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Ann. I. H. Poincaré – AN 29 (2012) 301–333

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Null controllability of viscous Hamilton–Jacobi equations ☆

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Received 29 April 2011; received in revised form 6 September 2011; accepted 2 November 2011

Available online 9 November 2011

Abstract

We study the problem of null controllability for viscous Hamilton–Jacobi equations in bounded domains of the Euclidean space in any space dimension and with controls localized in an arbitrary open nonempty subset of the domain where the equation holds. We prove the null controllability of the system in the sense that, every bounded (and in some cases uniformly continuous) initial datum can be driven to the null state in a sufficiently large time. The proof combines decay properties of the solutions of the uncontrolled system and local null controllability results for small data obtained by means of Carleman inequalities. We also show that there exists a waiting time so that the time of control needs to be large enough, as a function of the norm of the initial data, for the controllability property to hold. We give sharp asymptotic lower and upper bounds on this waiting time both as the size of the data tends to zero and infinity. These results also establish a limit on the growth of nonlinearities that can be controlled uniformly on a time independent of the initial data.

MSC: 49L25; 35B37; 93B05

Keywords: Null controllability; Viscous Hamilton–Jacobi equations; Decay estimates

1. Introduction

In this paper we address the problem of null controllability for the so-called viscous Hamilton–Jacobi equations. Focussing on a simple model example, we consider the system

$$\begin{cases} y_t - \Delta y + |\nabla y|^q = v\chi_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

☆ A. Porretta was partially supported by Gnampa project (2010) "Proprietà di regolarità in EDP nonlineari legate a problemi di controllo". E. Zuazua was partially supported by Grants MTM2008-03541 and MTM2011-29306 of the MICINN, Spain, ERC Advanced Grant FP7-246775 NUMERIWAVES, ESF Research Networking Programme OPTPDE and Grant PI2010-04 of the Basque Government.

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.11.002

where Ω is an open bounded set in $\mathbb{R}^N$, $N \geqslant 1$. We assume that $\omega \subset \Omega$ is an open set ($\omega \neq \Omega$), the control v belongs to $L^\infty((0,T)\times\omega)$, and the initial datum y_0 belongs to $L^\infty(\Omega)$ if $q \leqslant 2$, or to $C_0(\overline{\Omega})$ (i.e. y_0 is continuous in $\overline{\Omega}$ and $y_0 = 0$ on $\partial\Omega$) if $q > 2$. Note that the sign of the nonlinearity in (1.1) could be reversed changing y into $-y$, so that both cases are included in our analysis.

The problem of null controllability means, roughly speaking, that one can find a control v so that the evolution leads the state y from the initial condition y_0 to the final condition $y(T) = 0$. In order to make this question precise, we recall that there are significant differences between the cases $q \leqslant 2$ and $q > 2$ as far as the well-posedness of (1.1) is concerned. We only give here a rough summary and refer to Section 2 for precise statements and references. If $1 \leqslant q \leqslant 2$, problem (1.1) can be dealt with by only using the notions of weak solution ($y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$ such that $|\nabla y|^q \in L^1(Q_T)$ and the equation holds weakly) or strong solution (a weak solution such that $y \in L^p(0,T;W^{2,p}(\Omega))$ for every $p < \infty$ and $y, |\nabla y| \in L^\infty(Q_T)$). In particular, for every $y_0 \in L^\infty(\Omega)$ and every $v \in L^\infty((0,T)\times\omega)$ there exists a unique bounded weak solution of (1.1) which is defined globally in time, and is a strong solution for $t > 0$. Therefore, if $1 \leqslant q \leqslant 2$, we will say that the initial datum $y_0 \in L^\infty(\Omega)$ is controllable at time T if there exists $v \in L^\infty((0,T)\times\omega)$ such that the corresponding bounded weak solution y satisfies $y(T) = 0$. The system (1.1) will be said to be null controllable at time T if every $y_0 \in L^\infty(\Omega)$ is controllable at time T. Notice that we are referring to a *global* controllability property of the system, namely that it can be controlled to zero in a time which is uniform with respect to all initial data.

The situation is more delicate if $q > 2$ since, even if y_0 is regular, weak or strong solutions may not exist globally because of gradient blow-up (see [23]). On the other hand, if $v = 0$ and $y_0 \in C_0(\overline{\Omega})$, it is proved in [3] that a unique, globally defined solution exists in the sense of generalized viscosity solutions (see Section 2). Therefore, in the case $q > 2$ we define the property of null controllability in the following way. An initial data $y_0 \in C_0(\overline{\Omega}) \cap W^{2,p}(\Omega)$ (for every $p < \infty$) will be said to be controllable at time $T > 0$ if there exist $v \in L^\infty((0,T)\times\omega)$ and a strong solution y of (1.1) such that $y(T) = 0$. Note that the strong solution, if it exists, is unique. Finally, for every $y_0 \in C_0(\overline{\Omega})$ we denote by $\hat{y}$ the unique generalized viscosity solution of (1.1) when $v \equiv 0$; then, we will say that $y_0 \in C_0(\overline{\Omega})$ is controllable at time T if there exists $t_0 < T$ such that $\hat{y}(t_0) \in C_0(\overline{\Omega}) \cap W^{2,p}(\Omega)$ and is controllable at time T. As before, the system (1.1) will be said to be null controllable at time T if it is so for every $y_0 \in C_0(\overline{\Omega})$.

A wide literature exists nowadays concerning the problem of null controllability of parabolic equations. In particular, it is well known that the heat equation is (globally) null controllable at any time T, as well as a large class of linear parabolic equations (see e.g. [14,17]). The null controllability of semilinear equations is a much more delicate question. It was proved in [13,10] (see also [2], or [26] for the wave equation) that the semilinear equation

$$\begin{cases} y_t - \Delta y + f(x,y,\nabla y) = v\chi_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega \end{cases} \tag{1.2}$$

is still globally null controllable at any time T whenever the function $f(x,s,\xi)$ is locally Lipschitz continuous and satisfies the growth condition

$$\bigl|f(x,s,\xi)\bigr| \leqslant C\bigl(1 + |s|\bigl(\log\bigl(1+|s|\bigr)\bigr)^\gamma + |\xi|\bigl(\log\bigl(1+|\xi|\bigr)\bigr)^\alpha\bigr)$$

for some $\gamma < \frac{3}{2}$ and $\alpha < \frac{1}{2}$. In particular, in this range of growth of the nonlinearity, the semilinear system behaves very closely to the linear case, since not only every initial datum y_0 can be controlled to zero but also the control time can be any $T > 0$, independently of y_0. Unfortunately, it is not known whether the above condition on the growth of the nonlinearity $f(x,s,\xi)$ is necessary for the null controllability property to hold. However, in [13] the authors proved that the null controllability property fails for some $f = f(y)$ such that $f(y) \sim |y|\log(1+|y|)^\gamma$ as $|y| \to \infty$ with $\gamma > 2$; in that case, there are initial data which can never be controlled since the control mechanism is unable to avoid the blow-up of solutions. Similar counterexamples were missing as far as the growth of the nonlinearity with respect to ∇y is concerned.

The aim of our paper is to give a more clear picture of what happens in the case f depends on ∇y but not on y, and specifically for the model problem (1.1) when $q > 1$. On one hand, we prove that, for every $q > 1$, the null controllability property for the system (1.1) fails for all $T > 0$, i.e. that the system cannot be controlled in a time which is independent of the initial data. We actually prove, in more generality, that the same holds for the system (1.2) for a large class of nonlinearities, including the case $f = f(x,\nabla y) \sim |\nabla y|(\log(|\nabla y|))^\alpha$ with $\alpha > 1$. Notice that some gap with the positive results proved in [10] still remains, since it is known that, if $\alpha < \frac{1}{2}$, the null controllability property holds at any time T, independently of the size of the initial datum. We recall that a similar gap (though with different exponents ($\gamma < 3/2$ for positive results and $\gamma > 2$ for negative ones)) between negative and positive results also exists as far as the case $f = f(y)$ is concerned. Observe also that, as indicated in [11], the existing techniques for the control of parabolic equations based on Carleman inequalities will hardly lead to positive results in a larger class of nonlinearities, i.e. under a weaker restriction on γ.

On the other hand, as it happens in the case $f = f(y)$ under suitable sign conditions on the nonlinearity, we prove that any initial datum y_0 can be driven to zero in large time (depending on the size of the initial datum to be controlled). To this purpose, we use a typical strategy; on one hand, whatever $T > 0$ is, sufficiently small initial data are proved to be controllable; on the other hand, the decay properties of the uncontrolled system make it possible to reduce the problem, in a sufficiently large time, to the case of small data. As a consequence of this two-step argument, a waiting time is needed for the controllability of (1.1) to hold, which we estimate in terms of the norm $\|y_0\|_\infty$ of the initial datum y_0 to be controlled.

Our analysis of problem (1.1) can be summarized in the following result, expressing both the existence and the sharp estimates of the waiting time.

Theorem 1.1. *Let us consider problem* (1.1) *with* $q > 1$; *we set*

$$T(y_0) := \inf\big\{t > 0\colon\ (1.1) \text{ is null controllable at time } t\big\}$$

and

$$T(r) := \sup\big\{T(y_0),\ \|y_0\|_\infty \leqslant r\big\}.$$

Then we have $0 < T(r) < \infty$ *for every* $r > 0$, *and there exist positive constants* κ, K, λ, Λ (*only depending on* q, Ω, ω) *such that*

$$\frac{\kappa}{\ln(\frac{1}{r})} \leqslant T(r) \leqslant \frac{K}{\ln(\frac{1}{r})} \quad \text{as } r \to 0^+ \tag{1.3}$$

and

$$\lambda r \leqslant T(r) \leqslant \Lambda r \quad \text{as } r \to \infty. \tag{1.4}$$

Comparing with the controllability of the semilinear system in the case $f = f(y)$, which we mentioned before (see [13]), two main features appear from our result. In both cases, a sufficiently fast superlinear growth destroys the (global) null controllability property which is typical of the linear (or mildly superlinear) case. However, differently than in the superlinear case when $f = f(y)$, the maximum principle prevents the occurrence of blow-up in (1.1), and all initial data will be eventually controlled, even if in a delayed time. Indeed, the existence of a waiting time for the null controllability of (1.1) reminds of what happens in other cases of dissipative operators, e.g. when $f = f(y)$ has the "good sign" property (see [1]) or in the one-dimensional Burgers equation (see [12]), where a similar estimate as (1.3) was proved. In our context, the lower bound on the waiting time is obtained through the construction of barrier solutions, i.e. stationary solutions in a domain $\mathcal{O}$ which blow up at the boundary $\partial\mathcal{O}$ (sometimes called large solutions). Let us mention that, in case of nonlinear systems and specifically in absorption type problems (when the nonlinearity in (1.2) satisfies $f(y, \nabla y)y \geqslant 0$ for any $y \in \mathbb{R}$), the existence of universal barriers has often been used to construct some obstruction to controllability properties (see e.g. [1,14,9]). When dealing with (1.1), the existence of such barriers, and the failure of the controllability property, are strictly related to properties of the controlled stochastic dynamics underlying the PDE. Indeed, the possibility to represent the solution of (1.1) as the value function of a stochastic control problem suggests a possible rough explanation of the above controllability results, see Remark 4.3.

The construction of stationary barriers or, in an alternative viewpoint, the existence of universal local bounds, allow us to prove that the existence of the waiting time and the estimates (1.3)–(1.4) also hold for different nonlinearities, e.g. when $f(x, \nabla y) = h(|\nabla y|)$; however, such a construction requires a restriction on the growth of h, relying on the fact that $\int^\infty \frac{1}{h(s)}\,ds < \infty$. A significant example we deal with is the function $h(s) = s(\log(1+s))^\alpha$ with $\alpha > 1$. In order to clearly state this extension, let us consider the problem

$$\begin{cases} y_t - \Delta y + f(x, \nabla y) = v\chi_\omega & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

and let us restrict here to nonlinearities having at most a quadratic growth in the gradient, so that globally defined weak solutions are known to exist. Given $y_0 \in L^\infty(\Omega)$, we say that the system (1.5) is null controllable at time T if there exist $v \in L^\infty((0, T) \times \omega)$ and a bounded weak solution y of (1.5) such that $y(T) = 0$. Then, $T(y_0)$ and $T(r)$ are defined as in Theorem 1.1.

Theorem 1.2. *Assume that $f \in C^1(\Omega \times \mathbb{R}^N, \mathbb{R})$ is such that $f(x, 0) = 0$, $|D_\xi f(x, \xi)|$ is uniformly bounded if ξ is in a compact set, and there exist constants β, γ, γ_0, γ_1, L such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^N$,*

$$\exists \alpha > 1: \quad f(x, \xi) \geqslant \gamma|\xi|\big(\ln\big(|\xi|\big)\big)^\alpha \quad \forall \xi: |\xi| \geqslant L, \tag{1.6}$$

$$\big|f(x, \xi)\big| \leqslant \beta\big(1 + |\xi|^2\big), \tag{1.7}$$

$$f(x, \xi) - D_\xi f(x, \xi) \cdot \xi \leqslant \gamma_0, \tag{1.8}$$

$$\big|D_x f(x, \xi)\big| \leqslant \gamma_1|\xi|. \tag{1.9}$$

Then, the conclusion of Theorem 1.1 *holds true.*

As mentioned before, condition (1.6) is only one example of growth which ensures the lower bounds for the waiting time, that will be proved in more generality (see Theorem 4.2 below). On the contrary, assumptions (1.7)–(1.9) will be used for the positive result, namely the upper bounds of the waiting time. These latter ones strongly depend on the decay properties of the system. Note that assumption (1.9), though it is not the most general possible, allows us at least to include the presence of transport terms in the equation, as well as the case when $f(x, \xi) = h(|\xi|) + g(x, \xi)$, with g having a Lipschitz growth with respect to ξ.

Finally, the organization of the paper is the following. Section 2 is devoted to some preliminary notions and properties concerning viscous Hamilton–Jacobi equations which will be used later. In Section 3 we prove that the system (1.1) is controllable in large time, possibly depending on the initial datum y_0. To this purpose, a crucial role is played by the long time behavior when $v \equiv 0$; we discuss this problem and the decay properties of y in Appendix A, which may have its own interest. Actually, when $q > 2$ we prove not only that solutions of the uncontrolled system decay to zero but that they eventually become smooth, which justifies our definition of controllability for this case. In fact, we always let the control act only when solutions $y(t)$ belong to $W^{1,\infty}(\Omega)$ and are meant in a strong sense.

In Section 4 we prove the failure of the null controllability in arbitrary time $T > 0$, showing the necessity of some waiting time. Such a proof also provides a first estimate of the waiting time, and specifically the lower bound in (1.4). In Section 5, we refine our estimates for the short time range and we complete the proof of (1.3) and (1.4). Let us mention that, in Remark 5.2, we also give an interesting explicit estimate of the constants λ, Λ appearing in (1.4) for the case $q = 2$. We conclude Section 5 by proving Theorem 1.2 and discussing further generalizations for the superquadratic case.

In Section 6 we spend some words for the problem of controllability to trajectories, see Theorem 6.1. In particular, we prove that the system (1.1) is also exactly controllable to any trajectory. Moreover, similar estimates hold for the control time, at least when the trajectory is smooth. Further comments and remarks are devoted to the problem of approximate controllability and other open questions.

Last but not least, we give in Appendix A a direct proof of the exponential decay of solutions of the uncontrolled system, obtaining with a different approach (and in a slightly more general context) a result recently proved in [6] concerning the exact exponential decay rate for problem (1.1). Because of the importance of the long time behavior in such type of problems, we also give a self-contained proof of the exponential decay for more general nonlinearities.

2. Preliminaries

We recall a few definitions and preliminary results concerning problem (1.1). Set $Q_T = (0, T) \times \Omega$, where Ω is a bounded subset of $\mathbb{R}^N$; although it will be mostly unnecessary, we will assume Ω of class C^2 in order to make use of the classical regularity results. We denote by $C_0(\overline{\Omega})$ the functions which are continuous in $\overline{\Omega}$ and zero at the boundary $\partial\Omega$, and by $C_0^k(\overline{\Omega})$ the subset of functions being C^k in $\overline{\Omega}$. Let us consider a boundary value problem of the type

$$\begin{cases} y_t - \Delta y + f(t,x,\nabla y) = \chi(t,x) & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

We always assume, at least, that $f(t,x,\xi): Q_T \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function (i.e. measurable with respect to $(t,x) \in Q_T$ and continuous with respect to ξ), $\chi \in L^\infty(Q_T)$ and $y_0 \in L^\infty(\Omega)$.

By a bounded weak solution of (2.1), we mean a function $y \in L^\infty(Q_T)$ such that $y \in L^2(0,T;H_0^1(\Omega)) \cap C^0([0,T];L^2(\Omega))$, $f(t,x,\nabla y) \in L^1(Q_T)$ and the equation is satisfied in the weak sense, i.e.

$$-\int\limits_{Q_T} y\varphi_t\,dx\,dt + \int\limits_{Q_T} \nabla y\nabla\varphi\,dx\,dt + \int\limits_{Q_T} f(t,x,\nabla y)\varphi\,dx\,dt = \int\limits_{Q_T} \chi(t,x)\varphi\,dx\,dt + \int\limits_{\Omega} y_0\varphi(0)\,dx$$

for every $\varphi \in C_0^1([0,T]\times\overline{\Omega})$ such that $\varphi(T) = 0$.

By a strong solution of (2.1) we mean a weak solution which satisfies, in addition, $|\nabla y| \in L^\infty(Q_T)$, $y \in L^p(0,T;W^{2,p}(\Omega))$ for every $p < \infty$ and $y_t \in L^p(Q_T)$. Finally, we call y a classical solution in the case y is C^2 in the space variable, C^1 in the time variable and the equation holds pointwise (of course, in this case the nonlinearity f should be, at least, continuous).

When f is locally Lipschitz continuous with respect to ξ, the comparison principle and uniqueness hold in the class of strong solutions; indeed, strong solutions are Lipschitz in the space variable and a linearization argument yields easily a uniqueness result which is, of course, independent from the growth of f with respect to ξ. However, the existence of strong, or weak solutions, may depend on this growth.

When the nonlinearity f satisfies

$$\bigl|f(t,x,\xi)\bigr| \leqslant \gamma\bigl(1+|\xi|^2\bigr), \quad \gamma > 0,\ (t,x)\in Q_T,\ \xi \in \mathbf{R}^N \tag{2.2}$$

several well-posedness results exist for problem (2.1). If $y_0 \in C_0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ and f is locally Lipschitz continuous with respect to ξ, it is proved in [15, Chapter V, Thm. 6.3] that there exists a unique strong solution of (2.1). If $y_0 \in L^\infty(\Omega)$, the existence of a bounded weak solution is known even in much more generality (see e.g. [7,20]). Moreover, from [15, Chapter V], we know that, under assumption (2.2), any bounded weak solution is locally Hölder continuous, and even globally if y_0 is a Hölder continuous function. The uniqueness in the class of bounded weak solutions is more difficult to find in the literature. Let us give here a short, easy proof for the case that $f(t,x,\xi)$ is convex in the ξ variable, which in particular applies to problem (1.1). The following argument is taken from [4].

Proposition 2.1. *Assume that $f(t,x,\xi)$ satisfies* (2.2) *and in addition that $f(t,x,\xi)$ is convex with respect to ξ. Let $\chi \in L^\infty(Q_T)$, $y_0 \in L^\infty(\Omega)$. Then there exists a unique bounded weak solution of* (2.1).

Moreover, if y_1, y_2 are solutions corresponding to data χ_1, y_{01} and χ_2, y_{02} respectively, then $\chi_1 \leqslant \chi_2$ and $y_{01} \leqslant y_{02}$ imply $y_1 \leqslant y_2$ in Q_T.

Proof. Let $\varepsilon \in (0,1)$, and consider the function $u_\varepsilon := (1-\varepsilon)y_1$, which satisfies

$$(u_\varepsilon)_t - \Delta u_\varepsilon + (1-\varepsilon)f(t,x,\nabla y_1) \leqslant (1-\varepsilon)\chi_1.$$

Since $f(t,x,\xi)$ is convex in ξ, we have

$$f(t,x,\nabla y_2) \leqslant (1-\varepsilon)f(t,x,\nabla y_1) + \varepsilon f\left(t,x,\frac{\nabla y_2 - (1-\varepsilon)\nabla y_1}{\varepsilon}\right),$$

hence u_ε satisfies, in a weak sense,

$$(u_\varepsilon)_t - \Delta u_\varepsilon + f(t,x,\nabla y_2) \leqslant (1-\varepsilon)\chi_1 + \varepsilon f\left(t,x,\frac{\nabla y_2 - (1-\varepsilon)\nabla y_1}{\varepsilon}\right).$$

Note that the right-hand side is an L^1 function thanks to (2.2) and since $y_1, y_2 \in L^2(0,T; H_0^1(\Omega))$. Subtracting the equation of y_2, using that $\chi_1 \leqslant \chi_2$ we obtain

$$(u_\varepsilon - y_2)_t - \Delta(u_\varepsilon - y_2) \leqslant -\varepsilon\chi_1 + \varepsilon f\left(t, x, \frac{\nabla y_2 - (1-\varepsilon)\nabla y_1}{\varepsilon}\right).$$

Set $z_\varepsilon := \frac{u_\varepsilon - y_2}{\varepsilon}$; then, using (2.2), we deduce that z_ε is a bounded weak solution of

$$(z_\varepsilon)_t - \Delta z_\varepsilon \leqslant -\chi_1 + \gamma\big(1 + |\nabla z_\varepsilon|^2\big).$$

Moreover, since $y_{01} \leqslant y_{02}$, we have that $z_\varepsilon(0) \leqslant -y_{01}$. If $K = \|\chi_1\|_\infty + \gamma + \|y_{01}\|_\infty$, the function $z_\varepsilon - K(t+1)$ is a bounded weak solution of

$$\begin{cases} v_t - \Delta v \leqslant \gamma|\nabla v|^2, \\ v(0) \leqslant 0, \quad v_{(0,T)\times\partial\Omega} \leqslant 0 \end{cases}$$

and we deduce that $v \leqslant 0$ (using $e^{\gamma v^+} - 1$ as test function, which is justified by standard arguments). This means that $z_\varepsilon \leqslant K(t+1)$, so we proved that

$$y_1(1-\varepsilon) - y_2 \leqslant \varepsilon\big[\big(\|\chi_1\|_\infty + \gamma + \|y_{01}\|_\infty\big)(t+1)\big]$$

and letting $\varepsilon \to 0$ we conclude that $y_1 \leqslant y_2$. $\square$

As a consequence of Proposition 2.1, if $1 \leqslant q \leqslant 2$ problem (1.1) is well posed in the class of bounded weak solutions for any $y_0 \in L^\infty(\Omega)$ and any control $v \in L^\infty$. Moreover, applying to the function $ty(t,x)$ the results in [15, Chapter V], one deduces that y is a strong solution for $t > 0$, and even classical if $v = 0$.

When (2.2) does not hold, the situation is different as far as the existence of solutions is concerned. It is known that, even if the initial datum y_0 is C^1, it may not exist a global strong (or weak) solution because of a gradient blow-up which may occur at the boundary (see [23,24]).

In [3], a well-posedness result for (1.1) is proved if $v = 0$ in the class of generalized viscosity solutions satisfying the boundary condition in a relaxed sense. This means that $y \in C(\overline{Q}_T)$, $y(0) = y_0$, y is a viscosity solution in Q_T and satisfies the boundary condition in the sense that

$$\min\big\{y, y_t - \Delta y + |\nabla y|^q\big\} \leqslant 0 \quad \text{on } (0,T) \times \partial\Omega \tag{2.3}$$

and

$$\max\big\{y, y_t - \Delta y + |\nabla y|^q\big\} \geqslant 0 \quad \text{on } (0,T) \times \partial\Omega \tag{2.4}$$

where the inequalities for the operator are meant in the viscosity sense. As pointed out in [3], in the particular case of (1.1), every viscosity solution satisfies $y \leqslant 0$ at $(0,T) \times \partial\Omega$ (since y is a sub-solution of the heat equation), so (2.3) is always satisfied and (2.4) is the only condition required at the boundary, which plays a role whenever $y < 0$ at $(0,T) \times \partial\Omega$, a situation which may really happen (loss of classical boundary condition). We refer the reader to [3] for a true discussion and related references, while the basic tools in viscosity solutions theory can be found in [8].

It is proved in [3] that a comparison principle holds in the class of generalized viscosity solutions (so-called strong comparison result), and moreover there exists a unique generalized viscosity solution global in time. In the same paper, this result is also extended to the more general problem (2.1) under some technical structure conditions; moreover, the case $q \leqslant 2$ is also included, although in this situation viscosity solutions take (continuously) the boundary data and the uniqueness result falls in the usual theory.

Unfortunately, the above mentioned well-posedness result is proved in [3] in case of continuous data, whereas we need to consider controls v in (1.1) which are just bounded. As we will prove in Appendix A, the generalized viscosity solution of the uncontrolled system (with $v = 0$) will eventually become smooth and, in particular, a standard strong solution (note that strong solutions are still viscosity solutions). This justifies the definition of controllability which is given in the Introduction for the case $q > 2$. In other words, we use the well-posedness result for the uncontrolled system so that we may work in arbitrary time T. On the other hand, we let the control act when solutions have become sufficiently strong to use the Carleman estimates on the linearized problem. An alternative strategy could have been to extend the well-posedness result to L^p-viscosity solutions and to merely L^∞ data, possibly discontinuous. This seems reasonable but, of course, is far beyond the spirit of our work. It is quite natural in the control theory to use the regularizing properties of the uncontrolled system in order to reduce the problem to a smoother context.

3. Null controllability for large time

We prove here that any initial datum is controllable in a sufficiently large time. We start by proving that small (and smooth) initial data can be controlled to zero at time T, thanks to a fixed point argument relying on the controllability of linear equations. This kind of argument is by now classical for the controllability of semilinear problems, see e.g. [13,10].

Lemma 3.1. *Let $T > 0$ be any finite control time for problem* (1.1)*, and let $y_0 \in W^{2,p}(\Omega)$, with $p > N$, and such that $y_0 = 0$ on $\partial\Omega$. There exist positive constants c_0, c_1, M* (*independent of T*) *such that if*

$$\|y_0\|_{W^{2,p}(\Omega)} + \exp\left[c_0\left(T + \frac{1}{T}\right)\right]\|y_0\|_{L^2(\Omega)} \leqslant Me^{-c_1 T} \tag{3.1}$$

then y_0 is controllable at time T.

Remark 3.1. Note that the smoothness of the initial datum, which in particular belongs to $C_0^1(\overline{\Omega})$, and the smallness condition on y_0 in terms of T, will allow us here to obtain solutions of (1.1) in a strong sense, regardless whether $q \leqslant 2$ or $q > 2$.

Proof. Let us set

$$X = \left\{z \in C^0\left([0,T]; W^{1,\infty}(\Omega)\right):\ \|z\|_{C^0([0,T]; W^{1,\infty}(\Omega))} \leqslant 1\right\}.$$

Given $z \in X$, consider the linear null controllability problem:

$$\begin{cases} y_t - \Delta y + \nabla y \cdot \nabla z |\nabla z|^{q-2} = v\chi_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \\ y(T) = 0 & \text{in } \Omega \end{cases} \tag{$P_{z,v}$}$$

where v belongs to

$$V = \left\{v \in L^\infty\left((0,T)\times\omega\right):\ \|v\|_\infty \leqslant \exp\left[c_0\left(T + \frac{1}{T}\right)\right]\|y_0\|_{L^2(\Omega)}\right\},$$

with c_0 that will be defined below.

Using Theorem 3.1 in [10], there exists a constant c_0 such that, for every $z \in X$, problem $(P_{z,v})$ admits a solution for some $v \in V$. In other words, for every $z \in X$, the set

$$\mathcal{V}(z) := \left\{v \in V:\ \text{problem } (P_{z,v}) \text{ admits a solution } y\right\}$$

is not empty. Moreover, by linear estimates (see [15], and Lemma 2.1 in [10]) and using that $\|z(t)\|_{W^{1,\infty}(\Omega)} \leqslant 1$, we have that there exists a constant c_1 such that

$$\|y\|_{C^0([0,T]; W^{1,\infty}(\Omega))} \leqslant \exp\left[c_1(1+T)\right]\left(\|y_0\|_{W^{2,p}(\Omega)} + \|v\|_\infty\right)$$

and also, for some constant C_T depending on T,

$$\|y\|_{L^\infty((0,T); W^{2,p}(\Omega))} + \|y_t\|_{L^\infty((0,T); L^p(\Omega))} \leqslant C_T\left(\|y_0\|_{W^{2,p}(\Omega)} + \|v\|_\infty\right).$$

In particular, on account that $v \in V$, we have

$$\|y\|_{C^0([0,T]; W^{1,\infty}(\Omega))} \leqslant \exp\left[c_1(1+T)\right]\left(\|y_0\|_{W^{2,p}(\Omega)} + \exp\left[c_0\left(T + \frac{1}{T}\right)\right]\|y_0\|_{L^2(\Omega)}\right) \tag{3.2}$$

and also

$$\|y\|_{L^\infty((0,T);W^{2,p}(\Omega))} + \|y_t\|_{L^\infty((0,T);L^p(\Omega))} \leqslant C_T\left(\|y_0\|_{W^{2,p}(\Omega)} + \exp\left[c_0\left(T+\frac{1}{T}\right)\right]\|y_0\|_{L^2(\Omega)}\right). \tag{3.3}$$

Let us take $M = e^{-c_1}$. We define now the operator $\Phi : X \to 2^X$ so that

$$\Phi(z) = \{y \text{ sol. of } (P_{z,v}) \text{ for some } v \in \mathcal{V}(z)\}.$$

The assumption (3.1) on y_0 and (3.2) imply that, for every $z \in X$, $\Phi(z)$ is nonempty and is contained in X. We wish to apply Kakutani's fixed point theorem to the operator Φ. First of all, we recall that any $y \in \Phi(z)$ satisfies (3.3), hence if we set

$$K = \{y \in L^\infty((0,T); W^{2,p}(\Omega)),\ y_t \in L^\infty((0,T); L^p(\Omega)),\\ \|y\|_{L^\infty((0,T);W^{2,p}(\Omega))} + \|y_t\|_{L^\infty((0,T);L^p(\Omega))} \leqslant C_T M e^{-c_1 T}\}$$

we deduce by (3.1) that $\Phi(z) \subset K$ for every $z \in X$. Since $W^{2,p}(\Omega)$ is compactly embedded into $W^{1,\infty}(\Omega)$ when $p > N$, and obviously $W^{1,\infty}(\Omega) \subset L^p(\Omega)$, from well-known compactness results in parabolic spaces (see e.g. [22, Corollary 4]) we have that $K \cap X$ is a compact subset of X, and $\Phi(z) \subset K \cap X$ for all $z \in X$. It is easy to check that, for every $z \in X$, $\Phi(z)$ is a closed convex set. The convexity of $\Phi(z)$ follows from the convexity of V and the fact that a linear combination of solutions is a solution corresponding to the linear combinations of the corresponding controls. The fact that $\Phi(z)$ is closed is consequence of stability results; let $y_n \subset \Phi(z)$ be a sequence of solutions corresponding to controls $v_n \in \mathcal{V}(z)$ and converging to some y in X. There exists a subsequence (still denoted v_n) which converges to some $v \in L^\infty((0,T)\times\omega)$ in the weak-$*$ topology, and v also belongs to V. Since $y_n \in K$, extracting, if necessary, a further subsequence, we have that y_n will converge weakly in $L^p((0,T); W^{2,p}(\Omega))$ and strongly in X, so that its limit y will be a solution corresponding to the control v, hence $y \in \Phi(z)$. Therefore, we have proved that $\Phi(z)$ is a nonempty, closed and convex set contained in the compact set K for all $z \in X$. To conclude with Kakutani's theorem, we are only left to prove that, for every $\mu \in X'$, the mapping

$$z \in X \mapsto \sup_{y\in\Phi(z)} \langle \mu, y\rangle$$

is upper semi-continuous. To this purpose, let $z_n \to z$ in X; without loss of generality, assume that the sequence also satisfies (this is always true for at least a subsequence)

$$\lim_{n\to\infty} \sup_{y\in\Phi(z_n)} \langle\mu, y\rangle = \limsup_{n\to\infty} \sup_{y\in\Phi(z_n)} \langle\mu, y\rangle. \tag{3.4}$$

Let $y_n \in \Phi(z_n)$ be such that

$$\langle\mu, y_n\rangle \geqslant \sup_{y\in\Phi(z_n)} \langle\mu, y\rangle - \frac{1}{n}, \tag{3.5}$$

and let v_n be the corresponding associated controls. As before, we can extract subsequences, not relabeled, such that v_n converges in the weak-$*$ topology of L^∞ and y_n weakly converges in $L^p((0,T); W^{2,p}(\Omega))$ and strongly converges in X to some function $\tilde{y}$. Using that z_n strongly converges to z in X, it is possible to pass to the limit in the equation and we deduce that y is a solution of $(P_{z,v})$, hence $\tilde{y} \in \Phi(z)$. Since y_n converges in X, from (3.4)–(3.5) we deduce that

$$\sup_{y\in\Phi(z)} \langle\mu, y\rangle \geqslant \langle\mu, \tilde{y}\rangle \geqslant \limsup_{n\to\infty} \sup_{y\in\Phi(z_n)} \langle\mu, y\rangle$$

which proves the upper semi-continuity. Finally, we are in the position to apply Kakutani's fixed point theorem, hence Φ admits a fixed point, and we conclude. □

Remark 3.2. The same proof works for a nonlinearity of the type $f(t,x,\xi)$ which is C^1 in the ξ-variable and such that $f(t,x,0)=0$ and $\frac{\partial f}{\partial \xi}(t,x,\xi)$ is bounded in Q_T uniformly as $|\xi|$ is bounded. In this case we write $f(t,x,\nabla y)=\nabla y\cdot\int_0^1 \frac{\partial f}{\partial \xi}(t,x,s\nabla y)\,ds$ and we follow the above fixed point argument replacing in $(P_{z,v})$ the term $\nabla z|\nabla z|^{q-2}$ with $\int_0^1 \frac{\partial f}{\partial \xi}(t,x,s\nabla z)\,ds$. This latter term is uniformly bounded for $z\in X$, and the above proof can be applied with no differences. This shows that the controllability of $W^{2,p}(\Omega)$ small data holds in much generality, and moreover in the class of strong solutions.

We will now need the following result on the decay of solutions when the system (1.1) evolves without any control. Let us recall that, in case $q>2$, the initial datum is assumed to belong to $C_0(\overline{\Omega})$ in order to use the well-posedness result in [3] which ensures the existence of a unique global generalized viscosity solution. Such a solution, in general, need not be smooth and may even not assume the boundary data pointwise. Thus, in case $q>2$ next theorem contains at the same time both a regularity result (u becomes Lipschitz continuous and therefore a strong solution – actually smooth) and the decay estimate. In case $1\leqslant q\leqslant 2$, of course we can take $u_0\in L^\infty(\Omega)$; it is known that in this case $u(t)$ will belong to $C_0(\overline{\Omega})$, and is actually a smooth solution, as soon as $t>0$.

Lemma 3.2. *Let $q>1$. Assume that $u_0\in L^\infty(\Omega)$ if $1<q\leqslant 2$, while $u_0\in C_0(\overline{\Omega})$ if $q>2$. There exist positive constants K and λ, only depending on q and Ω, such that the solution u of*

$$\begin{cases} u_t-\Delta u+|\nabla u|^q=0 & \text{in } (0,T)\times\Omega,\\ u=0 & \text{on } (0,T)\times\partial\Omega,\\ u(0)=u_0 & \text{in } \Omega \end{cases} \tag{3.6}$$

satisfies, for every $t\geqslant K\|u_0\|_\infty$,

$$u(t)\in W^{1,\infty}(\Omega)\quad\text{and}\quad \|u(t)\|_\infty+\|\nabla u(t)\|_\infty\leqslant Ce^{-\lambda t} \tag{3.7}$$

where $C=C(q,\Omega)$.

Let us point out that, once $u(t)\in W^{1,\infty}(\Omega)$, then u becomes a classical solution of (3.6) even for $q>2$, which explains the importance of the gradient estimate above. The proof of Lemma 3.2 will be given in Appendix A. As far as the decay rate of the solution of (3.6) is concerned, a more precise result is proved in [6] assuming that either $1<q\leqslant 2$ or $q>2$ and $u_0\geqslant 0$. Under these assumptions, it is proved in [6] that there exists $\kappa=\kappa(u_0,\Omega,q)$ such that the solution of (3.6) satisfies

$$\|u(t)\|_\infty\leqslant \kappa e^{-\lambda_1 t},\qquad \|\nabla u(t)\|_\infty\leqslant \kappa\bigl(1+t^{-\frac12}\bigr)e^{-\lambda_1 t} \tag{3.8}$$

where λ_1 is the first eigenvalue of the Dirichlet problem for the Laplacian in Ω. However, in (3.8) there is no explicit estimate in terms of the initial datum u_0. For our next purposes, the statement of Lemma 3.2 is suitable to show that there is some time to wait, proportional to $\|u_0\|_\infty$, in order to have a uniform exponential decay. On the other hand, in Appendix A not only we give the proof of Lemma 3.2 but we also show how to get the sharp estimate (3.8) too; moreover, we will also prove the exponential decay for different nonlinearities. Let us now see what we can deduce in terms of controllability properties.

Theorem 3.1. *Let $q>1$. For every $y_0\in L^\infty(\Omega)$ if $1<q\leqslant 2$, respectively $y_0\in C_0(\overline{\Omega})$ if $q>2$, there exists $T(y_0)$ such that (1.1) is controllable at time $T(y_0)$.*

Proof. We know from Lemma 3.1 that there exist c_0, c_1, $M>0$ such that the system is controllable at time $T+\delta$ if

$$\|y(T)\|_{W^{2,p}(\Omega)}+\exp\left[c_0\left(\delta+\frac{1}{\delta}\right)\right]\|y(T)\|_{L^2(\Omega)}\leqslant Me^{-c_1\delta}.$$

We leave first the system evolve freely, taking $v=0$ up to $t=T$. Using Lemma 3.2 we have

$$\|y(T)\|_\infty+\|\nabla y(T)\|_\infty\leqslant Ce^{-\lambda T} \tag{3.9}$$

for any $T \geqslant K\|y_0\|_\infty$. Once a global bound is established on y and ∇y, we can apply the estimates for the second derivatives as well (see e.g. [15]); for example, for every $s \leqslant 1$ we have[1]

$$\big\|y(t+s)\big\|_{W^{2,p}} \leqslant C \sup_{\tau\in[t,t+s]}\left(\frac{\|y(\tau)\|_\infty}{s^2}+\frac{\|\nabla y(\tau)\|_\infty^q}{s}\right). \tag{3.10}$$

We deduce from (3.9) and (3.10) that $\|y(t)\|_{W^{2,p}(\Omega)}$ also decays exponentially, hence it is possible to find T such that

$$\big\|y(T)\big\|_{W^{2,p}(\Omega)}+\exp[2c_0]\big\|y(T)\big\|_{L^2(\Omega)} \leqslant Me^{-c_1}$$

and therefore the system is controllable at time $T+1$, taking $v=\tilde v\chi(T,T+1)$, where $\tilde v$ is provided by Lemma 3.1. □

Finally, in view of the previous result, we define

$$T(y_0) := \inf\big\{t>0\colon\ (1.1) \text{ is null controllable at time } t\big\}. \tag{3.11}$$

Thanks to Theorem 3.1, we know that $T(y_0)$ is finite. In the next section, we will see that $T(y_0)$ can actually be positive, justifying the terminology of waiting time. It will be interesting then to consider the quantity

$$T(r) := \sup\big\{T(y_0),\ \|y_0\|_\infty \leqslant r\big\}. \tag{3.12}$$

4. On the necessity of waiting time

In this section we prove that the system (1.1) cannot be null controllable in any time $T>0$, i.e. uniformly for all initial data. We are going to give two different proofs of this result. The first one points out the relation between the existence of stationary barrier functions and the failure of null controllability.

Theorem 4.1. *For any $r>0$, there exist $y_0 \in C_0(\overline{\Omega})$ such that $\|y_0\|_\infty = r$ and such that, regardless of the choice of the control v, any solution of* (1.1) *satisfies $y(t) \neq 0$ for every $t<r/c_0$, where $c_0 = c_0(q,\Omega,\omega)$.*

In particular, for every $q>1$ the system (1.1) *fails to be null controllable at any $T>0$.*

Remark 4.1. We recall (see Section 2) that, when $1<q\leqslant 2$, by a solution of (1.1) we refer to the unique bounded weak solution. When $q>2$, we refer to any possible solution of (1.1) such that, for some $t_0<T$, $v\equiv 0$ in $(0,t_0)$ and y is the unique generalized viscosity solution in $(0,t_0)$, $y(t_0)\in W^{2,p}(\Omega)\cap C_0(\overline{\Omega})$ and next y is a strong solution in (t_0,T).

---

[1] Estimate (3.10) can be deduced from [15] as follows. Applying [15, Chapter IV, Thm. 9.1] to the function $\tau y(x,t+\tau)$, for $\tau\in[0,s]$, one has

$$\big\|\tau y_\tau(t+\tau)\big\|_{L^p((0,s)\times\Omega)} \leqslant C\big[\|y\|_{L^\infty((t,t+s)\times\Omega)}+s\|\nabla y\|^q_{L^\infty((t,t+s)\times\Omega)}\big].$$

Next, using the equation of $w:=\tau^2 y_\tau(t+\tau)$, which satisfies

$$w_\tau - \Delta w + q|\nabla y|^{q-2}\nabla y\nabla w = 2\tau y_\tau, \quad \tau\in(0,s)$$

it follows that, for a sufficiently large p,

$$\|w\|_\infty \leqslant C\big\|\tau y_\tau(t+\tau)\big\|_{L^p((0,s)\times\Omega)},$$

where C can be chosen independent of y if, for example, we have $\|\nabla y\|_\infty \leqslant 1$ (this can be assumed due to (3.9)). Since $w(s)=s^2y_\tau(t+s)$ we deduce the pointwise estimate

$$\big\|y_\tau(t+s)\big\|_\infty \leqslant C \sup_{\tau\in[t,t+s]}\left(\frac{\|y(\tau)\|_\infty}{s^2}+\frac{\|\nabla y(\tau)\|_\infty^q}{s}\right)$$

and then by elliptic Calderon–Zygmund regularity (since $\Delta y(t+s) = |\nabla y(t+s)|^q + y_\tau(t+s)$) one obtains (3.10). This is not the sharpest possible estimate for the uncontrolled equation (1.1), but is largely sufficient to our purposes.

We stress that the proof below shows a local obstruction to null controllability which is completely independent from the control v and, in addition, applies to any notion of solution which satisfies a comparison principle, including the notions which we use of weak, viscosity or strong solutions. Roughly speaking, no matter one could define the null controllability at time T, there are initial data – which can be constructed as smooth as desired – for which it might fail.

Proof. Let us first deal with the case $1 < q \leqslant 2$. Take a smooth subset $\omega_0 \subset\subset \Omega$ such that $\omega_0 \cap \omega = \emptyset$. It is proved in [16, Section VI] that there exists a unique constant c_0 such that the problem

$$\begin{cases} -\Delta\varphi_0 + |\nabla\varphi_0|^q + c_0 = 0 & \text{in } \omega_0, \\ \lim\limits_{x\to\partial\omega_0} \varphi_0(x) = +\infty \end{cases} \tag{4.1}$$

admits a solution $\varphi_0 \in W^{2,p}_{loc}(\omega_0)$ for every $p < \infty$. Moreover, φ_0 is unique up to an additive constant; in the following, we fix φ_0 so that

$$\min_{\omega_0} \varphi_0 = 0.$$

In particular, we have that $\varphi_0 \geqslant 0$ in ω_0. Moreover, since φ_0 attains its minimum inside, the strong maximum principle (in this case, minimum!) implies that $c_0 > 0$. Consider now the function

$$y_0 = -r\chi_{\{x\in\omega_0:\ \varphi_0<r\}}. \tag{4.2}$$

Note that, since φ_0 is C^2 in ω_0, the set $\{x \in \omega_0:\ \varphi_0 < r\}$ is a C^2 open set compactly contained in ω_0. Observe that y_0 satisfies

$$y_0 \leqslant \varphi_0 - r \quad \text{in } \omega_0. \tag{4.3}$$

Since the function $z(t,x) = \varphi_0(x) - r + c_0 t$ satisfies

$$\begin{cases} z_t - \Delta z + |\nabla z|^q = 0 & \text{in } (0,T)\times\omega_0, \\ z \to +\infty & \text{on } (0,T)\times\partial\omega_0, \\ z(0) = \varphi_0 - r & \text{in } \omega_0 \end{cases}$$

we can use the comparison principle (see Proposition 2.1) in $\omega_0 \times (0,T)$ between y and z, and using (4.3), we deduce that $z(t,x) \geqslant y(t,x)$, i.e.

$$y(t,x) \leqslant \varphi_0(x) - r + c_0 t.$$

In particular, if x_0 is such that $\varphi_0(x_0) = \min_{\omega_0}\varphi_0 = 0$, we have $y(t,x_0) \leqslant c_0 t - r < 0$ as long as $t < r/c_0$. Similarly, if one takes an initial datum $\tilde{y}_0 \in C_0(\overline{\Omega})$ such that $-r \leqslant \tilde{y}_0 \leqslant y_0$, then the corresponding weak solution $\tilde{y}$ satisfies $\tilde{y} \leqslant y$ and the same conclusion holds for $\tilde{y}$. This shows that the same example can be constructed with smooth initial data.

Now, assume that $q > 2$. Still using the results in [16, Section VI], there exists a unique constant c_0 and a unique (up to addition of a constant) function $\varphi_0 \in C^2(\omega_0) \cap C^0(\overline{\omega}_0)$ which is solution of the equation

$$-\Delta\varphi_0 + |\nabla\varphi_0|^q + c_0 = 0 \quad \text{in } \omega_0, \tag{4.4}$$

and also satisfies the so-called state constraint boundary condition:

$$\text{for every } \psi \in C^2(\overline{\omega}_0), \quad \varphi_0 - \psi \text{ cannot have a local minimum at } x \in \partial\omega_0. \tag{4.5}$$

As before, we normalize φ_0 in a way that $\min_{\omega_0}\varphi_0 = 0$ and we define $z(t,x) = \varphi_0(x) - r + c_0 t$. Then we consider an initial datum $y_0 \in C_0(\overline{\Omega})$ such that $-r \leqslant y_0 \leqslant 0$ and $y_0 \equiv -r$ on ω_0. Let y be a generalized viscosity solution of (1.1) and, possibly, a strong solution in (t_0,T) for some $t_0 < T$. In particular, y belongs to $C([0,T]\times\overline{\omega}_0)$ and is a continuous viscosity solution in $(0,T)\times\omega_0$. Since z also belongs to $C([0,T]\times\overline{\omega}_0)$ and is a solution with state constraint boundary condition due to (4.5), in particular z is a super-solution of the generalized Dirichlet problem in $(0,T)\times\omega_0$ (actually, z is the maximal solution in $(0,T)\times\omega_0$). We can apply the comparison principle between generalized viscosity solutions [3], and, since in ω_0 we have $y_0 = -r \leqslant z(0,x)$, we deduce that $y \leqslant z$ in $(0,T)\times\omega_0$. Then we conclude as before.

Finally, if we are given any $T > 0$, we choose $r > c_0 T$; then the initial datum y_0 defined before cannot be driven to zero at time T. $\square$

Remark 4.2. It is interesting to give a closer look at the above construction in the case $q = 2$. Changing the unknown, setting $z = e^{-y}$, the system is transformed into the bilinear control problem

$$\begin{cases} z_t - \Delta z + zv\chi_\omega = 0 & \text{in } (0,T)\times\Omega, \\ z = 1 & \text{on } (0,T)\times\partial\Omega, \\ z(0) = e^{-y_0} & \text{in } \Omega \end{cases}$$

with the constraint that $z > 0$. The null controllability property for (1.1) means leading z to the constant state $z_1 \equiv 1$. The role of the barrier functions solutions of (4.1) is played here by eigenfunctions. The above argument becomes the following: take any $\omega_0 \subset \Omega \setminus \omega$ and φ_1 the first eigenfunction in ω_0 normalized with $\|\varphi_1\|_\infty = 1$. Take y_0 to be strictly negative in ω_0, e.g. let $\varepsilon > 0$ such that $e^{-y_0} > 1 + \varepsilon$ in ω_0. We have that $(1+\varepsilon)\varphi_1 e^{-\lambda_1 t}$ is a sub-solution in $\omega_0 \times (0,t)$, which vanishes on $\partial\omega_0$; we deduce by comparison that $z > (1+\varepsilon)\varphi_1 e^{-\lambda_1 t}$, hence $\{z(t) > 1\}$ has positive measure for $t < \frac{1}{\lambda_1}\log(1+\varepsilon)$.

Note in the above construction that ω_0 is an arbitrary subset of $\Omega \setminus \omega$. In the following, for any C^2 open set $\omega_0 \subset \Omega$, we denote by $c_0(\omega_0)$ the unique ergodic constant in ω_0 defined from (4.1) (if $1 < q \leqslant 2$) or (4.4)–(4.5) (if $q > 2$). We obtain then the following first estimate of the waiting time.

Corollary 4.1. *Let $T(r)$ be defined by* (3.12). *Then we have $T(r) \geqslant r/c_0(\omega_0)$ for every C^2 open set $\omega_0 \subset \Omega \setminus \omega$. In particular, if Ω and ω are of class C^2, we have*

$$T(r) \geqslant \frac{r}{c_0(\Omega \setminus \overline{\omega})}.$$

Proof. The first assertion follows from the construction of Theorem 4.1, on account of the fact that ω_0 was an arbitrary subset of $\Omega \setminus \omega$. If Ω and ω are C^2, then it is possible to define $c_0(\Omega \setminus \overline{\omega})$ by solving (4.1) in $\Omega \setminus \overline{\omega}$. Since the ergodic constant $c(\omega_0)$ is nonincreasing as the domain ω_0 increases (see e.g. [5,25]), we get

$$\frac{T(r)}{r} \geqslant \sup_{\omega_0 \subset \Omega \setminus \overline{\omega}} \frac{1}{c_0(\omega_0)} = \frac{1}{c_0(\Omega \setminus \overline{\omega})}. \qquad \square$$

We will see in the next section that the estimate provided above is far from optimal whenever r is small. On the other hand, for r large, we will show that the waiting time has actually the rate given by the above corollary.

Remark 4.3. It is known, by the dynamic programming principle, that the solution of (1.1) can be represented as the value function of a stochastic control problem. Precisely, if B_t is a standard Brownian motion in R^N (on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, P)$), let us denote by X_t the solution of the controlled stochastic differential equation

$$\begin{cases} dX_t = a_t\,dt + \sqrt{2}\,dB_t, \\ X_0 = x \in \Omega \end{cases}$$

where the control a_t belongs to a class $\mathcal{A}$ of admissible processes. Then the unique solution of (1.1) coincides, under some assumptions on $\mathcal{A}$, with the value function of the stochastic control problem, namely

$$y(t,x) = \inf_{a_t \in \mathcal{A}} \mathbb{E}_x \left[\int_0^{t\wedge\tau_x} \left(v(t-s, X_s)\chi_{[X_s \in \omega]} + c_q |a_s|^{\frac{q}{q-1}} \right) ds + y_0(X_t)\chi_{[t<\tau_x]} \right] \tag{4.6}$$

where $\mathbb{E}_x$ is the expectation conditioned to $X_0 = x$, c_q is a normalization constant and τ_x is the exit time variable

$$\tau_x = \inf\{t > 0:\ X_t \notin \Omega\}.$$

It is clear from the above representation that *the controllability property relies on the possibility that the process X_t reaches ω*. However the dynamics is influenced by the optimization procedure (which reflects the nonlinear character of (1.1)). It is proved in [16] that it is possible to constrain a Brownian motion inside some domain ω_0 using singular drifts (blowing up at $\partial\omega_0$), and the stationary barriers in ω_0 are the minimal cost to pay for realizing the constraint. Recall that y, by (4.6), is the minimal cost over all drifts (defined in the whole Ω). This suggests a rough explanation of the above result: namely, since there exist drifts which try to keep the process in $\Omega \setminus \omega$ as long as possible, the null controllability, if not avoided at all, will at least be delayed of some time.

We give now a different proof of Theorem 4.1 which does not use the results in [16] (in particular, it does not use the explicit stationary barriers) and, moreover, applies to the larger class of problems

$$\begin{cases} y_t - \Delta y + f(t,x,\nabla y) = v\chi_\omega & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{4.7}$$

We refer to Section 2 for the definition of weak solution of (4.7). Let us show that the controllability in arbitrary time fails depending on the behavior of $f(t,x,\xi)$ when $|\xi| \to \infty$. This kind of result is closer to the negative result proved in [13] for nonlinearities depending on y, relying on some underlying ODE's principle and the construction of test functions. However, in the case of nonlinearities depending on y, there are initial data which can never be controlled, while here the negative results mean that the control time cannot be uniform with respect to all data.

Theorem 4.2. *Assume that $f(t,x,\xi)$ satisfies*

$$\forall m > 0\ \exists c_m > 0: \quad \big|f(t,x,\xi)\big| \leqslant C_m \quad \forall (t,x) \in Q_T,\ \xi \in \mathbb{R}^N\colon |\xi| \leqslant m,$$

and moreover

$$\exists L > 0: \quad f(t,x,\xi) \geqslant h\big(|\xi|\big) \quad \forall (t,x) \in Q_T,\ \xi \in \mathbb{R}^N\colon |\xi| \geqslant L, \tag{4.8}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is some convex, increasing function such that

$$\int^{\infty} \frac{dt}{h(t)} < \infty, \tag{4.9}$$

and in addition

$$\limsup_{t\to\infty} \frac{t^2 h''(t)}{(th'(t) - h(t))} < \infty. \tag{4.10}$$

Then, for every $r > 0$ there exists $y_0 \in C_0(\overline{\Omega})$ with $\|y_0\|_\infty = r$ such that, for every control $v \in L^\infty((0,T)\times\omega)$, any weak solution y of (4.7) *satisfies $y(t) \neq 0$ for every $t < C_0 r$, where $C_0 = C_0(\Omega,\omega,f)$. In particular, the system* (4.7) *fails to be null controllable at any time $T > 0$.*

Remark 4.4. This result does not depend on the well-posedness of (4.7). Actually, the estimate is independent on what the control v might be and, especially, applies to *any* weak solution of (4.7) (and, a fortiori, to any stronger notion of solution). Moreover, the initial datum can also be chosen as smooth as desired.

Remark 4.5. Assumption (4.10) is only a technical condition that we will need in the construction of the test functions below, and we believe it unnecessary. On the other hand, condition (4.9) plays a key role; such a condition is usually required (and necessary) in order to have localized estimates in similar problems and existence of stationary barriers (see e.g. [18]). Of course, (4.8)–(4.10) are satisfied by the main example $f = |\nabla y|^q$. Note that (4.8) only concerns the behavior at infinity and only asks a bound from below, hence faster growths than powers are also included. Finally, the previous result includes the interesting case where $f(\nabla y) \sim |\nabla y|(\log(1+|\nabla y|))^\alpha$ as $|\nabla y| \to \infty$, with $\alpha > 1$, since the function $h(s) = s(\log(1+s))^\alpha$ satisfies (4.9)–(4.10) in this range. Notice that some gap with the positive results proved in [10] still remains, since it is known that the null controllability property holds for any T (independently of the initial datum) if $\alpha < 1/2$.

Proof. Take $\omega_0 \subset \Omega \setminus \omega$ and let $\eta \in C_c^\infty(\Omega \setminus \omega)$ be a cut-off function such that $\eta \equiv 1$ in ω_0. Multiply Eq. (4.7) by $\psi(\eta)$, where $\psi \in C^1(\mathbb{R})$ is to be fixed later (with $\psi(0) = 0$). We obtain

$$\int_\Omega y(t)\psi(\eta)\,dx + \int_0^t \int_\Omega f(\tau, x, \nabla y)\psi(\eta)\,dx\,d\tau = \int_\Omega y_0\psi(\eta)\,dx - \int_0^t \int_\Omega \nabla y \nabla\eta\psi'(\eta)\,dx\,d\tau,$$

which yields, using (4.8),

$$\int_\Omega y(t)\psi(\eta)\,dx + \int_0^t \int_\Omega h\big(|\nabla y|\big)\psi(\eta)\,dx\,d\tau \leqslant Kt\int_\Omega \psi(\eta)\,dx + \int_\Omega y_0\psi(\eta)\,dx\,d\tau - \int_0^t \int_\Omega \nabla y\nabla\eta\psi'(\eta)\,dx,$$

where $K = h(L) + c_L$, being $c_L = \sup_{Q_T \times \overline{B}_L(0)} |f|$. Young's inequality implies

$$\big|\nabla y\nabla\eta\psi'(\eta)\big| \leqslant \frac{1}{2}h\big(|\nabla y|\big)\psi(\eta) + \frac{1}{2}\psi(\eta)h^*\left(2|\nabla\eta|\frac{\psi'(\eta)}{\psi(\eta)}\right)$$

where h^* is the Legendre transform of the function h. Then we have

$$-\int_0^t \int_\Omega \nabla y\nabla\eta\psi'(\eta)\,dx\,d\tau \leqslant \frac{1}{2}\int_0^t \int_\Omega h\big(|\nabla y|\big)\psi(\eta)\,dx\,d\tau + \frac{1}{2}t\int_\Omega \psi(\eta)h^*\left(2|\nabla\eta|\frac{\psi'(\eta)}{\psi(\eta)}\right)dx.$$

Therefore we get

$$\int_\Omega y(t)\psi(\eta)\,dx \leqslant \int_\Omega y_0\psi(\eta)\,dx + t\left[K\int_\Omega \psi(\eta)\,dx + \frac{1}{2}\int_\Omega \psi(\eta)h^*\left(2|\nabla\eta|\frac{\psi'(\eta)}{\psi(\eta)}\right)dx\right],$$

hence we have obtained that

$$\int_\Omega y(t)\psi(\eta)\,dx \leqslant \int_\Omega y_0\psi(\eta)\,dx + tc_{\omega_0}$$

where

$$c_{\omega_0} = \left[K\int_\Omega \psi(\eta)\,dx + \frac{1}{2}\int_\Omega \psi(\eta)h^*\left(2|\nabla\eta|\frac{\psi'(\eta)}{\psi(\eta)}\right)dx\right]. \tag{4.11}$$

Assume that there exists a cut-off function $\psi(\eta)$ such that $c_{\omega_0} < \infty$. Then if we take some y_0 such that

$$\mathcal{I}(y_0) := \int_\Omega y_0\psi(\eta)\,dx < 0$$

we deduce that

$$\int_\Omega y(t)\psi(\eta)\,dx < 0 \quad \text{for } t < T_0 := \frac{|\mathcal{I}(y_0)|}{c_{\omega_0}}, \tag{4.12}$$

hence $y(t) \neq 0$ for $t < T_0$.

We are only left to show that, due to assumption (4.9), we can construct a ψ such that $c_{\omega_0} < \infty$. To this purpose, let us choose ψ as the solution of the ODE

$$\begin{cases} \psi' = \dfrac{1}{2\|\nabla\eta\|_\infty}\psi\big(h^*\big)^{-1}\left(\dfrac{1}{\psi}\right), \\ \psi(0) = 0. \end{cases}$$

We point out that ψ is well defined thanks to (4.9)–(4.10), and given by the implicit relation

$$\int_0^{\psi(r)} \frac{ds}{s(h^*)^{-1}(\frac{1}{s})} = \frac{r}{2\|\nabla\eta\|_\infty}.$$

Note that, by properties of the Legendre transform, we have

$$\int_0 \frac{ds}{s(h^*)^{-1}(\frac{1}{s})} = \int^\infty \frac{th''(t)}{(th'(t)-h(t))h'(t)}\,dt$$

hence, thanks to (4.10) and the convexity of h, we have

$$\int_0 \frac{ds}{s(h^*)^{-1}(\frac{1}{s})} \leqslant C\int^\infty \frac{1}{th'(t)}\,dt \leqslant C\int^\infty \frac{1}{h(t)}\,dt < \infty.$$

Therefore this choice of ψ is allowed, and we have

$$\int_\Omega \psi(\eta)h^*\left(2|\nabla\eta|\frac{\psi'(\eta)}{\psi(\eta)}\right)dx \leqslant |\Omega|$$

which implies that $c_{\omega_0} < \infty$.

Therefore, we proved that (4.12) holds true, and in particular the system is not controllable for $t < -\frac{1}{c_{\omega_0}}\int_\Omega y_0\psi(\eta)\,dx$. Choosing $y_0 \in C_0(\overline{\Omega})$: $-r \leqslant y_0 \leqslant -r\chi_{\omega_0}$, we have $\|y_0\|_\infty = r$, and

$$-\frac{1}{c_{\omega_0}}\int_\Omega y_0\psi(\eta)\,dx \geqslant \frac{r}{c_{\omega_0}}\psi(1)|\omega_0|.$$

We deduce that there exists a constant C_0 only depending on Ω, ω, h such that $T(r) \geqslant C_0 r$. $\square$

Remark 4.6. In the case $h(\xi) = |\xi|^q$, we can take $\psi(\eta) = |\eta|^{q'}$ where $q' = \frac{q}{q-1}$. The value of c_{ω_0} can be estimated explicitly in this case, being $c_{\omega_0} \leqslant C_q\int_\Omega |\nabla\eta|^{q'}\,dx$. In particular we obtain

$$T(y_0) \geqslant \frac{-\int_\Omega y_0\eta^{q'}\,dx}{C_q\int_\Omega |\nabla\eta|^{q'}\,dx}.$$

In the special case that $y_0 \leqslant 0$, we deduce

$$T(y_0) \geqslant \frac{\|y_0\|_{L^1(\omega_0)}}{C_q\int_\Omega |\nabla\eta|^{q'}\,dx}$$

and since this is true for any $\eta \geqslant \chi_{\omega_0}$ we obtain the explicit estimate

$$T(y_0) \geqslant \frac{\|y_0\|_{L^1(\omega_0)}}{C_q\,\mathrm{cap}_{q'}(\omega_0, \Omega\setminus\omega)}$$

where $\mathrm{cap}_{q'}(B,E)$ denotes the q'-capacity of a Borel set B with respect to a set E such that $B \subset E$. We recall that, for $r > 1$, the r-capacity of an open set A with respect to $E \supset A$ is defined as

$$\mathrm{cap}_r(A,E) = \inf\left\{\|\psi\|^r_{W_0^{1,r}(E)},\ \psi \in W_0^{1,r}(E):\ \psi \geqslant \chi_A \text{ a.e. in } E\right\}$$

and then such definition is extended to Borel subsets as

$$\mathrm{cap}_r(B,E) = \inf\left\{\mathrm{cap}_r(A),\ A \text{ open}:\ B \subset A \subset E\right\}.$$

Theorem 4.2 applies to weak solutions of (4.7). The following corollary is meant for the case when the growth of $f(t,x,\nabla u)$ with respect to ∇u does not allow to find global weak solutions but, on the contrary, there exists a global generalized viscosity solution (like for the case $q > 2$ in (1.1)). Sufficient conditions for this are given in [3] and are quite technical. We will just show, for completeness, that even if such a solution exists, it is not controllable. This also includes the case where a strong solution exists.

Corollary 4.2. *Assume that $f(t,x,\xi) \in C(\overline{Q}_T \times \mathbb{R}^N)$ satisfies $f(t,x,0) = 0$ and is locally Lipschitz continuous with respect to ξ (uniformly in Q_T), and moreover*

$$\exists L > 0: \quad f(t,x,\xi) \geqslant \gamma|\xi|^2 \quad \forall (t,x) \in Q_T,\ \xi \in \mathbb{R}^N:\ |\xi| \geqslant L. \tag{4.13}$$

Let y be a solution of (4.7) in the sense that, for some $t_0 < T$, $v \equiv 0$ in $(0,t_0)$ and y is a generalized viscosity solution in $(0,t_0)$, $y(t_0) \in C_0(\overline{\Omega}) \cap W^{2,p}(\Omega)$ for every $p < \infty$ and y is a strong solution in (t_0,T) for some control v.

For every $r > 0$, there exists $y_0 \in C_0(\overline{\Omega})$ such that $\|y_0\|_\infty = r$ and, if such a solution y exists, then $y(t) \neq 0$ for every $t < C_0 r$, where $C_0 = C_0(\Omega,\omega,f)$.

Proof. By the local Lipschitz continuity and (4.13), we have that f satisfies, for some constant $\beta > 0$,

$$f(t,x,\xi) \geqslant \gamma|\xi|^2 - \beta|\xi| \quad \forall (t,x,\xi) \in Q_T \times \mathbb{R}^N.$$

Consider the problem

$$\begin{cases} z_t - \Delta z + \gamma|\nabla z|^2 - \beta|\nabla z| = v\chi_\omega & \text{in } (0,T) \times \Omega, \\ z = 0 & \text{on } (0,T) \times \partial\Omega, \\ z(0) = y_0 & \text{in } \Omega. \end{cases} \tag{4.14}$$

There exists a unique global weak solution z of (4.14), and we can apply Theorem 4.2 to z. Then, for every $r > 0$ there exists $y_0 \in C_0(\overline{\Omega})$ such that $\|y_0\|_\infty = r$ and

$$\int_\Omega z(t)\psi(\eta)\,dx < 0 \quad \forall t < C_0 r, \tag{4.15}$$

with the notations used in the proof of Theorem 4.2. Let now y be a solution of (4.7) as in the statement. If $v = 0$ in $(0,t_0)$ and y is a generalized viscosity solution, then it is also a viscosity sub-solution of (4.14). Since $y_0 \in C_0(\overline{\Omega})$, we also have that the weak solution z is continuous up to $t = 0$ and is a classical solution in $(0,t_0)$. By the comparison principle between generalized viscosity sub/super-solutions, we deduce that $y \leqslant z$ in $(0,t_0)$. Next, the comparison also holds in (t_0,T) where y is a strong solution, and we conclude that $y \leqslant z$ in $(0,T)$. Therefore, (4.15) implies

$$\int_\Omega y(t)\psi(\eta)\,dx < 0 \quad \forall t < C_0 r,$$

which yields the conclusion. □

Remark 4.7. If we define the waiting time $T(r)$ as in (3.12) but with respect to problem (4.7) instead of (1.1), we can deduce from Theorem 4.2 and Corollary 4.2 the estimate $T(r) \geqslant C_0 r$ under either assumption (4.8)–(4.10) or (4.13).

Remark 4.8. Both Theorems 4.1 and 4.2 show that the waiting time is positive as soon as we choose initial data y_0 such that $y_0 < 0$ in some open set $\omega_0 \subset \Omega \setminus \omega$. It is an open question whether the null controllability property in arbitrary time T holds for nonnegative initial data.

5. Sharp estimates on the waiting time

In this section we estimate the rate of the waiting time in terms of the L^∞-norm of the initial data, in the two different cases whether $\|y_0\|_\infty$ tends to zero or infinity. Recall that $T(y_0)$ and $T(r)$ are defined in (3.11)–(3.12).

We start by improving, for short time range, the lower bound on $T(r)$ given in Corollary 4.1. The following construction is inspired by [12], where the authors prove a similar lower bound for the one-dimensional Burgers equation.

Proposition 5.1. *Let $T(r)$ be defined from* (3.11)–(3.12). *There exists a constant $\kappa > 0$ such that*

$$T(r) \geqslant \frac{\kappa}{\ln(\frac{1}{r})} \quad \textit{as } r \to 0^+.$$

Proof. Let ω_0 be a smooth open subset of $\Omega \setminus \overline{\omega}$, and let $\delta_0 > 0$ be such that the function $\operatorname{dist}(x, \partial\omega_0)$ is C^2 in $\{x \in \omega_0\colon \operatorname{dist}(x, \partial\omega_0) < \delta_0\}$. Take some $\delta < \frac{\delta_0}{2}$, to be fixed later small enough. We denote by $d(x)$ a function which is C^2 in ω_0, which coincides with $\operatorname{dist}(x, \partial\omega_0)$ in $\{x \in \omega_0\colon \operatorname{dist}(x, \partial\omega_0) < \delta\}$ and with a positive constant in $\{x \in \omega_0\colon d(x) > 2\delta\}$ and such that $d(x) \leqslant 1$, $|\nabla d(x)| \leqslant 1$ in ω_0. Such a choice is easily obtained through a smooth truncation of the distance function.

Let us consider the function

$$v(t, x) := e^{\frac{1}{d(x)}} e^{-\frac{\zeta(x)}{t}}, \quad (t, x) \in (0, T) \times \omega_0$$

where $\zeta \in C^2(\omega_0)$ will be fixed later. Computing we have

$$v_t - \Delta v = v\left[\frac{\zeta}{t^2} - \left(\frac{\nabla d}{d^2} + \frac{\nabla\zeta}{t}\right)^2 - \left(\frac{2|\nabla d|^2}{d^3} - \frac{\Delta d}{d^2} - \frac{\Delta\zeta}{t}\right)\right]$$

which implies

$$v_t - \Delta v \geqslant v\left[\frac{\zeta}{t^2} - 2\frac{|\nabla\zeta|^2}{t^2} + \frac{\Delta\zeta}{t}\right] - v\left[2\frac{|\nabla d|^2}{d^4} + \frac{2|\nabla d|^2}{d^3} - \frac{\Delta d}{d^2}\right]$$

and using that $|\nabla d| \leqslant 1$ and $|\Delta d|$ is bounded, and since $d(x) \leqslant 1$, we get

$$v_t - \Delta v \geqslant v\left[\frac{\zeta}{t^2} - 2\frac{|\nabla\zeta|^2}{t^2} + \frac{\Delta\zeta}{t} - \frac{C_0}{d^4}\right]$$

for some constant $C_0 > 0$. We fix now $\zeta = \Phi(d(x))$, for some smooth increasing function Φ such that $\Phi(0) = 0$ and in a way that ζ satisfies

$$|\nabla\zeta|^2 \leqslant \frac{\zeta}{4}, \qquad |\Delta\zeta| \leqslant \frac{\sqrt{\zeta}}{d^2}, \qquad \zeta = o\big(d^2(x)\big) \quad \text{as } d(x) \to 0. \tag{5.1}$$

To this purpose it is enough to ask that Φ satisfies $\Phi(s) = o(s^2)$ as $s \to 0^+$, $|\Phi'(s)|^2 \leqslant \frac{\Phi}{4}$ and $\Phi'' = o(\frac{\sqrt{\Phi}}{s^2})$, which is the case taking for instance $\Phi(s) \simeq s^3$. Thanks to (5.1) we have

$$2\frac{|\nabla\zeta|^2}{t^2} + \frac{|\Delta\zeta|}{t} \leqslant \frac{3}{4}\frac{\zeta}{t^2} + \frac{1}{d^4}$$

and we estimate

$$v_t - \Delta v \geqslant v\left[\frac{1}{4}\frac{\zeta}{t^2} - \frac{C_1}{d^4}\right].$$

On the other hand, since $\nabla\zeta \cdot \nabla d \geqslant 0$, we have

$$|\nabla v|^2 = v^2\left(\frac{\nabla d}{d^2} + \frac{\nabla\zeta}{t}\right)^2 \geqslant v^2\frac{|\nabla d|^2}{d^4} \geqslant \frac{v^2}{d^4}\chi_{\{d(x)<\delta\}}$$

hence

$$v_t - \Delta v + |\nabla v|^q \geqslant v\left[\frac{1}{4}\frac{\zeta}{t^2} - \frac{C_1}{d^4}\right] + \frac{v^q}{d^{2q}}\chi_{\{d(x)<\delta\}}.$$

We claim now that the right-hand side is nonnegative. This is of course true in the subset $\{(x,t)\colon \frac{C_1}{d^4} \leqslant \frac{1}{4}\frac{\zeta}{t^2}\}$. On the other hand, in the complement of this set we have $\frac{C_1}{d^4} > \frac{1}{4}\frac{\zeta}{t^2}$, which means that $d^4\zeta < 4C_1t^2$ hence

$$\frac{\zeta}{t} \leqslant 2\sqrt{C_1}\frac{\sqrt{\zeta}}{d(x)^2}.$$

Since $\zeta = \Phi(d) = o(d^2)$, we can choose δ small such that

$$\frac{\zeta}{t} \leqslant \frac{1}{2d(x)} \quad \forall x\colon d(x) < \delta.$$

This implies that $v \geqslant e^{\frac{1}{2d}}$ and so

$$v\frac{C_1}{d^4} \leqslant \frac{v^q}{d^{2q}} \quad \forall x\colon d(x) < \delta$$

provided δ is small enough. Assume now that $t \leqslant \frac{K}{\ln(\frac{1}{r})}$. Then, whenever $d^4\zeta < 4C_1t^2$ we have $d^4\Phi(d) < 4C_1K^2(\ln(\frac{1}{r}))^{-2}$ and we can choose r small enough so that this implies $d(x) < \delta$. In that way we conclude that

$$\text{if } \frac{C_1}{d^4} > \frac{1}{4}\frac{\zeta}{t^2}, \quad \text{then } v\frac{C_1}{d^4} \leqslant \frac{v^q}{d^{2q}}\chi_{\{d(x)<\delta\}}$$

if δ is fixed sufficiently small and r is small enough (depending on δ). All in all we have proved that

$$v\left[\frac{1}{4}\frac{\zeta}{t^2} - \frac{C_1}{d^4}\right] + \frac{v^q}{d^{2q}}\chi_{\{d(x)<\delta\}} \geqslant 0$$

so that v satisfies $v_t - \Delta v + |\nabla v|^q \geqslant 0$. Take now $y_0 \in C_0(\overline{\Omega}) \cap C^2(\overline{\Omega})$ such that $-r \leqslant y_0 \leqslant 0$ and $y_0 = -r$ in ω_0, and let y be the corresponding solution of (1.1). We use the comparison principle[2] in $\omega_0 \times (0,T)$ and we deduce

$$y(t,x) \leqslant v(t,x) - r, \quad (t,x) \in (0,T) \times \omega_0$$

and in particular

$$y \leqslant e^{\frac{1}{d(x)} - \frac{\zeta}{t}} - r \leqslant e^{\frac{1}{\delta} - \frac{\Phi(\delta)}{t}} - r \quad \forall x\colon \operatorname{dist}(x, \partial\omega_0) \geqslant \delta.$$

If $T = \frac{1}{2}\frac{\Phi(\delta)}{\ln(\frac{1}{r})}$ then we have

$$e^{\frac{1}{\delta} - \frac{\Phi(\delta)}{t}} - r \leqslant e^{\frac{1}{\delta}}r^2 - r < 0$$

if r is sufficiently small, hence we have $y(t,x) < 0$ for every $x\colon d(x) \geqslant \delta$ and for every $t \leqslant \frac{1}{2}\frac{\Phi(\delta)}{\ln(\frac{1}{r})}$, which means that $T(y_0) \geqslant \frac{1}{2}\frac{\Phi(\delta)}{\ln(\frac{1}{r})}$. Since $\|y_0\|_\infty = r$, this gives the conclusion. $\square$

In order to show the generality of the above estimate, as well as of the construction used to prove it, we extend the above result to general nonlinearities with similar growth as in Theorem 4.2.

Theorem 5.1. *Let us consider problem* (4.7). *Assume that* $f(t,x,\nabla y) \geqslant h(|\nabla y|)$ *for some function* $h(s) \in C^1([0,\infty))$ *such that* $h(0) = 0$ *and, for some* $L > 0$, $h(s)$ *is positive, increasing in* (L,∞) *and satisfies* $\int_L^\infty \frac{ds}{h(s)}\,ds < \infty$. *Moreover, assume that there exists a positive function* $\rho \in C^2(0,s_0)$ *such that* $\log\rho$ *is convex,* $\rho(s) \to \infty$ *as* $s \to 0^+$, *and* $4\rho'' \leqslant h(\frac{1}{2}|\rho'|)$ *in* $(0,s_0)$ *for some* $s_0 > 0$.

Then the conclusion of Proposition 5.1 *holds true* (*replacing problem* (1.1) *with* (4.7)).

[2] More precisely, we can use the comparison principle with $y_0 = -r - \varepsilon$ in ω_0 (so that $y < v - r$ in a ε-neighborhood of $t = 0$), and then conclude as $\varepsilon \to 0$. This is to get rid of the singularity of $v(t,x)$ at $\partial\omega_0 \cap \{t = 0\}$. We also recall that, in case $q > 2$, we use the comparison principle between viscosity solutions (only the standard form is needed here, since comparison at the boundary holds trivially). In case $q \leqslant 2$, since y_0 is smooth, the solution y is classical, and we are just comparing classical solutions which are ordered at the boundary (the usual maximum principle suffices).

Remark 5.1. Taking $\rho(s) = e^{\frac{1}{s}}$ we see that $f(t,x,\xi) = |\xi|^q$ satisfies the above assumption for every $q > 1$, which makes Proposition 5.1 a particular case of the present result. The assumption is also satisfied if $h(s) \sim s(\log s)^\alpha$ as $s \to \infty$, with $\alpha > 1$. In that case, the function $\rho(s) = e^{\lambda s^{-\frac{1}{\alpha-1}}}$ satisfies the required differential inequality for a suitable λ. We also observe that the assumption $\int^\infty \frac{ds}{h(s)}\,ds < \infty$ is a necessary condition for the existence of some function ρ with the desired properties.

Proof. We proceed as in the previous proof. Let $\omega_0 \subset \Omega \setminus \overline{\omega}$ and $d(x)$ be defined as in Proposition 5.1, and such that $d(x) < s_0$. We set $\varphi(x) = \rho(d(x))$, where ρ is given by the assumption, and $\zeta(x) = \Phi(d(x))$, where Φ is an increasing smooth function, to be suitably chosen later, such that $\Phi(0) = 0$. We consider the function

$$w(t,x) = \varphi(x) e^{-\frac{\zeta(x)}{t}}, \quad (t,x) \in (0,T) \times \omega_0.$$

Now w satisfies

$$w_t - \Delta w = e^{-\frac{\zeta}{t}} \left[\varphi \frac{\zeta}{t^2} - \Delta\varphi + \frac{2}{t} \nabla\varphi \nabla\zeta + \varphi \frac{\Delta\zeta}{t} - \varphi \frac{|\nabla\zeta|^2}{t^2} \right]$$

which yields

$$w_t - \Delta w \geqslant e^{-\frac{\zeta}{t}} \left[\varphi \frac{\zeta}{t^2} - \Delta\varphi - \frac{|\nabla\varphi|^2}{\varphi} + \varphi \frac{\Delta\zeta}{t} - 2\varphi \frac{|\nabla\zeta|^2}{t^2} \right].$$

The assumptions on h imply that there exists $L_0 > 0$ such that $h(s) + L_0 s$ is increasing. On the other hand, since Φ is increasing we have $\nabla\zeta \nabla d(x) \geqslant 0$, while ρ is decreasing and $\nabla\varphi = \rho'(d)\nabla d(x)$. Therefore, $|\nabla w| \geqslant e^{-\frac{\zeta}{t}} |\nabla\varphi|$, and we have

$$h\bigl(|\nabla w|\bigr) \geqslant -L_0|\nabla w| + h\bigl(e^{-\frac{\zeta}{t}}|\nabla\varphi|\bigr) + L_0 e^{-\frac{\zeta}{t}}|\nabla\varphi|,$$

and since $|\nabla w| \leqslant e^{-\frac{\zeta}{t}}[|\nabla\varphi| + \frac{\varphi}{t}|\nabla\zeta|]$ we get

$$\begin{aligned} w_t - \Delta w + h\bigl(|\nabla w|\bigr) \geqslant{}& e^{-\frac{\zeta}{t}} \left[\varphi \frac{\zeta}{t^2} - \Delta\varphi - \frac{|\nabla\varphi|^2}{\varphi} + \varphi \frac{\Delta\zeta}{t} - 2\varphi \frac{|\nabla\zeta|^2}{t^2} - L_0 \left(|\nabla\varphi| + \frac{\varphi}{t}|\nabla\zeta| \right) \right] \\ &+ h\bigl(e^{-\frac{\zeta}{t}}|\nabla\varphi|\bigr) + L_0 e^{-\frac{\zeta}{t}}|\nabla\varphi|. \end{aligned} \tag{5.2}$$

Assume that $\zeta = \Phi(d(x))$ satisfies

$$|\nabla\zeta|^2 \leqslant \frac{\zeta}{4}, \qquad |\Delta\zeta|^2 \leqslant \frac{1}{2}\zeta \frac{\rho''(d)}{\rho(d)}, \qquad \zeta = o\left(\frac{\rho(d)}{\rho''(d)} \right) \quad \text{as } d \to 0. \tag{5.3}$$

Then, recalling that $\varphi = \rho(d)$, we have

$$\begin{aligned} \varphi \frac{\Delta\zeta}{t} - 2\varphi \frac{|\nabla\zeta|^2}{t^2} - L_0 \frac{\varphi}{t}|\nabla\zeta| &\geqslant -\frac{3}{4}\varphi \frac{\zeta}{t^2} - 2\varphi \frac{|\Delta\zeta|^2}{\zeta} - \frac{L_0^2}{2}\varphi \\ &\geqslant -\frac{3}{4}\varphi \frac{\zeta}{t^2} - \rho''(d) - \frac{L_0^2}{2}\rho(d). \end{aligned}$$

Moreover, since $\log\rho$ is convex, we have $\rho\rho'' \geqslant (\rho')^2$, and in particular (since $|(\log\rho)'| = \frac{|\rho'|}{\rho} \to \infty$) we have $|\rho'| = o(\rho'')$, so that

$$\Delta\varphi + \frac{|\nabla\varphi|^2}{\varphi} = \left(\rho'' + \frac{(\rho')^2}{\rho} \right)|\nabla d|^2 + \rho'\Delta d \leqslant 2\rho'' + \rho'\Delta d = 2\rho'' + o\bigl(\rho''\bigr),$$

and as a whole we get

$$\begin{aligned} &\varphi \frac{\zeta}{t^2} - \Delta\varphi - \frac{|\nabla\varphi|^2}{\varphi} + \varphi \frac{\Delta\zeta}{t} - 2\varphi \frac{|\nabla\zeta|^2}{t^2} - L_0 \left(|\nabla\varphi| + \frac{\varphi}{t}|\nabla\zeta| \right) \\ &\quad \geqslant \frac{1}{4}\varphi \frac{\zeta}{t^2} - 3\rho''(d) - \frac{L_0^2}{2}\rho(d) - L_0\bigl|\rho'(d)\bigr| + o\bigl(\rho''(d)\bigr). \end{aligned}$$

Since $\rho = o(\rho''(d))$ as well as $|\rho'| = o(\rho'')$ as $d \to 0$, up to choosing s_0 eventually smaller we get from (5.2)

$$w_t - \Delta w + h\big(|\nabla w|\big) \geqslant e^{-\frac{\zeta}{t}}\left[\frac{1}{4}\varphi\frac{\zeta}{t^2} - 4\rho''(d)\right] + h\big(e^{-\frac{\zeta}{t}}|\nabla\varphi|\big) + L_0 e^{-\frac{\zeta}{t}}|\nabla\varphi|$$

and then, using that $|\nabla\varphi| = |\rho'(d)|$ if $d(x) < \delta$,

$$w_t - \Delta w + h\big(|\nabla w|\big) \geqslant e^{-\frac{\zeta}{t}}\left[\frac{1}{4}\varphi\frac{\zeta}{t^2} - 4\rho''(d)\right] + \big\{h\big(e^{-\frac{\zeta}{t}}\big|\rho'(d)\big|\big) + L_0 e^{-\frac{\zeta}{t}}\big|\rho'(d)\big|\big\}\chi_{\{d(x)<\delta\}}.$$

Now, if $4\rho''(d) \leqslant \frac{1}{4}\varphi\frac{\zeta}{t^2}$, the right-hand side is positive. Otherwise, if $4\rho''(d) > \frac{1}{4}\varphi\frac{\zeta}{t^2}$, this means that

$$\frac{\rho(d)}{\rho''(d)}\Phi(d) < 16t^2. \tag{5.4}$$

On one hand this implies that $\frac{\zeta}{t} \leqslant 4\sqrt{\zeta}(\frac{\rho''}{\rho})^{\frac{1}{2}}$; since by (5.3) $\zeta = o(\frac{\rho}{\rho''})$ as $d(x) \to 0$, we can find some δ in order to have

$$\frac{\zeta}{t} \leqslant \frac{1}{2} \leqslant \ln 2 \quad \forall x\colon d(x) < \delta$$

hence $e^{-\frac{\zeta}{t}} \geqslant \frac{1}{2}$. On the other hand, since $t \leqslant K(\ln(1/r))^{-1}$, we can choose r small enough in a way that (5.4) implies $d(x) < \delta$. Thus, we can deduce that, whenever $4\rho''(d) > \frac{1}{4}\varphi\frac{\zeta}{t^2}$, we have (using also the assumption on ρ)

$$e^{-\frac{\zeta}{t}}4\rho''(d) \leqslant 4\rho''(d) \leqslant h\left(\frac{1}{2}\big|\rho'(d)\big|\right) \leqslant h\big(e^{-\frac{\zeta}{t}}\big|\rho'(d)\big|\big)\chi_{\{d(x)<\delta\}}.$$

Finally, we conclude that $w_t - \Delta w + h(|\nabla w|) \geqslant 0$, provided we choose $\zeta = \Phi(d(x))$ satisfying (5.3). The existence of such a Φ can be proved as follows. Firstly, recall that $\frac{\rho''}{\rho} \to \infty$ (because $\frac{\rho''}{\rho} \geqslant (\frac{\rho'}{\rho})^2 \to \infty$); now take some positive increasing smooth function $g(s)$ such that $g(0) = 0$ and $g = o(\frac{\rho}{\rho''})$, and let $\Phi(s) = e^{-\frac{1}{g(s)}}$. Clearly we have $\Phi = o(g)$ hence $\zeta = o(\frac{\rho}{\rho''})$. Moreover we have $|\nabla\zeta|^2 \leqslant \zeta^2(\frac{g'}{g^2}) = o(\zeta)$ and similarly $(\Delta\zeta)^2 = o(\zeta)$ which clearly implies $(\Delta\zeta)^2 = o(\zeta\frac{\rho''(d)}{\rho(d)})$. Therefore ζ satisfies (5.3). We have therefore constructed a positive function w which blows up at $\partial\omega_0$ and is a super-solution of the equation $w_t - \Delta w + h(|\nabla w|) = 0$. Using the comparison principle on this latter equation, we conclude as in Proposition 5.1. □

We conclude now the estimates of the waiting time by proving the upper bounds.

Proposition 5.2. *Let us consider problem* (1.1) *with* $q > 1$, *and let* $T(r)$ *be defined from* (3.11)–(3.12). *We have*:

(i) *There exists* $K > 0$ *such that* $T(r) \leqslant \frac{K}{\ln(\frac{1}{r})}$ *as* $r \to 0^+$.
(ii) *There exists* $\Lambda > 0$ *such that* $T(r) \leqslant \Lambda r$ *as* $r \to \infty$.

Proof. Consider first the case $r \to 0$. We leave first the system evolve freely. Since $\|u_0\|_\infty \leqslant r$, it is proved (see Appendix A, (A.2)) that

$$\big|\nabla u(t)\big| \leqslant c\left(\frac{\|u_0\|_\infty}{t - kr}\right)^{\frac{1}{q}} \tag{5.5}$$

for every $t > kr$, where c, k only depend on q, Ω. In particular, $\|\nabla u(t)\|_\infty \leqslant c$ for every $t \geqslant 2kr$, and then u satisfies a heat equation where the right-hand side is bounded. Therefore classical estimates apply, and in particular (3.10) holds true, namely we have, for small $s > 0$,

$$\big\|u(t+s)\big\|_{W^{2,p}} \leqslant C \sup_{\tau\in[t,t+s]}\left(\frac{\|u(\tau)\|_\infty}{s^2} + \frac{\|\nabla u(\tau)\|_\infty^q}{s}\right). \tag{5.6}$$

Since $\|u\|_\infty \leqslant r$ and, from (5.5), $\|\nabla u(\tau)\|_\infty^q \leqslant c\frac{\|u_0\|_\infty}{t-kr}$ for every $\tau \geqslant t$, if we take some $\alpha < 1$ and $s = t = \frac{1}{2}r^\alpha$ we get

$$\left\|u\left(r^\alpha\right)\right\|_{W^{2,p}} \leqslant Cr^{1-2\alpha}$$

provided r is small. In particular, there exists $\alpha < 1$ such that $\|u(r^\alpha)\|_{W^{2,p}} \leqslant \frac{M}{2}e^{-c_1}$, where M, c_1 are given by Lemma 3.1. Therefore, if $T = \frac{K}{\ln(\frac{1}{r})}$ we have

$$\begin{aligned}\left\|u\left(r^\alpha\right)\right\|_{W^{2,p}} + \exp\left[c_0\left(T + \frac{1}{T}\right)\right]\left\|u\left(r^\alpha\right)\right\|_{L^2(\Omega)} &\leqslant \frac{M}{2}e^{-c_1} + \exp\left[c_0\left(\frac{K}{\ln(\frac{1}{r})} + \frac{1}{K}\ln\left(\frac{1}{r}\right)\right)\right]|\Omega|^{\frac{1}{2}}r \\ &\leqslant \frac{M}{2}e^{-c_1} + |\Omega|^{\frac{1}{2}}\exp\left[c_0\frac{K}{\ln(\frac{1}{r})}\right]r^{1-\frac{c_0}{K}}.\end{aligned}$$

Choosing $K > c_0$ we obtain that, for small r, we have

$$\left\|u\left(r^\alpha\right)\right\|_{W^{2,p}} + \exp\left[c_0\left(T + \frac{1}{T}\right)\right]\left\|u\left(r^\alpha\right)\right\|_{L^2(\Omega)} \leqslant Me^{-c_1} \leqslant Me^{-c_1 T}$$

hence the condition of Lemma 3.1 is satisfied and the system is controllable after a time $r^\alpha + T$. Since $r^\alpha + T \sim \frac{K}{\ln(\frac{1}{r})}$ as $r \to 0^+$, this proves (i).

Let us now consider the case that $\|u_0\|_\infty \leqslant r$, with $r \to \infty$. Again, we first take $v = 0$ and consider the system without control. Then, using Lemma 3.2 with $t = 2Kr$ we have

$$\left\|u(2Kr)\right\|_\infty \leqslant Ce^{-2\mu Kr}$$

for some $\mu > 0$. Next, in view of the above result on the time of control for small data, we deduce that the system is controllable after a time of the order of $\frac{1}{\ln(\frac{1}{Ce^{-2\mu Kr}})} \sim \frac{c}{r}$, hence all in all the system is controllable after a time $T \sim 2Kr + \frac{c}{r} = O(r)$. This proves (ii). □

We deduce now from the previous results

Proof of Theorem 1.1. From Corollary 4.1, it follows the lower bound in (1.4). The lower bound in (1.3) is proved in Proposition 5.1. Finally, Proposition 5.2 provides with the two bounds from above. □

Remark 5.2. It is interesting to consider the case $q = 2$ for which the decay estimate can be deduced from the linear one. Indeed, u is a solution of (3.6) with $q = 2$ if and only if $v = e^{-u} - 1$ is a solution of the heat equation with initial datum $e^{-u_0} - 1$. Consider the case $u_0 \leqslant 0$, which is the interesting one for the waiting time. Then we have, by linear theory,

$$0 \leqslant v(t) \leqslant Ce^{-\lambda_1 t}\left\|v(0)\right\|_\infty \leqslant Ce^{\|u_0\|_\infty - \lambda_1 t}$$

where C only depends on Ω. Hence

$$\left|u(t)\right| = \ln\left(1 + v(t)\right) \leqslant v(t) \leqslant Ce^{\|u_0\|_\infty - \lambda_1 t}$$

and in particular, if $T \geqslant \frac{\|u_0\|_\infty}{\lambda_1 - \varepsilon}$, we have

$$\left|u(T)\right| \leqslant Ce^{-\lambda_1 T + \|u_0\|_\infty} \leqslant Ce^{-\varepsilon T}$$

Assume now that $\|u_0\|_\infty \leqslant r$ with $r \to \infty$ and set $T = \frac{r}{\lambda_1 - \varepsilon}$. By Proposition 5.2 the system is controllable at time $T + \frac{K}{\ln(\frac{1}{e^{-\varepsilon T}C})}$ (provided εT is large enough) so that

$$T(u_0) \leqslant \frac{r}{\lambda_1 - \varepsilon} + \frac{K}{\varepsilon T - \ln C}$$

hence

$$T(r) \leqslant \frac{r}{\lambda_1 - \varepsilon} + \frac{K(\lambda_1 - \varepsilon)}{\varepsilon r - (\lambda_1 - \varepsilon)\ln C}$$

for any ε, r such that εr is large. Choosing some $\varepsilon_r \to 0$ such that $\varepsilon_r r \to \infty$ we deduce that

$$T(r) \leqslant \frac{r}{\lambda_1} + o(r) \quad \text{as } r \to \infty$$

and in particular $\limsup_{r\to\infty} \frac{T(r)}{r} \leqslant \frac{1}{\lambda_1(\Omega)}$. Recall that Corollary 4.1 implies that $\frac{T(r)}{r} \geqslant \frac{1}{c_0(\Omega\setminus\omega)}$, where $c_0(\Omega \setminus \omega)$ is the ergodic constant of the state constraint problem set in $\Omega \setminus \omega$ (i.e. (4.1) in $\Omega \setminus \omega$). However when $q = 2\, c_0$ coincides with the first eigenvalue, therefore in this case we obtain overall the following explicit estimate for the waiting time:

$$\frac{1}{\lambda_1(\Omega \setminus \omega)} \leqslant \frac{T(r)}{r} \leqslant \frac{1}{\lambda_1(\Omega)} + o(1) \quad \text{as } r \to \infty. \qquad \square \tag{5.7}$$

Let us conclude by discussing how the estimates of Theorem 1.1 can be extended to problem (4.7) for different nonlinearities f. Reviewing our previous results, we already established in more generality the two estimates from below (Theorems 4.2 and 5.1). In both cases, such estimates are proved assuming that $f(t,x,\xi) \geqslant h(|\xi|)$ for some h such that $\int^\infty \frac{ds}{h(s)} < \infty$, with some technical additional condition which applies to a large class of examples. The estimates from above require a bit more care since they depend on the large time behavior of solutions. As a model case, we consider the example where the nonlinearity is independent of time, has at most quadratic growth and behaves at least like $|\nabla y|(\log|\nabla y|)^\alpha$ when $|\nabla y| \to \infty$, with $\alpha > 1$. Namely, we consider problem (1.5) and we prove Theorem 1.2. Let us remind that, if f is convex in the ξ-variable, problem (1.5) is well-posed in the class of bounded weak solutions from Proposition 2.1. However, even when f is not convex, globally defined weak solutions are known to exist because of the natural growth condition.

Proof of Theorem 1.2. Thanks to (1.6), f satisfies the assumptions of Theorem 4.2 with $h(s) = s(\ln(1+s))^\alpha$, $\alpha > 1$. Then we conclude that there exists $c_0 > 0$ such that $T(r) \geqslant c_0 r$ for every $r > 0$. Using again (1.6) and the local Lipschitz character of f with respect to ξ, we have that $f(x,\xi) \geqslant |\xi|(\ln(1+|\xi|))^\alpha - L_0|\xi|$ for some $L_0 > 0$. In particular, the assumption of Theorem 5.1 is satisfied (see also Remark 5.1). Then, we conclude that, for some $\kappa > 0$, $T(r) \geqslant \kappa(\ln(1/r))^{-1}$ as $r \to 0$.

By means of Remark 3.2, the null controllability holds for suitably small $W^{2,p}$ initial data. Moreover, thanks to (1.7)–(1.9), we can use Proposition A.1 and we deduce that there exist $C, K, \lambda > 0$ such that $\|y(t)\|_\infty + \|\nabla y(t)\|_\infty \leqslant Ce^{-\lambda t}$ for all $t \geqslant Kr$ and for all weak solutions of (1.5) with $\|y_0\|_\infty \leqslant r$ and $v = 0$ in $(0,t)$. Henceforth, we can proceed exactly as in Proposition 5.2, leaving first the system evolve freely. Estimate (5.6) now takes the form

$$\|y(t+s)\|_{W^{2,p}} \leqslant C \sup_{\tau\in[t,t+s]} \left(\frac{\|y(\tau)\|_\infty}{s^2} + \frac{\|f(x,\nabla y)(\tau)\|_\infty}{s} \right).$$

From (A.20) we know that, if $t \geqslant Kr$ and $s \leqslant 1$, then

$$\left|\nabla y(t+s)\right|^2 \leqslant \frac{C}{s} \max\left\{\|y_0\|_\infty^2, \|y_0\|_\infty\right\}.$$

Choosing $t = s = \frac{1}{2} r^\alpha$, we get $|\nabla y(r^\alpha)|^2 \leqslant Cr^{1-\alpha}$, and since f is locally Lipschitz we get

$$\frac{\|f(x,\nabla y)(\tau)\|_\infty}{s} \leqslant Cr^{\frac{1-3\alpha}{2}}.$$

We deduce, as in Proposition 5.2, that there exists $\alpha > 0$ such that $\|u(r^\alpha)\|_{W^{2,p}}$ is suitably small. Then, we conclude in the same way that $T(r) \leqslant K(\ln(1/r))^{-1}$ when $r \to 0$. Next, using the exponential decay for large time, we also obtain $T(r) \leqslant \Lambda r$ when $r \to \infty$. $\square$

Remark 5.3. The case when $f(x,\xi)$ has a superquadratic growth with respect to ξ is more delicate because of the large time decay of solutions. For a superquadratic nonlinearity, it is necessary to prove first some gradient estimate in order to handle the lower order term and show that solutions are meant in a strong sense. In Proposition A.1, we give estimates for this purpose, and we indicate in Remark A.1 how to use such estimates for the case of viscosity solutions. The complete proof given for problem (1.1) when $q > 2$ should provide a significant example to handle this approach in detail.

6. Further results, extensions and final comments

6.1. Controllability to trajectories

Let us consider a trajectory $\hat{y}$ solution of

$$\begin{cases} \hat{y}_t - \Delta \hat{y} + |\nabla \hat{y}|^q = 0 & \text{in } (0,T)\times\Omega, \\ \hat{y} = 0 & \text{on } (0,T)\times\partial\Omega, \\ \hat{y}(0) = \hat{y}_0 & \text{in } \Omega, \end{cases} \tag{6.1}$$

where $\hat{y}_0 \in L^\infty(\Omega)$. A natural question is whether the system (1.1) can be controlled in a way that $y(T) = \hat{y}(T)$. With the tools developed so far, we can give an answer to this question. For simplicity, we only deal here with the case $1 < q \leqslant 2$ (see Remark 6.1 concerning the case $q > 2$).

Similarly as before, we say that (1.1) is controllable to the trajectory $\hat{y}$ at time T if there exists a control v such that $y(T) = \hat{y}(T)$. Then we define

$$T(y_0) = \inf\bigl\{t > 0\colon\ (1.1) \text{ is controllable to the trajectory } \hat{y} \text{ at time } t\bigr\}$$

and

$$T(r) = \sup\bigl\{T(y_0),\ \|y_0 - \hat{y}_0\|_\infty \leqslant r\bigr\}. \tag{6.2}$$

We point out that the two trajectories $\hat{y}$ and y, if left free of control, will both tend to zero, hence it is reasonable that one can be driven onto the other at least in long time. The interesting part of the next result lies in the estimate of the control time, depending on the distance between y_0 and $\hat{y}_0$ and not on the single independent decay of each solution. In order to get such estimate, we will assume that $\hat{y}_0$ is smooth, so that $\hat{y}$ is a classical solution of (6.1) since $t = 0$. In the general case, one can use that, even if $\hat{y}_0 \in L^\infty(\Omega)$, the solution $\hat{y}$ will become classical as soon as $t > 0$.

Theorem 6.1. *Assume that $\hat{y}_0 \in C^2_0(\overline{\Omega})$. Then $T(y_0)$ is finite for every y_0, and moreover estimates* (1.3) *and* (1.4) *hold for the waiting time $T(r)$ defined in* (6.2)*.*

Proof. Let us set $z = y - \hat{y}$. Then z solves

$$\begin{cases} z_t - \Delta z + |\nabla z + \nabla \hat{y}|^q - |\nabla \hat{y}|^q = v\chi_\omega & \text{in } (0,T)\times\Omega, \\ z = 0 & \text{on } (0,T)\times\partial\Omega, \\ z(0) = y_0 - \hat{y}_0 & \text{in } \Omega, \end{cases} \tag{6.3}$$

and we need to find a control v such that $z(T) = 0$. We set

$$f(t,x,\nabla z) = \bigl|\nabla z + \nabla \hat{y}(t,x)\bigr|^q - \bigl|\nabla \hat{y}(t,x)\bigr|^q. \tag{6.4}$$

First of all, we observe that f satisfies, for some constant $c_q > 0$ and $\gamma = \gamma(\|\nabla \hat{y}\|_\infty)$,

$$f(t,x,\nabla z) \geqslant c_q |\nabla z|^q - \gamma |\nabla z|.$$

Note that $\|\nabla \hat{y}\|_\infty$ is estimated only in terms of $\hat{y}_0$, and uniformly in time due to Lemma 3.2. Therefore, f satisfies the assumptions of Theorem 4.2 and of Theorem 5.1. Applying these results to z, we deduce the lower bounds for (6.2).

The upper bounds for the waiting time depend on the decay estimates. Let us suppose now that $v = 0$. Note that the function f satisfies, for some positive constants β, δ:

$$\bigl|f(t,x,\nabla z)\bigr| \leqslant \beta|\nabla z| + \delta|\nabla z|^q.$$

As a consequence of Step 1 of Proposition A.1 and the comparison principle, we deduce that there exist K, λ, $C > 0$ such that if $t \geqslant K\|y_0 - \hat{y}_0\|_\infty$, then we have

$$\bigl\|z(t)\bigr\|_\infty \leqslant Ce^{-\lambda t}.$$

We reason now like in Step 2 of Lemma 3.2. We set $w=(t-t_0)^{\frac{2}{q}}\frac{|\nabla z|^2}{z+c}$, with $c=2\|z_0\|_\infty$, where $z_0=y_0-\hat{y}_0$. We obtain

$$w_t-\Delta w\leqslant\frac{2}{q}\frac{w}{t-t_0}-\nabla w\cdot D_\xi f+2\frac{\nabla w\cdot\nabla z}{(z+c)}+(t-t_0)^{\frac{2}{q}}\left[(f-D_\xi f\cdot\nabla z)\frac{|\nabla z|^2}{(z+c)^2}-2\frac{D_xf\cdot\nabla z}{z+c}\right]$$

where f is computed at $(t,x,\nabla z)$. Since $f-D_\xi f\cdot\xi\leqslant-\frac{q-1}{2}|\xi|^q+c_q|\nabla\hat{y}|^q\leqslant-\frac{q-1}{2}|\xi|^q+\gamma$ and, similarly, $|D_xf|\leqslant c_q|\xi|^{q-1}+\gamma$, we have

$$\left[(f-D_\xi f\cdot\nabla z)\frac{|\nabla z|^2}{(z+c)^2}-2\frac{D_xf\cdot\nabla z}{z+c}\right]\leqslant-\frac{q-1}{2}\frac{|\nabla z|^2}{(z+c)^2}\big(|\nabla z|^q-\gamma\big)+\gamma\frac{|\nabla z|^q+|\nabla z|}{z+c}$$

where we denote with γ possibly different constants depending on q and $\|\nabla\hat{y}\|_\infty$. Now, assume that $\sup w>L\|z_0\|_\infty^{\frac{2}{q}-1}+M\frac{1+\|z_0\|_\infty}{\|z_0\|_\infty}(t-t_0)^{\frac{2}{q}}$ for some large L, M. Then, on the maximum point, we have

$$|\nabla z|^2\geqslant M\big(1+\|z_0\|_\infty\big)+L\left(\frac{\|z_0\|_\infty}{t-t_0}\right)^{\frac{2}{q}}\tag{6.5}$$

and in particular, if $M>1$, $|\nabla z|^2\geqslant 1+\frac{M}{3}(z+c)$. Hence we have $\frac{|\nabla z|^q+|\nabla z|}{z+c}\leqslant 2\frac{|\nabla z|^q}{z+c}\leqslant\frac{6}{M}\frac{|\nabla z|^{q+2}}{(z+c)^2}$ and, if M is large, we have

$$-\frac{q-1}{2}\frac{|\nabla z|^2}{(z+c)^2}\big(|\nabla z|^q-\gamma\big)+\gamma\frac{|\nabla z|^q+|\nabla z|}{z+c}\leqslant-\frac{q-1}{4}\frac{|\nabla z|^{2+q}}{(z+c)^2}$$

so that

$$w_t-\Delta w+\nabla w\cdot D_\xi f-2\frac{\nabla w\cdot\nabla z}{(z+c)}\leqslant\frac{2}{q}\frac{w}{t-t_0}-\frac{(q-1)}{4}(t-t_0)^{\frac{2}{q}}\frac{|\nabla z|^{2+q}}{(z+c)^2}<0$$

if the constant L in (6.5) is large enough. We deduce that such a maximum point cannot be in the interior, and, on account of the boundary estimate, which is as in Lemma 3.2, we conclude that

$$w\leqslant L\|z_0\|_\infty^{\frac{2}{q}-1}+M\frac{1+\|z_0\|_\infty}{\|z_0\|_\infty}(t-t_0)^{\frac{2}{q}}$$

which implies

$$|\nabla z|^2\leqslant 3L\left(\frac{\|z_0\|_\infty}{t-t_0}\right)^{\frac{2}{q}}+3M\big(1+\|z_0\|_\infty\big).\tag{6.6}$$

In particular, for every $t_0\geqslant K\|z_0\|_\infty$ and every $t\geqslant t_0+K\|z_0\|_\infty$, $\nabla z(t)$ is bounded and $|\nabla z(t)|^2\leqslant\tilde{M}(1+\|z_0\|_\infty)$.

Let us now suppose that $\|z_0\|_\infty=\|y_0-\hat{y}_0\|_\infty\leqslant r$, with $r\to 0$. Then we proved that $|\nabla z(t)|$ is bounded for $t\geqslant kr$. The nonlinearity $f(t,x,\nabla z)$ in (6.4) clearly satisfies (A.10) and (A.11) and, since ∇z is in a bounded set, then (A.12) is also satisfied with $\theta=q-1$, and using that $\hat{y}$ is C^2. We can now apply the estimates of Proposition A.1; in particular (A.21) implies that

$$\big|\nabla z(t+s)\big|^2\leqslant C_0(\gamma s)^{\frac{2}{1-\theta}}+\frac{C_1}{s}\big[\|z_0\|_\infty^2+(1+s)\|z_0\|_\infty\big]$$

for every $t\geqslant kr$. Taking $t=s=\frac{1}{2}r^\alpha$, using that $\|z_0\|_\infty\leqslant r$ and $\theta>0$, we can find some small $\alpha>0$ such that $|\nabla z(r^\alpha)|=o(r^\alpha)$ as $r\to 0$. Then, as in the proof of Theorem 1.2, we deduce that $\|z(r^\alpha)\|_{W^{2,p}(\Omega)}=o(1)$ as $r\to 0$, which is enough to prove the controllability of (6.3) for $T\leqslant C(\ln(1/r))^{-1}$. We just point out that, since the $W^{2,p}$-estimate also depends on the estimate on z_t, in this case such estimate also relies on $f_t(t,x,\nabla z)$. However, using that $|f_t(t,x,\nabla z)|\leqslant\gamma|\nabla z|^{q-1}$ (for some γ depending on $\|\hat{y}_t\|_\infty$ and $\|\nabla\hat{y}\|_\infty$), this term is estimated again with the gradient bound.

Finally, if $\|y_0-\hat{y}_0\|_\infty\leqslant r$, with $r\to\infty$, we proceed similarly. Taking $t\geqslant kr$ we have $\|z(t)\|_\infty\leqslant Ce^{-\lambda t}$, then rescaling (6.6) to estimate $\nabla z(t)$ and using that the system can be controlled in small time, we conclude in the usual way. □

Remark 6.1. In the case $q > 2$, we can show that the exact controllability to trajectories holds at least in long time. In fact, we know from Lemma 3.2 that the trajectory $\hat{y}$ will eventually become smooth, and then we can reason as above to deduce that $T(y_0)$ is finite for every $y_0 \in C_0(\overline{\Omega})$. On the other hand, in this way the control time will depend on $\|\hat{y}_0\|_\infty$ (because of the time needed to $\hat{y}$ to become smooth), and it is less clear that one gets the estimates on $T(r)$ as in Theorem 6.1.

6.2. Approximate controllability

We just make a remark on the question of approximate controllability for the system (1.1), namely whether, given some target function y_d and $\varepsilon > 0$, it is possible to control the system (1.1) so that $\|y(T) - y_d\|_X \leqslant \varepsilon$ for some suitable space X. A natural choice, for this kind of models, should be $X = C_0(\overline{\Omega})$; even if we choose a larger space X (hence a smaller norm), this problem is more delicate than controlling to trajectories. Of course, by density one can reason assuming y_d to be smooth. Even in this case, *the results of Section* 4 *imply that the approximate controllability cannot hold arbitrarily*. Indeed, the comparison results of Theorem 4.1 or Theorem 4.2 give suitable estimates on the solution y in $\Omega \setminus \omega$ which allow to estimate $\|y(t) - y_d\|_{L^\infty(\Omega\setminus\omega)}$ (as well as any other local norm) and to exclude that the approximate controllability can hold in that case. Giving a more detailed picture of the approximate controllability would anyway be an interesting fact.

6.3. Further comments and possible open problems

1. The case that $\omega = \Omega$ has been excluded by our analysis, since in this case the nonlinear effect can be removed. In fact, if $\omega = \Omega$, we can replace v with $f(t, x, \nabla y) + \tilde{v}$, reducing the problem to the controllability of the heat equation through some control $\tilde{v}$.

2. The estimate (5.7) in Remark 5.2 shows the precise role of the geometry of ω in the waiting time for large data. It seems possible that a similar estimate could hold if $q \neq 2$ with the ergodic constant c_0 in place of λ_1, namely that

$$\frac{1}{c_0(\Omega \setminus \omega)} \leqslant \frac{T(r)}{r} \leqslant \frac{1}{c_0(\Omega)} + o(1) \quad \text{as } r \to \infty,$$

where c_0 is the ergodic constant introduced in Section 3. Note that the estimate from below is already established in Corollary 4.1. Let us recall that if $q = 2$ we have $c_0 = \lambda_1$.

3. Our results do also apply for general second order parabolic equations provided enough regularity is imposed to have Carleman inequalities. In particular, we can replace the Laplace operator with a linear second order operator with Lipschitz coefficients.

4. As we mentioned at the beginning, the sign of the nonlinearity $|\nabla y|^q$ is not essential in our analysis, since one can change y into $-y$ in Eq. (1.1).

However, the necessity of the waiting time was proved, in Section 3, for data with suitable sign (see also Remark 4.8). Referring to (1.1), where the nonlinearity has positive sign, we showed necessity of waiting time for nonpositive data. It would be interesting to show the necessity of waiting time for positive data too. A similar feature can be found in the case of dissipative semilinear equations, see [1]. In general, one should note that the sign of the nonlinearity plays a role in the construction of universal positive or negative barriers.

Moreover, the coercivity and convexity of the Hamiltonian play somehow a role in the large time behavior of solutions of the uncontrolled system. Therefore, a more difficult analysis would concern the case of a nonlinearity changing sign (or convexity) as time–space vary.

Appendix A. The decay estimates

In this section we prove some estimates on the decay of the solutions of the uncontrolled system. We start by proving Lemma 3.2.

Proof of Lemma 3.2. Step 1. *There exists a constant $K = K(q, \Omega)$ such that $\|\nabla u(t)\|_{L^\infty(\partial\Omega)} \leqslant c_1 \frac{\|u_0\|_\infty}{t}$ for any $t \geqslant K\|u_0\|_\infty$, where c_1 only depends on Ω.*

To prove this claim, take $\zeta \in H_0^1(\Omega)$: $\Delta\zeta = -1$ and consider $v = -\mu\zeta + \frac{\mu}{2}(t-T)$. Then $v \in C^2(\overline{\Omega})$ and satisfies

$$v_t - \Delta v + |\nabla v|^q = -\frac{\mu}{2} + \mu^q |\nabla\zeta|^q \quad \text{in } (0,T)\times\Omega.$$

Choose $\mu = \frac{2\|u_0\|_\infty}{T}$, then we have

$$v(0) = -\mu\zeta - \|u_0\|_\infty \leqslant u_0, \qquad v_{|\partial\Omega\times(0,T)} \leqslant 0,$$

and v is a sub-solution provided $\mu^{q-1}\|\nabla\zeta\|_\infty^q \leqslant \frac{1}{2}$. By comparison principle, we deduce that $u(x,t) \geqslant v(x,t)$, and in particular

$$u(x,T) \geqslant v(x,T) = -\frac{2\|u_0\|_\infty}{T}\zeta$$

provided $\|u_0\|_\infty (2\|\nabla\zeta\|_\infty)^{q'} \leqslant T$. On the other hand we have $u \leqslant U$ where $U_t - \Delta U = 0$ with $U(0) = u_0$. This means that

$$-\frac{2\|u_0\|_\infty}{T}\zeta(x) \leqslant u(x,T) \leqslant U(x,T) \quad \forall T \geqslant \|u_0\|_\infty \big(2\|\nabla\zeta\|_\infty\big)^{q'}. \tag{A.1}$$

In particular, if $u \in C_0^1(\overline{\Omega})$ we deduce that

$$\big|\nabla u(T)\big| = \left|\frac{\partial u(T)}{\partial\nu}\right| \leqslant \frac{2\|u_0\|_\infty}{T}|\nabla\zeta| + \big|\nabla U(T)\big| \quad \text{on } \partial\Omega.$$

Since, by classical estimates on the heat equation (see e.g. [21]), we have

$$\big|\nabla U(t)\big| \leqslant C\big(1 + t^{-\frac{1}{2}}\big)e^{-\lambda_1 t}\|u_0\|_\infty \quad \forall t > 0$$

and since ζ is smooth, we deduce that

$$\big|\nabla u(x,T)\big| \leqslant \frac{\tilde{c}_1}{T}\|u_0\|_\infty \quad \forall x \in \partial\Omega,\ \forall T \geqslant \big(2\|\nabla\zeta\|_\infty\big)^{q'}\|u_0\|_\infty$$

and the claim is proved with $K = (2\|\nabla\zeta\|_\infty)^{q'}$. A minor point is only left in the case $q > 2$, since a priori we could not use the C^1 regularity of the solution u; however we will recover it with an approximation argument, detailed later.

Step 2. *Let $t_0 \geqslant K\|u_0\|_\infty$, where K is given by Step 1. We prove that*

$$\big|\nabla u(t)\big| \leqslant c_2\left(\frac{\|u_0\|_\infty}{t - t_0}\right)^{\frac{1}{q}} \quad \forall t > t_0 \tag{A.2}$$

for some constant c_2 only depending on q, Ω.

Here we proceed using a typical Bernstein's type argument (see e.g. [19] in a similar context). Assume for the moment that solutions are classical for $t > t_0$ (which is certainly true if $q \leqslant 2$, but not obvious if $q > 2$). Set

$$w = (t - t_0)^{\frac{2}{q}}\frac{|\nabla u|^2}{(u + c)}$$

where $c = 2\|u_0\|_\infty$. Recall that, by maximum principle, we have $\|u\|_\infty \leqslant \|u_0\|_\infty$, hence the choice of c implies

$$\frac{1}{3}\frac{1}{\|u_0\|_\infty} \leqslant \frac{1}{u + c} \leqslant \frac{1}{\|u_0\|_\infty}. \tag{A.3}$$

First of all observe that, since $t \geqslant t_0 \geqslant K\|u_0\|_\infty$, if $x \in \partial\Omega$ we have, by Step 1, and using that $q > 1$,

$$w \leqslant \frac{(t - t_0)^{\frac{2}{q}}}{2\|u_0\|_\infty}\left(c_1\frac{\|u_0\|_\infty}{t}\right)^2 \leqslant \frac{1}{2}c_1^2 t^{\frac{2}{q}-2}\|u_0\|_\infty \leqslant \frac{1}{2}c_1^2 K^{\frac{2}{q}-2}\|u_0\|_\infty^{\frac{2}{q}-1}. \tag{A.4}$$

Now we compute

$$w_t = \frac{2}{q}(t-t_0)^{\frac{2}{q}-1}\frac{|\nabla u|^2}{(u+c)} - (t-t_0)^{\frac{2}{q}}\frac{|\nabla u|^2}{(u+c)^2}u_t + (t-t_0)^{\frac{2}{q}}2\frac{\nabla u\cdot\nabla u_t}{(u+c)}$$

and

$$\Delta w \geqslant (t-t_0)^{\frac{2}{q}}\left[2\frac{\nabla(\Delta u)\cdot\nabla u}{(u+c)} - 2\frac{\nabla(|\nabla u|^2)\cdot\nabla u}{(u+c)^2} - \frac{|\nabla u|^2}{(u+c)^2}\Delta u + 2\frac{|\nabla u|^4}{(u+c)^3}\right]$$

hence using the equation of u

$$\begin{aligned} w_t - \Delta w \leqslant{}& \frac{2}{q}(t-t_0)^{\frac{2}{q}-1}\frac{|\nabla u|^2}{(u+c)} \\ &- (t-t_0)^{\frac{2}{q}}\left[q\frac{\nabla(|\nabla u|^2)\cdot\nabla u}{(u+c)}|\nabla u|^{q-2} - 2\frac{\nabla(|\nabla u|^2)\cdot\nabla u}{(u+c)^2} - \frac{|\nabla u|^{2+q}}{(u+c)^2} + 2\frac{|\nabla u|^4}{(u+c)^3}\right] \end{aligned}$$

which implies

$$\begin{aligned} w_t - \Delta w \leqslant{}& \frac{2}{q}(t-t_0)^{\frac{2}{q}-1}\frac{|\nabla u|^2}{(u+c)} \\ &- q\nabla w\cdot\nabla u|\nabla u|^{q-2} + 2\frac{\nabla w\cdot\nabla u}{(u+c)} - (t-t_0)^{\frac{2}{q}}(q-1)\frac{|\nabla u|^{2+q}}{(u+c)^2}. \end{aligned}$$

Therefore we get

$$w_t - \Delta w \leqslant -q\nabla w\cdot\nabla u|\nabla u|^{q-2} + 2\frac{\nabla w\cdot\nabla u}{(u+c)} - w\left[(q-1)\frac{|\nabla u|^q}{u+c} - \frac{2}{q}\frac{1}{t-t_0}\right]. \tag{A.5}$$

Now, assume that $\sup w > L\|u_0\|_\infty^{\frac{2}{q}-1}$, where $L > \frac{1}{2}c_1^2K^{\frac{2}{q}-2}$; by (A.4) this means that the maximum of w is attained inside Ω on some point where we have $(t-t_0)^{\frac{2}{q}}\frac{|\nabla u|^2}{u+c} > L\|u_0\|_\infty^{\frac{2}{q}-1}$. Hence, at this point we have

$$\begin{aligned} w_t - \Delta w &\leqslant -w\left[(q-1)\frac{|\nabla u|^q}{u+c} - \frac{2}{q}\frac{1}{t-t_0}\right] \\ &< -\frac{w}{t-t_0}\left[(q-1)L^{\frac{q}{2}}\left(\frac{\|u_0\|_\infty}{u+c}\right)^{1-\frac{q}{2}} - \frac{2}{q}\right]. \end{aligned}$$

Thanks to (A.3), if L is sufficiently large (only depending on q) we have $w_t - \Delta w < 0$ which is impossible on an interior maximum point. This means that $w \leqslant L\|u_0\|_\infty^{\frac{2}{q}-1}$, which gives

$$|\nabla u|^2 \leqslant L\|u_0\|_\infty^{\frac{2}{q}-1}\frac{(u+2\|u_0\|_\infty)}{(t-t_0)^{\frac{2}{q}}} \leqslant 3L\frac{\|u_0\|_\infty^{\frac{2}{q}}}{(t-t_0)^{\frac{2}{q}}}$$

hence (A.2).

Now, if $1 < q \leqslant 2$ the above computation is justified since the unique solution is smooth for any $t > 0$.

If $q > 2$, we proceed in the following way: we start by solving an approximating problem

$$\begin{cases} (u_n)_t - \Delta u_n + H_n(\nabla u_n) = 0 & \text{in } \Omega\times(0,\infty), \\ u_n = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u_n(0) = u_0 & \text{in } \Omega \end{cases} \tag{A.6}$$

where $H_n(\xi) = |\xi|^2\psi_n(|\xi|^{q-2})$, being ψ_n some bounded increasing smooth function such that $\psi_n(s) \leqslant s$ and $\psi_n(s) \to s$ as $s \to \infty$. Since H_n has quadratic growth, problem (A.6) admits a unique classical solution u_n which is globally defined. Since $0 \leqslant H_n(\nabla u_n) \leqslant |\nabla u_n|^q$, the estimate obtained in the first step (see (A.1)) remains valid for u_n and in particular the claim of Step 1 holds true. Next we use Bernstein's argument for the function $w_n = (t-t_0)\frac{|\nabla u_n|^2}{u_n+c}$;

this argument works similarly as before using that $H_n(\xi) - D_\xi H_n \cdot \xi \leqslant -|\xi|^2 \psi_n(|\xi|^{q-2})$. In this way[3] one obtains that $u_n(t)$ is bounded in $W^{1,\infty}(\Omega)$ if $t > t_0$. Since $H_n(\nabla u_n)$ is bounded, using classical estimates for the heat equation we deduce that u_n is bounded in $L^p((t,T); W^{2,p}(\Omega))$ for every $p < \infty$ and $t_0 < t < T$, and all the stronger estimates of the classical theory hold. In particular, $u_n(t)$ is relatively compact in $C_0^1(\overline{\Omega})$ for every $t > t_0$. On the other hand, by stability results of the viscosity solutions theory, namely the half-relaxed limits method (see e.g. [8] and references therein), it can be proved that u_n converges (locally uniformly in $(0,T) \times \Omega$) to the unique generalized viscosity solution u of (3.6). We deduce that $u(t) \in C_0^1(\overline{\Omega})$ for $t > t_0$, so u is actually a classical solution for $t > t_0$ and the above computations are justified even if $q > 2$. In particular, u satisfies estimates (A.1) and (A.2) in a classical sense.

Step 3. By Step 2, we have, for any $t > t_0 \geqslant K\|u_0\|_\infty$,

$$0 = u_t - \Delta u + |\nabla u|^q \leqslant u_t - \Delta u + |\nabla u| \left(c_2^q \frac{\|u_0\|_\infty}{(t - t_0)} \right)^{\frac{1}{q'}}.$$

Take now any $\lambda < \lambda_1$, where λ_1 is the first eigenvalue of the Laplacian in Ω with Dirichlet boundary conditions. There exists a (unique) positive $\psi \in H_0^1(\Omega)$ such that

$$-\Delta\psi = \lambda\psi + 1 \quad \text{in } \Omega.$$

We set $v = -\Lambda\psi e^{-\lambda(t-t_1)}$, where $t_1 = \hat{K}\|u_0\|_\infty$ and $\Lambda = \frac{2}{\hat{K}}$, with $\hat{K} > K$ to be chosen below. First observe that, using (A.1) from Step 1, we have

$$\frac{-u(t_1)}{\psi} \leqslant \frac{2\|u_0\|_\infty}{t_1} \frac{\zeta}{\psi}$$

and since $\psi \geqslant \zeta$ we have, by the choice of t_1 and Λ,

$$\frac{-u(t_1)}{\psi} \leqslant \frac{2}{\hat{K}} = \Lambda$$

hence $v(t_1) = -\Lambda\psi \leqslant u(t_1)$. Next, we have

$$v_t - \Delta v + |\nabla v| \left(c_2^q \frac{\|u_0\|_\infty}{(t - t_0)} \right)^{\frac{1}{q'}} = -\Lambda e^{-\lambda(t-t_1)} \left[-\lambda\psi - \Delta\psi - |\nabla\psi| \left(c_2^q \frac{\|u_0\|_\infty}{(t - t_0)} \right)^{\frac{1}{q'}} \right]$$

hence, choosing t large such that $|\nabla\psi| (c_2^q \frac{\|u_0\|_\infty}{(t-t_0)})^{\frac{1}{q'}} \leqslant 1$, we have that v is a sub-solution, and by comparison we deduce that $u(t) \geqslant -\Lambda\psi e^{-\lambda(t-t_1)}$ for $t > t_1$. This choice means that

$$|\nabla\psi|^{q'} c_2^q \|u_0\|_\infty \leqslant (t - t_0)$$

which holds for $t \geqslant t_1 = \hat{K}\|u_0\|_\infty$ if e.g. we take $t_0 = K\|u_0\|_\infty$ and $\hat{K} = K + c_2^q \|\nabla\psi\|_\infty^{q'}$. Therefore we proved that, for this choice of $\hat{K}$, Λ (only depending on Ω, q, λ) we have

$$u(t) \geqslant -Ce^{-\lambda(t-t_1)} \quad \forall t \geqslant t_1 = \hat{K}\|u_0\|_\infty$$

where $C = C(q, \Omega, \lambda) > 0$. In particular, we have

$$u(t) \geqslant -Ce^{-\frac{\lambda}{2}t} \quad \forall t \geqslant 2\hat{K}\|u_0\|_\infty.$$

On the other hand we have, by comparison with the heat potential,

$$u(t) \leqslant \tilde{C}e^{-\lambda_1 t}\|u_0\|_\infty \leqslant \tilde{C}\frac{t}{2\hat{K}}e^{-\lambda_1 t} \quad \forall t \geqslant 2\hat{K}\|u_0\|_\infty$$

where $\tilde{C}$ only depends on Ω. Then we globally obtain the desired estimate on $\|u(t)\|_\infty$. Since (A.2) implies

[3] An alternative way is to prove estimate (A.8), as we do later, only using that $H_n(\xi) - D_\xi H_n \cdot \xi \leqslant 0$.

$$\left|\nabla u(t+s)\right| \leqslant c_2\left(\frac{\|u(s)\|_\infty}{t-K\|u(s)\|_\infty}\right)^{\frac{1}{q}} \quad \forall s>0,\ t>K\left\|u(s)\right\|_\infty,$$

if $t \geqslant 2K\|u_0\|_\infty$ then we also have $t \geqslant 2K\|u(t)\|_\infty$ and we can choose $s=t$ obtaining

$$\left|\nabla u(2t)\right| \leqslant c_2\left(\frac{2\|u(t)\|_\infty}{t}\right)^{\frac{1}{q}} \quad \forall t>2K\|u_0\|_\infty.$$

The estimate on $u(t)$ then implies the exponential decay for $\|\nabla u(t)\|_\infty$ as well. □

As a matter of fact, we obtained a stronger result in the above proof because, in Step 3, any $\lambda<\lambda_1$ was admitted, up to choosing the constant $\hat{K}$ accordingly. Following this remark, we can prove the sharp asymptotic rate, which is the same as for the heat equation. In this way we provide a new proof, with a different approach, of the following result of [6]. More precisely, in the case $q>2$, we extend their result by considering any $u_0 \in C_0(\overline{\Omega})$, with no sign or smallness condition required.

Theorem A.1. *Let $q>1$, and let $u_0 \in C_0(\overline{\Omega})$. There exists a positive constant K such that the solution u of* (3.6) *satisfies*

$$\left\|u(t)\right\|_\infty+\sqrt{t}\left\|\nabla u(t)\right\|_\infty \leqslant Ke^{-\lambda_1 t} \quad \text{as } t\to\infty, \tag{A.7}$$

where $K=C(q,\Omega,u_0)$.

Proof. We proved above (see Step 3) that for any $\lambda<\lambda_1$ there exist constants κ_1,κ_2, depending on Ω, λ, q such that

$$\left\|u(t)\right\|_\infty \leqslant \kappa_1 e^{-\lambda t} \quad \forall t \geqslant \kappa_2\|u_0\|_\infty.$$

We refine now Bernstein's estimate on ∇u, by proving that

$$\left|\nabla u(t)\right| \leqslant c_2\frac{\|u_0\|_\infty}{\sqrt{t-t_0}} \quad \forall t>t_0, \tag{A.8}$$

where $t_0 \geqslant K\|u_0\|_\infty$, K is given by Step 1 of the previous proof. Set

$$w=(t-t_0)|\nabla u|^2+\big(u-\|u_0\|_\infty\big)^2.$$

First of all, since $t \geqslant t_0$, we note that, by Step 1, if $x \in \partial\Omega$ we have

$$w \leqslant c_1^2\frac{t-t_0}{t^2}\|u_0\|_\infty^2+\|u_0\|_\infty^2 \leqslant \|u_0\|_\infty^2\left(\frac{c_1^2}{t_0}+1\right). \tag{A.9}$$

Now we compute

$$w_t=|\nabla u|^2+2(t-t_0)\nabla u\nabla u_t+2\big(u-\|u_0\|_\infty\big)u_t$$

and

$$\Delta w \geqslant 2(t-t_0)\nabla u\nabla(\Delta u)+2\big(u-\|u_0\|_\infty\big)\Delta u+2|\nabla u|^2$$

hence using the equation of u

$$\begin{aligned} w_t-\Delta w &\leqslant -|\nabla u|^2-2\big(u-\|u_0\|_\infty\big)|\nabla u|^q-2(t-t_0)\nabla u\nabla\big(|\nabla u|^q\big) \\ &= -|\nabla u|^2+2(q-1)\big(u-\|u_0\|_\infty\big)|\nabla u|^q-q|\nabla u|^{q-2}\nabla u\nabla w \end{aligned}$$

which implies, since $u \leqslant \|u_0\|_\infty$,

$$w_t-\Delta w+q|\nabla u|^{q-2}\nabla u\nabla w \leqslant 0.$$

Applying the maximum principle and using (A.9) we deduce

$$w \leqslant \left\|w(t_0)\right\|_\infty+\max_{\partial\Omega\times(t_0,T)} w \leqslant \|u_0\|_\infty^2\left(\frac{c_1^2}{t_0}+5\right)$$

which implies (A.8) with $c_2 = \sqrt{\frac{c_1^2}{t_0} + 5}$. Thanks to (A.8), if s is large enough we have $|\nabla u(s+1)| \leqslant c\|u(s)\|_\infty$ and so we deduce that, for any $\lambda < \lambda_1$,

$$\left\|\nabla u(t)\right\|_\infty \leqslant \kappa_1 e^{-\lambda t} \quad \forall t \geqslant \kappa_2 \|u_0\|_\infty$$

for possibly different constants κ_1, κ_2 depending on λ. Since $q > 1$, now we fix $\lambda < \lambda_1$ such that $\lambda q > \lambda_1$; then $|\nabla u(t)|^q \leqslant C(e^{-\mu t})$ as $t \to \infty$ for some $\mu > \lambda_1$. Since $u_t - \Delta u = -|\nabla u|^q$, by the classical representation of the heat semigroup we deduce that u satisfies the same asymptotic estimates of the linear case. □

We conclude this appendix by showing how the method used provides an exponential decay for possibly different nonlinearities.

Proposition A.1. *Assume that $f : \mathbb{R} \times \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a C^1 function such that $f(t,x,0) = 0$, $|D_\xi f(t,x,\xi)|$ is bounded (uniformly in $\mathbb{R} \times \Omega$) when ξ is in a compact set, and*

$$\exists \sigma > 0,\ q > 1: \quad \left|f(t,x,\xi)\right| \leqslant \sigma\left(1 + |\xi|^q\right), \tag{A.10}$$

$$\exists \delta \geqslant 0: \quad f(t,x,\xi) - D_\xi f(t,x,\xi) \cdot \xi \leqslant \delta, \tag{A.11}$$

$$\exists \gamma \geqslant 0,\ \theta \in (0,1]: \quad \left|D_x f(t,x,\xi)\right| \leqslant \gamma |\xi|^\theta, \tag{A.12}$$

for every $(t,x) \in \mathbb{R} \times \Omega$, $\xi \in \mathbb{R}^N$.

Assume that $u_0 \in L^\infty(\Omega)$ and u is a weak solution if $q \leqslant 2$ (respectively, $u_0 \in C_0^1(\overline{\Omega})$ and u is a strong solution if $q > 2$) of

$$\begin{cases} u_t - \Delta u + f(t,x,\nabla u) = 0 & \text{in } (0,\infty) \times \Omega, \\ u = 0 & \text{on } (0,\infty) \times \partial\Omega, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{A.13}$$

There exist positive constants C, K and λ, only depending on f and Ω, such that, for every $t \geqslant K\|u_0\|_\infty$,

$$\left\|u(t)\right\|_\infty + \left\|\nabla u(t)\right\|_\infty \leqslant C e^{-\lambda t}. \tag{A.14}$$

The dependence of C, K, λ on f is through the constants σ, q, δ, γ, θ and $\sup\{|D_\xi f(t,x,\xi)|,\ (t,x) \in \mathbb{R} \times \Omega,\ |\xi| \in [0,1]\}$.

Remark A.1. In the superquadratic case, the statement can be used as an a priori estimate, and, whenever a strong comparison result holds for generalized viscosity solutions of (A.13), then the estimate will hold for the unique such solution. In fact, for any f satisfying (A.10)–(A.12), it is possible to construct an approximation f_n which is subquadratic and preserves the same assumptions with uniform constants. All weak solutions u_n corresponding to f_n will satisfy estimate (A.14) (uniform in n) and so is the case for any possible limit u. In particular, if (A.13) possesses a unique generalized viscosity solution, then this solution is Lipschitz continuous for $t > K\|u_0\|_\infty$ and satisfies (A.14).

In order to construct such an approximation, let us suppose that $f \geqslant 0$ (otherwise replace f with $f + \sigma(1 + |\xi|^q)$) and take $\varphi(s) = (1 + s_+^2)^{\frac{1}{2}} - 1$. Then $\varphi \in C^1(\mathbb{R})$, $\varphi(s) \equiv 0$ if $s \leqslant 0$ and $\varphi(s) \geqslant s - 1$; the function $f_n(t,x,\xi) = f(t,x,\xi) - \varphi(f(t,x,\xi) - n|\xi|^2)$ converges to f pointwise and satisfies all the required properties.

Proof. Step 1. *The conclusion of Lemma 3.2 holds true for the model problem*

$$v_t - \Delta v + a|\nabla v| + b|\nabla v|^q = 0 \quad \text{in } \Omega \times (0,\infty), \tag{A.15}$$

where a, $b > 0$, complemented with initial and Dirichlet boundary conditions.

We point out the only changes needed in the proof. The conclusion of Step 1 of Lemma 3.2 is obtained in the same way except for using now the function $\zeta \in H_0^1(\Omega) \cap L^\infty(\Omega)$ which solves $-\Delta\zeta = a|\nabla\zeta| + 1$ in Ω. For any $a > 0$, such a function ζ exists and is unique, and satisfies $\zeta \in C_0^1(\overline{\Omega})$. Then, we have the pointwise estimate (A.1) which yields the boundary gradient estimate. Step 2 of Lemma 3.2 is applied to the function v without significant changes.

It is enough to remark that we derive the equation in a neighborhood of the maximum point of $w = (t - t_0)^{\frac{2}{q}} \frac{|\nabla v|^2}{(v+c)}$, hence $|\nabla v| \neq 0$ and the term $a|\nabla v|$ can be differentiated. The estimate (A.2) follows. Finally, we proceed as in Step 3 by choosing now ψ such that

$$-\Delta\psi - a|\nabla\psi| = \lambda\psi + 1,$$

for some $\lambda > 0$. This is possible for some $\lambda > 0$ since the operator $-\Delta(\cdot) - a|\nabla(\cdot)|$ satisfies the maximum principle, hence its first eigenvalue is positive. Therefore, we conclude that, for every $a, b > 0$, there exists some $\lambda > 0$ such that

$$\left|v(t)\right| \leqslant Ce^{-\lambda t} \quad \forall t \geqslant K\|u_0\|_\infty$$

for any solution of the Cauchy–Dirichlet problem related to (A.15), where C, K λ only depend on q, a, b, Ω.

Step 2. Let us consider a general function f. Using (A.10) and the fact that $f(t,x,0) = 0$, we have $|f(t,x,\xi)| \leqslant a|\xi| + b|\xi|^q$ for every $\xi \in R^N$, for some constants $a, b > 0$, where a depends on $\sup\{|D_\xi f(t,x,\xi)|, (t,x) \in \mathbb{R} \times \Omega, |\xi| \in [0,1]\}$. Therefore, any solution of (A.13) is such that both u and $-u$ are super-solutions of (A.15). Applying the comparison principle, we have $\pm u \geqslant v$, hence we deduce that u satisfies $|u(x,T)| \leqslant \frac{2\|u_0\|_\infty}{T}\zeta(x)$ and we conclude as in Step 1 of Lemma 3.2 that

$$\left\|\nabla u(t)\right\|_{L^\infty(\partial\Omega)} \leqslant c_1 \frac{\|u_0\|_\infty}{t} \quad \forall t \geqslant K\|u_0\|_\infty \tag{A.16}$$

where c_1, K only depend on a, b, q, Ω.

Still using the comparison principle, and the exponential decay proved for (A.15), we have that there exists $\lambda > 0$ such that

$$\left\|u(t)\right\|_\infty \leqslant Ce^{-\lambda t} \quad \forall t \geqslant K\|u_0\|_\infty. \tag{A.17}$$

We are only left with the gradient estimate. To this purpose, we proceed as in Theorem A.1 applying Bernstein's argument to the function

$$w = (t - t_0)|\nabla u|^2 + \mu\left(u - \|u_0\|_\infty\right)^2,$$

with $\mu > 0$ to be fixed. Note that (A.16) implies, for every $t > t_0 \geqslant K\|u_0\|_\infty$:

$$w \leqslant \|u_0\|_\infty^2\left(\frac{c_1^2}{t_0} + \mu\right) \leqslant \frac{c_1^2}{K}\|u_0\|_\infty + \mu\|u_0\|_\infty^2 \quad \forall x \in \partial\Omega. \tag{A.18}$$

We consider the equation of w for $t \in (t_0, t_0 + s)$. Using (A.13), we obtain now (we avoid to write the argument of f, which is $(t, x, \nabla u)$)

$$w_t - \Delta w \leqslant (1 - 2\mu)|\nabla u|^2 - D_\xi f \cdot \nabla w - 2\mu\left(u - \|u_0\|_\infty\right)(f - D_\xi f \cdot \nabla u) - 2(t - t_0)\nabla u \cdot D_x f$$

which yields, using (A.11) and (A.12),

$$w_t - \Delta w + D_\xi f \cdot \nabla w \leqslant (1 - 2\mu)|\nabla u|^2 + 4\mu\delta\|u_0\|_\infty + 2\gamma(t - t_0)|\nabla u|^{1+\theta}.$$

Since $t \leqslant t_0 + s$, we have $w \leqslant s|\nabla u|^2 + 4\mu\|u_0\|_\infty^2$, hence we get

$$w_t - \Delta w + D_\xi f \cdot \nabla w + \frac{(2\mu - 1)}{s}w \leqslant \frac{4\mu(2\mu - 1)}{s}\|u_0\|_\infty^2 + 4\mu\delta\|u_0\|_\infty + 2\gamma(t - t_0)|\nabla u|^{1+\theta}, \tag{A.19}$$

provided $2\mu - 1 > 0$. If $\theta = 1$, last term is smaller than $2\gamma w$; we take $\mu = 1 + \gamma s$ and we deduce that $w \leqslant 4\mu(2\mu - 1)\|u_0\|_\infty^2 + 4\mu\delta s\|u_0\|_\infty$ on any internal maximum point. On account of (A.18), the value at $t = t_0$ and the choice of μ, we conclude that

$$w \leqslant C(1 + \gamma s)^2\left[\|u_0\|_\infty^2 + \|u_0\|_\infty\right] \quad \forall t \in [t_0, t_0 + s].$$

Since $w \geqslant (t - t_0)|\nabla u|^2$, when $t = t_0 + s$ we deduce

$$\left|\nabla u(t_0 + s)\right|^2 \leqslant \frac{C}{s}(1 + \gamma s)^2\left[\|u_0\|_\infty^2 + \|u_0\|_\infty\right], \tag{A.20}$$

for every $t_0 \geqslant K\|u_0\|_\infty$. Rescaling the estimate for $u(t+\cdot)$ we deduce that

$$\left|\nabla u(t+t_0+s)\right|^2 \leqslant \frac{C}{s}(1+\gamma s)^2\left[\left\|u(t)\right\|_\infty^2+\left\|u(t)\right\|_\infty\right],$$

for every $t_0 \geqslant K\|u(t)\|_\infty$. In particular, if $t > K\|u_0\|_\infty$, then $t \geqslant K\|u(t)\|_\infty$; taking $t_0 = t = s$ and using (A.17) we deduce the exponential decay for $|\nabla u|$.

If $\theta < 1$ in (A.12), we take $\mu = 1$ and we deduce from (A.19) that

$$w \leqslant C\left[s(\gamma s)^{\frac{2}{1-\theta}}+\|u_0\|_\infty^2+(1+s)\|u_0\|_\infty\right] \quad \forall t \in [t_0, t_0+s],$$

which yields

$$\left|\nabla u(t_0+s)\right|^2 \leqslant C_0(\gamma s)^{\frac{2}{1-\theta}}+\frac{C_1}{s}\left[\|u_0\|_\infty^2+(1+s)\|u_0\|_\infty\right]. \tag{A.21}$$

Now, let $t \geqslant 2K\|u_0\|_\infty$. We have

$$\left|\nabla u(t+t_0+s)\right|^2 \leqslant C_0(\gamma s)^{\frac{2}{1-\theta}}+\frac{C_1}{s}\left[\left\|u(t)\right\|_\infty^2+(1+s)\left\|u(t)\right\|_\infty\right]$$

for every $t_0 \geqslant K\|u(t)\|_\infty$. Now, if $t \geqslant KCe^{-\lambda t}+e^{-\frac{\lambda}{2}t}$, then we are allowed to take $t_0 = t - e^{-\frac{\lambda}{2}t}$ (since $t_0 \geqslant K\|u(t)\|_\infty$) and $s = e^{-\frac{\lambda}{2}t}$, and we get

$$\left|\nabla u(2t)\right|^2 \leqslant \tilde{C}_0 e^{-\tilde{\lambda} t}, \tag{A.22}$$

for some $\tilde{\lambda} > 0$. On the other hand, if $t \leqslant KCe^{-\lambda t}+e^{-\frac{\lambda}{2}t}$, this means that t as well as $\|u_0\|_\infty$ are bounded; using (A.21) with $t_0 = K\|u_0\|_\infty$ and $s = t - t_0$, and using that $t - t_0 \geqslant K\|u_0\|_\infty$, we obtain (A.22) again. In any case, we have proved (A.14) for (possibly different) constants C, λ, K, only depending on f, Ω. □

References

[1] S. Anita, D. Tataru, Null controllability for the dissipative semilinear heat equation, Appl. Math. Optim. 46 (2–3) (2002) 97–105.

[2] V. Barbu, Exact controllability of the superlinear heat equation, Appl. Math. Optim. 42 (2000) 73–89.

[3] G. Barles, F. Da Lio, On the generalized Dirichlet problem for viscous Hamilton–Jacobi equations, J. Math. Pures Appl. (9) 83 (1) (2004) 53–75.

[4] G. Barles, A. Porretta, Uniqueness for unbounded solutions to stationary viscous Hamilton–Jacobi equations, Ann. Scuola Norm. Sup. Pisa 5 (2006) 107–136.

[5] G. Barles, A. Porretta, T. Tabet Tchamba, On the large time behavior of solutions of the Dirichlet problem for subquadratic viscous Hamilton–Jacobi equations, J. Math. Pures Appl. 94 (2010) 497–519.

[6] S. Benachour, S. Dabuleanu-Hapca, P. Laurencot, Decay estimates for a viscous Hamilton–Jacobi equation with homogeneous Dirichlet boundary conditions, Asymptot. Anal. 51 (3–4) (2007) 209–229.

[7] L. Boccardo, F. Murat, J.P. Puel, Existence results for some quasilinear parabolic equations, Nonlinear Anal. 13 (1989) 373–392.

[8] M. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1) (1992).

[9] J.I. Díaz, Approximate controllability for some nonlinear parabolic problems, in: System Modelling and Optimization, Compiègne, 1993, in: Lecture Notes in Control and Inform. Sci., vol. 197, Springer, London, 1994, pp. 128–143.

[10] A. Doubova, E. Fernández-Cara, M. González-Burgos, E. Zuazua, On the controllability of parabolic systems with a nonlinear term involving the state and the gradient, SIAM J. Control Optim. 41 (3) (2002) 798–819.

[11] Th. Duyckaerts, X. Zhang, E. Zuazua, On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials, Ann. Inst. H. Poincaré Analyse Nonlin. 25 (2008) 1–41.

[12] E. Fernández-Cara, S. Guerrero, Null controllability of the Burgers system with distributed controls, Systems Control Lett. 56 (2007) 366–372.

[13] E. Fernández-Cara, E. Zuazua, Null and approximate controllability for weakly blowing up semilinear heat equations, Ann. Inst. H. Poincaré Analyse Nonlin. 17 (5) (2000) 583–616.

[14] A. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes, vol. 34, Seoul National University, Korea, 1996.

[15] O.A. Ladyzenskaya, V.A. Solonnikov, N.N. Uraltzeva, Linear and Quasilinear Equations of Parabolic Type, Trans. Math. Monogr., vol. 23, 1967.

[16] J.M. Lasry, P.L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, Math. Ann. 283 (4) (1989) 583–630.

[17] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1995) 335–356.

[18] T. Leonori, Large solutions for a class of nonlinear elliptic equations with gradient terms, Adv. Nonlinear Stud. 7 (2) (2007) 237–269.

[19] P.L. Lions, Regularizing effects for first-order Hamilton Jacobi equations, Appl. Anal. 20 (3–4) (1985) 283–307.
[20] L. Orsina, M.M. Porzio, $L^\infty(Q)$-estimate and existence of solutions for some nonlinear parabolic equations, Boll. Un. Mat. Ital. B 6 (1992) 631–647.
[21] F. Rothe, Global Solutions of Reaction–Diffusion Systems, Springer-Verlag, 1984.
[22] J. Simon, Compact sets in $L^p(0, T; B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.
[23] P. Souplet, Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions, Differential Integral Equations 15 (2) (2002) 237–256.
[24] P. Souplet, Q.S. Zhang, Global solutions of inhomogeneous Hamilton–Jacobi equations, J. Anal. Math. 99 (2006) 355–396.
[25] T. Tabet Tchamba, Large time behavior of solutions of viscous Hamilton–Jacobi equations with superquadratic Hamiltonian, Asymptot. Anal. 66 (3–4) (2010) 161–186.
[26] E. Zuazua, Exact controllability for the semilinear wave equation in one space dimension, Ann. Inst. H. Poincaré Analyse Nonlin. 10 (1993) 109–129.