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Ann. I. H. Poincaré – AN 28 (2011) 965–981

ANNALES DE L'INSTITUT HENRI POINCARÉ
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Symmetry and nonexistence of positive solutions of elliptic equations and systems with Hardy terms

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Received 19 November 2010; accepted 15 July 2011

Available online 17 August 2011

Abstract

We prove some symmetry property for equations with Hardy terms in cones, without any assumption at infinity. We also show symmetry property and nonexistence of entire solutions of some elliptic systems with Hardy weights.

MSC: 35B06; 35B08; 35B09

Keywords: Hardy–Sobolev inequality; Caffarelli–Kohn–Nirenberg inequality; Lane–Emden system; Nonexistence

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, be a smooth domain and $\mathcal{D}^{1,2}(\Omega)$ denote as the completion of $C_c^\infty(\Omega)$, the set of smooth functions with compact support in Ω, under the norm $\|u\|_{\mathcal{D}^{1,2}(\Omega)} := (\int_\Omega |\nabla u|^2)^{1/2}$. The Hardy–Sobolev inequality [4,14] asserts that for $t \in [0,2]$ and $2^*(t) := \frac{2(n-t)}{n-2}$, there exists $C > 0$ such that for all $u \in \mathcal{D}^{1,2}(\Omega)$

$$C\left(\int_\Omega \frac{|u|^{2^*(t)}}{|x|^t}\,dx\right)^{\frac{2}{2^*(t)}} \leqslant \int_\Omega |\nabla u|^2\,dx. \tag{1}$$

The best constant of (1) is defined by

$$C_t(\Omega) := \inf\left\{\int_\Omega |\nabla u|^2\,dx \;\middle|\; u \in \mathcal{D}^{1,2}(\Omega),\ \int_\Omega \frac{|u|^{2^*(t)}}{|x|^t}\,dx = 1\right\}.$$

If $t = 0$, (1) becomes the classical Sobolev inequality. The best constant $C_0(\mathbb{R}^n)$ and extremal functions of Sobolev inequality have been obtained explicitly by Aubin [1] and Talenti [31]. Moreover $C_0(\Omega) = C_0(\mathbb{R}^n)$ for any Ω and $C_0(\Omega)$ is never attained unless $\mathrm{cap}(\Omega) = \mathbb{R}^n$ (see, e.g., [30]). If $t = 2$, (1) is the classical Hardy inequality which is known not to possess extremal functions.

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.07.003

The best constant $C_t(\Omega)$ for $0 < t < 2$ is delicate, which depends on the properties of Ω. In the entire case, $C_t(\mathbb{R}^n)$ was first computed in [16] and extremal functions were identified by Lieb in [21]. For a general domain Ω it was shown by Ghoussoub and Yuan in [14] that if 0 is in the interior of Ω, then $C_t(\Omega) = C_t(\mathbb{R}^n)$ and $C_t(\Omega)$ is achieved if $\Omega = \mathbb{R}^n$. However, things are different when 0 is on the boundary of Ω, which was first studied by H. Egnell [11]. Egnell considered open cones of the form $\mathcal{C} := \{x \in \mathbb{R}^n\colon\ x = r\theta,\ \theta \in \Sigma \text{ and } r > 0\}$ where Σ is a connected domain on the unit sphere S^{n-1} in $\mathbb{R}^n$, and proved that $C_t(\mathcal{C})$ is achieved for any $0 < t < 2$ even if $\overline{\mathcal{C}} \neq \mathbb{R}^n$. So $C_t(R^n_+)$ is achieved where $R^n_+ \triangleq \{x \in \mathbb{R}^n\colon\ x_n > 0\}$ is the upper half space. The upper half space, is of special interest, since it was identified in [12,17,18] as the limiting space after blow-up in the case where Ω is bounded and $\partial\Omega$ is smooth at 0. The curvature of the boundary at 0 then plays important roles. It was proved by Ghoussoub and Robert in [13] that $C_t(\Omega)$ $(0 < t < 2)$ is achieved if the mean curvature of $\partial\Omega$ at 0 is negative. Complementarily, due to Pohozaev identity, nonexistence occurs if Ω is star-shaped with respect to 0.

We consider rotationally symmetric cones Ω_a which are defined by

$$\Omega_a \triangleq \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^+\colon\ x_n > a|x'|,\ \text{where constant } a \geqslant 0\}. \tag{2}$$

By Egnell's theorem, $C_t(\Omega_a)$ is attained $\forall t \in (0, 2)$, i.e. the equation

$$\begin{cases} -\Delta u(x) = \dfrac{u^{2^*(t)-1}(x)}{|x|^t}, & \text{in } \Omega_a, \\ u \geqslant 0, & \text{in } \Omega_a, \\ u = 0, & \text{on } \partial\Omega_a \end{cases} \tag{3}$$

always has a least energy solution in $\mathcal{D}^{1,2}(\Omega_a)$ for any $0 < t < 2$. One natural question is whether all the solutions (not only the least energy ones) of Eq. (3) have corresponding symmetry. We give an affirmative answer in Theorem 1.1. In the following we use $\mathcal{D}^{1,2}_{loc}(\Omega)$ to denote the set of functions u which satisfy, on all compact set K of Ω, $u \in L^{2n/(n-2)}(K)$ and $\nabla u \in L^2(K)$.

Theorem 1.1. *If $u \in \mathcal{D}^{1,2}_{loc}(\overline{\Omega}_a)$ is a solution of* (3)*, then $u(x', x_n)$ is symmetric under rotations around the x_n axis. Namely, $u(x', x_n) = u(\tilde{x}', x_n)$ if $|x'| = |\tilde{x}'|$. Moreover $u(x', x_n) \leqslant u(\tilde{x}', x_n)$ if $|x'| \geqslant |\tilde{x}'|$.*

Remark 1.1. When $a = 0$, i.e. $\Omega_0 = \mathbb{R}^n_+$, the symmetry property was proved by Ghoussoub and Robert in [13] under the assumptions that $u \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{\mathbb{R}^n_+})$ and $\limsup_{|x|\to+\infty} |x|^{n-1}u(x) < \infty$. Theorem 1.1 does not make any assumption on u near infinity.

A generalization of Hardy–Sobolev inequality (1) is the following Caffarelli–Kohn–Nirenberg inequality [4], which asserts that for all $w \in C^\infty_c(\mathbb{R}^n)$, there is a constant $C > 0$ such that

$$C\Bigl(\int_{\mathbb{R}^n} |x|^{-\beta q}|w|^q\,dx\Bigr)^{\frac{2}{q}} \leqslant \int_{\mathbb{R}^n} |x|^{-2\alpha}|\nabla w|^2\,dx \tag{4}$$

where

$$-\infty < \alpha < \frac{n-2}{2}, \qquad 0 \leqslant \beta - \alpha \leqslant 1 \quad \text{and} \quad q = \frac{2n}{n-2+2(\beta-\alpha)}.$$

The best constants and minimizers to Caffarelli–Kohn–Nirenberg inequality (4) have been extensively studied. We refer to [7,8,17,18] and references therein. The minimizers $w(x)$ of (4) are closely related (see, e.g. [8]) to the least energy solution of the following equation:

$$\begin{cases} -\Delta u(x) = b\dfrac{u^{2^*(t)-1}(x)}{|x|^t} + \dfrac{u^{2^*(s)-1}(x)}{|x|^s}, & \text{in } \Omega, \\ u \geqslant 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where b is a constant and $0 \leqslant s < t \leqslant 2$. When $\Omega = \Omega_a$ it has been proved by Bartsch, Peng and Zhang in [2] that Eq. (5) always has a least energy solution if $0 < s < t = 2$ and $b < (n-2)^2/4$. The existence of entire solutions of (5), i.e. $\Omega = \mathbb{R}^n_+$, has been proved by Musina in [24] when $s = 0$, $t = 2$, $0 < b < (n-2)^2/4$; by Hsia, Lin and Wadade in [17] when $s = 0 < t < 2$, $b > 0$; and by Li and Lin in [18] when $0 < s < t < 2$, $b \in \mathbb{R}$.

Our proof of Theorem 1.1 can also be applied to obtain symmetry property of solutions of Eq. (5) with $b > 0$.

Theorem 1.2. *If $u \in \mathcal{D}^{1,2}_{loc}(\overline{\Omega}_a)$ is a solution of* (5) *in Ω_a with $b > 0$, then $u(x', x_n)$ is symmetric under rotations around the x_n axis. Namely, $u(x', x_n) = u(\tilde{x}', x_n)$ if $|x'| = |\tilde{x}'|$. Moreover $u(x', x_n) \leqslant u(\tilde{x}', x_n)$ if $|x'| \geqslant |\tilde{x}'|$.*

Remark 1.2. When $\Omega = \mathbb{R}^n_+$ and under the assumption that $u \in H^1_0(\Omega)$, the completion of $C^\infty_c(\Omega)$ under the norm $\|u\|_{H^1(\Omega)} := (\int_\Omega |\nabla u|^2 + u^2)^{1/2}$, the symmetry property was obtained in [17] in the case that $s = 0 < t < 2$, $b > 0$, and in [8] in the case that $0 < s < t = 2$, $b < (n-2)^2/4$ (can be negative). Theorem 1.2 does not make any assumption on u near infinity, but with the assumption that $b > 0$.

We extend the above Eq. (3) to the following Lane–Emden systems with Hardy weights:

$$\begin{cases} -\Delta u(x) = |x|^{-s} v^p(x), & x \in \mathbb{R}^n, \\ -\Delta v(x) = |x|^{-t} u^q(x), & x \in \mathbb{R}^n. \end{cases} \tag{6}$$

We first recall the case $s = t = 0$ which has been studied by many authors. It has been conjectured that, see for example de Figueiredo and Felmer [9], the following critical hyperbola:

$$\frac{n}{p+1} + \frac{n}{q+1} = n - 2, \quad p > 0,\ q > 0 \tag{7}$$

is the dividing curve for existence and nonexistence of solutions of system (6). This conjecture was verified for positive radial solutions (see, e.g. Mitidieri [23], Serrin and Zou [27,28]). de Figueiredo and Felmer [9] proved that system (6) has no positive solutions provided that

$$0 < p, q \leqslant \frac{n+2}{n-2}, \qquad (p, q) \neq \left(\frac{n+2}{n-2}, \frac{n+2}{n-2}\right).$$

See also [23] and [26] for other nonexistence results.

Recently the general case for $s \neq 0$ and/or $t \neq 0$ has been investigated independently by de Figueiredo et al. [10] and Liu and Yang [22], where it is indicated that the dividing curve between existence and nonexistence is given by the following "weighted" critical hyperbola:

$$\frac{n-s}{p+1} + \frac{n-t}{q+1} = n - 2, \quad p > 0,\ q > 0. \tag{8}$$

Both papers considered the Dirichlet problem of system (6) in a bounded smooth domain, via an approach of fractional Sobolev spaces. See also [5], where nonexistence of solutions and existence of symmetric solutions in balls were studied.

We say $u \geqslant 0$ and $v \geqslant 0$ are weak solutions of system (6) if $(u, v) \in \mathcal{D}^{1,2}_{loc}(\mathbb{R}^n) \times \mathcal{D}^{1,2}_{loc}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \nabla u \nabla \phi = \int_{\mathbb{R}^n} |x|^{-s} v^p \phi, \quad \forall \phi \in C^\infty_c(\mathbb{R}^n),$$

$$\int_{\mathbb{R}^n} \nabla v \nabla \phi = \int_{\mathbb{R}^n} |x|^{-t} u^q \phi, \quad \forall \phi \in C^\infty_c(\mathbb{R}^n).$$

In the rest of the paper we will always assume system (6) satisfies

$$0 \leqslant s, t < 2, \qquad (s, t) \neq (0, 0), \qquad 0 < p \leqslant 2^*(s) - 1 \quad \text{and} \quad 0 < q \leqslant 2^*(t) - 1. \tag{9}$$

Theorem 1.3. *If $u, v \in \mathcal{D}^{1,2}_{loc}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $u, v \geqslant 0$ are weak solutions of system* (6) *then u, v are radial. Namely, $u(x_1) = u(x_2)$, $v(x_1) = v(x_2)$ if $|x_1| = |x_2|$. Moreover $u(x_1) \leqslant u(x_2)$, $v(x_1) \leqslant v(x_2)$ if $|x_1| \geqslant |x_2|$.*

Remark 1.3. If $(p, q) \neq (2^*(s) - 1, 2^*(t) - 1)$ and $u, v \in \mathcal{D}^{1,2}_{loc}(\mathbb{R}^n)$ are weak solutions then $u, v \in C^\alpha(\mathbb{R}^n)$ for some $\alpha > 0$. See Appendix A for details.

Consequently, we have the following nonexistence result.

Theorem 1.4. *If $u, v \geqslant 0$ are weak solutions of system* (6)*, then $u \equiv v \equiv 0$ provided that*:

$$0 < p, q < \frac{n+2-s-t}{n-2+|s-t|}. \tag{10}$$

Remark 1.4. Under condition (10), (p, q) is below "weighted" hyperbola (8).

We can also obtain symmetry property for system (6) in the upper half space with Dirichlet boundary condition. It can be viewed as, like before, the limiting spaces after blow-up in the case where the boundaries of domains are smooth at 0.

$$\begin{cases} -\Delta u(x) = |x|^{-s} v^p(x), & \text{in } \mathbb{R}^n_+, \\ -\Delta v(x) = |x|^{-t} u^q(x), & \text{in } \mathbb{R}^n_+, \\ u, v > 0, & \text{in } \mathbb{R}^n_+, \\ u = v = 0, & \text{on } \partial\mathbb{R}^n_+. \end{cases} \tag{11}$$

Theorem 1.5. *If $u, v \in \mathcal{D}^{1,2}_{loc}(\overline{\mathbb{R}^n_+})$ satisfy* (11) *with $p, q > 1$, then $u(x', x_n)$ and $v(x', x_n)$ are symmetric under rotations around the x_n axis. Moreover, $u(|x'|, x_n) \leqslant u(|\tilde{x}'|, x_n)$ and $v(|x'|, x_n) \leqslant v(|\tilde{x}'|, x_n)$ if $|x'| \geqslant |\tilde{x}'|$.*

As remarked in [5], systems of type (6) are related to the double weighted Hardy–Littlewood–Sobolev inequality (see e.g. Stein and Weiss [29] and Lieb [21]).

The proofs of our theorems use the method of moving spheres, a variant of the method of moving planes which are developed through the works of Alexandrov, Serrin [25] and Gidas, Ni and Nirenberg [15]. We make use of ideas in the proof of Liouville-type theorems given in [20,19,6], to fully exploit the conformal invariance of the problems. We also make use of the "narrow domain idea" from Berestycki and Nirenberg [3].

2. Equations with Hardy terms

2.1. *Proof of Theorem 1.1 if $\Omega_a = \mathbb{R}^n_+$*

We first consider the case when $a = 0$, i.e. $\Omega_a = \mathbb{R}^n_+$. We make a remark about regularity of u. By standard elliptic estimates,

$$u \in C^\infty\big(\overline{\mathbb{R}^n_+}\backslash 0\big).$$

If $u(x_0) = 0$ at some point x_0 in $\mathbb{R}^n_+$, then u is identically zero, by the strong maximum principle. Hence we always assume that

$$u(x) > 0.$$

For any $\bar{x}' \in \mathbb{R}^{n-1}$, define

$$\bar{x} = \big(\bar{x}', -1\big).$$

Let

$$u_{\bar{x},\lambda}(x) := \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n-2} u\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right) \tag{12}$$

be the Kelvin transformation of u with respect to the ball $B(\bar{x}, \lambda)$ with center $\bar{x}$ and radius λ. By direct computations, we have for $x \in B^+(\bar{x}, \lambda) \triangleq B(\bar{x}, \lambda) \cap \mathbb{R}^n_+$,

$$-\Delta u_{\bar{x},\lambda}(x) = \left(\frac{\lambda^2|x|}{|x-\bar{x}|^2|\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}|}\right)^t \frac{u^{2^*(t)-1}_{\bar{x},\lambda}(x)}{|x|^t}.$$

We start with a lemma, which is similar to Lemma 4 in [6].

Lemma 2.1. $\forall\lambda \in (1, |\bar{x}|)$, *we have*

$$\frac{\lambda^2|x|}{|x-\bar{x}|^2|\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}|} \geqslant 1 \tag{13}$$

for all $x \in B^+(\bar{x}, \lambda)$.

Proof. The proof is the same as that of Lemma 4 in [6]. We include it here for convenience. (13) is equivalent to

$$\lambda^2|x| \geqslant \left|\lambda^2 x + \left(|x-\bar{x}|^2 - \lambda^2\right)\bar{x}\right|,$$

which is equivalent to (by taking square on both sides)

$$-2\lambda^2 x \cdot \bar{x} \leqslant \left(|x-\bar{x}|^2 - \lambda^2\right)|\bar{x}|^2,$$

which is equivalent to

$$-2\lambda^2 (x-\bar{x}) \cdot \bar{x} \leqslant \left(|x-\bar{x}|^2 + \lambda^2\right)|\bar{x}|^2.$$

The last inequality holds since

$$-2\lambda^2 (x-\bar{x}) \cdot \bar{x} \leqslant 2\lambda^2|\bar{x}||x-\bar{x}| \leqslant 2|\bar{x}|^2\lambda|x-\bar{x}| \leqslant \left(|x-\bar{x}|^2 + \lambda^2\right)|\bar{x}|^2. \qquad \square$$

Lemma 2.2. $\forall|\bar{x}'| > 2$, *there exists* $\lambda_0(\bar{x}) > 1$ *such that for any* $\lambda \in (1, \lambda_0(\bar{x}))$,

$$u_{\bar{x},\lambda}(x) \geqslant u(x), \quad \forall x \in B^+(\bar{x}, \lambda).$$

Proof. By Lemma 2.1,

$$-\Delta u_{\bar{x},\lambda}(x) \geqslant \frac{u_{\bar{x},\lambda}^{2^*(t)-1}(x)}{|x|^t}, \quad \forall x \in B^+(\bar{x}, \lambda).$$

Thus

$$-\Delta\left(u_{\bar{x},\lambda}(x) - u\right) \geqslant \frac{u_{\bar{x},\lambda}^{2^*(t)-1}(x) - u^{2^*(t)-1}(x)}{|x|^t}, \quad \forall x \in B^+(\bar{x}, \lambda). \tag{14}$$

Denote

$$w_\lambda = u_{\bar{x},\lambda}(x) - u, \qquad w_\lambda^- = \max\{0, -w_\lambda\}.$$

We first require that $1 < \lambda_0(\bar{x}) < \sqrt{2}$, then we have $|x| > 1$, $\forall x \in B^+(\bar{x}, \lambda)$. In the following part, as well as in the proof of Lemma 2.3 below, we make use of the "narrow domain idea" from Berestycki and Nirenberg [3].

Multiplying both sides of (14) by w_λ^- then integrating on $B^+(\bar{x}, \lambda)$, we have, using $w_\lambda^- = 0$ on $\partial(B^+(\bar{x}, \lambda))$ and the mean value theorem,

$$\begin{aligned}
\int\limits_{B^+(\bar{x},\lambda)} \left|\nabla w_\lambda^-\right|^2 dx &\leqslant \int\limits_{B^+(\bar{x},\lambda)} \frac{u^{2^*(t)-1}(x) - u_{\bar{x},\lambda}^{2^*(t)-1}(x)}{|x|^t} w_\lambda^- \\
&\leqslant \frac{n+2}{n-2} \int\limits_{B^+(\bar{x},\lambda)} u^{2^*(t)-2}\left(w_\lambda^-\right)^2 \\
&\leqslant C(n, \bar{x})\left|B^+(\bar{x}, \lambda)\right|^{\frac{2}{n}} \left\|w_\lambda^-\right\|^2_{L^{\frac{2n}{n-2}}(B^+(\bar{x},\lambda))} \\
&\leqslant C(n, \bar{x})\left|B^+(\bar{x}, \lambda)\right|^{\frac{2}{n}} \int\limits_{B^+(\bar{x},\lambda)} \left|\nabla w_\lambda^-\right|^2 dx,
\end{aligned}$$

where $C(n,\bar{x})$ denotes various constants depending only on n and $\bar{x}$. Now we can choose $\lambda_0(\bar{x}) > 1$ but very close to 1, then $C(n,\bar{x})|B^+(\bar{x},\lambda)|^{\frac{2}{n}}$ is small, and we have

$$\int_{B^+(\bar{x},\lambda)} \left|\nabla w_\lambda^-\right|^2 dx < \frac{1}{2} \int_{B^+(\bar{x},\lambda)} \left|\nabla w_\lambda^-\right|^2 dx.$$

This implies $\nabla w_\lambda^- = 0$ in $B^+(\bar{x},\lambda)$. Since $w_\lambda^- = 0$ on $\partial(B^+(\bar{x},\lambda))$, $w_\lambda^- = 0$ in $B^+(\bar{x},\lambda)$. $\square$

Define

$$\bar{\lambda}(\bar{x}) := \sup\bigl\{\mu \bigm| 1 < \mu < |\bar{x}|, \text{ and } u_{\bar{x},\lambda}(x) \geqslant u(x),\ \forall x \in B^+(\bar{x},\lambda),\ \forall 1 < \lambda < \mu\bigr\}.$$

By Lemma 2.2, $\bar{\lambda}(\bar{x})$ is well defined for all $|\bar{x}'| > 2$. Moreover $1 < \bar{\lambda}(\bar{x}) \leqslant |\bar{x}|$.

Lemma 2.3. $\forall |\bar{x}'| > 2$, $\bar{\lambda}(\bar{x}) = |\bar{x}|$. *Namely,*

$$u_{\bar{x},\lambda}(x) \geqslant u(x), \quad \forall x \in B^+(\bar{x},\lambda),\ \forall 0 < \lambda \leqslant |\bar{x}|. \tag{15}$$

Proof. We argue by contradiction. Suppose that $\bar{\lambda} = \bar{\lambda}(\bar{x}) < |\bar{x}|$ for some $\bar{x}$. Then

$$u_{\bar{x},\bar{\lambda}}(x) \geqslant u(x), \quad \forall x \in B^+(\bar{x},\bar{\lambda}).$$

Since $u_{\bar{x},\bar{\lambda}}(x) > u(x)$ on $B(\bar{x},\bar{\lambda}) \cap \partial\mathbb{R}^n_+$, we have, by the strong maximum principle,

$$u_{\bar{x},\bar{\lambda}}(x) > u(x), \quad \forall x \in B^+(\bar{x},\bar{\lambda}).$$

For $\delta > 0$ small, which will be fixed later, let

$$K := \bigl\{x \in B^+(\bar{x},\bar{\lambda}) \bigm| \mathit{dist}\bigl(x, \partial\bigl(B^+(\bar{x},\bar{\lambda})\bigr)\bigr) \geqslant \delta\bigr\}.$$

Then

$$b := \min_K w_{\bar{\lambda}} > 0.$$

Consider $\bar{\lambda} < \lambda < \bar{\lambda} + \epsilon < (\bar{\lambda} + |\bar{x}|)/2$, where the value of $\epsilon = \epsilon(\delta) < \delta$ is chosen so that

$$w_\lambda > \frac{b}{2} \quad \text{on } K,\ \forall \bar{\lambda} < \lambda < \bar{\lambda} + \epsilon < \bigl(\bar{\lambda} + |\bar{x}|\bigr)/2. \tag{16}$$

We use the "narrow domain techniques" again. Multiplying w_λ^- to (14) and using integration by parts on $(B^+(\bar{x},\lambda))\backslash K$, then we have

$$\begin{aligned}
\int_{(B^+(\bar{x},\lambda))\backslash K} \left|\nabla w_\lambda^-\right|^2 dx &\leqslant C(n,\bar{\lambda},\bar{x}) \int_{B^+(\bar{x},\lambda)} u^{2^*(t)-2}\bigl(w_\lambda^-\bigr)^2 \\
&\leqslant C(n,\bar{\lambda},\bar{x})\bigl|\bigl(B^+(\bar{x},\lambda)\bigr)\backslash K\bigr|^{\frac{2}{n}} \bigl\|w_\lambda^-\bigr\|^2_{L^{\frac{2n}{n-2}}(B^+(\bar{x},\lambda)\backslash K)} \\
&\leqslant C(n,\bar{\lambda},\bar{x})\bigl|\bigl(B^+(\bar{x},\lambda)\bigr)\backslash K\bigr|^{\frac{2}{n}} \int_{B^+(\bar{x},\lambda)\backslash K} \left|\nabla w_\lambda^-\right|^2 dx,
\end{aligned}$$

where $C(n,\bar{\lambda},\bar{x})$ denotes various constants depending only on n, $\bar{\lambda}$ and $\bar{x}$. Now we can fix the value of δ so that $C(n,\bar{\lambda},\bar{x})|(B^+(\bar{x},\lambda))\backslash K|^{\frac{2}{n}} < \frac{1}{2}$. Then we obtain, as before, $w_\lambda^- = 0$ on $(B^+(\bar{x},\lambda))\backslash K$, i.e.

$$w_\lambda \geqslant 0, \quad \forall x \in \bigl(B^+(\bar{x},\lambda)\bigr)\backslash K,\ \forall \bar{\lambda} < \lambda < \bar{\lambda} + \epsilon.$$

This and (16) contradict the definition of $\bar{\lambda}$. $\square$

Proof of Theorem 1.1. For any $x_1 < 0 < \bar{x}_1$, we set $\bar{x} = (\bar{x}_1, 0, \dots, 0, -1)$ and $\lambda = |\bar{x}| = \sqrt{|\bar{x}_1|^2 + 1}$. (15) with $x = (x_1, 0, \dots, 0, x_n)$ leads to, after sending $\bar{x}_1 \to \infty$,

$$u(-x_1, 0, \dots, 0, x_n) \geqslant u(x_1, 0, \dots, 0, x_n), \quad \forall x_n > 0.$$

By the symmetry of the equation, $u_O(x) := u(Ox', x_n)$ satisfies the same equation for any orthogonal matrix $O \in O(n-1)$, so we have

$$u_O(-x_1, 0, \dots, 0, x_n) \geqslant u_O(x_1, 0, \dots, 0, x_n), \quad \forall x_n > 0,$$

which implies u is symmetric under rotations around the x_n axis.

For any $0 < a < x_1 < \bar{x}_1$, we set $\bar{x} = (\bar{x}_1, 0, \dots, 0, -1)$ and $\lambda = \sqrt{|\bar{x}_1 - a|^2 + 1}$. (15) with $x = (x_1, 0, \dots, 0, x_n)$ leads to, after sending $\bar{x}_1 \to \infty$,

$$u(2a - x_1, 0, \dots, 0, x_n) \geqslant u(x_1, 0, \dots, 0, x_n), \quad \forall x_n > 0.$$

This implies $u(|x'|, x_n) \leqslant u(|\tilde{x}'|, x_n)$ if $|x'| \geqslant |\tilde{x}'|$. Theorem 1.1 is proved when $\Omega_a = \mathbb{R}^n_+$. □

2.2. The case $\Omega_a \neq \mathbb{R}^n_+$

We do a small modification of the method used in Section 2.1. For any $0 \neq \bar{x}' \in \mathbb{R}^{n-1}$, define

$$\bar{x} = (\bar{x}', 0).$$

Note that $\bar{x} \notin \Omega_a$. Let $u_{\bar{x},\lambda}$ be the Kelvin transformation (12) of u with respect to the ball $B(\bar{x}, \lambda)$ with center $\bar{x}$ and radius λ.

Lemma 2.4. *If $\lambda \leqslant |\bar{x}'|$ and $x = (x', x_n) \in \Omega_a \cap B(\bar{x}, \lambda)$, then*

$$\bar{x} + \frac{\lambda^2 (x - \bar{x})}{|x - \bar{x}|^2} \in \Omega_a. \tag{17}$$

Proof. (17) is equivalent to

$$\lambda^2 x_n > a\left|\lambda^2 x' + \left(|x - \bar{x}|^2 - \lambda^2\right)\bar{x}'\right|,$$

which is equivalent to (by taking square on both sides)

$$\lambda^4 x_n^2 > a^2\left(\lambda^4 |x'|^2 + 2\lambda^2\left(|x - \bar{x}|^2 - \lambda^2\right)x' \cdot \bar{x}' + \left(|x - \bar{x}|^2 - \lambda^2\right)^2 |\bar{x}'|^2\right).$$

So it suffices to show

$$-2\lambda^2\left(x' - \bar{x}'\right) \cdot \bar{x}' \leqslant \left(|x' - \bar{x}'|^2 + \lambda^2\right)|\bar{x}'|^2.$$

The last inequality holds since

$$-2\lambda^2\left(x' - \bar{x}'\right) \cdot \bar{x}' \leqslant 2\lambda^2 |\bar{x}'||x' - \bar{x}'| \leqslant 2|\bar{x}'|^2 \lambda |x' - \bar{x}'| \leqslant \left(|x' - \bar{x}'|^2 + \lambda^2\right)|\bar{x}'|^2. \qquad \square$$

Proof of Theorem 1.1. By Lemma 2.4, $u_{\bar{x},\lambda}(x)$ is well defined in Ω_a. Thus we can run exactly the same procedure as that in Section 2.1, replacing $B^+(x, \lambda)$ by $B(x, \lambda) \cap \Omega_a$, to get

$$u_{\bar{x},\lambda}(x) \geqslant u(x), \quad \forall x \in B(\bar{x}, \lambda) \cap \Omega_a,\ \forall 0 < \lambda \leqslant |\bar{x}|.$$

This implies, as before, that $u(|x'|, x_n) = u(|\tilde{x}'|, x_n)$ if $|x'| = |\tilde{x}'|$, and $u(|x'|, x_n) \leqslant u(|\tilde{x}'|, x_n)$ if $|x'| \geqslant |\tilde{x}'|$. Theorem 1.1 is proved. □

2.3. Other symmetric domains

Another symmetric domain for Eq. (3) other than cones would be the unit upper half cylinder:

$$S := \{(x', x_n \in \mathbb{R}^n): |x'| < 1,\ x_n > 0\}.$$

If $0 \leqslant u \in \mathcal{D}^{1,2}_{loc}(\bar{S})$ is a solution of Eq. (3) in S we can also prove

Proposition 2.1. *u is symmetric under rotations around x_n axis.*

Proof. In this case the moving sphere method is not suitable since the Kelvin transformation of S with respect to large balls will no longer stay in S itself. Fortunately we can apply one version of moving plan method. As before it suffices to show

$$u(x_1, \dots, x_{n-1}, x_n) = u(x_1, \dots, -x_{n-1}, x_n). \tag{18}$$

We choose moving hyperplanes like the following:

$$l_k = \{(x_1, \dots, x_{n-1}, x_n) \mid x_i \in \mathbb{R},\ i = 1, \dots, n-2;\ x_n + 1 = k\, x_{n-1}\}.$$

l_k passes through $(0, \dots, 0, -1)$ but never passes 0. Denote

$$\Sigma_k = \{x \in S:\ x_n + 1 < k\, x_{n-1}\}.$$

When $k > 1$, $\Sigma_k \neq \emptyset$ and it is bounded. For any $x \in \Sigma_k$, let x^{l_k} be the reflection point of x with respect to l_k and define $u_{l_k}(x) = u(x^{l_k})$. Direct computations yield that

$$-\Delta u_{l_k} \geqslant \frac{u_{l_k}^{2^*(t)-1}}{|x|^t}, \quad \forall x \in \Sigma_k.$$

Then the proof of Theorem 1.1 would be applied to prove (18). □

2.4. Scalar equation with multiple Hardy terms

Proof of Theorem 1.2. If we examine the proof of Theorem 1.1, we note that the number of Hardy terms with positive coefficients does not interfere with the moving sphere method and "narrow domain techniques". Actually the same proof works for the following equation with $b_i \geqslant 0$, $0 \leqslant t_i \leqslant 2$, $i = 1, \dots, m$:

$$\begin{cases} -\Delta u = \sum_{i=1}^{m} b_i |x|^{-t_i} u^{2^*(t_i)-1} + |x|^{-s} u^{2^*(s)-1}, & \text{in } \Omega_a, \\ u \geqslant 0, & \text{in } \Omega_a, \\ u = 0, & \text{on } \partial\Omega_a \end{cases} \tag{19}$$

for any positive integer m provided that it admits a solution. □

3. Lane–Emden systems with Hardy weights

3.1. Radial symmetry

In this section we will prove Theorem 1.3. First by standard elliptic regularity theory,

$$u, v \in C^\infty(\mathbb{R}^n \backslash \{0\}).$$

Recall that in Theorem 1.3 we assume

$$u, v \in C(\mathbb{R}^n).$$

Suppose $u(x_0) = 0$ for some point $x_0 \in \mathbb{R}^n$. If $x_0 \neq 0$ then by the strong maximum principle and continuity, $u(x) \equiv 0$ in $\mathbb{R}^n$. If $x_0 = 0$ and $u(x) > 0$, $\forall x \neq 0$, noting that $-\Delta u(x) \geqslant 0$ in $\mathbb{R}^n \setminus \{0\}$ and $\{0\}$ has zero (Newtonian) capacity, we have

$$u(0) = \liminf_{x \to 0} u(x) > 0.$$

So $u(x) > 0$ or $u(x) \equiv 0$. In view of system (6) $v \equiv 0$ if $u \equiv 0$. Thus either $u \equiv v \equiv 0$ or $u, v > 0$ in $\mathbb{R}^n$. Hence we always assume that

$$u, v > 0 \quad \text{in } \mathbb{R}^n.$$

For $0 \neq \bar{x} \in \mathbb{R}^n$, we let $u_{\bar{x},\lambda}$ and $v_{\bar{x},\lambda}$ be the Kelvin transformation (12) of u and v with respect to the ball $B(\bar{x}, \lambda)$.

Lemma 3.1. $\forall \lambda \in (0, |\bar{x}|)$, *we have*

$$\frac{\lambda^2 |x|}{|x - \bar{x}|^2 |\bar{x} + \frac{\lambda^2 (x - \bar{x})}{|x - \bar{x}|^2}|} \leqslant 1 \tag{20}$$

for all $x \in \mathbb{R}^n \setminus B(\bar{x}, \lambda)$.

The proof of Lemma 3.1 is the same as that of Lemma 2.1.

Lemma 3.2. *For every* $\bar{x} \in \mathbb{R}^n \setminus \{0\}$, *there exists* $\lambda_0(\bar{x}) > 0$ *such that for all* $0 < \lambda < \lambda_0(\bar{x})$, *we have* $u_{\bar{x},\lambda}(x) \leqslant u(x)$, $v_{\bar{x},\lambda}(x) \leqslant v(x)$, $\forall |x - \bar{x}| > \lambda$.

Proof. Since u, v are super-harmonic, by the maximum principle

$$\liminf_{|x| \to \infty} \big(|x|^{n-2} u(x)\big) > 0, \qquad \liminf_{|x| \to \infty} \big(|x|^{n-2} v(x)\big) > 0.$$

Noting that u and v are smooth near $\bar{x}$ and continuous in the whole space, the rest of the proof is the same as that of Lemma 2.1 in [19]. $\square$

Define

$$\bar{\lambda}(\bar{x}) := \sup\big\{\mu \;\big|\; 0 < \mu < |\bar{x}|, \text{ and } u_{\bar{x},\lambda}(x) \leqslant u(x),\ v_{\bar{x},\lambda}(x) \leqslant v(x) \text{ for all } |x - \bar{x}| \geqslant \lambda,\ 0 < \lambda < \mu\big\}.$$

By Lemma 3.2, $\bar{\lambda}(\bar{x})$ is well defined for all $\bar{x} \neq 0$. Moreover $0 < \bar{\lambda}(\bar{x}) \leqslant |\bar{x}|$.

Lemma 3.3. *For all* $\bar{x} \neq 0$, $\bar{\lambda}(\bar{x}) = |\bar{x}|$. *Namely,*

$$u_{\bar{x},\lambda}(x) \leqslant u(x), \qquad v_{\bar{x},\lambda}(x) \leqslant v(x), \quad \forall x \in \mathbb{R}^n \setminus B(\bar{x}, \lambda),\ \forall 0 < \lambda \leqslant |\bar{x}|. \tag{21}$$

Proof. We prove by contradiction arguments. Suppose the contrary, that there exists $\bar{x} \in \mathbb{R}^n \setminus \{0\}$ such that $0 < \bar{\lambda}(\bar{x}) < |\bar{x}|$. Without loss of generality we assume $s > 0$. Using Lemma 3.2 and $p \leqslant 2^*(s) - 1$ we have, for any $\lambda \in (0, |\bar{x}|)$,

$$-\Delta u_{\bar{x},\lambda}(x) \leqslant \frac{1}{|x|^s} u_{\bar{x},\lambda}^p, \quad \forall |x - \bar{x}| > \lambda.$$

Indeed, for $|x - \bar{x}| > \lambda$,

$$\begin{aligned} -\Delta u_{\bar{x},\lambda}(x) &= \left(\frac{\lambda}{|x - \bar{x}|}\right)^{n+2} \left(-\Delta u\left(\bar{x} + \frac{\lambda^2 (x - \bar{x})}{|x - \bar{x}|^2}\right)\right) \\ &= \left(\frac{\lambda}{|x - \bar{x}|}\right)^{n+2} \frac{1}{|\bar{x} + \frac{\lambda^2 (x - \bar{x})}{|x - \bar{x}|^2}|^s} v^p\left(\bar{x} + \frac{\lambda^2 (x - \bar{x})}{|x - \bar{x}|^2}\right) \end{aligned}$$

$$= \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n+2-p(n-2)} \frac{|x|^s}{|\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}|^s} \frac{1}{|x|^s} v^p_{\bar{x},\lambda}(x)$$
$$\leqslant \left(\frac{\lambda}{|x-\bar{x}|}\right)^{2s} \frac{|x|^s}{|\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}|^s} \frac{1}{|x|^s} v^p_{\bar{x},\lambda}(x)$$
$$\leqslant \frac{1}{|x|^s} v^p_{\bar{x},\lambda}(x)$$

where $p \leqslant 2^*(s)-1$ is used in the first inequality and Lemma 3.1 is used in the second inequality.

By the definition of $\bar{\lambda}(\bar{x})$,

$$v(x) - v_{\bar{x},\bar{\lambda}(\bar{x})}(x) \geqslant 0, \quad \forall |x-\bar{x}| > \bar{\lambda}(\bar{x}).$$

Hence

$$-\Delta\big(u(x) - u_{\bar{x},\bar{\lambda}(\bar{x})}(x)\big) \geqslant \frac{1}{|x|^s}\big(v^p(x) - v^p_{\bar{x},\bar{\lambda}(\bar{x})}(x)\big) \geqslant 0,$$

when $x \neq 0$ and $|x-\bar{x}| > \bar{\lambda}(\bar{x})$.

Since $\bar{\lambda}(\bar{x}) < |\bar{x}|$, all u, $u_{\bar{x},\bar{\lambda}(\bar{x})}$, v and $v_{\bar{x},\bar{\lambda}(\bar{x})}$ are smooth near $\partial B(\bar{x},\bar{\lambda}(\bar{x}))$. If there exists $x_0 \neq 0$, $|x_0-\bar{x}| > \bar{\lambda}(\bar{x})$ such that $u(x_0) = u_{\bar{x},\lambda(\bar{x})}(x_0)$, then by the strong maximum principle and continuity $u(x) = u_{\bar{x},\bar{\lambda}(\bar{x})}(x)$ for all $|x-\bar{x}| \geqslant \bar{\lambda}(\bar{x})$. Hence for $x \neq 0$ and $|x-\bar{x}| > \bar{\lambda}(\bar{x})$,

$$\frac{1}{|x|^s} v^p(x) = -\Delta u(x) = -\Delta u_{\bar{x},\lambda}(x) \leqslant \left(\frac{\lambda}{|x-\bar{x}|}\right)^{2s} \frac{|x|^s}{|\bar{x}+\frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}|^s} \frac{1}{|x|^s} v^p_{\bar{x},\lambda}(x).$$

By the proof of Lemma 3.1, there exists y with $|y-\bar{x}| > \bar{\lambda}(\bar{x})$ such that $v(y) < v_{\bar{x},\bar{\lambda}(\bar{x})}(y)$. But this contradicts the definition of $\bar{\lambda}(\bar{x})$. So we have

$$u(x) > u_{\bar{x},\bar{\lambda}(\bar{x})}(x), \quad \forall |x-\bar{x}| > \bar{\lambda}(\bar{x}) \text{ with } x \neq 0.$$

Since $-\Delta(u(x) - u_{\bar{x},\bar{\lambda}(\bar{x})}(x)) \geqslant 0$ in $B^c(\bar{x},\bar{\lambda}(\bar{x}))\backslash\{0\}$ and $\{0\}$ has zero (Newtonian) capacity, we have

$$u(x) > u_{\bar{x},\bar{\lambda}(\bar{x})}(x), \quad \forall |x-\bar{x}| > \bar{\lambda}(\bar{x}).$$

By Hopf lemma and the compactness of $\partial B(\bar{x},\bar{\lambda}(\bar{x}))$ we have

$$\frac{\partial}{\partial\nu}(u - u_{\bar{x},\bar{\lambda}(\bar{x})})|_{\partial B(\bar{x},\bar{\lambda}(\bar{x}))} \geqslant b > 0$$

where ν denotes the out normal of $\partial B(\bar{x},\bar{\lambda}(\bar{x}))$ and b is a positive constant. With noting that $u(x)$ is uniformly continuous in any compact set of $\mathbb{R}^n$, we can show, by exactly the same proof of Lemma 2.2 in [19], that there exists $\epsilon_1 > 0$ such that

$$u(x) - u_{\bar{x},\lambda}(x) > 0, \quad \forall \bar{\lambda}(\bar{x}) \leqslant \lambda < \bar{\lambda}(\bar{x}) + \epsilon_1,\ |x-\bar{x}| \geqslant \lambda. \tag{22}$$

Similarly, there exists $\epsilon_2 > 0$ such that

$$v(x) - v_{\bar{x},\lambda}(x) > 0, \quad \forall \bar{\lambda}(\bar{x}) \leqslant \lambda < \bar{\lambda}(\bar{x}) + \epsilon_2,\ |x-\bar{x}| \geqslant \lambda. \tag{23}$$

Estimates (22) and (23) violate the definition of $\bar{\lambda}(\bar{x})$. □

Proof of Theorem 1.3. Denote $0_{n-1} = (0,\dots,0) \in \mathbb{R}^{n-1}$. For $x_1 < 0 < \bar{x}_1$, we set $\bar{x} = (\bar{x}_1, 0_{n-1})$, $\lambda = |\bar{x}| = \bar{x}_1$. (21) with $x = (x_1, 0_{n-1})$ leads to, after sending $\bar{x}_1 \to \infty$,

$$u(-x_1, 0_{n-1}) \leqslant u(x_1, 0_{n-1}), \qquad v(-x_1, 0_{n-1}) \leqslant v(x_1, 0_{n-1}). \tag{24}$$

For $0 < x_1 < a < \bar{x}_1$, we set $\bar{x} = (\bar{x}_1, 0_{n-1})$, $\lambda = \bar{x}_1 - a$. (21) with $x = (x_1, 0_{n-1})$ leads to, after sending $\bar{x}_1 \to \infty$,

$$u(2a - x_1, 0_{n-1}) \leqslant u(x_1, 0_{n-1}), \qquad v(2a - x_1, 0_{n-1}) \leqslant v(x_1, 0_{n-1}). \tag{25}$$

By the symmetry of the system, $u_O(x) := u(Ox)$ and $v_O(x) := v(Ox)$ satisfy the same system for any orthogonal matrix $O \in O(n)$. Hence u_O and v_O satisfy (24) which implies u and v are radial about the origin; and satisfy (25) which implies that $u(x_1) \leqslant u(x_2)$, $v(x_1) \leqslant v(x_2)$ if $|x_1| \geqslant |x_2|$. $\square$

3.2. Nonexistence

Let $u, v \geqslant 0$ be solutions of system (6). For simplicity, we denote $u(r)$ as $u(x)$ and $v(x)$ as $v(r)$ for $|x| = r$, since both of them are radial functions. Then $u(r)$ and $v(r)$ satisfy, in $\mathbb{R}^n \backslash \{0\}$,

$$\begin{cases} \left(r^{n-1}u'(r)\right)' = -r^{n-1-s}v^p(r), \\ \left(r^{n-1}v'(r)\right)' = -r^{n-1-t}u^q(r). \end{cases} \tag{26}$$

Here $'$ means the differentiation with respect to r.

To get nonexistence, first we derive a Pohozaev-type identity for (26).

Lemma 3.4. *Solutions u and v of system* (26) *satisfy, for $R > 0$,*

$$\begin{aligned} &R^n u'v' + \frac{R^{n-s}}{p+1}v^{p+1} + \frac{R^{n-t}}{q+1}u^{q+1} + \frac{n-t}{p+1}R^{n-1}v'u + \left(n-2-\frac{n-t}{p+1}\right)R^{n-1}vu' \\ &\quad = \left(\frac{n-s}{p+1} + \frac{n-t}{q+1} - (n-2)\right)\int_0^R r^{n-s-1}v^{p+1}\,dr. \end{aligned} \tag{27}$$

Proof. Multiplying $rv'(r)$ to the first equation of (26) and integrating from ε to R, we have

$${}^n u'(r)v'(r)\big|_\varepsilon^R - \int_\varepsilon^R r^{n-1}u'v' - \int_\varepsilon^R r^n u'v''\,dr = -\frac{r^{n-s}v^{p+1}(r)}{p+1}\bigg|_\varepsilon^R + \frac{n-s}{p+1}\int_\varepsilon^R r^{n-s-1}v^{p+1}\,dr. \tag{28}$$

Similarly, multiplying $ru'(r)$ to the second one and integrating from ε to R, we obtain

$$r^n u'(r)v'(r)\big|_\varepsilon^R - \int_\varepsilon^R r^{n-1}u'v' - \int_\varepsilon^R r^n v'u''\,dr = -\frac{r^{n-t}v^{q+1}(r)}{q+1}\bigg|_\varepsilon^R + \frac{n-t}{q+1}\int_\varepsilon^R r^{n-t-1}u^{q+1}\,dr. \tag{29}$$

Adding (28) to (29), and using

$$\int_\varepsilon^R \left(u'v'\right)' = r^n u'v'\big|_\varepsilon^R - \int_\varepsilon^R nr^{n-1}u'v'$$

it yields

$$\begin{aligned} &\left(r^n u'v' + \frac{r^{n-s}}{p+1}v^{p+1} + \frac{r^{n-t}}{q+1}u^{q+1}\right)\bigg|_\varepsilon^R \\ &\quad = \frac{n-s}{p+1}\int_\varepsilon^R r^{n-s-1}v^{p+1}\,dr + \frac{n-t}{q+1}\int_\varepsilon^R r^{n-t-1}u^{q+1}\,dr - (n-2)\int_\varepsilon^R r^{n-1}u'v'\,dr. \end{aligned}$$

On the other hand, we multiply v to the first equation of (26) and u to the second one and integrate from ε to R, to get

$$r^{n-1}u'v\big|_\varepsilon^R - \int_\varepsilon^R r^{n-1}u'v'\,dr = -\int_\varepsilon^R r^{n-1-s}v^{p+1},$$

$$r^{n-1}v'u\big|_\varepsilon^R - \int_\varepsilon^R r^{n-1}u'v'\,dr = -\int_\varepsilon^R r^{n-1-t}u^{q+1}.$$

A direct substitution yields that

$$\left(\frac{r^{n-s}}{p+1}v^{p+1}+\frac{r^{n-t}}{q+1}u^{q+1}+\frac{n-t}{p+1}r^{n-1}v'u+\left(n-2-\frac{n-t}{p+1}\right)r^{n-1}vu'\right)\bigg|_{\varepsilon}^{R}$$
$$=-\left(r^{n}u'v'\right)\big|_{\varepsilon}^{R}+\left(\frac{n-s}{p+1}+\frac{n-t}{q+1}-(n-2)\right)\int_{\varepsilon}^{R}r^{n-s-1}v^{p+1}\,dr. \tag{30}$$

Since $u, v \in H^1(B(0,1))$,

$$\liminf_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}}u'(\varepsilon)=0 \quad\text{and}\quad \liminf_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}}v'(\varepsilon)=0. \tag{31}$$

Taking $\liminf_{\varepsilon\to 0}$ on both sides of (30), and with the help of (31) we obtain (27). □

We will use the Pohozaev identity (27) to prove Theorem 1.4.

Proof of Theorem 1.4. Step 1: We need some rates of u, v, u' and v' near infinity. First we obtain some rates of u and v, following from [26]. We denote C as various positive constants that depend only on n, s, t, p and q. Note that u, v are continuous near 0 under condition (10), by Remark 1.3.

Clearly, with the help of (31),

$$\begin{aligned}
r^{n-1}u'(r) &= -\int_0^r \tau^{n-1-s}v^p(\tau)\,d\tau \\
&\leqslant -\int_0^{r/2} \tau^{n-1-s}v^p(\tau)\,d\tau \\
&\leqslant -v^p(r/2)\int_0^{r/2} \tau^{n-1-s}\,d\tau \\
&= -v^p(r/2)\frac{r^{n-s}}{(n-s)2^{n-s}}
\end{aligned}$$

where $p>0$ and that v is non-increasing are used. Hence

$$u'(r)\leqslant -\frac{r^{1-s}}{(n-s)2^{n-s}}v^p(r/2). \tag{32}$$

Integrating (32) from r to $2r$, we have

$$\begin{aligned}
u(r)-u(2r) &\geqslant C\int_r^{2r}\tau^{1-s}v^p(\tau/2)\,d\tau \\
&\geqslant Cv^p(r)\int_r^{2r}\tau^{1-s}\,d\tau \\
&\geqslant Cv^p(r)r^{2-s}
\end{aligned}$$

which implies

$$u(r)\geqslant Cr^{2-s}v^p(r). \tag{33}$$

Similarly

$$v(r)\geqslant Cr^{2-t}u^q(r). \tag{34}$$

Connecting (33) and (34), it yields

$$u(r) \geqslant Cr^{2-s}v^p(r) \geqslant Cr^{2-s}r^{(2-t)p}u^{pq}(r). \tag{35}$$

If $pq \leqslant 1$, then either $u \equiv 0$ which implies $v \equiv 0$, or

$$u^{1-pq}(0) \geqslant u^{1-pq}(r) \geqslant Cr^{2-s}r^{(2-t)p} \to \infty \quad \text{as } r \to \infty$$

which contradicts that $u \in C(\mathbb{R}^n)$ (Remark 1.3). So

$$u(x) \equiv v(x) \equiv 0 \quad \text{if } pq \leqslant 1.$$

In the following we assume that $pq > 1$. Then from (33) and (34) we have

$$u(r) \leqslant Cr^{\frac{s-2+p(t-2)}{pq-1}}, \qquad v(r) \leqslant Cr^{\frac{t-2+q(s-2)}{pq-1}}. \tag{36}$$

Secondly, we obtain some estimates for $u'(r)$ and $v'(r)$ from (36).

$$\begin{aligned}\left|r^{n-1}u'(r)\right| &= \left|-\int_0^r \tau^{n-1-s}v^p(\tau)\,d\tau\right| \\ &\leqslant v^p(0) + \int_1^r \tau^{n-1-s+\frac{p}{pq-1}(t-2+q(s-2))}\,d\tau \\ &\leqslant v^p(0) + C + Cr^{n-s+\frac{p}{pq-1}(t-2+q(s-2))}\log r\end{aligned}$$

where we put $\log r$ in case that the last exponent is zero. Hence

$$\left|u'(r)\right| \leqslant \left(v^p(0)+C\right)r^{1-n} + Cr^{1-s+\frac{p}{pq-1}(t-2+q(s-2))}\log r. \tag{37}$$

Similarly we can have an estimate for $v'(r)$:

$$\left|v'(r)\right| \leqslant \left(u^q(0)+C\right)r^{1-n} + Cr^{1-t+\frac{q}{pq-1}(s-2+p(t-2))}\log r. \tag{38}$$

Step 2: Via direct computations, the left-handed side of (27) will go to zero as $R \to +\infty$ if the following three inequalities are all satisfied:

(i) $n+1-s+\frac{p}{pq-1}(t-2+q(s-2))+1-t+\frac{q}{pq-1}(s-2+p(t-2)) < 0$;
(ii) $n-s+\frac{p+1}{pq-1}(t-2+q(s-2)) < 0$;
(iii) $n-t+\frac{q+1}{pq-1}(s-2+p(t-2)) < 0$.

In the following we let

$$\alpha = \min(s,t), \qquad \beta = \max(s,t).$$

Clearly

$$\frac{p+1}{pq-1}\big(2-t+q(2-s)\big) \geqslant \frac{(p+1)(q+1)}{pq-1}(2-\beta) \quad \text{and} \quad n-\alpha > n-s.$$

This implies that (ii) holds if

$$\frac{(p+1)(q+1)}{pq-1}(2-\beta) > n-\alpha. \tag{39}$$

Similar (iii) holds if (39) is true.

Furthermore,

$$\frac{p}{pq-1}\big(2-t+q(2-s)\big)+\frac{q}{pq-1}\big(2-t+p(2-s)\big)\geqslant\left(1+\frac{(p+1)(q+1)}{pq-1}\right)(2-\beta)$$

from which we can see that (i) holds if (39) is satisfied.

Since $pq>1$, $\max(p,q)>1$ and consequently

$$\frac{(p+1)(q+1)}{pq-1}\geqslant\frac{\max(p,q)+1}{\max(p,q)-1}.$$

So (39) is satisfied provided that

$$\frac{\max(p,q)+1}{\max(p,q)-1}>\frac{n-\alpha}{2-\beta},$$

which is equivalent to

$$\max(p,q)<\frac{n+2-\beta-\alpha}{n-2+\beta-\alpha}=\frac{n+2-s-t}{n-2+|s-t|}. \tag{40}$$

So under condition (10), the left-handed side of Pohozaev identity (27) will go to zero as $R\to+\infty$, which implies $v\equiv 0$. So $u\equiv 0$. Theorem 1.4 is proved. □

3.3. *Half space*

In this section, we prove Theorem 1.5. Recall that we assume

$$1\leqslant p\leqslant 2^*(s)-1,\qquad 1\leqslant q\leqslant 2^*(t)-1.$$

The following proof is a small modification of the proof of Theorem 1.1. From elliptic estimates and the maximum principle,

$$u,v\in C^\infty\big(\overline{\mathbb{R}^n_+}\backslash 0\big),\quad u(x)>0,\ v(x)>0.$$

For any $\bar{x}'\in\mathbb{R}^{n-1}$, let $\bar{x}=(\bar{x}',-1)$ and $u_{\bar{x},\lambda}(x)$, $v_{\bar{x},\lambda}(x)$ be the Kelvin transformation of u and v.

By Lemma 2.1 and direction computations, we have $\forall\lambda>1$, $\forall x\in B^+(\bar{x},\lambda)\triangleq B(\bar{x},\lambda)\cap\mathbb{R}^n_+$,

$$\begin{cases}-\Delta\big(u_{\bar{x},\lambda}(x)-u(x)\big)\geqslant\dfrac{v^p_{\bar{x},\lambda}(x)-v^p(x)}{|x|^s},\\[2mm] -\Delta\big(v_{\bar{x},\lambda}(x)-v(x)\big)\geqslant\dfrac{u^q_{\bar{x},\lambda}(x)-u^q(x)}{|x|^t}.\end{cases} \tag{41}$$

Lemma 3.5. $\forall|\bar{x}'|>2$, *there exists* $\lambda_0(\bar{x})>1$ *such that for all* $\lambda\in(1,\lambda_0(\bar{x}))$, *we have* $u_{\bar{x},\lambda}(x)\geqslant u(x)$, $v_{\bar{x},\lambda}(x)\geqslant v(x)$, $\forall x\in B^+(\bar{x},\lambda)$.

Proof. Denote

$$w_\lambda(u)=u_{\bar{x},\lambda}(x)-u,\qquad w_\lambda^-(u)=\max\big\{0,-w_\lambda(u)\big\},$$
$$w_\lambda(v)=v_{\bar{x},\lambda}(x)-v,\qquad w_\lambda^-(v)=\max\big\{0,-w_\lambda(v)\big\}.$$

We first require that $1<\lambda_0(\bar{x})<\sqrt{2}$, then we have $|x|>1$, $\forall x\in B^+(\bar{x},\lambda)$.

Multiplying $w_\lambda^-(u)$ to (41) and integrating on $B^+(\bar{x},\lambda)$, we have, using the mean value theorem,

$$\int_{B^+(\bar{x},\lambda)}\big|\nabla w_\lambda^-(u)\big|^2\,dx\leqslant C(n,\bar{x})\int_{B^+(\bar{x},\lambda)}v^{p-1}w_\lambda^-(v)w_\lambda^-(u)$$
$$\leqslant C(n,\bar{x})\big|B^+(\bar{x},\lambda)\big|^{\frac{2}{n}}\big\|\nabla w_\lambda^-(u)\big\|_{L^2(B^+(\bar{x},\lambda))}\big\|\nabla w_\lambda^-(v)\big\|_{L^2(B^+(\bar{x},\lambda))},$$

where we used $p \geqslant 1$. Similarly,

$$\int\limits_{B^+(\bar{x},\lambda)} \left|\nabla w_\lambda^-(v)\right|^2 dx \leqslant C(n,\bar{x})\left|B^+(\bar{x},\lambda)\right|^{\frac{2}{n}} \left\|\nabla w_\lambda^-(u)\right\|_{L^2(B^+(\bar{x},\lambda))} \left\|\nabla w_\lambda^-(v)\right\|_{L^2(B^+(\bar{x},\lambda))}.$$

Now we can choose $\lambda_0(\bar{x}) > 1$ but very close to 1, to force $w_\lambda^-(u) = w_\lambda^-(v) = 0$ in $B^+(\bar{x},\lambda)$. □

Define

$$\bar{\lambda}(\bar{x}) := \sup\left\{\mu \;\middle|\; 1 < \mu < |\bar{x}|,\ \text{and } u_{\bar{x},\lambda}(x) \geqslant u(x),\ v_{\bar{x},\lambda}(x) \geqslant v(x) \text{ for all } x \in B^+(\bar{x},\lambda),\ \forall 1 < \lambda < \mu\right\}.$$

By Lemma 3.5, $\bar{\lambda}(\bar{x})$ is well defined for all $|\bar{x}'| > 2$. Moreover $1 < \bar{\lambda}(\bar{x}) \leqslant |\bar{x}|$.

Lemma 3.6. $\forall |\bar{x}'| > 2$, $\bar{\lambda}(\bar{x}) = |\bar{x}|$. *Namely,*

$$u_{\bar{x},\lambda}(x) \geqslant u(x), \qquad v_{\bar{x},\lambda}(x) \geqslant v(x), \quad \forall x \in B^+(\bar{x},\lambda),\ \forall 0 < \lambda \leqslant |\bar{x}|. \tag{42}$$

Proof. The proof is similar to that of Lemma 2.3. The only difference has been shown in the proof of Lemma 3.5. We omit the details here. □

Proof of Theorem 1.5. By Lemmas 3.5 and 3.6, the proof is the same as proof of Theorem 1.1. □

Acknowledgements

The author would like to express his deepest gratitude to Professor Y.Y. Li for encouragements and suggestions. The work is supported by Rutgers University School of Art and Science Excellence Fellowship.

Appendix A. Regularity

In this appendix we will prove Remark 1.3. The proof is based on standard Sobolev embeddings and bootstrap arguments.

Proposition A.1. *Under* (9) *but* $(p,q) \neq (2^*(s)-1, 2^*(t)-1)$, *and if* $u, v \in \mathcal{D}^{1,2}_{loc}(\mathbb{R}^n)$ *are weak solutions of system* (6) *then* $u, v \in C^\alpha(\mathbb{R}^n)$ *for some* $\alpha > 0$.

Proof. We only need to show that u, v are Hölder continuous near 0. Without loss of generality we assume $p < 2^*(s) - 1$. Denote, for simplicity, L^P as $L^P(D)$ where D is the unit ball centered at 0. By Sobolev embedding $v \in L^{2n/(n-2)}$.

Suppose $v \in L^{A_k}$ with $A_1 = \frac{2n}{n-2}$. We will use bootstrap to find out A_{k+1}. We choose a decreasing and positive sequence $\{\varepsilon_k\} \to 0$ and will fix ε_1 small in Claim 1 below. Since $|x|^{-s} \in L^{(n-\varepsilon_k)/s}$, $|x|^{-t} \in L^{(n-\varepsilon_k)/t}$, by Hölder inequality

$$\frac{v^p}{|x|^s} \in L^{B_k} \quad \text{where } B_k = \frac{(n-\varepsilon_k)A_k}{sA_k + p(n-\varepsilon_k)}.$$

By $W^{2,P}$ theory and the first equation of system (6),

$$u \in W^{2,B_k} \hookrightarrow L^{B_k^{**}} \quad \text{if } \frac{1}{B_k^{**}} := \frac{1}{B_k} - \frac{2}{n} > 0.$$

This implies

$$\frac{u^q}{|x|^t} \in L^{C_k} \quad \text{where } C_k = \frac{(n-\varepsilon_k)B_k^{**}}{tB_k^{**} + q(n-\varepsilon_k)}.$$

By $W^{2,P}$ theory and the second equation of system (6),

$$v \in W^{2,C_k} \hookrightarrow L^{C_k^{**}} \quad \text{if } \frac{1}{C_k^{**}} := \frac{1}{C_k} - \frac{2}{n} > 0.$$

Define

$$A_{k+1} = C_k^{**}.$$

Combining the formulas together we have

$$\frac{1}{A_{k+1}} = q\left(\frac{p}{A_k} + \frac{s}{n-\varepsilon_k} - \frac{2}{n}\right) + \frac{t}{n-\varepsilon_k} - \frac{2}{n}. \tag{A.1}$$

Claim 1. *$A_{k+1} > A_k$ if $B_k, C_k < n/2$.*

Proof of Claim 1. By our assumption that $p < 2^*(s) - 1$,

$$(n-2)pq < (n+2-2s)q \leqslant n+2-2t-2qs+4q,$$

which implies

$$\frac{pq-1}{A_1} < \frac{2}{n} - \frac{qs}{n} - \frac{t}{n} + \frac{2q}{n}.$$

Fix an ε_1 to be small enough such that

$$\max\left(0, \frac{pq-1}{A_1}\right) < \frac{2}{n} - \frac{qs}{n-\varepsilon_1} - \frac{t}{n-\varepsilon_1} + \frac{2q}{n}. \tag{A.2}$$

By (A.1) Claim 1 is equivalent to

$$\frac{pq-1}{A_k} < \frac{2}{n} - \frac{qs}{n-\varepsilon_k} - \frac{t}{n-\varepsilon_k} + \frac{2q}{n}. \tag{A.3}$$

We can see from (A.2) that (A.3) is satisfied. Claim 1 is proved. □

Claim 2. *After finite steps, either $C_k \geqslant n/2$ or $B_k \geqslant n/2$.*

Proof of Claim 2. If not, the sequence $\{A_k\}$ is increasing by Claim 1. Denote $A = \lim_{k\to\infty} A_k$ (which could be $+\infty$). Letting $k \to \infty$ in (A.1), we have

$$\frac{pq-1}{A} = \frac{2}{n} - \frac{qs}{n} - \frac{t}{n} + \frac{2q}{n}.$$

Noting that

$$A \geqslant \frac{2n}{n-2},$$

we have

$$(n-2)p \geqslant \frac{n+2-2t}{q} - 2s + 4 > n+2-2s,$$

which violates our assumption about p. Claim 2 is proved. □

By Claim 2 and Sobolev embeddings, we immediately get that either $u \in L^\gamma$, $\forall \gamma < \infty$ or $v \in L^\gamma$, $\forall \gamma < \infty$. Hence $u, v \in C^\alpha$ for some $\alpha > 0$ by the $W^{2,P}$ theory. □

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