x_{2}}^{2}+u_{x_{1} x_{2}}^{1}\right)=\frac{1}{\varepsilon^{2}} u^{2}\left(1-|u|^{2}\right) \tag{2.3}
\end{align*}
$$

if $\underline{k}=k_{b}$ in $\Omega$, with $u=g$ on $\partial \Omega$. For each $\varepsilon>0$, there is an upper and a lower bound that can be obtained for the integral (1.3).

## Proposition 2.1.

$$
\begin{equation*}
\bar{J}_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+C_{2}\left(\Omega, g, d, k_{s}, k_{b}\right) \tag{2.4}
\end{equation*}
$$

for a minimizer $u_{\varepsilon}$ and

$$
\begin{equation*}
\bar{J}_{\varepsilon}(u) \geqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)-C_{1}\left(\Omega, g, d, k_{s}, k_{b}\right) \tag{2.5}
\end{equation*}
$$

for any function $u \in H_{g}^{1}$ where $C_{1}$ and $C_{2}$ are positive constants.

Proof. For the upper estimate, this proof is similar to the proof of Lemma 2.1 in [3]. Consider the case when $\Omega=$ $B_{R}(0)=B_{R}$ and $g(x)=\beta \frac{x}{|x|}, \beta= \pm 1$ if $\underline{k}=k_{s}$ or $\beta= \pm i$ if $\underline{k}=k_{b}$. We drop the $\varepsilon$ for notational purposes. Denote the values

$$
\begin{equation*}
\bar{I}_{\beta}(\varepsilon, R)=\inf _{u \in H_{g}^{1}}\left\{\int_{B_{R}(0)} \bar{j}_{\varepsilon}(u, \nabla u) d x\right\} \tag{2.6}
\end{equation*}
$$

Let $\bar{I}_{\beta}(t)=\bar{I}_{\beta}(t, 1)$. By using a change of variables, $\bar{I}_{\beta}(\varepsilon, R)=\bar{I}_{\beta}\left(1, \frac{R}{\varepsilon}\right)=\bar{I}_{\beta}\left(\frac{\varepsilon}{R}\right)$. Then, using the same method as in the proof of Theorem 3.1 in [1], noting that $\operatorname{div}\left(\frac{i x}{|x|}\right)=\operatorname{curl}\left(\frac{x}{|x|}\right)=0$, we have $\bar{I}_{\beta}\left(t_{1}\right) \leqslant \underline{k} \pi \ln \left(\frac{t_{2}}{t_{1}}\right)+\bar{I}_{\beta}\left(t_{2}\right)$ for all $t_{1} \leqslant t_{2}$.

Fix $d$ points, $a_{1}, a_{2}, \ldots, a_{d}, a_{n} \neq a_{m}$ for $n \neq m$ in $\Omega$ and $R>0$ such that

$$
\overline{B_{R}\left(a_{n}\right)} \subset \Omega \quad \text { for each } n \quad \text { and } \quad \overline{B_{R}\left(a_{n}\right)} \cap \overline{B_{R}\left(a_{m}\right)}=\emptyset \quad \text { for every } n \neq m .
$$

Let $\Omega_{R}=\Omega \backslash \bigcup_{n=1}^{d} B_{R}\left(a_{n}\right)$ and consider $\bar{g}(x): \partial \Omega_{R} \rightarrow \mathbb{S}^{1}$ such that

$$
\bar{g}(x)= \begin{cases}g(x), & \text { if } x \in \partial \Omega, \\ e^{i \theta}, & \text { if } x=a_{j}+R e^{i \theta} \in \partial B_{R}\left(a_{j}\right) \text { for some } j \text { and } \underline{k}=k_{s}, \\ i e^{i \theta}, & \text { if } x=a_{j}+\operatorname{Re} e^{i \theta} \in \partial B_{R}\left(a_{j}\right) \text { for some } j \text { and } \underline{k}=k_{b} .\end{cases}
$$

Note that $\operatorname{deg}\left(\bar{g}, \partial \Omega_{R}\right)=0$, hence there exists a smooth function $v: \bar{\Omega}_{R} \rightarrow \mathbb{S}^{1}$ such that $\left.v\right|_{\partial \Omega_{R}}=\bar{g}$. Then by the above claim for $0<\varepsilon<R$,

$$
\begin{aligned}
\bar{J}_{\varepsilon}(u) & \leqslant \int_{\Omega_{R}} \bar{j}_{\varepsilon}(v, \nabla v) d x+\sum_{i=1}^{d} \bar{I}_{\beta}(\varepsilon, R) \\
& \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+C_{2}
\end{aligned}
$$

Define

$$
\begin{equation*}
F_{\varepsilon}(u)=F_{\varepsilon}(u ; \Omega)=\frac{1}{2} \int_{\Omega}\left(\underline{k}|\nabla u|^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x . \tag{2.7}
\end{equation*}
$$

Now, let $u \in H_{g}^{1}$ be any function. Note that $\bar{J}_{\varepsilon}(u) \geqslant F_{\varepsilon}(u)$. The lower bound (2.5) holds for a minimizer $v_{\varepsilon} \in H_{g}^{1}$ for $F_{\varepsilon}(\cdot)$ and is proved in [1] for the case that $\Omega$ is star shaped. The general case follows from [3] and [1]. (See [4] and [5] for alternative proofs.) It follows that the lower bound holds for any $u \in H_{g}^{1}$.

The following corollary is a direct result of the previous proposition, utilizing a method described in [5].
Corollary 2.1. If $u$ is a minimizer for $\bar{J}_{\varepsilon}(u)$, then there exists $C_{4}=C_{4}\left(g, \Omega, k_{s}, k_{b}, d\right)$ such that

$$
\begin{align*}
& \int_{\Omega}\left(\left(k_{b}-k_{s}\right)(\operatorname{curl} u)^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x \leqslant C_{4} \quad \text { if } \underline{k}=k_{s}, \\
& \int_{\Omega}\left(\left(k_{s}-k_{b}\right)(\operatorname{div} u)^{2}+\frac{1}{2 \varepsilon^{2}}\left(1-|u|^{2}\right)^{2}\right) d x \leqslant C_{4} \quad \text { if } \underline{k}=k_{b} . \tag{2.8}
\end{align*}
$$

Proof. For clarity, we provide a short proof. From Proposition 2.1, we have that for any minimizer $u_{\varepsilon} \in H_{g}^{1}, \bar{J}_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant$ $k \pi d \ln \left(\varepsilon^{-1}\right)+C_{2}$. Let $F_{2 \varepsilon}(u)$ be as defined in the proof of Proposition 2.1. Then $F_{2 \varepsilon}\left(u_{\varepsilon}\right) \geqslant \underline{k} \pi d \ln \left(\varepsilon^{-1}\right)-C_{3}$. Thus (2.8) follows from considering the expression $\bar{J}_{\varepsilon}\left(u_{\varepsilon}\right)-F_{2 \varepsilon}\left(u_{\varepsilon}\right)$.

Using the estimate from the corollary and the strong ellipticity of the system in (2.2) and (2.3), we obtain several bounds for minimizers over the entire domain for small $\varepsilon$. Let $x \in \bar{\Omega}$ and set

$$
\tilde{\Omega}=\{y: \varepsilon y+x \in \Omega\} \quad \text { and } \quad \tilde{u}(y)=u_{\varepsilon}(\varepsilon y+x) \quad \text { for } y \in \tilde{\Omega} .
$$

Then from (2.8) we have that $\|\tilde{u}\|_{4 ; \tilde{\Omega} \cap B_{1}(0)} \leqslant C$ where $C$ is independent of $x \in \bar{\Omega}$ and $0<\varepsilon \leqslant 1$. If we express (2.2) and (2.3) as $\mathscr{L}_{\underline{k}} u=\varepsilon^{-2} \mathbf{f}(u)$, where $\mathbf{f}(u)=u\left(1-|u|^{2}\right)$, then $\mathscr{L}_{\underline{k}}$ is a second order strongly elliptic operator with constant coefficients. Moreover we have that $\mathscr{L}_{\underline{k}} \tilde{u}=\mathbf{f}(\tilde{u})$ on $\tilde{\Omega} \cap B_{1}(0)$. Based on the $L^{4}$ a priori estimate, the ellipticity of the operator $\mathscr{L}_{\underline{k}}$, and the smoothness of both the boundary data and $\partial \Omega$, the proof of the next proposition follows just as the proof for Lemma 3.1 in [15].

Proposition 2.2. There exists a constant $C_{5}=C_{5}\left(g, \Omega, k_{s}, k_{b}, d\right)$ so that if $u_{\varepsilon}$ is a minimizer to $\bar{J}_{\varepsilon}(u)$, then

$$
\left|u_{\varepsilon}\right|, \varepsilon\left|\nabla u_{\varepsilon}\right| \leqslant C_{5} \quad \text { in } \Omega, \text { for } 0<\varepsilon<1
$$

From Proposition 2.1 and the definition of $\bar{j}_{\varepsilon}(u, \nabla u)$, we get that

$$
\begin{equation*}
F_{\varepsilon}\left(u_{\varepsilon}\right) \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+K \tag{2.9}
\end{equation*}
$$

for every $0<\varepsilon<1$ and minimizer $u_{\varepsilon}$ to $\bar{J}_{\varepsilon}(\cdot)$. With this estimate we are able to apply a structure and compactness theorem to the sequence of minimizers for $\bar{J}_{\varepsilon}(u)$.

Proposition 2.3. There exist constants $\delta>0$ and $C$ depending on $K, \underline{k}, g$, and $\Omega$ so that for any sequence of functions $u_{\varepsilon} \in H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ with $\varepsilon \downarrow 0$ satisfying (2.9) there exists a subsequence $\left\{u_{\varepsilon_{\ell}}\right\}$, points $\left\{a_{1}, \ldots, a_{d}\right\} \subset \Omega$, and a function $h(x) \in H^{1}(\Omega)$ so that

$$
\min \left\{\operatorname{dist}\left(a_{m}, \partial \Omega\right),\left|a_{m}-a_{n}\right|, m \neq n, 1 \leqslant m, n \leqslant d\right\} \geqslant \delta, \quad\|h\|_{H^{1}} \leqslant C
$$

and

$$
u_{\varepsilon_{\ell}} \rightarrow \prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{i(h(x)+\zeta(x))}
$$

where the convergence of $\left\{u_{\varepsilon_{\ell}}\right\}$ is weak in $H_{\text {loc }}^{1}\left(\bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\} ; \mathbb{C}\right)$ and strong in $L^{2}(\Omega ; \mathbb{C})$.

Proposition 2.3 is due to Fanghua $\operatorname{Lin}[18,14,19,20]$ if $\Omega$ is simply connected. In this case the limit takes the form $\prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{i h(x)}$. We modify his arguments below to prove it for general $\Omega$. We first need a lemma.

Lemma 2.1. There is a constant $K^{\prime}$, depending only on $K, \underline{k}, g$, and $\Omega$ so that if $u$ satisfies (2.9) then $w^{*}(x) u(x)=$ $e^{-i \zeta(x)} u(x)$ satisfies

$$
F_{\varepsilon}\left(w^{*} u\right) \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+K^{\prime}
$$

Proof. For $a \in \mathbb{C}$ we denote $a^{*}$ as the complex conjugate of $a$. Writing $w u=e^{i \zeta} u$. We have

$$
F_{\varepsilon}(w u)=F_{\varepsilon}(u)+\underline{k} \int_{\Omega} \mathscr{I} m\left\{u^{*} \nabla \zeta \cdot \nabla u\right\} d x+\frac{k}{2} \int_{\Omega}|\nabla \zeta|^{2}|u|^{2} d x
$$

Since $\operatorname{deg}(w u ; \partial \Omega)=d$ we have the lower bound

$$
F_{\varepsilon}(w u) \geqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)-C
$$

where $C$ depends on $\left.w u\right|_{\partial \Omega}=w g$. Second, since $\zeta$ is smooth and fixed in $\bar{\Omega}$ we have that

$$
\int_{\Omega}|\nabla \zeta|^{2}|u|^{2} d x \leqslant M \int_{\Omega}|u|^{2} d x \leqslant M_{1}\left(\varepsilon^{2} \ln \left(\frac{1}{\varepsilon}\right)+1\right)
$$

With these two estimates, together with our hypothesis on $F_{\varepsilon}(u)$ we get

$$
-\int_{\Omega} \mathscr{I} m\left\{u^{*} \nabla \zeta \cdot \nabla u\right\} d x \leqslant C
$$

where $C$ depends only on $K, \underline{k}, g$, and $\Omega$. Finally we can expand and estimate

$$
\begin{aligned}
F_{\varepsilon}\left(w^{*} u\right) & =F_{\varepsilon}(u)-\underline{k} \int_{\Omega} \mathscr{I} m\left\{u^{*} \nabla \zeta \cdot \nabla u\right\} d x+\frac{k}{2} \int_{\Omega}|\nabla \zeta|^{2}|u|^{2} d x \\
& \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+K^{\prime} .
\end{aligned}
$$

Proof of Proposition 2.3. Let $u_{\varepsilon} \in H_{g}^{1}\left(\Omega ; \mathbb{R}^{2}\right)$ with $\varepsilon \downarrow 0$ satisfying (2.9). Set $z_{\varepsilon}=w^{*} u_{\varepsilon}$. Then the winding number for $\left.z_{\varepsilon}\right|_{\Gamma_{j}}$ is 0 for $1 \leqslant j \leqslant k$ and we can extend $z_{\varepsilon}$ onto $\Lambda_{j}$ as a function in $H^{1}\left(\Lambda_{j} ; \mathbb{S}^{1}\right)$ that is independent of $\varepsilon$ for each $j$. We also have that the winding number for $\left.z_{\varepsilon}\right|_{\Gamma_{0}}$ is $d$. Thus setting $\left.z_{\varepsilon}\right|_{\Gamma_{0}}=\tilde{g}$, we have that the sequence $\left\{z_{\varepsilon}\right\} \subset H_{\tilde{g}}^{1}(D ; \mathbb{C})$ and that

$$
F_{\varepsilon}\left(z_{\varepsilon} ; D\right) \leqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+K^{\prime \prime}
$$

where $K^{\prime \prime}$ is independent of $\varepsilon$. We can apply the proposition for the simply connected domain $D$; see [14]. We find a subsequence $\left\{z_{\varepsilon_{\ell}}\right\}$, points $\left\{a_{1}, \ldots, a_{d}\right\} \subset D$, and $h \in H^{1}(D)$ so that

$$
z_{\varepsilon_{\ell}} \rightarrow z_{*}(x)=\prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{i h(x)} \quad \text { in } D .
$$

Thus since $u_{\varepsilon_{\ell}}=w z_{\varepsilon_{\ell}}$ we have that

$$
u_{\varepsilon_{\ell}} \rightarrow \prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{i(h(x)+\zeta(x))} \quad \text { in } \Omega .
$$

We know that the $\left\{a_{1}, \ldots, a_{d}\right\}$ are uniformly bounded away from each other and $\Gamma_{0}$. It remains to show that they are bounded away from $\bigcup_{j=1}^{k} \bar{\Lambda}_{j}$. If this is not so, then we can find a case with $a_{\ell} \in \bar{\Lambda}_{j}$ for some $\ell$ and $j$. We choose $r>0$ sufficiently small so that $\bar{B}_{r}\left(a_{\ell}\right) \cap\left\{a_{n}: n \neq \ell\right\}=\emptyset$. By construction $z_{\varepsilon_{\ell}}(x)$ for $x \in \Lambda_{j}$ are independent of $\varepsilon_{\ell}$ and in $H^{1}\left(\Lambda_{j}\right)$. Thus $z_{*} \in H^{1}\left(\Lambda_{j}\right)$. On the other hand we have

$$
z_{*}(x)=\frac{x-a_{\ell}}{\left|x-a_{\ell}\right|} \tilde{z}(x) \quad \text { such that } \quad|\tilde{z}(x)|=1 \quad \text { for } x \in B_{r}\left(a_{\ell}\right) . \quad \text { Moreover } \tilde{z} \in H^{1}\left(B_{r}\left(a_{\ell}\right)\right) .
$$

Thus $z_{*} \notin H^{1}\left(\Lambda_{j}\right)$ and this is a contradiction.
For $\rho>0$ set $\Omega_{\rho}=\Omega \backslash \bigcup_{m=1}^{d} \bar{B}_{\rho}\left(a_{m}\right)$.
Proposition 2.4. Let $\left\{u_{\varepsilon \ell}\right\}$ be a sequence of minimizers converging to $u_{*}(x)=\prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{i(h(x)+\zeta(x))}$ in $L^{2}(\Omega)$. Then for each $\rho>0$ the convergence is in $H^{1}\left(\Omega_{\rho}\right)$ and $\varepsilon_{\ell}^{-2} \int_{\Omega_{\rho}}\left(1-\left|u_{\varepsilon_{\ell}}\right|^{2}\right)^{2} d x \rightarrow 0$ as $\ell \rightarrow \infty$. Moreover, $u_{*}$ is a local minimizer for the limiting energy in $H^{1}\left(\Omega_{\rho}, \mathbb{S}^{1}\right)$.

Proof. The proof is similar to the proof of Lemma 3.9 in [15]. For notational purposes we write $\left\{u_{\ell}\right\}=\left\{u_{\varepsilon_{\ell}}\right\}$, where $\varepsilon_{\ell}$ is a subsequence of $\varepsilon$ and $\varepsilon \rightarrow 0$. By Proposition 2.3, $u_{\ell} \rightharpoonup u_{*}$ in $H^{1}\left(\Omega_{\rho}\right)$ for every $\rho>0$ and $u_{\ell}$ is a local minimizer for $\int_{\Omega_{\rho}} \bar{j}_{\varepsilon}(u, \nabla u) d x$. As in the proof of Lemma 3.9 of [15], choose $\bar{x} \in \bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}$, assuming first that $\bar{x} \notin \partial \Omega$ and let $\bar{d}=\bar{d}(\bar{x})$ be such that $\bar{B}_{2 \bar{d}}=\bar{B}_{2 \bar{d}}(\bar{x}) \subset \Omega \backslash\left\{a_{1}, \ldots, a_{d}\right\}$. Set $\omega(x)=h(x)+\zeta(x)+\sum_{n=1}^{d} \theta_{a_{n}}(x)$ where $\frac{x-a_{m}}{\left|x-a_{m}\right|}=e^{i \theta_{a_{m}}(x)}$. Then with out loss of generality $\omega$ is single valued in $B_{2 \bar{d}}$. Furthermore $\omega \in H^{1}\left(B_{2 \bar{d}}\right)$ by

Proposition 2.3 and $u_{*}(x)=e^{i \omega(x)}$ on $\bar{B}_{2 \bar{d}}$. From Corollary 2.1, Proposition 2.3, and as in the proof of Lemma 3.9 of [15], for a subsequence $\left\{u_{\ell^{\prime}}\right\}$ that we do not relabel, there exists a radius $d$ such that $\bar{d} \leqslant d \leqslant 2 \bar{d}$ and for which

$$
\begin{equation*}
\frac{1}{2} \int_{\partial B_{d}}\left(\underline{k}\left|\partial_{\tau} u_{\ell}\right|^{2}+\frac{1}{2 \varepsilon_{\ell}^{2}}\left(1-\left|u_{\ell}\right|^{2}\right)^{2}\right) d \sigma \leqslant C_{1} . \tag{2.10}
\end{equation*}
$$

It follows that $\left\{u_{\ell}\right\}$ converges to $u_{*}$ uniformly on $\partial B_{d}$ and weakly in $H^{1}\left(\partial B_{d}\right)$. Note that $\left.\operatorname{deg} u_{*}\right|_{\partial B_{d}}=0$. Therefore $\left.\operatorname{deg} u_{\ell}\right|_{\partial B_{d}}=0$ for sufficiently large $\ell$. Thus $u_{\ell}(x)$ can be expressed as $u_{\ell}(x)=\left|u_{\ell}(x)\right| e^{i \omega_{\ell}(x)}$ such that $\left|u_{\ell}(x)\right| \neq 0$ for every $x \in \partial B_{d}$ where $\omega_{\ell}$ converges to $\omega$ uniformly on $\partial B_{d}$ and weakly in $H^{1}\left(\partial B_{d}\right)$ as well.

Define, for each $\ell$, the function $\Phi_{\ell}(r, \theta):=\phi_{\ell}(r)\left(\omega_{\ell}(\theta)-\omega(d, \theta)\right)$, where $d \geqslant r=|x-\bar{x}|$,

$$
\phi_{\ell}(r)= \begin{cases}1 & \text { if } r=d, \\ 0 & \text { if } r<r_{\ell}<d,\end{cases}
$$

$\phi_{\ell}$ is a smooth cut-off function and $\left|\nabla \phi_{\ell}\right| \leqslant \frac{1}{d-r_{\ell}}$, where $r_{\ell}$ is a sequence of radii such that $r_{\ell} \rightarrow d$ so that

$$
\int_{\partial B_{d}} \frac{\left|\omega_{\ell}-\omega\right|^{2}}{\left(r_{\ell}-d\right)} d \sigma \rightarrow 0
$$

By construction and the fact that $\omega_{\ell} \rightharpoonup \omega$ in $H^{1}\left(\partial B_{d}\right)$, we get that $\Phi_{\ell} \rightarrow 0$ in $H^{1}\left(B_{d}\right)$ and $\Phi_{\ell} \rightarrow 0$ uniformly in $B_{d}$. Let $\bar{u}$ minimize $\int_{B_{d}} \bar{j}(u, \nabla u) d x$ in the set $\left\{u \in H^{1}\left(B_{d} ; \mathbb{S}^{1}\right): u=u_{*}\right.$ on $\left.\partial B_{d}\right\}$. Consider the function $e^{i \Phi_{\ell}} \bar{u}$. Then we have $e^{i \Phi_{\ell}} \bar{u} \rightarrow \bar{u}$ in $H^{1}\left(B_{d}\right)$, and uniformly in $\bar{B}_{d}$. Now, we construct comparison functions

$$
\hat{u}_{\ell}=\left|\hat{u}_{\ell}\right| e^{i \Phi_{\ell}} \bar{u} \quad \text { on } B_{d}
$$

where

$$
\begin{aligned}
& \left|\hat{u}_{\ell}\right|=1 \quad \text { on } B_{d-\varepsilon_{\ell}}, \\
& \left|\hat{u}_{\ell}\right|=\left|u_{\ell}\right| \quad \text { on } \partial B_{d},
\end{aligned}
$$

and for each $\theta$, define $\left|\hat{u}_{\ell}\right|(|x-\bar{x}|, \theta)$ to be linear for $d-\varepsilon_{\ell} \leqslant|x-\bar{x}| \leqslant d$. This gives that $\hat{u}_{\ell}=u_{\ell}$ on $\partial B_{d}$. By construction, we get that $\hat{u}_{\ell} \rightarrow \bar{u}$ uniformly in $\overline{B_{d}}, \hat{u}_{\ell} \rightarrow \bar{u}$ in $H^{1}\left(B_{d}\right)$, and $\varepsilon_{\ell}^{-2} \int_{B_{d}}\left(1-\left|\hat{u}_{\ell}\right|^{2}\right)^{2} d x \rightarrow 0$ as $\ell \rightarrow \infty$. These limits imply that

$$
\lim _{\ell \rightarrow 0} \int_{B_{d}} \bar{j}_{\varepsilon_{\ell}}\left(\hat{u}_{\ell}, \nabla \hat{u}_{\ell}\right) d x=\int_{B_{d}} \bar{j}(\bar{u}, \nabla \bar{u}) d x:=\bar{J}(\bar{u})
$$

where

$$
\bar{j}(u, \nabla u)= \begin{cases}k_{s}|\nabla u|^{2}+\left(k_{b}-k_{s}\right)(\operatorname{curl} u)^{2} & \text { if } \underline{k}=k_{s}, \\ k_{b}|\nabla u|^{2}+\left(k_{s}-k_{b}\right)(\operatorname{div} u)^{2} & \text { if } \underline{k}=k_{b}\end{cases}
$$

This notion for $\bar{j}(u, \nabla u)$ will be used throughout the rest of the work. Then by the lower semicontinuity of the integral $\int_{\Omega} \bar{j}(u, \nabla u) d x$,

$$
\begin{aligned}
\int_{B_{d}} \bar{j}\left(u_{*}, \nabla u_{*}\right) d x & \leqslant \liminf _{\ell \rightarrow \infty} \int_{B_{d}} \bar{j}_{\varepsilon_{\ell}}\left(u_{\ell}, \nabla u_{\ell}\right) d x \leqslant \limsup _{\ell \rightarrow \infty} \int_{B_{d}} \bar{j}_{\varepsilon_{\ell}}\left(u_{\ell}, \nabla u_{\ell}\right) d x \\
& \leqslant \lim _{\ell \rightarrow \infty} \int_{B_{d}} \bar{j}_{\varepsilon_{\ell}}\left(\hat{u}_{\ell}, \nabla \hat{u}_{\ell}\right) d x=\int_{B_{d}} \bar{j}(\bar{u}, \nabla \bar{u}) d x \\
& \leqslant \int_{B_{d}} \bar{j}\left(u_{*}, \nabla u_{*}\right) d x .
\end{aligned}
$$

This implies that

$$
\lim _{\ell \rightarrow \infty} \int_{B_{d}} \bar{j}_{\varepsilon_{\ell}}\left(u_{\ell}, \nabla u_{\ell}\right) d x=\int_{B_{d}} \bar{j}\left(u_{*}, \nabla u_{*}\right) d x .
$$

Since $u_{\ell} \rightharpoonup u_{*}$ in $H^{1}\left(B_{d} ; \mathbb{C}\right)$, then by [21], $u_{\ell} \rightarrow u_{*}$ in $H^{1}\left(B_{d} ; \mathbb{C}\right)$. A further consequence of the convergence of the integrals and the strong convergence in $H^{1}\left(B_{d} ; \mathbb{C}\right)$ is

$$
\lim _{\ell \rightarrow \infty} \varepsilon_{\ell}^{-2} \int_{B_{d}}\left(1-\left|u_{\ell}\right|^{2}\right)^{2} d x=0
$$

We have also showed that $u_{*}$ is a minimizer to $\bar{J}$ in $H_{u_{*}}^{1}\left(B_{d} ; \mathbb{S}^{1}\right)$. We have proved our assertions for a subsequence of the original sequence on $B_{d} \supset B_{\bar{d}}$, for some radius $d$. It follows that we have established the assertions for the full sequence on $B_{d}$.

Finally, suppose $\bar{x} \in \partial \Omega$. Then let $U_{\bar{x}}$ be a neighborhood of $\bar{x}$ such that there exists a smooth diffeomorphism $\psi(x)$ defined on $B_{2 \bar{d}}$ such that $\psi(\bar{x})=0$ and

$$
\psi: U_{\bar{x}} \rightarrow B_{2 \bar{d}}^{+}=\left\{(x, y): x^{2}+y^{2}<(2 \bar{d})^{2}, y>0\right\} .
$$

Then we can carry out the same argument in $B_{2 \bar{d}}^{+}$, with $\psi(\bar{x})$, push back into $U_{\bar{x}}$, and then argue as in the previous case.

The next proposition shows that the norms of the minimizers converge uniformly to 1 outside of any positive radius distance away from the vortices. The proof is similar to the proof of Lemma 3.10 in [15].

Proposition 2.5. Let $\left\{u_{\ell}\right\}$ be a sequence of minimizers with $\varepsilon_{\ell} \rightarrow 0$, converging to $u_{*}(x)=\prod_{m=1}^{d} \frac{x-a_{m}}{\left|x-a_{m}\right|} e^{(i h(x)+\zeta(x))}$ in $L^{2}(\Omega)$. Then for each $\rho>0,\left|u_{\ell}\right| \rightarrow 1$ uniformly in $\bar{\Omega}_{\rho}$.

Proof. Fix $\rho>0$ and assume that there exists a $\delta>0$, a subsequence $\left\{u_{\ell}\right\}$ (that we do not relabel), and a sequence of points $\left\{x_{\ell}\right\} \subset \bar{\Omega}_{\rho}$ so that $\left|1-\left|u_{\ell}\left(x_{\ell}\right)\right|\right| \geqslant \delta$. From Proposition 2.2 there is a $c(\delta)>0$ so that $\left|1-\left|u_{\ell}(x)\right|\right| \geqslant \frac{\delta}{2}$ for $x \in B_{c \varepsilon_{\ell}}\left(x_{\ell}\right) \cap \Omega$. It follows that

$$
\varepsilon_{\ell}^{-2} \int_{\Omega_{\rho}}\left(1-\left|u_{\varepsilon_{\ell}}\right|^{2}\right)^{2} d x \geqslant \varepsilon_{\ell}^{-2} \int_{B_{c \varepsilon_{\ell}}\left(x_{\ell}\right) \cap \Omega_{\rho}}\left(1-\left|u_{\varepsilon_{\ell}}\right|^{2}\right)^{2} d x \geqslant \bar{C}>0
$$

where $\bar{C}$ is independent of $\ell$. We have seen from Proposition 2.4 however that the left side tends to 0 as $\ell \rightarrow \infty$ and this leads to a contradiction.

The next two propositions prove higher regularity on the sequence of minimizers on compact subsets of the domain away from the vortices. For the sequence of minimizers $\left\{u_{\ell}\right\}$ that converges in $H_{l o c}^{1}\left(\bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}\right)$, we have that $\left|u_{\ell}\right|$ converges uniformly to 1 on every $K \Subset \bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}$. The bulk term of the energy has a non-degenerate minimum when $|u|=1$. The elastic term of the energy is strongly elliptic, as well as quadratic in the gradient term. Due to these facts, the proofs of the following propositions follow from the proofs of Lemma 3.11 and Lemma 3.12 from [15] respectively.

Proposition 2.6. Let $\left\{u_{\left.\varepsilon_{\ell}\right\}}\right\}$ be a sequence of minimizers for (1.1) in $H_{g}^{1}(\Omega)$ converging in $H_{l o c}^{1}\left(\bar{\Omega} \backslash\left\{a_{1}, a_{2}, \ldots, a_{d}\right\}\right)$ as $\varepsilon_{\ell} \rightarrow 0$. Then, for $K \Subset \bar{\Omega} \backslash\left\{a_{1}, a_{2}, \ldots, a_{d}\right\}$, there exist constants $\ell_{0}$ and $C$ such that if $\ell \geqslant \ell_{0}$, then

$$
\left\|D^{2} u_{\varepsilon_{\ell}}\right\|_{2 ; K} \leqslant C
$$

Proposition 2.7. Let $u_{\varepsilon_{\ell}}$ be the sequence of minimizers as in the previous lemma. For each $n>2$ and set $K \Subset$ $\Omega \backslash\left\{a_{1}, a_{2}, \ldots, a_{d}\right\}$, there are constants $C$ and $\ell_{0}$ such that

$$
\left\|u_{\varepsilon_{\ell}}\right\|_{n, 2 ; K} \leqslant C \quad \text { for } \ell \geqslant \ell_{0} .
$$

A consequence of Propositions 2.6 and 2.7 is that for each compact subset $K$ of $\Omega \backslash\left\{a_{1}, \ldots, a_{d}\right\}$ and for each $n \geqslant 2$, we have the entire converging subsequence of minimizers $\left\{u_{\varepsilon_{\ell}}\right\} \in H^{n}(K)$ and bounded. This implies that $u_{*} \in H^{n}(K)$. Now, by Sobolev's Theorem and Arzelà-Ascoli Theorem, we have the following corollary.

Corollary 2.2. Let $\left\{u_{\varepsilon_{\ell}}\right\}$ be a sequence of minimizers converging to $u_{*}$. Then for any $0<\alpha<1$, and each integer $m$

$$
u_{\varepsilon_{\ell}} \rightarrow u_{*} \quad \text { in } C_{l o c}^{\alpha}\left(\bar{\Omega} \backslash\left\{a_{1}, \ldots, a_{d}\right\}\right) \text { and } C_{l o c}^{m}\left(\Omega \backslash\left\{a_{1}, \ldots, a_{d}\right\}\right)
$$

as $\ell \rightarrow \infty$.
Proof of Theorem A. Let $\left\{u_{\varepsilon}\right\}$ be a sequence of minimizers to (1.1) such that $\varepsilon \downarrow 0$. Then we know it is also a minimizer of (1.3) for each $\varepsilon$. By applying Proposition 2.3, it follows that there exists a subsequence $\left\{u_{\varepsilon_{\ell}}\right\}$, a function $h$, and points $\left\{a_{1}, \ldots, a_{d}\right\}$ such that

$$
u_{\varepsilon_{\ell}}(x) \rightarrow \prod_{n=1}^{d} \frac{x-a_{n}}{\left|x-a_{n}\right|} e^{i(h(x)+\zeta(x))}=u_{*}(x) \quad \text { in } H_{l o c}^{1}\left(\bar{\Omega} \backslash\left\{a_{1}, a_{2}, \ldots, a_{d}\right\}\right)
$$

For each $\rho>0$ it follows from Corollary 2.2 that we have $u_{\varepsilon_{\ell}} \rightarrow u_{*}$ in $C^{\alpha}\left(\bar{\Omega}_{\rho}\right)$ for every $0<\alpha<1$, and $u_{\varepsilon_{\ell}} \rightarrow u_{*}$ in $C^{m}\left(\bar{\Omega}_{\rho}\right)$ for every integer $m \geqslant 0$, where $\Omega_{\rho}=\Omega \backslash \bigcup_{n=1}^{d} \bar{B}_{\rho}\left(a_{n}\right)$.

Remark 1. To be definite we point out that Theorem A applies to the case $k_{s}=k_{b}>0$, and that in this case our arguments are extensions of those from [1,3,14] that allow us to treat the case of a multiply connected domain. Moreover, in this case the nature of $h(x)$ is distinct. Indeed from Proposition 2.4 we have that $h$ is harmonic in $\Omega \backslash\left\{a_{1}, \ldots, a_{d}\right\}$ and $h \in H^{1}(\Omega)$. As such $h$ is harmonic in $\Omega$. (This observation originates in [1].)

In the next section we show that in contrast with the case $k_{s}=k_{b}$, the values of $h$ are pinned at $\left\{a_{1}, \ldots, a_{d}\right\}$ if $k_{s} \neq k_{b}$.

## 3. Class of functions for each configuration of points

Assume that $k_{s} \neq k_{b}$. Define the set

$$
T=\left\{\mathbf{b}=\left(b_{1}, \ldots, b_{d}\right) \in \Omega^{d}: b_{n} \neq b_{m} \text { for } n \neq m\right\} .
$$

Fix a configuration $\mathbf{b} \in T$. We can choose $f \in C^{3}(\partial \Omega)$ so that

$$
g(x)=e^{i\left(f(x)+\sum_{n=1}^{d} \theta_{b_{n}}(x)+\zeta(x)\right)} \quad \text { for } x \in \partial \Omega
$$

where $\theta_{b_{n}}$ is such that $\frac{x-b_{n}}{\left|x-b_{n}\right|}=e^{i \theta_{b_{n}}(x)}$ for $x \neq b_{n}$. Note that $\left.f\right|_{\Gamma_{\ell}}$ is uniquely determined, $\bmod 2 \pi$, for each component $\Gamma_{\ell}$ of $\partial \Omega$. Let $\phi \in H^{1}(\Omega)$ and set

$$
v(x)=v(\mathbf{b}, \phi)(x)=e^{i\left(\phi(x)+\sum_{n=1}^{d} \theta_{b_{n}}(x)+\zeta(x)\right)} .
$$

We define the set

$$
A(\mathbf{b})= \begin{cases}\left\{\phi \in H^{1}(\Omega): v(\mathbf{b}, \phi)=g \text { on } \partial \Omega \text { and } \int_{\Omega}(\operatorname{curl} v)^{2} d x<\infty\right\} & \text { if } k_{s}<k_{b}, \\ \left\{\phi \in H^{1}(\Omega): v(\mathbf{b}, \phi)=g \text { on } \partial \Omega \text { and } \int_{\Omega}(\operatorname{div} v)^{2} d x<\infty\right\} & \text { if } k_{b}<k_{s} .\end{cases}
$$

It follows from Lemma 1.1 of [22] that $\phi=f+2 \pi t_{\ell}$ on each component $\Gamma_{\ell}$ of $\partial \Omega$, for some $t_{\ell} \in \mathbb{Z}$, for each $\phi \in A(\mathbf{b})$. We prove in Proposition 4.1 that $A(\mathbf{b})$ is nonempty. Let $\rho>0$ be such that $B_{\rho}\left(b_{n}\right) \subset \Omega$ and such that $\left\{\bar{B}_{\rho}\left(b_{n}\right)\right\}$ are pair-wise disjoint. For $h \in A(\mathbf{b})$ define the function $h_{n}(x)=h(x)+\sum_{m \neq n} \theta_{b_{m}}(x)+\zeta(x)$ for $x \in B_{\rho}\left(b_{n}\right)$. Set $v=v(\mathbf{b}, h)$. From the definition of $A(\mathbf{b})$, we have

$$
\begin{aligned}
\int_{B_{\rho}\left(b_{n}\right)}(\operatorname{curl} v)^{2} d x \leqslant C \quad \text { if } k_{s}<k_{b}, \\
\int_{B_{\rho}\left(b_{n}\right)}(\operatorname{div} v)^{2} d x \leqslant C \quad \text { if } k_{b}<k_{s},
\end{aligned}
$$

for each $1 \leqslant n \leqslant d$ such that $C=C(v)<\infty$. On $B_{\rho}\left(b_{n}\right)$ we have that $v=\frac{x-b_{n}}{\left|x-b_{n}\right|} e^{i h_{n}}=\cos \left(h_{n}\right) \frac{x-b_{n}}{\left|x-b_{n}\right|}+$ $\sin \left(h_{n}\right) i \frac{x-b_{n}}{\left|x-b_{n}\right|}$. Since $h_{n} \in H^{1}\left(B_{\rho}\left(b_{n}\right)\right)$, curl $\frac{x-b_{n}}{\left|x-b_{n}\right|}=\operatorname{div} i \frac{x-b_{n}}{\left|x-b_{n}\right|}=0$, and $\operatorname{div} \frac{x-b_{n}}{\left|x-b_{n}\right|}=\operatorname{curl} i \frac{x-b_{n}}{\left|x-b_{n}\right|}=\frac{1}{\left|x-b_{n}\right|}$, we obtain using Young's Inequality that

$$
\begin{align*}
& \int_{B_{\rho}\left(b_{n}\right)} \frac{\sin ^{2}\left(h_{n}\right)}{\left|x-b_{n}\right|^{2}} d x \leqslant C \quad \text { if } k_{s}<k_{b}, \\
& \int_{B_{\rho}\left(b_{n}\right)} \frac{\cos ^{2}\left(h_{n}\right)}{\left|x-b_{n}\right|^{2}} d x \leqslant C \quad \text { if } k_{b}<k_{s} . \tag{3.1}
\end{align*}
$$

From (3.1), we obtain the following proposition.
Proposition 3.1. Let $\mathbf{b} \in T$ and $h \in A(\mathbf{b})$. Then we have

$$
\frac{1}{\left|\partial B_{\rho}\left(b_{n}\right)\right|} \int_{\partial B_{\rho}\left(b_{n}\right)} h d \sigma \rightarrow \alpha_{n}-\sum_{m \neq n} \theta_{b_{m}}\left(b_{n}\right)-\zeta\left(b_{n}\right) \quad \text { as } \rho \rightarrow 0,
$$

for each $1 \leqslant n \leqslant d$ where

$$
\alpha_{n}= \begin{cases}c_{n} \pi & \text { for some } c_{n} \in \mathbb{Z} \text { if } k_{s}<k_{b} \\ \frac{\left(2 c_{n}+1\right) \pi}{2} & \text { for some } c_{n} \in \mathbb{Z} \text { if } k_{s}<k_{b}\end{cases}
$$

Proof. Let $h_{n}(x)=h(x)+\sum_{m \neq n} \theta_{b_{m}}(x)+\zeta(x)$ and

$$
\bar{h}_{n}(\rho)=\frac{1}{\left|\partial B_{\rho}\left(b_{n}\right)\right|} \int_{\partial B_{\rho}\left(b_{n}\right)} h_{n} d \sigma
$$

Define the functions $\omega^{n}(x)=\sin \left(h_{n}(x)\right), \bar{\omega}^{n}(\rho)=\sin \left(\bar{h}_{n}(\rho)\right)$ if $k_{s}<k_{b}$, and $\omega^{n}(x)=\cos \left(h_{n}(x)\right), \bar{\omega}^{n}(\rho)=$ $\cos \left(\bar{h}_{n}(\rho)\right)$ if $k_{b}<k_{s}$. Set $\omega_{\rho}^{n}(y)=\omega^{n}\left(\rho y+b_{n}\right)$ for $0<\rho<\rho_{0}$ where $\rho_{0}$ is such that $B_{2 \rho_{0}}\left(b_{n}\right) \Subset \Omega \backslash \bigcup_{m \neq n} B_{2 \rho_{0}}\left(b_{m}\right)$. With $h_{n} \in H^{1}\left(B_{\rho_{0}}\left(b_{n}\right)\right)$ and (3.1) we get

$$
\begin{equation*}
\lim _{\rho \rightarrow 0}\left\|\omega_{\rho}^{n}\right\|_{1,2 ; B_{1}}^{2}=\lim _{\rho \rightarrow 0} \int_{B_{\rho}\left(b_{n}\right)}\left(\left|\nabla \omega^{n}\right|^{2}+\frac{\left|\omega^{n}\right|^{2}}{\rho^{2}}\right) d x=0 \tag{3.2}
\end{equation*}
$$

Note that $\bar{h}_{n}(\rho) \in C\left(\left(0, \rho_{0}\right]\right)$. If the proposition is false there exists a constant $\delta_{0}>0$ and a sequence of radii $\left\{\rho_{k}\right\}$ such that $\rho_{k} \rightarrow 0$ and

$$
\begin{align*}
& \left|\bar{h}_{n}\left(\rho_{k}\right)-t \pi\right| \geqslant \delta_{0}>0 \quad \text { if } k_{s}<k_{b} \\
& \left|\bar{h}_{n}\left(\rho_{k}\right)-\frac{(2 t+1) \pi}{2}\right| \geqslant \delta_{0}>0 \quad \text { if } k_{b}<k_{s} \tag{3.3}
\end{align*}
$$

for every $t \in \mathbb{Z}$. Since $\int_{B_{2}}\left|\nabla_{y}\left(h_{n}\left(\rho y+b_{n}\right)\right)\right|^{2} d y \rightarrow 0$ as $\rho \rightarrow 0$, we have

$$
\begin{equation*}
\lim _{\rho \rightarrow 0}\left\{\sup _{1 / 2 \leqslant s \leqslant 1} \int_{\partial B_{1}}\left|h_{n}\left(s \rho y+b_{n}\right)-\bar{h}_{n}(\rho)\right|^{2} d \sigma_{y}\right\}=0 . \tag{3.4}
\end{equation*}
$$

Then from the Lipschitz continuity of the sine and cosine functions and (3.4) we have

$$
\begin{equation*}
\lim _{\rho_{k} \rightarrow 0}\left\{\sup _{1 / 2 \leqslant s \leqslant 2} \int_{\partial B_{1}}\left|\omega_{s \rho_{k}}^{n}(y)-\bar{\omega}^{n}\left(\rho_{k}\right)\right|^{2} d \sigma_{y}\right\}=0 . \tag{3.5}
\end{equation*}
$$

From (3.3) we get

$$
\begin{equation*}
\left|\left(\bar{\omega}^{n}\left(\rho_{k}\right)\right)\right| \geqslant C_{3}>0 \tag{3.6}
\end{equation*}
$$

for each $k$. Then, from (3.5) and (3.6) we get for every $s \in[1 / 2,1]$ and every $k$ sufficiently large that

$$
\int_{\partial B_{s}}\left|\omega_{\rho_{k}}^{n}\right|^{2} d \sigma \geqslant C_{4}>0 .
$$

This implies

$$
\begin{aligned}
\int_{B_{\rho_{k}\left(b_{n}\right)}} \frac{\left|\omega^{n}\right|^{2}}{\rho_{k}^{2}} d x & \geqslant \int_{B_{\rho_{k}\left(b_{n}\right) \backslash B_{\rho_{k} / 2}\left(b_{n}\right)}} \frac{\left|\omega^{n}\right|^{2}}{\rho_{k}^{2}} d x \\
& =\int_{\frac{1}{2}}^{1} \int_{\partial B_{s}}\left|\omega_{\rho_{k}}^{n}\right|^{2} d \sigma d s \geqslant \frac{C_{4}}{4}
\end{aligned}
$$

which is a contradiction.
Using the notation from Proposition 3.1 we have the following corollary.
Corollary 3.1. Let $h_{n}(x)=h(x)+\sum_{m \neq n} \theta_{b_{m}}(x)+\zeta(x)$ for $h \in A(\mathbf{b})$. Then for $1 \leqslant n \leqslant d$

$$
\lim _{\rho \rightarrow 0} \int_{B_{\rho}\left(b_{n}\right) \backslash B_{\rho / 2}\left(b_{n}\right)}\left(\left|\nabla h_{n}\right|^{2}+\frac{\left(h_{n}-\alpha_{n}\right)^{2}}{\rho^{2}}\right) d x=0
$$

and

$$
h_{n}\left(\rho y+b_{n}\right) \rightarrow \alpha_{n} \quad \text { in } L^{2}\left(\mathbb{S}^{1}\right) \text { as } \rho \rightarrow 0 .
$$

From the definition of $\alpha_{n}$ then, for each $\phi \in A(\mathbf{b})$, if $k_{s}<k_{b}$ and $v(x)=v(\mathbf{b}, \phi)(x)$ we have $v\left(\rho y+b_{n}\right) \rightarrow \pm y$ as $\rho \rightarrow 0$ in $L^{2}\left(\partial B_{1}(0), \mathbb{S}^{1}\right)$, giving rise to a pure splay pattern near each defect. If $k_{b}<k_{s}$ it follows that $v\left(\rho y+b_{n}\right) \rightarrow$ $\pm i y$ in $L^{2}\left(\partial B_{1}(0), \mathbb{S}^{1}\right)$, giving rise to a pure bend pattern near each defect.

## 4. Construction and properties of the renormalized energy

Denote the integrals $J_{\varepsilon}(u ; A)=\int_{A} j_{\varepsilon}(u, \nabla u) d x$ and $\bar{J}_{\varepsilon}(u ; A)=\int_{A} \bar{j}_{\varepsilon}(u, \nabla u) d x$, with $A \subset \Omega$, noting that $J_{\varepsilon}(u ; \Omega)=J_{\varepsilon}(u)$. Given any configuration $\mathbf{b} \in T$, take a function $\phi \in A(\mathbf{b})$ and define the function $v$ as in the previous section. Then, for a given $\rho>0$, denoting $\Omega_{\rho}=\Omega \backslash \bigcup_{n=1}^{d} \bar{B}_{\rho}\left(b_{n}\right)$, we have

$$
\bar{J}_{\varepsilon}\left(v ; \Omega_{\rho}\right)=\bar{J}\left(v ; \Omega_{\rho}\right)=\int_{\Omega_{\rho}} \bar{j}(v, \nabla v) d x
$$

where

$$
v=e^{i\left(\phi+\zeta+\sum_{n=1}^{d} \theta_{b_{n}}\right)} \quad \text { in } \Omega_{\rho} .
$$

Then, we have

$$
\frac{\underline{k}}{2} \int_{\Omega_{\rho}}|\nabla v|^{2} d x=\frac{k}{2} \int_{\Omega_{\rho}}\left(|\nabla \phi|^{2}+2 \nabla \phi \cdot\left(\nabla \zeta+\sum_{n=1}^{d} \nabla \theta_{b_{n}}\right)+\left|\nabla \zeta+\sum_{n=1}^{d} \nabla \theta_{b_{n}}\right|^{2}\right) d x .
$$

Recall that $\theta_{b_{n}}(x)$ is the harmonic conjugate of $\ln \left(\left|x-b_{n}\right|\right), \zeta(x)$ is the harmonic conjugate of $-\sum_{\ell=1}^{k} d_{\ell} \ln \left(\left|x-y_{\ell}\right|\right)$ in $\Omega_{\rho}$, and $\left\{y_{1}, \ldots, y_{k}\right\} \subset(\bar{\Omega})^{c}$. Define a function $G_{\mathbf{b}}(x)=\sum_{n=1}^{d} \ln \left(\left|x-b_{n}\right|\right)-\sum_{\ell=1}^{k} d_{\ell} \ln \left(\left|x-y_{\ell}\right|\right)$. Hence, using integration by parts, we obtain

$$
\begin{aligned}
\frac{\bar{k}}{2} \int_{\Omega_{\rho}}|\nabla v|^{2} d x= & \frac{k}{2} \int_{\Omega_{\rho}}|\nabla \phi|^{2} d x+\frac{k}{2} \int_{\partial \Omega}\left(\left(\partial_{\nu} G_{\mathbf{b}}\right) G_{\mathbf{b}}-2\left(\partial_{\tau} G_{\mathbf{b}}\right) \phi\right) d \sigma \\
& -\sum_{m \neq n} \underline{k} \pi \ln \left(\left|b_{n}-b_{m}\right|\right)+\sum_{n=1}^{d} \sum_{\ell=1}^{k} \underline{k} \pi d_{\ell} \ln \left(\left|b_{n}-y_{\ell}\right|\right)+\underline{k} \pi d \ln \left(\frac{1}{\rho}\right) \\
& +o_{\rho}(1) .
\end{aligned}
$$

We also have on the boundary, $v(x)=e^{i\left(\phi(x)+\sum \theta_{b_{n}}(x)+\zeta(x)\right)}=g(x)$. Using integration by parts again, we have

$$
\begin{aligned}
\frac{\bar{k}}{2} \int_{\Omega_{\rho}}|\nabla v|^{2} d x= & \frac{k}{2} \int_{\Omega_{\rho}}|\nabla \phi|^{2} d x+\frac{\underline{k}}{2} \int_{\partial \Omega}\left(2 G_{\mathbf{b}}\left(g \times \partial_{\tau} g\right)-\left(\partial_{\nu} G_{\mathbf{b}}\right) G_{\mathbf{b}}\right) d \sigma \\
& -\sum_{m \neq n} \underline{k} \pi \ln \left(\left|b_{n}-b_{m}\right|\right)+\sum_{n=1}^{d} \sum_{\ell=1}^{k} \underline{k} \pi d_{\ell} \ln \left(\left|b_{n}-y_{\ell}\right|\right)+\underline{k} \pi d \ln \left(\frac{1}{\rho}\right) \\
& +o_{\rho}(1) .
\end{aligned}
$$

This implies that

$$
\begin{align*}
J\left(v ; \Omega_{\rho}\right) & =\bar{J}\left(v ; \Omega_{\rho}\right)+\underline{k} \pi d \\
& =\underline{k} \pi d \ln \left(\frac{1}{\rho}\right)+\underline{k} W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, \phi, k_{s}, k_{b}\right)+o_{\rho}(1) \tag{4.1}
\end{align*}
$$

where $W(\mathbf{b})$ and $\mathscr{H}\left(\mathbf{b}, h, k_{s}, k_{b}\right)$ are defined as in (1.6) and (1.7) respectively. Notice that $\mathscr{H}\left(\mathbf{b}, \phi, k_{s}, k_{b}\right) \geqslant$ $\mathscr{H}(\mathbf{b}, \phi, \underline{k}, \underline{k})$ and that $\underline{k} W(\mathbf{b})+H(\mathbf{b}, \underline{k}, \underline{k})$ is simply $\underline{k}$ times the renormalized energy for the Ginzburg-Landau energy studied in [1] for the case when $\Omega$ is simply connected. It is proved there that the latter tends to infinity as either $\mathbf{b} \rightarrow \partial \Omega^{d}$ or $\left|b_{n}-b_{m}\right| \rightarrow 0$. These properties can be seen to hold here by examining the first and third terms on the right side of (1.6) respectively. Thus they hold for our renormalized energy as well. To minimize the energy then, the vortices must be distinct and stay within the domain $\Omega$. Now for each configuration, we will show that there is a function that will minimize the renormalized energy.

Proposition 4.1. Assume that $k_{s} \neq k_{b}$. Let $\mathbf{b} \in T$ be a configuration in $\Omega^{d}$. Then, there exists a function $h_{\mathbf{b}}(x) \in A_{\mathbf{b}}$ such that

$$
\begin{aligned}
\underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right) & =\min _{\phi \in A(\mathbf{b})}\left(\underline{k} W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, \phi, k_{s}, k_{b}\right)\right) \\
& =\underline{k} W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, h_{\mathbf{b}}, k_{s}, k_{b}\right) .
\end{aligned}
$$

Proof. We first point out that $A(\mathbf{b}) \neq \emptyset$. Indeed let $\phi \in H^{1}(\Omega)$ such that $v(\mathbf{b}, \phi)=g$ on $\partial \Omega$. Let $\left\{\bar{B}_{\rho}\left(b_{n}\right) ; n=\right.$ $1, \ldots, d\}$ be a nonintersecting collection of closed disks that are contained in $\Omega$. One can always modify $\phi$ so that $\phi(x)=\alpha_{n}-\sum_{m \neq n} \theta_{b_{m}}(x)-\zeta(x)$ for $x \in \bar{B}_{\rho}\left(b_{n}\right)$, for each $n$, for some $\alpha_{n}$ as defined in Proposition 3.1. The resulting function $\phi \in A(\mathbf{b})$.

Since $W(\mathbf{b})$ is independent of the particular $\phi \in A(\mathbf{b})$ we only need to minimize $\mathscr{H}\left(\mathbf{b}, \cdot, k_{s}, k_{b}\right)$. Let $\left\{h_{n}\right\} \subset A(\mathbf{b})$ be a minimizing sequence for $\mathscr{H}\left(\mathbf{b}, \cdot, k_{s}, k_{b}\right)$. By definition of the integral, we can subtract an integer multiple of $2 \pi$ from each $h_{n}$, so that without a loss of generality $\left.h_{n}\right|_{\Gamma_{0}}=f$ for each $n$. Then Poincaré's inequality can be applied so that we have $\left\|h_{n}\right\|_{2 ; \Omega} \leqslant C\left(\left\|\nabla h_{n}\right\|_{2 ; \Omega}+1\right)$. Since $\left\{h_{n}\right\}$ is a minimizing sequence for $\mathscr{H}\left(\mathbf{b}, \cdot, k_{s}, k_{b}\right)$ it follows that $\left\|h_{n}\right\|_{1,2 ; \Omega} \leqslant C_{0}$ for some constant $C_{0}$. Thus there exists a function $h_{0} \in H^{1}(\Omega)$ satisfying $v\left(\mathbf{b}, h_{0}\right)=g$ on $\partial \Omega$ and such that $h_{n_{k}} \rightharpoonup h_{0}$ in $H^{1}(\Omega)$ and $h_{n_{k}} \rightarrow h_{0}$ almost everywhere in $\Omega$. This gives

$$
\begin{equation*}
\liminf _{k \rightarrow \infty} \int_{\Omega}\left|\nabla h_{n_{k}}\right|^{2} d x \geqslant \int_{\Omega}\left|\nabla h_{0}\right|^{2} d x . \tag{4.2}
\end{equation*}
$$

This also gives $e^{i h_{n_{k}}(x)} \rightharpoonup e^{i h_{0}(x)}$ in $H^{1}(\Omega ; \mathbb{C})$. Let $\prod_{n=1}^{d} \frac{x-b_{n}}{\left|x-b_{n}\right|} e^{i\left(h_{n_{k}}(x)+\zeta(x)\right)}=w_{n_{k}}(x)$ and $\prod_{n=1}^{d} \frac{x-b_{n}}{\left|x-b_{n}\right|} \times$ $e^{i\left(h_{0}(x)+\zeta(x)\right)}=w_{0}(x)$. Then we have $w_{n_{k}} \rightarrow w_{0}$ in $L^{2}(\Omega, \mathbb{C})$. Let $z_{k}=\operatorname{curl} w_{n_{k}}\left(\operatorname{div} w_{n_{k}}\right)$ if $k_{s}<k_{b}\left(k_{b}<k_{s}\right)$.

Then, we have $\left\|z_{k}\right\|_{2 ; \Omega} \leqslant C(\mathbf{b})$. Hence, there exists a subsequence, relabeled as $z_{k}$, such that $z_{k} \rightharpoonup z_{0}$ in $L^{2}(\Omega)$. This implies that $z_{0}=\operatorname{curl} w_{0}\left(\operatorname{div} w_{0}\right)$ and we have

$$
\begin{array}{ll}
\liminf _{k \rightarrow \infty} \int_{\Omega}\left(\operatorname{curl} w_{n_{k}}\right)^{2} d x \geqslant \int_{\Omega}\left(\operatorname{curl} w_{0}\right)^{2} d x & \text { if } k_{s}<k_{b}, \\
\liminf _{k \rightarrow \infty} \int_{\Omega}\left(\operatorname{div} w_{n_{k}}\right)^{2} d x \geqslant \int_{\Omega}\left(\operatorname{div} w_{0}\right)^{2} d x & \text { if } k_{b}<k_{s} . \tag{4.3}
\end{array}
$$

Then combining (4.2) and (4.3), we get

$$
\begin{aligned}
\min _{h \in H_{v}^{1}}\left(W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, h, k_{s}, k_{b}\right)\right) & =\liminf _{k \rightarrow \infty}\left(W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, h_{n_{k}}, k_{s}, k_{b}\right)\right) \\
& \geqslant W(\mathbf{b})+\mathscr{H}\left(\mathbf{b}, h_{0}, k_{s}, k_{b}\right)
\end{aligned}
$$

giving us the result of the proof, with $h_{\mathbf{b}}:=h_{0}$.
By the previous proposition, we can utilize the notation as in (1.5) so that there is no dependency on the choice of the function $h$. Even though it will not be used in this work, one can show that $k W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right)$ is continuous on $T$. This is done in [23].

Proposition 4.2. Let $h$ be the polar function appearing in the definition of $u_{*}$ from Theorem A. Then $h \in A(\mathbf{a})$.
Proof. Consider the configuration $\mathbf{a}=\left(a_{1}, \ldots, a_{d}\right)$ and the function $u_{*}=\prod_{n=1}^{d} \frac{x-a_{n}}{\left|x-a_{n}\right|} e^{i(h(x)+\zeta(x))}$. Let $\left\{u_{\ell}\right\}$ be a sequence of minimizers to $J_{\varepsilon_{\ell}}(\cdot)$ for each $\varepsilon_{\ell}$. Then from Corollary 2.1 we get $\int_{\Omega}\left(\operatorname{curl} u_{\ell}\right)^{2} d x \leqslant \tilde{C}$ or $\int_{\Omega}\left(\operatorname{div} u_{\ell}\right)^{2} d x \leqslant \tilde{C}$, where $\tilde{C}$ has no dependency on $\varepsilon$. From Proposition $2.3, u_{\ell} \rightarrow u_{*}$ in $L^{2}(\Omega ; \mathbb{C})$ and we have that the distributions $\left\{\operatorname{curl} u_{\ell}\right\}\left(\left\{\operatorname{div} u_{\ell}\right\}\right)$ are uniformly bounded in $L^{2}(\Omega)$. Hence, as in the proof of Proposition 4.1, we get

$$
\begin{aligned}
& \int_{\Omega}\left(\operatorname{curl} u_{*}\right)^{2} d x \leqslant \liminf _{\ell \rightarrow \infty} \int_{\Omega}\left(\operatorname{curl} u_{\ell}\right)^{2} d x \leqslant C_{0}, \quad \text { if } k_{s}<k_{b}, \\
& \int_{\Omega}\left(\operatorname{div} u_{*}\right)^{2} d x \leqslant \liminf _{\ell \rightarrow \infty} \int_{\Omega}\left(\operatorname{div} u_{\ell}\right)^{2} d x \leqslant C_{0}, \quad \text { if } k_{b}<k_{s} .
\end{aligned}
$$

From the definition of each $u_{\ell}$ on the boundary, we have $\left.u_{*}\right|_{\partial \Omega}=g$, giving us $v(\mathbf{b}, h)=g$ on $\partial \Omega$. From Proposition $2.3, h \in H^{1}(\Omega)$. Therefore, $h \in A(\mathbf{a})$ and must satisfy the result of Proposition 3.1.

## 5. Energy away from the vortices

We must first analyze the following minimum problem before proving Theorem B. Let $\beta \in \mathbb{C}$, such that

$$
\beta= \begin{cases} \pm 1 & \text { if } k_{s}<k_{b} \\ \pm i & \text { if } k_{b}<k_{s}\end{cases}
$$

and recall from the proof of Proposition 2.1 the expression

$$
\bar{I}_{\beta}\left(\frac{\varepsilon}{R}\right)=\inf _{u \in H_{g}^{1}}\left\{\int_{B_{R}(0)} \bar{j}_{\varepsilon}(u, \nabla u) d x\right\}
$$

where

$$
g(x)=\beta \frac{x}{|x|} \quad \text { for }|x|=R
$$

Proposition 5.1. The function $\bar{I}_{\beta}(\tau)+\underline{k} \pi \ln (\tau)$ is a nondecreasing function of $\tau$ for $\tau>0$ such that $\gamma:=$ $\lim _{\tau \rightarrow 0}\left\{\bar{I}_{\beta}(\tau)+\underline{k} \pi \ln (\tau)\right\}>-\infty$.

Proof. The argument is similar to that in [1]. It is shown within the proof of Proposition 2.1 that the expression $\bar{I}_{\beta}(\tau)+\underline{k} \pi \ln (\tau)$ is monotone nondecreasing and bounded below. The existence of the finite one-sided limit at $\tau=0$ follows from these two properties.

Proof of Theorem B. Assume that $k_{s} \neq k_{b}$. We argue in a similar manner as in [15] and Chapter 8 of [1]. Those works however, used the fact that the polar function $h_{\mathbf{b}}(x)$ is smooth (since it is harmonic for the case $k_{s}=k_{b}$ ). Here we appeal to Corollary 3.1 to control $h_{\mathbf{b}}$. Let $\mathbf{b}=\left(b_{1}, \ldots, b_{n}\right)$ be a configuration in $\Omega^{d}$, where $b_{n} \neq b_{m}$ for $m \neq n$, and $\Omega_{\rho}=\Omega \backslash \bigcup_{n=1}^{d} \bar{B}_{\rho}\left(b_{n}\right)$. For this configuration, set

$$
w_{\mathbf{b}}(x)=\prod_{n=1}^{d} \frac{x-b_{n}}{\left|x-b_{n}\right|} e^{i\left(h_{\mathbf{b}}(x)+\zeta(x)\right)}
$$

where $h_{\mathbf{b}} \in A(\mathbf{b})$ satisfies Proposition 4.1. Then, using (4.1) we find that,

$$
\begin{equation*}
\int_{\Omega_{\rho}} \bar{j}\left(w_{\mathbf{b}}, \nabla w_{\mathbf{b}}\right) d x=\underline{k} \pi d \ln \left(\frac{1}{\rho}\right)+\underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right)-\underline{k} \pi d+o_{\rho}(1) \tag{5.1}
\end{equation*}
$$

as $\rho \rightarrow 0$. We next construct comparison functions using the configuration point $\mathbf{b}$. For $0<\varepsilon \ll \rho \ll 1$, define

$$
\tilde{u}_{\ell}(x)= \begin{cases}w_{\mathbf{b}}(x) & \text { for } x \in \Omega_{\rho} \\ e^{i q_{n}(x)} \frac{x-b_{n}}{\left|x-b_{n}\right|} & \text { for } x \in B_{\rho}\left(b_{n}\right) \backslash B_{\rho / 2}\left(b_{n}\right) \\ z_{n}\left(x-b_{n}\right) & \text { for } x \in B_{\rho / 2}\left(b_{n}\right)\end{cases}
$$

where $z_{n}$ minimizes $\bar{J}_{\varepsilon_{\ell}}\left(\cdot ; B_{\rho / 2}(0)\right)$, with

$$
\left.z_{n}\left(x-b_{n}\right)\right|_{\partial B_{\rho / 2}\left(b_{n}\right)}=e^{i \alpha_{n}} \frac{x-b_{n}}{\left|x-b_{n}\right|}
$$

Here $\alpha_{n}$ is determined from $h_{\mathbf{b}}$ by way of Corollary 3.1 such that

$$
\alpha_{n}= \begin{cases}c_{n} \pi & \text { for some } c_{n} \in \mathbb{Z} \text { if } k_{s}<k_{b} \\ \frac{\left(2 c_{n}+1\right) \pi}{2} & \text { for some } c_{n} \in \mathbb{Z} \text { if } k_{b}<k_{s}\end{cases}
$$

Then by Proposition 5.1, we have

$$
\begin{equation*}
\int_{B_{\rho / 2}(0)} \bar{j}_{\varepsilon}\left(z_{n}, \nabla z_{n}\right) d x=\underline{k} \pi \ln \left(\frac{\rho}{2 \varepsilon}\right)+\gamma+o_{\varepsilon}(1) \tag{5.2}
\end{equation*}
$$

as $\varepsilon \rightarrow 0$ for each $\rho>0$. Define $\tilde{h}_{n}(x)=h_{\mathbf{b}}(x)+\sum_{m \neq n} \theta_{b_{m}}(x)+\zeta(x)$ and $q_{n}$ such that

$$
\left.q_{n}(x)\right|_{\partial B_{\rho}}=\tilde{h}_{n}(x),\left.\quad q_{n}(x)\right|_{\partial B_{\rho / 2}}=\alpha_{n}
$$

such that $q_{n}$ is linear in radial directions centered at $b_{n}$. From Corollary 3.1 there exist $\rho_{\ell} \downarrow 0$ so that

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \int_{\partial B_{\rho_{\ell}}\left(b_{n}\right)}\left(\rho_{\ell}\left|\nabla \tilde{h}_{n}\right|^{2}+\frac{\left(\tilde{h}_{n}-\alpha_{n}\right)^{2}}{\rho_{\ell}}\right) d \sigma=0 \tag{5.3}
\end{equation*}
$$

With this property we have that

$$
\tilde{u}_{\ell}(x)=e^{i q_{n}(x)} \frac{x-b_{n}}{\left|x-b_{n}\right|}=\left(e^{i\left(q_{n}(x)-\alpha_{n}\right)}\right)\left(e^{i \alpha_{n}} \frac{x-b_{n}}{\left|x-b_{n}\right|}\right) \quad \text { for } x \in B_{\rho_{\ell}}\left(b_{n}\right) \backslash B_{\rho_{\ell} / 2}\left(b_{n}\right)
$$

satisfies

$$
\begin{equation*}
\int_{B_{\rho_{\ell}} \backslash B_{\rho_{\ell} / 2}} \bar{j}_{\varepsilon}\left(\tilde{u}_{\ell}, \nabla \tilde{u}_{\ell}\right) d x=\underline{k} \pi \ln (2)+o_{\rho}(1) \tag{5.4}
\end{equation*}
$$

uniformly in $\varepsilon$. Then from (5.1), (5.2), and (5.4) we have for minimizers $u_{\varepsilon}$ that

$$
\begin{aligned}
J_{\varepsilon}\left(u_{\varepsilon}\right) & =\bar{J}_{\varepsilon}\left(u_{\varepsilon}\right)+\underline{k} \pi d \leqslant \bar{J}_{\varepsilon}\left(\tilde{u}_{\ell}\right)+\underline{k} \pi d \\
& =\underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)+\underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right)+d \gamma+o_{\rho}(1)+o_{\varepsilon}(1) .
\end{aligned}
$$

Thus

$$
\begin{equation*}
\limsup _{\varepsilon \rightarrow 0}\left(J_{\varepsilon}\left(u_{\varepsilon}\right)-\underline{k} \pi d \ln \left(\frac{1}{\varepsilon}\right)\right) \leqslant \underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right)+d \gamma . \tag{5.5}
\end{equation*}
$$

We next obtain an estimate from below. Let a be a limiting configuration as in Theorem A. Using Propositions 2.4 and 4.2 we can find a sequence of radii $\left\{\rho_{\ell}\right\}$ and a subsequence of minimizers $\left\{u_{\varepsilon_{k, \ell}}\right\}$ (that we label $\left\{u_{k}\right\}$ ) so that (5.3) holds at $\mathbf{a}$, and in addition

$$
u_{k} \rightarrow u_{*} \quad \text { in } H^{1}\left(\partial B_{\rho_{\ell}}\left(a_{n}\right)\right) \quad \text { and } \quad \frac{1-\left|u_{k}\right|^{2}}{\varepsilon_{k}} \rightarrow 0 \quad \text { in } L^{2}\left(\partial B_{\rho_{\ell}}\left(a_{n}\right)\right) \text { as } k \rightarrow \infty
$$

for each $\ell$ and $n$. It follows that we can construct, in a similar fashion as before, functions $\tilde{u}_{n k \ell} \in H^{1}\left(B_{2 \rho_{\ell}}\left(a_{n}\right) \backslash\right.$ $\left.B_{\rho_{\ell}}\left(a_{n}\right) ; \mathbb{C}\right)$ so that

$$
\left.\tilde{u}_{n k \ell}\right|_{\partial B_{\rho_{\ell}}\left(a_{n}\right)}=u_{k} \quad \text { and }\left.\quad \tilde{u}_{n k \ell}\right|_{\partial B_{2 \rho_{\ell}}\left(a_{n}\right)}=e^{i \alpha_{n}} \frac{x-a_{n}}{\left|x-a_{n}\right|},
$$

satisfying

$$
\int_{B_{2 \rho_{\ell}} \backslash B_{\rho_{\ell}}} \bar{j}_{\varepsilon_{k}}\left(\tilde{u}_{n k \ell}, \nabla \tilde{u}_{n k \ell}\right) d x=\underline{k} \pi \ln (2)+o_{\rho}(1)+o_{\varepsilon}(1) .
$$

From this and Proposition 5.1 we see that

$$
\begin{aligned}
\underline{k} \pi \ln \left(\frac{\rho_{\ell}}{\varepsilon_{k}}\right)+\gamma & \leqslant \bar{J}_{\varepsilon_{k}}\left(u_{k} ; B_{\rho_{\ell}}\right)+\bar{J}_{\varepsilon_{k}}\left(\tilde{u}_{n k \ell} ; B_{2 \rho_{\ell}} \backslash B_{\rho_{\ell}}\right)-\underline{k} \pi \ln (2)+o_{\rho}(1)+o_{\varepsilon}(1) \\
& =\bar{J}_{\varepsilon_{k}}\left(u_{k} ; B_{\rho_{\ell}}\right)+o_{\rho}(1)+o_{\varepsilon}(1)
\end{aligned}
$$

We use Proposition 2.4 and (4.1) to determine the asymptotic nature for $\bar{J}_{\varepsilon_{k}}\left(u_{k} ; \Omega_{\rho_{\ell}}\right)$. These two estimates give

$$
\begin{aligned}
J_{\varepsilon_{k}}\left(u_{k}\right) & \geqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon_{k}}\right)+\underline{k} W(\mathbf{a})+\mathscr{H}\left(\mathbf{a}, h, k_{s}, k_{b}\right)+d \gamma+o_{\rho}(1)+o_{\varepsilon}(1) \\
& \geqslant \underline{k} \pi d \ln \left(\frac{1}{\varepsilon_{k}}\right)+\underline{k} W(\mathbf{a})+H\left(\mathbf{a}, k_{s}, k_{b}\right)+d \gamma+o_{\rho}(1)+o_{\varepsilon}(1) .
\end{aligned}
$$

Choosing subsequences $\left\{\rho_{\ell}\right\}$, $\left\{\varepsilon_{\ell}\right\}$ such that $\varepsilon_{\ell} \ll \rho_{\ell} \rightarrow 0$ as $\ell \rightarrow \infty$ allows us to compare this to (5.5). Since $\mathbf{b}$ was arbitrary, we have that

$$
\underline{k} W(\mathbf{a})+H\left(\mathbf{a}, k_{s}, k_{b}\right) \leqslant \underline{k} W(\mathbf{a})+\mathscr{H}\left(\mathbf{a}, h, k_{s}, k_{b}\right) \leqslant \underline{k} W(\mathbf{b})+H\left(\mathbf{b}, k_{s}, k_{b}\right)
$$

for any configuration $\mathbf{b} \in T$. Hence the configuration a from Proposition 2.3 minimizes the renormalized energy, with $\mathbf{a} \in T$. If we set $\mathbf{a}=\mathbf{b}$, we obtain that the function $h$ from Proposition 2.3 minimizes the renormalized energy for the configuration a. Finally we see that

$$
\lim _{\ell \rightarrow \infty}\left(J_{\varepsilon_{\ell}}\left(u_{\varepsilon_{\ell}}\right)-\underline{k} \pi d \ln \left(\frac{1}{\varepsilon_{\ell}}\right)\right)=\underline{k} W(\mathbf{a})+H\left(\mathbf{a}, k_{s}, k_{b}\right)+d \gamma .
$$

Remark 2. To be complete, we point out that Theorem B holds in the case $k=k_{s}=k_{b}$ as well. In this case the renormalized energy is as in (1.5) and (1.6), such that $H(\mathbf{b}, k, k)=\frac{1}{2} \int_{\Omega} k\left|\nabla h_{\mathbf{b}}\right|^{2} d x$, where $v\left(\mathbf{b}, h_{\mathbf{b}}\right)=g$ on $\partial \Omega$ and $h_{\mathbf{b}}$ is a harmonic function that minimizes this energy subject to this boundary condition. If $\Omega$ is multiply connected the proof proceeds just as in [1], once one knows that a configuration a obtained from a sequence of minimizers $\left\{u_{\ell}\right\}$ is in $T$. This follows from Proposition 2.3.

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