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Ann. I. H. Poincaré – AN 28 (2011) 357–384

ANNALES DE L'INSTITUT HENRI POINCARÉ
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Two soliton collision for nonlinear Schrödinger equations in dimension 1

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Received 14 June 2009; received in revised form 16 January 2011; accepted 28 January 2011

Available online 24 February 2011

Abstract

We study the collision of two solitons for the nonlinear Schrödinger equation $i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi$, $F(\xi) = -2\xi + O(\xi^2)$ as $\xi \to 0$, in the case where one soliton is small with respect to the other. We show that in general, the two soliton structure is not preserved after the collision: while the large soliton survives, the small one splits into two outgoing waves that for sufficiently long times can be controlled by the cubic NLS: $i\psi_t = -\psi_{xx} - 2|\psi|^2\psi$.

0. Introduction

In this paper we consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + F\big(|\psi|^2\big)\psi, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \tag{0.1}$$

where F is a smooth function that satisfies $F(\xi) = -2\xi + O(\xi^2)$, as $\xi \to 0$.

This equation possesses solutions of special form – solitary waves (or, shortly, solitons):

$$e^{i\Phi(x,t)}\varphi\big(x - b(t), E\big),$$
$$\Phi(x,t) = \omega t + \gamma + \frac{1}{2}vx, \qquad b(t) = vt + c, \qquad E = \omega + \frac{v^2}{4} > 0,$$

where $\omega, \gamma, c, v \in \mathbb{R}$ are constants and φ is the ground state that is a smooth positive even exponentially decreasing solution of the equation

$$-\varphi_{xx} + E\varphi + F\big(\varphi^2\big)\varphi = 0, \quad \varphi \in H^1. \tag{0.2}$$

In this paper we shall be concerned with the solutions of (0.1) that behave as $t \to -\infty$ like a sum of two nonlinearly stable solitons

$$e^{i\Phi_0}\varphi\big(x - b_0(t), E_0\big) + e^{i\Phi_1}\varphi\big(x - b_1(t), E_1\big),$$

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.02.002

$\Phi_j = \omega_j t + \gamma_j + \frac{1}{2}v_j x,\ b_j(t) = v_j t,\ v_1 - v_0 \neq 0$, our goal being to understand the collision between the solitons and to determine what happens after. We show that in the case where $E_1 \equiv \varepsilon^2 \ll 1$ (depending on $v_1 - v_0$ and E_0) the collision leads to the splitting of the small soliton into two outgoing parts, that at least up to the times $t \sim \varepsilon^{-2}|\ln \varepsilon|$ propagate independently according to the cubic NLS:

$$i\psi_t = -\psi_{xx} - 2|\psi|^2\psi. \tag{0.3}$$

The splitting of the small soliton is essentially controlled by the flow linearized around the "large" one: in the interaction region a small amplitude soliton behaves as a slowly modulated plane wave $\varepsilon e^{-iv_1^2 t/4 + iv_1 x/2}$ and is splitted by the large soliton into a reflected and a transmitted parts accordingly to the linear scattering theory. For the first time this phenomenon was observed by J. Holmer, J. Marzuola, M. Zworski [3,4] in the context of soliton–potential interaction for the cubic NLS with an external delta potential:

$$i\psi_t = -\psi_{xx} + \delta(x)\psi - 2|\psi|^2\psi,$$

see also [10].

To control the solution in the post-interaction region $\varepsilon^{-1-\delta} \leqslant t \leqslant \delta\varepsilon^{-2}|\ln \varepsilon|$ one invokes the orbital stability argument combined with the integrability of (0.3), again in the spirit of [3,4].

The structure of this paper is briefly as follows. It consists of three sections. In the first section we introduce some preliminary objects and state the main results. The second and the third ones contain the complete proofs of the indicated results, some technical details being removed to Appendix A.

1. Background and statement of the results

1.1. Assumptions on F

Consider the nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + F\big(|\psi|^2\big)\psi, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

We assume the following.

Hypothesis (H1). F is a C^∞ function, $F(\xi) = -2\xi + O(\xi^2)$, as $\xi \to 0$, and satisfies:

$$F(\xi) \geqslant -C\xi^q, \quad C > 0,\ \xi \geqslant 1,\ q < 2. \tag{1.2}$$

It is well known, see [1] and references therein, that under assumption (H1) the Cauchy problem for Eq. (1.1) is globally well posed in H^1: for any $\psi_0 \in H^1(\mathbb{R})$ there exists a unique solution $\psi \in C(\mathbb{R}, H^1)$ of (1.1) satisfying $\psi(0) = \psi_0$. Furthermore, for all $t \in \mathbb{R}$ one has the conservation of L_2 norm, of energy and of momentum:

$$\big\|\psi(t)\big\|_2 = \|\psi_0\|_2, \tag{1.3}$$

$$H\big(\psi(t)\big) = \int dx\,\big(\big|\psi_x(x,t)\big|^2 + U\big(\big|\psi(x,t)\big|^2\big)\big) = H(\psi_0), \qquad U(\xi) = \int_0^\xi ds\, F(s), \tag{1.4}$$

$$P\big(\psi(t)\big) = \int dx\,(\psi_x\bar\psi - \bar\psi_x\psi) = P(\psi_0). \tag{1.5}$$

Recall also that conservation of energy (1.4) and of L_2 norm (1.3) combined with (1.2) imply an a priori bound on the H^1 norm of the solution:

$$\big\|\psi(t)\big\|_{H^1} \leqslant c\big(\|\psi_0\|_{H^1}\big)\|\psi_0\|_{H^1},$$

with some smooth function $c : \mathbb{R}_+ \to \mathbb{R}_+$.

Furthermore, if in addition $x\psi_0 \in L_2$, then $x\psi \in C(\mathbb{R}, L_2)$ and

$$\big\|x\psi(t)\big\|_2 \leqslant c\big(\|\psi_0\|_{H^1}\big)\big(\|x\psi_0\|_2 + |t|\|\psi_0\|_{H^1}\big). \tag{1.6}$$

Set $\mathcal{U}(\xi, E) = E\xi + U(\xi)$. We denote by $\mathcal{A}$ the set of E, $E > 0$, such that the function $\xi \to \mathcal{U}(\xi, E)$ has a positive root and $\mathcal{U}_\xi(\xi_0, E) \neq 0$, where $\xi_0 = \xi_0(E)$ is the smallest positive root. Note that under assumption (H1) this condition is always verified for E sufficiently small.

It is easily seen that for $E \in \mathcal{A}$, (0.2) has a unique smooth positive even exponentially decreasing solution $\varphi(x, E)$: $\varphi(x, E) \sim C(E)e^{-\sqrt{E}|x|}$. Moreover, as $E \to 0$, $\varphi(x, E)$ admits an asymptotic expansion of the following form:

$$\varphi(x, E) = \varepsilon\hat{\varphi}(\varepsilon x, \varepsilon), \qquad \hat{\varphi}(y, \varepsilon) = \sum_{k=0} \varepsilon^{2k}\varphi_k(y), \qquad \varepsilon = \sqrt{E},$$

$$\varphi_0(y) = \frac{1}{\operatorname{ch} y}, \qquad -\varphi_0'' + \varphi_0 - 2\varphi_0^3 = 0,$$

$$\left|\varphi_k(y)\right| \leqslant C_k e^{-|y|}. \tag{1.7}$$

Asymptotic expansion (1.7) holds in the sense:

$$\left|\hat{\varphi}(y, \varepsilon) - \sum_{k=0}^{N} \varepsilon^{2k}\varphi_k(y)\right| \leqslant C_N \varepsilon^{2N+2} e^{-|y|},$$

and can be differentiated any number of times with respect to y.

We shall call the functions $w(x, \sigma) = \exp(i\beta + ivx/2)\varphi(x - b, E)$, $\sigma = (\beta, E, b, v) \in \mathbb{R}^4$ by soliton states. $w(x, \sigma(t))$ is a solitary wave solution iff $\sigma(t)$ satisfies the system:

$$\beta' = E - \frac{v^2}{4}, \qquad E' = 0, \qquad b' = v, \qquad v' = 0. \tag{1.8}$$

1.2. Linearization

Consider the linearization of Eq. (1.1) on a soliton $w(x, \sigma(t))$:

$$\psi \sim w + \chi,$$

$$i\chi_t = \left(-\partial_x^2 + F\left(|w|^2\right)\right)\chi + F'\left(|w|^2\right)\left(|w|^2\chi + w^2\bar{\chi}\right).$$

Introducing the function $\vec{f}$:

$$\vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \qquad \chi(x, t) = \exp(i\Phi) f(y, t),$$

$$\Phi = \beta(t) + \frac{vx}{2}, \qquad y = x - b(t),$$

one gets

$$i\vec{f}_t = L(E)\vec{f}, \qquad L(E) = L_0(E) + V(E), \qquad L_0(E) = \left(-\partial_y^2 + E\right)\sigma_3,$$

$$V(E) = V_1(E)\sigma_3 + iV_2(E)\sigma_2,$$

$$V_1 = F\left(\varphi^2\right) + F'\left(\varphi^2\right)\varphi^2, \qquad V_2(E) = F'\left(\varphi^2\right)\varphi^2.$$

Here σ_2, σ_3 are the standard Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider L as an operator in $L_2(\mathbb{R} \to \mathbb{C}^2)$ defined on the domain where L_0 is self-adjoint. L satisfies the relations

$$\sigma_3 L \sigma_3 = L^*, \qquad \sigma_1 L \sigma_1 = -L,$$

where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The continuous spectrum of $L(E)$ fills up two semi-axes $(-\infty, -E]$ and $[E, \infty)$. In addition $L(E)$ may have finite and finite dimensional point spectrum on the real and imaginary axes.

Zero is always a point of the discrete spectrum. One can indicate two eigenfunctions

$$\vec{\xi}_0 = \varphi\begin{pmatrix}1\\-1\end{pmatrix}, \qquad \vec{\xi}_1 = \varphi_y\begin{pmatrix}1\\1\end{pmatrix},$$

and two generalized eigenfunctions

$$\vec{\xi}_2 = -\varphi_E\begin{pmatrix}1\\1\end{pmatrix}, \qquad \vec{\xi}_3 = -\frac{1}{2}y\varphi\begin{pmatrix}1\\-1\end{pmatrix},$$

$$L\vec{\xi}_j = 0, \qquad L\vec{\xi}_{2+j} = \vec{\xi}_j, \quad j = 0, 1.$$

Let $M(E)$ be the generalized null space of the operator $L(E)$.

Proposition 1.1. *The spectrum of $L(E)$ is real and the vectors $\vec{\xi}_j$, $j = 0, \dots, 3$, span the subspace M iff*

$$\frac{d}{dE}\|\varphi(E)\|_2^2 > 0. \tag{1.9}$$

See [12,2] for example.

Note that under assumption (H1) condition (1.9) is verified at least for E sufficiently small.

We denote by $\mathcal{A}_0$ the set of E, $E \in \mathcal{A}$, such that (1.9) holds.

Consider the evolution operator e^{-itL}. One has the following proposition.

Proposition 1.2. *For $E \in \mathcal{A}_0$, one has*

$$\|e^{-itL(E)}P(E)f\|_{H^1(\mathbb{R})} \leqslant C\|f\|_{H^1(\mathbb{R})}, \tag{1.10}$$

where $P(E) = I - P_0(E)$, $P_0(E)$ being the spectral projection of $L(E)$ corresponding to the zero eigenvalue:

$$\operatorname{Ker} P_0 = (\sigma_3 M)^{\perp}, \qquad \operatorname{Ran} P_0 = M.$$

The constant C here is uniform with respect to E in compact subsets of $\mathcal{A}_0$.

The proof of this assertion can be found in [12].

Next we introduce some notions related to the scattering problem for the operator $L(E)$.

Consider the equation

$$Lf = \lambda f, \quad \lambda \in (-\infty, -E] \cup [E, +\infty). \tag{1.11}$$

Since $\sigma_1 L = -L\sigma_1$, it suffices to consider the solutions for $\lambda \geqslant E$. In [2] (see also [6]) a basis of solutions f_j, $j = 1, \dots, 4$, with the standard behavior $e^{\pm ikx}\binom{1}{0}$, $e^{\pm\mu x}\binom{0}{1}$, $\lambda = E + k^2$, $\mu = \sqrt{2E + k^2} > 0$, as $x \to +\infty$, was constructed. We collect here some properties of these solutions that we shall need later, all the details and the proofs can be found in [2]. The solutions $f_j(x, k)$, $j = 1, \dots, 4$, are smooth functions of x and k with the following asymptotics as $x \to +\infty$:

$$f_{1,2}(x, k) = e^{\pm ikx}\left[\begin{pmatrix}1\\0\end{pmatrix} + O\left(\left(1 + |k|\right)^{-1}e^{-\gamma x}\right)\right], \tag{1.12}$$

$$f_{3,4}(x, k) = e^{\mp\mu x}\left[\begin{pmatrix}0\\1\end{pmatrix} + O\left(\left(1 + |k|\right)^{-1}e^{-\gamma x}\right)\right], \tag{1.13}$$

uniformly with respect to $k \in \mathbb{R}$.

The asymptotic representations (1.12), (1.13) can be differentiated with respect to x and k any number of times.

Here and below we use γ as a general notation for the positive constants that may change from line to line.

Furthermore, one can choose f_j in such a way that

$$f_1(x, -k) = f_2(x, k), \qquad f_{3,4}(x, -k) = f_{3,4}(x, k),$$
$$\overline{f_1(x, k)} = f_2(x, k), \qquad \overline{f_{3,4}(x, k)} = f_{3,4}(x, k), \quad k \in \mathbb{R},$$
$$w(f_1, f_2) = 2ik, \qquad w(f_1, f_3) = 0,$$
$$w(f_1, f_4) = 0, \qquad w(f_3, f_4) = -2\mu.$$

Here $w(f,g)$ stands for the Wronskian:

$$w(f,g)=\langle f',g\rangle_{\mathbb{R}^2}-\langle f,g'\rangle_{\mathbb{R}^2},$$

$w(f,g)$ does not depend on x if f and g are solutions of (1.11).

The solutions with standard behavior as $x\to-\infty$ can be obtained by using the fact that the operator L is invariant under the change of variable $x\to-x$. Let

$$g_j(x,k)=f_j(-x,k),\quad j=1,\dots,4.$$

In addition to scalar Wronskian we shall also use matrix Wronskian

$$W(F,G)=F^{t'}G-F^tG',$$

where F and G are 2×2 matrices composed of pairs of solutions. The matrix Wronskian does not depend on x.

Consider the matrix solutions

$$\begin{aligned}&F_1=(f_1,f_3),\qquad F_2=(f_2,f_4),\qquad G_1=(g_2,g_4),\qquad G_2=(g_1,g_3),\\&G_1(x,k)=F_2(-x,k),\qquad G_2(x,k)=F_1(-x,k).\end{aligned}\tag{1.14}$$

For $k\neq 0$ we have

$$F_1=G_1A+G_2B,\qquad G_2=F_2A+F_1B,\tag{1.15}$$

where A and B are constant matrices given by:

$$\begin{aligned}&A^t(2ik\mathbf{p}-2\mu\mathbf{q})=W(F_1,G_2),\\&-B^t(2ik\mathbf{p}-2\mu\mathbf{q})=W(F_1,G_1).\end{aligned}$$

Here

$$\mathbf{p}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad \mathbf{q}=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

As $|k|\to+\infty$,

$$A(k)=I+O(k^{-1}),\qquad B(k)=O(k^{-1}).$$

It was shown in [2] that for $k\neq 0$, $\det A(k)=0$ if and only if $E+k^2$ is an eigenvalue of $L(E)$.

For $k\in\mathbb{R}$, $E+k^2\notin\sigma_p(L(E))$, one can define the solutions

$$\mathcal{F}(x,k)=F_1(x,k)A^{-1}(k)e,\qquad \mathcal{G}(x,k)=\mathcal{F}(-x,k),\qquad e=\begin{pmatrix}1\\0\end{pmatrix}.$$

It follows from (1.14), (1.15) that $\mathcal{F}$, which is bounded as $x\to+\infty$ by the definition of F_1, is also bounded as $x\to-\infty$:

$$\mathcal{F}=\big[\overline{G_2}+G_2BA^{-1}\big]e.$$

More precisely, $\mathcal{F}$ satisfies

$$\begin{aligned}&\mathcal{F}(x,k)=se^{ikx}e+O\big((1+|k|)^{-1}e^{-\gamma x}\big),\quad x\to+\infty,\\&\mathcal{F}(x,k)=\big(e^{ikx}+r(k)e^{-ikx}\big)e+O\big((1+|k|)^{-1}e^{\gamma x}\big),\quad x\to-\infty.\end{aligned}$$

Here the coefficients s and r are defined by the relations

$$\mathbf{p}A^{-1}e=se,\qquad \mathbf{p}BA^{-1}e=re.$$

One has

$$|s|^2+|r|^2=1,\qquad r\bar{s}+s\bar{r}=0,\qquad \mathcal{F}=\bar{s}^{-1}\overline{\mathcal{G}}+rs^{-1}\mathcal{G}.$$

We call the solutions $\mathcal{F}(x,k)$, $\mathcal{G}(x,k)$ solutions of the scattering problem, $s(k)$ and $r(k)$ being respectively the transmission and reflection coefficients.

1.3. Description of the problem

We are interested in the solutions that as $t \to -\infty$ are given by a two soliton state:

$$\psi = w\big(x, \sigma_0(t)\big) + w\big(x, \sigma_1(t)\big) + o_{H^1}(1), \quad t \to -\infty,$$

where $\sigma_j(t) = (\beta_j(t), E_j, b_j(t), v_j)$, $j = 0, 1$, are two solutions of (1.8). One can always suppose that

$$b_0(t) = v_0 = 0, \qquad \beta_1(0) = b_1(0) = 0, \qquad v_1 \geqslant 0.$$

Assume that

(H2) $v_1 > 0$;
(H3) $E_0 \in \mathcal{A}_0$;
(H4) $m(E_0 + v_1^2/4) \notin \sigma_p(L(E_0))$ for $m = 1, 2$;
(H5) $\epsilon \equiv \sqrt{E_1}$ *is sufficiently small.*[1]

Then one has the following proposition.

Proposition 1.3. *Under assumptions* (H1), (H2), (H3), (H5), *Eq.* (1.1) *has a unique solution* $\psi \in C(\mathbb{R}, H^1)$ *such that*

$$\big\|\psi(t) - w\big(\cdot, \sigma_0(t)\big) - w\big(\cdot, \sigma_1(t)\big)\big\|_{H^1} = O\big(e^{\gamma\varepsilon t}\big), \quad t \to -\infty.$$

See Appendix A for the proof.

Towards the understanding of the behavior of the above solutions for large positive times we have the following partial result.

Theorem 1.1. *Assume* (H1)–(H5). *Let* $\psi(t)$ *be a solution of* (1.1) *given by Proposition* 1.3. *Then, for* $\varepsilon^{-1-\delta} \leqslant t \leqslant \delta\varepsilon^{-2}|\ln\varepsilon|$, $0 < \delta$ *sufficiently small,* $\psi(t)$ *admits the representation*

$$\psi(t) = w\big(\cdot, \sigma(t)\big) + \psi_+(t) + \psi_-(t) + h(t), \qquad \sigma(t) = \big(\beta(t), E_0, b(t), V_0\big).$$

Here

(i) $V_0 = 2\varepsilon\kappa$, $\kappa = \frac{v_1|r(v_1/2)|^2}{\|\varphi(E_0)\|_2^2}$,

$$\big|\beta(t) - \beta_0(t)\big|, \big|b(t) - tV_0\big| \leqslant C\varepsilon^2 t.$$

(ii) $\psi_\pm(x,t) = e^{-iv_1^2 t/4 \pm i v_1 x/2}\varepsilon\zeta^\pm(\varepsilon(x \mp v_1 t), \varepsilon^2 t)$, *where* $\zeta^\pm(y,\tau)$ *is the solution of the Cauchy problem*:

$$i\zeta^\pm_\tau = -\zeta^\pm_{yy} - 2\big|\zeta^\pm\big|^2\zeta^\pm,$$
$$\zeta^+\big|_{\tau=0} = \alpha_+\varphi_0(y),$$
$$\zeta^-\big|_{\tau=0} = \alpha_- e^{2i\kappa S(y)}\varphi_0(y),$$
$$\alpha_+ = s(v_1/2), \qquad \alpha_- = r(v_1/2), \qquad S(y) = \ln(\operatorname{ch} y) + y + \ln 2,$$

$s(k)$, $r(k)$ being the transmission and reflection coefficients of the operator $L(E_0)$.

(iii) $$\big\|h(t)\big\|_{H^1} \leqslant C\varepsilon^{3/2}e^{C\varepsilon^2 t}.$$

Remark. As $\varepsilon^2 t = \tau \to +\infty$, $\zeta^\pm(\tau)$ resolves into solitons plus radiation. More precisely, one has

$$\big\|\zeta^\pm(\tau)\big\|_\infty = O\big(\langle\tau\rangle^{-1/2}\big), \quad \text{if } a_\pm < \frac{1}{4},$$

$$\big\|\zeta^\pm(\tau)\big\|_\infty = O\big(\ln(2+\tau)\langle\tau\rangle^{-1/2}\big), \quad \text{if } a_\pm = \frac{1}{4},$$

[1] "Sufficiently small" assumes constants that depend only on v_1 and E_0.

and finally for $\frac{1}{4} < a_\pm < \frac{9}{4}$,

$$\zeta^\pm(y,\tau) = e^{-iv_\pm^2 t/4 + i\mu_\pm^2 t + iv_\pm y/2 + i\gamma_\pm}\mu_\pm\varphi_0\big(\mu_\pm(y - v_\pm\tau)\big) + O_{L_\infty}\big(\langle\tau\rangle^{-1/2}\big),$$

with some $\gamma_\pm \in \mathbb{R}$ and $\mu_\pm = 2\sqrt{a_\pm} - 1$. Here

$$a_+ = |\alpha_+|^2, \qquad a_- = |\alpha_-|^2 - \kappa^2,$$
$$v_+ = 0, \qquad v_- = 4\kappa.$$

See [3–5,11], for example.

1.4. Outline of the proof

To prove Theorem 1.1 we split the analysis into two parts by considering separately the time interval $(-\infty, T]$, $T = \varepsilon^{-1-\delta}$, $\delta > 0$ sufficiently small, where the interaction takes place, and the post-interaction region $[T, \delta\varepsilon^{-2}|\ln\varepsilon|)$, our main tool being the construction of a suitable approximate solution in the collision region in the spirit of [3,4, 8–10]. The large time behavior is then controlled by combining the perturbation arguments of [3,4] with the orbital stability techniques [12], see also [7]. We now describe briefly the main steps of the corresponding constructions.

Step 1. Re-parametrisation of the flow. We start by decomposing the solution $\psi(t)$ as

$$\psi(x,t) = e^{i\beta(t) + iv(t)x/2}\big(\varphi\big(x - b(t), E(t)\big) + f\big(x - b(t), t\big)\big), \tag{1.16}$$

the time-dependent parameters $\sigma(t) = (\beta(t), E(t), b(t), v(t))$ being determined by the orthogonality conditions

$$\big\langle \vec{f}(t), \sigma_3\vec{\xi}_k\big(E(t)\big)\big\rangle = 0, \quad k = 0, \ldots, 3. \tag{1.17}$$

Here $\vec{f} = \binom{f}{\bar{f}}$, $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\mathbb{R} \to \mathbb{C}^2)$. The decomposition (1.16), (1.17) satisfies the following orbital stability bounds: for all $t \in \mathbb{R}$,

$$\big\|f(t)\big\|_{H^1} \leqslant C\varepsilon^{1/2}, \qquad \big|v(t)\big| + \big|E(t) - E_0\big| \leqslant C\varepsilon,$$

see Lemma 2.1 below.

Rewriting (1.1) in terms of $f(t)$ and $\sigma(t)$ one gets a system of coupled equations of the form:

$$i\vec{f}_t = L\big(E(t)\big)\vec{f} + D\big(E(t), \vec{f}\big),$$
$$\lambda = G\big(E(t), \vec{f}\big),$$

where $\lambda = (\beta' - E, E', b' - v, v')$ and D, G are some nonlinear functions that are at least quadratic in f.

Step 2. Construction of an approximate solution. Motivated by the fact that $\sigma(t) - \sigma_0(t)$ is a slow function of t, we perform a kind of "normal form analysis" and construct iteratively the main order terms of $f(t)$ and $\sigma(t)$ for $t \leqslant T$ as a power expansion in ε, see Sections 2.3, 2.4, 2.5. At this stage the considerations are formal.

Step 3. Control of the remainder. The remainder is controlled by energy estimates of the linearized flow close to a solitary wave. The success of this strategy relies crucially on the fact that the formal expansions of f and σ have been pushed to a sufficiently large order in ε. As a final outcome of our analysis we obtain a complete description of the solution on the time interval $t \leqslant T$ up to the terms of order $\varepsilon^{3/2}$ and, in particular, prove that for $\varepsilon^{-1}|\ln\varepsilon|^2 \leqslant t \leqslant \varepsilon^{-1-\delta}$, it resolves into a modulated solution $w(\cdot,\sigma(t))$ plus two outgoing waves $\psi_\pm(t)$ of the cubic NLS:

$$\psi(t) = w\big(\cdot,\sigma(t)\big) + \psi_+(t) + \psi_-(t) + O_{H^1}\big(\varepsilon^{3/2}\big),$$

$\sigma(t) = (\beta(t), E(t), b(t), v(t))$ with $b(t)$ satisfying $|b(t) - tV_0| \leqslant C\varepsilon^2 t$.

Step 4. Control of the solution in the post-interaction region $\varepsilon^{-1-\delta} \leqslant t \leqslant \delta\varepsilon^{-2}|\ln\varepsilon|$. We are now left with the following Cauchy problem

$$i\psi_t = -\psi_{xx} + F\big(|\psi|^2\big)\psi, \quad x \in \mathbb{R},\ t \geqslant T,$$
$$\psi|_{t=T} = w\big(\cdot, \sigma(T)\big) + \psi_+(T) + \psi_-(T) + O_{H^1}\big(\varepsilon^{3/2}\big).$$

We use once more the representation (1.16), (1.17) further decomposing $\chi(x,t) = e^{i\beta(t)+iv(t)x/2} f(x-b(t),t)$ as

$$\chi(t) = \psi_+(t) + \psi_-(t) + r(t).$$

To control $r(t)$ we employ again the energy estimates of the linearized flow around a solitary wave. In this way we are able to show that

$$\big\|r(t)\big\|_{H^1} \leqslant C\varepsilon^{3/2} e^{C\varepsilon^2 t}$$

up to the times $t \leqslant \delta\varepsilon^{-2}|\ln\varepsilon|$. This result relies heavily on the fact that for $\varepsilon^{-1-\delta} \leqslant t \leqslant \delta\varepsilon^{-2}|\ln\varepsilon|$, $\psi_+(t)$ and $\psi_-(t)$ remain decoupled from each other and from the large soliton $w(\cdot,\sigma(t))$ in all orders of ε.

2. Pre-interaction and interaction region: $t \leqslant T$

2.1. Decomposition of the solution

We represent $\psi(t)$ as a sum

$$\psi(x,t) = w\big(x, \sigma(t)\big) + \chi(x,t). \tag{2.1}$$

Here $\sigma(t) = (\beta(t), E(t), b(t), v(t))$ is an arbitrary trajectory in the set of admissible values of parameters.

We fix the decomposition (2.1) by imposing the orthogonality conditions

$$\big\langle \vec{f}(t), \sigma_3 \vec{\xi}_k\big(E(t)\big)\big\rangle = 0, \quad k = 0, \dots, 3. \tag{2.2}$$

Here

$$\vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \qquad \chi(x,t) = \exp(i\Phi) f(y,t),$$
$$\Phi = \beta(t) + vx/2, \qquad y = x - b(t).$$

The existence of such a decomposition is guaranteed by the following lemma.

Lemma 2.1. *System* (2.1), (2.2) *has a unique C^1 solution $\sigma(t)$, $\sigma(t) - \sigma_0(t) = O(e^{\gamma\varepsilon t})$ as $t \to -\infty$. Moreover, one has for all $t \in \mathbb{R}$*

$$\big\|\chi(t)\big\|_{H^1} \leqslant C\varepsilon^{1/2}, \tag{2.3}$$
$$\big|v(t)\big| + \big|E(t) - E_0\big| \leqslant C\varepsilon. \tag{2.4}$$

Proof. This lemma is a standard consequence of the orbital stability of $w(\cdot,\sigma_0(t))$, combined with Proposition 1.3 (see [2,7,12] for example), estimate (2.4) being due to the conservation of mass and momentum. Indeed, by the conservation of $\|\psi(t)\|_2$ and the orthogonality condition $(\mathrm{re}\, f, \varphi) = 0$, we have

$$\big\|\varphi\big(E(t)\big)\big\|_2^2 - \big\|\varphi(E_0)\big\|_2^2 = \big\|w\big(\cdot, \sigma_1(t)\big)\big\|_2^2 - \big\|\chi(t)\big\|_2^2. \tag{2.5}$$

Recall that

$$\frac{d}{dE}\big\|\varphi(E)\big\|_2^2\bigg|_{E=E_0} > 0,$$

and, by the orbital stability, $E(t)$ is close to E_0. Therefore by (2.5), (2.3) we obtain

$$\big|E(t) - E_0\big| \leqslant C\varepsilon.$$

Similarly, by the conservation of momentum and the orthogonality condition $(\mathrm{im}\, f, \varphi_y) = 0$ one has

$$iv(t)\big\|\psi(t)\big\|_2^2 + P\big(f(t)\big) = P\big(w\big(\cdot, \sigma_1(t)\big)\big).$$

As a consequence,

$$\big|v(t)\big| \leqslant C\varepsilon. \qquad \square$$

2.2. Differential equations

We start by rewriting (1.1) as an equation for $f(t)$:

$$i\vec{f}_t = L\big(E(t)\big)\vec{f} + D, \qquad D = D_0 + D_1 + D_2 + D_3,$$
$$D_0 = l(\sigma)\vec{\xi}_0(y,E), \qquad l(\sigma) = \gamma' + \frac{1}{2}v'y + ic'\partial_y\sigma_3 - iE'\partial_E\sigma_3,$$
$$\beta(t) = \beta_0(t) + \int_{-\infty}^{t} ds\left(E(s) - E_0 - \frac{v^2(s)}{4} - \frac{v'(s)b(s)}{2}\right) + \gamma(t),$$
$$b(t) = q(t) + c(t), \qquad q(t) = \int_{-\infty}^{t} ds\, v(s),$$
$$D_1 = l_1\big(\sigma(t)\big)\sigma_3\vec{f}, \qquad l_1(\sigma) = \gamma' + \frac{1}{2}v'y + ic'\partial_y\sigma_3,$$
$$D_2 = D_2\big(E(t), f\big),$$
$$D_2(E,f) = F\big(|\varphi+f|^2\big)\begin{pmatrix}\varphi+f\\ -\varphi-\bar{f}\end{pmatrix} - F\big(\varphi^2\big)\varphi\begin{pmatrix}1\\ -1\end{pmatrix} - V(E)\vec{f} - F\big(|f|^2\big)\begin{pmatrix}f\\ -\bar{f}\end{pmatrix}, \qquad \varphi = \varphi(E),$$
$$D_3 = F\big(|f|^2\big)\begin{pmatrix}f\\ -\bar{f}\end{pmatrix}. \tag{2.6}$$

Substituting the expression for f_t from (2.6) into the derivative of the orthogonality conditions, one gets

$$\big(\mathcal{B}_0(E) + \mathcal{B}_1(E,f)\big)\lambda = g(E,f), \qquad \lambda = \big(E', v', \gamma', c'\big)^t,$$
$$\mathcal{B}_0(E) = \operatorname{diag}\big(ie(E), n(E), e(E), -in(E)\big), \qquad e = \frac{d}{dE}\|\varphi\|_2^2, \qquad n = \frac{1}{2}\|\varphi\|_2^2,$$
$$g(E,f) = \big(\langle D_2 + D_3, \sigma_3\vec{\xi}_j(E)\rangle\big)^t_{j=0,\dots,3}. \tag{2.7}$$

At last, the matrix $\mathcal{B}_1(E,f)$ is defined by the relation

$$\mathcal{B}_1(E,f)\lambda = -\big(\langle \vec{f}, l(\sigma)\vec{\xi}_j(E)\rangle\big)^t_{j=0,\dots,3}.$$

In principle, Eqs. (2.7) can be solved with respect to λ and together with (2.6) constitute a complete system for σ and $\vec{f}$:

$$i\vec{f}_t = L\big(E(t)\big)\vec{f} + D\big(E(t), \vec{f}\big), \tag{2.8}$$
$$\lambda = G\big(E(t), \vec{f}\big). \tag{2.9}$$

Note that both D and G are at least quadratic in f. In particular, $\lambda(t)$ admits the estimate

$$\big|\lambda(t)\big| \leqslant C\varepsilon, \quad t \in \mathbb{R}. \tag{2.10}$$

Note also that (2.8), (2.9) imply that $\sigma \in C^2$ and

$$\big|\lambda'(t)\big| \leqslant C\varepsilon, \quad t \in \mathbb{R}. \tag{2.11}$$

The remainder of this section is devoted to the proof of the following proposition.

Proposition 2.2. *For $t \leqslant \varepsilon^{-1-\delta}$, $0 < \delta$ sufficiently small, the decomposition* (2.1), (2.2) *satisfies*:

$$\big|E(t) - E_0\big| + \big|v(t) - v_0(t)\big| \leqslant C\varepsilon^2\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}}, \tag{2.12}$$
$$v_0(t) = \varepsilon\kappa\big(\operatorname{th}(\varepsilon v_1 t) + 1\big), \tag{2.13}$$
$$\big|\beta(t) - \beta_0(t)\big| + \big|b(t) - 2\kappa v_1^{-1}S(\varepsilon v_1 t)\big| \leqslant C\varepsilon(1 + \varepsilon t)\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}}, \tag{2.14}$$

$$\|\chi(t)\|_\infty \leqslant C\varepsilon, \tag{2.15}$$

$$\|\chi(t) - w(\cdot, \sigma_1(t))\|_{H^1} \leqslant \varepsilon^{1/2} \frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}. \tag{2.16}$$

Moreover, for $\varepsilon^{-1}|\ln\varepsilon|^2 \leqslant t \leqslant \varepsilon^{-1-\delta}$, $\chi(t)$ *resolves into outgoings waves*:

$$\chi(t) = \psi_+(t) + \psi_-(t) + O_{H^1}(\varepsilon^{3/2}). \tag{2.17}$$

2.3. Main order terms of χ

In this and next two sections we exhibit an explicit ansatz that we expect to describe adequately the main order terms of $\chi(t)$ and $\sigma(t)$ in the region $t \leqslant \varepsilon^{-1-\delta}$, δ being a small positive number to be fixed later. At this stage the considerations are formal.

We start by rewriting (2.6) in terms of $g(x,t) = e^{-i\beta_0(t)}\chi(x,t)$:

$$i\vec{g}_t = H(t)\vec{g} + N, \qquad \vec{g} = \begin{pmatrix} g \\ \bar{g} \end{pmatrix}. \tag{2.18}$$

Here

$$\begin{aligned}
&H(t) = H_{\theta(t),q(t)}, \qquad H_{\theta,q} = (-\partial_x^2 + E_0)\sigma_3 + e^{i\theta\sigma_3}V(x-q, E_0)e^{-i\theta\sigma_3},\\
&\theta(t) = \beta(t) - \beta_0(t),\\
&N = N_0 + N_1 + N_2 + N_3,\\
&N_j = e^{i\sigma_3\tilde{\Phi}}D_j, \qquad \tilde{\Phi} = \Phi - \beta_0(t), \quad j = 0,2,3, \qquad N_1 = \mathcal{V}(t)\vec{g},\\
&\mathcal{V}(x,t) = e^{i\sigma_3\tilde{\Phi}}V(x-b(t), E(t))e^{-i\sigma_3\tilde{\Phi}} - e^{i\theta(t)\sigma_3}V(x-q(t), E_0)e^{-i\theta(t)\sigma_3}.
\end{aligned} \tag{2.19}$$

Notice that the operator $H_{\theta,q}$ is unitary equivalent to $L(E_0)$:

$$H_{\theta,q} = T_{\theta,q}L(E_0)T_{\theta,q}^*, \qquad (T_{\theta,q}f)(x) = e^{i\theta\sigma_3}f(x-q).$$

Note also that by (2.3), (2.4), (2.10), θ and q are slow functions of t:

$$\theta'(t), q'(t) = O(\varepsilon). \tag{2.20}$$

This fact is a starting point of our analysis that on the heuristic level can be summarized as follows. In a first approximation we neglect N and consider the linear problem

$$i\vec{g}_t = H(t)\vec{g}, \tag{2.21}$$

$$\|e^{i\beta_0(t)}g(t) - w(\cdot, \sigma_1(t))\|_{H^1} = O(e^{\gamma\varepsilon t}), \quad t \to -\infty. \tag{2.22}$$

Using (2.20) we solve (2.21), (2.22) approximatively by exhibiting an explicit profile $g_0(x,t)$ that satisfies (2.22) and solves (2.21) up to the terms of order ε^2, see (2.23) below. We expect the difference $g(t) - g_0(t)$ to be of order $\varepsilon^{3/2}$ in H^1 and of order ε^2 in L^∞. To determine the main order terms of σ we inject the profile g_0 into (2.9) and solve (again approximatively)

$$(E', v', \gamma', c') = G(E, f_0),$$

where f_0 is defined by

$$g_0(x,t) = e^{i\tilde{\Phi}(x,t)}f_0(x-b(t), t).$$

As a result, we get an explicit trajectory $(E_0, v_0(t), \gamma_0(t), c_0(t))$, that we expect to satisfy

$$|E - E_0|, |c(t) - c_0(t)|, |\gamma(t) - \gamma_0(t)|, |v(t) - v_0(t)| = O(\varepsilon^2),$$

at least for $t \lesssim \varepsilon^{-1}$, see Section 2.4. We next return to the full equation (2.18), inject g_0 and $(E_0, v_0(t), \gamma_0(t), c_0(t))$ into N and construct a better approximation g^{ap} of g by adding the corrections g_j, $j = 1, 2$:

$$g^{ap} = g_0 + g_1 + g_2,$$

where the increments g_1 and g_2, both of order $\varepsilon^{3/2}$ in H^1 and of order ε^2 in L^∞ successively improve the error in the regions $|x-q(t)|\gg 1$ and $x\approx q(t)|$ respectively. As a result, the profile g^{ap} solves (2.18) up to an error of order ε^3 in H^1, see Section 2.6.

On the technical level this scheme is realised as follows. Consider the profile $g_0(x,t)$:

$$\begin{aligned}
&\vec{g}_0(x,t)=e^{i\Omega(t)}\eta(x,t)+e^{-i\Omega(t)}\sigma_1\overline{\eta(x,t)},\qquad \vec{g}_0=\begin{pmatrix}g_0\\ \bar{g}_0\end{pmatrix},\\
&\eta(x,t)=\varepsilon\hat{\varphi}\big(\varepsilon(x-v_1t),\varepsilon\big)T_{\theta(t),q(t)}\mathcal{F}(x,v_1/2)-\varepsilon\big[\hat{\varphi}\big(\varepsilon(x-v_1t),\varepsilon\big)-e^{iv_1(q(\rho(x,t))-q(t))}\hat{\varphi}\big(\varepsilon(x+v_1t),\varepsilon\big)\big]\\
&\qquad\times\Theta_-\big(x-q(t)\big)\frac{r(v_1/2)}{s(v_1/2)}T_{\theta(t),q(t)}\mathcal{G}(x,v_1/2),\\
&\Omega=\beta_1(t)-\beta(t)+v_1q(t)/2,\qquad \rho(x,t)=\big(x-q(t)\big)v_1^{-1}+t.
\end{aligned}\tag{2.23}$$

Here $\mathcal{F}(x,k)$, $\mathcal{G}(x,k)$ are the solutions of the scattering problem associated to the operator $L(E_0)$, Θ_- is a cut-off function: $\Theta_-\in C^\infty(\mathbb{R})$, $\Theta_-(\xi)=1$ for $\xi\leqslant -2$, $\Theta_-(\xi)=0$ for $\xi\geqslant -1$.

Notice that:

(i) for $|x|\leqslant\varepsilon^{-\nu}$, $|q(t)|\leqslant\varepsilon^{-\nu}$, $\nu<1$, the main order term of $\vec{g}_0(x,t)$ is given by the expression:

$$\vec{g}_0(x,t)\sim T_{\theta(t),q(t)}Z(x,t),\tag{2.24}$$

$$Z(x,t)=\varepsilon\varphi_0(\varepsilon v_1t)\big[e^{i\Omega(t)}\mathcal{F}(x,v_1/2)+e^{-i\Omega(t)}\sigma_1\overline{\mathcal{F}(x,v_1/2)}\big],\tag{2.25}$$

(ii) as $x\to\pm\infty$, g_0 has the following asymptotics:

$$\begin{aligned}
&g_0(x,t)\sim e^{i(\beta_1(t)-\beta_0(t))}\eta_\pm(x,t),\\
&\eta_+(x,t)=e^{iv_1x/2}\varepsilon s\hat{\varphi}\big(\varepsilon(x-v_1t),\varepsilon\big),\\
&\eta_-(x,t)=e^{iv_1x/2}\varepsilon\hat{\varphi}\big(\varepsilon(x-v_1t),\varepsilon\big)+e^{-iv_1x/2+iv_1q(\rho(x,t))}\varepsilon r\hat{\varphi}\big(\varepsilon(x+v_1t),\varepsilon\big).
\end{aligned}\tag{2.26}$$

As a consequence, for all $t\in\mathbb{R}$ one has

$$\begin{aligned}
&\big\|g_0(t)\big\|_{H^1}\leqslant C\varepsilon^{1/2},\qquad \big\|\partial_x^l g_0(t)\big\|_\infty\leqslant C\varepsilon,\quad l=0,1,\\
&\big\|e^{i\beta_0(t)}g_0(t)-w\big(\cdot,\sigma_1(t)\big)\big\|_{H^1}\leqslant C\varepsilon^{1/2}\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}},
\end{aligned}\tag{2.27}$$

and for any $\gamma_1>0$, $l=0,1$,

$$\begin{aligned}
&\big\|e^{-\gamma_1|x-q(t)|}\partial_x^l g_0(t)\big\|_\infty\leqslant C\varepsilon e^{-\gamma\varepsilon|t|},\\
&\big\|e^{-\gamma_1|x-q(t)|}\partial_x^l\big(g_0(t)-T_{\theta(t),q(t)}Z(t)\big)\big\|_\infty\leqslant C\varepsilon^2\langle q(t)\rangle e^{-\gamma\varepsilon|t|}.
\end{aligned}\tag{2.28}$$

Here and in what follows $q(t)$ is assumed to satisfy

$$\big|q(t)\big|\leqslant C\varepsilon^{-1}.\tag{2.29}$$

As a first approximation of $\vec{g}(t)$ we take the expression $\vec{g}_0(t)$. The difference $\vec{g}^{(1)}(t)=\vec{g}(t)-\vec{g}_0(t)$ solves

$$i\vec{g}_t^{(1)}=H(t)\vec{g}^{(1)}+N^{(1)},\tag{2.30}$$

where

$$N^{(1)}=N+N_4,\qquad N_4=-i\partial_t\vec{g}_0+H(t)\vec{g}_0.\tag{2.31}$$

We expect $N^{(1)}$ to be of order ε^2 in H^1.

2.4. Main order terms of $\sigma(t)$

Consider the r.h.s. of (2.9). We expect that locally

$$\vec{f}(y,t) \approx Z(y,t),$$

and therefore the main order term of G is given by the expression

$$G_0 = \mathcal{B}_0^{-1}(E_0)g_0, \qquad g_0 = \bigl(\bigl\langle Q, \sigma_3\xi_j(E_0)\bigr\rangle\bigr)^t_{j=0,\dots,3}, \tag{2.32}$$

Q being the quadratic in Z part of the expression $D_2(E_0, Z)$. One can write it down explicitly:

$$Q = c_1\begin{pmatrix} Z(Z+2\bar{Z}) \\ -\bar{Z}(\bar{Z}+2Z)\end{pmatrix} + c_2\begin{pmatrix} \bar{Z}^2 \\ -Z^2\end{pmatrix},$$
$$c_1 = F'\bigl(\varphi^2\bigr)\varphi + \frac{1}{2}F''\bigl(\varphi^2\bigr)\varphi^3,$$
$$c_2 = \frac{1}{2}F''\bigl(\varphi^2\bigr)\varphi^3, \qquad \varphi = \varphi(E_0). \tag{2.33}$$

Substituting (2.25) into (2.33) one gets

$$Q = \varepsilon^2\varphi_0^2(\varepsilon v_1 t)\bigl[e^{2i\Omega(t)}Q_+(E_0) + e^{-2i\Omega(t)}Q_-(E_0) + Q_0(E_0)\bigr],$$
$$Q_+ = c_1\begin{pmatrix} v(v+u) \\ -u(u+2v)\end{pmatrix} + c_2\begin{pmatrix} u^2 \\ -v^2\end{pmatrix}, \qquad Q_- = -\sigma_1\overline{Q_+},$$
$$Q_0 = 2c_1\begin{pmatrix} |v|^2+|u|^2+v\bar{u} \\ -|v|^2-|u|^2-u\bar{v}\end{pmatrix} + 2c_2\begin{pmatrix} u\bar{v} \\ -v\bar{u}\end{pmatrix}, \qquad \mathcal{F} = \begin{pmatrix} v \\ u\end{pmatrix}.$$

As a consequence,

$$G_0 = \varepsilon^2\varphi_0^2(\varepsilon v_1 t)\bigl[e^{2i\Omega(t)}\kappa^+(E_0) + e^{-2i\Omega(t)}\kappa^-(E_0) + \kappa^0(E_0)\bigr],$$
$$\kappa^l = \bigl(\kappa_0^l, \kappa_1^l, \kappa_2^l, \kappa_3^l\bigr)^t,$$
$$\kappa^l = \mathcal{B}_0^{-1}(E_0)g_0^l, \qquad g_0^l = \bigl(\bigl\langle Q_l(E_0), \sigma_3\xi_j(E_0)\bigr\rangle\bigr)^t_{j=0,\dots,3}, \quad l = +,-,0. \tag{2.34}$$

The expressions $\langle Q_0(E_0), \sigma_3\xi_j(E_0)\rangle$, $j = 0, 1$, and, therefore, two first components of κ^0 can be computed explicitly. Indeed,

$$\begin{aligned}\bigl\langle Q_0(E), \sigma_3\xi_0(E)\bigr\rangle &= -i\,\mathrm{im}\bigl\langle V(E)\mathcal{F}, \mathcal{F}\bigr\rangle \\ &= \lim_{R\to+\infty}\mathrm{im}\bigl\langle\sigma_3\mathcal{F}', \mathcal{F}\bigr\rangle_{\mathbb{C}^2}\Big|_{-R}^{R} \\ &= \lim_{R\to+\infty}\mathrm{im}\bigl\langle\mathcal{F}', \mathcal{F}\bigr\rangle_{\mathbb{C}^2}\Big|_{-R}^{R} = 0.\end{aligned}$$

As a consequence,

$$\kappa_0^0 = 0. \tag{2.35}$$

Consider $\langle Q_0(E_0), \sigma_3\xi_1(E_0)\rangle$. One has

$$\begin{aligned}\langle Q_0, \sigma_3\xi_1\rangle &= \langle V_x\mathcal{F}, \sigma_3\mathcal{F}\rangle \\ &= -\lim_{R\to+\infty}\int_{-R}^{R} dx\,\bigl\langle\bigl(H - \bigl(E+k^2\bigr)\bigr)\mathcal{F}_x, \sigma_3\mathcal{F}\bigr\rangle_{\mathbb{C}^2} \\ &= \lim_{R\to+\infty}\langle\mathcal{F}_{xx}, \mathcal{F}\rangle_{\mathbb{C}^2} - \langle\mathcal{F}_x, \mathcal{F}_x\rangle_{\mathbb{C}^2}\Big|_{-R}^{R} \\ &= -2k^2\bigl|s(k)\bigr|^2 + 2k^2\bigl(1 + \bigl|r(k)\bigr|^2\bigr) \\ &= 4k^2\bigl|r(k)\bigr|^2, \quad k = v_1/2,\end{aligned}$$

which together with (2.7) implies

$$\kappa_1^0 = \frac{2v_1^2|r(v_1/2)|^2}{\|\varphi(E_0)\|_2^2} = 2v_1\kappa. \tag{2.36}$$

We write

$$G = G_0 + G_R \tag{2.37}$$

and decompose vector $(E(t), v(t), \gamma(t), c(t))$ as

$$\begin{aligned} &E(t) = E_0 + E_R(t), \qquad v(t) = v_0(t) + v_R(t), \\ &\gamma(t) = \gamma_0(t) + \gamma_R(t), \qquad c(t) = c_0(t) + c_R(t), \end{aligned} \tag{2.38}$$

where $v_0(t), \gamma_0(t), c_0(t)$ are defined by

$$\big(0, v_0(t), \gamma_0(t), c_0(t)\big) = \varepsilon^2\kappa^0 \int_{-\infty}^{t} ds\, \varphi_0^2(\varepsilon v_1 s) = \varepsilon v_1^{-1}\kappa^0\big(\mathrm{th}(\varepsilon v_1 t) + 1\big). \tag{2.39}$$

The remainder $(E_R(t), v_R(t), \gamma_R(t), c_R(t))$ satisfies

$$\begin{aligned} &\lambda_R = \varepsilon^2\varphi_0^2(\varepsilon v_1 t)\big[e^{2i\Omega(t)}\kappa^+(E_0) + e^{2i\Omega(t)}\kappa^-(E_0)\big] + G_R, \\ &\lambda_R = \big(E_R'(t), v_R'(t), \gamma_R'(t), c_R'(t)\big). \end{aligned} \tag{2.40}$$

We expect that $G_R = O(\varepsilon^3)$ and therefore $(E_R(t), v_R(t), \gamma_R(t), c_R(t))$ is of order ε^2 (at least for $t \lesssim \varepsilon^{-1}$).

Furthermore, for $q(t) = \int_{-\infty}^t ds\, v(s)$ we will need a refined decomposition

$$\begin{aligned} &q(t) = q_0(t) + q_R(t), \qquad q_0(t) = q_{00}(t) + q_{01}(t), \\ &q_{00}(t) = \int_{-\infty}^{t} ds\, v_0(s) = 2\kappa v_1^{-1} S(\varepsilon v_1 t), \\ &q_{01}(t) = \varepsilon^2 \int_{-\infty}^{t} ds\, (t-s)\varphi_0^2(\varepsilon v_1 s)\big[e^{2i\Omega_1(s)}\kappa_1^+ + e^{-2i\Omega_1(s)}\bar{\kappa}_1^+\big], \\ &\Omega_1(t) = \Omega(t) + \theta(t) = \beta_1(t) - \beta_0(t) + \frac{v_1 q(t)}{2}, \end{aligned} \tag{2.41}$$

$q_R(t)$ being the remainder. We will see later that for $t \lesssim \varepsilon^{-1}$,

$$\big|q_R(t)\big| = O(\varepsilon), \qquad \big|q_R'(t)\big| = O\big(\varepsilon^2\big), \qquad \big|q_R''(t)\big| + \big|q_R'''(t)\big| = O\big(\varepsilon^3\big).$$

Notice also that due to (2.4), one has for all $t \in \mathbb{R}$

$$\begin{aligned} &\big|q_{01}(t)\big| \leqslant C\varepsilon(1 + \varepsilon t_+)\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}}, \\ &\big|q_{01}'(t)\big| \leqslant C\varepsilon\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}}, \\ &\big|q_{01}''(t)\big|, \big|q_{01}'''(t)\big| \leqslant C\varepsilon^2 e^{-\gamma\varepsilon|t|}. \end{aligned}$$

Here and below $t_+ = \max\{t, 0\}$.

2.5. Second approximation of g

We return now to (2.30) and consider the main order terms of $N^{(1)}$. They are generated by the expression

$$N_{10}+N_{20}+N_{30}+N_4,$$

where

$$\begin{aligned}
&N_{10}=T_{\theta(t),q(t)}\mathcal{V}_0(t)\vec{Z},\\
&\mathcal{V}_0(x,t)=-c_0(t)V_x(x,E_0)+\frac{i}{2}v_0(t)(x+q)\big[\sigma_3,V(x,E_0)\big],\\
&N_{20}=T_{\theta(t),q(t)}Q(t),\\
&N_{30}=F\big(|g_0|^2\big)\begin{pmatrix}g_0\\-\bar{g}_0\end{pmatrix}.
\end{aligned}\tag{2.42}$$

The direct calculations show that

$$\begin{aligned}
&N_{10}+N_{30}+N_4=T_{\theta(t),q(t)}L_0+L_\infty+R,\\
&L_0=e^{i\Omega(t)}\mathcal{L}_0-e^{-i\Omega(t)}\sigma_1\overline{\mathcal{L}_0},\\
&L_\infty=e^{i(\beta_1(t)-\beta_0(t))\sigma_3}\left(\Theta_+(x-q)\begin{pmatrix}r_+\\-\bar{r}_+\end{pmatrix}+\Theta_-(x-q)\begin{pmatrix}r_-\\-\bar{r}_-\end{pmatrix}\right),\\
&\Theta_+=1-\Theta_-.
\end{aligned}\tag{2.43}$$

Here

$$\begin{aligned}
&r_\pm(x,t)=\big(-i\partial_t-\partial_x^2-v_1^2/4+\varepsilon^2\big)\eta_\pm+F\big(|\eta_\pm|^2\big)\eta_\pm,\\
&\mathcal{L}_0(x,t)=\varepsilon^2\varphi_0(\varepsilon v_1t)\big(W_1(x)+q(t)W_2(x)\big)+\varepsilon^2\varphi_0'(\varepsilon v_1t)\big(W_3(x)+q(t)W_4(x)\big),
\end{aligned}\tag{2.44}$$

W_j, $j=1,\dots,4$, being some smooth exponentially decaying functions of x only. It is possible to give the explicit expressions for them but they are useless for our purpose.

At last, the remainder R satisfies

$$|R|+|\partial_xR|\leqslant C\varepsilon\Big(\varepsilon^2\langle q\rangle+\big|\theta'(t)\big|+\big|v_R(t)\big|+\sup_{\tau\leqslant t}\big|q''(\tau)\big|+\sup_{\tau\leqslant t}\big|q'''(\tau)\big|\Big)e^{-\gamma|x-q|-\varepsilon\gamma|t|}.\tag{2.45}$$

Notice that L_0 and L_∞ are of size $O_{H^1}(\varepsilon^2)$ and $O_{H^1}(\varepsilon^{5/2})$ respectively, while R is expected to be of order ε^3 at least for $t\lesssim\varepsilon^{-1}$.

We now construct a correction to g_0 that allows to solve (2.30) up to an error of order ε^3 in H^1. This is done by a two step procedure. We first construct a correction g_1, that allows to improve the approximation at infinity. For this purpose we further develop $r_\pm$ using (1.7), (2.38), (2.41):

$$\begin{aligned}
&r_\pm=r_\pm^0+r_\pm',\\
&r_+^0(x,t)=-2\varepsilon^3e^{iv_1x/2}\big(|s|^2-1\big)s\varphi_0^3\big(\varepsilon(x-v_1t)\big),\\
&r_-^0(x,t)=\sum_{j=0}^3e^{iv_1(2j-3)x/2}A_j(x,t),\\
&A_0=-2\varepsilon^3r^2e^{i2v_1q(\rho)}\varphi_0^2\big(\varepsilon(x+v_1t)\big)\varphi_0\big(\varepsilon(x-vt)\big),\\
&A_1=A_{11}+A_{12}+A_{13}+A_{14},\\
&A_{11}=-4\varepsilon^3re^{iv_1q(\rho)}\varphi_0^2\big(\varepsilon(x-v_1t)\big)\varphi_0\big(\varepsilon(x+v_1t)\big),\\
&A_{12}=-2\varepsilon^3r\big(|r|^2-1\big)e^{iv_1q(\rho)}\varphi_0^3\big(\varepsilon(x+v_1t)\big)\\
&\qquad+\varepsilon re^{iv_1q(\rho)}\big[\big(v_0^2(\rho)-iv_1^{-1}v_0'(\rho)\big)\varphi_0\big(\varepsilon(x+v_1t)\big)-2i\varepsilon v_0(\rho)\varphi_0'\big(\varepsilon(x+v_1t)\big)\big],\\
&A_{13}=-i\varepsilon rv_1^{-1}q_{01}''(\rho)e^{iv_1q(\rho)}\varphi_0\big(\varepsilon(x+v_1t)\big),
\end{aligned}$$

$$A_{14} = -\varepsilon r v_1^{-1} e^{iv_1 q(\rho)} v_0(t) v_0(\rho) \varphi_0\big(\varepsilon(x+vt)\big),$$
$$A_2 = -4\varepsilon^3 |r|^2 \varphi_0^2\big(\varepsilon(x+vt)\big)\varphi_0\big(\varepsilon(x-vt)\big),$$
$$A_3 = -2\varepsilon^3 \bar{r} e^{-iv_1 q(\rho)} \varphi_0\big(\varepsilon(x+vt)\big)\varphi_0^2\big(\varepsilon(x-vt)\big),$$
$$\rho = v_1^{-1}\big(x - q(t)\big) + t. \tag{2.46}$$

At last, the remainders $r'_\pm$ satisfy

$$\big\|\Theta_\pm(x-q) r'_\pm\big\|_{H^1(\mathbb{R})} \leqslant C\varepsilon^{1/2} \frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}} \Big(\varepsilon^4 + \sup_{\tau\leqslant t}\big|q''_R(\tau)\big| + \sup_{\tau\leqslant t}\big|q'''_R(\tau)\big| + \varepsilon \sup_{\tau\leqslant t}\big|v_R(\tau)\big| + \varepsilon \sup_{\tau\leqslant t}\big|v'_R(\tau)\big|\Big). \tag{2.47}$$

As a consequence, we expect that

$$\big\|\Theta_\pm(x-q) r'_\pm\big\|_{H^1(\mathbb{R})} \lesssim \varepsilon^{7/2}, \quad t \lesssim \varepsilon^{-1}.$$

Set

$$g_1(x,t) = e^{i(\beta_1(t)-\beta_0(t))\sigma_3}\left[\Theta_+(x-q)\begin{pmatrix} H^+(x,t) \\ \overline{H^+(x,t)}\end{pmatrix} + \Theta_-(x-q)\begin{pmatrix} H^-(x,t) \\ \overline{H^-(x,t)}\end{pmatrix}\right],$$
$$H^+(x,t) = -i\frac{x}{v_1} r_+^0(x,t),$$
$$H^-(x,t) = \sum_{j=0}^{3} H_j^-(x,t),$$
$$H_0^- = -e^{-3iv_1 x/2}\frac{A_0(x,t)}{2v_1^2},$$
$$H_1^- = H_{11}^- + H_{12}^- + H_{13}^- + H_{14}^-,$$
$$H_{11}^- = -2i\varepsilon^2 r v_1^{-1} e^{-iv_1 x/2 + iv_1 q(\rho)} \varphi_0\big(\varepsilon(x+v_1 t)\big)\big(\mathrm{th}\big(\varepsilon(x+v_1 t)\big) + \mathrm{th}\big(\varepsilon(x-v_1 t)\big)\big),$$
$$H_{12}^- = i v_1^{-1} e^{-iv_1 x/2} x A_{12}(x,t),$$
$$H_{13}^- = -\frac{4v_1^2}{(4E_0+v_1^2)^2} e^{-iv_1 x/2} A_{13}(x,t),$$
$$H_{14}^- = i\varepsilon r v_1^{-1} e^{-iv_1 x/2 + iv_1 q(\rho)} v_0(\rho)\big(q_{00}(t) - q_{00}\big(v_1^{-1}x + t\big)\big)\varphi_0\big(\varepsilon(x+v_1 t)\big),$$
$$H_2^- = 2i\varepsilon^2 v_1^{-1}|r|^2 e^{iv_1 x/2}\varphi_0\big(\varepsilon(x-v_1 t)\big)\big(\mathrm{th}\big(\varepsilon(x+v_1 t)\big) + 1\big),$$
$$H_3^- = -e^{3iv_1 x/2}\frac{A_3(x,t)}{2v_1^2}. \tag{2.48}$$

It is not difficult to check that H^+ gives an approximate solution of the equation $i\partial_t H^+ = (-\partial_x^2 - v_1^2/4)H^+ + r_+^0$ in the sense that

$$\big\|\Theta_+(x-q)\big(\big(-i\partial_t - \partial_{xx} + v_1^2/4 - \varepsilon^2\big)H^+ + r_+^0\big)\big\|_{H^1(\mathbb{R})} \leqslant C(1+\varepsilon t_+)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}\varepsilon^{7/2}. \tag{2.49}$$

Similarly, for H_j^- one has

$$\big\|\Theta_-(x-q)\big(\big(-i\partial_t - \partial_{xx} + v_1^2/4 - \varepsilon^2\big)H_j^- + e^{iv_1(2j-3)x/2}A_j\big)\big\|_{H^1(\mathbb{R})} \leqslant C(1+\varepsilon t_+)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}\Big[\varepsilon^{7/2} + \varepsilon^{3/2}\sup_{\tau\leqslant t}\big|q''_R(\tau)\big| + \varepsilon^{3/2}\sup_{\tau\leqslant t}\big|q'''_R(\tau)\big|\Big]. \tag{2.50}$$

Notice also that

$$\|g_1(t)\|_{H^1} \leqslant C\varepsilon^{3/2}(1+\varepsilon t_+)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}, \tag{2.51}$$

$$\|\partial_x^l g_1(t)\|_\infty \leqslant C\varepsilon^2(1+\varepsilon t_+)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}, \quad l=0,1,$$

$$\|\Theta_-(x-q)\big(H^- - H^-_{12} - H^-_{14}\big)\|_{H^1} \leqslant C\varepsilon^{3/2}\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}, \tag{2.52}$$

and for any $\gamma_1 > 0$, $l=0,1$, one has

$$\|e^{-\gamma_1|x-q|}\partial_x^l g_1(t)\|_\infty \leqslant C\varepsilon^2 e^{-\gamma\varepsilon|t|},$$
$$\|e^{-\gamma_1|x-q|}\partial_x^l H^+\|_\infty, \|e^{-\gamma_1|x-q|}\partial_x^l\big(H^- - \tilde H^-\big)\|_\infty \leqslant Ce^{-\gamma\varepsilon|t|}\varepsilon^3\langle q\rangle, \tag{2.53}$$

where

$$\tilde H^-(x,t) = \frac{2i\varepsilon^2}{v_1}e^{iv_1x/2}|r|^2\varphi_0(\varepsilon v_1 t)\big(1+\mathrm{th}(\varepsilon v_1 t)\big).$$

Write $g^{(1)} = g_1 + g^{(2)}$. Then $g^{(2)}$ solves

$$i\vec g^{(2)}_t = H(t)\vec g^{(2)} + N^{(2)}, \tag{2.54}$$

where

$$N^{(2)} = N - i\partial_t(\vec g_0 + \vec g_1) + H(t)(\vec g_0 + \vec g_1). \tag{2.55}$$

It follows from (2.43), (2.45), (2.46), (2.47), (2.49), (2.50), (2.53) that

$$N^{(2)} = T_{\theta,q}L_1 + T_{\theta,q}Q + N_0 + N_1' + N_2' + N_3' + R'. \tag{2.56}$$

Here

$$N_j' = N_j - N_{0j}, \quad j=1,2,3,$$
$$L_1 = L_0 + V(x,E_0)\Theta_-(x)e^{i\Omega(t)\sigma_3}\begin{pmatrix}\tilde H_-(x,t)\\ \overline{\tilde H_-(x,t)}\end{pmatrix}$$
$$- iv_1\Theta_-'(x)e^{i\Omega(t)\sigma_3}\begin{pmatrix}\tilde H_-(x,t)\\ \overline{\tilde H_-(x,t)}\end{pmatrix} - \Theta_-''(x)\sigma_3e^{i\Omega(t)\sigma_3}\begin{pmatrix}\tilde H_-(x,t)\\ \overline{\tilde H_-(x,t)}\end{pmatrix}. \tag{2.57}$$

Notice that L_1 can be written in the form

$$L_1 = e^{i\Omega(t)}\mathcal L_1 + e^{-i\Omega(t)}\sigma_1\overline{\mathcal L_1},$$

where

$$\mathcal L_1(x,t) = \varepsilon^2\varphi_0(\varepsilon v_1 t)\big(W_1(x) + q(t)W_2(x)\big) + \varepsilon^2\varphi_0'(\varepsilon v_1 t)\big(W_3(x)+q(t)W_4(x)\big),$$

with some new functions W_j, $j=1,\dots,4$, of the same type as in (2.43), (2.44).

At last, R' is the remainder that satisfies:

$$\begin{aligned}\|R'(t)\|_{H^1} \leqslant{}& C\big(1+|q(t)|\big)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}\\ &\times\Big[\varepsilon^3 + \varepsilon^{3/2}|\theta'(t)| + \varepsilon^{3/2}\sup_{\tau\leqslant t}|v_R(\tau)| + \varepsilon^{1/2}\sup_{\tau\leqslant t}|q_R''(\tau)| + \varepsilon^{1/2}\sup_{\tau\leqslant t}|q_R'''(\tau)|\Big]\\ &+ Ce^{-\gamma\varepsilon|t|}\varepsilon\big(|v_R(t)| + |\theta'(t)|\big).\end{aligned} \tag{2.58}$$

We assume here that $\delta \leqslant \frac12$. Accordingly to our heuristics we expect that

$$\|R'(t)\|_{H^1} \lesssim \varepsilon^3, \quad t \lesssim \varepsilon^{-1}.$$

The next step is to construct a correction g_2 that allows to remove from the r.h.s. of (2.54) the ε^2-order terms $T_{\theta,q}L_1$ and $T_{\theta,q}Q$. Following (2.23) we define

$$\vec{g}_2(x,t) = \vec{g}_{2,0}(x,t) + \vec{g}_{2,1}(x,t) + \vec{g}_{2,2}(x,t),$$
$$\vec{g}_{2,0}(x,t) = -\varepsilon^2 \varphi_0^2(\varepsilon v_1 t) T_{\theta,q} L^{-1}(E_0) P(E_0) Q_0,$$
$$\vec{g}_{2,j}(x,t) = \varepsilon^2 \bigl[e^{ij\Omega(t)} \eta_j(x,t) + e^{-ij\Omega(t)} \sigma_1 \overline{\eta_j(x,t)}\bigr], \quad j = 1, 2,$$
$$\eta_1(x,t) = \sum_{j=1,\dots,4} \psi_j(x,t) T_{\theta(t),q(t)} \mathcal{E}_j(x),$$
$$\eta_2(x,t) = \psi_5(x,t) T_{\theta(t),q(t)} \mathcal{E}_5(x).$$

Here

$$\mathcal{E}_j(x) = -\bigl(L(E_0) - E_0 - v_1^2/4 - i0\bigr)^{-1} W_j(x), \quad j = 1, \dots, 4,$$
$$\mathcal{E}_5(x) = -\bigl(L(E_0) - 2E_0 - v_1^2/2 - i0\bigr)^{-1} P(E_0) Q_+,$$
$$\psi_j(x,t) = \Theta_+(x-q)\rho_j(x - v_1 t, t) + \Theta_-(x-q) e^{iv_1 q_{00}(xv_1^{-1}+t) - iv_1 q_{00}(t)} \rho_j\bigl((x + v_1 t), t\bigr), \quad j = 1, \dots, 4,$$
$$\rho_1(\xi,t) = \varphi_0(\varepsilon\xi), \qquad \rho_3(\xi,t) = -\varphi_0'(\varepsilon\xi),$$
$$\rho_{j+1}(\xi,t) = \rho_j(\xi,t)\bigl(q_{00}\bigl(\xi v_1^{-1}\bigr) + q_{01}(t) + q_R(t)\bigr), \quad j = 1, 3,$$
$$\psi_5(x,t) = \Theta_+(x-q)\varphi_0^2\bigl(\varepsilon(x - v_2 t)\bigr) e^{i\Delta_+} + \Theta_-(x-q)\varphi_0^2\bigl(\varepsilon(x + v_2 t)\bigr) e^{i\Delta_-},$$
$$\Delta_\pm = \frac{1}{2}(v_1 \mp v_2)\bigl(q_{00}\bigl(t \mp x v_2^{-1}\bigr) - q_{00}(t)\bigr), \qquad v_2 = 2\sqrt{E_0 + v_1^2/2}.$$

It is not difficult to check that

$$\bigl\|g_{2,0}(t)\bigr\|_{H^1} \leqslant C\varepsilon^2 e^{-\gamma\varepsilon|t|},$$
$$\bigl\|g_{2,1}(t)\bigr\|_{H^1} \leqslant C\varepsilon^{3/2}\bigl(1 + \bigl|q(t) - q_{00}(t)\bigr|\bigr)\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}},$$
$$\bigl\|g_{2,2}(t)\bigr\|_{H^1} \leqslant C\varepsilon^{3/2}\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}},$$
$$\bigl\|\partial_x^l g_2\bigr\|_\infty \leqslant C\varepsilon^2\bigl(1 + \bigl|q(t) - q_{00}(t)\bigr|\bigr)\frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}},$$
$$\bigl\|e^{-\gamma_1|x-q|}\partial_x^l g_2\bigr\|_\infty \leqslant C\varepsilon^2\langle q\rangle e^{-\gamma\varepsilon|t|}, \quad l = 0, 1,$$
$$\bigl\|-i\partial_t \vec{g}_2 + H(t)\vec{g}_2 + T_{\theta,q}L_1 + T_{\theta,q}P(E_0)Q\bigr\|_{H^1}$$
$$\leqslant C\langle q\rangle \frac{e^{\gamma\varepsilon t}}{1 + e^{\gamma\varepsilon t}}\bigl(\varepsilon^{3/2}\bigl|\theta'(t)\bigr| + \varepsilon^{3/2}\bigl|v_R(t)\bigr| + \varepsilon^3\bigr). \tag{2.59}$$

2.6. Final equations

We now summarize the above constructions. Set

$$f = f^{ap} + f^r, \qquad f^{ap} = f_0 + f_1 + f_2 \tag{2.60}$$

where f_j are defined by

$$g_j(x,t) = e^{i\tilde{\Phi}(x,t)} f_j\bigl(x - b(t), t\bigr).$$

Then $f^r(y,t)$ solves

$$i\vec{f}_t^r = L\bigl(E(t)\bigr)\vec{f}^r + l_1\bigl(\sigma(t)\bigr)\vec{f}^r + \mathcal{D},$$
$$\mathcal{D} = \mathcal{D}_0 + \mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3 + R'', \tag{2.61}$$
$$\mathcal{D}_0 = D_0 + P_0\bigl(E(t)\bigr)D_2\bigl(E(t), f(t)\bigr),$$
$$\mathcal{D}_1 = e^{-i\tilde{\Phi}\sigma_3}\mathcal{V}(t)e^{i\tilde{\Phi}\sigma_3}\vec{f}^{ap} - e^{-i\tilde{\Phi}\sigma_3}T_{\theta(t),q(t)}\mathcal{V}_0(t)Z, \tag{2.62}$$
$$\mathcal{D}_2 = P(E)D_2\bigl(E(t), f(t)\bigr) - e^{-i\tilde{\Phi}\sigma_3 + i\theta(t)\sigma_3}\bigl(P(E_0)Q\bigr)\bigl(\cdot + c(t), t\bigr), \tag{2.63}$$

$$\mathcal{D}_3 = D_3 - F\big(|f_0|^2\big)\begin{pmatrix} f_0 \\ -\bar{f}_0 \end{pmatrix}, \tag{2.64}$$

where R'' admits the same estimate as R':

$$\begin{aligned}\big\|R''(t)\big\|_{H^1} &\leqslant C\big\langle q(t)\big\rangle \frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}} \\ &\quad\times\Big[\varepsilon^3 + \varepsilon^{3/2}\big|\theta'(t)\big| + \varepsilon^{3/2}\sup_{\tau\leqslant t}\big|v_R(\tau)\big| + \varepsilon^{1/2}\sup_{\tau\leqslant t}\big|q_R''(\tau)\big| + \varepsilon^{1/2}\sup_{\tau\leqslant t}\big|q_R'''(\tau)\big|\Big] \\ &\quad + Ce^{-\gamma\varepsilon|t|}\varepsilon\big(\big|v_R(t)\big| + \big|\theta'(t)\big|\big).\end{aligned} \tag{2.65}$$

We finally represent f^r as a sum:

$$\vec{f}^r(t) = h(t) + k(t), \qquad h(t) = P\big(E(t)\big)\vec{f}^r(t),$$
$$k(t) = P_0\big(E(t)\big)\vec{f}^r(t) = \sum_j k_j(t)\xi_j\big(E(t)\big).$$

Then $h(t)$ and $k(t)$ satisfy

$$i\vec{h}_t = L\big(E(t)\big)\vec{h} + l_1\big(\sigma(t)\big)\vec{h} + \mathcal{D}',$$
$$\mathcal{D}' = P(E)\mathcal{D} + \mathcal{D}_4, \qquad \mathcal{D}_4 = \big[P(E), l_1(\sigma)\big]\vec{f}^r + iE'(t)\partial_E P(E)\vec{f}^r, \tag{2.66}$$
$$\langle\vec{k}, \sigma_3\xi_j\rangle = -\big\langle\vec{f}^{ap}, \sigma_3\xi_j\big\rangle. \tag{2.67}$$

It follows from (2.23), (2.27), (2.28), (2.51), (2.59) that

$$\|f_0\|_{H^1} \leqslant C\varepsilon^{1/2}, \tag{2.68}$$
$$\|f_j\|_{H^1} \leqslant C\varepsilon^{3/2}\big(\langle q\rangle + \varepsilon t_+\big)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}, \quad j=1,2, \tag{2.69}$$
$$\|f_0\|_{W^{1,\infty}} \leqslant C\varepsilon, \tag{2.70}$$
$$\|f_j\|_{W^{1,\infty}} \leqslant C\varepsilon^2\big(\langle q\rangle + \varepsilon t_+\big)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}, \quad j=1,2, \tag{2.71}$$

and for any $\gamma_1 > 0$,

$$\big\|e^{-\gamma_1|y|}\partial_y^l(f_0 - Z)\big\|_\infty \leqslant C\big(\varepsilon^2\langle q\rangle + \varepsilon\big|c(t)\big|\big)e^{-\gamma\varepsilon|t|}, \quad l=0,1, \tag{2.72}$$
$$\big\|e^{-\gamma_1|y|}\partial_y^l f_j\big\|_\infty \leqslant C\varepsilon^2\langle q\rangle e^{-\gamma\varepsilon|t|}, \quad j=1,2,\ l=0,1. \tag{2.73}$$

Here we have assumed that

$$\big|c(t)\big| \leqslant C. \tag{2.74}$$

Combining (2.67), (2.72), (2.73) one gets

$$\big\|k(t)\big\| \leqslant C\big(\varepsilon^2\langle q\rangle + \varepsilon\big|c(t)\big|\big)e^{-\gamma\varepsilon|t|}. \tag{2.75}$$

2.7. Estimates of $\mathcal{D}'$

Consider $\mathcal{D}_1$. It follows from (2.28), (2.73), (2.74) that

$$\|\mathcal{D}_1\|_{H^1} \leqslant Ce^{-\gamma\varepsilon|t|}\langle q\rangle^2\big(\varepsilon^3 + \varepsilon\big(|E_R| + |c_R| + |v_R|\big)\big). \tag{2.76}$$

For $\mathcal{D}_2$ one has

$$\begin{aligned}\|\mathcal{D}_2\|_{H^1} &\leqslant C\Big(\Big(\varepsilon^3\langle q\rangle + \varepsilon^2\big|c(t)\big| + \varepsilon\big\|e^{-\gamma_0|y|}(f-Z)\big\|_{H^1}\Big)e^{-\gamma\varepsilon|t|} \\ &\quad + \big\|e^{-\gamma_0|y|}(f-Z)\big\|_{H^1}^2 + \big\|e^{-\gamma_0|y|}f\big\|_{H^1}^3\Big),\end{aligned}$$

with some $\gamma_0 > 0$. Combining this inequality with (2.72), (2.73), (2.75), one gets

$$\|\mathcal{D}_2\|_{H^1} \leqslant C\Big(\Big(\varepsilon^3\langle q\rangle + \varepsilon^2\big|c(t)\big| + \varepsilon\|h\|_{H^1}\Big)e^{-\gamma\varepsilon|t|} + \|h\|_{H^1}^2\Big). \tag{2.77}$$

Consider $\mathcal{D}_3$. From (2.68)–(2.72) one has

$$\left\| F\left(\left|f^{ap}+k\right|^2\right)\left(f^{ap}+k\right) - F\left(|f_0|^2\right) f_0 \right\|_{H^1} \leqslant C\varepsilon^3\left(\langle q\rangle + \varepsilon t_+\right)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}},$$

so that

$$\begin{aligned}\|\mathcal{D}_3\|_{H^1} &\leqslant C\left(\left\| F\left(\left|f^{ap}+k\right|^2\right)\left(f^{ap}+k\right) - F\left(|f_0|^2\right) f_0 \right\|_{H^1} + \left\|f^{ap}+k\right\|^2_{W^{1,\infty}}\|h\|_{H^1} + \|h\|^3_{H^1}\right)\\ &\leqslant C\left(\varepsilon^3\left(\langle q\rangle + \varepsilon t_+\right)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}} + \varepsilon^2\|h\|_{H^1} + \|h\|^3_{H^1}\right).\end{aligned} \tag{2.78}$$

For $\mathcal{D}_4$ one has

$$\|\mathcal{D}_4\|_{H^1} \leqslant C\left|\lambda(t)\right|\left(\varepsilon e^{-\gamma\varepsilon|t|} + \|h\|_{H^1}\right).$$

By (2.7),

$$|\lambda| \leqslant C\left(\varepsilon^2 e^{-\gamma\varepsilon|t|} + \|h\|^2_{H^1}\right). \tag{2.79}$$

Therefore,

$$\|\mathcal{D}_4\|_{H^1} \leqslant C\left(\varepsilon^3 e^{-\gamma\varepsilon|t|} + \|h\|^3_{H^1}\right). \tag{2.80}$$

Combining (2.76), (2.77), (2.78), (2.80), one gets

$$\begin{aligned}\left\|P(E)\mathcal{D}'\right\|_{H^1} \leqslant C\Big(&\varepsilon e^{-\gamma\varepsilon|t|}\langle q\rangle^2\left[\left|\theta'(t)\right| + \left|E_R(t)\right| + \left|c_R(t)\right| + \left|v_R(t)\right|\right]\\ &+ \left(\langle q\rangle^2 + \varepsilon t_+\right)\frac{e^{\gamma\varepsilon t}}{1+e^{\gamma\varepsilon t}}\Big[\varepsilon^3 + \varepsilon^{3/2}\Big(\left|\theta'(t)\right| + \sup_{\tau\leqslant t}\left|v_R(\tau)\right|\Big)\\ &+ \varepsilon^{1/2}\Big(\sup_{\tau\leqslant t}\left|q_R''(\tau)\right| + \sup_{\tau\leqslant t}\left|q_R'''(\tau)\right|\Big)\Big]\\ &+ e^{-\gamma\varepsilon|t|}\varepsilon\|h\|_{H^1} + \varepsilon^2\|h\|_{H^1} + \|h\|^2_{H^1}\Big).\end{aligned} \tag{2.81}$$

2.8. *Some estimates of the trajectory $\sigma(t)$*

Consider (2.37). By definition of G_R one has

$$|G_R| \leqslant C\left(\varepsilon e^{-\gamma\varepsilon|t|}\left\|e^{-\gamma_0|y|}(f-Z)\right\|_2 + \left\|e^{-\gamma_0|y|}(f-Z)\right\|_2^2 + \varepsilon\left\|e^{-\gamma_0|y|}f\right\|_2^2 + \left\|e^{-\gamma_0|y|}f\right\|^3_{H^1}\right).$$

It is also not difficult to check that a similar estimate holds for $\frac{d}{dt}G_R$:

$$\left|\frac{d}{dt}G_R\right| \leqslant C\left(\varepsilon e^{-\gamma\varepsilon|t|}\left\|e^{-\gamma_0|y|}(f-Z)\right\|_{H^1} + \left\|e^{-\gamma_0|y|}(f-Z)\right\|^2_{H^1} + \varepsilon\left\|e^{-\gamma_0|y|}f\right\|^2_{H^1} + \left\|e^{-\gamma_0|y|}f\right\|^3_{H^1}\right).$$

Combining these inequalities with (2.72), (2.73), (2.75) one gets

$$|G_R|, \left|\frac{d}{dt}G_R\right| \leqslant C\left(\left(\varepsilon^3\langle q\rangle + \varepsilon^2|c|\right)e^{-\gamma\varepsilon|t|} + \varepsilon e^{-\gamma\varepsilon|t|}\|h\|_{H^1} + \|h\|^2_{H^1}\right). \tag{2.82}$$

Therefore, integrating (2.40) one obtains

$$\begin{aligned}&\left|E_R(t)\right|, \left|v_R(t)\right|, \left|c_R(t)\right|\\ &\quad\leqslant C\left(\varepsilon^2 e^{-\gamma\varepsilon|t|} + \int_{-\infty}^{t} ds\,\left(\varepsilon^3\left\langle q(s)\right\rangle + \varepsilon^2\left|c(s)\right|\right)e^{-\gamma\varepsilon|s|} + \varepsilon e^{-\gamma\varepsilon|s|}\left\|h(s)\right\|_{H^1} + \left\|h(s)\right\|^2_{H^1}\right).\end{aligned} \tag{2.83}$$

2.9. Bootstrap arguments

Define

$$m(t) = \sup_{\tau < t} \| h(\tau) \|_{H^1} \left(1 + e^{-\delta_1 \varepsilon \tau}\right),$$

$\delta_1 > 0$. In this subsection we prove

Proposition 2.3. *For δ_1 sufficiently small (independent of ε) and for $t \leqslant \varepsilon^{-1-\delta}$, $\delta \leqslant 1/8$, one has*

$$m(t) \leqslant C\varepsilon^2 (1 + \varepsilon t_+)^{7/2}. \tag{2.84}$$

The proof of this proposition is by the bootstrap argument based on the following lemma.

Lemma 2.4. *Assume that in addition to* (2.29), (2.74) *one has*

$$m(t) \leqslant \varepsilon^{3/2}. \tag{2.85}$$

Then

$$\begin{aligned} &m(t) \leqslant C\varepsilon^2 (1 + \varepsilon t_+)^{7/2}, \\ &\left|q(t)\right| \leqslant C(1 + \varepsilon t_+) \frac{e^{2\delta_1 \varepsilon t}}{1 + e^{2\delta_1 \varepsilon t}}, \\ &\left|c(t)\right| \leqslant C\varepsilon, \end{aligned} \tag{2.86}$$

provided $t \leqslant \varepsilon^{-3/2}$ and ε is sufficiently small.

Since (2.29), (2.74), (2.85) do hold as $t \to -\infty$, Lemma 2.4 implies Proposition 2.3 by a standard continuity argument.

The rest of the subsection is devoted to the proof of Lemma 2.4 which relies on the energy type estimate on $h(t)$.

Proof of Lemma 2.4. Combining (2.79) with (2.85) one gets

$$|\lambda| \leqslant C\left[\varepsilon^2 e^{-\gamma\varepsilon|t|} + \varepsilon^3 \frac{e^{2\delta_1 \varepsilon t}}{1 + e^{2\delta_1 \varepsilon t}}\right], \tag{2.87}$$

which gives

$$\begin{aligned} &\left|v(t)\right|, \left|c(t)\right| \leqslant C\varepsilon \frac{e^{2\delta_1 \varepsilon t}}{1 + e^{2\delta_1 \varepsilon t}}, \\ &\left|q(t)\right| \leqslant C(1 + \varepsilon t_+) \frac{e^{2\delta_1 \varepsilon t}}{1 + e^{2\delta_1 \varepsilon t}}. \end{aligned} \tag{2.88}$$

Injecting (2.88) into (2.83) one obtains

$$\left|E_R(t)\right|, \left|v_R(t)\right|, \left|c_R(t)\right| \leqslant C\left(\varepsilon^2 (1 + \varepsilon t_+) \frac{e^{2\delta_1 \varepsilon t}}{1 + e^{2\delta_1 \varepsilon t}} + \int_{-\infty}^{t} ds\, \varepsilon e^{-\gamma\varepsilon|s|} \left\|h(s)\right\|_{H^1}\right). \tag{2.89}$$

Consider $\theta(t) = \beta(t) - \beta_0(t)$. One has

$$\left|\theta'(t)\right| \leqslant C\left(\left|\lambda(t)\right| \left\langle q(t)\right\rangle + v^2 + |E_R|\right).$$

Therefore by (2.87)–(2.89),

$$\left|\theta'(t)\right| \leqslant C\left(\varepsilon^2\frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}+|v_R|+|E_R|\right), \tag{2.90}$$

$$\left|\theta(t)\right| \leqslant C\left(\varepsilon(1+\varepsilon t_+)^2\frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}+\int_{-\infty}^{t} ds\,\varepsilon(t-s)e^{-\gamma\varepsilon|s|}\left\|h(s)\right\|_{H^1}\right)$$
$$\leqslant C\varepsilon^{1/2}(1+\varepsilon t_+)^2\frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}. \tag{2.91}$$

By definition of $q_R(t)$ (see (2.41)) one has

$$\left|q_R''\right| \leqslant C\left(|G_R|+\varepsilon^2e^{-\gamma\varepsilon|t|}\left|\theta(t)\right|\right),$$
$$\left|q_R'''\right| \leqslant C\left(\left|\frac{d}{dt}G_R\right|+\varepsilon^2e^{-\gamma\varepsilon|t|}\left(\left|\theta(t)\right|+\varepsilon\right)\right),$$

which together with (2.82), (2.85), (2.88), (2.91) gives

$$\left|q_R''(t)\right|+\left|q_R'''(t)\right| \leqslant C\varepsilon^{5/2}\frac{e^{2\delta_1\varepsilon v_1 t}}{1+e^{2\delta_1\varepsilon v_1 t}}. \tag{2.92}$$

Combining (2.81), (2.89), (2.90), (2.92) one gets finally

$$\left\|P(E)\mathcal{D}'\right\|_{H^1} \leqslant C\left(\varepsilon^3(1+\varepsilon t_+)^2\frac{e^{2\delta_1\varepsilon t}}{1+2e^{\delta_1\varepsilon t}}+\varepsilon^2\|h\|_{H^1}\right.$$
$$\left.+\varepsilon e^{-\gamma\varepsilon|t|}\|h\|_{H^1}+\varepsilon^2e^{-\gamma\varepsilon|t|}\int_{-\infty}^{t} ds\,e^{-\gamma\varepsilon|s|}\left\|h(s)\right\|_{H^1}\right). \tag{2.93}$$

Here we have used $t \leqslant \varepsilon^{-3/2}$.

We now perform the energy type estimates on h. We compute

$$\frac{d}{dt}\left\langle L\left(E(t)\right)\vec{h}(t), \sigma_3\vec{h}(t)\right\rangle$$
$$= \mathrm{im}\left\langle\left[L(E), l_1(\sigma)\sigma_3\right]\vec{h}, \sigma_3\vec{h}\right\rangle+2\,\mathrm{im}\left\langle P(E)\mathcal{D}', \sigma_3L(E)\vec{h}\right\rangle+E'(t)\left(\langle\vec{h},\vec{h}\rangle+\langle V_E\vec{h},\sigma_3\vec{h}\rangle\right). \tag{2.94}$$

Since $[L(E), l_1(\sigma)\sigma_3] = -v'\partial_y+[V(E), l_1(\sigma)\sigma_3]$, one gets from (2.94), (2.93), (2.87):

$$\frac{d}{dt}\left\langle L\left(E(t)\right)\vec{h}(t), \sigma_3\vec{h}(t)\right\rangle \leqslant C\left(\left|\lambda(t)\right|\left\|h(t)\right\|_{H^1}^2+\left\|P(E)\mathcal{D}'\right\|_{H^1}\|h\|_{H^1}\right)$$
$$\leqslant C\left(\varepsilon^5\alpha_0(\varepsilon t)+\varepsilon\alpha(\varepsilon t)\|h\|_{H^1}^2+\varepsilon^2\alpha(\varepsilon t)\int_{-\infty}^{t} ds\,\alpha(\varepsilon s)\left\|h(s)\right\|_{H^1}^2\right), \tag{2.95}$$

where

$$\alpha_0(\tau) = (1+\tau_+)^6\frac{e^{2\delta_1\tau}}{1+e^{2\delta_1\tau}},$$
$$\alpha(\tau) = (1+\tau_+)^{-2}\frac{e^{2\delta_1\tau}}{1+e^{2\delta_1\tau}}.$$

Since under assumption (H2), one has

$$\left\langle L\left(E(t)\right)\vec{h}(t), \sigma_3\vec{h}(t)\right\rangle \geqslant C\|h\|_{H^1}^2$$

(see [12], for example), one can rewrite (2.95) as

$$\frac{d}{dt}y(t) \leqslant C\left(\varepsilon^5\alpha_0(\varepsilon t)+\varepsilon\alpha(\varepsilon t)y(t)+\varepsilon^2\alpha(\varepsilon t)\int_{-\infty}^{t} ds\,\alpha(\varepsilon s)y(s)\right),$$

where $y(t) = \langle L(E(t))\vec{h}(t), \sigma_3\vec{h}(t)\rangle$. Integrating this inequality and using the boundedness of α in $L_1(\mathbb{R})$ one gets

$$y(t) \leqslant C\varepsilon^5 \int_{-\infty}^{t} ds\, \alpha_0(\varepsilon s) \leqslant C\varepsilon^4(1+\varepsilon t_+)^7 \frac{e^{2\delta_1 \varepsilon t}}{1+e^{2\delta_1\varepsilon t}}. \tag{2.96}$$

In terms of h, (2.96) takes the form

$$\|h(t)\|_{H^1} \leqslant C\varepsilon^2(1+\varepsilon t_+)^{7/2} \frac{e^{\delta_1\varepsilon t}}{1+e^{\delta_1\varepsilon t}},$$

which completes the proof of Lemma 2.4. □

2.10. Proof of Proposition 2.2

We are now in position to finish the proof of Proposition 2.2. By (2.75), (2.83), (2.84), (2.88), (2.90) one has for any $\varepsilon t \leqslant \varepsilon^{-1/8}$:

$$\begin{aligned}
&|v(t)|, |c(t)| \leqslant C\varepsilon \frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}},\\
&|E_R(t)|, |v_R(t)|, |c_R(t)| \leqslant C\varepsilon^2 \frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}, &(2.97)\\
&|q(t)| \leqslant C(1+\varepsilon t_+) \frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}},\\
&|q(t) - q_{00}(t)| \leqslant C\varepsilon(1+\varepsilon t_+) \frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}, &(2.98)\\
&|\theta(t)| \leqslant C\varepsilon(1+\varepsilon t_+) \frac{e^{2\delta_1\varepsilon t}}{1+e^{2\delta_1\varepsilon t}}, &(2.99)\\
&\|f^r(t)\|_{H^1} \leqslant C\varepsilon^2(1+\varepsilon t_+)^4 \frac{e^{\delta_1\varepsilon t}}{1+e^{\delta_1\varepsilon t}}, &(2.100)
\end{aligned}$$

which together with (2.27), (2.69), (2.70) gives (2.12), (2.15).

To prove (2.17) we return to the explicit formulas for g_j, $j = 0, 1$. Consider g_0. It follows directly from (2.23), (2.98) that for $(\ln\varepsilon)^2 \leqslant \varepsilon t \leqslant \varepsilon^{-1/8}$,

$$\begin{aligned}
e^{i(\beta_0(t)-\beta_1(t))} g_0(x,t) &= \varepsilon r e^{-iv_1x/2+iv_1q_{00}(\rho(x,t))}\varphi_0(\varepsilon(x+v_1t)) + \varepsilon s e^{iv_1x/2}\varphi_0(\varepsilon(x-v_1t)) + O_{H^1}(\varepsilon^{3/2})\\
&= \varepsilon r e^{-iv_1x/2+iv_1q_{00}(v_1^{-1}x+t)}\bigl(1 - iv_1^{-1}q_{00}(t)v_0(v_1^{-1}x+t)\bigr)\varphi_0(\varepsilon(x+v_1t))\\
&\quad + \varepsilon s e^{iv_1x/2}\varphi_0(\varepsilon(x-v_1t)) + O_{H^1}(\varepsilon^{3/2}).
\end{aligned} \tag{2.101}$$

By (2.52), (2.98) one has for g_1:

$$\begin{aligned}
e^{i(\beta_0(t)-\beta_1(t))} g_1(x,t) &= i\varepsilon v_1^{-1} r e^{-iv_1x/2+iv_1q_{00}(v_1^{-1}x+t)} v_0(v_1^{-1}x+t) q_{00}(t)\varphi_0(\varepsilon(x+v_1t))\\
&\quad - itr_+^0(x,t) - ite^{-iv_1x/2}\tilde{A}_{12}(x,t) + O_{H^1}(\varepsilon^{3/2}),
\end{aligned} \tag{2.102}$$

where

$$\begin{aligned}
\tilde{A}_{12}(x,t) &= -2\varepsilon^3 r(|r|^2-1)e^{iv_1q_{00}(v_1^{-1}x+t)}\varphi_0^3(\varepsilon(x+v_1t))\\
&\quad + \varepsilon r e^{iv_1q_{00}(v_1^{-1}x+t)}\bigl[\bigl(v_0^2(v_1^{-1}x+t) - iv_1^{-1}v_0'(v_1^{-1}x+t)\bigr)\varphi_0(\varepsilon(x+v_1t))\\
&\quad - 2i\varepsilon v_0(v_1^{-1}x+t)\varphi_0'(\varepsilon(x+v_1t))\bigr]
\end{aligned}$$

and r_+^0 is given by (2.45). Notice that

$$\tilde{A}_{12}(x,t) = e^{iv_1x/2}\bigl(-ia_t - a_{xx} - (v_1^2/4 - \varepsilon^2)a - 2|a|^2a\bigr), \tag{2.103}$$

where

$$a(x,t)=\varepsilon r e^{-iv_1x/2+iv_1q_{00}(v_1^{-1}x+t)}\varphi_0\big(\varepsilon(x+v_1t)\big).$$

Combining (2.59), (2.98), (2.100), (2.101), (2.102), (2.103) and still under assumption $(\ln\varepsilon)^2\leqslant\varepsilon t\leqslant\varepsilon^{-1/8}$ one gets

$$\chi(x,t)=\chi_+(x,t)+\chi_-(x,t)+O_{H^1}\big(\varepsilon^{3/2}\big), \tag{2.104}$$

where

$$\begin{aligned}
&\chi_\pm(x,t)=e^{-iv_1^2t/4\pm iv_1x/2}\varepsilon\tilde\chi_\pm\big(\varepsilon(x\mp v_1t),\varepsilon^2t\big),\\
&\tilde\chi_\pm(y,\tau)=\zeta_0^\pm(y)+i\tau\big(\partial_y^2\zeta_0^\pm(y)+2\big|\zeta_0^\pm(y)\big|^2\zeta_0^\pm(y)\big),\\
&\zeta_0^+=s\varphi_0(y),\\
&\zeta_0^-=re^{2i\kappa S(y)}\varphi_0(y).
\end{aligned} \tag{2.105}$$

Introducing $\zeta^\pm(y,\tau)$ as being the solution of the following Cauchy problem

$$\begin{aligned}
&i\zeta_\tau^\pm=-\zeta_{yy}^\pm-2\big|\zeta^\pm\big|^2\zeta^\pm,\\
&\zeta^\pm\big|_{\tau=0}=\zeta_0^\pm
\end{aligned}$$

one can rewrite (2.104), (2.105) in the form

$$\begin{aligned}
&\chi(x,t)=\psi_+(x,t)+\psi_-(x,t)+O_{H^1}\big(\varepsilon^{3/2}\big),\\
&\psi_\pm(x,t)=e^{-iv_1^2t/4\pm iv_1x/2}\varepsilon\zeta^\pm\big(\varepsilon(x\mp v_1t),\varepsilon^2t\big),
\end{aligned} \tag{2.106}$$

which concludes the proof of Proposition 2.2.

3. Post-interaction region: $\varepsilon^{-1-\delta}\leqslant t\leqslant\delta|\ln\varepsilon|\varepsilon^{-2}$

3.1. Ansatz

To control the dynamics in the post-interaction regime $\varepsilon^{-1-\delta}=T\leqslant t$ we still use the representation (2.1), (2.2) further decomposing f as

$$f(y,t)=f_+(y,t)+f_-(y,t)+r(y,t),$$

where $f_\pm$ are defined by

$$\psi_\pm(x,t)=e^{i\Phi(x,t)}f_\pm\big(x-b(t),t\big).$$

In terms of r orthogonality conditions (2.2) take the form

$$\langle\vec r,\sigma_3\xi_j\rangle=-\langle\vec f_++\vec f_-,\sigma_3\xi_j\rangle,\quad j=0,\dots,3. \tag{3.1}$$

Notice also that $\zeta^\pm(\tau)$ are bounded in H^1 uniformly in $\tau\in\mathbb{R}$. As a consequence, one has

$$\big\|f_\pm(t)\big\|_{H^1}\leqslant C\varepsilon^{1/2},\quad t\in\mathbb{R},$$

and by Lemma 2.1, the same is true for $r(t)$:

$$\big\|r(t)\big\|_{H^1}\leqslant C\varepsilon^{1/2},\quad t\in\mathbb{R}. \tag{3.2}$$

Note also that by Proposition 2.2:

$$\big\|r(T)\big\|_{H^1}\leqslant C\varepsilon^{3/2}. \tag{3.3}$$

Our goal now will be to improve (3.2) by showing that, in fact, one has

Proposition 3.1. *For $\varepsilon^{-1-\delta} \leqslant t \leqslant \delta|\ln\varepsilon|\varepsilon^{-2}$, $\delta > 0$ sufficiently small, one has*

$$\|r(t)\|_{H^1} \leqslant C\varepsilon^{3/2}e^{C\varepsilon^2 t}. \tag{3.4}$$

The proof of (3.4) relies on the orbital stability type arguments combined with the spacial localisation properties of $\zeta^{\pm}$ described by the following lemma.

Lemma 3.2. *For any $k, l \in \mathbb{N}$ one has*

$$\begin{aligned}&\|\partial_y^k \zeta^{\pm}(\tau)\|_2 \leqslant C,\\ &\|\langle y\rangle^l \partial_y^k \zeta^{\pm}(\tau)\|_2 \leqslant C\langle\tau\rangle^{m(k,l)}, \quad \tau \in \mathbb{R}.\end{aligned} \tag{3.5}$$

The proof of this lemma can be found in [3].

As a consequence of (3.5) one gets for any k, l:

$$\begin{aligned}&\|f_{\pm}(t)\|_{W^{k,\infty}} \leqslant C\varepsilon,\\ &\|f_+(t)f_-(t)\|_{H^k} \leqslant C\varepsilon^l, \quad \varepsilon^{-1-\delta} \leqslant t \leqslant C\varepsilon^{-2}|\ln\varepsilon|.\end{aligned} \tag{3.6}$$

By (2.4), (2.12) one has

$$|b(t)| \leqslant C(1+\varepsilon t), \tag{3.7}$$

which combined with (3.5) implies for any l, k

$$\|\langle y\rangle^{-l}\partial_y^k f_{\pm}(t)\|_2 \leqslant C\varepsilon^l, \tag{3.8}$$

provided $\varepsilon^{-1-\delta} \leqslant t \leqslant C\varepsilon^{-2}|\ln\varepsilon|$.

Applying (3.8) to (3.1) one gets for any l

$$\left|\langle\vec{r}(t), \sigma_3\xi_j\big(E(t)\big)\rangle\right| \leqslant C\varepsilon^l, \quad \varepsilon^{-1-\delta} \leqslant t \leqslant C\varepsilon^{-2}|\ln\varepsilon|,\ j = 0, \ldots, 3. \tag{3.9}$$

Consider

$$\psi^0(t) = \psi(t) - \psi_+(t) - \psi_-(t) = e^{i\Phi}\big(\varphi\big(x-b(t), E(t)\big) + r\big(x-b(t), t\big)\big).$$

It solves

$$\begin{aligned}&i\psi^0_t = -\psi^0_{xx} + F\big(|\psi^0|^2\big)\psi^0 + \mathcal{R},\\ &\mathcal{R}(x,t) = e^{i\Phi}\big(\mathcal{R}_0\big(x-b(t),t\big) + \mathcal{R}_1\big(x-b(t),t\big)\big),\\ &\mathcal{R}_0 = F\big(|\varphi+f_++f_-|^2\big)(\varphi+f_++f_-) - F\big(\varphi^2\big)\varphi + 2|f_+|^2 f_+ + 2|f_-|^2 f_+,\\ &\mathcal{R}_1 = F\big(|\varphi+f_++f_-+r|^2\big)(\varphi+f_++f_-+r) - F\big(|\varphi+f_++f_-|^2\big)(\varphi+f_++f_-)\\ &\qquad - F\big(|\varphi+r|^2\big)(\varphi+r) + F\big(\varphi^2\big)\varphi.\end{aligned} \tag{3.10}$$

Lemma 3.3. *$\psi^0(t)$ satisfies the following "conservation laws":*

$$\left|\frac{d}{dt}\|\psi^0(t)\|_2^2\right| + \left|\frac{d}{dt}H\big(\psi^0(t)\big)\right| + \left|\frac{d}{dt}P\big(\psi^0(t)\big)\right| \leqslant C\big[\varepsilon^7 + \varepsilon^2\|r(t)\|_{H^1}^2 + \varepsilon\|r(t)\|_{H^1}^3\big]. \tag{3.11}$$

Proof. It follows from (3.10), that

$$\begin{aligned}&\frac{d}{dt}\|\psi^0(t)\|_2^2 = 2\,\mathrm{im}\big(\mathcal{R}, \psi^0\big),\\ &\frac{d}{dt}H\big(\psi^0(t)\big) = 2\,\mathrm{im}\big(\psi^0_x, \mathcal{R}_x\big) - 2\,\mathrm{im}\big(F\big(|\psi^0|^2\big)\psi^0, \mathcal{R}\big),\\ &\frac{d}{dt}P\big(\psi^0(t)\big) = -4\,\mathrm{re}\big(\mathcal{R}_x, \psi^0\big).\end{aligned} \tag{3.12}$$

By (3.6), (3.7) one has

$$\|\mathcal{R}_0\|_{H^1} \leqslant C\varepsilon^{9/2},$$

and for any $\gamma_1 > 0$, any l,

$$\big\|e^{-\gamma_1|y|}\mathcal{R}_0\big\|_{H^1} \leqslant C\varepsilon^l, \tag{3.13}$$

$\varepsilon^{-1-\delta} \leqslant t \leqslant C\varepsilon^{-2}|\ln\varepsilon|$.

Consider $\mathcal{R}_1$. It is easy to check that

$$\|\mathcal{R}_1\|_{H^1} \leqslant C\big(\|r\|_{H^1}\big(\big\|e^{-\gamma|y|}(f_+ + f_-)\big\|_{W^{1,\infty}} + \|f_+ + f_-\|^2_{W^{1,\infty}}\big) + \|r\|^2_{H^1}\|f_+ + f_-\|_{W^{1,\infty}}\big),$$

and for any $\gamma_1 > 0$,

$$\big\|e^{-\gamma_1|y|}\mathcal{R}_1\big\|_{H^1} \leqslant C\big(\big\|e^{-\gamma_1|y|}f_+\big\|_{H^1} + \big\|e^{-\gamma_1|y|}f_-\big\|_{H^1}\big)\|r\|_{H^1}.$$

Combining these inequalities with (3.6), (3.7) one gets

$$\begin{aligned}&\|\mathcal{R}_1\|_{H^1} \leqslant C\big(\varepsilon^2\|r\|_{H^1} + \varepsilon\|r\|^2_{H^1}\big),\\ &\big\|e^{-\gamma_1|y|}\mathcal{R}_1\big\|_{H^1} \leqslant C\varepsilon^l,\end{aligned} \tag{3.14}$$

$\varepsilon^{-1-\delta} \leqslant t \leqslant C\varepsilon^{-2}|\ln\varepsilon|$.

From (3.12)–(3.14) one deduces immediately

$$\left|\frac{d}{dt}\big\|\psi^0(t)\big\|_2^2\right|, \left|\frac{d}{dt}H\big(\psi^0(t)\big)\right|, \left|\frac{d}{dt}P\big(\psi^0(t)\big)\right| \leqslant C\big[\varepsilon^7 + \varepsilon^2\big\|r(t)\big\|^2_{H^1} + \varepsilon\big\|r(t)\big\|^3_{H^1}\big]. \qquad \square$$

As a direct consequence of Lemma 3.3 one gets the following estimate on the parameters:

$$\big|E - E(T)\big|, \big|v - v(T)\big| \leqslant C\left(\varepsilon^3 + \big\|r(t)\big\|_2^2 + \int_T^t ds\,\big(\varepsilon^2\big\|r(s)\big\|^2_{H^1} + \varepsilon\big\|r(s)\big\|^3_{H^1}\big)\right), \tag{3.15}$$

$T = \varepsilon^{-1-\delta}$. Indeed, it follows from (3.8) that for any l

$$\begin{aligned}&\big\|\psi^0(t)\big\|_2^2 = \big\|\varphi\big(E(t)\big)\big\|_2^2 + \big\|r(t)\big\|_2^2 + O\big(\varepsilon^l\big),\\ &P\big(\psi^0(t)\big) = P\big(r(t)\big) + iv\big(\big\|\varphi\big(E(t)\big)\big\|_2^2 + \big\|r(t)\big\|_2^2\big) + O\big(\varepsilon^l\big).\end{aligned}$$

Combined with (3.11) this gives (3.15).

Consider the expression $\mathcal{H}(\psi^0(t))$, where the functional $\mathcal{H}$ is defined by

$$\mathcal{H}(\psi) = H(\psi) + E_0\|\psi\|_2^2.$$

One has

$$\begin{aligned}\mathcal{H}\big(\psi^0(t)\big) &= H\big(\varphi(E) + r\big) - \frac{iv}{2}P\big(\varphi(E) + r\big) + \big(E_0 + v^2/4\big)\big\|\varphi(E) + r\big\|_2^2\\ &= \mathcal{H}_0(t) + \big\langle\sigma_3 L(E)\vec{r}, \vec{r}\big\rangle + \mathcal{H}_1(t),\end{aligned} \tag{3.16}$$

where

$$\mathcal{H}_0(t) = H\big(\varphi\big(E(t)\big)\big) + \big(E_0 + v^2(t)/4\big)\big\|\varphi\big(E(t)\big)\big\|_2^2$$

and $\mathcal{H}_1(t)$ satisfies

$$\big|\mathcal{H}_1(t)\big| \leqslant C\big(\varepsilon^l + \big(|v| + |E - E_0|\big)\|r\|^2_{H^1} + \|r\|^3_{H^1}\big), \quad \forall l,$$

or applying (3.2), (2.4)

$$\big|\mathcal{H}_1(t)\big| \leqslant C\big(\varepsilon^l + \varepsilon^{1/2}\|r\|^2_{H^1}\big). \tag{3.17}$$

Notice also that due to (H2), (3.9)

$$\|r\|_{H^1}^2 \leqslant C\big(\langle\sigma_3 L(E)\vec{r},\vec{r}\rangle + \varepsilon^l\big). \tag{3.18}$$

From (3.11) one has

$$\left|\frac{d}{dt}\mathcal{H}(t)\right| \leqslant C\big[\varepsilon^7 + \varepsilon^2\big\|r(t)\big\|_{H^1}^2 + \varepsilon\big\|r(t)\big\|_{H^1}^3\big].$$

Integrating this inequality and using (3.16)–(3.18) one gets

$$\big\|r(t)\big\|_{H^1}^2 \leqslant C\left(\varepsilon^3 + \big|\mathcal{H}_0(t) - \mathcal{H}_0(T)\big| + \int_T^t ds\,\big(\varepsilon^2\big\|r(s)\big\|_{H^1}^2 + \varepsilon\big\|r(s)\big\|_{H^1}^3\big)\right). \tag{3.19}$$

For the difference $\mathcal{H}_0(t) - \mathcal{H}_0(T)$ we have

$$\big|\mathcal{H}_0(t) - \mathcal{H}_0(T)\big| \leqslant C\big(\big|E(t) - E(T)\big|^2 + \big|v^2(t) - v^2(T)\big| + \big(\big|E_0 - E(T)\big| + v^2\big)\big|E(t) - E(T)\big|\big),$$

which by Lemma 2.1 implies

$$\big|\mathcal{H}_0(t) - \mathcal{H}_0(T)\big| \leqslant C\varepsilon\big(\big|E - E(T)\big| + \big|v(t) - v(T)\big|\big). \tag{3.20}$$

Combining (3.20), (3.19), (3.15) one obtains finally

$$\big\|r(t)\big\|_{H^1}^2 \leqslant C\left(\varepsilon^3 + \int_T^t ds\,\big(\varepsilon^2\big\|r(s)\big\|_{H^1}^2 + \varepsilon\big\|r(s)\big\|_{H^1}^3\big)\right). \tag{3.21}$$

To deduce (3.4) from (3.21) we use once more the bootstrap argument. Assume

$$\big\|r(t)\big\|_{H^1} < \varepsilon. \tag{3.22}$$

Note that by (3.2), (3.22) is verified for t sufficiently close to T. Under bootstrap assumption (3.22), (3.21) becomes

$$\big\|r(t)\big\|_{H^1}^2 \leqslant C\left(\varepsilon^3 + \varepsilon^2\int_T^t ds\,\big\|r(s)\big\|_{H^1}^2\right)$$

and gives the desired bound

$$\big\|r(t)\big\| \leqslant C\varepsilon^{3/2}e^{C\varepsilon^2 t}.$$

Combined with the standard continuity arguments this concludes the proof of Proposition 3.1.

3.2. Estimates of the trajectory

To finish the proof of Theorem 1.1 it remains to analyse the trajectory $\sigma(t)$. Combining (3.4), (3.15) one gets

$$\big|E(t) - E(T)\big|, \big|v(t) - v(T)\big| \leqslant C\varepsilon^3 e^{C\varepsilon^2 t}, \tag{3.23}$$

which together with (2.12) implies

$$\big|E(t) - E_0\big|, \big|v(t) - V_0\big| \leqslant C\varepsilon^2, \qquad V_0 = 2\varepsilon\kappa. \tag{3.24}$$

Furthermore, from (2.7), (3.8), (3.4), (3.24), (2.12) one has

$$\begin{aligned} &\big|b' - V_0\big| \leqslant C\varepsilon^2, \qquad \big|b(T) - V_0 T\big| \leqslant C\varepsilon^{1-\delta},\\ &\left|\beta' - \beta_0' + \frac{(vb)'}{2}\right| \leqslant C\varepsilon^2, \qquad \big|\beta(T) - \beta_0(T)\big| \leqslant C\varepsilon^{1-\delta}. \end{aligned} \tag{3.25}$$

As a consequence,

$$\big|b(t) - V_0 t\big|, \big|\beta(t) - \beta_0(t)\big| \leqslant C\varepsilon^2 t, \quad T \leqslant t \leqslant \delta\varepsilon^{-2}|\ln\varepsilon|, \tag{3.26}$$

which concludes the proof of Theorem 1.1.

Appendix A. Proof of Proposition 1.1

The proof is by the standard fixed point arguments. We write the solution as the sum

$$\psi(t) = w_0(t) + w_1(t) + \chi(t), \qquad w_j(t) = w(\cdot, \sigma_j(t)), \quad j = 0, 1.$$

Then $\vec{\chi}(t) = \binom{\chi}{\bar{\chi}}$ solves

$$i\vec{\chi}_t = -\sigma_3\vec{\chi}_{xx} + \mathcal{V}(w_0(t))\vec{\chi} + N(\chi), \tag{A.1}$$

where

$$\begin{aligned}
&\mathcal{V}(w_0) = \big(F(|w_0|^2) + F'(|w_0|^2)|w_0|^2\big)\sigma_3 + F'(|w_0|^2)\begin{pmatrix} 0 & w_0^2 \\ -\bar{w}_0^2 & 0 \end{pmatrix} = e^{i\beta_0\sigma_3}V(E_0)e^{-i\beta_0\sigma_3},\\
&N(\chi) = N_0(\chi) + N_1(\chi),\\
&N_0(\chi) = \mathcal{V}(w_0)\begin{pmatrix} w_1 \\ \bar{w}_1 \end{pmatrix} + F(|w_0 + w_1 + \chi|^2)\begin{pmatrix} w_0 + w_1 + \chi \\ -\bar{w}_0 - \bar{w}_1 - \bar{\chi} \end{pmatrix} - \mathcal{V}(w_0)\begin{pmatrix} w_1 + \chi \\ \bar{w}_1 + \bar{\chi} \end{pmatrix}\\
&\qquad - F(|w_0|^2)\begin{pmatrix} w_0 \\ -\bar{w}_0 \end{pmatrix} - F(|w_1 + \chi|^2)\begin{pmatrix} w_1 + \chi \\ -\bar{w}_1 - \bar{\chi} \end{pmatrix},\\
&N_1(\chi) = F(|w_1 + \chi|^2)\begin{pmatrix} w_1 + \chi \\ -\bar{w}_1 - \bar{\chi} \end{pmatrix} - F(|w_1|^2)\begin{pmatrix} w_1 \\ -\bar{w}_1 \end{pmatrix}.
\end{aligned}$$

For χ satisfying $\|\chi\|_{H^1} \leqslant C$, one has

$$\begin{aligned}
&\|N_0(\chi)\|_{H^1} \leqslant C\big(\varepsilon e^{\varepsilon\gamma t} + \|\chi\|_{H^1}^2\big),\\
&\|N_0(\chi_1) - N_0(\chi_2)\|_{H^1} \leqslant C\|\chi_1 - \chi_2\|_{H^1}\big(\varepsilon e^{\varepsilon\gamma t} + \|\chi_1\|_{H^1} + \|\chi_2\|_{H^1}\big),\\
&\|N_1(\chi)\|_{H^1} \leqslant C\big(\varepsilon^2\|\chi\|_{H^1} + \|\chi\|_{H^1}^3\big),\\
&\|N(\chi_1) - N(\chi_2)\|_{H^1} \leqslant C\|\chi_1 - \chi_2\|_{H^1}\big(\varepsilon^2 + \|\chi_1\|_{H^1}^2 + \|\chi_2\|_{H^1}^2\big),\\
&\big|\langle N_1(\chi), \sigma_3 e^{i\beta_0(t)\sigma_3}\xi_j(E_0)\rangle\big| \leqslant C\big(\varepsilon e^{\varepsilon\gamma t} + \|\chi\|_{H^1}^2\big),\\
&\big|\langle N_1(\chi_1) - N(\chi_2), \sigma_3 e^{i\beta_0(t)\sigma_3}\xi_j(E_0)\rangle\big| \leqslant C\|\chi_1 - \chi_2\|_{H^1}\big(\varepsilon e^{\varepsilon\gamma t} + \|\chi_1\|_{H^1} + \|\chi_2\|_{H^1}\big).
\end{aligned} \tag{A.2}$$

We rewrite (A.1) as an integral equation

$$\begin{aligned}
&\chi(t) = \mathcal{J}(\chi)(t) \equiv -i\int_{-\infty}^{t} ds\,\mathcal{U}(t,s)N(\chi(s)),\\
&\mathcal{U}(t,s) = e^{i\beta_0(t)\sigma_3}e^{-i(t-s)L(E_0)}e^{-i\beta_0(t)\sigma_3}
\end{aligned}$$

and view $\mathcal{J}$ is a mapping in the space $C((-\infty, T], H^1)$, $T < 0$, equipped with the norm

$$|||\chi||| = \varepsilon \sup_{\tau \leqslant T} e^{-\gamma\varepsilon\tau}\|\chi(\tau)\|_{H^1}.$$

It follows from (1.10), (A.2) that

$$\begin{aligned}
&|||\mathcal{J}(\chi)||| \leqslant K\big(1 + \varepsilon^{-3}e^{\gamma\varepsilon T}|||\chi|||^2 + \varepsilon|||\chi|||\big),\\
&|||\mathcal{J}(\chi_1) - \mathcal{J}(\chi_2)||| \leqslant K_1|||\chi_1 - \chi_2|||\big(\varepsilon + \varepsilon^{-1}e^{\gamma\varepsilon T} + \varepsilon^{-3}e^{\gamma\varepsilon T}\big(|||\chi_1||| + |||\chi_2|||\big)\big),
\end{aligned}$$

provided $\|\chi\|_{H^1} \leqslant C$. This means that $\mathcal{J}$ is a contraction of the ball $|||\chi||| \leqslant 2K$ into itself, provided both ε and $\varepsilon^{-3}e^{\gamma\varepsilon T}$ are sufficiently small. Consequently, it has a unique fixed point χ that satisfies

$$\|\chi(t)\|_{H^1} \leqslant C\varepsilon^{-1}e^{\gamma\varepsilon t}, \quad t \leqslant T.$$

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