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On symmetry of nonnegative solutions of elliptic equations

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Abstract

We consider the Dirichlet problem for a class of fully nonlinear elliptic equations on a bounded domain Ω. We assume that Ω is symmetric about a hyperplane H and convex in the direction perpendicular to H. By a well-known result of Gidas, Ni and Nirenberg and its generalizations, all positive solutions are reflectionally symmetric about H and decreasing away from the hyperplane in the direction orthogonal to H. For nonnegative solutions, this result is not always true. We show that, nonetheless, the symmetry part of the result remains valid for nonnegative solutions: any nonnegative solution u is symmetric about H. Moreover, we prove that if $u \not\equiv 0$, then the nodal set of u divides the domain Ω into a finite number of reflectionally symmetric subdomains in which u has the usual Gidas–Ni–Nirenberg symmetry and monotonicity properties. We also show several examples of nonnegative solutions with a nonempty interior nodal set.

Résumé

Nous considérons le problème de Dirichlet pour une classe d'équations elliptiques complètement non-linéaires sur un domaine borné Ω. Nous supposons que Ω est symétrique par rapport à un hyperplan H et convexe dans la direction perpendiculaire à H. Par un résultat bien connu de Gidas, Ni et Nirenberg ainsi que par ses généralisations, toutes les solutions positives sont symétriques par rapport à une réflextion de H et décroissent à partir de leur distance de l'hyperplan dans la direction orthogonale à H. Pour les solutions non-négatives, ce résultat n'est pas toujours vrai. Nous montrons que, néanmoins, le résultat sur la symétrie reste valable pour les solutions positives : toute solution non-négative u est symétrique par rapport à H. En outre, nous montrons que si $u \not\equiv 0$, alors l'ensemble nodal de u divise le domaine Ω en un nombre fini de sous-domaines symétriques sous réflextion dans lesquels u possède la symétrie habituelle de Gidas–Ni–Nirenberg et des propriétés de monotonie. Nous montrons aussi plusieurs exemples de solutions positives avec un ensemble nodal intérieur non vide.

MSC: 35J60; 35B06; 35B05

1. Introduction

In this paper we consider nonlinear elliptic problems of the form

$$F\big(x, u, Du, D^2u\big) = 0, \quad x \in \Omega, \tag{1.1}$$

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$$u = 0, \quad x \in \partial\Omega. \tag{1.2}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ which is convex in one direction and reflectionally symmetric about a hyperplane orthogonal to that direction. We choose the coordinate system such that the direction is $e_1 := (1, 0, \ldots, 0)$ (that is, Ω is convex in x_1) and the symmetry hyperplane is given by

$$H_0 = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1}: \ x_1 = 0\}.$$

The nonlinearity F is assumed to be sufficiently regular, elliptic, and symmetric (see the next section for the precise hypotheses), so that in particular the equation is invariant under the reflection in H_0. For example, the semilinear problem

$$\Delta u + f(x', u) = 0, \quad x = (x_1, x') \in \Omega, \tag{1.3}$$

$$u = 0, \quad x \in \partial\Omega, \tag{1.4}$$

where $f : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}$ is continuous in all variables and Lipschitz in u, is an admissible problem for our results without any additional assumption on f.

By a celebrated theorem of Gidas, Ni, and Nirenberg [18] and its generalization to nonsmooth domains given by Berestycki and Nirenberg [5] (see also Dancer's result in [12]), each positive (classical) solution u of (1.1), (1.2) is even in x_1:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega), \tag{1.5}$$

and decreasing with increasing $|x_1|$:

$$u_{x_1}(x_1, x') < 0 \quad ((x_1, x') \in \Omega, x_1 > 0). \tag{1.6}$$

This result was proved using the method of moving hyperplanes introduced by Alexandrov [1] and further developed and applied in a symmetry problem by Serrin [30]. We refer the reader to the surveys [4,23,26,27], or the more recent paper [9], for perspectives on this theorem, related results, and many other references.

The above symmetry and monotonicity theorem is not valid in general if the solution u is assumed to be nonnegative, rather than strictly positive. A standard counterexample is given by the function $u(x) = 1 + \cos x$ considered as a solution of $u'' + u - 1 = 0$ on $\Omega = (-(2k+1)\pi, (2k+1)\pi)$, $k \in \mathbb{N}$. Note that, in this example, u is still has several symmetry properties. First of all it is even in x. Further, in each interval between two successive zeros of u, u is symmetric about the center of the interval and decreasing away from that center. Actually, it is not hard to prove these symmetry and monotonicity properties for nonnegative solutions of any problem (1.3), (1.4) in one space dimension (in the one-dimensional case, $\Omega = (-\ell, \ell)$ for some $\ell > 0$, and there is no variable x' in the equation). In the semilinear case, this can be shown by elementary phase plane analysis; a more involved argument which also applies to one-dimensional quasilinear problems is given in [29, pp. 192–193].

Surprisingly perhaps, similar symmetry statements for nonnegative solutions turn out to be valid in any dimension. Specifically, we prove in this paper that any nonnegative solution u of (1.1), (1.2) has to be even in x_1 and, if $u \not\equiv 0$ and u is not strictly positive in Ω, the nodal set of u divides Ω into a finite number of reflectionally symmetric subdomains in which u has the usual Gidas–Ni–Nirenberg symmetry and monotonicity properties. Thus nonnegative solutions have a similar symmetry structure as in one dimension. See Theorem 2.2 below for the complete statement including an additional information on global symmetries of nonnegative solutions (statement (ii) of the theorem, also see Remark 4.6). As in one dimension, it is possible for a nonnegative nonzero solution of (1.1), (1.2) to have interior zeros in Ω so that its graph has several "bumps." We shall return to the question of existence of multi-bump solutions in another part of this introduction.

We remark that symmetry properties of nonnegative, possibly multi-bump, solutions were studied by Brock in [6]. He considered a class of variational problems and established a local symmetry property for each nonnegative solution. It says that for each subdomain U of Ω in which $u > 0$ and $u_{x_1} > 0$, the graph of u contains a part reflectionally symmetric to $\{(x, u(x)): \ x \in U\}$. The global symmetry of u (evenness in x_1) and the precise information about the Gidas–Ni–Nirenberg symmetry in nodal domains of u in not proved in [6]. Also, it is in the nature of the continuous Steiner symmetrization employed in [6] that the variational structure of the problems is required, hence fully nonlinear equations cannot in general be treated by that method. On the other hand, the method requires only mild regularity assumptions and it also applies to some degenerate elliptic problems.

The proof of our symmetry result is based on the method of moving hyperplanes in which we introduce a sequence of rest points. The outline is as follows. We start moving the hyperplanes from the right in the usual way and continue as long as a certain positivity condition is satisfied. The position beyond which the process cannot be continued any more is the first rest point. At that point, we remove a part of the domain Ω in such a way that the process can be resumed and continued to a next rest point. Repeating this procedure a finite number of times, removing an additional part of the domain at each rest point, we eventually reach the central position given by the symmetry hyperplane H_0. Using an analogous procedure with hyperplanes moving from the left, we then show that u is even in x_1 in a subdomain of Ω. Invoking a unique continuation theorem, we conclude that (1.5) holds. Examining the boundaries of the sets removed from Ω in the above process, we then obtain the remaining conclusions.

We remark that the unique continuation has already been used in symmetry problems, see for example [13,24]. In order to be able to apply it, we need a stronger regularity assumption on the nonlinearity F than in the symmetry results for positive solutions, see Section 2.

Having established the symmetry properties of nonnegative solutions, our next concern is the existence of nonnegative nonzero solutions with a nonempty nodal set in Ω. Using the one-dimensional example mentioned above, it is not difficult to find such solutions for some problems with separable variables (see Example 2.4 below). However, it is not at all obvious that such solutions exist for other problems in higher dimension. In fact, there are several results ruling out the existence of such solutions in certain situations. Consider, for example, problem (1.3), (1.4), where Ω is of class C^2 (and symmetric as above) and f is independent of x'. Then it has been proved that nonnegative solutions are necessarily positive if Ω is a ball [8] (see also [14,16] for the proof and a discussion of this result), or, more generally, if Ω is strictly convex [10], or if the unit normal vector field on $\partial\Omega \setminus H_0$ is nowhere parallel to H_0 [21]. In the recent paper [28], we proved that the positivity result holds without any additional condition on Ω, as long as it is of class C^2. For nonsmooth domains, a sufficient condition for the strict positivity of nonnegative nonzero solutions was given in [15]. It requires, roughly speaking, that for any $\delta > 0$ there be a two-dimensional wedge W, such that if the tip of W is translated to any point of $\partial\Omega$ with $x_1 \geqslant \delta$, then W is contained in $\bar{\Omega}$. Note that a rectangle, or a rectangle with smoothed out corners, does not satisfy the geometric condition of [15]. The results of [15] apply to Eqs. (1.3) (and to a class of fully nonlinear equations (1.1)), which may be spatially nonhomogeneous if they satisfy additional symmetry assumptions. Eq. (1.3) with a general f depending on x' is not admissible.

In examples given in the next section, we consider a special form of problem (1.3), (1.4) in which f is an affine function of u. In addition to the equation with separable variables, we show other examples where nontrivial nonnegative solutions with interior zeros exist, see Examples 2.4–2.6. The nodal sets of the corresponding solutions are indicated in Figs. 1–3. In the first two examples, the nodal sets consist of (possibly intersecting) line segments. In the last example, the interior nodal set is given by non-flat analytic curves. We remark that there does not seem to be a straightforward way to construct examples with curved nodal lines. In particular, a small perturbation of a rectangle which would merely smooth out the "corners" will never lead to such an example. Proposition 2.7 below in particular implies that if Ω is a smooth convex domain in $\mathbb{R}^2$ and $\partial\Omega$ has a flat vertical part, then problem (1.3), (1.4) has no nontrivial nonnegative solutions with interior zeros, regardless of how $f(x', u)$ is chosen.

It is not surprising that the existence of nontrivial nonnegative solutions with interior zeros for a problem (1.3), (1.4) poses restrictions on the domain Ω. This is related to some results on overdetermined problems. Specifically, assuming that Ω is piecewise smooth, one can show that if u is a nonnegative nonzero solution of (1.3), (1.4) which is not strictly positive, then u satisfies a partly overdetermined problem on some subdomain $G \subset \Omega$: it satisfies the Dirichlet boundary condition on the whole boundary ∂G and the Neumann boundary condition on a part S of ∂G. If $f = f(u)$, G is smooth, and $S = \partial G$, then, by a well-known result of Serrin [30], G is necessarily a ball. In [28], this result was instrumental for proving the strict positivity of nonnegative nonzero solutions of spatially homogeneous problems (1.3), (1.4) on smooth domains. It is not clear whether a similarly general positivity result can be proved for nonsmooth domains or for nonhomogeneous equations on smooth domains. The symmetry result of Serrin does not seem to have an immediate bearing on this problem for, first, it is not clear to what extent it remains valid for nonsmooth domains; second, it may not be valid at all if S is a proper subset of ∂G (see [17]); and, third, it does not apply to nonhomogeneous equations even if the domain G is smooth. Currently, we do not have a very good understanding of domains Ω that admit the existence of nonnegative nonzero solutions with interior zeros for some problems (1.1), (1.2) or for restricted classes of such problems. This presents interesting problems for further research.

2. Main results and examples

It is a *standing hypothesis* throughout the paper that $\Omega \subset \mathbb{R}^N$ is a bounded domain which is convex in x_1 and symmetric about the hyperplane $H_0 = \{x = (x_1, x') \in \mathbb{R}^N\colon\ x_1 = 0\}$:

$$\{(-x_1, x')\colon\ (x_1, x') \in \Omega\} = \Omega.$$

We now formulate hypotheses on the nonlinearity F. Let $\mathcal{S}$ denote the space of $N \times N$ symmetric (real) matrices. We consider a function $F : (x, u, p, q) \mapsto F(x, u, p, q)$ defined on $\bar{\Omega} \times \mathcal{B}$, where $\mathcal{B}$ is an open convex set in $\mathbb{R} \times \mathbb{R}^N \times \mathcal{S}$, which is invariant under the transformation Q defined by

$$\begin{aligned} &Q(u, p, q) = (u, -p_1, p_2, \dots, p_N, \bar{q}),\\ &\bar{q}_{ij} = \begin{cases} -q_{ij} & \text{if exactly one of } i, j \text{ equals } 1,\\ q_{ij} & \text{otherwise.} \end{cases} \end{aligned} \tag{2.1}$$

We assume that F satisfies the following conditions:

(F1) (Regularity) F is continuous on $\bar{\Omega} \times \mathcal{B}$ and Lipschitz in (u, p, q): there is $\beta_0 > 0$ such that

$$\bigl|F(x, u, p, q) - F(x, \tilde{u}, \tilde{p}, \tilde{q})\bigr| \leqslant \beta_0\bigl|(u, p, q) - (\tilde{u}, \tilde{p}, \tilde{q})\bigr| \quad \bigl((x, u, p, q), (x, \tilde{u}, \tilde{p}, \tilde{q}) \in \bar{\Omega} \times \mathcal{B}\bigr). \tag{2.2}$$

Moreover, F is differentiable with respect to q on $\Omega \times \mathcal{B}$.

(F2) (Ellipticity) There is a constant $\alpha_0 > 0$ such that

$$F_{q_{ij}}(x, u, p, q)\xi_i\xi_j \geqslant \alpha_0|\xi|^2 \quad \bigl((x, u, p, q) \in \Omega \times \mathcal{B},\ \xi \in \mathbb{R}^N\bigr). \tag{2.3}$$

Here and below we use the summation convention (summation over repeated indices). For example, in the above formula the left-hand side represents the sum over $i, j = 1, \dots, N$.

(F3) (Symmetry) F is independent of x_1 and for any $(x, u, p, q) \in \Omega \times \mathcal{B}$ one has

$$F\bigl(x, Q(u, p, q)\bigr) = F(x, u, p, q) \quad \bigl(= F\bigl((0, x'), u, p, q\bigr)\bigr).$$

We consider classical solutions u of (1.1), (1.2). By this we mean functions $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$\bigl(u(x), Du(x), D^2u(x)\bigr) \in \mathcal{B} \quad (x \in \Omega)$$

and (1.1), (1.2) are satisfied everywhere.

The above hypotheses are sufficient for positive solutions of (1.1), (1.2) to be even in x_1 [5]. To deal with nonnegative solutions we need additional hypotheses. In the fully nonlinear case, we assume a Lipschitz continuity condition on F and u:

(HFU) For $i, j = 1, \dots, N$, the derivatives $F_{q_{ij}}$ and $u_{x_ix_j}$ are Lipschitz continuous on $\Omega \times \mathcal{B}$ and Ω, respectively.

If Eq. (1.1) is quasilinear, that is,

$$F(x, u, p, q) = A_{ij}(x, u, p)q_{ij} + f(x, u, p) \quad \bigl((x, u, p) \in (\bar{\Omega} \times \mathcal{B})\bigr) \tag{2.4}$$

for some functions A_{ij} and f, then no additional regularity of u is needed, we only require an extra regularity assumption on the A_{ij}:

(HA) The functions A_{ij}, $i, j = 1, \dots, N$, are Lipschitz in (x, u, p).

The reason for the assumption (HFU), or (HA), is an application of a unique continuation theorem, which is essential for our arguments. Note that (NFU) excludes some classical examples of fully nonlinear elliptic operators, as considered in [9]. In particular, we do not treat viscosity solutions of equations involving the extremal Pucci operator.

Under the above conditions, we have the following symmetry result.

Theorem 2.1. *Assume that* (F1)–(F3) *hold and let u be a nonnegative solution of* (1.1), (1.2). *Further assume that* (HFU) *holds or F is of the form* (2.4) *and* (HA) *holds. Then u is even in x_1*:

$$u(-x_1, x') = u(x_1, x') \quad ((x_1, x') \in \Omega). \tag{2.5}$$

This result follows from the next theorem which contains additional information on the solution. For the formulation of that theorem and for the rest of the paper, we need the following notation. For any $\lambda \in \mathbb{R}$ and any open set $U \subset \Omega$, we set

$$\begin{aligned} &H_\lambda := \{x \in \mathbb{R}^N\colon\ x_1 = \lambda\},\\ &\Sigma_\lambda^U := \{x \in U\colon\ x_1 > \lambda\},\\ &\tilde{\Sigma}_\lambda^U := \{x \in U\colon\ x_1 < \lambda\},\\ &\Gamma_\lambda^U := H_\lambda \cap U,\\ &\ell^U := \sup\{x_1 \in \mathbb{R}\colon\ (x_1, x') \in U \text{ for some } x' \in \mathbb{R}^{N-1}\}. \end{aligned} \tag{2.6}$$

When $U = \Omega$, we often omit the superscript Ω and simply write Σ_λ for Σ_λ^Ω, ℓ for ℓ^U, etc.

Let P_λ stand for the reflection in the hyperplane H_λ. Note that since Ω is convex in x_1 and symmetric in the hyperplane H_0, $P_\lambda(\Sigma_\lambda) \subset \Omega$ for each $\lambda \in [0, \ell)$ and Σ_0 is connected (for $\lambda > 0$, Σ_λ may not be connected).

For any function z on $\bar{\Omega}$, we define z^λ and $V_\lambda z$ by

$$\begin{aligned} &z^\lambda(x) = z(P_\lambda x) = z(2\lambda - x_1, x'),\\ &V_\lambda z(x) = z^\lambda(x) - z(x) \quad (x = (x_1, x') \in \bar{\Sigma}_\lambda). \end{aligned} \tag{2.7}$$

Theorem 2.2. *Assume that* (F1)–(F3) *hold and let u be a nonnegative solution of* (1.1), (1.2). *Further assume that* (HFU) *holds or F is of the form* (2.4) *and* (HA) *holds. Then either $u \equiv 0$ (hence, necessarily, $F(\cdot, 0, 0, 0) \equiv 0$) or else there exist $m \in \mathbb{N}$ and constants $\lambda_1, \dots, \lambda_m$ with the following properties*:

(i) $0 = \lambda_m < \lambda_{m-1} < \cdots < \lambda_1 < \ell$.
(ii) *For $i = 1, \dots, m$, $V_{\lambda_i} u \equiv 0$ on a connected component of Σ_{λ_i}. In particular, as Σ_0 is connected, $V_0 u \equiv 0$ in Σ_0, that is, u is even in x_1.*
(iii) *There are open mutually disjoint open sets $G_i \subset \Omega$, $i = 1, \dots, m$, with G_m possibly empty, such that the following statements are true*:
  (a) $\emptyset \neq G_i \subset \Sigma_0$ $(i = 1, \dots, m-1)$.
  (b) $\bar{\Omega} = \bar{G}_m \cup \bigcup_{i=1}^{m-1} (\bar{G}_i \cup P_0(\bar{G}_i))$.
  (c) *For $i = 1, \dots, m$, the set G_i is convex in x_1 and $P_{\lambda_i}(G_i) = G_i$.*
  (d) *For $i = 1, \dots, m$, one has $u > 0$ in G_i, $u = 0$ on ∂G_i, $V_{\lambda_i} u \equiv 0$ in G_i, and $u_{x_1} < 0$ in $\Sigma_{\lambda_i}^{G_i}$.*

Remark 2.3. (i) In case $m = 1$, (ii) and (iii) give the usual symmetry and monotonicity properties of u. In fact, u is positive in Ω in that case. In the general case, (ii), (iii) show that the nodal set of u, $u^{-1}(0)$, divides Ω into a finite number of open reflectionally symmetric subsets $G_m, G_i, P_0(G_i)$, $i = 1, \dots, m-1$, in each of which u is positive and has the usual Gidas–Ni–Nirenberg symmetry and monotonicity properties. By (d), the nodal set $u^{-1}(0)$ is given by the boundaries of these open sets, hence, by (b) and (c), it is a finite union of sets of the form $P_{\lambda_{i_k}} P_{\lambda_{i_{k-1}}} \dots P_{\lambda_{i_1}}(M)$, where $k \leqslant m$, $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant m$, and $M \subset \partial\Omega$ is a connected component of $\partial\Sigma_{\lambda_{i_1}} \cap \partial\Omega$.

(ii) From the proof of Theorem 2.2 one can find estimates on the number m. For example, if Σ_λ is connected for each $\lambda > 0$, then $\lambda_i - \lambda_{i+1} \geqslant d$ $(i = 1, \dots, m-1)$, where d is a positive constant determined only by N, the diameter of Ω, and the quantities α_0 and β_0 appearing in hypotheses (F1), (F2). This implies that $m \leqslant [\ell/d] + 1$, where $[\cdot]$ stands for the integer part.

We next show examples of nonnegative solutions for which $m > 1$. We consider two-dimensional problems of the form

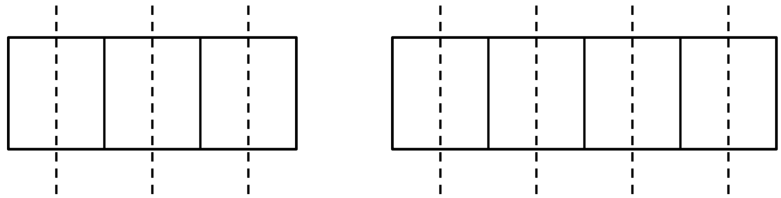

Fig. 1. The nodal set (solid lines) and hyperplanes H_{λ_i} (dashed lines) for the solutions u_1, u_2 in Example 2.4.

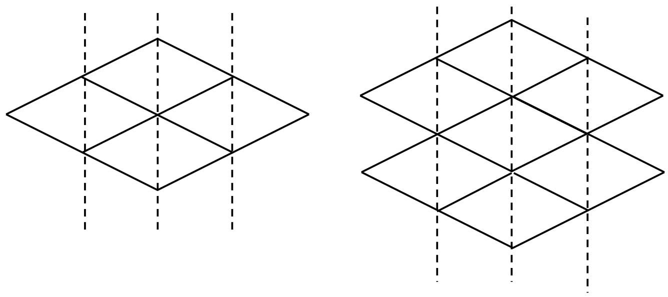

Fig. 2. The nodal set (solid lines) and hyperplanes H_{λ_i} (dashed lines) for solutions in Example 2.5.

$$\Delta u + \mu u + h(y) = 0, \quad (x, y) \in \Omega, \tag{2.8}$$

$$u = 0, \quad (x, y) \in \partial\Omega. \tag{2.9}$$

The domain $\Omega \subset \mathbb{R}^2$ satisfies the standing symmetry and convexity hypothesis and, as it is two-dimensional, we use the simplified notation $(x, y) = (x_1, x')$. This linear nonhomogeneous problem, μ being a positive constant and h a continuous function of y only, is of the form (1.3), (1.4). For suitable Ω, μ, and h, we find a nonnegative solution u with interior nodal curves. In the constructions of these examples, we always use as a crucial ingredient an eigenfunction of the Laplacian with a suitable nodal set (see in particular the computations in Section 5.2). One can probably construct other examples using different eigenfunctions. An example on a smooth domain, if there is any, would be particularly interesting.

Example 2.4. Let $\mu = 2$, $h(y) = -\sin y$, $u_1(x, y) := (1 + \cos x)\sin y$, and $u_2(x, y) := (1 - \cos x)\sin y$. Then, for any $k \in \mathbb{N}$, the functions u_1 and u_2 are nonnegative solutions of (2.8), (2.9) on $\Omega = (-(2k+1)\pi, (2k+1)\pi) \times (0, \pi)$ and $\Omega = (-2k\pi, 2k\pi) \times (0, \pi)$, respectively. Their interior nodal set is formed by vertical lines, hence the sets G_i are rectangles. We have $G_m \neq \emptyset$ for u_1 and $G_m = \emptyset$ for u_2 (the symmetry hyperplane H_0 itself is a nodal line of u_2). See Fig. 1.

Example 2.5. Let $\mu = 16/3$, $h(y) = -(32/3)\sin^2(2y)$,

$$u(x, y) := \left(\cos\frac{2x}{\sqrt{3}} - \cos 2y\right)^2.$$

The nodal lines of u are given by $y = \pm x/\sqrt{3} + k\pi$, $k \in \mathbb{Z}$, and the function u is a nonnegative solution of (2.8), (2.9) on any symmetric domain whose boundary consists of segments from these lines. Fig. 2 shows some possibilities. In this case the sets G_i consist of parallelograms. The set G_m is nonempty, but it has several connected components.

Example 2.6. In this example, Ω and the nodal curves of u are as in Fig. 3. The domains G_2, G_1 are not translations of one another, as in the previous examples, and the interior nodal lines of u are non-flat analytic curves. The definition of Ω, μ, and h is not so explicit here and we leave the construction for Section 5.2.

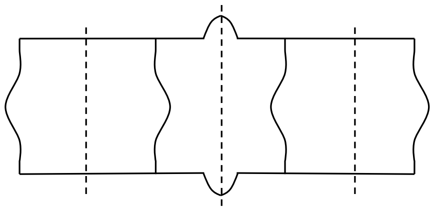

Fig. 3. The domain, nodal lines, and symmetries for Example 2.6.

The construction for Example 2.6, as given in Section 5.2, is a bit involved and it is not derived from Example 2.4 via a small perturbation of the rectangle. The next proposition in particular shows that a small perturbation of the rectangle which simply smooths out the "corners" could never provide similar examples.

Proposition 2.7. *Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be any Lipschitz function with Lipschitz constant γ. There is a positive constant $\delta=\delta(\gamma)$ with the following property. If Ω is a bounded C^1 domain in $\mathbb{R}^2$ such that*

(a) *Ω is symmetric in H_0 and convex in the direction of the x-axis,*
(b) *Σ_λ is connected for each $\lambda\in(0,\ell)$, and*
(c) *the set $\{(x,y)\in\partial\Omega:\ x>\ell-\delta\}$ contains a (nontrivial) line segment parallel to the y-axis,*

then any nonnegative solution of (1.3), (1.4) *is either identical to* 0 *or strictly positive in Ω.*

See Section 5.1 for the proof and some generalizations.

3. Preliminaries on linear equations

In this section we collect basic theorems on linear elliptic equations that we use in the proofs of the symmetry results. Let $G\subset\mathbb{R}^N$ be an open bounded set. Consider a linear equation of the form

$$a_{ij}(x)v_{x_ix_j}+b_i(x)v_{x_i}+c(x)v=0,\quad x\in G,\tag{3.1}$$

where

(L1) a_{ij}, b_i, c are measurable functions on G and there are positive constants α_0, β_0 such that

$$\begin{aligned}&\left|a_{ij}(x)\right|,\left|b_i(x)\right|,\left|c(x)\right|<\beta_0\quad(i,j=1,\dots,N,\ x\in G),\\&a_{ij}(x)\xi_i\xi_j\geqslant\alpha_0|\xi|^2\quad\big(\xi\in\mathbb{R}^N,\ x\in G\big).\end{aligned}$$

By a solution of (3.1) we mean a strong solution, that is, a function $v\in W^{2,N}_{\mathrm{loc}}(G)$ such that (3.1) is satisfied almost everywhere in G. In the following proposition, $|G|$ stands for the Lebesgue measure on $\mathbb{R}^N$.

Proposition 3.1. *Assume that* (L1) *holds and let $v\in W^{2,N}(G)$ be a solution of* (3.1).

(i) *If $v\geqslant 0$ in G and G is connected then either $v\equiv 0$ or $v>0$ in G.*
(ii) *Assume that $v\in C^1(\bar B)$, where B is a ball in G, and $x_0\in\partial G\cap\bar B$. If $v>0$ in B and $v(x_0)=0$, then $\partial v/\partial\eta<0$ at x_0, where η is an outer normal vector to ∂B at x_0.*
(iii) *Assume that $v\in C(\bar G)$. There is $\delta_0>0$ depending only on N, α_0, β_0 such that the relation $v\geqslant 0$ on ∂G implies $v\geqslant 0$ in $\bar G$, provided one of the following two conditions is satisfied*:
    (a) *$|G|<\delta_0$,*
    (b) *$G\subset\{x\in\mathbb{R}^N:\ m-\delta_0\leqslant x\cdot e\leqslant m\}$ for some unit vector $e\in\mathbb{R}^N$ and some $m\in\mathbb{R}$.*

Statements (i), (ii) are the standard strong maximum principle and Hopf boundary lemma for nonnegative solutions. Statement (iii) is the maximum principle for small or narrow domains (see [5,7]). Note that no sign condition on the coefficient c is needed for these results.

For the next proposition we need the leading coefficients to be more regular.

(L2) The functions a_{ij}, $i, j = 1, \dots, N$ are Lipschitz on G.

Proposition 3.2. *Assume that* (L1), (L2) *hold and G is connected. Let $v \in W^{2,N}(G)$ be a solution of* (3.1).

(i) *If $v \equiv 0$ in a nonempty open subset of G, then $v \equiv 0$ in G.*
(ii) *Let $S \subset G$ be a $C^{1,1}$ hypersurface and $\eta : S \to \mathbb{R}^N$ a nowhere tangent vector field on S. If v is of class C^1 near S and $v = \partial v/\partial \eta = 0$ on S, then $v \equiv 0$ in G.*

Statement (i) is a (weak) unique continuation theorem (see [22, Theorem 17.2.6], for example, for a stronger result that implies (i)). Statement (ii), the uniqueness for the Cauchy problem for elliptic equations, follows from (i) when one redefines the solution such that it becomes locally identical to 0 on one side of S (see [25, p. 60 and Section VI.40]; we remark that since the leading coefficients are Lipschitz continuous, it does not matter whether the equation is in the divergence or nondivergence form).

4. Proof of Theorem 2.2

Throughout this section we assume that the hypotheses of Theorem 2.2 are satisfied. We use notation from the previous section, see (2.6).

We frequently rely, usually without further notice, on the following standard observations concerning a linearization of Eq. (1.1) via reflections. Let U be a (not necessarily symmetric) open subset of Ω and $\lambda \geqslant 0$. If $\Sigma_\lambda^U \neq \emptyset$, then, by (F3), u^λ satisfies the same equation as u in $\Sigma_\lambda^U \subset \Omega$. Hence, for any $x \in \Sigma_\lambda^U$ we have (omitting the argument x of $u, u^\lambda, Du, \dots$)

$$0 = F\big(x, u, Du, D^2u^\lambda\big) - F\big(x, u, Du, D^2u\big) - F\big(x, u, Du, D^2u^\lambda\big) + F\big(x, u^\lambda, Du^\lambda, D^2u^\lambda\big).$$

Using the Hadamard's formulas in the integral form (which is legitimate in view of (F1)), we obtain that $v = V_\lambda u$ solves on $G = \Sigma_\lambda^U$ a linear equation (3.1) with coefficients depending on λ. By (F1), (F2), these coefficients satisfy condition (L1) of Section 3, with constants α_0 and β_0 as in (F1), (F2) (hence, independent of λ). Moreover, thanks to (HFU) or to (HA) if F is of the form (2.4), the leading coefficient a_{ij} is Lipschitz hence (L2) is satisfied as well. Now, if in addition $u = 0$ on ∂U, then, since $u \geqslant 0$, $v = V_\lambda u$ also satisfies

$$v(x) \geqslant 0, \quad x \in \partial\Sigma_\lambda^U \setminus \Gamma_\lambda^U. \tag{4.1}$$

Of course, on the remaining part of $\partial\Sigma_\lambda^U$, Γ_λ^U, we have the following condition

$$v(x) = 0, \quad x \in \Gamma_\lambda^U. \tag{4.2}$$

4.1. Three technical lemmas

In preparation for the process of moving hyperplanes, we prove the following three lemmas.

Lemma 4.1. *Let U be a nonempty open subset of Ω with $\ell^U > 0$ and let $\lambda \in [0, \ell^U)$. If D is a connected component of Σ_λ^U such that $V_\lambda u \geqslant 0$ on D, then either $V_\lambda u \equiv 0$ on D or $V_\lambda u(x) > 0$ for each $x \in D$. In the latter case one has*

$$\partial_{x_1} u(x) < 0 \quad \big(x \in \Gamma_\lambda^U \cap \partial D\big). \tag{4.3}$$

Proof. This follows directly from statements (i), (ii) of Proposition 3.1 (applied to $v = V_\lambda u$) and the fact that $\partial_{x_1} u(x) = -2\partial_{x_1}(V_\lambda u(x))$ on Γ_λ^U. □

Lemma 4.2. *Let U be a nonempty open subset of Ω which is convex in x_1 and such that $\ell^U > 0$. Let J be an open interval in $(0, \ell^U)$. If there is a ball $B \subset U$ such that*

$$B \subset \Sigma_\lambda^U \quad (\lambda \in J) \quad \text{and} \quad V_\lambda u(x) = 0 \quad (x \in B,\ \lambda \in J), \tag{4.4}$$

then $u \equiv 0$ in Ω.

Proof. Differentiating the identity

$$u\big(2\lambda - x_1, x'\big) - u\big(x_1, x'\big) = 0 \quad \big(x = \big(x_1, x'\big) \in B\big)$$

with respect to λ, we obtain $u_{x_1}(2\lambda - x_1, x') = 0$ $(x \in B)$. This means that, fixing any $\lambda \in J$, $u_{x_1} \equiv 0$ on the ball $\tilde{B} := P_\lambda B$. Consider now the function $w(x) = u(x_1 + \epsilon, x') - u(x)$, where $\epsilon > 0$ is small. Just like $V_\lambda u$, w satisfies a linear elliptic equation (3.1) on $G = \Omega \cap (\Omega - \epsilon e_1)$ with coefficients satisfying (L1), (L2). Since u is constant on $\tilde{B}$, for each sufficiently small ϵ the function w vanishes on the nonempty open set $\tilde{B} \cap (\tilde{B} - \epsilon e_1)$. Hence, by Proposition 3.2, $w \equiv 0$ on the connected component of $\Omega \cap (\Omega - \epsilon e_1)$ containing $\tilde{B} \cap (\tilde{B} - \epsilon e_1)$. Since this is true for any small $\epsilon > 0$, u is constant in Ω and the Dirichlet boundary condition forces $u \equiv 0$. □

In the next lemma, $\delta_0 = \delta_0(N, \lambda_0, \beta_0)$ is a positive constant as in Proposition 3.1(iii).

Lemma 4.3. *Assume that $u \not\equiv 0$. Let U be as in Lemma 4.2 and let $\lambda > 0$. Further assume that $u = 0$ on ∂U and $V_\lambda u(x) > 0$ for each $x \in K$, where $K \subset \Sigma_\lambda^U$ is a compact set such that $|\Sigma_\lambda^U \setminus K| < \delta_0$. Then for each $\tilde{\lambda} \leqslant \lambda$ sufficiently close to λ one has $V_{\tilde{\lambda}} u(x) > 0$ for each $x \in \Sigma_{\tilde{\lambda}}^U$.*

Note that we allow $K = \emptyset$ in which case we assume $|\Sigma_\lambda^U| < \delta_0$ (the assumption $V_\lambda u(x) > 0$ for each $x \in K$ is trivially satisfied). The possibility $\Sigma_\lambda^U = \emptyset$ (that is, $\lambda \geqslant \ell^U$) is also allowed. For easy reference, we single out the case $|\Sigma_\lambda^U| < \delta_0$ in the following corollary.

Corollary 4.4. *Assume that $u \not\equiv 0$. Let U be as in Lemma 4.2 and $\lambda > 0$. If $u = 0$ on ∂U and $|\Sigma_\lambda^U| < \delta_0$, then for each $\tilde{\lambda} \leqslant \lambda$ sufficiently close to λ (and in particular for $\tilde{\lambda} = \lambda$) one has $V_{\tilde{\lambda}} u(x) > 0$ for each $x \in \Sigma_{\tilde{\lambda}}^U$.*

Proof of Lemma 4.3. The first part of the proof goes by a standard application the maximum principle on small domains (cp. [5]).

Since $V_\lambda u > 0$ on the compact set K, for each $\tilde{\lambda}$ sufficiently close to λ one has $V_{\tilde{\lambda}} u > 0$ on K. Say this is true for all $\tilde{\lambda} \in [\lambda - \epsilon, \lambda + \epsilon]$, where ϵ is a small positive constant. (Although the statement of the lemma concerns values $\tilde{\lambda} \leqslant \lambda$ only, for technical reasons it is useful to consider values $\tilde{\lambda} > \lambda$, as well.) Making $\epsilon > 0$ smaller, if necessary, we also have $|\Sigma_{\tilde{\lambda}}^U \setminus K| < \delta_0$ for each $\tilde{\lambda} \in [\lambda - \epsilon, \lambda + \epsilon]$. Using (4.2), (4.1) with λ replaced with $\tilde{\lambda}$, together with the condition $V_{\tilde{\lambda}} u > 0$ on K, we obtain $V_{\tilde{\lambda}} u \geqslant 0$ on the boundary of $\Sigma_{\tilde{\lambda}}^U \setminus K$. Proposition 3.1(iii) then implies that $V_{\tilde{\lambda}} u$ is nonnegative in $\Sigma_{\tilde{\lambda}}^U \setminus K$, hence in $\Sigma_{\tilde{\lambda}}^U$.

Next we note that the inequality $V_{\tilde{\lambda}} u \geqslant 0$ in $\Sigma_{\tilde{\lambda}}^U$ implies that

$$u_{x_1}(x) = -2\partial_{x_1}\big(V_{\tilde{\lambda}} u(x)\big) \leqslant 0 \quad \big(x \in \Gamma_{\tilde{\lambda}}^U,\ \tilde{\lambda} \in [\lambda - \epsilon, \lambda + \epsilon]\big).$$

Hence u is nonincreasing in x_1 in the set $\Sigma_{\lambda-\epsilon}^U \setminus \Sigma_{\lambda+\epsilon}^U$.

We now prove that $V_{\tilde{\lambda}} u$ is strictly positive in $\Sigma_{\tilde{\lambda}}^U$ for each $\tilde{\lambda} \in (\lambda - \epsilon, \lambda]$. Fix any $\tilde{\lambda} \in (\lambda - \epsilon, \lambda]$ and let D be any connected component of $\Sigma_{\tilde{\lambda}}^U$. By Lemma 4.1, $V_{\tilde{\lambda}} u > 0$ in D or $V_{\tilde{\lambda}} u \equiv 0$ in D. We only need to rule out the latter. Since $\lambda - \epsilon < \tilde{\lambda} < \lambda + \epsilon$, if $\bar{B} \subset D$ is a closed ball whose radius is small enough and center close enough to $H_{\tilde{\lambda}}$, then the following inclusion holds for $\hat{\lambda} = \tilde{\lambda}$:

$$P_{\hat{\lambda}}(\bar{B}) \cup \bar{B} \subset \Sigma_{\lambda+\epsilon}^U \setminus \Sigma_{\lambda-\epsilon}^U. \tag{4.5}$$

Then, for any $\hat{\lambda} \in (\tilde{\lambda}, \lambda + \epsilon]$ sufficiently close to $\tilde{\lambda}$, (4.5) remains valid and also $\bar{B} \subset \Sigma_{\hat{\lambda}}^U$. By Lemma 4.2, we can choose $\hat{\lambda} \in (\tilde{\lambda}, \lambda + \epsilon]$ with these properties such that, in addition, $V_{\hat{\lambda}} u \not\equiv 0$ in B (remember that we are assuming that

$u \not\equiv 0$). Since $V_{\hat\lambda} u \geqslant 0$ in $\Sigma^U_{\hat\lambda}$, Lemma 4.1 yields $V_{\hat\lambda} u > 0$ in B. It follows, since $u_{x_1} \leqslant 0$ in $\Sigma^U_{\lambda+\epsilon} \setminus \Sigma^U_{\lambda+\epsilon}$, that the following holds for the center $z = (z_1, z')$ of B:

$$V_{\tilde\lambda} u(z) = u\big(2\tilde\lambda - z_1, z'\big) - u(z) \geqslant u\big(2\hat\lambda - z_1, z'\big) - u(z) = V_{\hat\lambda} u(z) > 0.$$

This rules out the identity $V_{\tilde\lambda} u \equiv 0$ in D, hence $V_{\tilde\lambda} u > 0$ in D. Since D is an arbitrary connected component of $\Sigma^U_{\tilde\lambda}$, we have $V_{\tilde\lambda} u > 0$ in $\Sigma^U_{\tilde\lambda}$. □

4.2. *Moving the hyperplanes from the right: the definition of the rest points*

Since the conclusion of Theorem 2.2 holds trivially if $u \equiv 0$, we assume that $u \not\equiv 0$.

We now carry out a process of moving hyperplanes. We first move H_λ from the right towards H_0, stopping and changing the underlying domain at certain values $\lambda = \lambda_i$ (the rest points). Set

$$\lambda_0 := \ell, \qquad D_0 := \emptyset, \qquad U_0 := \Omega. \tag{4.6}$$

Our goal is to define a sequence λ_i, finite or infinite for now, and corresponding domains U_i in such a way that the following statements hold true for $i = 1, 2, \ldots.$

(p1) $0 \leqslant \lambda_i < \lambda_{i-1}$.
(p2) U_i is convex in x_1.
(p3) One has

$$V_\lambda u(x) > 0 \quad \big(x \in \Sigma^{U_{i-1}}_\lambda,\ \lambda \in (\lambda_i, \lambda_{i-1}]\big), \tag{4.7}$$

$$u(x) > 0 \quad \text{and} \quad u_{x_1}(x) < 0 \quad \big(x \in \Sigma^{U_{i-1}}_{\lambda_i}\big). \tag{4.8}$$

(p4) On each connected component of $\Sigma^{U_{i-1}}_{\lambda_i}$ one has either $V_{\lambda_i} u \equiv 0$ or $V_{\lambda_i} u > 0$; we denote by D_i the union of all connected components of $\Sigma^{U_{i-1}}_{\lambda_i}$ on which $V_{\lambda_i} u \equiv 0$, hence

$$V_{\lambda_i} u \equiv 0 \quad (x \in D_i), \tag{4.9}$$

$$V_{\lambda_i} u(x) > 0 \quad \big(x \in \Sigma^{U_{i-1}}_{\lambda_i} \setminus D_i\big). \tag{4.10}$$

(p5) If $\lambda_i > 0$, then $V_{\lambda_i} u \equiv 0$ on at least one connected component of $\Sigma^{U_{i-1}}_{\lambda_i}$, that is, $D_i \neq \emptyset$.
(p6) If $\lambda_i > 0$, then

$$U_i = U_{i-1} \setminus \big(\bar D_i \cup P_{\lambda_i}(\bar D_i)\big), \tag{4.11}$$

$$\Sigma^{U_i}_{\lambda_i} = \Sigma^{U_{i-1}}_{\lambda_i} \setminus \bar D_i, \tag{4.12}$$

and

$$u(x) = 0 \quad (x \in \partial U_i). \tag{4.13}$$

Before continuing, we remark that since D_i consists of connected components of $\Sigma^{U_{i-1}}_{\lambda_i}$, (4.11) and (4.12) can be equivalently written as follows

$$U_i = U_{i-1} \setminus \big(D_i \cup P_{\lambda_i}(\bar D_i)\big), \tag{4.14}$$

$$\Sigma^{U_i}_{\lambda_i} = \Sigma^{U_{i-1}}_{\lambda_i} \setminus D_i. \tag{4.15}$$

Starting with (4.6), suppose that for some $k \geqslant 0$ the values $\lambda_0, \ldots, \lambda_k$ and domains $U_0, \ldots, U_k$ have been defined such that (p1)–(p6) hold for $i = 1, \ldots, k$. We consider the definition of the (finite) sequence λ_i complete if $\lambda_k = 0$. In the opposite case, $\lambda_k > 0$, we continue by setting

$$\lambda_{k+1} := \inf\big\{\mu \in (0, \lambda_k]:\ V_\lambda u(x) > 0 \text{ for all } x \in \Sigma^{U_k}_\lambda \text{ and } \lambda \in [\mu, \lambda_k]\big\}. \tag{4.16}$$

Let us verify, first of all, that λ_{k+1} is well defined, that is, the set in (4.16) is nonempty. We claim that the set actually contains an interval, which also gives $\lambda_{k+1} < \lambda_k$. If $k = 0$ (and $\lambda_k = \ell$), this follows directly from Corollary 4.4, where we take $U = U_0 = \Omega$. If $k > 0$, the claim follows from (4.10), (4.12) upon an application of Lemma 4.3, where we take $U = U_k$, $\lambda = \lambda_k$, and choose a compact subset $K \subset \Sigma_{\lambda_k}^{U_k} = \Sigma_{\lambda_k}^{U_{k-1}} \setminus D_k$ whose complement has small measure. Note that the application of Lemma 4.3 is justified by (p2) and (4.13). Let us add a clarifying remark. It is possible here that $\Sigma_\lambda^{U_k} = \emptyset$ for $\lambda \leqslant \lambda_k$ close to λ_k, or even for all $\lambda \in (0, \lambda_k]$ (this possibility may occur if $D_k = \Sigma_{\lambda_k}^{U_{k-1}}$). Of course, if $\Sigma_\lambda^{U_k} = \emptyset$, then the condition that $V_\lambda u(x) > 0$ for all $x \in \Sigma_\lambda^{U_k}$ is trivially satisfied, so this possibility requires no extra attention.

We have thus correctly defined $\lambda_{k+1} < \lambda_k$. We next set $i := k + 1$ and verify that, with a suitably defined U_{k+1}, statements (p1)–(p6) hold.

Since $\lambda_{k+1} \geqslant 0$, (p1) holds. From the definition of λ_{k+1} and Lemma 4.1 (using again $u_{x_1} = -2\partial_{x_1}(V_\lambda u)$ for $x_1 = \lambda$), we obtain properties (4.7), (4.8), hence (p3) holds. By (4.16) and continuity of u, $V_{\lambda_{k+1}} u \geqslant 0$ in $\Sigma_{\lambda_{k+1}}^{U_k}$. Hence (p4) holds by Lemma 4.1. We denote, in accordance with (p4), by D_{k+1} the union of all connected components of $\Sigma_{\lambda_{k+1}}^{U_k}$ on which $V_{\lambda_{k+1}} u \equiv 0$.

If $\lambda_{k+1} = 0$, we set

$$U_{k+1} = U_k \setminus \left(\overline{\Sigma_0^{U_k}} \cup P_0\left(\overline{\Sigma_0^{U_k}}\right)\right). \tag{4.17}$$

This set is clearly convex in x_1 (since U_k is), hence (p2) holds, and there is nothing to be verified in (p5), (p6).

We continue assuming that $\lambda_{k+1} > 0$. We set

$$U_{k+1} := U_k \setminus \left(\bar{D}_{k+1} \cup P_{\lambda_{k+1}}(\bar{D}_{k+1})\right), \tag{4.18}$$

which makes (4.11) and (4.12) valid. Also, U_{k+1} is convex in x_1, by the definition of D_{k+1} and the convexity of U_k, hence (p2) holds. We have

$$u = 0 \quad \text{on } \partial U_k \supset (\partial D_{k+1}) \setminus H_{\lambda_{k+1}} \tag{4.19}$$

and, as $V_{\lambda_i} u \equiv 0$ in D_{k+1}, also

$$u = 0 \quad \text{on } P_{\lambda_k}(\partial D_{k+1}) \setminus H_{\lambda_{k+1}}. \tag{4.20}$$

This implies (4.13), hence (p6) holds.

It remains to prove (p5). Assume that $D_{k+1} = \emptyset$. Then, by (4.10), $V_{\lambda_{k+1}} u > 0$ on $\Sigma_{\lambda_{k+1}}^{U_k}$ and an application of Corollary 4.4 with $\lambda := \lambda_{k+1} > 0$, $U = U_k$ immediately gives a contradiction to the definition of λ_{k+1}. Thus $D_{k+1} \neq \emptyset$ and (p5) is proved. Below, the following additional information will be useful:

$$|D| \geqslant \delta_0 \quad \text{for each connected component } D \text{ of } D_{k+1}, \tag{4.21}$$

where $\delta_0 > 0$ is as in Lemma 4.3. To prove (4.21), we apply Corollary 4.4 with $\lambda := \lambda_{k+1} > 0$ and

$$U := D \cup P_{\lambda_{k+1}}(D) \cup \left(\Gamma_{\lambda_{k+1}}^{U_k} \cap \partial D\right).$$

Since D is a connected component of D_{k+1}, hence of $\Sigma_{\lambda_{k+1}}^{U_k}$, the set U is clearly open and convex in x_1, and $u = 0$ on ∂D (cf. (4.19), (4.20)). If $|D| < \delta_0$, then Corollary 4.4 gives $V_{\lambda_{k+1}} u > 0$ in $\Sigma_{\lambda_{k+1}}^U = D$, a contradiction to the definition of D_{k+1}. Hence (4.21) holds.

This completes the definition of the sequence $\{\lambda_i\}_{i=0}^m$ and the corresponding domains U_i, such that (p1)–(p6) hold true. Obviously, $\ell > \lambda_i > 0$ for $i = 1, \dots, m-1$ and either $m = \infty$ or else $m < \infty$ and $\lambda_m = 0$.

We next rule out the possibility $m = \infty$. Assume it holds. Using (p5) and (4.21), for each $k = 0, 1, \dots,$ we find a connected component E_{k+1} of D_{k+1} with $|E_{k+1}| \geqslant \delta_0$. But by (p4) and (4.11), the sets D_{k+1}, $k = 0, 1, \dots$ are mutually disjoint and we have a contraction to the boundedness of Ω.

Thus $m < \infty$ and $\lambda_m = 0$.

It is obvious that

$$\Omega = U_0 \supset U_1 \supset U_2 \supset \cdots \supset U_m, \tag{4.22}$$

where all the inclusions, except possibly the last one, are strict. From the relation

$$U_m = U_{m-1} \setminus \left(\overline{\Sigma_0^{U_{m-1}}} \cup P_0\left(\overline{\Sigma_0^{U_{m-1}}}\right)\right) \tag{4.23}$$

(which is the same as (4.17) with $k+1=m$) we see that $U_m = U_{m-1}$ if and only if $\Sigma_0^{U_{m-1}} = \emptyset$. Also from (4.23),

$$U_m \cap \Sigma_0 = \emptyset. \tag{4.24}$$

4.3. *The definition of the symmetric open sets G_i*

Set for $i = 1, \dots, m$

$$G_i := U_{i-1} \setminus \bar{U}_i. \tag{4.25}$$

We prove that the conclusion of Theorem 2.2 holds for these sets.

Obviously, the sets G_i are open. In view of (4.22), they are mutually disjoint and $G_1, \dots, G_{m-1}$ are nonempty. The set G_m is empty if and only if $\Sigma_0^{U_{m-1}} = \emptyset$. By (4.22) and (4.24),

$$\bar{\Sigma}_0 \subset \bar{G}_m \cup \bigcup_{i=1}^{m-1} \bar{G}_i. \tag{4.26}$$

This implies statement (b) of Theorem 2.2.

In view of (4.11), (4.14), for $i = 1, \dots, m-1$ we can write G_i as

$$G_i = D_i \cup P_{\lambda_i}(D_i) \cup \left(\Gamma_{\lambda_i}^{U_{i-1}} \cap \partial D_i\right). \tag{4.27}$$

In particular, $\Sigma_{\lambda_i}^{G_i} = D_i$. For G_m, we have from (4.23)

$$G_m = \Sigma_0^{U_{m-1}} \cup P_0\left(\Sigma_0^{U_{m-1}}\right) \cup \Gamma_0^{U_{m-1}}. \tag{4.28}$$

These relations imply that for $i = 1, \dots, m$, one has $P_{\lambda_i}(G_i) = G_i$ and, since U_{i-1} is convex in x_1 and D_i consists of connected components of $\Sigma_{\lambda_i}^{U_{i-1}}$, G_i is also convex in x_1. This verifies statement (c) of Theorem 2.2.

Now we show that for $i = 1, \dots, m-1$ one has

$$u > 0 \quad \text{and} \quad V_{\lambda_i} u \equiv 0 \quad \text{in } G_i, \qquad u_{x_1} < 0 \quad \text{in } \Sigma_{\lambda_i}^{G_i} = D_i, \tag{4.29}$$

$$u = 0 \quad \text{on } \partial G_i. \tag{4.30}$$

Indeed, properties (4.29) follow directly from (4.8), (4.9), and (4.30) follows from (4.13) and the fact that $V_{\lambda_i} u \equiv 0$ in G_i. This verifies statement (d) of Theorem 2.2 for $i < m$. Next, we claim that if $G_m \neq \emptyset$, then

$$u > 0 > u_{x_1} \quad \text{and} \quad V_0 u \geqslant 0 \quad \text{in } \Sigma_0^{G_m} \text{ (consequently, } u > 0 \text{ in } G_m\text{)}, \tag{4.31}$$

$$u = 0 \quad \text{on } \partial G_m \cap \Sigma_0, \tag{4.32}$$

$$\text{either } V_0 u \equiv 0 \quad \text{in } \Sigma_0^{G_m} \quad \text{or else} \quad u_{x_1}(x) < 0 \quad \text{for some } x \in \Gamma_0^{G_m}. \tag{4.33}$$

Indeed, relations (4.31) follow from (p3) and the continuity of u, and (4.32) is a consequence of (4.28) and identity (4.13) for $i = m-1$. If there is a connected component E of $\Sigma_0^{G_m}$ on which $V_0 u \not\equiv 0$, then, by Lemma 4.1, $u_{x_1}(x) < 0$ on $\Gamma_0^{G_m} \cap \partial E$. By the convexity of G_m in x_1, $\Gamma_0^{G_m} \cap \partial E \neq \emptyset$ which shows that (4.33) holds.

We have thus verified all the relations in statement (d) of Theorem 2.2 except $V_0 u \equiv 0$ in $\Sigma_0^{G_m}$.

Next, using the symmetry identity in (4.29) and recalling that $\Sigma_{\lambda_i}^{G_i} = D_i \neq \emptyset$ for $i = 1, \dots, m-1$, we apply Proposition 3.2 (unique continuation), to conclude that

$$V_{\lambda_i} u \equiv 0 \quad \text{in a connected component of } \Sigma_{\lambda_i} \left(= \Sigma_{\lambda_i}^{\Omega}\right). \tag{4.34}$$

This proves statement (ii) of Theorem 2.2 for $i < m$.

To complete the proof of Theorem 2.2, it remains to show that

(r1) for $i = 1, \dots, m-1$, one has $G_i \subset \Sigma_0$,
(r2) (4.34) hold for $i = m$, that is, $V_0 u \equiv 0$ in Ω.

4.4. The completion of the proof: moving the hyperplanes from the left

To prove (r1), (r2), we first carry out an analogous moving plane procedure from the left. We find $n \geqslant 1$, values $-\ell < \mu_1 < \mu_2 < \cdots < \mu_n = 0$, and mutually disjoint open sets $W_1, \ldots, W_n$ (the analogs of the sets G_i) with the following properties (recall that $\tilde{\Sigma}_0$ was defined in (2.6)).

(w1) $W_1, \ldots, W_{n-1}$ are nonempty.
(w2) For $i = 1, \ldots, n$, the set W_i is convex in x_1 and $P_{\mu_i}(W_i) = W_i$.
(w3) $\mathrm{cl}(\tilde{\Sigma}_0) \subset \bar{W}_m \cup \bigcup_{i=1}^{n-1} \bar{W}_i$.
(w4) For $i = 1, \ldots, n-1$, one has

$$u > 0 \quad \text{and} \quad V_{\mu_i} u \equiv 0 \quad \text{in } W_i, \qquad u_{x_1} > 0 \quad \text{in } \tilde{\Sigma}_{\mu_i}^{W_i}, \quad \text{and} \quad u = 0 \quad \text{on } \partial W_i. \tag{4.35}$$

(w5) If $W_n \neq \emptyset$, then

$$u > 0 \quad \text{in } W_n, \qquad V_0 u \geqslant 0 \quad \text{and} \quad u_{x_1} > 0 \quad \text{in } \tilde{\Sigma}_0^{W_n}, \tag{4.36}$$

$$u = 0 \quad \text{on } \partial W_n \cap \tilde{\Sigma}_0, \tag{4.37}$$

$$\text{either } V_0 u \equiv 0 \quad \text{in } \tilde{\Sigma}_0^{W_n} \quad \text{or else} \quad u_{x_1}(x) > 0 \quad \text{for some } x \in \Gamma_0^{W_n}. \tag{4.38}$$

(w6) For $i = 1, \ldots, n-1$,

$$V_{\mu_i} u \equiv 0 \quad \text{in a connected component of } \tilde{\Sigma}_{\mu_i}^{\Omega}. \tag{4.39}$$

We now claim that there is a nonempty, (relatively) open subset $\tilde{\Gamma}_0$ of $\Gamma_0 = H_0 \cap \Omega$ such that

$$u_{x_1}(x) = 0 \quad (x \in \tilde{\Gamma}_0). \tag{4.40}$$

Assume for a while this is true. Then the function $v = V_0 u$ vanishes on $\tilde{\Gamma}_0$ together with v_{x_1}. Applying Proposition 3.2 to v, we obtain $V_0 u \equiv 0$ in Ω, hence (r2) holds. We next show that

$$G_i \subset \Sigma_0 \quad (i = 1, \ldots, m-1). \tag{4.41}$$

Assume that (4.41) fails for some $1 \leqslant k \leqslant m-1$. Then, by convexity of G_k in x_1, $G_k \cap H_0 \neq \emptyset$. Fix a point $x^0 \in G_k \cap H_0$ and observe that the relations $x_1^0 = 0 < \lambda_k$ and (4.29) imply $u_{x_1}(x^0) > 0$, in contradiction to (4.40).

We have thus shown that (4.40) implies that (r1), (r2) hold. Therefore, the proof of Theorem 2.2 will be complete if we prove (4.40). For this aim, we fix any λ with $\max\{\lambda_1, -\mu_1\} < \lambda < \ell$. Let $\tilde{\Gamma}_0$ be the set of all $x \in \Gamma_0$ with the property that the line through x orthogonal to H_0 intersects the set Σ_λ (by symmetry, it then also intersects $\tilde{\Sigma}_{-\lambda}$). Clearly, $\tilde{\Gamma}_0 \neq \emptyset$ and it is open in the relative topology of H_0. We prove that (4.40) holds. We go by contradiction: assume there is $x^0 \in \tilde{\Gamma}_0$ such that $u_{x_1}(x^0) \neq 0$. Specifically, we assume that

$$u_{x_1}\bigl(x^0\bigr) > 0. \tag{4.42}$$

The case $u_{x_1}(x^0) < 0$ can be ruled out in an analogous way.

By (4.26), $x^0 \in \bar{G}_k$ for some $k \in \{1, \ldots, m\}$. We cannot have $x^0 \in \partial G_k$, for that would give $u(x^0) = 0$ (see (4.30), (4.32)), which is impossible by (4.42) and the nonnegativity of u in Ω. Further, by (4.42) and (4.31), $x^0 \notin G_m$. Hence, $x^0 \in G_k$, for some $k < m$.

Let T be the line through x^0 orthogonal to H_0. There is $q > 0$ such that

$$T \cap \Omega = \bigl\{x^0 + s e_1 \colon s \in (-q, q)\bigr\}.$$

Define

$$\varphi(s) = u\bigl(x^0 + s e_1\bigr) \quad \bigl(s \in (-q, q)\bigr). \tag{4.43}$$

Obviously, $\varphi \geqslant 0$ and $\varphi(\pm q) = 0$. We claim that φ has also the following properties.

(c1) The set

$$M := \bigl\{s \in (-q, q) \colon \varphi(s) > 0\bigr\}$$

has only finitely many connected components (which, of course, are open intervals).

(c2) Let J be any connected component of M and let s_J be the center of J. Then

$$\varphi'(s) > 0 \quad (s \in J, s < s_J), \tag{4.44}$$

$$\varphi'(s) < 0 \quad (s \in J, s > s_J), \tag{4.45}$$

and $\varphi(2s_J - s) - \varphi(s) = 0$ for all $s \in (-q, q)$ for which the left-hand side is defined.

To prove this, let J be any connected component of M. Assume first that $0 \notin J$. Consider the case $J \subset (-q, 0)$, so that the line segment $T_J := \{x^0 + se_1\colon s \in J\}$ is contained $\tilde{\Sigma}_0$. Clearly, u is positive on T_J and it vanishes at the end points of T_J. By (w3), there is $1 \leqslant i \leqslant n$ such that $T_J \subset W_i$ (note that T_J cannot intersect the boundary of any of the sets W_i as u would have to vanish at the intersection, see (4.35), (4.37)). The facts that $T_J \subset \tilde{\Sigma}_0$ and that u vanishes at the end points of T_J, in conjunction with (4.36), imply that we cannot have $T_J \subset W_n$. Hence $i < n$ and the end points of T_J are in the boundary of W_i. Statement (c2) now follows from (w6). The unique critical point of φ in J is μ_i in this case.

Similarly one considers the case $J \subset (0, q)$. If $0 \in J$, then $x^0 \in T_J \cap G_k$. Therefore $T_J \subset G_k$ and, since $k \leqslant m - 1$, (c2) is proved as above.

Next we note that it is impossible that for two different components J_1, J_2 of M the segments T_{J_1}, T_{J_2} belong to the same set G_j (and the same goes for W_j). Indeed, that would imply, by convexity of G_j in x_1, that G_j contains an end point of T_{J_1} (and of T_{J_2}) contradicting the positivity of u in G_j, see (4.29), (4.31). This proves (c1).

Observe also that $\varphi > 0$ near the boundary point q, that is, $q \in \bar{M}$. This follows from the facts that T intersects the set $\Sigma_\lambda \subset \Sigma_{\lambda_1}$ (recall that $\max\{\lambda_1, -\mu_1\} < \lambda < \ell$) and that $u > 0$ in Σ_{λ_1} (see (4.8)). Similarly, $\varphi > 0$ near $-q$, hence $-q, q \in \bar{M}$. We now use the following elementary lemma, whose proof is given at the end of this section.

Lemma 4.5. *Let $\varphi \in C[-q, q] \cap C^1(-q, q)$ be such that $\varphi \geqslant 0$, $\varphi(\pm q) = 0$,* (c1), (c2) *are satisfied, and $-q, q \in \bar{M}$. If the number of the connected components of M is even, assume in addition that there is $\sigma \in (-q, q) \setminus M$ such that $\varphi(2\sigma - s) - \varphi(s) = 0$ for all $s \in (-q, q)$ for which the left-hand side is defined. Then $\varphi'(0) = 0$.*

By this lemma, we already have a contradiction to (4.42) if the number of the connected components of M is odd. If it is even, we need to verify the additional symmetry hypothesis.

Let $B \subset \tilde{\Gamma}_0$ be a (relatively) open neighborhood of x^0 in H_0 such that $u_{x_1}(\tilde{x}^0) > 0$ for all $\tilde{x}^0 \in B$. For each $\tilde{x}^0 \in B$, define $\tilde{q}$, $\tilde{M}$, $\tilde{\varphi}$ in the same way as q, M, φ were defined for x^0. Obviously, conditions (c1), (c2) remain valid with q, M, φ replaced with $\tilde{q}$, $\tilde{M}$, $\tilde{\varphi}$ and $-\tilde{q}, \tilde{q} \in \operatorname{cl} \tilde{M}$. As for φ, the critical points of $\tilde{\varphi}|_{\tilde{M}}$, that is, the centers of the connected components of $\tilde{M}$, are all contained in the set $\{\mu_1, \dots, \mu_n\} \cup \{\lambda_1, \dots, \lambda_m\}$, hence there is at most $n + m$ of them. Therefore, there exist $\hat{x}^0 \in B$ and a positive integer $p \leqslant n + m$ such that $\tilde{\varphi}|_{\tilde{M}}$ has at most p critical points for each $\tilde{x}^0 \in B$ and it has exactly p of them if $\tilde{x}^0 = \hat{x}^0$. Let $s_1 < \cdots < s_p$ be the critical points of $\tilde{\varphi}|_{\tilde{M}}$ for $\tilde{x}^0 = \hat{x}^0$. Clearly, there is a neighborhood $\tilde{B} \subset B$ of $\hat{x}^0$ in H_0 such that for each $\tilde{x}^0 \in \tilde{B}$ the function $\tilde{\varphi}|_{\tilde{M}}$ has at least p critical points, hence it must have exactly p of them. Moreover, as these critical points must be contained in the finite set $\{\mu_1, \dots, \mu_n\} \cup \{\lambda_1, \dots, \lambda_m\}$, making the neighborhood $\tilde{B}$ smaller, if necessary, we achieve that for each $\tilde{x}^0 \in \tilde{B}$ the critical points of $\tilde{\varphi}|_{\tilde{M}}$ coincide with $s_1, \dots, s_m$. If p is odd, then Lemma 4.5 gives a contradiction to $u_{x_1}(\tilde{x}^0) > 0$, as above. Assume that p is even. Using (c2), and the condition $-\tilde{q}, q \in \operatorname{cl} \tilde{M}$, we obtain that the connected components of $\tilde{M}$ centered at $s_1, s_3, \dots, s_{p-1}$ have all the same length, which is equal to $2(s_1 + \tilde{q})$. Likewise, the connected components centered at $s_2, s_4, \dots, s_p$ have the length $2(\tilde{q} - s_p)$. Moreover, the closed intervals between any two successive connected components of $\tilde{M}$ have all the same length which we denote by $d \geqslant 0$ (these closed intervals reduce to points if $d = 0$). Summing up the lengths of all these intervals and the lengths of the connected components of $\tilde{M}$ we obtain

$$\frac{p}{2}\big(2(s_1 + \tilde{q}) + 2(\tilde{q} - s_p)\big) + (p - 1)d.$$

This must be equal to $2\tilde{q}$, the total length of $(-\tilde{q}, \tilde{q})$. From this we compute

$$\sigma := 2s_p - \tilde{q} - \frac{d}{2} = 2s_p - \frac{p}{(p-1)}\big(2(s_p - s_1)\big).$$

Note that the right-hand side is independent of $\tilde{x}^0 \in \tilde{B}$ (although $\tilde{q}$ and d were defined as functions of $\tilde{x}^0$). Since σ is a point between the two connected components of $\tilde{M}$ centered at s_p and s_{p-1}, we have $\tilde{\varphi}(\sigma) = 0$. This shows that $u = 0$ (and consequently $u_{x_1} = 0$) on the set $\tilde{\Gamma}_\sigma := \{\tilde{x}^0 + \sigma e_1: \tilde{x}^0 \in \tilde{B}\}$, which is open in the relative topology of H_σ. Similarly as above with H_0, applying Proposition 3.2 to $V_\sigma u$ we obtain that $V_\sigma u \equiv 0$ in each connected component of $\Omega \cap P_\sigma(\Omega)$ intersecting $\tilde{\Gamma}_\sigma$. For each $\tilde{x}^0 \in \tilde{B}$ this means that the function $\tilde{\varphi}$ satisfies the additional symmetry condition of Lemma 4.5. Therefore $\tilde{\varphi}'(0) = 0$, which is a contradiction to $u_{x_1}(\tilde{x}^0) > 0$.

We have thus proved that (4.42) leads to a contradiction in all cases. The proof of Theorem 2.2 is complete.

Proof of Lemma 4.5. Let p be the number of critical points of φ in M. This number is finite by (c1), (c2). By (c2), these critical points are the points of local maxima of u around which φ is (globally) even. Elementary considerations using the symmetries show that the following is true. If p is odd, then one of the critical points of φ is necessarily at 0, and if p is even, then $\varphi(0) = 0$. Since $\varphi \geqslant 0$, we have $\varphi'(0) = 0$ in either case. □

We remark, that the possibility that there are intervals of zeros of φ between components of M is not excluded in the proof of Lemma 4.5. It is conceivable that such a degenerate possibility occurs with φ as in (4.43) if $\partial\Omega$ contains a segment of the line T. By reflection, this segment could then become a part of the boundary of some sets G_j and W_i.

Remark 4.6. Having proved Theorem 2.2, consider again the function φ defined as in (4.43) for an arbitrary $x^0 \in \Omega \cap H_0$. This function has properties (c1), (c2) and now we also know that it is even (around $s = 0$). Considering the symmetries of φ, as in the proof of Lemma 4.5, one can also show that the nodal intervals of φ (that is, the connected components of M) have at most two lengths. More specifically, arranging the intervals in a finite sequence $J_1, J_2, \ldots$ such that their centers are increasing, the even-indexed intervals have the same lengths and the odd-indexed intervals have the same lengths. If the number of the nodal intervals is even, then they all have equal lengths.

5. Planar domains

5.1. Domains with partially flat boundaries

In this section we first prove Proposition 2.7 and then mention a generalization.

Proof of Proposition 2.7. Assume the hypotheses of the proposition to be satisfied. We prove that the conclusion holds with $\delta := \delta_0$, where δ_0 is as in Lemma 4.3, with $\alpha_0 = 1$ and $\beta_0 = \gamma$ (the Lipschitz constant of f).

Let $u \not\equiv 0$ be a solution of (1.3), (1.4). We need to rule out $m > 1$, where m is as in Theorem 2.2. Suppose it is the case, so that $\lambda_1 > 0$. From Lemma 4.3 and the proof of Theorem 2.2 it follows that $\lambda_1 < \ell - \delta$. Since $V_{\lambda_1} u \equiv 0$ in Σ_{λ_1} (recall that Σ_{λ_1} is connected by assumption), the Dirichlet boundary condition implies that $u = 0$ on $C := \Omega \cap P_{\lambda_1}(\partial\Omega)$. Of course, since $u \geqslant 0$ in Ω, we have $\nabla u = 0$ on C also. By assumption, $\partial\Omega \cap \Sigma_{\lambda_1}$ contains a nontrivial closed segment of a vertical line. Let S_0 be a maximal vertical line segment in $\partial\Omega \cap \Sigma_{\lambda_1}$ and let $S := P_{\lambda_1} S_0$. Then S is a closed segment of a vertical line H_λ for some $\lambda \in [2\lambda_1 - \ell, \lambda_1)$. Moreover, since $\partial\Omega$ is of class C^1 and Ω is convex in x, $S \subset \Omega$ hence $S \subset C$. Let $z_0 = (x_0, y_0) \in S$ be an end point of S. By the maximality of S_0, C is not a part of H_λ in any neighborhood of z_0 (see Fig. 4). Of course, $S \subset C$ implies that the C^1 curve C is tangent to H_λ at z_0.

Since u and ∇u vanish on S, the function $v = V_\lambda u$ vanishes on S together with v_x. Therefore, by Proposition 3.2(ii), $v \equiv 0$ in Σ_λ. This implies that $u_x = 0$ on $H_\lambda \cap \Omega$. Hence, the nodal set $w^{-1}(0)$ of the function $w := u_x$ contains the segment $H_\lambda \cap \Omega$ and the C^1 curve C. However, w is a nontrivial (strong) solution of the equation

$$\Delta w + f_u\big(y, u(x, y)\big) w = 0, \quad (x, y) \in \Omega,$$

as one can verify in a standard way, applying interior elliptic estimates to the function $(u(x + \epsilon, y) - u(x, y))/\epsilon$ and taking the limit as $\epsilon \to 0$. The structure of the nodal set of w, as described above, contradicts well-know results on local asymptotics of nontrivial solutions of such linear equations near their zeros: there are no two nodal curves through z_0, which are different in any neighborhood of z_0 and tangent at z_0 (see [19] or [20, Theorem 2.1]). □

The assumption that Σ_λ is connected for each λ can be removed if one assumes instead that each connected component of the set $\{(x, y) \in \partial\Omega: x > \ell - \delta\} = \partial\Omega \cap \Sigma_{\ell-\delta}$ contains a vertical line segment. The above proof is

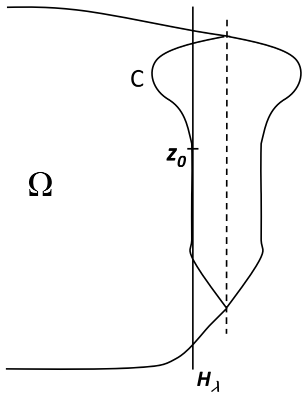

Fig. 4. The nodal curves $H_\lambda \cap \Omega$ and C of u_x are tangent at z_0.

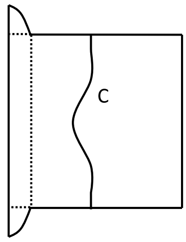

Fig. 5. Domain Ω_2 and curve C (the dotted lines indicate the rectangle in Ω_1 which was replaced).

easily adapted to that case using the facts that $V_{\lambda_1}u = 0$ in a connected component of Σ_{λ_1} and that, since $\lambda_1 \leqslant \ell - \delta$, one has $\Sigma_{\lambda_1} \supset \Sigma_{\ell-\delta}$.

5.2. Computations for Example 2.6

The domain Ω for Example 2.6 will be found in several steps. We start with the square $\Omega_1 = (0,\pi) \times (-\pi/2, \pi/2)$. The second eigenvalue of the negative Dirichlet Laplacian, $-\Delta$, on Ω_1 (further, we just say "the second eigenvalue on Ω_1") is $\mu_0 = 5$ and $\psi_0(x,y) = \sin(2x)\cos y$ is one of the corresponding eigenfunctions. This eigenfunction is even in y and in the class of such functions, μ_0 is a simple eigenvalue.

We now perturb Ω_1 near the y-axis by replacing the narrow rectangle $\{(x,y) \in \Omega_1\colon\ 0 < x < \epsilon,\ |y| < \pi/2\}$, ϵ being a small positive constant, with the set $\{(x,y)\colon\ 0 < x < \eta(y),\ |y| < \pi/2 + \epsilon\}$, where η is a smooth function on $[-\pi/2-\epsilon, \pi/2+\epsilon]$ with

$$\begin{aligned}
&\eta(-y) = \eta(y) > 0 \quad \big(y \in (-\pi/2-\epsilon, \pi/2+\epsilon)\big),\\
&\eta'(y) > 0 \quad \big(y \in [-\pi/2-\epsilon, -\pi/2)\big),\\
&\eta\big(\pm(\pi/2+\epsilon)\big) = 0, \qquad \eta(y) = \epsilon \quad \big(y \in [-\pi/2, \pi/2]\big),
\end{aligned}$$

see Fig. 5.

The new domain obtained this way is denoted by Ω_2. It is still symmetric about the x-axis. If $\epsilon \in (0, \pi/4)$ is sufficiently small, the second eigenvalue μ on Ω_2 in the space of functions even in y is close to $\mu_0 = 5$. Moreover, the corresponding eigenfunction ψ can be taken close to ψ_0, at least in the $H^1(\mathbb{R}^2)$-norm, when both eigenfunctions are extended by 0 outside their respective domains (see [2,3,11], for example). Using elliptic interior and boundary estimates, one can further show that ψ is close to ψ_0 in $C^2([\pi/4, 3\pi/4] \times [-\pi/2, \pi/2])$, as only flat parts of the boundary are involved. From there it is not difficult to verify (we omit the details) that the nodal set of ψ consists of $\partial\Omega_2$ and a C^1 curve $C = \{(\xi(y), y)\colon\ y \in [-\pi/2, \pi/2]\}$ in $(\pi/4, 3\pi/4) \times [-\pi/2, \pi/2]$, see Fig. 5. Moreover, the function ξ is even and it is analytic in $(-\pi/2, \pi/2)$, by the implicit function theorem, since ψ is analytic and $\nabla\psi \neq 0$

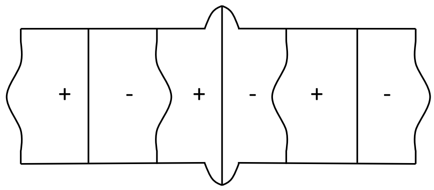

Fig. 6. Ω and the nodal domains of the eigenfunction v.

along the nodal curve in Ω_2. (It can also be proved that C is orthogonal to $\partial\Omega_2$ at the two points where it meets it, but this is not used below.)

We next show that the nodal curve C is not flat (hence, it has no flat part by analyticity). Assume C is a line segment. Since ξ is even, this means that $\xi \equiv c = \text{const}$. But then ψ is the principal eigenfunction on the rectangle $(c,\pi)\times(-\pi/2,\pi/2)$, μ being the principal eigenvalue. Since the principal eigenvalue is simple, for suitable constants α, β, we have

$$\psi(x,y) = \big(\alpha \sin(x\sqrt{\mu}) + \beta\cos(x\sqrt{\mu})\big)\cos y$$

in $(c,\pi)\times(-\pi/2,\pi/2)$. By analyticity, this identity is valid in the whole of Ω_2. In particular, $\psi(x,\pm\pi/2)=0$ for all $x\in(0,\pi)$, a contradiction. So C is not flat.

We now enlarge Ω_2 by first adding to it the reflection of $\{(x,y)\in\Omega_2\colon \xi(y)<x\leqslant\pi\}$ in the line $\{x=\pi\}$ and then adding the reflection of that extended domain in $\{x=0\}$. See Fig. 6. We denote the resulting domain by Ω. Taking also the odd extensions of ψ across the lines $\{x=\pi\}$, $\{x=0\}$, we obtain an eigenfunction w on Ω with the same eigenvalue μ. The nodal set of w consists of $\partial\Omega$, the curves C, P_0C, and the intersections of Ω with the lines $\{x=-\pi\}$, $\{x=0\}$, and $\{x=\pi\}$. By definition, w is odd about these three lines and it is even in y. We set $v=-w$ so that the signs of v in its nodal domains are as indicated in Fig. 6. Define

$$u(x,y) = \int_0^x v(s,y)\,ds - m(y) \quad \big((x,y)\in\bar{\Omega}\big), \tag{5.1}$$

where $m\in C^2[-\pi/2-\epsilon,\pi/2+\epsilon]$ is an even function such that

$$m(y) = \int_0^{\xi(y)} v(s,y)\,ds \quad \big(|y|\leqslant\pi/2\big), \tag{5.2}$$

$$\int_0^{\eta(y)} v(s,y)\,ds < m(y) < 0 \quad \big(\pi/2<|y|<\pi/2+\epsilon/2\big), \tag{5.3}$$

$$m(y) = 0 \quad \big(|y|\geqslant\pi/2+\epsilon/2\big). \tag{5.4}$$

Note that $\int_0^{\eta(y)} v(s,y)\,ds<0$ for $y\in(-\pi/2-\epsilon,-\pi/2]\cup[\pi/2,\pi/2+\epsilon)$ so that (5.3) is meaningful. To see that a function m with these properties exists, we need to show that identity (5.2) defines a C^2 function on $[-\pi/2,\pi/2]$ and that (5.2) is compatible with conditions (5.3). Obviously, m is smooth in $(-\pi/2,\pi/2)$ and continuous on $[-\pi/2,\pi/2]$. For any $y\in(-\pi/2,\pi/2)$,

$$m'(y) = v\big(\xi(y),y\big)\xi'(y) + \int_0^{\xi(y)} v_y(s,y)\,ds = \int_0^{\xi(y)} v_y(s,y)\,ds$$

and

$$\begin{aligned}
m''(y) &= v_y\big(\xi(y), y\big)\xi'(y) + \int_0^{\xi(y)} v_{yy}(s, y)\,ds \\
&= v_y\big(\xi(y), y\big)\xi'(y) - \int_0^{\xi(y)} \big(\mu v(s, y) + v_{xx}(s, y)\big)\,ds \\
&= v_y\big(\xi(y), y\big)\xi'(y) - \mu m(y) + v_x(0, y) - v_x\big(\xi(y), y\big).
\end{aligned}$$

Since v is smooth up to the boundary near $(\pm\pi/2, \xi(\pm\pi/2))$, and ξ is C^1 on $[-\pi/2, \pi/2]$, we see that $m \in C^2[-\pi/2, \pi/2]$. Next,

$$\lim_{y \nearrow \frac{\pi}{2}} m(y) = \int_0^{\xi(\frac{\pi}{2})} v\left(s, \frac{\pi}{2}\right) ds = \int_0^{\eta(\frac{\pi}{2})} v\left(s, \frac{\pi}{2}\right) ds,$$

since $v = 0$ on the segment $\{(x, \pi/2)\colon\ \eta(\pi/2) < x < \xi(\pi/2)\}$ of the boundary. On the same segment, the normal derivative v_y is positive, as $v < 0$ in Ω near it (see Fig. 6). Therefore,

$$\lim_{y \nearrow \frac{\pi}{2}} m'(y) = \int_0^{\xi(\frac{\pi}{2})} v_y\left(s, \frac{\pi}{2}\right) ds > \int_0^{\eta(\frac{\pi}{2})} v_y\left(s, \frac{\pi}{2}\right) ds = \left(\int_0^{\eta(y)} v(s, y)\,ds\right)'\Bigg|_{y=\frac{\pi}{2}-}.$$

This shows the compatibility of (5.2) and (5.3) at $y = \pi/2$. Since m is even in $[-\pi/2, \pi/2]$, we get the compatibility at $y = -\pi/2$, as well. Hence a function m with the indicated properties exists.

Clearly, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and

$$\begin{aligned}
\Delta u(x, y) &= v_x(x, y) + \int_0^x v_{yy}(s, y)\,ds - m''(y) \\
&= v_x(x, y) - \int_0^x \big(v_{xx}(s, y) + \mu v(s, y)\big)\,ds - m''(y) \\
&= v_x(0, y) - \mu u(x, y) - \mu m(y) - m''(y).
\end{aligned}$$

Hence u solves (2.8) with $h(y) := v_x(0, y) - \mu m(y) - m''(y)$. This a continuous x-independent function on Ω. We will shortly make a final perturbation of the domain Ω, shrinking it a little near the y-axis. With that perturbation we will achieve that h is continuous on $\bar{\Omega}$ and that the boundary condition (2.9) is satisfied.

It is straightforward to verify, using the oddness properties of v, that u is even in x, $u \geqslant 0$ in $\{(x, y) \in \Omega\colon\ |y| \leqslant \pi/2\}$, and that u vanishes on $\{(x, y) \in \partial\Omega\colon\ |x| \geqslant \epsilon\}$ and on the curves C, P_0C. Also

$$\begin{aligned}
&u(0, y) > 0 \quad \big(\pi/2 \leqslant |y| < \pi/2 + \epsilon/2\big), \qquad u\big(0, \pm(\pi/2 + \epsilon/2)\big) = 0, \\
&u\big(\eta(y), y\big) < 0 \quad \big(\pi/2 < |y| \leqslant \pi/2 + \epsilon/2\big), \qquad u\big(\eta(\pm\pi/2), \pm\pi/2\big) = 0.
\end{aligned}$$

Therefore, for each y with $\pi/2 \leqslant |y| \leqslant \pi/2 + \epsilon/2$ there is $\tilde{\eta}(y) \in [0, \eta(y)]$ such that

$$\begin{aligned}
&\tilde{\eta}(\pm\pi/2) = \eta(\pm\pi/2), \qquad \tilde{\eta}\big(\pm(\pi/2 + \epsilon/2)\big) = 0, \\
&0 < \tilde{\eta}(y) < \eta(y) \quad \big(\pi/2 < |y| < \pi/2 + \epsilon/2\big), \\
&u\big(\pm\tilde{\eta}(y), y\big) = 0.
\end{aligned}$$

The value $\tilde{\eta}(y)$ is unique and depends continuously on y, since $u_x(x, y) = v(x, y) < 0$ for $x \in (0, \eta(y))$. We now shrink Ω in $\Omega \cap \{(x, y)\colon\ |y| \geqslant \pi/2\}$ such that the nodal curves $\{(\pm\tilde{\eta}(y), y)\colon\ \pi/2 \leqslant |y| \leqslant \pi/2 + \epsilon/2\}$ become a part of its boundary, in place of $\{(\pm\eta(y), y)\colon\ \pi/2 \leqslant |y| \leqslant \pi/2 + \epsilon\}$. On this smaller domain Ω, $u \geqslant 0$ is a solution of (2.8), (2.9) and it has the interior nodal curves C, P_0C.

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