# On the multiple existence of semi-positive solutions for a nonlinear Schrödinger system 

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#### Abstract

The paper concerns multiplicity of vector solutions for nonlinear Schrödinger systems, in particular of semi-positive solutions. New variational techniques are developed to study the existence of this type of solutions. Asymptotic behaviors are examined in various parameter regimes including both attractive and repulsive cases.


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## 0. Introduction

In this paper, we consider the following nonlinear Schrödinger systems:

$$
\begin{array}{ll}
-\Delta u+\lambda_{1} u=\mu_{1} u^{3}+\beta u v^{2} & \text { in } \Omega, \\
-\Delta v+\lambda_{2} v=\mu_{2} v^{3}+\beta u^{2} v & \text { in } \Omega,  \tag{*}\\
& u, v \in H_{0}^{1}(\Omega) .
\end{array}
$$

Here $\Omega$ is a bounded domain in $\mathbf{R}^{n}(n \leqslant 3)$ and $\lambda_{i}, \mu_{i}>0$ for $i=1$, 2. In this paper, we show the multiple existence of semi-positive solutions ( $u_{k}, v_{k}$ ) for $(*)$. As there may be semi-trivial solutions (which are zero for some components) we call a solution non-trivial if every component is non-zero. Here we say a non-trivial solution $(u, v)$ is a semipositive solution for $(*)$ if and only if it satisfies $u>0$ or $v>0$ in $\Omega$.

For positive solutions (which means $u>0$ and $v>0$ in $\Omega$ ) of nonlinear Schrödinger systems, there has been extensive work in recent years (cf. [1-7,11,13,15-22,24,27-30] and their references). In particular, we refer to results of [13] which partially inspire our work of the current paper. Dancer, Wei and Weth [13] showed that the a priori bounds of positive solutions and the multiplicity of positive solutions of nonlinear Schrödinger systems are complementary to each other depending on the parameter regimes. They showed the existence of a priori bounds of positive solutions

[^0]for some nonlinear Schrödinger systems which contain (*). Applying their result to ( $*$ ), when $\beta>-\sqrt{\mu_{1} \mu_{2}}$, there exists a constant $C=C\left(\beta, \mu_{1}, \mu_{2}, \Omega\right)$ such that $\|u\|_{L^{\infty}(\Omega)},\|v\|_{L^{\infty}(\Omega)} \leqslant C$ for any positive solutions (u,v). On the other hand, when $\lambda_{1}=\lambda_{2}=\mu_{1}=\mu_{2}=1$ in $(*)$, they showed the multiple existence of positive solutions of $(*)$. More precisely, when $\beta \leqslant-1,(*)$ has an unbounded sequence of positive solutions $\left(u_{k}\right)_{k=1}^{\infty}$ such that
$$
\left\|u_{k}\right\|_{L^{\infty}(\Omega)}+\left\|v_{k}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty \quad \text { as } k \rightarrow \infty .
$$

These positive solutions were given by minimax method from making use of a symmetry $\sigma(u, v)=(v, u)$. That is, the variational functional $I_{\beta}(u, v)$ associated with $(*)$ satisfies $I_{\beta}(\sigma(u, v))=I_{\beta}(u, v)$ for $\sigma(u, v)=(v, u)$. This multiplicity result was recovered and generalized to the non-symmetric case of $\mu_{1} \neq \mu_{2}$ by using a bifurcation method in [5] in which an unbounded sequence of positive solutions was established for $\beta \leqslant-\sqrt{\mu_{1} \mu_{2}}$ when the domain is radial.

For nonlinear Schrödinger systems $(*)$ with $\lambda_{1}=\lambda_{2}=\mu_{1}=\mu_{2}=1$, these results suggest that $\beta=-\sqrt{\mu_{1} \mu_{2}}$ is the threshold that divides the existence of a priori bounds of positive solutions and the existence of an unbounded sequence of positive solutions. In this paper, we consider the existence and multiplicity of semi-positive solution of $(*)$. A natural question is to examine the coupling constant $\beta$ and to find the coupling value that separates the a priori bounds and infinitely many semi-positive solutions. Our results suggest that $\beta=0$ is the threshold dividing the existence of a priori bounds of semi-positive solutions and the existence of an unbounded sequence of semi-positive solutions. This is the main motivation of the current work. We also study the asymptotic properties of semi-positive solutions when $\beta \rightarrow 0$ and $\beta \rightarrow \infty$, and establish multiplicity results of semi-positive solutions in these regimes.

When $\beta<0$, we get infinite many semi-positive solutions of $(*)$ by the following theorem.
Theorem 0.1. Let $\beta<0$. Then (*) has a sequence of solutions ( $u_{k}, v_{k}$ ) such that

$$
u_{k}>0, \quad\left\|u_{k}\right\|_{L^{\infty}(\Omega)}+\left\|v_{k}\right\|_{L^{\infty}(\Omega)} \rightarrow \infty \text { as } k \rightarrow \infty .
$$

Moreover, if $\beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, 0\right)$, then $v_{k}$ must change sign for large $k$.
When $\beta>0$ is small, we get multiplicity of semi-positive solutions of $(*)$ as follows.
Theorem 0.2. For given $k \in \mathbf{N}$, there exists $\beta_{k}>0$ such that, for any $\beta \in\left(0, \beta_{k}\right)$, we have $k$ semi-positive solutions ( $u_{i}, v_{i}$ ) of $(*)$ with $u_{i}>0$ in $\Omega(i=1,2, \ldots, k)$.

Roughly speaking, our semi-positive solutions are given by making use of a symmetry $\sigma(u, v)=(u,-v)$. That is, it is essential that the variational functional $I_{\beta}(u, v)$ satisfies $I_{\beta}(u, v)=I_{\beta}(u,-v)$. More generally, we develop an abstract framework in Section 2. We consider the following situation. Let $H$ be a Hilbert space and suppose that $\sigma: H \rightarrow H$ satisfies

$$
\begin{align*}
& \sigma^{2}=i d_{H}  \tag{0.1}\\
& \sigma \neq i d_{H} \tag{0.2}
\end{align*}
$$

Then, for $C^{1}$-manifold $M \subset H$ which does not contain fix points of $\sigma$ and $C^{1}$-functional $J: M \rightarrow \mathbf{R}$ satisfying $J(\sigma(u))=J(u)$ and some conditions, we can prove the multiple existence of the critical values of $J$. For details, see Section 2. We point out that generalizations and variants of the genus theory have been established recently in [ $9,10,26$ ]. Refs. [ 9,10 ] were for existence of multiple vector solutions of some elliptic systems. Ref. [26] was on existence of multiple sign-changing vector solutions with each component sign-changing for systems like (*) in the defocussing case (i.e., $\mu_{j} \leqslant 0$ ). In the general perspective we use partial symmetry for variants of the genus theory in this paper.

Next, we consider the asymptotic behavior of semi-positive solutions as $\beta \rightarrow 0$. To state our result about the asymptotic behavior, we need the following notations: for $J_{2}(v)=\left(4 \mu_{2}\|v\|_{L^{4}(\Omega)}^{4}\right)^{-1}: \Sigma_{2}=\left\{\left.v \in H_{0}^{1}(\Omega)\left|\int_{\Omega}\right| \nabla v\right|^{2}+\right.$ $\left.\lambda_{2}|u|^{2} d x=1\right\} \rightarrow \mathbf{R}$, we define symmetric mountain pass values $b_{n}^{2}(n \in \mathbf{N} \cup\{0\})$ by

$$
\begin{aligned}
& b_{n}^{2}=\inf _{\gamma_{2} \in \Gamma_{n}^{2}} \max _{\theta \in S^{n}} J_{2}\left(\gamma_{2}(\theta)\right), \\
& \Gamma_{n}^{2}=\left\{\gamma_{2}(\theta) \in C\left(S^{n}, \Sigma_{2}\right) \mid \gamma_{2}(-\theta)=-\gamma_{2}(\theta) \text { for all } \theta \in S^{n}\right\},
\end{aligned}
$$

where $S^{n}=\left\{\theta=\left(\theta_{1}, \ldots, \theta_{n+1}\right) \in \mathbf{R}^{n+1}| | \theta \mid=1\right\}$. Now, we show the following theorem.

Theorem 0.3. For given $k \in \mathbf{N}$, there exists $\beta_{k}^{\prime}>0$ such that, for any $\beta \in\left(-\beta_{k}^{\prime}, \beta_{k}^{\prime}\right)$, we have $k$ solutions ( $u_{i, \beta}, v_{i, \beta}$ ) of $(*)$ with $u_{i, \beta}>0$ in $\Omega(i=1,2, \ldots, k)$ and $\left(u_{i, \beta}, v_{i, \beta}\right)$ satisfy the following: extracting a subsequence $\beta_{j} \rightarrow 0$, we have

$$
\left(u_{i, \beta_{j}}, v_{i, \beta_{j}}\right) \rightarrow\left(u_{i, 0}, v_{i, 0}\right) \quad \text { in } H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) .
$$

Here $u_{i, 0}$ is a positive least energy solution of

$$
\begin{align*}
& -\Delta u+\lambda_{1} u=\mu_{1} u^{3} \quad \text { in } \Omega, \\
& \quad u \in H_{0}^{1}(\Omega) . \tag{0.3}
\end{align*}
$$

$v_{i, 0}$ is a solution of

$$
\begin{align*}
& -\Delta v+\lambda_{2} v=\mu_{2} v^{3} \quad \text { in } \Omega, \\
& v \in H_{0}^{1}(\Omega) . \tag{0.4}
\end{align*}
$$

In particular, $v_{i, 0}$ corresponds to the critical value $b_{i}^{2}$ which is given by a symmetric mountain pass theorem.
Remark 0.4. The functional $J_{2}(v): \Sigma_{2} \rightarrow \mathbf{R}$ corresponds to (0.4). In fact, for a critical point $v_{0}$ of $J_{2}$, $\left(\sqrt{\mu_{2}}\left\|v_{0}\right\|_{L^{4}(\Omega)}^{2}\right)^{-1} v_{0}$ is a non-trivial solution of (0.4).

Remark 0.5. The semi-positive solutions ( $u_{i, \beta}, v_{i, \beta}$ ) in Theorem 0.3 may be different from the semi-positive solutions ( $u_{i}, v_{i}$ ) in Theorem 0.1 or Theorem 0.2.

Next, we consider the semi-positive solutions for the case $\beta$ is large. In [18], Liu and Wang showed that, for given $k \in \mathbf{N}$, there exists $\bar{\beta}^{\prime}{ }_{k}>0$ such that, for any $\beta>\bar{\beta}^{\prime}{ }_{k},(*)$ has at least $k$ solutions. In this paper, we get multiplicity of semi-positive solutions of ( $*$ ) as follows.

Theorem 0.6. For given $k \in \mathbf{N}$, there exists $\bar{\beta}_{k}>0$ such that, for any $\beta>\bar{\beta}_{k}$, (*) has at least $k$ semi-positive solutions $\left(u_{i, \beta}, v_{i, \beta}\right)$ with $u_{i, \beta}>0$ in $\Omega(i=1,2, \ldots, k)$.

We study the asymptotic behavior as $\beta \rightarrow \infty$. For the solution $\left(u_{i, \beta}, v_{i, \beta}\right)$ of Theorem $0.6,\left(\sqrt{\beta} u_{i, \beta}, \sqrt{\beta} v_{i, \beta}\right)$ is bounded in $H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega)$ as $\beta \rightarrow \infty$. (See Section 7.) Thus, extracting a subsequence $\beta_{j} \rightarrow \infty$, we expect that $\left(\sqrt{\beta_{j}} u_{i, \beta_{j}}, \sqrt{\beta_{j}} v_{i, \beta_{j}}\right)$ approaches to a solution of

$$
\begin{gather*}
-\Delta u+\lambda_{1} u=u v^{2} \quad \text { in } \Omega, \\
-\Delta v+\lambda_{2} v=u^{2} v \quad \text { in } \Omega, \\
u, v \in H_{0}^{1}(\Omega) . \tag{0.5}
\end{gather*}
$$

Here, we remark that ( 0.5 ) does not have semi-trivial solutions. In fact, letting $(0, v)$ be a solution of $(0.5)$, we also have $v=0$ from the second equation of ( 0.5 ). For the limiting equation ( 0.5 ), we have the following:

Theorem 0.7. Eq. (0.5) has infinitely many semi-positive solutions ( $u_{k}, v_{k}$ ) such that $u_{k}>0$ in $\Omega$ and

$$
\begin{equation*}
\left\|u_{k}\right\|_{L^{\infty}}(\Omega)+\left\|v_{k}\right\|_{L^{\infty}}(\Omega) \rightarrow \infty \quad \text { as } k \rightarrow \infty . \tag{0.6}
\end{equation*}
$$

Moreover, when $\lambda_{1}=\lambda_{2}, v_{k}$ must change sign for large $k \in \mathbf{N}$.
Remark 0.8. The solutions $\left(u_{k}, v_{k}\right)$ of Theorem 0.7 are characterized by values $e_{k, \infty}$ which are defined as follows. Let $N=\left\{(u, v) \in H_{0}^{1}(\Omega) \times\left. H_{0}^{1}(\Omega)\left|\int_{\mathbf{R}^{N}}\right| \nabla u\right|^{2}+|\nabla v|^{2}+\lambda_{1}|u|^{2}+\lambda_{2}|v|^{2} d x=1, u_{+} v \not \equiv 0\right\}, \tilde{J}_{\infty}(u, v)=$ $\left(8\left\|u_{+} v\right\|_{L^{2}(\Omega)}^{2}\right)^{-1}$. We define $e_{k, \infty}(k \in \mathbf{N} \cup\{0\})$ by

$$
e_{k, \infty}=\inf \left\{c \in \mathbf{R} \mid \gamma\left(\left[\tilde{J}_{\infty} \leqslant c\right]_{N}\right) \geqslant k\right\} .
$$

Here $\gamma$ is a genus corresponding to $\sigma(u, v)=(u,-v)$ which is defined in Section 2.

Remark 0.9. When $\lambda_{1}=\lambda_{2}=\lambda>0$, all positive solutions $(u, v)$ of ( 0.5 ) must satisfy $u=v$. In fact, $u-v$ satisfies

$$
-\Delta(u-v)+\lambda(u-v)=u v(v-u) .
$$

Multiplying $u-v$ and integrating over $\Omega$ the above equation, we have

$$
\int_{\Omega}|\nabla(u-v)|^{2}+\lambda(u-v)^{2} d x=-\int_{\Omega} u v(u-v)^{2} d x
$$

Thus we have $u=v$. We also remark that there exist a priori bounds of $-\Delta u+\lambda u=u^{3}$ in $\Omega$ and $u=0$ on $\partial \Omega$. Therefore, when $\lambda_{1}=\lambda_{2}=\lambda>0$, (0.6) implies that $v_{k}$ is a sign-changing solution for large $k \in \mathbf{N}$. When $\lambda_{1} \neq \lambda_{2}$ we do not know whether $v_{k}$ changes sign.

Now, we get the following theorem about the asymptotic behavior as $\beta \rightarrow \infty$.
Theorem 0.10. For given $k \in \mathbf{N}$, let $\left(u_{k, \beta}, v_{k, \beta}\right)$ be a family of solutions of $(*)$ which are given in Theorem 0.6 . Then there exist a subsequence $\beta_{j} \rightarrow \infty$ and $\left(u_{k, \infty}, v_{k, \infty}\right) \in H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega)$ such that

$$
\left(\sqrt{\beta_{j}} u_{k, \beta_{j}}, \sqrt{\beta_{j}} v_{k, \beta_{j}}\right) \rightarrow\left(u_{k, \infty}, v_{k, \infty}\right) \quad \text { in } H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) .
$$

Here $\left(u_{k, \infty}, v_{k, \infty}\right)$ is a solution of ( 0.5 ) and corresponds to critical value $e_{k, \infty}$.
We devote the next four sections to the proofs of our theorems. For the case $\beta \leqslant 0$ or the case $\beta>0$ small, we reduce the functional $I_{\beta}(u, v)$ to a functional $J_{\beta}(u, v)$ defined on a subset of a torus $\Sigma_{1} \times \Sigma_{2}$ in Section 1. On the other hand, for the case $\beta>0$ is large, we reduce the functional $\tilde{I}_{\beta}(u, v)$ to a functional $\tilde{J}_{\beta}(u, v)$ defined on a subset of the sphere $\Sigma$ in Section 6. In Section 2, we give an abstract theory for the multiple existence of the critical values of $C^{1}$-functional $J: M \rightarrow \mathbf{R}$ satisfying $J(\sigma(u))=J(u)$. We will get most of our multiple existence of semi-positive solutions by using these abstract results. In Section 3, we will show Theorem 0.1 and Theorem 0.2. In Sections 4-5, we will prove Theorem 0.3. To show this, we apply the method from [25]. In Sections 6-7, we will show Theorems 0.6 , 0.7 and 0.10.

## 1. The functional setting for the case $\beta \leqslant 0$ or the case $\beta>0$ small

To prove the existence of semi-positive solutions $(u, v)$ with $u>0$, we seek critical points of the following functional

$$
I_{\beta}(u, v)=\frac{1}{2}\left(\|u\|_{\lambda_{1}}^{2}+\|v\|_{\lambda_{2}}^{2}\right)-\frac{1}{4}\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}\right)-\frac{\beta}{2}\left\|u_{+} v\right\|_{2}^{2}: H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \rightarrow \mathbf{R} .
$$

Here we use notations $u_{+}=\max \{u, 0\}, u_{-}=\min \{u, 0\}$ and

$$
\|u\|_{\lambda}^{2}=\int_{\Omega}|\nabla u|^{2}+\lambda u^{2} d x, \quad\|u\|_{p}^{p}=\int_{\Omega}|u|^{p} d x .
$$

For a critical point $(u, v)$ of $I_{\beta}(u, v)$, the positivity of $u$ comes from the following proposition.
Proposition 1.1. Let $(u, v)$ be a critical point of $I_{\beta}(u, v)$ with $u \neq 0$. Then we have $u>0$ in $\Omega$.
Proof. Let $(u, v)$ be a critical point of $I_{\beta}(u, v)$. Then $\nabla I_{\beta}(u, v)\left(u_{-}, 0\right)=\left\|u_{-}\right\|_{\lambda_{1}}^{2}=0$. Thus we have $u_{+} \equiv u \geqslant 0$. Now, for $\beta \leqslant 0, u$ satisfies

$$
-\Delta u+\left(\lambda_{1}-\beta v^{2}\right) u=\mu_{1} u^{3} \geqslant 0 .
$$

For $\beta>0, u$ satisfies

$$
-\Delta u+\lambda_{1} u=\left(\mu_{1} u^{2}+\beta v^{2}\right) u \geqslant 0 .
$$

Since the maximum principle works for $u$ in both cases, we have $u>0$ in $\Omega$.

We set $\Sigma_{i}=\left\{u \in H_{0}^{1}(\Omega) \mid\|u\|_{\lambda_{i}}=1\right\}$ for $i=1,2$. We remark that there exists $C_{1}>0$ such that

$$
\begin{equation*}
\|u\|_{4},\|v\|_{4}<C_{1} \quad \text { for all }(u, v) \in \Sigma_{1} \times \Sigma_{2} \tag{1.1}
\end{equation*}
$$

To seek non-trivial critical points of $I_{\beta}(u, v)$, sometimes one may reduce $I_{\beta}(u, v)$ to a functional defined on a Nehari manifold with co-dimension 2. In this paper, we reduce $I_{\beta}(u, v)$ to a functional defined on an open subset of torus $\Sigma_{1} \times \Sigma_{2}$. Since we also consider a perturbation problem for $\beta$ (Theorem 0.3), it is easy to treat a domain which does not depend on $\beta$. This is the main reason to reduce the functional to one on the torus but not on a Nehari manifold.

### 1.1. The reduction to a functional on a torus

When $\beta \in \mathbf{R}$, we set

$$
N_{\beta}=\left\{\begin{array}{l|l}
(u, v) \in \Sigma_{1} \times \Sigma_{2} & \begin{array}{l}
g_{1}(u, v):=\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}>0, \\
g_{2}(u, v):=\mu_{1}\left\|u_{+}\right\|_{4}^{4}-\beta\left\|u_{+} v\right\|_{2}^{2}>0, \\
g_{3}(u, v):=\mu_{2}\|v\|_{4}^{4}-\beta\left\|u_{+} v\right\|_{2}^{2}>0
\end{array}
\end{array}\right\} .
$$

From the Hölder inequality, we see that

$$
N_{\beta}= \begin{cases}\left\{(u, v) \in \Sigma_{1} \times \Sigma_{2} \mid g_{1}(u, v)>0\right\}, & \beta \in\left(-\infty,-\sqrt{\mu_{1} \mu_{2}}\right], \\ \left\{(u, v) \in \Sigma_{1} \times \Sigma_{2} \mid u_{+} \not \equiv 0\right\}, & \beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, 0\right], \\ \left\{(u, v) \in \Sigma_{1} \times \Sigma_{2} \mid g_{2}(u, v)>0, g_{3}(u, v)>0\right\}, & \beta \in(0, \infty) .\end{cases}
$$

We remark that, for all $\beta \in \mathbf{R},(u, v) \in N_{\beta}$ implies $g_{1}(u, v)>0$ and $u_{+} \not \equiv 0$. We can define a functional $J_{\beta}(u, v)$ on $N_{\beta}$ by the following proposition.

Proposition 1.2. For any $(u, v) \in N_{\beta}$, a function

$$
(s, t) \mapsto I_{\beta}(s u, t v): \mathbf{R}_{+}^{2} \rightarrow \mathbf{R}
$$

has a unique maximum point $\left(s_{\beta}(u, v), t_{\beta}(u, v)\right)$. Moreover, setting

$$
J_{\beta}(u, v)=\sup _{s, t>0} I_{\beta}(s u, t v),
$$

we have

$$
\begin{align*}
J_{\beta}(u, v) & =\frac{1}{4}\left(s_{\beta}(u, v)^{2}+t_{\beta}(u, v)^{2}\right)  \tag{1.2}\\
& =\frac{1}{4}\left(\mu_{1} s_{\beta}(u, v)^{4}\left\|u_{+}\right\|_{4}^{4}+\mu_{2} t_{\beta}(u, v)^{4}\|v\|_{4}^{4}+2 \beta s_{\beta}(u, v)^{2} t_{\beta}(u, v)^{2}\left\|u_{+} v\right\|_{2}^{2}\right)  \tag{1.3}\\
& =\frac{1}{4} \cdot \frac{\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}-2 \beta\left\|u_{+} v\right\|_{2}^{2}}{\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}} \tag{1.4}
\end{align*}
$$

and
(i) $s_{\beta}(u, v), t_{\beta}(u, v): N \rightarrow \mathbf{R}_{+}$are $C^{1}$-functions.
(ii) $J_{\beta}(u, v): N_{\beta} \rightarrow \mathbf{R}$ is a $C^{1}$-function.
(iii) If $(u, v) \in N_{\beta}$ is a critical point of $J_{\beta}(u, v)$, then $\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right)$ is a non-trivial critical point of $I_{\beta}(u, v)$.
(iv) $J_{\beta}(u, v)$ satisfies $(P S)$-condition.

Proof. For any $(u, v) \in N_{\beta}$, we set

$$
f(s, t)=I_{\beta}(s u, t v): \mathbf{R}_{+}^{2} \rightarrow \mathbf{R} .
$$

Differentiating $f(s, t)$, we have

$$
\begin{aligned}
& \frac{\partial f}{\partial s}(s, t)=s-s^{3} \mu_{1}\left\|u_{+}\right\|_{4}^{4}-s t^{2} \beta\left\|u_{+} v\right\|_{2}^{2} \\
& \frac{\partial f}{\partial t}(s, t)=t-t^{3} \mu_{2}\|v\|_{4}^{4}-s^{2} t \beta\left\|u_{+} v\right\|_{2}^{2}
\end{aligned}
$$

Thus critical points $(s, t)$ of $f(s, t)$ satisfy

$$
\left[\begin{array}{cc}
\mu_{1}\left\|u_{+}\right\|_{4}^{4}, & \beta\left\|u_{+} v\right\|_{2}^{2} \\
\beta\left\|u_{+} v\right\|_{2}^{2}, & \mu_{2}\|v\|_{4}^{4}
\end{array}\right]\left[\begin{array}{c}
s^{2} \\
t^{2}
\end{array}\right]=\left[\begin{array}{l}
1 \\
1
\end{array}\right] .
$$

Here, noting $\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}>0$, we have

$$
\begin{align*}
{\left[\begin{array}{l}
s^{2} \\
t^{2}
\end{array}\right] } & =\frac{1}{\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}}\left[\begin{array}{cc}
\mu_{2}\|v\|_{4}^{4}, & -\beta\left\|u_{+} v\right\|_{2}^{2} \\
-\beta\|u\|_{2}^{2}, & \mu_{1}\left\|u_{+}\right\|_{4}^{4}
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right] \\
& =\frac{1}{\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}}\left[\begin{array}{c}
\mu_{2}\|v\|_{4}^{4}-\beta\left\|u_{+} v\right\|_{2}^{2} \\
\mu_{1}\left\|u_{+}\right\|_{4}^{4}-\beta\left\|u_{+} v\right\|_{2}^{2}
\end{array}\right] . \tag{1.5}
\end{align*}
$$

Since $(u, v) \in N_{\beta}, f(s, t)$ has a unique critical point $\left(s_{0}, t_{0}\right)=\left(s_{\beta}(u, v), t_{\beta}(u, v)\right)$. Next, to show $\left(s_{0}, t_{0}\right)$ is a maximum point, we calculate the second derivatives of $f(s, t)$.

$$
\begin{aligned}
& \frac{\partial^{2} f}{\partial s^{2}}(s, t)=1-3 s^{2} \mu_{1}\left\|u_{+}\right\|_{4}^{4}-t^{2} \beta\left\|u_{+} v\right\|_{2}^{2}=\frac{1}{s} \frac{\partial f}{\partial s}(s, t)-2 s^{2} \mu_{1}\left\|u_{+}\right\|_{4}^{4}, \\
& \frac{\partial^{2} f}{\partial t \partial s}(s, t)=-2 s t \beta\left\|u_{+} v\right\|_{2}^{2}, \\
& \frac{\partial^{2} f}{\partial t^{2}}(s, t)=1-3 t^{2} \mu_{2}\|v\|_{4}^{4}-s^{2} \beta\left\|u_{+} v\right\|_{2}^{2}=\frac{1}{t} \frac{\partial f}{\partial t}(s, t)-2 t^{2} \mu_{2}\|v\|_{4}^{4} .
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
A & =\frac{\partial^{2} f}{\partial s^{2}}\left(s_{0}, t_{0}\right)=-2 s_{0}^{2} \mu_{1}\left\|u_{+}\right\|_{4}^{4} \\
B & =\frac{\partial^{2} f}{\partial t \partial s}\left(s_{0}, t_{0}\right)=-2 \beta s_{0} t_{0}\left\|_{+} v\right\|_{2}^{2} . \\
C & =\frac{\partial^{2} f}{\partial t^{2}}\left(s_{0}, t_{0}\right)=-2 t_{0}^{2} \mu_{2}\|v\|_{4}^{4} .
\end{aligned}
$$

Since $A<0$ and $A C-B^{2}=4 s_{0}^{2} t_{0}^{2}\left(\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}\right)>0,\left(s_{0}, t_{0}\right)$ is a maximum point of $f(s, t)$. Thus, by direct calculations, we get (1.2)-(1.4).

Next we show (i). To show (i), we use the implicit function theorem. We consider the following function:

$$
\mathbf{F}(s, t, u, v)=\left[\begin{array}{c}
F(s, t, u, v) \\
G(s, t, u, v)
\end{array}\right]=\left[\begin{array}{c}
\frac{\partial f}{\partial s}(s, t) \\
\frac{\partial f}{\partial t}(s, t)
\end{array}\right]: \mathbf{R}_{+}^{2} \times N_{\beta} \rightarrow \mathbf{R}^{2} .
$$

Now, for any $(u, v) \in N_{\beta}$, we have

$$
\begin{aligned}
& \mathbf{F}\left(s_{0}, t_{0}, u, v\right)=\mathbf{0}, \\
& {\left[\begin{array}{ll}
\frac{\partial F}{\partial s}\left(s_{0}, t_{0}, u, v\right) & \frac{\partial F}{\partial t}\left(s_{0}, t_{0}, u, v\right) \\
\frac{\partial G}{\partial s}\left(s_{0}, t_{0}, u, v\right) & \frac{\partial F}{\partial t}\left(s_{0}, t_{0}, u, v\right)
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
B & C
\end{array}\right] .}
\end{aligned}
$$

Thus from the implicit function theorem, we can easily see the $C^{1}$-property of $\left(s_{0}, t_{0}\right)=\left(s_{\beta}(u, v), t_{\beta}(u, v)\right)$.
We show (ii). Noting

$$
J_{\beta}(u, v)=I_{\beta}\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right)
$$

we can easily find that $J_{\beta}(u, v)$ is a $C^{1}$-function. Moreover we have

$$
\begin{align*}
\nabla_{u} J_{\beta}(u, v) \varphi= & \nabla_{u} I_{\beta}\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right)\left(\nabla_{u} s_{\beta}(u, v) \varphi u+s_{\beta}(u, v) \varphi\right) \\
& +\nabla_{v} I_{\beta}\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right) \nabla_{u} t_{\beta}(u, v) \varphi v \\
= & \nabla_{u} I_{\beta}\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right) s_{\beta}(u, v) \varphi  \tag{1.6}\\
\nabla_{v} J_{\beta}(u, v) \psi= & \nabla_{v} I_{\beta}\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right) t_{\beta}(u, v) \psi . \tag{1.7}
\end{align*}
$$

Thus, if $(u, v) \in N_{\beta}$ is a critical point of $J_{\beta}(u, v)$, then $\left(s_{\beta}(u, v) u, t_{\beta}(u, v) v\right)$ is a non-trivial critical point of $I_{\beta}(u, v)$ and we get (iii).

Finally, we show (iv). If $\left(u_{n}, v_{n}\right) \in N_{\beta}$ is a (PS)-sequence for $J_{\beta}$, then $J_{\beta}\left(u_{n}, v_{n}\right)$ are bounded and this means the boundedness of $\left(s_{\beta}\left(u_{n}, v_{n}\right), t_{\beta}\left(u_{n}, v_{n}\right)\right)$ from (1.2). Thus from (1.6)-(1.7), $\left(s_{\beta}\left(u_{n}, v_{n}\right) u_{n}, t_{\beta}\left(u_{n}, v_{n}\right) v_{n}\right)$ is also a (PS)-sequence for $I_{\beta}$. Since $I_{\beta}(u, v)$ satisfies (PS)-condition, $J_{\beta}(u, v)$ also satisfies (PS)-condition.

From (1.2), for all $\beta \in \mathbf{R}$, it is obvious that $J_{\beta}(u, v)$ is bounded from below. Moreover, we have the following proposition.

Proposition 1.3. When $\beta<0$, we have

$$
\begin{equation*}
\liminf _{(u, v) \in N_{\beta}, \operatorname{dist}\left\{(u, v), \partial N_{\beta}\right\} \rightarrow 0} J_{\beta}(u, v)=\infty \tag{1.8}
\end{equation*}
$$

Proof. For any sequence $\left(\left(u_{n}, v_{n}\right)\right)_{n=1}^{\infty} \subset N_{\beta}$ with $g_{1}\left(u_{n}, v_{n}\right) \rightarrow 0(n \rightarrow \infty)$, we need to show $J_{\beta}\left(u_{n}, v_{n}\right) \rightarrow \infty$ $(n \rightarrow \infty)$. Since $\left\|u_{n}\right\|_{\lambda_{1}}=\| \| v_{n} \|_{\lambda_{2}}=1$, for some $u_{0}, v_{0} \in H_{0}^{1}(\Omega)$, we may assume

$$
u_{n} \rightarrow u_{0}, \quad v_{n} \rightarrow v_{0} \quad \text { strongly in } L^{4}(\Omega)
$$

Here if $g_{2}\left(u_{0}, v_{0}\right)+g_{3}\left(u_{0}, v_{0}\right)>0$, then it is obvious that $(1.8)$ holds. Thus we assume $g_{2}\left(u_{0}, v_{0}\right)+g_{3}\left(u_{0}, v_{0}\right)=0$. Since $\beta<0$, we have $u_{0}=v_{0}=0$ and we find $\left\|u_{n}\right\|_{4}^{4} \rightarrow 0,\left\|v_{n}\right\|_{4}^{4} \rightarrow 0$ as $n \rightarrow \infty$. Since $J_{\beta}(u, v)$ is written by (1.4), we get (1.8).

Remark 1.4. From Proposition 1.3 , when $\beta<0$, the behavior of $J_{\beta}(u, v)$ in the neighborhood of $\partial N_{\beta}$ does not disturb deformation arguments. When $\beta>0$, it is complicated by the behavior of $J_{\beta}(u, v)$ in the neighborhood of $\partial N_{\beta}$ and we cannot expect the property like (1.8). But for $\beta>0$ small, $J_{\beta}(u, v)$ satisfies the property like (1.8) on a proper subset $M_{\delta} \subset N_{\beta}$. (See Proposition 1.9.)

### 1.2. The case $\beta>0$ small

For $\delta>0$, we set

$$
M_{\delta}=\left\{(u, v) \in \Sigma_{1} \times \Sigma_{2} \mid \mu_{1}\left\|u_{+}\right\|_{4}^{4}>\delta, \mu_{2}\|v\|_{4}^{4}>\delta\right\}
$$

We remark that $M_{\delta} \neq \emptyset$ if $\delta<\frac{1}{4 b_{0}}$ where $b_{0}$ is given by

$$
\begin{equation*}
b_{0}=\min \left\{b_{0}^{1}, b_{0}^{2}\right\}>0, \quad b_{0}^{1}=\inf _{u \in \Sigma_{1}} \frac{1}{4 \mu_{1}\|u\|_{4}^{4}}>0, \quad b_{0}^{2}=\inf _{v \in \Sigma_{2}} \frac{1}{4 \mu_{2}\|v\|_{4}^{4}}>0 \tag{1.9}
\end{equation*}
$$

Here $b_{0}^{i}(i=1,2)$ is a least energy level of (1.15) and (1.17) respectively. (See Remark 1.8.) We also remark that $M_{\delta}$ is independent of $\beta$.

Lemma 1.5. For any given $\delta \in\left(0, \frac{1}{4 b_{0}}\right)$, there exists $\beta_{\delta} \in\left(0, \sqrt{\mu_{1} \mu_{2}}\right)$ such that

$$
M_{\delta} \subset N_{\beta} \quad \text { for all } \beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, \beta_{\delta}\right)
$$

Proof. When $\beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, 0\right), M_{\delta} \subset N_{\beta}$ is obvious. For $\delta \in\left(0, \frac{1}{4 b_{0}}\right)$, we choose $\beta_{\delta}>0$ satisfying $\delta>\beta_{\delta} C_{1}^{4}$. Here $C_{1}$ is a constant given in (1.1). Then it holds

$$
\mu_{1}\left\|u_{+}\right\|_{4}^{4}>\delta>\beta_{\delta} C_{1}^{4} \geqslant \beta\left\|u_{+} v\right\|_{2}^{2} \quad \text { for all }(u, v) \in M_{\delta}, \beta \in\left[0, \beta_{\delta}\right)
$$

By a similar way, we have $\mu_{2}\|v\|_{4}^{4}>\beta\left\|u_{+} v\right\|_{2}^{2}$. Thus we get $M_{\delta} \subset N_{\beta}$ for all $\beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, \beta_{\delta}\right)$.

From Lemma 1.5, $J_{\beta}(u, v)$ is defined on $M_{\delta}$.
Lemma 1.6. For any given $\delta \in\left(0, \frac{1}{4 b_{0}}\right)$, there exists a constant $C_{\delta}>0$ which does not depend on $\beta$ such that

$$
\begin{equation*}
s_{\beta}(u, v) \leqslant C_{\delta}, \quad t_{\beta}(u, v) \leqslant C_{\delta} \quad \text { for all }(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right) . \tag{1.10}
\end{equation*}
$$

Here $\beta_{\delta}$ was given in Lemma 1.5. Moreover it holds

$$
\begin{equation*}
\left(s_{\beta}(u, v), t_{\beta}(u, v)\right) \rightarrow\left(\frac{1}{\sqrt{\mu_{1}}\left\|u_{+}\right\|_{4}^{2}}, \frac{1}{\sqrt{\mu_{2}}\|v\|_{4}^{2}}\right) \quad \text { uniformly for }(u, v) \in M_{\delta} \text { as } \beta \rightarrow 0 \tag{1.11}
\end{equation*}
$$

Proof. Suppose $(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right)$. Since $s_{\beta}(u, v)$ was written by (1.5), we have

$$
s_{\beta}(u, v)^{2}=\frac{\mu_{2}\|v\|_{4}^{4}-\beta\left\|u_{+} v\right\|_{2}^{2}}{\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{2}} \leqslant \frac{\left(\mu_{2}+\beta_{\delta}\right) C_{1}^{4}}{\left(\mu_{1} \mu_{2}-\beta_{\delta}^{2}\right) \frac{\delta^{2}}{\mu_{1} \mu_{2}}}
$$

Here $C_{1}$ is a constant given in (1.1) and we have used the fact that $\mu_{1}\left\|u_{+}\right\|_{4}^{4}, \mu_{2}\|v\|_{4}^{4} \geqslant \delta$ for all $(u, v) \in M_{\delta}$. And we also have

$$
s_{\beta}(u, v)^{2} \rightarrow \frac{1}{\mu_{1}\left\|u_{+}\right\|_{4}^{4}} \quad \text { uniformly for }(u, v) \in M_{\delta} \text { as } \beta \rightarrow 0 .
$$

Since $t_{\beta}(u, v)$ also was similarly written by (1.5), we obtain (1.10) and (1.11).
Proposition 1.7. For any given $\delta \in\left(0, \frac{1}{4 b_{0}}\right)$, there exists a constant $c_{\delta}(\beta)$ with $c_{\delta}(\beta) \rightarrow 0($ as $\beta \rightarrow 0)$ such that $J_{\beta}(u, v)$ satisfies

$$
\begin{align*}
& \left|J_{\beta}(u, v)-J_{1}(u)-J_{2}(v)\right| \leqslant c_{\delta}(\beta) \quad \text { for all }(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right),  \tag{1.12}\\
& \left\|\nabla_{u} J_{\beta}(u, v)-\nabla J_{1}(u)\right\|_{\lambda_{1 *}} \leqslant c_{\delta}(\beta) \quad \text { for all }(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right),  \tag{1.13}\\
& \left\|\nabla_{v} J_{\beta}(u, v)-\nabla J_{2}(v)\right\|_{\lambda_{2} *} \leqslant c_{\delta}(\beta) \quad \text { for all }(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right), \tag{1.14}
\end{align*}
$$

where, for $i=1,2, J_{i}(u)=\frac{1}{4 \mu_{i}\|u\|_{4}^{4}}, T_{u} \Sigma_{i}=\left\{v \in H_{0}^{1}(\Omega) \mid\langle u, v\rangle_{\lambda_{i}}=0\right\}$ and

$$
\left\|\nabla J_{i}(u)\right\|_{\lambda_{i} *}=\sup _{v \in T_{u} \Sigma_{i},\|v\| \|_{\lambda_{i}}=1}\left|\nabla J_{i}(u) v\right|
$$

Remark 1.8. For any $u \in \Sigma_{1}$ with $u_{+} \neq 0$, a function $s \mapsto I_{1}(s u)=\frac{s^{2}}{2}-\frac{s^{4}}{4} \mu_{1}\left\|u_{+}\right\|_{4}^{4}$ has a maximum value at a unique maximum point $s=\frac{1}{\sqrt{\mu_{1}\left\|u_{+}\right\|_{4}^{2}}}$ and we can write as follows

$$
\begin{align*}
& J_{1}(u)=\sup _{s>0} I_{1}(s u)=\frac{1}{4 \mu_{1}\left\|u_{+}\right\|_{4}^{4}},  \tag{1.15}\\
& \nabla J_{1}(u) \varphi=-\frac{1}{\mu_{1}\left\|u_{+}\right\|_{4}^{8}} \int_{\Omega} u_{+}^{3} \varphi d x \quad \text { for all } \varphi \in T_{u} \Sigma_{1} . \tag{1.16}
\end{align*}
$$

By a similar way, for any $u \in \Sigma_{2}$, a function $t \mapsto I_{2}(t u)=\frac{t^{2}}{2}-\frac{t^{4}}{4} \mu_{2}\|v\|_{4}^{4}$ has a unique maximum point and we have

$$
\begin{align*}
& J_{2}(u)=\sup _{t>0} I_{2}(t v)=\frac{1}{4 \mu_{2}\|v\|_{4}^{4}},  \tag{1.17}\\
& \nabla J_{2}(v) \psi=-\frac{1}{\mu_{2}\|v\|_{4}^{8}} \int_{\Omega} v^{3} \psi d x \quad \text { for all } \psi \in T_{v} \Sigma_{2} . \tag{1.18}
\end{align*}
$$

Proof of Proposition 1.7. From (1.4), (1.15) and (1.17), we can directly calculate $J_{\beta}(u, v)-J_{1}(u)-J_{2}(v)$ as follows:

$$
J_{\beta}(u, v)-J_{1}(u)-J_{2}(v)=\frac{1}{4} \cdot \frac{\beta\left\|u_{+} v\right\|_{2}^{2}}{\mu_{1} \mu_{2}\left\|u_{+}\right\|_{4}^{4}\|v\|_{4}^{4}-\beta^{2}\left\|u_{+} v\right\|_{2}^{4}}\left(\frac{\beta\left\|u_{+} v\right\|_{2}^{2}}{\mu_{1}\left\|u_{+}\right\|_{4}^{4}}+\frac{\beta\left\|u_{+} v\right\|_{2}^{2}}{\mu_{2}\|v\|_{4}^{4}}-2\right) .
$$

For $(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right)$, we have

$$
\begin{equation*}
\left|J_{\beta}(u, v)-J_{1}(u)-J_{2}(v)\right| \leqslant \frac{C_{1}^{4}|\beta|}{4\left(\mu_{1} \mu_{2}-\beta^{2}\right) \frac{\delta^{2}}{\mu_{1} \mu_{2}}}\left(\frac{C_{1}^{4}|\beta|}{\delta}+\frac{C_{1}^{4}|\beta|}{\delta}+2\right) . \tag{1.19}
\end{equation*}
$$

Here $C_{1}$ is a constant given in (1.1) and we have used the fact that $\mu_{1}\left\|u_{+}\right\|_{4}^{4}, \mu_{2}\|v\|_{4}^{4} \geqslant \delta$ for all $(u, v) \in M_{\delta}$. From (1.19), we get (1.12). Next we calculate $\nabla_{u} J_{\beta}(u, v) \varphi-\nabla J_{1}(u) \varphi$ for any $\varphi \in T_{u} \Sigma_{1}$. From (1.6),

$$
\nabla_{u} J_{\beta}(u, v) \varphi=-s_{\beta}(u, v)^{4} \mu_{1} \int_{\Omega} u_{+}^{3} \varphi d x-\beta s_{\beta}(u, v)^{2} t_{\beta}(u, v)^{2} \int_{\Omega} u_{+} v^{2} \varphi d x
$$

Combining (1.16), we have

$$
\begin{aligned}
\left|\nabla_{u} J_{\beta}(u, v) \varphi-\nabla J_{1}(u) \varphi\right| & \leqslant\left|s_{\beta}(u, v)^{4}-\frac{1}{\mu_{1}^{2}\left\|u_{+}\right\|_{4}^{8}}\right| \mu_{1} \int_{\Omega} u_{+}^{3}|\varphi| d x+|\beta| s_{\beta}(u, v)^{2} t_{\beta}(u, v)^{2} \int_{\Omega} u_{+} v^{2}|\varphi| d x \\
& \leqslant\left|s_{\beta}(u, v)^{4}-\frac{1}{\mu_{1}^{2}\left\|u_{+}\right\|_{4}^{8}}\right| \mu_{1} C_{1}^{4}\|\varphi\|_{\lambda_{1}}+|\beta| C_{\delta}^{4} C_{1}^{4}\|\varphi\| \lambda_{1} .
\end{aligned}
$$

We obtain (1.13) from the above inequality and Lemma 1.6. (1.14) also holds from a similar calculation.
For small $\beta>0$, the following proposition plays a role similar to Proposition 1.3.
Proposition 1.9. For any $\beta \in\left(-\beta_{\delta}, \beta_{\delta}\right)$, we have

$$
\begin{align*}
& \sup _{(u, v) \in M_{\delta}} J_{\beta}(u, v) \leqslant \frac{1}{2 \delta}+c_{\delta}(\beta),  \tag{1.20}\\
& \inf _{(u, v) \in \partial M_{\delta}} J_{\beta}(u, v) \geqslant \frac{1}{4 \delta}+b_{0}-c_{\delta}(\beta) . \tag{1.21}
\end{align*}
$$

Here $b_{0}$ was given in (1.9).
Proof. From Proposition 1.7, for $(u, v) \in M_{\delta}, \beta \in\left(-\beta_{\delta}, \beta_{\delta}\right)$, we have

$$
J_{1}(u)+J_{2}(v)-c_{\delta}(\beta) \leqslant J_{\beta}(u, v) \leqslant J_{1}(u)+J_{2}(v)+c_{\delta}(\beta) .
$$

We remark that

$$
\inf _{u \in \Sigma_{1}, u_{+} \neq 0} J_{1}(u) \geqslant b_{0}^{1} \geqslant b_{0}, \quad \inf _{v \in \Sigma_{2}} J_{2}(v) \geqslant b_{0}^{2} \geqslant b_{0} .
$$

Here $(u, v) \in \partial M_{\delta}$ implies $J_{1}(u)=\frac{1}{4 \delta}$ or $J_{2}(v)=\frac{1}{4 \delta}$ and $(u, v) \in M_{\delta}$ implies $J_{1}(u) \leqslant \frac{1}{4 \delta}$ or $J_{2}(v) \leqslant \frac{1}{4 \delta}$. Therefore we get (1.20) and (1.21).

## 2. The multiplicity of critical values for $\sigma$-invariant functionals

In this section, we construct abstract theories to get the multiple existence of critical points of functionals having symmetry $J(\sigma(u))=J(u)$ where $u$ is in a Hilbert space and $\sigma$ satisfies ( 0.1$)-(0.2)$. To do so, we construct a genus type index for the symmetry $\sigma$. In [23] or [13], the authors constructed the genus type index for $\sigma(-u)=u$ in the scaler case or $\sigma(u, v)=(v, u)$ in the vector case respectively.

In this section, let $H$ be a Hilbert space and $\sigma: H \rightarrow H$ be a bounded linear operator satisfying (0.1)-(0.2). Setting $H_{0}=\{u \in H \mid \sigma(u)=u\}, H_{0}$ is a subspace composed of fixed points of $\sigma$. Here $H_{0} \neq H$ from (0.2). We also set $H_{1}=H_{0}^{\perp} \neq\{0\}$. For any $u \in H$, we uniquely write $u=u_{0}+u_{1},\left(u_{0}, u_{1}\right) \in H_{0} \oplus H_{1}$. Then, from (0.1)-(0.2), we have

$$
\sigma\left(u_{0}+u_{1}\right)=u_{0}-u_{1} \quad \text { for all } u=u_{0}+u_{1} \in H_{0}+H_{1} .
$$

For this $\sigma: H \rightarrow H$, we define a genus as follows:
Definition 2.1. For any $\sigma$-invariant closed set $A \subset H \backslash H_{0}, \gamma(A)$ is the least integer $n$ such that there exists a function $g \in C\left(A, \mathbf{R}^{n} \backslash\{0\}\right)$ with

$$
\begin{equation*}
g(\sigma(u))=-g(u) \quad \text { for all } u \in A \tag{2.1}
\end{equation*}
$$

If there is no such $g$, we define $\gamma(A)=\infty$. We also define $\gamma(\emptyset)=0$.
Here, when $g$ satisfies (2.1), we say $g$ is a $\sigma$-odd function. When $J \in C(A, \mathbf{R})$ satisfies

$$
J(\sigma(u))=J(u) \quad \text { for all } u \in A,
$$

we say $J$ is a $\sigma$-invariant functional or a $\sigma$-even functional. When $h \in C(A, H)$ satisfies

$$
h(\sigma(u))=\sigma(h(u)) \quad \text { for all } u \in A,
$$

we say $h$ is $\sigma$-equivariant.
The following theorem is the main theorem in this section:
Theorem 2.2. Let $M \subset H \backslash H_{0}$ be a $\sigma$-invariant $C^{1}$-manifold and $J: M \rightarrow \mathbf{R}$ be a $\sigma$-even $C^{1}$-functional satisfying (PS)-condition. Moreover, we assume that

$$
\begin{align*}
& \inf _{u \in M} J(u)>-\infty,  \tag{2.2}\\
& \liminf _{u \in M, \operatorname{dist}\{u, \partial M\} \rightarrow 0} J(u)=\infty, \tag{2.3}
\end{align*}
$$

and, for any $k \in \mathbf{N}$, there exists $\psi \in C\left(S^{k}, M\right)$ with $\psi(-x)=\sigma(\psi(x))$. Then $J$ has an unbounded nondecreasing sequence of critical values $\left(c_{k}\right)_{k=1}^{\infty}$. Here $c_{k}$ is defined by

$$
\begin{align*}
& c_{k}=\inf \left\{c \in \mathbf{R} \mid \gamma\left([J \leqslant c]_{M}\right) \geqslant k\right\}, \\
& {[J \leqslant c]_{M}=\{u \in M \mid J(u) \leqslant c\} .} \tag{2.4}
\end{align*}
$$

Firstly we state the properties of our genus. These are similar to the properties of the genus type index constructed in [23] or [13].

Lemma 2.3. Let $A, B \subset H \backslash H_{0}$ be $\sigma$-invariant closed sets. Then we have:
(i) If $A \subset B$, then $\gamma(A) \leqslant \gamma(B)$.
(ii) $\gamma(A \cup B) \leqslant \gamma(A)+\gamma(B)$.
(iii) If $h \in C\left(A, H \backslash H_{0}\right)$ satisfies $h(\sigma(u))=\sigma(h(u))$, then $\gamma(A) \leqslant \gamma(h(A))$.
(iv) $\gamma(\overline{A \backslash B}) \geqslant \gamma(A)-\gamma(B)$.
(v) If $\gamma(A)>1$, then $A$ is an infinite set.
(vi) If $A$ is a compact set, then $\gamma(A)<\infty$. Moreover there exists $\sigma$-invariant neighborhood of $N$ of $A$ in $M$ such that $\gamma(A)=\gamma(\bar{N})$.
(vii) If $\psi \in C\left(S^{n}, H \backslash H_{0}\right)$ satisfies $\psi(-u)=\sigma(\psi(u))$, then $\gamma\left(\psi\left(S^{n}\right)\right) \geqslant n+1$.

Proof. First of all, we show (iii). If $\gamma(h(A))=\infty$, (iii) is trivial. Supposing $\gamma(h(A))=m<\infty$, there exists $\sigma$-odd function $g \in C\left(h(A), \mathbf{R}^{m} \backslash\{0\}\right)$. Then $(g \circ h) \in C\left(A, \mathbf{R}^{m} \backslash\{0\}\right)$ satisfies $(g \circ h)(\sigma(u))=g(\sigma(h(u)))=$ $-(g \circ h)(u)$. Thus we have $\gamma(A) \leqslant m=\gamma(h(A))$ and (iii) holds. We get (i), taking an inclusion map $i d_{A} \in C(A, B)$ in (iii). Next, we show (v). When $A$ is a finite set, $A$ is written by $A=\left\{u_{1}, \ldots, u_{k}, \sigma\left(u_{1}\right), \ldots, \sigma\left(u_{k}\right)\right\}$ where $u_{1}, \ldots, u_{k}, \sigma\left(u_{1}\right), \ldots, \sigma\left(u_{k}\right)$ are different from each other. Then we have $g \in C(A, \mathbf{R} \backslash\{0\})$ such that $g\left(x_{i}\right)=1$, $g\left(\sigma\left(x_{i}\right)\right)=-1$ for $i=1, \ldots, k$. Thus we find $\gamma(A)=1$. This implies (v).

Next, we show (ii). Supposing $\gamma(A)=n<\infty, \gamma(B)=m<\infty$, there exist $\sigma$-odd functions $g \in C\left(A, \mathbf{R}^{n} \backslash\{0\}\right)$ and $h \in C\left(B, \mathbf{R}^{m} \backslash\{0\}\right)$. By the extension theorem of Tietze, we have $\hat{g}, \hat{h} \in C(H, H)$ such that $\hat{g}(u)=g(u)$ for all $u \in A$ and $\hat{h}(u)=h(u)$ for all $u \in B$. Here, set

$$
\tilde{g}(u)=\frac{\hat{g}(u)-\hat{g}(\sigma(u))}{2}, \quad \tilde{h}(u)=\frac{\hat{h}(u)-\hat{h}(\sigma(u))}{2}
$$

Then $\tilde{g}$ and $\tilde{h}$ are $\sigma$-odd and also an extension of $g$ and $h$ respectively. Since $f=\left(\left.\tilde{g}\right|_{A \cup B},\left.\tilde{h}\right|_{A \cup B}\right) \in C\left(A \cup B, \mathbf{R}^{n+m} \backslash\right.$ $\{0\}$ ) also $\sigma$-odd, we get $\gamma(A \cup B) \leqslant n+m=\gamma(A)+\gamma(B)$. (iv) easily follows from (i) and (ii).

Next, we show (vi). For any $u \in A$, we set $T_{u}=B_{d_{u} / 2}(u) \cup B_{d_{u} / 2}(\sigma(u))$ where $d_{u}=\operatorname{dist}\left\{u, H_{0}\right\}>0$. Then we have $\gamma\left(T_{u}\right)=1$. Since $A$ is compact and $\left\{T_{u} \mid u \in A\right\}$ are open covering of $A$, for finite $u_{1}, \ldots, u_{k} \in A$, we have $A \subset \bigcup_{i=1}^{k} T_{u_{i}}$. From (ii), we get $\gamma(A) \leqslant k$. Next, we show later part of (vi). We remark that letting $N_{\delta}(A)$ be $\delta$-neighborhood of $A$ in $M, N_{\delta}(A)$ is $\sigma$-invariant and $N_{\delta}(A) \subset H \backslash H_{0}$ for small $\delta>0$. Supposing $\gamma(A)=n$, there exists a $\sigma$-odd function $g \in C\left(A, \mathbf{R}^{n} \backslash\{0\}\right)$. By a similar way to show (iii), we have $\sigma$-odd function $\tilde{g} \in C\left(N_{\delta}(A), \mathbf{R}^{n} \backslash\{0\}\right)$. Thus we get $\gamma\left(N_{\delta}(A)\right) \leqslant n=\gamma(A)$. On the other hand, $A \subset N_{\delta}(A)$ implies $\gamma\left(N_{\delta}(A)\right) \geqslant \gamma(A)$. Thus we get $\gamma\left(N_{\delta}(A)\right)=\gamma(A)$.

Finally we show (vii). By a contradiction, we assume $\gamma\left(\psi\left(S^{n}\right)\right) \leqslant n$. Then there exists a $\sigma$-odd function $g \in$ $\left.C\left(\psi\left(S^{n}\right)\right), \mathbf{R}^{n} \backslash\{0\}\right)$. Here $g \circ \psi \in C\left(S^{n}, \mathbf{R}^{n} \backslash\{0\}\right)$ is an odd function but this contradicts the Borsuk-Ulam theorem. Thus we obtain (vii).

Proposition 2.4. Let $M \subset H \backslash H_{0}$ be a $\sigma$-invariant $C^{1}$-manifold and $J: M \rightarrow \mathbf{R}$ be a $\sigma$-even $C^{1}$-functional satisfying (PS)-condition. Moreover, we assume that

$$
\begin{equation*}
\liminf _{u \in M, \operatorname{dist}\{u, \partial M\} \rightarrow 0} J(u)=d \leqslant \infty \tag{2.5}
\end{equation*}
$$

Then, for any $c<d$ and $\delta>0$, there exist $\epsilon>0$ and $\eta:[0,1] \times[J \leqslant c+\epsilon]_{M} \rightarrow[J \leqslant c+\epsilon]_{M}$ such that

$$
\begin{align*}
& \eta(0, u)=u \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M}  \tag{2.6}\\
& \eta(1, u) \in\left[J \leqslant c_{k}-\epsilon\right]_{M} \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M} \backslash N_{\delta}\left(K_{c}\right)  \tag{2.7}\\
& \eta(1, \sigma(u))=\sigma(\eta(1, u)) \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M} \tag{2.8}
\end{align*}
$$

Here $K_{c}=\left\{u \in M \mid J(u)=c, J^{\prime}(u)=0\right\}$ and $N_{\delta}\left(K_{c}\right)$ is $\delta$-neighborhood of $K_{c}$ in $M$.
Proof. For any $u \in M$, we uniquely write $u=u_{0}+u_{1} \in H_{0}+H_{1}$ and $J(\sigma(u))$ is also uniquely written as $J(\sigma(u))=$ $J\left(u_{0}-u_{1}\right)$. Since $J: M \rightarrow \mathbf{R}$ is $\sigma$-even, we also have

$$
J\left(u_{0}-u_{1}\right)=J\left(u_{0}+u_{1}\right) \quad \text { for all } u=u_{0}+u_{1} \in H_{0}+H_{1}
$$

Therefore, noting $\nabla_{u}=\nabla_{u_{0}}+\nabla_{u_{1}}$, we obtain

$$
\begin{equation*}
\nabla J(\sigma(u)) \varphi=\sigma(\nabla J(u)) \varphi=\nabla J(u) \sigma(\varphi) \tag{2.9}
\end{equation*}
$$

Constructing a deformation flow $\eta:[0,1] \times[J \leqslant c+\epsilon]_{M} \rightarrow[J \leqslant c+\epsilon]_{M}$ by a standard way, it is obvious that $\eta$ satisfies (2.6)-(2.7). In addition, (2.8) holds from (2.9).

Proof of Theorem 2.2. Firstly we show (i). By a contradiction, we suppose that $c_{k}$ is not a critical point. From the definition of $c_{k}$, for any $\epsilon>0$, we have $\gamma\left(\left[J \leqslant c_{k}+\epsilon\right]_{M}\right) \geqslant k$. Applying Proposition 2.4 for $c=c_{k}$ and $K_{c_{k}}=\emptyset$, there exist $\epsilon>0$ and $\eta:[0,1] \times\left[J \leqslant c_{k}+\epsilon\right]_{M} \rightarrow\left[J \leqslant c_{k}+\epsilon\right]_{M}$ such that

$$
\begin{align*}
& \eta(0, u)=u \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M}  \tag{2.10}\\
& \eta(1, u) \in\left[J \leqslant c_{k}-\epsilon\right]_{M} \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M}  \tag{2.11}\\
& \eta(1, \sigma(u))=\sigma(\eta(1, u)) \quad \text { for all } u \in\left[J \leqslant c_{k}+\epsilon\right]_{M} \tag{2.12}
\end{align*}
$$

From (2.12) and (iii) of Lemma 2.3, we have

$$
\begin{equation*}
\gamma\left(\left[J \leqslant c_{k}+\epsilon\right]_{M}\right) \leqslant \gamma\left(\eta\left(1,\left[J \leqslant c_{k}+\epsilon\right]_{M}\right)\right) . \tag{2.13}
\end{equation*}
$$

From (2.11) and (i) of Lemma 2.3, we have

$$
\begin{equation*}
\gamma\left(\eta\left(1,\left[J \leqslant c_{k}+\epsilon\right]_{M}\right)\right) \leqslant \gamma\left(\left[J \leqslant c_{k}-\epsilon\right]_{M}\right) . \tag{2.14}
\end{equation*}
$$

Combining (2.13)-(2.14), we get $\gamma\left(\left[J \leqslant c_{k}-\epsilon\right]_{M}\right) \geqslant \gamma\left(\left[J \leqslant c_{k}+\epsilon\right]_{M}\right) \geqslant k$ and this contradicts the definition of $c_{k}$. Thus $c_{k}$ is a critical point. (ii) is obvious from the definition of $c_{k}$. Next we show (iii). By a contradiction, we suppose that $c_{k} \rightarrow \bar{c}<\infty$ as $k \rightarrow \infty$. Since $J$ satisfies (PS)-condition, $K_{\bar{c}}=\left\{u \in M \mid J(u)=\bar{c}, J^{\prime}(u)=0\right\}$ is a compact set. Thus, from (vi) of Lemma 2.3, there exists a $\sigma$-invariant neighborhood of $N_{\delta}\left(K_{\bar{c}}\right)$ such that $\gamma\left(K_{\bar{c}}\right)=\gamma\left(\overline{N_{\delta}\left(K_{\bar{c}}\right)}\right)=$ $q<\infty$. Applying Proposition 2.4 for $c=\bar{c}$ and $K_{\bar{c}}$, there exist $\epsilon>0$ and $\eta:[0,1] \times[J \leqslant \bar{c}+\epsilon]_{M} \rightarrow[J \leqslant \bar{c}+\epsilon]_{M}$ such that

$$
\begin{align*}
& \eta(0, u)=u \quad \text { for all } u \in\left[J \leqslant \bar{c}+\epsilon_{0}\right]_{M}  \tag{2.15}\\
& \eta(1, u) \in[J \leqslant \bar{c}-\epsilon]_{M} \quad \text { for all } u \in[J \leqslant \bar{c}+\epsilon]_{M} \backslash N_{\delta}\left(K_{\bar{c}}\right),  \tag{2.16}\\
& \eta(1, \sigma(u))=\sigma(\eta(1, u)) \quad \text { for all } u \in[J \leqslant \bar{c}+\epsilon]_{M} \tag{2.17}
\end{align*}
$$

Since $c_{k} \rightarrow \bar{c}<\infty$ as $k \rightarrow \infty$, there exists $k_{0}$ such that

$$
\bar{c}-\frac{\epsilon}{2}<c_{k} \leqslant \bar{c} \quad \text { for all } k \geqslant k_{0}
$$

From the definition $c_{k_{0}+q}$, we have $\gamma\left(\left[J \leqslant c_{k_{0}+q}+\epsilon\right]_{M}\right) \geqslant k_{0}+q$. Then, using (i), (iii) and (iv) of Lemma 2.3, we have

$$
\begin{aligned}
\gamma\left(\left[J \leqslant c_{k_{0}}-\frac{\epsilon}{2}\right]_{M}\right) & \geqslant \gamma\left([J \leqslant \bar{c}-\epsilon]_{M}\right) \\
& \geqslant \gamma\left(\eta\left(1,\left[J \leqslant c_{k_{0}+q}+\epsilon\right]_{M} \backslash N_{\delta}\left(K_{\bar{c}}\right)\right)\right) \\
& \geqslant \gamma\left(\eta\left(1,\left[J \leqslant c_{k_{0}+q}+\epsilon\right]_{M}\right)-\gamma\left(\overline{N_{\delta}\left(K_{\bar{c}}\right)}\right)\right) \\
& \geqslant\left(k_{0}+q\right)-q=k_{0}
\end{aligned}
$$

This is a contradiction to the definition of $c_{k_{0}}$. Thus we see that $c_{k} \rightarrow \infty$ as $k \rightarrow \infty$.
By a similar way to Theorem 2.2, we get the following theorem.
Theorem 2.5. Let $M \subset H \backslash H_{0}$ be a $\sigma$-invariant $C^{1}$-manifold and $J: M \rightarrow \mathbf{R}$ be a $\sigma$-even $C^{1}$-functional satisfying (PS)-condition. Moreover, we assume that

$$
\begin{align*}
& \inf _{u \in M} J(u)>-\infty  \tag{2.18}\\
& \liminf _{u \in M,}, \operatorname{dist}\{u, \partial M\} \rightarrow 0 \tag{2.19}
\end{align*} J(u)=d<\infty,
$$

and, for some $k \in \mathbf{N}$, there exists $\psi \in C\left(S^{k}, M\right)$ with $\psi(-x)=\sigma(\psi(x))$ such that $\sup _{x \in S^{k}} J(\psi(x))<d$. Then $J(u)$ has at least $k$ critical points.

Proof. We define $c_{i}(1 \leqslant i \leqslant k)$ as (2.4). Then we see that $c_{1} \leqslant c_{2} \leqslant \cdots \leqslant c_{k}(<d)$ are critical values of $J(u)$. Moreover, if $c_{i}=c_{i+1}=\cdots=c_{i+q}$ holds, then $\gamma\left(K_{c_{i}}\right) \geqslant q+1$. This is shown by a similar way to show $c_{k} \rightarrow \infty$ in the proof of Lemma 2.3. From (v) of Lemma 2.3, $\gamma\left(K_{c_{i}}\right) \geqslant q+1 \geqslant 2$ implies $K_{c_{i}}$ is an infinite set. Thus we get Theorem 2.5.

## 3. Proofs of Theorem 0.1 and Theorem 0.2

In this section, we will give the proofs of Theorem 0.1 and Theorem 0.2 by using abstract theories for $\sigma(u, v)=$ $(u,-v): H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \rightarrow H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega)$. To apply our abstract theory, we need the following lemma.

Lemma 3.1. Suppose $\beta<0$. For any $k \in \mathbf{N}$, there exists $\psi \in C\left(S^{k}, N_{\beta}\right)$ such that $\psi(-v)=\sigma(\psi(v))$.

Proof. We choose non-empty open sets $\Omega_{1}, \Omega_{2} \subset \Omega$ with $\Omega_{1} \cap \Omega_{2}=\emptyset$. We also choose $u_{0} \in H_{0}^{1}\left(\Omega_{1}\right)$ such that $\left\|u_{0}\right\| \|_{\lambda_{1}}=1$ and $u_{0+} \not \equiv 0$. Let $W_{k}$ be a $k$-dimensional subspace of $H_{0}^{1}\left(\Omega_{2}\right)$. Then it is obvious that $\mu_{1} \mu_{2}\left\|u_{0+}\right\|_{4}^{4}\|v\|_{4}^{4}-$ $\beta^{2}\left\|u_{0+} v\right\|_{2}^{4}>0$ for all $v \in S^{k}:=\left\{v \in W_{k} \mid\|v\|_{\lambda_{2}}=1\right\}$. Thus, setting $\psi(v)=\left(u_{0}, v\right), \psi(v)$ satisfies $\psi(v) \in N_{\beta}$ for all $v \in S^{k}$ and $\psi(-v)=\left(u_{0},-v\right)=\sigma(\psi(v))$.

Here, we give the proof of Theorem 0.1.
Proof of Theorem 0.1. Suppose $\beta<0$. We apply Theorem 2.2 for $H=H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega), \sigma(u, v)=(u,-v), M=$ $N_{\beta}, J(u)=J_{\beta}(u, v)$. Firstly, we will check that the assumptions of Theorem 2.2 hold. From Proposition 1.3, we have

$$
\underset{(u, v) \in N_{\beta}, \operatorname{dist}\left\{(u, v), \partial N_{\beta}\right\} \rightarrow 0}{\liminf } J_{\beta}(u, v)=\infty
$$

Moreover, from Lemma 3.1, for any $k \in \mathbf{N}$, there exists $\psi \in C\left(S^{k}, N_{\beta}\right)$ such that $\psi(-u)=\sigma(\psi(u))$. Therefore the assumptions of Theorem 2.2 hold and $J_{\beta}$ has a sequence of critical values $\left(c_{k}\right)_{k=1}^{\infty}$ such that $c_{k} \rightarrow \infty$ as $k \rightarrow \infty$. Let $\left(u_{k}, v_{k}\right)$ be a critical point of $J_{\beta}$ corresponding to $c_{k}$ and we set $\left(U_{k}, V_{k}\right)=\left(s_{\beta}\left(u_{k}, v_{k}\right) u_{k}, t_{\beta}\left(u_{k}, v_{k}\right) v_{k}\right)$. Then, from (iii) of Proposition 1.2, $\left(U_{k}, V_{k}\right)$ is a non-trivial critical point of $I_{\beta}$. From Proposition 1.1, we see $U_{k}>0$ in $\Omega$. Moreover, from (1.3) and $\beta<0$, we find

$$
\left(\mu_{1}\left\|U_{k}\right\|_{\infty}^{4}+\mu_{2}\left\|V_{k}\right\|_{\infty}^{4}\right)|\Omega| \geqslant \mu_{1}\left\|U_{k}\right\|_{4}^{4}+\mu_{2}\left\|V_{k}\right\|_{4}^{4} \geqslant 4 c_{k} \rightarrow \infty \quad(k \rightarrow \infty) .
$$

Thus we get $\left\|U_{k}\right\|_{\infty}+\left\|V_{k}\right\|_{\infty} \rightarrow \infty$. On the other hand, when $\beta>-\sqrt{\mu_{1} \mu_{2}}$, there exists a priori bound of positive solution of $(*)$ by a result of [13]. Thus, when $\beta \in\left(-\sqrt{\mu_{1} \mu_{2}}, 0\right), V_{k}$ must change sign for large $k$. Now, the proof of Theorem 0.1 is complete.

Next, we show Theorem 0.2. To prove Theorem 0.2, we need the following lemma.
Lemma 3.2. For any given $k \in \mathbf{N}$, there exist $\delta_{k}>0, \beta_{k}>0$ and $\psi \in C\left(S^{k}, M_{\delta_{k}}\right)$ with $\psi(-v)=\sigma(\psi(v))$ such that

$$
\begin{equation*}
\sup _{v \in S^{k}} J_{\beta}(\psi(v)) \leqslant d=\inf _{(u, v) \in \partial M_{\delta_{k}}} J_{\beta}(u, v) \quad \text { for all } \beta \in\left(-\beta_{k}, \beta_{k}\right) \tag{3.1}
\end{equation*}
$$

Proof. Let $W_{k}$ be $k$-dimensional subspace of $H_{0}^{1}(\Omega)$ such that

$$
W_{1} \subset W_{2} \subset \cdots \subset W_{k} \subset W_{k+1} \subset \cdots .
$$

For any given $k \in \mathbf{N}$, we choose small $\delta_{k}>0$ satisfying

$$
\mu_{2}\|v\|_{4}^{4}>4 \delta_{k} \quad \text { for all } v \in S^{k}:=\left\{v \in W_{k} \mid\|v\|_{\lambda_{2}}=1\right\} .
$$

We remark that $\delta_{k}$ also satisfies $4 \delta_{k} \in\left(0, \frac{1}{4 b_{0}}\right)$. For this $\delta_{k}>0$, from Proposition 1.9 and Proposition 1.7, there exists $\beta_{k}=\beta_{\delta_{k}}>0$ such that, for all $\beta \in\left(-\beta_{k}, \beta_{k}\right)$, we have

$$
\begin{aligned}
& \sup _{(u, v) \in M_{4 \delta_{k}}} J_{\beta}(u, v) \leqslant \frac{1}{8 \delta_{k}}+c_{\delta_{k}}(\beta), \\
& \inf _{(u, v) \in \partial M_{\delta_{k}}} J_{\beta}(u, v) \geqslant \frac{1}{4 \delta_{k}}+b_{0}-c_{\delta_{k}}(\beta), \\
& \left|2 c_{\delta_{k}}(\beta)\right|<\frac{1}{8 \delta_{k}}-b_{0} .
\end{aligned}
$$

Here we choose $u_{0} \in H_{0}^{1}(\Omega)$ such that $\left\|u_{0}\right\| \|_{\lambda_{1}}=1$ and $\left\|u_{0+}\right\|_{4}^{4} \geqslant 4 \delta_{k}$. Setting $\psi(v)=\left(u_{0}, v\right), \psi(v)$ satisfies

$$
\begin{aligned}
& \psi(v) \in M_{4 \delta_{k}} \subset M_{\delta_{k}} \quad \text { for all } v \in S^{k}, \\
& \psi(-v)=\left(u_{0},-v\right)=\sigma(\psi(v))
\end{aligned}
$$

Then $\psi(v)$ satisfies (3.1) and we get Lemma 3.2.

Now, we give the proof of Theorem 0.2.
Proof of Theorem 0.2. From Lemma 3.2, for any given $k \in \mathbf{N}$, there exist $\delta_{k}>0, \beta_{k}>0$ and $\psi \in C\left(S^{k}, M_{\delta_{k}}\right)$ with $\psi(-v)=\sigma(\psi(v))$ such that

$$
\sup _{v \in S^{k}} J_{\beta}(\psi(v)) \leqslant d=\inf _{(u, v) \in \partial M_{\delta_{k}}} J_{\beta}(u, v) \quad \text { for all } \beta \in\left(-\beta_{k}, \beta_{k}\right) .
$$

Here, setting $H=H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega), \sigma(u, v)=(u,-v), M=M_{\delta_{k}}, J(u)=J_{\beta}(u, v)\left(0<\beta<\beta_{k}\right)$, the assumptions of Theorem 2.5 hold. Thus $J_{\beta}$ has at least $k$ critical points. In conclusion from Proposition 1.2, we get Theorem 0.2.

## 4. The asymptotic behavior of some critical values of $J_{\beta}$

In this section, for $J_{\beta}(u, v)$, we will define the mountain pass values corresponding to solutions in Theorem 0.3. Firstly, for $J_{2}(v)$, we define symmetric mountain pass values $b_{n}^{2}(n \in \mathbf{N} \cup\{0\})$ by

$$
\begin{aligned}
& b_{n}^{2}=\inf _{\gamma_{2} \in \Gamma_{n}^{2}} \max _{\theta \in S^{n}} J_{2}\left(\gamma_{2}(\theta)\right), \\
& \Gamma_{n}^{2}=\left\{\gamma_{2}(\theta) \in C\left(S^{n}, \Sigma_{2}\right) \mid \gamma_{2}(-\theta)=-\gamma_{2}(\theta) \text { for all } \theta \in S^{n}\right\},
\end{aligned}
$$

where $S^{n}=\left\{\theta=\left(\theta_{1}, \ldots, \theta_{n+1}\right) \in \mathbf{R}^{n+1}| | \theta \mid=1\right\}$. Then, from the symmetric mountain pass theory for $J_{2}, b_{n}^{2}$ satisfies the following:
(i) $b_{n}^{2}$ is a critical value of $J_{2}$. In particular, $b_{0}^{2}$ is a least energy level of $J_{2}$.
(ii) $b_{0}^{2}<b_{1}^{2} \leqslant b_{2}^{2} \leqslant \cdots \leqslant b_{n}^{2} \leqslant b_{n+1}^{2} \leqslant \cdots$.
(iii) $b_{n}^{2} \rightarrow \infty$ as $n \rightarrow \infty$.

Now, from Lemma 3.2, for any given $k \in N$, there exist $\delta_{k}>0, \beta_{k}>0$ and $\psi \in C\left(S^{k}, M_{\delta_{k}}\right)$ with $\psi(-v)=\sigma(\psi(v))$ such that

$$
\sup _{v \in S^{k}} J_{\beta}(\psi(v)) \leqslant d=\inf _{(u, v) \in \partial M_{\delta_{k}}} J_{\beta}(u, v) \quad \text { for all } \beta \in\left(-\beta_{k}, \beta_{k}\right)
$$

We fix $k \in N, \delta_{k}>0$ and $\beta_{k}>0$ as above. Here, for $\beta \in\left(-\beta_{k}, \beta_{k}\right)$, we define minimax values $d_{i, \beta}(i=1,2, \ldots, k)$ of $J_{\beta}(u, v)$ by the following:

$$
\begin{align*}
& d_{i, \beta}=\inf _{g \in \Gamma_{i}} \max _{\theta \in S^{i}} J_{\beta}(\gamma(\theta)), \\
& \Gamma_{i}=\left\{\gamma(\theta) \in C\left(S^{i}, M_{\delta_{k}}\right) \mid \gamma(-\theta)=\sigma(\gamma(\theta)) \text { for all } \theta \in S^{i}\right\} . \tag{4.1}
\end{align*}
$$

We remark that $\Gamma_{i} \neq \emptyset$ by the existence of $\psi$. We show that $d_{i, \beta}$ satisfies the following proposition.
Proposition 4.1. For $i=1,2, \ldots, k$, we have:
(i) $d_{i, \beta}$ is a critical value of $J_{\beta}(u, v)$ for $\beta \in\left(-\beta_{k}, \beta_{k}\right)$.
(ii) $d_{i, \beta} \rightarrow b_{0}^{1}+b_{i}^{2}$ as $\beta \rightarrow 0$.

Proof. Firstly we show (i). By a contradiction, we suppose that $d_{i, \beta}$ is not a critical point. For $\epsilon_{0}>0$, there exists $\gamma \in \Gamma_{i}$ such that $\sup _{\theta \in S^{i}} J_{\beta}(\gamma(\theta)) \leqslant d_{i, \beta}+\epsilon_{0}$. Here, applying Proposition 2.4, we have small $\epsilon \in\left(0, \epsilon_{0}\right)$ and $\eta$ : $[0,1] \times\left[J_{\beta} \leqslant d_{i, \beta}+\epsilon\right]_{M_{\delta_{k}}} \rightarrow\left[J_{\beta} \leqslant d_{i, \beta}+\epsilon\right]_{M_{\delta_{k}}}$ such that

$$
\begin{align*}
& \eta(0, u)=u \text { for all } u \in\left[J_{\beta} \leqslant d_{i, \beta}+\epsilon\right]_{M_{\delta_{k}}},  \tag{4.2}\\
& \eta(1, u) \in\left[J_{\beta} \leqslant d_{i, \beta}-\epsilon\right]_{M_{\delta_{k}}} \text { for all } u \in\left[J_{\beta} \leqslant d_{i, \beta}+\epsilon\right]_{M_{\delta_{k}}},  \tag{4.3}\\
& \eta(1, \sigma(u))=\sigma(\eta(1, u)) \quad \text { for all } u \in\left[J_{\beta} \leqslant d_{i, \beta}+\epsilon\right]_{M_{\delta_{k}}} \tag{4.4}
\end{align*}
$$

Setting $\tilde{\gamma}(\theta)=\eta(1, \gamma(\theta))$, we have $\tilde{\gamma} \in \Gamma_{i}$ and $\sup _{\theta \in S^{i}} J_{\beta}(\tilde{\gamma}(\theta)) \leqslant d_{i, \beta}-\epsilon$. This contradicts the definition of $d_{i, \beta}$. Thus $d_{i, \beta}$ is a critical point.

Next, we show (ii). From Proposition 1.7, we have

$$
J_{1}(u)+J_{2}(v)-c_{\delta_{k}}(\beta) \leqslant J_{\beta}(u, v) \leqslant J_{1}(u)+J_{2}(v)+c_{\delta_{k}}(\beta)
$$

for all $(u, v) \in M_{\delta_{k}}$ and $\beta \in\left(-\beta_{k}, \beta_{k}\right)$. For any $\epsilon>0$, we choose $\gamma_{2} \in \Gamma_{i}^{2}$ such that

$$
\max _{\theta \in S^{i}} J_{2}\left(\gamma_{2}(\theta)\right) \leqslant b_{i}^{2}+\epsilon
$$

Setting $\gamma(\theta)=\left(u_{0}, \gamma_{2}(\theta)\right)$ where $u_{0}$ is a minimizer of $J_{1}(u)$, then we have $\gamma(\theta) \in \Gamma_{i}$ and

$$
\begin{equation*}
d_{i, \beta} \leqslant \max _{\theta \in S^{i}} J_{\beta}(\gamma(\theta)) \leqslant J_{1}\left(u_{0}\right)+\max _{\theta \in S^{i}} J_{2}\left(\gamma_{2}(\theta)\right)+c_{\delta_{k}}(\beta) \leqslant b_{0}^{1}+b_{i}^{2}+\epsilon+c_{\delta_{k}}(\beta) \tag{4.5}
\end{equation*}
$$

On the other hand, we choose $\gamma \in \Gamma_{i}$ such that

$$
\max _{\theta \in S^{i}} J_{\beta}(\gamma(\theta)) \leqslant d_{i, \beta}+\epsilon
$$

Writing $\gamma(\theta)=\left(\gamma_{1}(\theta), \gamma_{2}(\theta)\right) \in \Sigma_{1} \times \Sigma_{2}$, we have $\gamma_{2}(\theta) \in \Gamma_{i}^{2}$ and

$$
\begin{equation*}
b_{0}^{1}+b_{i}^{2} \leqslant J_{1}\left(\gamma_{1}(\theta)\right)+\max _{\theta \in S^{i}} J_{2}\left(\gamma_{2}(\theta)\right) \leqslant \max _{\theta \in S^{i}} J_{\beta}(\gamma(\theta))+c_{\delta_{k}}(\beta) \leqslant d_{i, \beta}+\epsilon+c_{\delta_{k}}(\beta) \tag{4.6}
\end{equation*}
$$

From (4.5)-(4.6), we have

$$
\left|d_{i, \beta}-\left(b_{0}^{1}+b_{i}^{2}\right)\right| \leqslant \epsilon+c_{\delta_{k}}(\beta)
$$

Since $\epsilon>0$ is arbitrary and $c_{\delta_{k}}(\beta) \rightarrow 0$ as $\beta \rightarrow 0$, we obtain (ii).

## 5. Proof of Theorem 0.3

In this section, we will complete the proof of Theorem 0.3 . For $i \in\{1,2, \ldots, k\}$, we show the following proposition.
Proposition 5.1. For any $\epsilon>0$, there exists $\beta_{k}^{\prime}>0$ such that, for all $|\beta|<\beta_{k}^{\prime}, J_{\beta}(u, v)$ has critical points in $A_{\beta}^{\epsilon}$ which are defined by

$$
A_{\beta}^{\epsilon}=\left\{\begin{array}{l|l}
(u, v) \in M_{\delta_{k}} & \begin{array}{l}
d b_{0}^{1} \leqslant J_{1}(u) \leqslant b_{0}^{1}+\epsilon \\
b_{0}^{1}+b_{i}^{2}-\epsilon \leqslant J_{\beta}(u, v) \leqslant b_{0}^{1}+b_{i}^{2}+\epsilon
\end{array}
\end{array}\right\}
$$

We remark that $A_{\beta}^{\epsilon}$ is an invariant set for $\sigma(u, v)=(u,-v)$ and $A_{\beta}^{\epsilon} \neq \emptyset$. If Proposition 5.1 holds, then we get Theorem 0.3 as follows:

Proof of Theorem 0.3. From Proposition 5.1, for all $\epsilon>0$ and $|\beta|<\beta_{k}^{\prime}$, there exists critical point $\left(u_{i, \beta}, v_{i, \beta}\right)$ of $J_{\beta}(u, v)$ which satisfies

$$
\begin{aligned}
& b_{0}^{1} \leqslant J_{1}\left(u_{i, \beta}\right) \leqslant b_{0}^{1}+\epsilon \\
& b_{0}^{1}+b_{i}^{2}-\epsilon \leqslant J_{\beta}\left(u_{i, \beta}, v_{i, \beta}\right) \leqslant b_{0}^{1}+b_{i}^{2}+\epsilon
\end{aligned}
$$

Since $\epsilon>0$ is arbitrary, from Proposition 1.7 , we see that $u_{i, \beta}, v_{i, \beta}$ satisfy

$$
\begin{aligned}
& J_{1}\left(u_{i, \beta}\right) \rightarrow b_{0}^{1} \quad J_{1}^{\prime}\left(u_{i, \beta}\right) \rightarrow 0 \quad \text { as } \beta \rightarrow 0, \\
& J_{2}\left(v_{i, \beta}\right) \rightarrow b_{i}^{2} \quad J_{2}^{\prime}\left(v_{i, \beta}\right) \rightarrow 0 \quad \text { as } \beta \rightarrow 0 .
\end{aligned}
$$

Thus, after extracting subsequence $\beta_{j} \rightarrow 0$, there exist $u_{i, 0} \in \Sigma_{1}$ and $v_{i, 0} \in \Sigma_{2}$ which are critical points of $J_{1}(u)$ and $J_{2}(v)$ respectively, such that

$$
\begin{array}{lll}
u_{i, \beta_{j}} \rightarrow u_{i, 0} & \text { as } \beta_{j} \rightarrow 0, & J_{1}\left(u_{i, 0}\right)=b_{0}^{1},
\end{array} J_{1}^{\prime}\left(u_{i, 0}\right)=0, ~ v_{2}\left(v_{i, 0}\right)=b_{i}^{2}, \quad J_{2}^{\prime}\left(v_{i, 0}\right)=0
$$

From (5.1)-(5.2), Proposition 1.1 and Proposition 1.2, we get Theorem 0.3.

In what follows, we will show Proposition 5.1 by a contradiction. If Proposition 5.1 does not hold, then there exist $\epsilon_{0}>0$ and a sequence $\beta_{j} \rightarrow 0$ such that $J_{\beta_{j}}(u, v)$ does not have critical points in $A_{\beta_{j}}^{\epsilon_{0}}$.

Here, we remark that a set of critical values of $J_{1}(u)$ is nowhere dense. Thus there exists $b_{0}^{1}+\frac{1}{3} \epsilon_{0}<a_{0}<a_{1}<$ $b_{0}^{1}+\epsilon_{0}$ such that $J_{1}(u)$ does not have critical points in $\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}$.

Remark 5.2. Fučík, Kučera, Nečas, Souček and Souček [14] gave a result for the Morse-Sard theorem in infinite dimensional setting. Since $\tilde{I}_{1}(u)=\frac{1}{2}\|u\|_{\lambda_{1}}^{2}-\frac{1}{4} \mu_{1}\|u\|_{4}^{4}: H_{0}^{1}(\Omega) \rightarrow \mathbf{R}$ is analytic and satisfies (PS)-condition, the set of critical values of $\tilde{I}_{1}(u)$ is measure zero and closed. Thus the set of critical values of $\tilde{I}_{1}(u)$ is nowhere dense. This implies the nowhere denseness of the set of critical values of $J_{1}(u)$. Moreover there exist further results of Dancer [12] and Cao and Noussair [8] about when critical values of $I_{1}(u)$ are isolated.

Since there are not critical points of $J_{1}(u)$ in $\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}$, we set

$$
\begin{equation*}
\rho_{0}=\inf _{u \in\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}} \mid\left\|\nabla J_{1}(u)\right\|_{\lambda_{1 *}}>0 . \tag{5.3}
\end{equation*}
$$

Then we have the following lemma.
Lemma 5.3. . For sufficiently small $\backslash \beta_{j} \mid>0$, we have the following: for any $(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}$ with $u \in\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]$, there exists $(X, Y) \in T_{u} \Sigma_{1} \times T_{v} \Sigma_{2}$ such that

$$
\begin{aligned}
& \|X\|_{\lambda_{1}}=1, \quad Y=0, \\
& \nabla J_{1}(u) X \geqslant \frac{\rho_{0}}{2}, \quad \nabla J_{\beta_{j}}(u, v)(X, Y) \geqslant \frac{\rho_{0}}{2} .
\end{aligned}
$$

Proof. Let $(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}$ with $u \in\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}$. From (5.3), we see that there exists $X \in T_{u} \Sigma_{1}$ such that

$$
\nabla J_{1}(u) X \geqslant \frac{3 \rho_{0}}{4} .
$$

From Proposition 1.7, choosing small $\left|\beta_{j}\right|>0$ such that $c_{\delta_{k}}\left(\beta_{j}\right)<\frac{\rho_{0}}{4}$, we have

$$
\nabla J_{\beta_{j}}(u, v)(X, 0) \geqslant \nabla J_{1}(u) X-c_{\delta_{k}}\left(\beta_{j}\right)\|X\|_{\lambda_{1}} \geqslant \frac{\rho_{0}}{2} .
$$

Thus we get Lemma 5.3.
Lemma 5.4. For small $\left|\beta_{j}\right|>0$, there exists a vector field $(u, v) \mapsto(X(u, v), Y(u, v)): A_{\beta_{j}}^{\epsilon_{0}} \rightarrow T_{u} \Sigma_{1} \times T_{v} \Sigma_{2}$ such that:
(i) $\|X(u, v)\|_{\lambda_{1}}^{2}+\|\mid Y(u, v)\|_{\lambda_{2}}^{2}=1$ and $(X(u, v), Y(u, v))$ are Lipschitz continuous.
(ii) $(X(\sigma(u, v)), Y(\sigma(u, v)))=\sigma(X(u, v), Y(u, v))$.
(iii) There exists $\mu_{j}>0$ such that $\nabla J_{\beta_{j}}(u, v)(X(u, v), Y(u, v)) \geqslant \mu_{j}$ for all $(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}$.
(iv) For any $(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}$ with $u \in\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}$, we have $\nabla J_{1}(u) X(u, v) \geqslant \frac{\rho_{0}}{2}$ and $\nabla J_{\beta_{j}}(u, v)(X(u, v)$, $Y(u, v)) \geqslant \frac{\rho_{0}}{2}$.

Proof. Since $J_{\beta_{j}}(u, v)$ does not have critical points in $A_{\beta_{j}}^{\epsilon_{0}}$, there exists $\mu_{j}>0$ such that

$$
\begin{equation*}
\mu_{j}=\inf _{(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}} \mid\left\|\nabla J_{\beta_{j}}(u, v)\right\|_{*}>0 . \tag{5.4}
\end{equation*}
$$

We also remark that $\nabla J_{\beta_{j}}(\sigma(u, v))=\sigma\left(\nabla J_{\beta_{j}}(u, v)\right)$. Thus from (5.4) and Lemma 5.3, we can construct a vector field with desired properties.

Here we consider the following ODE:

$$
\begin{aligned}
& \frac{d \eta_{1}}{d t}=-X\left(\eta_{1}, \eta_{2}\right), \quad \frac{d \eta_{2}}{d t}=-Y\left(\eta_{1}, \eta_{2}\right), \\
& \eta_{1}\left(0 ; u_{0}, v_{0}\right)=u_{0}, \quad \eta_{2}\left(0 ; u_{0}, v_{0}\right)=v_{0} .
\end{aligned}
$$

Then deformation flow $\eta(t,(u, v))=\left(\eta_{1}(t,(u, v)), \eta_{1}(t,(u, v))\right)$ satisfies the following:
(i) When $\eta(t,(u, v)) \in A_{\beta_{j}}^{\epsilon_{0}}$, we have $\frac{d}{d t} J_{\beta_{j}}(\eta(t,(u, v))) \leqslant-\mu_{j}$.
(ii) When $\eta(t,(u, v)) \in A_{\beta_{j}}^{\epsilon_{0}} \cap\left[a_{0} \leqslant J_{1} \leqslant a_{1}\right]_{\Sigma_{1}}$, we have $\frac{d}{d t} J_{\beta_{j}}(\eta(t,(u, v))) \leqslant-\frac{\rho_{0}}{2}$ and $\frac{d}{d t} J_{1}\left(\eta_{1}(t,(u, v))\right) \leqslant-\frac{\rho_{0}}{2}$.

From (ii), we see that, for $(u, v) \in A_{\beta_{j}}^{\epsilon_{0}}$ with $J_{1}(u)<b_{0}^{1}+\frac{1}{3} \epsilon_{0}$, when $\eta(t,(u, v))$ passes through $\partial A_{\beta_{j}}^{\epsilon_{0}}, \eta(t,(u, v))$ must satisfy $J_{\beta_{j}}(\eta(t,(u, v)))=b_{0}^{1}+b_{i}^{2}-\epsilon_{0}$. Moreover, from (i), $\eta(t,(u, v))$ must pass through $\partial A_{\beta_{j}}^{\epsilon_{0}}$ for finite time. Now, we complete the proof of Proposition 5.1.

Completion of the proof of Proposition 5.1. By the definition of $b_{0}^{1}$ and $b_{i}^{2}$, we can choose $u_{0} \in \Sigma_{1}$ and $\gamma_{2}(\theta) \in \Gamma_{i}^{2}$ such that

$$
\begin{aligned}
& J_{1}\left(u_{0}\right)<b_{0}^{1}+\frac{1}{3} \epsilon_{0}, \\
& \max _{\theta \in S^{i}} J_{2}\left(\gamma_{2}(\theta)\right)<b_{i}^{2}+\frac{1}{3} \epsilon_{0} .
\end{aligned}
$$

We set

$$
\gamma(\theta)=\left(u_{0}, \gamma_{2}(\theta)\right) \in \Gamma_{i} .
$$

Since $d_{i, \beta_{j}} \rightarrow b_{0}^{1}+b_{i}^{2}$ as $\beta_{j} \rightarrow 0$, for sufficiently small $\left|\beta_{j}\right|>0$, we have

$$
\max _{\theta \in S^{i}} J_{\beta_{j}}(\gamma(\theta))<b_{0}^{1}+b_{i}^{2}+\epsilon_{0} .
$$

Moreover $J_{\beta_{j}}(\gamma(\theta)) \geqslant b_{0}^{1}+b_{i}^{2}-\epsilon_{0}$ implies $\gamma(\theta) \in A_{\beta_{j}}^{\epsilon_{0}}$. For large $t>0$, we set

$$
\tilde{\gamma}(\theta)=\left(\eta_{1}(t ; \gamma(\theta)), \eta_{2}(t ; \gamma(\theta))\right) .
$$

Then we have $\tilde{\gamma}(\theta) \in \Gamma_{i}$ and

$$
\max _{\theta \in S^{i}} J_{\beta_{j}}(\tilde{\gamma}(\theta))<b_{0}^{1}+b_{i}^{2}-\epsilon_{0} .
$$

This is a contradiction for (4.1) and Proposition 4.1. Thus Proposition 5.1 holds and we complete the proofs of our theorems.

## 6. The setting for large $\beta$ and the proofs of Theorem 0.6 and Theorem 0.7

To prove Theorem 0.6 and Theorem 0.10 , we seek critical points of the following functional

$$
\tilde{I}_{\beta}(u, v)=\frac{1}{2}\left(\|u\|_{\lambda_{1}}^{2}+\|v\|_{\lambda_{2}}^{2}\right)-\frac{1}{4 \beta}\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}\right)-\frac{1}{2}\left\|u_{+} v\right\|_{2}^{2}: H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \rightarrow \mathbf{R} .
$$

Here, when $\beta=\infty$, we regard $\tilde{I}_{\infty}(u, v)$ as

$$
\tilde{I}_{\infty}(u, v)=\frac{1}{2}\left(\|u\|_{\lambda_{1}}^{2}+\|v\|_{\lambda_{2}}^{2}\right)-\frac{1}{2}\left\|u_{+} v\right\|_{2}^{2}: H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \rightarrow \mathbf{R} .
$$

We remark that if $(u, v)$ is a critical point of $\tilde{I}_{\beta}(u, v)$ for $\beta \in(0, \infty)$ then $(u / \sqrt{\beta}, v / \sqrt{\beta})$ is a solution of $(*)$ and if $u \neq 0$ we have $u>0$ in $\Omega$ from Proposition 1.1. Similarly, if $(u, v)$ is a critical point of $\tilde{I}_{\infty}(u, v)$, then $(u, v)$ is a solution of (0.5). We set

$$
\begin{aligned}
& \Sigma=\left\{(u, v) \in H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) \mid\|u\|_{\lambda_{1}}^{2}+\|v\|_{\lambda_{2}}^{2}=1\right\}, \\
& \Sigma_{+}=\left\{(u, v) \in \Sigma \mid\left\|u_{-}\right\| \lambda_{\lambda_{1}}<1\right\}, \\
& N=\left\{(u, v) \in \Sigma \mid u_{+} v \not \equiv 0\right\} .
\end{aligned}
$$

For $\beta \in(0, \infty]$, we define a functional $\tilde{J}_{\beta}(u, v)$ as follows.
Proposition 6.1. Suppose $\beta \in(0, \infty]$. For any $(u, v) \in \Sigma_{+}$if $\beta<\infty,(u, v) \in N$ if $\beta=\infty$, a function

$$
t \mapsto \tilde{I}_{\beta}(t u, t v): \mathbf{R}_{+} \rightarrow \mathbf{R}
$$

has a unique maximum point

$$
\tilde{t}_{\beta}(u, v)= \begin{cases}\sqrt{\beta}\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}+2 \beta\left\|u_{+} v\right\|_{2}^{2}\right)^{-\frac{1}{2}}, & \beta \in(0, \infty), \\ \left(\sqrt{2}\left\|u_{+} v\right\|_{2}\right)^{-1}, & \beta=\infty .\end{cases}
$$

Moreover, setting

$$
\tilde{J}_{\beta}(u, v)=\sup _{t>0} \tilde{I}_{\beta}(t u, t v)= \begin{cases}\frac{\beta}{4\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}+2 \beta\left\|u_{+}\right\|_{2}^{2}\right)}, & \beta \in(0, \infty), \\ \frac{1}{8}\left\|u_{+} v\right\|_{2}^{2}, & \beta=\infty,\end{cases}
$$

we have:
(i) $\tilde{t}_{\beta}(u, v): \Sigma_{+} \rightarrow \mathbf{R}_{+}$is a $C^{1}$-function.
(ii) $\tilde{J}_{\beta}(u, v): \Sigma_{+} \rightarrow \mathbf{R}$ is a $C^{1}$-function.
(iii) If $(u, v) \in \Sigma_{+}$is a critical point of $\tilde{J}_{\beta}(u, v)$, then $\left(\tilde{t}_{\beta}(u, v) u, \tilde{f}_{\beta}(u, v) v\right)$ is a non-trivial critical point of $\tilde{I}_{\beta}(u, v)$.
(iv) $\tilde{J}_{\beta}(u, v)$ satisfies PS-condition.

Proof. For $\beta \in(0, \infty]$, from direct calculations, we can write $\tilde{t}_{\beta}(u, v)$ and $\tilde{J}_{\beta}(u, v)$ explicitly. Thus, from those representations, we see that (i)-(ii) hold. (iii)-(iv) also are very standard.

We seek critical points of $\tilde{J}_{\beta}$ in $N=\left\{(u, v) \in \Sigma, \mid u_{+} v \not \equiv 0\right\} \subset \Sigma_{+}$.
Proposition 6.2. When $\beta \in(0, \infty]$, we have

$$
\begin{equation*}
\liminf _{(u, v) \in N, \operatorname{dist}\{(u, v), \partial N\} \rightarrow 0} \tilde{J}_{\beta}(u, v)=\beta b_{0} . \tag{6.1}
\end{equation*}
$$

Here $b_{0}$ was given in (1.9) and, when $\beta=\infty$, we regard $\beta b_{0}$ as $\infty$. In particular, $\tilde{J}_{\beta}(u, v)<\beta b_{0}$ implies $u_{+} v \neq 0$.
Proof. When $\beta=\infty$, (6.1) clearly holds. Thus we suppose $\beta \in(0, \infty)$. For any sequence $\left(\left(u_{n}, v_{n}\right)\right)_{n=1}^{\infty} \subset N$ with $\left\|u_{n+} v_{n}\right\|_{2} \rightarrow 0(n \rightarrow \infty)$, we should show $\liminf _{n \rightarrow \infty} \tilde{J}_{\beta}\left(u_{n}, v_{n}\right) \geqslant \beta b_{0}$. Since $\left\|u_{n}\right\| \lambda_{\lambda_{1}}+\left\|v_{n}\right\| \lambda_{\lambda_{2}}=1$, there exist subsequence $n_{j} \rightarrow \infty$ and some $u_{0}, v_{0} \in H_{0}^{1}(\Omega)$ such that

$$
u_{n_{j}+} \rightarrow 0, \quad v_{n_{j}} \rightarrow v_{0} \quad \text { strongly in } L^{4}(\Omega) .
$$

Here if $u_{0}=v_{0}=0$, then it is obvious that $\lim _{n_{j} \rightarrow \infty} \tilde{J}_{\beta}\left(u_{n_{j}}, v_{n_{j}}\right)=\infty$. On the other hand, if $u_{0}=0, v_{0} \neq 0$, we have

$$
\lim _{n_{j} \rightarrow \infty} \tilde{J}_{\beta}\left(u_{n_{j}}, v_{n_{j}}\right)=\frac{\beta}{4 \mu_{2}\left\|v_{0}\right\|_{4}^{4}} \geqslant \beta b_{0} .
$$

Thus we assume $\left(u_{0}, v_{0}\right) \in V=\left\{u, v \in \Sigma \mid u_{+} \neq 0, v \neq 0, u v=0\right\}$. Then we can also show

$$
\begin{equation*}
\inf _{(u, v) \in V} \tilde{J}_{\beta}(u, v)=\inf _{(u, v) \in \partial V} \tilde{J}_{\beta}(u, v) \geqslant \beta b_{0} . \tag{6.2}
\end{equation*}
$$

In fact, letting $\left(u_{*}, v_{*}\right) \in V$ be a minimizer of $\inf _{(u, v) \in V} \tilde{J}_{\beta}(u, v)$, then $\left(u_{*}, v_{*}\right)$ is a solution of $(*)$ with $\beta=0$ and $\tilde{J}_{\beta}\left(u_{*}, v_{*}\right) \geqslant 2 \beta b_{0}$. Thus $\inf _{(u, v) \in V} \tilde{J}_{\beta}(u, v)$ does not have minimizers in $V$ and we get (6.2). Thus we get Proposition 6.2.

Next, we give the proofs of Theorem 0.6 and Theorem 0.7. To show these theorems, we need the following lemma.
Lemma 6.3. For any given $k \in \mathbf{N}$, there exist $\bar{\beta}_{k} \in(0, \infty)$ and $\psi \in C\left(S^{k}, N\right)$ with $\psi(-u)=\sigma(\psi(u))$ such that

$$
\begin{equation*}
\sup _{u \in S^{k}} \tilde{J}_{\beta}(\psi(u)) \leqslant \beta b_{0} \quad \text { for all } \beta \in\left(\bar{\beta}_{k}, \infty\right] . \tag{6.3}
\end{equation*}
$$

Proof. Let $W_{k}$ be $k$-dimensional subspaces of $H_{0}^{1}(\Omega)$ such that $W_{1} \subset W_{2} \subset \cdots \subset W_{k} \subset W_{k+1} \subset \cdots$. For any given $k \in \mathbf{N}$, we set $S^{k}:=\left\{u \in W_{k} \mid\|u\|_{\lambda_{2}}=1\right\}$ and define $\psi(u): S^{k} \rightarrow N$ by

$$
\psi(u)=\left(\frac{|u|}{\sqrt{2}\|u\|_{\lambda_{1}}}, \frac{u}{\sqrt{2}}\right) .
$$

Here we choose $\bar{\beta}_{k}$ satisfying

$$
\bar{\beta}_{k} b_{0} \geqslant \sup _{u \in S^{k}} \frac{\|u\|_{\lambda_{1}}^{2}}{2\|u\|_{4}^{4}} .
$$

Then $\psi(u)$ satisfies $\psi(-u)=\sigma(\psi(u))$ and

$$
\tilde{J}_{\beta}(\psi(u)) \leqslant \frac{\|u\|_{\lambda_{1}}^{2}}{2\|(|u| u)\|_{2}^{2}} \leqslant \bar{\beta}_{k} b_{0} \quad \text { for all } \beta \in\left(\bar{\beta}_{k}, \infty\right] .
$$

Thus we get Lemma 6.3.
Now, we show Theorem 0.6.

Proof of Theorem 0.6. From Lemma 6.3, for any given $k \in \mathbf{N}$, there exist $\bar{\beta}_{k}>0$ and $\psi \in C\left(S^{k}, N\right)$ with $\psi(-v)=$ $\sigma(\psi(v))$ such that

$$
\sup _{v \in S^{k}} J_{\beta}(\psi(v)) \leqslant \beta b_{0} \quad \text { for all } \beta>\bar{\beta}_{k} .
$$

Thus, from Theorem 2.5, $\tilde{J}_{\beta}$ has at least $k$ critical values $e_{1, \beta} \leqslant e_{2, \beta} \leqslant \cdots \leqslant e_{k, \beta}$. Here $e_{i, \beta}$ is defined as follows:

$$
\begin{equation*}
e_{i, \beta}=\inf \left\{c \in \mathbf{R} \mid \gamma\left(\left[\tilde{J}_{\beta} \leqslant c\right]_{N}\right) \geqslant i\right\} . \tag{6.4}
\end{equation*}
$$

Let $\left(u_{i, \beta}, v_{i, \beta}\right)$ be a critical point corresponding to critical value $e_{i, \beta}$ of $\tilde{J}_{\beta}(u, v)$. We set $\left(U_{i, \beta}, V_{i, \beta}\right)=$ $\left(\frac{1}{\sqrt{\beta}} t_{\beta}\left(u_{i, \beta}, v_{i, \beta}\right) u_{i, \beta}, \frac{1}{\sqrt{\beta}} t_{\beta}\left(u_{i, \beta}, v_{i, \beta}\right) v_{i, \beta}\right)$. Then $\left(U_{i, \beta}, V_{i, \beta}\right)$ are solutions of $(*)$ and we get Theorem 0.6.

Finally, we give the proof of Theorem 0.7.
Proof of Theorem 0.7. Firstly we remark that $\liminf _{(u, v) \in N, \operatorname{dist}\{(u, v), \partial N\} \rightarrow 0} \tilde{J}_{\infty}(u, v)=\infty$ from Proposition 6.2. And, from Lemma 6.3, for any $k \in \mathbf{N}$, there exists $\psi \in C\left(S^{k}, N\right)$ with $\psi(-v)=\sigma(\psi(v))$. Thus, from Theorem 2.2, $\tilde{J}_{\infty}$ has a sequence of critical values $\left(e_{k, \infty}\right)_{k=1}^{\infty}$ such that $e_{k, \infty} \rightarrow \infty$ as $k \rightarrow \infty$. Here $e_{k, \infty}$ is defined by

$$
\begin{equation*}
e_{k, \infty}=\inf \left\{c \in \mathbf{R} \mid \gamma\left(\left[\tilde{J}_{\infty} \leqslant c\right]_{N}\right) \geqslant k\right\} . \tag{6.5}
\end{equation*}
$$

Let $\left(u_{k}, v_{k}\right)$ be a critical point of $\tilde{J}_{\infty}$ corresponding to $e_{k, \infty}$ and we set $\left(U_{k}, V_{k}\right)=\left(\tilde{t}_{\infty}\left(u_{k}, v_{k}\right) u_{k}, \tilde{t}_{\infty}\left(u_{k}, v_{k}\right) u_{k}\right)$. Then $\left(U_{k}, V_{k}\right)$ is a solution of (0.5) and we find

$$
\begin{aligned}
\left\|U_{k+} V_{k}\right\|_{\infty}^{2}|\Omega| \geqslant\left\|U_{k+} V_{k}\right\|_{2}^{2} & =\tilde{t}_{\infty}\left(u_{k}, v_{k}\right)^{4}\left\|u_{k+} v_{k}\right\|_{2}^{2} \\
& =8 \tilde{J}_{\infty}\left(u_{k}, v_{k}\right)=8 e_{k, \infty} \rightarrow \infty \quad(k \rightarrow \infty) .
\end{aligned}
$$

From the above inequality, we get $\left\|U_{k}\right\|_{\infty}+\left\|V_{k}\right\|_{\infty} \rightarrow \infty$. Moreover, from observation in Remark 0.9 , when $\lambda_{1}=\lambda_{2}$, $v_{k}$ must change sign for large $k$. From the above results, the proof of Theorem 0.7 is complete.

## 7. The asymptotic behavior as $\boldsymbol{\beta} \rightarrow \infty$

In this section, we consider the asymptotic behavior of solutions which were given in Theorem 0.6 as $\beta \rightarrow \infty$. In what follows, we fix a $k \in \mathbf{N}$ and let $\left(u_{k, \beta}, v_{k, \beta}\right)$ be a family of critical points of $\tilde{J}_{\beta}(u, v)$ corresponding to critical value $e_{k, \beta}$. Here $e_{k, \beta}$ was defined in (6.4). The following theorem is the main theorem in this section.

Theorem 7.1. There exists a subsequence $\beta_{j} \rightarrow \infty$ such that

$$
\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right) \rightarrow\left(u_{k, \infty}, v_{k, \infty}\right) \quad \text { in } H_{0}^{1}(\Omega) \times H_{0}^{1}(\Omega) .
$$

Here $\left(u_{k, \infty}, v_{k, \infty}\right)$ is a critical point of $\tilde{J}_{\infty}(u, v)$ corresponding to the critical value $e_{k, \infty}$. Here $e_{k, \infty}$ was defined in (6.5).

We remark that Theorem 0.10 easily follows from Theorem 7.1.
Proof of Theorem 0.10. For the $\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right)$ and $\left(u_{k, \infty}, v_{k, \infty}\right)$ in Theorem 7.1, we set

$$
\begin{aligned}
& \left(U_{k, \beta_{j}}, V_{k, \beta_{j}}\right)=\left(\frac{1}{\sqrt{\beta_{j}}} t_{\beta_{j}}\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right) u_{k, \beta_{j}}, \frac{1}{\sqrt{\beta_{j}}} t_{\beta_{j}}\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right) v_{k, \beta_{j}}\right), \\
& \left(U_{k, \infty}, V_{k, \infty}\right)=\left(t_{\infty}\left(u_{k, \infty}, v_{k, \infty}\right) u_{k, \infty}, t_{\infty}\left(u_{k, \infty}, v_{k, \infty}\right) v_{k, \infty}\right) .
\end{aligned}
$$

Then $\left(U_{k, \beta_{j}}, V_{k, \beta_{j}}\right)$ are solutions of $(*)$ obtained in Theorem 0.6 and $\left(\sqrt{\beta_{j}} U_{k, \beta_{j}}, \sqrt{\beta_{j}} V_{k, \beta_{j}}\right)$ converges to ( $U_{k, \infty}, V_{k, \infty}$ ) which is a solution of ( 0.5 ) corresponding to critical value $e_{k, \infty}$. These complete the proof of Theorem 0.10.

In the rest of this section, we will show Theorem 7.1. We need the following lemmas.
Lemma 7.2. For any $M>0$, we have

$$
\begin{equation*}
\frac{1}{8 M}-\frac{1}{2 \beta}\left(\mu_{1}+\mu_{2}\right) C_{1}^{4} \leqslant\left\|u_{+} v\right\|_{2}^{2} \leqslant C_{1}^{4} \quad \text { for all }(u, v) \in\left[J_{\beta} \leqslant M\right]_{N} . \tag{7.1}
\end{equation*}
$$

Here $C_{1}$ was given in (1.1).
Proof. Since $J_{\beta}(u, v) \leqslant M$ is equivalent to $\frac{1}{8 M} \leqslant \frac{1}{\beta}\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}\right)+2\left\|u_{+} v\right\|_{2}^{2}$, we easily get Lemma 7.2.
Lemma 7.3. For any $M>0$ and $\epsilon>0$, there exists $\beta_{M}^{\prime \prime}=\beta_{M}^{\prime \prime}(\epsilon)>0$ such that, for all $\beta \geqslant \beta_{M}^{\prime \prime}$, we have

$$
J_{\beta}(u, v)<J_{\infty}(u, v) \leqslant J_{\beta}(u, v)+\epsilon \quad \text { for all }(u, v) \in\left[J_{\beta} \leqslant M\right]_{N} .
$$

Proof. By a direct computation, we have

$$
\begin{equation*}
\tilde{J}_{\beta}(u, v)=\tilde{J}_{\infty}(u, v)-\frac{\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}}{8\left(\mu_{1}\left\|u_{+}\right\|_{4}^{4}+\mu_{2}\|v\|_{4}^{4}+2 \beta\left\|u_{+} v\right\|_{2}^{2}\right)\left\|u_{+} v\right\|_{2}^{2}} . \tag{7.2}
\end{equation*}
$$

Thus it is trivial that $\tilde{J}_{\beta}(u, v)<\tilde{J}_{\infty}(u, v)$. For any $M>0$, from Lemma 7.2, $J_{\beta}(u, v) \leqslant M$ implies (7.1). From (7.1) and (7.2), we get Lemma 7.3.

To show Theorem 7.1, the following proposition is essential.
Proposition 7.4. We have $e_{k, \beta} \leqslant e_{k, \infty}$ and

$$
e_{k, \beta} \rightarrow e_{k, \infty} \quad \text { as } \beta \rightarrow \infty
$$

Here $e_{k, \beta}$ and $e_{k, \infty}$ were defined in (6.4) and (6.5) respectively.

Proof. Firstly, we show $e_{k, \beta} \leqslant e_{k, \infty}$. Since $\tilde{J}_{\beta}(u, v)<\tilde{J}_{\infty}(u, v)$, we have $\left[J_{\infty} \leqslant c\right]_{N} \subset\left[J_{\beta} \leqslant c\right]_{N}$ for any $c \in \mathbf{R}$. From the definitions of $e_{k, \infty}$ and (i) of Lemma 2.3, for any $\epsilon>0$, we have

$$
\gamma\left(\left[J_{\beta} \leqslant e_{k, \infty}+\epsilon\right]_{N}\right) \geqslant \gamma\left(\left[J_{\infty} \leqslant e_{k, \infty}+\epsilon\right]_{N}\right) \geqslant k .
$$

This implies $e_{k, \beta} \leqslant e_{k, \infty}+\epsilon$ and, since $\epsilon>0$ is arbitrary, we get $e_{k, \beta} \leqslant e_{k, \infty}$. Next we show $e_{k, \beta} \rightarrow e_{k, \infty}$ as $\beta \rightarrow \infty$. From Lemma 7.3, for $M=e_{k, \infty}+1$ and any $\epsilon \in\left(0, \frac{1}{2}\right)$, there exists $\beta_{M}^{\prime \prime}>0$ such that for all $\beta \geqslant \beta_{M}^{\prime \prime}$ we have [ $\left.J_{\beta} \leqslant c_{k, \beta}+\epsilon\right]_{N} \subset\left[J_{\infty} \leqslant c_{k, \beta}+2 \epsilon\right]_{N}$. From the definitions of $e_{k, \beta}$ and (i) of Lemma 2.3, we get

$$
\gamma\left(\left[J_{\infty} \leqslant e_{k, \beta}+2 \epsilon\right]_{N}\right) \geqslant \gamma\left(\left[J_{\beta} \leqslant e_{k, \beta}+\epsilon\right]_{N}\right) \geqslant k .
$$

Thus we have $e_{k, \infty} \leqslant e_{k, \beta}+2 \epsilon$ for all $\beta \geqslant \beta_{M}^{\prime \prime}$. Combining $e_{k, \beta} \leqslant e_{k, \infty}$, we get $e_{k, \beta} \rightarrow e_{k, \infty}$ as $\beta \rightarrow \infty$.
Now we give the proof of Theorem 7.1.
Proof of Theorem 7.1. Let $\left(u_{k, \beta}, v_{k, \beta}\right)$ be a family of critical points of $\tilde{J}_{\beta}(u, v)$ corresponding to critical value $e_{k, \beta}$. Then there exist $u_{k, \infty}, v_{k, \infty} \in H_{0}^{1}(\Omega)$ and a subsequence $\left(\beta_{j}\right)_{j=1}^{\infty}$ such that

$$
u_{k, \beta_{j}} \rightarrow u_{k, \infty}, \quad v_{k, \beta_{j}} \rightarrow v_{k, \infty} \quad \text { weakly in } H_{0}^{1}(\Omega) \text { and strongly in } L^{4}(\Omega) \text { as } \beta_{j} \rightarrow \infty .
$$

Here, from Lemma 7.2, we see that $\left(u_{k, \infty}\right)_{+} \neq 0, v_{k, \infty} \neq 0$. Thus $t_{\beta_{j}}\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right)$ also converges to $t_{\infty}=$ $\left(\sqrt{2}\left\|\left(u_{k, \infty}\right)+v_{k, \infty}\right\|_{2}^{2}\right)^{-1}$ and $\left(t_{\infty} u_{k, \infty}, t_{\infty} v_{k, \infty}\right)$ is a critical point of $\tilde{I}_{\infty}(u, v)$. Now, if we show

$$
\begin{equation*}
\left\|u_{k, \infty}\right\|\left\|\lambda_{1}+\right\| v_{k, \infty}\| \|_{\lambda_{2}}=1 \tag{7.3}
\end{equation*}
$$

then our proof is complete. In fact, if (7.3) holds, then $u_{k, \beta_{j}}$ and $v_{k, \beta_{j}}$ strongly converge to $u_{k, \infty}$ and $v_{k, \infty}$ in $H_{0}^{1}(\Omega)$, respectively. Moreover, from Proposition 7.4, we have $\tilde{J}_{\infty}\left(u_{k, \infty}, v_{k, \infty}\right)=e_{k, \infty}$. Thus Theorem 7.1 obviously holds.

We will show (7.3). Since $\left(t_{\infty} u_{k, \infty}, t_{\infty} v_{k, \infty}\right)$ is a critical point of $\tilde{I}_{\infty}(u, v)$, we have

$$
\begin{equation*}
\left\|u_{k, \infty}\right\|\left\|_{\lambda_{1}}+\right\| v_{k, \infty}\left\|\lambda_{\lambda_{2}}=2 t_{\infty}^{2}\right\|\left(u_{k, \infty}\right)+v_{k, \infty} \|_{2}^{2} . \tag{7.4}
\end{equation*}
$$

On the other hand, from the representation of $t_{\beta_{j}}\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right)$ in Proposition 6.1, we have

$$
\begin{equation*}
1=t_{\beta_{j}}\left(u_{k, \beta_{j}}, v_{k, \beta_{j}}\right)^{2}\left[\frac{1}{\beta}\left(\mu_{1}\left\|u_{k, \beta_{j}}\right\|_{4}^{4}+\mu_{2}\left\|v_{k, \beta_{j}}\right\|_{4}^{4}\right)+2\left\|\left(u_{k, \beta_{j}}\right)_{+} v_{k, \beta_{j}}\right\|_{2}^{2}\right] . \tag{7.5}
\end{equation*}
$$

From (7.4) and (7.5), we get (7.3) and Theorem 7.1 holds.

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