

# A sharp lower bound for the first eigenvalue on Finsler manifolds <sup>☆</sup>

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This paper is dedicated to Professor S.S. Chern on the occasion of his 100th birthday

## Abstract

In this paper, we give a sharp lower bound for the first (nonzero) Neumann eigenvalue of Finsler-Laplacian in Finsler manifolds in terms of diameter, dimension, weighted Ricci curvature.

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## 1. Introduction

The study of the first (nonzero) eigenvalue of Laplacian in Riemannian manifolds plays an important role in differential geometry. The first result on this subject, due to Lichnerowicz [11], says that for an  $n$ -dimensional smooth compact manifold without boundary, the first eigenvalue  $\lambda_1$  can be estimated below by  $\frac{n}{n-1}K$ , provided that its  $Ric \geq K > 0$ . In this case, Obata [13] established a rigidity result, asserting the optimality of Lichnerowicz' estimate. Namely,  $\lambda_1 = K$  if and only if  $M$  is isometric to the  $n$ -dimensional sphere with constant curvature  $\frac{1}{n-1}K$ . When  $K = 0$ , Li–Yau [9,10] developed a method, which depends on the gradient estimate of the eigenfunctions, to give the lower bound of the first eigenvalue via diameter  $d$ , precisely,  $\lambda_1 \geq \frac{\pi^2}{2d^2}$ . Their method had been improved by Zhong–Yang [24] to obtain  $\lambda_1 \geq \frac{\pi^2}{d^2}$ , which is optimal in the sense that equality can be attained for one-dimensional circle. Very recently, Hang and Wang showed that  $\lambda_1 > \frac{\pi^2}{d^2}$  in [7], if the dimension  $n > 1$ . These results also hold true when  $M$  is a manifold with convex boundary. When  $M$  is a convex domain in  $\mathbb{R}^n$ , this is a classical result of Payne–Weinberger [17]. Later Chen–Wang [5] and Bakry–Qian [3] combined these results into a same framework, and gave estimates for the first eigenvalue of very general elliptic symmetric operators, via diameter and Ricci curvature. This sharp estimate on Riemannian manifolds has been also generalized to Alexandrov spaces by Qian–Zhang–Zhu [18].

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Finsler geometry attracts many attentions in recent years, since it has broader applications in nature science. Simultaneously Finsler manifold is one of the most natural metric measure spaces, which plays an important role in many aspects in mathematics. There exists a natural Laplacian on Finsler manifolds, which we call here *Finsler-Laplacian*. Unlike the usual Laplacian, the Finsler-Laplacian is a nonlinear operator. The objective of this paper is to study the lower bound for the first (nonzero) eigenvalue of this Finsler-Laplacian on Finsler manifolds. In [14] Ohta introduced the weighted Ricci curvature  $Ric_N$  for  $N \in [n, \infty]$  of Finsler manifolds, following the work of Lott–Villani [12] and Sturm [20] on metric measure space. He proved the equivalence of the lower boundedness of the  $Ric_N$  and the curvature-dimension conditions  $CD(K, N)$  in [12,20]. As a byproduct, he obtained a Lichnerowicz type estimate on the first eigenvalue of Finsler-Laplacian under the assumption  $Ric_N \geq K > 0$ . Another interesting type of eigenvalue estimates was obtained by Ge–Shen in [6], namely the Faber–Krahn type inequality for the first Dirichlet eigenvalue of the Finsler-Laplacian holds. See also [4] and [22]. Recently, we proved in [21] the Li–Yau–Zhong–Yang type sharp estimate for a so-called anisotropic Laplacian on a Minkowski space, which could be viewed as the simplest, but interesting and important case of non-Riemannian Finsler manifolds. In this paper, we shall generalize the results in [21] to general Finsler manifolds. Moreover, as in [5] and [3], we shall put the Li–Yau–Zhong–Yang type and the Lichnerowicz type sharp estimates into a uniform framework.

Our main result of this paper is

**Theorem 1.1.** *Let  $(M^n, F, m)$  be an  $n$ -dimensional compact Finsler measure space, equipped with a Finsler structure  $F$  and a smooth measure  $m$ , without boundary or with a convex boundary. Assume that  $Ric_N \geq K$  for some real numbers  $N \in [n, +\infty]$  and  $K \in \mathbb{R}$ . Let  $\lambda_1$  be the first (nonzero) Neumann eigenvalue of the Finsler-Laplacian  $\Delta_m$ , i.e.,*

$$-\Delta_m u = \lambda_1 u, \quad \text{in } M, \tag{1}$$

with a Neumann boundary condition

$$\nabla u(x) \in T_x(\partial M), \tag{2}$$

if  $\partial M$  is not empty. Then

$$\lambda_1 \geq \lambda_1(K, N, d), \tag{3}$$

where  $d$  is the diameter of  $M$ ,  $\lambda_1(K, N, d)$  represents the first (nonzero) eigenvalue of the 1-dimensional problem

$$v'' - T(t)v' = -\lambda_1(K, N, d)v \quad \text{in } \left(-\frac{d}{2}, \frac{d}{2}\right), \quad v'\left(-\frac{d}{2}\right) = v'\left(\frac{d}{2}\right) = 0, \tag{4}$$

with  $T(t)$  varying according to different values of  $K$  and  $N$ .  $T$  is explicitly defined by

$$T(t) = \begin{cases} \sqrt{(N-1)K} \tan\left(\sqrt{\frac{K}{N-1}}t\right), & \text{for } K > 0, 1 < N < \infty, \\ -\sqrt{-(N-1)K} \tanh\left(\sqrt{-\frac{K}{N-1}}t\right), & \text{for } K < 0, 1 < N < \infty, \\ 0, & \text{for } K = 0, 1 < N < \infty, \\ Kt, & \text{for } N = \infty. \end{cases} \tag{5}$$

The precise definition of the Finsler measure space, convex boundary, diameter  $d$ , weighted Ricci curvature  $Ric_N$ , gradient vector field  $\nabla$ , Finsler-Laplacian  $\Delta_m$  will be given in Section 2 below.

Equivalently, Theorem 1.1 gives an optimal Poincaré inequality in Finsler manifolds.

**Theorem 1.2.** *Under the same assumptions as in Theorem 1.1, we have*

$$\int_M F^2(\nabla u) dm \geq \lambda_1(K, N, d) \int_M (u - \bar{u})^2 dm, \tag{6}$$

where  $\bar{u}$  is the average of  $u$ .

In the case of  $K > 0$  and  $N = n$ , Theorem 1.1 sharpens the Lichnerowicz type estimate given by Ohta [14], since by Meyer theorem,  $d \leq \frac{\pi}{\sqrt{(n-1)K}}$ , and  $\lambda_1(K, N, d) \geq \lambda_1(K, n, \frac{\pi}{\sqrt{(n-1)K}}) = \frac{nK}{n-1}$ . In the case of  $K = 0$  and  $N = n$ ,

Theorem 1.1 gives the Li–Yau–Zhong–Yang type sharp estimate for the Finsler-Laplacian, since  $\lambda_1(0, n, d) = \frac{\pi^2}{d^2}$ . We remark that the Minkowski space  $(\mathbb{R}^n, F)$  equipped with the  $n$ -dimensional Lebesgue measure satisfies that  $Ric_N \geq K$  with  $K = 0$  and  $N = n$  (see e.g. [23] theorem on page 908 and [14, Theorem 1.2]), hence Theorem 1.1 covers the estimate in [21].

Our proof goes along the line of Bakry–Qian [3]. The technique is based on a comparison theorem on the gradient of the first eigenfunction with that of a one-dimensional (1-D) model function (Theorem 3.1), which was developed by Kröger [8] and improved by Chen–Wang [5] and Bakry–Qian [3]. By using a Bochner–Weitzenböck formula established recently by Ohta–Sturm [16], we find that the one-dimensional model coincides with that in the Riemannian case, as presented in Theorem 1.1. It should be not so surprising, because when we consider  $F$  in  $\mathbb{R}$ , it can only be two pieces of linear functions. Since the 1-D model has been extensively studied in [3], it also eases our situation, although we deal with a nonlinear operator. One difficulty arises when we deal with the Neumann boundary problem, since the convexity of boundary could not be directly applied due to the difference between the metric induced from the boundary itself and the metric induced from the gradient of the first eigenfunction. We will establish some equivalence between them (see Lemmas 3.1 and 3.2) to overcome this difficulty. Another ingredient is a comparison theorem on the maxima of eigenfunction with that of the 1-D model function (Theorem 3.2). Everything in [3] works except the boundedness of the Hessian of eigenfunctions around a critical point (since the eigenfunction is only  $C^{1,\alpha}$  among  $M$ ), which was used to prove (25). Here we avoid the use of the Hessian of eigenfunctions by using the comparison theorem on the gradient. For the rest we follow step by step the work of Bakry–Qian [3] to get Theorem 1.1.

This paper is organized as follows. In Section 2, the fundamentals in Finsler geometry is briefly introduced and the recent work of Ohta–Sturm is reviewed. We shall first prove the comparison theorem on the gradient and on the maxima of the eigenfunction and then Theorem 1.1 in Section 3.

## 2. Preliminaries on Finsler geometry

In this section we briefly recall the fundamentals of Finsler geometry, as well as the recent developments on the analysis of Finsler geometry by Ohta–Sturm [14–16]. For Finsler geometry, we refer to [1] and [19].

### 2.1. Finsler structure and Chern connection

Let  $M^n$  be a smooth, connected  $n$ -dimensional manifold. A function  $F : TM \rightarrow [0, \infty)$  is called a *Finsler structure* if it satisfies the following properties:

- (i)  $F$  is  $C^\infty$  on  $TM \setminus \{0\}$ ;
- (ii)  $F(x, tV) = tF(x, V)$  for all  $(x, y) \in TM$  and all  $t > 0$ ;
- (iii) for every  $(x, V) \in TM \setminus \{0\}$ , the matrix

$$g_{ij}(V) := \frac{\partial^2}{\partial V_i \partial V_j} \left( \frac{1}{2} F^2 \right) (x, V)$$

is positive definite.

Such a pair  $(M^n, F)$  is called a *Finsler manifold*. A Finsler structure is said to be *reversible* if, in addition,  $F$  is even. Otherwise  $F$  is nonreversible. By a *Finsler measure space* we mean a triple  $(M^n, F, m)$  constituted with a smooth, connected  $n$ -dimensional manifold  $M$ , a Finsler structure  $F$  on  $M$  and a measure  $m$  on  $M$ .

For  $x_1, x_2 \in M$ , the *distance function* from  $x_1$  to  $x_2$  is defined by

$$d(x_1, x_2) := \inf_{\gamma} \int_0^1 F(\dot{\gamma}(t)) dt,$$

where the infimum is taken over all  $C^1$ -curves  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = x_1$  and  $\gamma(1) = x_2$ . Note that the distance function may not be symmetric unless  $F$  is reversible. A  $C^\infty$ -curve  $\gamma : [0, 1] \rightarrow M$  is called a *geodesic* if  $F(\dot{\gamma})$  is constant and it is locally minimizing. The *diameter* of  $M$  is defined by

$$d := \sup_{x,y \in M} d(x, y).$$

The forward and backward open balls are defined by

$$B^+(x, r) := \{y \in M: d(x, y) < r\}, \quad B^-(x, r) := \{y \in M: d(y, x) < r\}.$$

We denote  $B^\pm(x, r) := B^+(x, r) \cup B^-(x, r)$ .

For every nonvanishing vector field  $V$ ,  $g_{ij}(V)$  induces a Riemannian structure  $g_V$  of  $T_x M$  via

$$g_V(X, Y) = \sum_{i,j=1}^n g_{ij}(V) X^i Y^j, \quad \text{for } X, Y \in T_x M.$$

In particular,  $g_V(V, V) = F^2(V)$ .

Let  $\pi : TM \setminus \{0\} \rightarrow M$  the projection map. The pull-back bundle  $\pi^*TM$  admits a unique linear connection, which is the Chern connection. The Chern connection is determined by the following structure equations, which characterize ‘‘torsion freeness’’:

$$D_X^V Y - D_Y^V X = [X, Y] \tag{7}$$

and ‘‘almost  $g$ -compatibility’’

$$Z(g_V(X, Y)) = g_V(D_Z^V X, Y) + g_V(X, D_Z^V Y) + C_V(D_Z^V V, X, Y) \tag{8}$$

for  $V \in TM \setminus \{0\}$ ,  $X, Y, Z \in TM$ . Here

$$C_V(X, Y, Z) := C_{ijk}(V) X^i Y^j Z^k = \frac{1}{4} \frac{\partial^3 F^2}{\partial V^i \partial V^j \partial V^k}(V) X^i Y^j Z^k$$

denotes the Cartan tensor and  $D_X^V Y$  the covariant derivative with respect to reference vector  $V \in TM \setminus \{0\}$ . We mention here that  $C_V(V, X, Y) = 0$  due to the homogeneity of  $F$ . In terms of the Chern connection, a geodesic  $\gamma$  satisfies  $D_{\dot{\gamma}}^{\dot{\gamma}} \dot{\gamma} = 0$ . For local computations in Finsler geometry, we refer to [19].

### 2.2. Hessian and Finsler-Laplacian

We shall introduce the Finsler-Laplacian on Finsler manifolds. First of all, we recall the notion of the Legendre transform.

Given a Finsler structure  $F$  on  $M$ , there is a natural dual norm  $F^*$  on the cotangent space  $T^*M$ , which is defined by

$$F^*(x, \xi) := \sup_{F(x,V) \leq 1} \xi(V) \quad \text{for any } \xi \in T_x^*M.$$

One can show that  $F^*$  is also a Minkowski norm on  $T^*M$  and

$$g_{ij}^*(\xi) := \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left( \frac{1}{2} F^{*2} \right) (x, \xi)$$

is positive definite for every  $(x, \xi) \in T^*M \setminus \{0\}$ .

The Legendre transform is defined by the map  $l : T_x M \rightarrow T_x^*M$ :

$$l(V) := \begin{cases} g_V(V, \cdot) & \text{for } V \in T_x M \setminus \{0\}, \\ 0 & \text{for } V = 0. \end{cases}$$

One can verify that  $F(V) = F^*(l(V))$  for any  $V \in TM$  and  $g_{ij}^*(x, l(V))$  is the inverse matrix of  $g_{ij}(x, V)$ .

Let  $u : M \rightarrow \mathbb{R}$  be a smooth function on  $M$  and  $Du$  be its differential 1-form. The gradient of  $u$  is defined as  $\nabla u(x) := l^{-1}(Du(x)) \in T_x M$ . Denote  $M_u := \{Du \neq 0\}$ . Locally we can write in coordinates

$$\nabla u = \sum_{i,j=1}^n g^{ij}(x, \nabla u) \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j} \quad \text{in } M_u.$$

The Hessian of  $u$  is defined by using Chern connection as

$$\nabla^2 u(X, Y) = g_{\nabla u}(D_X^{\nabla u} \nabla u, Y). \tag{9}$$

One can show that  $\nabla^2 u(X, Y)$  is symmetric, see [22] and [16]. Indeed, using (7) and (8) and noticing that  $C_{\nabla u}(\nabla u, X, Y) = 0$ , we have

$$\begin{aligned} g_{\nabla u}(D_X^{\nabla u} \nabla u, Y) &= X(g_{\nabla u}(\nabla u, Y)) - g_{\nabla u}(\nabla u, D_X^{\nabla u} Y) \\ &= XY(u) - g_{\nabla u}(\nabla u, D_Y^{\nabla u} X + [X, Y]) \\ &= YX(u) + [X, Y](u) - g_{\nabla u}(\nabla u, D_Y^{\nabla u} X) - [X, Y](u) \\ &= Y(g_{\nabla u}(\nabla u, X)) - g_{\nabla u}(\nabla u, D_Y^{\nabla u} X) = g_{\nabla u}(D_Y^{\nabla u} \nabla u, X). \end{aligned}$$

In order to define a Laplacian on Finsler manifolds, we need a measure  $m$  (or a volume form  $dm$ ) on  $M$ . From now on, we consider the Finsler measure space  $(M, F, m)$  equipped with a fixed smooth measure  $m$ . Let  $V \in TM$  be a smooth vector field on  $M$ . The divergence of  $V$  with respect to  $m$  is defined by

$$\operatorname{div}_m V \, dm = d(V \lrcorner dm),$$

where  $V \lrcorner dm$  denotes the inner product of  $V$  with the volume form  $dm$ . In a local coordinate  $(x^i)$ , expressing  $dm = e^\Phi dx^1 dx^2 \dots dx^n$ , we can write  $\operatorname{div}_m V$  as

$$\operatorname{div}_m V = \sum_{i=1}^n \left( \frac{\partial V^i}{\partial x^i} + V^i \frac{\partial \Phi}{\partial x^i} \right).$$

A Laplacian, which is called the *Finsler-Laplacian*, can now be defined by

$$\Delta_m u := \operatorname{div}_m(\nabla u).$$

We remark that the Finsler-Laplacian is better to be viewed in a weak sense that for  $u \in W^{1,2}(M)$ ,

$$\int_M \phi \Delta_m u \, dm = - \int_M D\phi(\nabla u) \, dm \quad \text{for } \phi \in C_c^\infty(M).$$

The relationship between  $\Delta_m u$  and  $\nabla^2 u$  is that

$$\Delta_m u + D\Psi(\nabla u) = \operatorname{tr}_{g_{\nabla u}}(\nabla^2 u) = \sum_{i=1}^n \nabla^2 u(e_i, e_i),$$

where  $\Psi$  is defined by  $dm = e^{-\Psi(V)} d \operatorname{Vol}_{g_V}$  and  $\{e_i\}$  is an orthonormal basis of  $T_x M$  with respect to  $g_{\nabla u}$ . See e.g. [22, Lemma 3.3].

Given a vector field  $V$ , the *weighted Laplacian* is defined on the weighted Riemannian manifold  $(M, g_V, m)$  by

$$\Delta_m^V u := \operatorname{div}_m(\nabla^V u),$$

where

$$\nabla^V u := \begin{cases} \sum_{i,j=1}^n g^{ij}(x, V) \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j} & \text{for } x \in M_u, \\ 0 & \text{for } x \in M \setminus M_u. \end{cases}$$

Similarly, the weighted Laplacian can be viewed in a weak sense that for  $u \in W^{1,2}(M)$ . We note that  $\Delta_m^{\nabla u} u = \Delta_m u$ .

### 2.3. Finsler manifolds with boundary

Assume that  $(M, F, m)$  is a Finsler measure space with boundary  $\partial M$ , then we shall view  $\partial M$  as a hypersurface embedded in  $M$ .  $\partial M$  is also a Finsler manifold with a Finsler structure  $F_{\partial M}$  induced by  $F$ . For any  $x \in \partial M$ , there exists exactly two unit *normal vectors*  $\nu$ , which are characterized by

$$T_x(\partial M) = \{V \in T_x M: g_\nu(\nu, V) = 0, g_\nu(\nu, \nu) = 1\}.$$

Throughout this paper, we choose the normal vector that points outward  $M$ . Note that, if  $\nu$  is a normal vector,  $-\nu$  may be not a normal vector unless  $F$  is reversible.

The normal vector  $\nu$  induces a volume form  $dm_\nu$  on  $\partial M$  from  $dm$  by

$$V \lrcorner dm = g_\nu(\nu, V) dm_\nu, \quad \text{for all } V \in T(\partial M).$$

One can check that Stokes theorem holds (see [19, Theorem 2.4.2])

$$\int_M \operatorname{div}_m(V) dm = \int_{\partial M} g_\nu(\nu, V) dm_\nu.$$

We recall the convexity of the boundary of  $M$ .

The *normal curvature*  $\Lambda_\nu(V)$  at  $x \in \partial M$  in a direction  $V \in T_x(\partial M)$  is defined by

$$\Lambda_\nu(V) = g_\nu(\nu, D_{\dot{\gamma}}^V \dot{\gamma}(0)), \tag{10}$$

where  $\gamma$  is the unique local geodesic for the Finsler structure  $F_{\partial M}$  on  $\partial M$  induced by  $F$  with the initial data  $\gamma(0) = x$  and  $\dot{\gamma}(0) = V$ .

$M$  is said to has *convex boundary* if for any  $x \in \partial M$ , the normal curvature  $\Lambda_\nu$  at  $x$  is nonpositive in any directions  $V \in T_x(\partial M)$ . We remark that the convexity of  $M$  means that  $D_{\dot{\gamma}}^V \dot{\gamma}(0)$  lies at the same side of  $T_x M$  as  $M$ . Hence the choice of normal is not essential for the definition of convexity. (See Lemma 3.2 below.) There are several equivalent definitions of convexity, see for example [2] and [19].

### 2.4. Weighted Ricci curvature

The Ricci curvature of Finsler manifolds is defined as the trace of the flag curvature. Explicitly, given two linearly independent vectors  $V, W \in T_x M \setminus \{0\}$ , the *flag curvature* is defined by

$$\mathcal{K}^V(V, W) = \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where  $R^V$  is the *Chern curvature* (or *Riemannian curvature*):

$$R^V(X, Y)Z = D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X, Y]}^V Z.$$

Then the *Ricci curvature* is defined by

$$\operatorname{Ric}(V) := \sum_{i=1}^{n-1} \mathcal{K}^V(V, e_i),$$

where  $e_1, \dots, e_{n-1}, \frac{V}{F(V)}$  form an orthonormal basis of  $T_x M$  with respect to  $g_V$ .

We recall the definition of the weighted Ricci curvature on Finsler manifolds, which was introduced by Ohta in [14], motivated by the work of Lott–Villani [12] and Sturm [20] on metric measure spaces.

**Definition 2.1.** (See [14].) Given a unit vector  $V \in T_x M$ , let  $\eta: [-\varepsilon, \varepsilon] \rightarrow M$  be the geodesic such that  $\dot{\eta}(0) = V$ . Decompose  $m$  as  $m = e^{-\Psi} d \operatorname{vol}_\eta$  along  $\eta$ , where  $\operatorname{vol}_\eta$  is the volume form of  $g_\eta$  as a Riemannian metric. Then

$$\operatorname{Ric}_n(V) := \begin{cases} \operatorname{Ric}(V) + (\Psi \circ \eta)''(0) & \text{if } (\Psi \circ \eta)'(0) = 0, \\ -\infty & \text{otherwise;} \end{cases}$$

$$\operatorname{Ric}_N(V) := \operatorname{Ric}(V) + (\Psi \circ \eta)''(0) - \frac{(\Psi \circ \eta)'(0)^2}{N - n}, \quad \text{for } N \in (n, \infty),$$

$$\operatorname{Ric}_\infty(V) := \operatorname{Ric}(V) + (\Psi \circ \eta)''(0).$$

For  $c \geq 0$  and  $N \in [n, \infty]$ , define

$$\operatorname{Ric}_N(cV) := c^2 \operatorname{Ric}_N(V).$$

Ohta proved in [14] that, for  $K \in \mathbb{R}$ , the bound  $\operatorname{Ric}_N(V) \geq K F^2(V)$  is equivalent to Lott–Villani and Sturm’s *weak curvature-dimension condition*  $CD(K, N)$ .

### 2.5. Bochner–Weitzenböck formula

The following Bochner–Weitzenböck type formula, established by Ohta–Sturm in [16], plays an important role in this paper.

**Theorem 2.1.** (See [16, Theorem 3.6].) Given  $u \in W_{loc}^{2,2}(M) \cap C^1(M)$  with  $\Delta_m u \in W_{loc}^{1,2}(M)$ , we have

$$-\int_M D\eta \left( \nabla^{\nabla u} \left( \frac{F^2(x, \nabla u)}{2} \right) \right) dm = \int_M \eta \left\{ D(\Delta_m u)(\nabla u) + Ric_\infty(\nabla u) + \|\nabla^2 u\|_{HS(\nabla u)}^2 \right\} dm$$

as well as

$$-\int_M D\eta \left( \nabla^{\nabla u} \left( \frac{F^2(x, \nabla u)}{2} \right) \right) dm \geq \int_M \eta \left\{ D(\Delta_m u)(\nabla u) + Ric_N(\nabla u) + \frac{(\Delta_m u)^2}{N} \right\} dm$$

for any  $N \in [n, \infty]$  and all nonnegative functions  $\eta \in W_c^{1,2}(M) \cap L^\infty(M)$ . Here  $\|\nabla^2 u\|_{HS(\nabla u)}^2$  denotes the Hilbert–Schmidt norm with respect to  $g_{\nabla u}$ .

Based on Bochner–Weitzenböck formula, a similar argument as Bakry–Qian [3, Theorem 6], leads to a refined inequality, which was referred to as an extended curvature-dimension inequality there. Another direct proof was also given in [21, Lemma 2.3].

**Theorem 2.2.** Assume that  $Ric_N \geq K$  for some  $N \in [n, \infty]$  and some  $K \in \mathbb{R}$ . Given  $u \in W_{loc}^{2,2}(M) \cap C^1(M)$  with  $\Delta_m u \in W_{loc}^{1,2}(M)$ , we have

$$\begin{aligned} -\int_M D\eta \left( \nabla^{\nabla u} \left( \frac{F^2(x, \nabla u)}{2} \right) \right) dm &\geq \int_M \eta \left\{ D(\Delta_m u)(\nabla u) + K F(\nabla u)^2 + \frac{(\Delta_m u)^2}{N} \right. \\ &\quad \left. + \frac{N}{N-1} \left( \frac{\Delta_m u}{N} - \frac{D(F^2(x, \nabla u))(\nabla u)}{2F^2(x, \nabla u)} \right)^2 \right\} dm \end{aligned} \tag{11}$$

for any  $N \in [n, \infty]$  and all nonnegative functions  $\eta \in W_c^{1,2}(M) \cap L^\infty(M)$ .

### 3. Proof of Theorem 1.1

We first remark that a weak eigenfunction  $u \in W^{1,2}(M)$  of Finsler-Laplacian defined in (1) has regularity that  $u \in C^{1,\alpha}(M) \cap W^{2,2}(M) \cap C^\infty(M_u)$  (see [6]).

Let us recall the 1-D models  $L_{K,N}$  described in [3]. Let  $K \in \mathbb{R}$  and  $N \in (1, \infty]$ .

(i) For  $K > 0$  and  $1 < N < \infty$ ,  $L_{K,N}$  is defined on  $(-\frac{\pi}{2\sqrt{K/(N-1)}}, \frac{\pi}{2\sqrt{K/(N-1)}})$  by

$$L_{K,N}(v)(t) = v'' - \sqrt{K(N-1)} \tan\left(\sqrt{\frac{K}{N-1}}t\right)v';$$

(ii) For  $K < 0$  and  $1 < N < \infty$ ,  $L_{K,N}$  is defined on  $(-\infty, 0) \cup (0, \infty)$  by

$$L_{K,N}(v)(t) = v'' - \sqrt{-K(N-1)} \coth\left(\sqrt{-\frac{K}{N-1}}t\right)v'$$

and on  $(-\infty, \infty)$  by

$$L_{K,N}(v)(t) = v'' - \sqrt{-K(N-1)} \tanh\left(\sqrt{-\frac{K}{N-1}}t\right)v';$$

(iii) For  $K = 0$  and  $1 < N < \infty$ ,  $L_{K,N}$  is defined on  $(-\infty, 0) \cup (0, \infty)$  by

$$L_{K,N}(v)(t) = v'' + \frac{N-1}{t}v'$$

and on  $(-\infty, \infty)$  by

$$L_{K,N}(v)(t) = v'';$$

(iv) For  $K \neq 0$  and  $N = \infty$ ,  $L_{K,N}$  is defined on  $(-\infty, \infty)$  by

$$L_{K,N}(v)(t) = v'' - Ktv';$$

(v) For  $K = 0$  and  $N = \infty$ ,  $L_{K,N}$  is defined on  $(-\infty, \infty)$  by

$$L_{K,N}(v)(t) = v'' - cv'$$

for any constant  $c$ .

For convenience, we write  $L_{K,N}(v)(t) = v'' - T(t)v'$ . It is easy to check that  $T' = K + \frac{T^2}{N-1}$ . Denote by  $\mu_{K,N}$  the invariant measure associated with  $L_{K,N}$ , that is, a measure satisfying  $\int_a^b L_{K,N}(v) d\mu_{K,N} = 0$  for  $v'(a) = v'(b) = 0$ . For instance, in the case (i),  $d\mu_{K,N} = \cos^{N-1}(\sqrt{\frac{K}{N-1}}t) dt$ .

The following gradient comparison theorem plays the most crucial role in the proof of our main theorem.

**Theorem 3.1.** *Let  $(M, F, m)$  and  $\lambda_1$  be as in Theorem 1.1 and  $u$  be the eigenfunction. Let  $v$  be a solution of the 1-D model problem on some interval  $(a, b)$ :*

$$L_{K,N}(v) = -\lambda_1 v, \quad v'(a) = v'(b) = 0, \quad v' > 0. \tag{12}$$

Assume that  $[\min u, \max u] \subset [\min v, \max v]$ , then

$$F(x, \nabla u(x)) \leq v'(v^{-1}(u(x))). \tag{13}$$

**Proof.** First, since  $\int_M u = 0$ ,  $\min u < 0$  while  $\max u > 0$ . We may assume that  $[\min u, \max u] \subset (\min v, \max v)$  by multiplying  $u$  by a constant  $0 < c < 1$ . If we prove the result for this  $u$ , then letting  $c \rightarrow 1$  implies the original statement.

Under the condition  $[\min u, \max u] \subset (\min v, \max v)$ ,  $v^{-1}$  is smooth on a neighborhood  $U$  of  $[\min u, \max u]$ .

Consider  $P(x) = \psi(u)(\frac{1}{2}F^2(x, \nabla u) - \phi(u))$ , where  $\psi, \phi \in C^\infty(U)$  are two positive smooth functions to be determined later. We first consider the case that  $P$  attains its maximum at  $x_0 \in M$ , then study the case that  $x_0 \in \partial M$  if  $\partial M$  is not empty.

*Case 1.*  $P$  attains its maximum at  $x_0 \in M$ .

Due to the lack of regularity of  $u$ , we shall compute in the distributional sense. Let  $\eta$  be any nonnegative function in  $W_c^{1,2}(M) \cap L^\infty(M)$ . We first compute  $-\int_M D\eta(\nabla^{\nabla u} P) dm$ .

$$\begin{aligned} -\int_M D\eta(\nabla^{\nabla u} P) dm &= -\int_M \left( \frac{\psi'}{\psi} P - \psi \phi' \right) D\eta(\nabla u) + \psi D\eta \left( \nabla^{\nabla u} \left( \frac{1}{2} F^2(x, \nabla u) \right) \right) dm \\ &= \int_M -D \left[ \left( \frac{\psi'}{\psi} P - \psi \phi' \right) \eta \right] (\nabla u) + \eta D \left( \frac{\psi'}{\psi} P - \psi \phi' \right) (\nabla u) \\ &\quad - D(\psi \eta) \left( \nabla^{\nabla u} \left( \frac{1}{2} F^2(x, \nabla u) \right) \right) + \eta D\psi \left( \nabla^{\nabla u} \left( \frac{1}{2} F^2(x, \nabla u) \right) \right) dm \\ &:= I + II + III + IV. \end{aligned}$$



By using  $Du(\nabla u) = F^2(x, \nabla u) = 2\left(\frac{P}{\psi} + \phi\right)$  and  $\Delta_m u = -\lambda_1 u$  in weak sense, we compute

$$\begin{aligned} I &= \int_M -\lambda_1 u \left( \frac{\psi'}{\psi} P - \psi \phi' \right) \eta \, dm, \\ II &= \int_M \eta \left[ \left( \left( \frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} \right) P - \psi \phi'' - \psi' \phi' \right) Du(\nabla u) + \frac{\psi'}{\psi} DP(\nabla u) \right] dm \\ &= \int_M \eta \left[ 2 \left( \left( \frac{\psi''}{\psi} - \frac{\psi'^2}{\psi^2} \right) P - \psi \phi'' - \psi' \phi' \right) \left( \frac{P}{\psi} + \phi \right) + \frac{\psi'}{\psi} DP(\nabla u) \right] dm, \\ IV &= \int_M \eta \psi' \left[ \frac{1}{\psi} Du(\nabla^{\nabla u} P) + \left( -\frac{\psi'}{\psi^2} P + \phi' \right) Du(\nabla u) \right] dm \\ &= \int_M \left\{ 2\eta \psi' \left( -\frac{\psi'}{\psi^2} P + \phi' \right) \left( \frac{P}{\psi} + \phi \right) + \text{terms of } DP(\nabla u) \right\} dm. \end{aligned}$$

For the term III, we apply the refined integral Bochner–Weitzenböck formula (11) to derive

$$\begin{aligned} III &\geq \int_M \psi \eta \left[ D(\Delta_m u)(\nabla u) + K F^2 + \frac{(\Delta_m u)^2}{N} + \frac{N}{N-1} \left( \frac{\Delta_m u}{N} - \frac{D(F^2(x, \nabla u))(\nabla u)}{2F^2(x, \nabla u)} \right)^2 \right] dm \\ &= \int_M \psi \eta \left[ 2(K - \lambda_1) \left( \frac{P}{\psi} + \phi \right) + \frac{\lambda_1^2 u^2}{N} + \frac{N}{N-1} \left( \frac{-\lambda_1 u}{N} - \left( -\frac{\psi'}{\psi^2} P + \phi' \right) - \frac{1}{\psi F^2} DP(\nabla u) \right)^2 \right] dm \\ &= \int_M \psi \eta \left[ 2(K - \lambda_1) \left( \frac{P}{\psi} + \phi \right) + \frac{\lambda_1^2 u^2}{N-1} + \frac{N}{N-1} \left( -\frac{\psi'}{\psi^2} P + \phi' \right)^2 \right. \\ &\quad \left. + \frac{2}{N-1} \lambda_1 u \left( -\frac{\psi'}{\psi^2} P + \phi' \right) + \text{terms of } DP(\nabla u) \right] dm. \end{aligned}$$

Combining all we obtain

$$\begin{aligned} - \int_M D\eta(\nabla^{\nabla u} P) \, dm &\geq \int_M \eta \left\{ \frac{1}{\psi} \left[ 2\frac{\psi''}{\psi} - \left( 4 - \frac{N}{N-1} \right) \frac{\psi'^2}{\psi^2} \right] P^2 \right. \\ &\quad + \left[ 2\phi \left( \frac{\psi''}{\psi} - 2\frac{\psi'^2}{\psi^2} \right) - \frac{N+1}{N-1} \frac{\psi'}{\psi} \lambda u - \frac{2N}{N-1} \frac{\psi'}{\psi} \phi' + 2(K - \lambda_1) - 2\phi'' \right] P \\ &\quad + \psi \left[ \frac{1}{N-1} \lambda_1^2 u^2 + \frac{N+1}{N-1} \lambda_1 u \phi' + \frac{N}{N-1} \phi'^2 + 2(K - \lambda_1) \phi - 2\phi \phi'' \right] \\ &\quad \left. + \text{terms of } DP(\nabla u) \right\} dm \\ &:= - \int_M \{ a_1 P^2 + a_2 P + a_3 + \text{terms of } DP(\nabla u) \} dm. \end{aligned} \tag{14}$$

Therefore,

$$\Delta_m^{\nabla u} P + \text{terms of } DP(\nabla u) = a_1 P^2 + a_2 P + a_3 \tag{15}$$

holds in the distributional sense in  $M$ .

We claim that at the maximum point  $x_0$  of  $P$ ,

$$a_1 P^2 + a_2 P + a_3 \leq 0. \tag{16}$$

In fact, if not, then in a neighborhood  $\mathcal{U}$  of  $x_0$ ,  $a_1 P^2 + a_2 P + a_3 > 0$ . It follows from (15) that the function  $P$  is a strict subsolution to an elliptic operator in  $\mathcal{U}$ . By maximum principle,  $P(x_0) < \max_{\partial\mathcal{U}} P$ , which contradicts the maximality of  $P(x_0)$ .

It is interesting to see that the coefficients  $a_i, i = 1, 2, 3$ , coincide with that appeared in the Riemannian case (see e.g. [3, Lemma 1]). The next step is to choose suitable positive functions  $\psi$  and  $\phi$  such that  $a_1, a_2 > 0$  everywhere and  $a_3 = 0$ , which has already been done in [3]. For completeness, we sketch the main idea here.

Choose  $\phi(u) = \frac{1}{2}v'(v^{-1}(u))^2$ , where  $v$  is a solution of 1-D problem (12). One can compute that

$$\phi'(u) = v''(v^{-1}(u)), \quad \phi''(u) = \frac{v'''}{v'}(v^{-1}(u)).$$

Set  $t = v^{-1}(u)$  and  $u = v(t)$  then

$$\begin{aligned} \frac{a_3(t)}{\psi} &= \frac{1}{N-1}\lambda_1^2 v^2 + \frac{N+1}{N-1}\lambda_1 v v'' + \frac{N}{N-1}v''^2 + (K - \lambda_1)v'^2 - v'v''' \\ &= -v'(v'' - T v' + \lambda_1 v)' + \frac{1}{N-1}(v'' - T v' + \lambda_1 v)(N v'' + T v' + \lambda_1 v) = 0. \end{aligned}$$

Here we have used that  $T$  satisfies  $T' = K + \frac{T^2}{N-1}$ . For  $a_1, a_2$ , we introduce

$$X(t) = \lambda_1 \frac{v(t)}{v'(t)}, \quad \psi(u) = \exp\left(\int h(v(t))\right), \quad f(t) = -h(v(t))v'(t).$$

With these notations, we have

$$\begin{aligned} f' &= -h'v'^2 + f(T - X), \\ v'|_{v^{-1}a_1}^2 \psi &= 2f(T - X) - \frac{N-2}{N-1}f^2 - 2f' := 2(Q_1(f) - f'), \\ a_2 &= f\left(\frac{3N-1}{N-1}T - 2X\right) - 2T\left(\frac{N}{N-1}T - X\right) - f^2 - f' := Q_2(f) - f'. \end{aligned}$$

We may now use Corollary 3 in [3], which says that there exists a bounded function  $f$  on  $[\min u, \max u] \subset (\min v, \max v)$  such that  $f' < \min\{Q_1(f), Q_2(f)\}$ .

In view of (16), we know that by our choice of  $\psi$  and  $\phi$ ,  $P(x_0) \leq 0$ , and hence  $P(x) \leq 0$  for every  $x \in M$ , which leads to (13).

Case 2.  $\partial M \neq \emptyset$  and  $x_0 \in \partial M$ .

To handle this case, we need to define a new normal vector field on  $\partial M$ , that is normal with respect to the Riemannian metric  $g_{\nabla u}$ . To be more general, for every  $X \in TM$ , there is a unique normal vector field  $\nu_X$  such that

$$g_X(\nu_X, Y) = 0 \quad \text{for any } Y \in T(\partial M), \quad g_X(\nu_X, \nu_X) = 1, \quad g_\nu(\nu, \nu_X) > 0. \tag{17}$$

A simple calculation shows that

$$g_X(\nu, \nu_X) > 0. \tag{18}$$

Indeed, let  $\nu_X = Z + av$  for some  $a \in \mathbb{R}$  and  $Z \in T(\partial M)$ . (17) tells that  $a > 0$ . Hence  $g_X(\nu, \nu_X) = g_X(\frac{1}{a}(\nu_X - Z), \nu_X) = \frac{1}{a} > 0$ .

The following lemma follows directly from the definition of  $\nu$  and  $\nu_X$ .

**Lemma 3.1.** *Let  $X, Y \in TM$ . Then*

$$g_\nu(\nu, Y) = 0 \iff Y \in T(\partial M) \iff g_X(\nu_X, Y) = 0. \quad \square$$

Define four sets

$$T_{\pm}^\nu M := \{Y \in TM: g_\nu(\nu, Y) > 0 (< 0)\}$$

and

$$T_{\pm}^{\nu_X} M := \{Y \in TM: g_X(\nu_X, Y) > 0 (< 0)\}.$$

We have the following simple but important observation, which may be familiar to experts.

**Lemma 3.2.**  $T_+^{\nu} M = T_+^{\nu_X} M, \quad T_-^{\nu} M = T_-^{\nu_X} M.$

**Proof.** We first claim that either  $T_+^{\nu} M \subset T_+^{\nu_X} M$  or  $T_+^{\nu} M \subset T_-^{\nu_X} M$ . Otherwise, there are two vector fields  $Y_1, Y_2 \in T_+^{\nu} M$ , such that  $g_X(\nu_X, Y_1) > 0$  and  $g_X(\nu_X, Y_2) < 0$ . Then by the continuity of  $g_X(\nu_X, \cdot)$  in  $T_+^{\nu} M$ , there exists  $\bar{Y} \in T_+^{\nu} M$  with  $g_X(\nu_X, \bar{Y}) = 0$ , which means  $g_{\nu}(\nu, \bar{Y}) = 0$  from Lemma 3.1. A contradiction. Taking into consideration that  $\nu \in T_+^{\nu_X} M$ , we see that  $T_+^{\nu} M \subset T_+^{\nu_X} M$ . A similar argument implies that  $T_+^{\nu_X} M \subset T_+^{\nu} M$ . The second equivalence follows in a similar way.  $\square$

Return to the case when  $P$  attains its maximum at  $x_0 \in \partial M$ . If  $\nabla u(x_0) = 0$ , nothing needs to be proved. Thus we assume  $x_0 \in M_u$ . Recall that  $P \in C^\infty(M_u)$ . Since  $\nu_{\nabla u}$  points outward due to its definition, by taking normal derivative of  $P$  with respect to  $\nu_{\nabla u}$ , we have

$$DP(\nu_{\nabla u})(x_0) \geq 0.$$

On one hand, the Neumann boundary condition  $\nabla u \in T(\partial M)$  implies that

$$g_{\nabla u}(\nu_{\nabla u}, \nabla u)(x) = 0,$$

or equivalently,

$$Du(\nu_{\nabla u})(x) = 0 \quad \text{for } x \in \partial M.$$

Thus we have

$$DP(\nu_{\nabla u})(x_0) = \frac{1}{2} \psi(u)(D(F^2(\nabla u))(\nu_{\nabla u}))(x_0). \tag{19}$$

On the other hand, using (8) and the symmetry of  $\nabla^2 u$ , we have

$$D(F^2(\nabla u))(\nu_{\nabla u}) = D(g_{\nabla u}(\nabla u, \nabla u))(\nu_{\nabla u}) = 2g_{\nabla u}(D_{\nu_{\nabla u}}^{\nabla u}(\nabla u), \nabla u) = 2g_{\nabla u}(D_{\nabla u}^{\nabla u}(\nabla u), \nu_{\nabla u}). \tag{20}$$

By the convexity of  $\partial M$ , for any  $X \in T(\partial M)$ ,  $g_{\nu}(\nu, D_X^X X) \leq 0$ . In particular, set  $X = \nabla u$ , we know that

$$g_{\nu}(\nu, D_{\nabla u}^{\nabla u}(\nabla u)) \leq 0. \tag{21}$$

It follows from Lemmas 3.1 and 3.2 that (21) is equivalent to

$$g_{\nabla u}(\nu_{\nabla u}, D_{\nabla u}^{\nabla u}(\nabla u)) \leq 0. \tag{22}$$

Combining (19), (20) and (22), we conclude that  $DP(\nu_{\nabla u})(x_0) \leq 0$ , and hence  $DP(\nu_{\nabla u})(x_0) = 0$ . The tangent derivatives of  $P$  obviously vanish due to its maximality. Hence we have also

$$\nabla P(x_0) = 0.$$

Thus the proof for Case 1 works in this case. This finishes the proof of Theorem 3.1.  $\square$

Another ingredient is a comparison theorem for the maxima of the eigenfunctions.

**Theorem 3.2.** Let  $(M, F, m), \lambda_1$  be as in Theorem 1.1 and  $1 < N < \infty$ . Let  $v = v_{K,N}$  be a solution of the 1-D model problem on some interval  $(a, b)$   $L_{K,N} v = -\lambda_1 v$ , with initial data  $v(a) = -1, v'(a) = 0$ , where

$$a = \begin{cases} -\frac{\pi}{2\sqrt{K/(N-1)}} & \text{for } K > 0, \\ 0 & \text{for } K \leq 0 \end{cases}$$

and  $b = b(a)$  be the first number after  $a$  with  $v'(b) = 0$ . Denote  $m_{K,N} = v_{K,N}(b) = \max(v)$ . Assume that  $\lambda_1 > \max\{\frac{KN}{N-1}, 0\}$  and  $\min(u) = -1$ . Then  $\max u \geq m_{K,N}$ .

**Proof.** We argue by contradiction. Suppose  $\max(u) < m_{K,N}$ . Then  $[\min u, \max u] \subset [\min v, \max v]$ . The condition  $\lambda_1 > \max\{\frac{KN}{N-1}, 0\}$  ensures that

$$b \leq \begin{cases} \frac{\pi}{2\sqrt{K/(N-1)}} & \text{for } K > 0, \\ \infty & \text{for } K \leq 0, \end{cases}$$

which in turn ensures that  $v' > 0$  in  $(a, b)$ . Hence we could apply Theorem 3.1 for  $u$  and  $v$ .

The same argument as Theorem 12 in [3] implies that the ratio

$$R(c) = \frac{\int_{\{u \leq c\}} u \, dm}{\int_{\{v \leq c\}} v \, d\mu_{K,N}}$$

is increasing on  $[\min(u), 0]$  and decreasing on  $[0, \max(u)]$ . Therefore, for  $c \leq -\frac{1}{2}$ , we have that

$$m(\{u \leq c\}) \leq 2 \int_{\{u \leq c\}} |u| \, dm \leq 2R(0) \int_{\{v \leq c\}} |v| \, d\mu_{K,N} \leq 2R(0)\mu_{K,N}(\{v \leq c\}). \tag{23}$$

Let  $c = -1 + \varepsilon$  for  $\varepsilon > 0$  small. A simple calculation gives that  $v''(a) = \frac{\lambda_1}{N}$ . Hence for  $t$  close to  $a$ ,  $v''(t)$  has positive lower and upper bound. Together with  $v'(a) = 0$ , we see that  $v(t) - v(a) \geq C(t - a)^2$ . Thus if  $t \in \{v \leq -1 + \varepsilon\}$ , then  $t \in (a, a + C\varepsilon^{\frac{1}{2}})$ . It follows that

$$\mu_{K,N}(\{v \leq -1 + \varepsilon\}) \leq \mu_{K,N}((a, a + C\varepsilon^{\frac{1}{2}})) \leq C\varepsilon^{N/2}. \tag{24}$$

On the other hand, we shall prove that

$$m(\{u \leq -1 + \varepsilon\}) \geq m(B^\pm(x_0, C\varepsilon^{\frac{1}{2}})). \tag{25}$$

Let  $x_0 \in M$  be such that  $u(x_0) = -1$ . For any  $x \in B^\pm(x_0, \delta)$  with  $\delta$  small,  $u(x)$  is close to  $-1$  and  $s := v^{-1}(u(x))$  is close to  $a$ . Thus we see again from the upper bound of  $v''$  and  $v'(a) = 0$  that  $v'(s) \leq C(s - a)$ . Therefore, we have from Theorem 3.1 that  $F(x, \nabla u(x)) \leq v'(v^{-1}(u(x))) \leq C(s - a)$  and  $F(x, \nabla v^{-1}(u(x))) = (v^{-1})'(u(x))F(x, \nabla u(x)) \leq 1$ . In turn, we get

$$s - a = v^{-1}(u(x)) - v^{-1}(u(x_0)) \leq F(\tilde{x}, \nabla v^{-1}(u(\tilde{x})))\delta \leq \delta,$$

and

$$u(x) \leq u(x_0) + F(\tilde{x}, \nabla u(\tilde{x}))\delta \leq -1 + C(s - a)\delta \leq -1 + C\delta^2,$$

for some  $\tilde{x}, \tilde{\tilde{x}} \in B^\pm(x_0, \delta)$ . Let  $\varepsilon = C\delta^2$ , we conclude  $B^\pm(x_0, \delta) \subset \{u \leq -1 + \varepsilon\}$ , which implies (25).

Combining (23), (24) and (25), we see that there exists some constant  $C > 0$  such that

$$m(B^\pm(x_0, r)) \leq Cr^N. \tag{26}$$

This will lead to a contradiction. In fact, since  $\max(u) < m_{K,N}$  and  $m_{K,N}$  is continuous with respect to  $(K, N)$ , we also have that  $\max(u) < m_{K,N'}$  for any  $N' > N$  close to  $N$ . Argued as before, we will obtain (26) with  $N'$  instead of  $N$ , i.e.

$$m(B^\pm(x_0, r)) \leq Cr^{N'}. \tag{27}$$

However, the volume comparison theorem for Finsler manifolds under the assumption of lower bound for  $Ric_N$  (see [14, Theorem 7.3]), implies that  $m(B^\pm(x_0, r)) \geq Cr^N$  for  $r > 0$  small. A contradiction to (27). The previous argument also works in the case  $x_0 \in \partial M$ . The proof is completed.  $\square$

Besides the comparison theorem on the gradient and maxima, in order to prove Theorem 1.1, we also need some properties of the 1-D models, which has been extensively studied in [3]. We refer to [3] for the elementary properties, meanwhile we list two of them, one presents the full range of the maximum function  $m_{K,N}$ , the other reveals that the central interval has the lowest first Neumann eigenvalue.

**Lemma 3.3.** (See [3, Section 3].) Assume  $1 < N < \infty$  ( $N = \infty$  resp.) and fix  $\lambda > \max\{\frac{KN}{N-1}, 0\}$ . Let  $v, m$  be as in Theorem 3.2. Then for any  $k \in [m, \frac{1}{m}]$  ( $(0, \infty)$ , resp.), there exists an interval which has the first Neumann eigenvalue  $\lambda$  and a corresponding eigenfunction  $\tilde{v}$  such that  $\min \tilde{v} = -1$ ,  $\max \tilde{v} = k$ .

**Lemma 3.4.** (See [3, Theorem 13].) Let  $\lambda_1(K, N, a, b)$  denotes the first Neumann eigenvalue of  $L_{K,N}$  on the interval  $(a, b)$ . Then  $\lambda_1(K, N, a, b) \geq \lambda_1(K, N, -\frac{b-a}{2}, \frac{b-a}{2}) = \lambda_1(K, N, b-a)$ .

We now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Without loss of generality, we may assume that  $\min u = -1$  and  $0 < \max u := k \leq 1$ . It was shown by Ohta [14, Corollary 8.5], that  $\lambda_1 \geq \frac{NK}{N-1}$  in the case of  $K > 0$ . Choose  $\tilde{K} < K$  close to  $K$ , we have  $\lambda_1 > \max\{\frac{\tilde{K}N}{N-1}, 0\}$ . Therefore, Theorem 3.2 and Lemma 3.3 imply that there exists an interval  $[a, b]$  which has the first Neumann eigenvalue  $\lambda_1$  and a corresponding eigenfunction  $v$  such that  $\min v = -1 = \min u$ ,  $\max v = \max u = k$ . Choose  $x_1, x_2 \in M$  with  $u(x_1) = \min u$ ,  $u(x_2) = k$  and  $\gamma(t) : [0, 1] \rightarrow M$  the minimal geodesic from  $x_1$  to  $x_2$ . Consider the subset  $I$  of  $[0, 1]$  such that  $\frac{d}{dt}u(\gamma(t)) \geq 0$ . By using Theorem 3.1, we have

$$\begin{aligned} d &\geq \int_0^1 F(\dot{\gamma}(t)) dt \geq \int_I F(\dot{\gamma}(t)) dt \geq \int_0^1 \frac{1}{F^*(Du)} Du(\dot{\gamma}(t)) dt = \int_{-1}^k \frac{1}{F(\nabla u)} du \\ &\geq \int_{-1}^k \frac{1}{v'(v^{-1}(u))} du = \int_a^b dt = b - a. \end{aligned}$$

A general property says that  $\lambda_1(\tilde{K}, N, d)$  is monotone decreasing with respect to  $d$ . Hence  $\lambda_1(\tilde{K}, N, b-a) \geq \lambda_1(\tilde{K}, N, d)$ . Finally, it follows from Lemma 3.4 that

$$\lambda_1 \geq \lambda_1(\tilde{K}, N, b-a) \geq \lambda_1(\tilde{K}, N, d).$$

By letting  $\tilde{K} \rightarrow K$ , we get the conclusion  $\lambda_1 \geq \lambda_1(K, N, d)$ .  $\square$

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