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Gradient flow of the Chapman–Rubinstein–Schatzman model for signed vortices

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Dedicated to the memory of Michelle Schatzman (1949–2010)

Abstract

We continue the study of Ambrosio and Serfaty (2008) [4] on the Chapman–Rubinstein–Schatzman–E evolution model for superconductivity, viewed as a gradient flow on the space of measures equipped with the quadratic Wasserstein structure. In Ambrosio and Serfaty (2008) [4] we considered the case of positive (probability) measures, while here we consider general real measures, as in the physical model. Understanding the evolution as a gradient flow in this context gives rise to several new questions, in particular how to define a "Wasserstein" distance for signed measures. We generalize the minimizing movement scheme of Ambrosio et al. (2005) [3] in this context, we show the entropy argument of Ambrosio and Serfaty (2008) [4] still carries through, and derive an evolution equation for the measure which contains an error term compared to the Chapman–Rubinstein–Schatzman–E model. Moreover, we also show the same applies to a very similar dissipative model on the whole plane.

1. Introduction

In [6], Chapman, Rubinstein and Schatzman (see also E [10]) derived formally the following mean-field model for the evolution of the density of vortices in a type-II superconductor under the effect of an external magnetic field, in the limit where the Ginzburg–Landau parameter κ tends to $+\infty$ and the number of vortices becomes large:

$$\begin{cases} \dfrac{d}{dt}\mu(t) - \operatorname{div}\big(\nabla h_{\mu(t)}\big|\mu(t)\big|\big) = 0 & \text{in } (0,+\infty)\times\Omega, \\ \mu(0) = \mu_0 & \text{at } t = 0 \end{cases} \tag{1.1}$$

where h_μ is given by

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$$\begin{cases} -\Delta h_\mu + h_\mu = \mu & \text{in } \Omega, \\ h_\mu = 1 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Type-II superconductors, submitted to an external field, have a very particular response: they "repel" the applied field, which only penetrates through "vortices". In the above, Ω is the two-dimensional domain occupied by the superconducting sample, μ is a signed measure representing the vortex density (vortices are punctual objects which carry a quantized topological degree, which can be positive or negative) and h_μ is the magnetic field induced in the sample. The boundary condition $h_\mu = 1$ corresponds to the effect of an external magnetic whose intensity is here normalized to 1. h_μ can be viewed as a potential generated by the vortices through the relation (1.2).

In [4] we studied the problem in the case where μ is a positive measure, which one can normalize to be a probability measure on $\overline{\Omega}$. Here we examine the signed measure case. More precisely we look for a solution $\mu(t)$ to the continuity equation

$$\frac{d}{dt}\mu(t) - \operatorname{div}\big(\chi_\Omega \nabla h_{\mu(t)} \big|\mu(t)\big|\big) = 0 \quad \text{in } \mathcal{D}'\big((0,+\infty)\times\mathbb{R}^2\big) \tag{1.3}$$

with the initial datum $\mu(0) = \mu_0$ in $\mathcal{M}(\overline{\Omega}) \cap H^{-1}(\Omega)$, where $\mathcal{M}(\overline{\Omega})$ denotes the space of bounded Radon measures on $\overline{\Omega}$.

Let us recall the definition of the well-known quadratic Wasserstein distance between two probability measures μ and ν on $\mathbb{R}^n$:

$$W_2(\mu,\nu) = \inf_{\gamma\in\Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} |x-y|^2\, d\gamma(x,y) \right)^{\frac{1}{2}} \tag{1.4}$$

where $\Gamma(\mu,\nu)$ denotes the set of probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ which have marginals μ and ν. The key point in [4], was to view (1.1)–(1.2) as the gradient flow of the energy functional (related to the standard Ginzburg–Landau functional, see [19,20])

$$\Phi_\lambda(\mu) = \frac{\lambda}{2}|\mu|(\Omega) + \frac{1}{2}\int_\Omega |\nabla h_\mu|^2 + |h_\mu - 1|^2, \quad \lambda \geqslant 0, \tag{1.5}$$

for the above quadratic Wasserstein W_2 structure on the space of probability measures on $\overline{\Omega}$, and then to apply the framework of [3] (inspired from the seminal papers [11,17]) for constructing gradient flows in the Wasserstein spaces, which consists in minimizing recursively

$$\nu \mapsto \Phi_\lambda(\nu) + \frac{W_2^2(\mu_k,\nu)}{2\tau}, \tag{1.6}$$

and then passing to the limit as $\tau \to 0$. This specific problem posed several difficulties:

- the natural energy space was $P(\overline{\Omega}) \cap H^{-1}(\Omega)$ (where P denotes probability measures) and not the space of absolutely continuous measures;
- for no $\alpha \in \mathbb{R}$ the energy functional (1.5) is α-displacement convex in that space;
- the case of measures on a bounded domain with the possibility of mass entering or exiting the domain is nonstandard.

The results of [4] can be summarized as follows:

- if the initial datum is in L^∞ then there is existence for a strong formulation of the equation, and uniqueness until some mass reaches the boundary;
- for general finite energy initial data, there is existence (but in general no uniqueness) of solutions to the equation in a weak sense, obtained as the limit of a time-discrete "minimizing movement" scheme, and satisfying an energy-dissipation relation;
- there exists a family of "entropy" functionals which decrease along the flow, and ensure that if the initial datum is in L^p, the solution remains in L^p for all time;

- in addition, in [13] a global uniqueness result is proved in the case of a convex domain Ω. The difficulty is the potential presence of mass on $\partial\Omega$, and this result is obtained through a very precise formulation of boundary conditions, an issue that we are not going to address in this paper.

Here we would like to pursue the same strategy of viewing (1.1) as a gradient flow, but on the space of signed measures. Note that while there have been numerous studies of PDE's viewed as gradient flows on the Wasserstein spaces of probability measures (most of the time for absolutely continuous measures and α-displacement convex functionals), see [3,21,22] and the references therein, there has been absolutely *no* such study in the case of signed measures. This is an open field which we believe to be natural since physical models, such as this one, sometimes also involve signed or charged densities. As we shall see, our study raises as many open questions as it solves.

The first question that arises is to define an analogue of the Wasserstein distance on signed measures (which have equal integrals). While this is obvious in the case of the 1-Wasserstein (or Kantorovich–Rubinstein) distance, it turns out to be really nontrivial for exponents $p > 1$, as we shall see in Section 2. The first naive attempt one can make is to define the distance between the signed measures μ and ν such that $\mu(\overline{\Omega}) = \nu(\overline{\Omega})$ by

$$\mathbb{W}_2(\mu,\nu) := W_2\big(\mu^+ + \nu^-, \nu^+ + \mu^-\big),$$

where μ^+, ν^+ (resp. μ^-, ν^-) are the positive (resp. negative) parts of the measures μ and ν. (Here, by assumption, $\mu^+ + \nu^-$ and $\mu^- + \nu^+$ are two positive measures of same mass, one can easily extend the definition of the standard W_2 distance to that case.) We will study the properties of $\mathbb{W}_2$ in Section 2. It turns out that this definition has two major flaws: the "distance" defined this way is not lower semi-continuous, and examples show that the triangle inequality can be violated! One can then think of several ways to fix these problems, obtained by relaxing the definition in various ways, which will make the "distance" lower semi-continuous. However, it is still not at all obvious that the triangle inequality holds. So in the end, we postponed the definition of a "canonical" 2-Wasserstein distance on signed measures, which we believe to be a problem of independent interest, to future work. In the meantime, an expansion of this discussion including several possible concurrent definitions is described in the proceedings paper [14]. For our purposes of applying the minimizing movement scheme (1.6), it suffices to build a "pseudo-distance" which is lower semi-continuous and bounded from below by a distance.

One of the advantages of the Wasserstein variational approach is the possibility of handling measure-valued solutions and nonsmooth velocity fields. Here, even thinking of a mildly regular $\mu(t)$, we do not have regularity of the velocity in (1.3), which is always multiplied by the sign of $\mu(t)$. This prevents the application of the DiPerna–Lions theory [8] of flows associated to weakly differentiable vector fields.

We then proceed similarly as in [4]. The weak formulation of [4] cannot be used for signed measures, since it uses Delort's convergence theorem [7] which holds only for positive measures. So we have to assume L^p $(p \geqslant 4)$ integrability of the initial data, and it turns out that the entropy argument of [4] still carries through (although in a not completely obvious way), and ensures that L^p integrability is preserved along the discrete flow. The Euler–Lagrange equation for discrete minimizers μ_τ can be derived as in [4]. However, taking the limit as the time-step $\tau \to 0$, one is confronted with two difficulties: first, there are no standard *limiting velocity* results as in the positive case for writing down a continuity equation. However, we can still pass to the limit "by hand", without using the general theory of [3]. Second and more importantly, the positive and negative parts μ_τ^+ and μ_τ^- of the discrete minimizer have weak limits μ^+ and μ^-, but these weak limits are not necessarily mutually singular, so it is not clear that $|\mu_\tau|$ converge weakly to $\mu^+ + \mu^-$. This kind of strong convergence of the scheme is an open problem of independent interest, that we hope to address in a future paper: at least in principle, one can hope that strong compactness properties hold for the transport-cancellation mechanism, when good bounds on the velocity (as in our case) are present, so that cancellation is encoded only in the discrete scheme, and does not happen in the limit (see also the additional remarks at the end of the introduction). We also emphasize that the role of the energy term $\lambda|\mu|/2$ in (1.5) is not completely clear: while in [4] it was a null-Lagrangian, thought to have an influence only on the rate of mass dissipation through the boundary, here it probably has an influence also in cancellations in the discrete scheme. Because of potential cancellations in the limit, we obtain an evolution equation with a limit term ϱ which is really the limit of $\mu^+ + \mu^-$ above and could be thought as a kind of defect measure:

Theorem 1.1. *Let $\mu^0 \in L^4(\Omega)$. The minimizing movement scheme produces a signed measure $\mu(t) \in L^4(\Omega)$ which satisfies $\mu(0) = \mu_0$ and*

$$\frac{d}{dt}\mu(t) - \operatorname{div}\bigl(\chi_\Omega \nabla h_{\mu(t)} \varrho(t)\bigr) = 0 \quad \text{in } \mathcal{D}'\bigl((0,+\infty) \times \mathbb{R}^2\bigr), \tag{1.7}$$

where $\varrho(t)$ is a suitable positive measure satisfying $\varrho(t) \geqslant |\mu(t)|$ in Ω.

We can check however that when $\mu(0) \geqslant 0$ the measure $\varrho(t)$ is equal to $\mu(t)$, so we retrieve at least the result of [4], and we conjecture for the reasons explained before that the scheme can be improved to obtain $\varrho(t) = |\mu(t)|$ for all $t \geqslant 0$. Actually our methods yield more, namely the system of PDE's

$$\begin{cases} \dfrac{d}{dt}\varrho^+(t) - \operatorname{div}\bigl(\chi_\Omega \nabla h_{\mu(t)} \varrho^+(t)\bigr) = -\sigma(t), \\ \dfrac{d}{dt}\varrho^-(t) + \operatorname{div}\bigl(\chi_\Omega \nabla h_{\mu(t)} \varrho^-(t)\bigr) = -\sigma(t). \end{cases} \tag{1.8}$$

This system, where $\varrho^\pm$ need not be the positive and negative part of a signed measure ϱ, has an interesting structure in its own right, the coupling being due to the negative term $-\sigma(t) \leqslant 0$ and in the velocity field, since $\mu(t) = \varrho^+(t) - \varrho^-(t)$. At this level, it would be nice to understand under which assumptions the system preserves orthogonality of ϱ^+ and ϱ^- in time.

Let us finally turn to the case of the whole plane and comment on related results in the literature. Models very similar to (1.1) were previously studied (see references in [4], like [9,12,16]). In particular Lin and Zhang [12], Masmoudi and Zhang [16] studied the equation

$$\frac{d}{dt}\mu(t) + \operatorname{div}\bigl(\nabla \Delta^{-1}\mu(t)\bigl|\mu(t)\bigr|\bigr) = 0 \tag{1.9}$$

in the whole $\mathbb{R}^2$, which can be viewed as a dissipative version of the Euler equation in vorticity formulation. Lin and Zhang focused on the positive measure case, and Masmoudi and Zhang on the signed case. They do not use the gradient flow approach, but find solutions by passing to the limit in some approximating PDEs. In [12] results analogous to those we described from [4] were proven. In [16] they construct solutions to Eq. (1.9) but assuming some $W^{1,p}$ regularity of the initial measure which is used crucially and ensures good compactness properties (so, in this case the transport-cancellation mechanism has good compactness properties). Thus existence of solutions in the general measure case, or in the L^p case, is still open. In Section 6, we study (1.9) in the whole plane, and extend our gradient-flow approach to that case. This poses a slight difficulty since the obvious energy functional that should replace Φ_λ, which is $\frac{1}{2}\int_{\mathbb{R}^2} |\nabla h_\mu|^2$, is in general infinite because of the logarithmic behavior of the Green's kernel in dimension 2. To remedy this, we introduce a "renormalized" way of computing the energy. We then show that the entropy argument of [4] can still be adapted to that energy, so that all the results of [4] and those of the present paper are valid for that infinite-plane model as well. In the case of positive measures, this retrieves solutions of (1.9).

The paper is organized as follows: in Section 2 we investigate potential distances on signed measures, give counter-examples for $\mathbb{W}_2$ and present several alternative costs that we use later.

In Section 3 we present the time-stepping discrete minimization (or "minimizing movement") scheme and derive the Euler–Lagrange equation satisfied by discrete minimizers.

In Section 4 we prove the same entropy result as in [4], in the signed case, ensuring that L^p regularity is preserved along the discrete flow, and that the product $\mu \nabla h_\mu$ makes sense.

In Section 5 we pass to the limit in the discrete minimizers, and obtain the limit evolution equation.

In Section 6 we examine the model (1.9), introduce the renormalized formulation of the energy, and discuss how to adapt our methods to that case.

2. Transport cost for signed measures

When trying to generalize the theory of Wasserstein gradient flows to signed measures, as we mentioned the first difficulty arises at the theoretical point of view: there is no standard definition of p-Wasserstein distance on signed measures. Moreover, we do not know how to rephrase the characterization of absolutely continuous curves in the space of measures by means of continuity equations given in [3].

On the other hand, we can take advantage of the flexibility of the minimizing movements approach. Indeed, the minimization problem

$$\min_{\nu\in\mathfrak{X}} \phi(\nu) + \frac{1}{2\tau} d^2(\nu,\mu), \quad \mu \in \mathfrak{X} \tag{2.1}$$

makes sense in any metric space $\mathfrak{X}$, d being the corresponding distance, where $\phi : \mathfrak{X} \to \mathbb{R}$. On top of that, it is not strictly needed for the functional d appearing in (2.1) to be a distance. In fact, often the important thing is its behavior on small scales, when $\nu \sim \mu$. As in the seminal paper [1], one could also use a non-triangular or non-symmetric object. Actually, we are going to make use of a functional d which, though not a distance, is bounded from below by a distance.

We begin with the definition of the ambient space. Let $\mathcal{M}(\overline{\Omega})$ denote the set of bounded Radon measures over $\overline{\Omega}$. We endow $\mathcal{M}(\overline{\Omega})$ with the standard weak (or narrow) convergence, given by the duality with continuous and bounded functions. Let us define the following measure subset of $\mathcal{M}(\overline{\Omega})$:

$$\mathcal{M}_{\kappa,M}(\overline{\Omega}) := \big\{\mu \in \mathcal{M}(\overline{\Omega}):\ \mu(\overline{\Omega}) = \kappa,\ |\mu|(\overline{\Omega}) \leqslant M\big\}, \tag{2.2}$$

where $\kappa \in \mathbb{R}$.

In the sequel we will often make use of the Hahn decomposition for a real measure μ, identifying its positive and negative parts, so that $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are two positive measures. This decomposition is minimal in the sense that, for any other pair of positive measures σ^1, σ^2 such that $\sigma^1 - \sigma^2 = \mu$, there hold $\mu^+ \leqslant \sigma^1$ and $\mu^- \leqslant \sigma^2$. Moreover, if μ is a positive measure, we will say that μ_0 is a submeasure of μ if $\mu_0(A) \leqslant \mu(A)$ for any μ-measurable set A.

A way which seems at first glance natural for defining a 2-Wasserstein distance in $\mathcal{M}_{\kappa,M}(\overline{\Omega})$ is the following.

2.1. *First cost*

Let $\mu, \nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$. Define

$$\mathbb{W}_2(\mu,\nu) := W_2\big(\mu^+ + \nu^-, \nu^+ + \mu^-\big), \tag{2.3}$$

where W_2 is as in (1.4) (but naturally extended from probability measures to nonnegative measures with a fixed total mass, possibly different from 1). It is immediate to check that, if μ and ν are nonnegative, $\mathbb{W}_2$ reduces to the Wasserstein distance between positive measures of a given mass κ on $\overline{\Omega}$. The functional $\mathbb{W}_2$ accounts for the cost of transporting signed measures, and some heuristics on its behavior are worthy. We notice that, when transporting a signed measure μ, its positive and negative masses may change (only $\int \mu$ is fixed, as in (2.2)). So, in order to connect μ to ν, it may be convenient to transport some part of μ^+ onto μ^-, this correspond to auto-annihilation of mass. On the other hand, if the total variation of ν is larger than that of μ, one expects that, in the transport given by $\mathbb{W}_2$, a nonzero part will come from moving some part of ν^- to ν^+. From the dynamic point of view, this corresponds to some fake zero charge mass which is created and separated into positive and negative mass, while being transported at a certain cost.

Remark 2.1. This framework fits the physical problem we are investigating, since we expect that vortices with opposite degrees can interact like dipoles and cancel each other. Also it is in principle possible that dipoles be created ex-nihilo.

Although it is immediate to verify that $\mathbb{W}_2$ is symmetric and vanishes if and only if $\mu = \nu$, $\mathbb{W}_2$ is not a distance. Indeed, the following example shows that the triangle inequality fails. On the real line, let $\mu = \delta_0$, $\nu = \delta_4$ and $\eta = \delta_1 - \delta_2 + \delta_3$. Clearly $\mathbb{W}_2(\mu,\nu) = W_2(\mu,\nu) = 4$. But the optimal transport plan between $\mu^+ + \eta^-$ and $\eta^+ + \mu^-$ is given by $\delta_0 \times \delta_1 + \delta_2 \times \delta_3$, so that

$$\mathbb{W}_2(\mu,\eta) = \sqrt{\int_{\mathbb{R}} |x-y|^2\, d(\delta_0 \times \delta_1) + \int_{\mathbb{R}} |x-y|^2\, d(\delta_2 \times \delta_3)} = \sqrt{2}.$$

Symmetrically, $\mathbb{W}_2(\nu,\eta)=\sqrt{2}$, so that

$$\mathbb{W}_2(\mu,\nu) > \mathbb{W}_2(\mu,\eta)+\mathbb{W}_2(\nu,\eta).$$

On the other hand, we notice that if $\gamma\in\Gamma_0(\mu^+ + \nu^-, \nu^+ + \mu^-)$ where Γ_0 denotes the set of optimal transport plans, by Hölder inequality we have

$$\left(\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\,d\gamma\right)^{1/2} \geqslant \sqrt{\frac{1}{2M}}\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\gamma, \tag{2.4}$$

but

$$\mathbb{W}_1(\mu,\nu) := W_1\big(\mu^+ + \nu^-, \nu^+ + \mu^-\big) = \inf_{\gamma\in\Gamma(\mu^+ + \nu^-, \nu^+ + \mu^-)} \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\,d\gamma \tag{2.5}$$

is indeed a distance between signed measures. This can be seen by the well-known Kantorovich duality formula, that gives (see for example [21])

$$W_1\big(\mu^+ + \nu^-, \nu^+ + \mu^-\big) = \sup_{\varphi\in\mathrm{Lip}(\Omega),\, \|\varphi\|_{\mathrm{Lip}}\leqslant 1} \int_{\Omega} \varphi\, d(\mu-\nu). \tag{2.6}$$

Clearly, by looking at the right-hand side, we have a distance. Notice in addition that $\mathbb{W}_1$ is not sensitive to the addition of equal masses in the source and in the target (a feature typical of 1-distances), since (2.6) readily gives

$$\mathbb{W}_1(\mu,\nu) = \mathbb{W}_1(\mu+\sigma, \nu+\sigma), \quad \forall\sigma\in\mathcal{M}\big(\mathbb{R}^2\big). \tag{2.7}$$

It is worth analyzing some other features of $\mathbb{W}_2$. In the next proposition we see that $\mathbb{W}_2$ "metrizes" the weak topology of $\mathcal{M}_{\kappa,M}(\overline{\Omega})$.

Proposition 2.2. *Let μ_n, μ belong to $\mathcal{M}_{\kappa,M}(\overline{\Omega})$. Then $\mu_n \rightharpoonup \mu$ if and only if $\mathbb{W}_2(\mu_n,\mu)\to 0$.*

Proof. Assume that $\mu_n \rightharpoonup \mu$. Since $\mu_n^+(\overline{\Omega}) \leqslant M$ and $\mu_n^-(\overline{\Omega}) \leqslant M$, by tightness there exists a subsequence (n_k) such that $\mu_{n_k}^+ \rightharpoonup \sigma^+$ and $\mu_{n_k}^- \rightharpoonup \sigma^-$, with $\sigma^+ - \sigma^- = \mu$. By continuity of the Wasserstein distance, for each limit point we have $W_2(\mu_n^+ + \mu^-, \mu^+ + \mu_n^-) \to W_2(\sigma^+ + \mu^-, \mu^+ + \sigma^-) = 0$.

Assume that $\mathbb{W}_2(\mu_n,\mu)\to 0$, that is $W_2(\mu_n^+ + \mu^-, \mu^+ + \mu_n^-) \to 0$. Since W_2 metrizes the weak convergence, there exists a positive measure ϑ such that $\mu_n^+ + \mu^- \rightharpoonup \vartheta$ and $\mu_n^- + \mu^+ \rightharpoonup \vartheta$, hence $\mu_n^+ - \mu_n^- \rightharpoonup \mu^+ - \mu^- = \mu$. □

We have seen that $\mathbb{W}_2$ is not a distance. With a similar simple construction, it is possible to see that the map $\nu \mapsto \mathbb{W}_2(\nu,\mu)$ is not weakly l.s.c. in $\mathcal{M}_{\kappa,M}(\Omega)$. For instance, let $\mu = \delta_{-1} - \delta_1$ and $\nu_n = \delta_{-1/n} - \delta_{1/n}$, so that $\nu_n \rightharpoonup \nu = 0$. Clearly $\mathbb{W}_2(\nu^+ + \mu^-, \mu^+ + \nu^-) = W_2(\mu^-, \mu^+) = 2$. But, as $n\to\infty$, $\liminf \mathbb{W}_2(\nu_n^+ + \mu^-, \mu^+ + \nu_n^-) = \liminf \sqrt{2}(n-1)/n = \sqrt{2}$. The point is that if $\nu_n \rightharpoonup \nu$ in $\mathcal{M}_{\kappa,M}(\overline{\Omega})$, then (ν_n^+) and (ν_n^-) are tight, but the limits are not in general ν^+ and ν^- (in the example above, they are not zero). In order to overcome this problem, we can consider a kind of relaxation of $\mathbb{W}_2$. More details are given in the proceedings paper [14].

In order to deal with optimal transport plans between signed measures, consider partitions of the positive and negative parts of ν and μ of the form

$$\begin{aligned} &\mu_0^+ + \mu_1^+ = \mu^+, \qquad \mu_0^- + \mu_1^- = \mu^-,\\ &\nu_0^+ + \nu_1^+ = \nu^+, \qquad \nu_0^- + \nu_1^- = \nu^-, \end{aligned} \tag{2.8}$$

where all the terms are positive measures. Some compatibility conditions have to be taken into account, and precisely

$$\nu_0^+(\overline{\Omega}) = \mu_0^+(\overline{\Omega}), \qquad \mu_0^-(\overline{\Omega}) = \nu_0^-(\overline{\Omega}), \qquad \mu_1^-(\overline{\Omega}) = \mu_1^+(\overline{\Omega}), \qquad \nu_1^+(\overline{\Omega}) = \nu_1^-(\overline{\Omega}). \tag{2.9}$$

μ_0^+ and μ_0^- correspond to the parts that will move to ν_0^+, ν_0^- respectively and μ_1^+, μ_1^- (resp. ν_1^+, ν_1^-) to the self-cancelling parts.

Of course there are many partitions of this kind. Moreover, we have the following

Lemma 2.3 *(Splitting of the optimal plan). Let $\gamma \in \Gamma_0(\nu^+ + \mu^-, \mu^+ + \nu^-)$. Then there exists a partition of the form* (2.8)–(2.9) *such that γ can be written as the sum of four plans γ_+^+, γ_-^-, γ_-^+, γ_+^- satisfying*

$$\begin{aligned} &\gamma_+^+ \in \Gamma_0(\nu_0^+, \mu_0^+), \qquad \gamma_-^- \in \Gamma_0(\mu_0^-, \nu_0^-), \\ &\gamma_-^+ \in \Gamma_0(\mu_1^-, \mu_1^+), \qquad \gamma_+^- \in \Gamma_0(\nu_1^+, \nu_1^-). \end{aligned} \tag{2.10}$$

Proof. Let $\vartheta_1 = \nu^+ + \mu^-$ and $\vartheta_2 = \mu^+ + \nu^-$. It is clear that ν^+ and μ^- are both absolutely continuous with respect to ϑ_1. Let $f_1, g_1 \in L^1(\mathbb{R}^2, \vartheta_1)$ denote the respective densities. Similarly, let f_2, g_2 be the densities of ν^- and μ^+ with respect to ϑ_2, so that

$$\nu^+ = f_1\vartheta_1, \qquad \mu^- = g_1\vartheta_1, \qquad \mu^+ = g_2\vartheta_2, \qquad \nu^- = f_2\vartheta_2.$$

Clearly $f_1 + g_1 = f_2 + g_2 = 1$, so that we can write

$$\gamma = (f_1 \circ \pi^1)(g_2 \circ \pi^2)\gamma + (f_1 \circ \pi^1)(f_2 \circ \pi^2)\gamma + (g_1 \circ \pi^1)(g_2 \circ \pi^2)\gamma + (g_1 \circ \pi^1)(f_2 \circ \pi^2)\gamma. \tag{2.11}$$

Notice that

$$\begin{aligned} &\pi^1_\#\big((f_1 \circ \pi^1)(g_2 \circ \pi^2)\gamma\big) = f_1\, \pi^1_\#\big((g_2 \circ \pi^2)\gamma\big) \leqslant f_1\, \pi^1_\#\gamma = f_1\vartheta_1 = \nu^+, \\ &\pi^2_\#\big((f_1 \circ \pi^1)(g_2 \circ \pi^2)\gamma\big) = g_2\, \pi^2_\#\big((f_1 \circ \pi^1)\gamma\big) \leqslant g_2\, \pi^2_\#\gamma = g_2\vartheta_2 = \mu^+. \end{aligned}$$

Moreover,

$$\pi^1_\#\big((f_1 \circ \pi^1)(g_2 \circ \pi^2)\gamma\big) + \pi^1_\#\big((f_1 \circ \pi^1)(f_2 \circ \pi^2)\gamma\big) = f_1\, \pi^1_\#\big((g_2 \circ \pi^2 + f_2 \circ \pi^2)\gamma\big) = f_1\, \pi^1_\#\gamma = \nu^+.$$

With the analogous computations for the other terms in the right-hand side of (2.11), we see that the marginals of the four plans therein are submeasures of ν^+, μ^-, μ^+, ν^- satisfying (2.8)–(2.9). Hence, in (2.11) γ is written as the sum of four plans on a partition of the desired form. Moreover, each of these plans is optimal, since their sum is. □

2.2. Second cost

In order to deal with a sequence of measures with decreasing total mass, we introduce the following simplified version of $\mathbb{W}_2$. Let $\mu, \nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$ with $|\nu|(\overline{\Omega}) \leqslant |\mu|(\overline{\Omega})$. Define

$$\mathcal{W}_2^2(\nu, \mu) = \inf_{\substack{\nu_n \to \nu \\ \nu_n^+(\overline{\Omega}) = \mu^+(\overline{\Omega}) \\ \nu_n^-(\overline{\Omega}) = \mu^-(\overline{\Omega})}} \Big\{ \liminf_{n\to\infty} \big(W_2^2(\nu_n^+, \mu^+) + W_2^2(\nu_n^-, \mu^-)\big)\Big\}. \tag{2.12}$$

This way, we see that we may cancel mass only between μ^+ and μ^-. This will correspond to the fact that in the evolution we allow for mass cancellation, but not for mass creation.

By its very definition, the map $\nu \mapsto \mathcal{W}_2^2(\nu, \mu)$ is lower semi-continuous. Moreover, since any weak limit point of ν_n^+, ν_n^- is a couple σ^+, σ^- satisfying $\sigma^+ - \sigma^- = \nu$, $\mathcal{W}_2^2(\nu, \mu)$ can also be written as

$$\inf_{\substack{\sigma^+ - \sigma^- = \nu \\ \sigma^+(\overline{\Omega}) = \mu^+(\overline{\Omega}) \\ \sigma^-(\overline{\Omega}) = \mu^-(\overline{\Omega})}} \big\{ W_2^2(\sigma^+, \mu^+) + W_2^2(\sigma^-, \mu^-)\big\}. \tag{2.13}$$

Tightness and semi-continuity of the standard Wasserstein distance show that there exists an optimal couple ϑ^+, ϑ^- such that

$$\mathcal{W}_2^2(\nu, \mu) = W_2^2(\vartheta^+, \mu^+) + W_2^2(\vartheta^-, \mu^-), \tag{2.14}$$

where $\vartheta^+ - \vartheta^- = \nu$.

$\mathcal{W}_2$ is not symmetric. But symmetry is not a key point, since we are going to compute the costs corresponding to subsequent time-steps: an evolution problem has a natural time direction. To connect this definition with the previous ones, we can easily show the following

Proposition 2.4. *Let $\mu,\nu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$ and $|\nu|(\overline{\Omega})\leqslant|\mu|(\overline{\Omega})$. Then*

$$\mathcal{W}_2(\nu,\mu)\geqslant\sqrt{\frac{1}{2M}}\mathbb{W}_1(\mu,\nu). \tag{2.15}$$

Proof. Let ϑ^1, ϑ^2 be the optimal couple for $\mathcal{W}_2$, so that the infimum in (2.13) is attained. Let $\gamma^1\in\Gamma_0(\mu^+,\vartheta^+)$, $\gamma^2\in\Gamma_0(\mu^-,\vartheta^-)$. Then $(\gamma^2)^{-1}\in\Gamma_0(\vartheta^-,\mu^-)$ and $\gamma^1+(\gamma^2)^{-1}\in\Gamma(\mu^++\vartheta^-,\vartheta^++\mu^-)$. Hence

$$\begin{aligned}\mathcal{W}_2^2(\mu,\nu)=W_2^2\big(\mu^+,\vartheta^+\big)+W_2^2\big(\mu^-,\vartheta^-\big)&=\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|^2\,d\big(\gamma^1+\gamma^2\big)(x,y)\\&=\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|^2\,d\big(\gamma^1+\big(\gamma^2\big)^{-1}\big)(x,y)\\&\geqslant W_2^2\big(\mu^++\vartheta^-,\vartheta^++\mu^-\big).\end{aligned}$$

Exploiting (2.4) and (2.7) we get the thesis. □

Notation 2.5. We let ϑ_1 denote the common part of ϑ^+ and ϑ^-, so that $\vartheta^+=\nu^++\vartheta_1$ and $\vartheta^-=\nu^-+\vartheta_1$. Moreover, we let $\gamma^+\in\Gamma_0(\vartheta^+,\mu^+)$ and $\gamma^-\in\Gamma_0(\vartheta^-,\mu^-)$ be the two optimal transport plans corresponding to $\mathcal{W}_2$. Thanks to (a simplified version of) Lemma 2.3, we can write these plans as

$$\gamma^+=\gamma_0^++\gamma_1^+\quad\text{and}\quad\gamma^-=\gamma_0^-+\gamma_1^-$$

where

$$\gamma_0^+\in\Gamma_0\big(\nu^+,\mu_0^+\big),\qquad\gamma_1^+\in\Gamma_0\big(\vartheta_1,\mu_1^+\big),\qquad\gamma_0^-\in\Gamma_0\big(\nu^-,\mu_0^-\big),\qquad\gamma_1^-\in\Gamma_0\big(\vartheta_1,\mu_1^-\big), \tag{2.16}$$

and $\mu_0^++\mu_1^+=\mu^+$ and $\mu_0^-+\mu_1^-=\mu^-$.

3. Fine characterization of discrete minimizers

The functional we are going to analyze is (1.5), defined on signed measures. Notice that Φ_0 is weakly lower semi-continuous, as shown in [4]. As a consequence, the full Φ_λ is still lower semi-continuous, since $\mu\mapsto|\mu|(\Omega)$ is. Moreover, Proposition 2.1 of [4] works in the same way also in this case, giving the representation formula

$$\Phi_\lambda(\mu)=\frac{1}{2}\big(\lambda|\mu|(\Omega)+|\Omega|\big)+\sup_{h-1\in H_0^1(\Omega)}\left\{\int\limits_\Omega(h-1)\,d\mu-\frac{1}{2}\int\limits_\Omega|\nabla h|^2+|h|^2\right\}, \tag{3.1}$$

the supremum being attained at $h=h_\mu$. By means of (3.1), we deduce some standard inequalities, as discussed in [4]: for any couple of real measures μ, ν there holds

$$\Phi_\lambda(\mu)-\Phi_\lambda(\nu)\geqslant\frac{\lambda}{2}|\mu|(\Omega)-\frac{\lambda}{2}|\nu|(\Omega)+\int\limits_\Omega(h_\nu-1)\,d(\mu-\nu). \tag{3.2}$$

On the other hand,

$$\Phi_0(\mu)-\Phi_0(\nu)=\nu(\Omega)-\mu(\Omega)+\frac{1}{2}\int\limits_\Omega(h_\mu+h_\nu)\,d(\mu-\nu). \tag{3.3}$$

We are concerned with the discrete time-stepping minimization problem: given $\mu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$, solve

$$\min_{\nu\in\mathcal{M}_{\kappa,M}(\overline{\Omega}),\,|\nu|(\overline{\Omega})\leqslant|\mu|(\overline{\Omega})}\Phi_\lambda(\nu)+\frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu). \tag{3.4}$$

In the sequel, given any signed measure $\mu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$, we denote by $\hat{\mu}$ its restriction to Ω (i.e. $\hat{\mu}=\chi_\Omega\mu$) and by $\tilde{\mu}$ its restriction to $\partial\Omega$, so that we have the orthogonal decomposition $\mu=\hat{\mu}+\tilde{\mu}$.

In order to derive an Euler–Lagrange equation, as in [4], we introduce a perturbed, regularized functional. Let

$$\Phi_\lambda^\delta(\nu) = \Phi_\lambda(\hat{\nu}) + \delta \int_\Omega |\hat{\nu}|^4 \tag{3.5}$$

if $\hat{\nu} \ll \mathcal{L}^2$ and $+\infty$ otherwise. We have the following result:

Lemma 3.1. *The perturbed minimization problem*

$$\min_{\nu \in \mathcal{M}_{\kappa,M}(\overline{\Omega}),\, |\nu|(\overline{\Omega}) \leqslant |\mu|(\overline{\Omega})} \Phi_\lambda^\delta(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu) \tag{3.6}$$

has a solution μ_τ^δ, the family μ_τ^δ has limit points both for the strong H^{-1} topology and the $\mathcal{M}(\overline{\Omega})$ weak topology, $\delta \int_\Omega (\hat{\mu}_\tau^\delta)^4 \to 0$ as $\delta \to 0$, and any limit point μ_τ as $\delta \to 0$ solves (3.4).

Proof. The existence of μ_τ^δ is given by the direct method, as for the existence of μ_τ, using the crucial fact that $\mathcal{W}_2(\cdot,\mu)$ is lower semi-continuous. Let M_δ be the minimum in (3.6) and let M be the minimum of the functional in (3.4). It is clear that $M_\delta \geqslant M$; on the other hand, $\Phi_\lambda^\delta \to \Phi_\lambda$ as $\delta \to 0$ at any admissible point ν such that $\hat{\nu} \in L^4(\Omega)$. Thus

$$\limsup_{\delta \downarrow 0} M_\delta \leqslant \Phi_\lambda(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu)$$

for all ν in $\mathcal{M}_{\kappa,M}(\overline{\Omega})$ such that $|\nu|(\overline{\Omega}) \leqslant |\mu|(\overline{\Omega})$ and $\hat{\nu} \in L^4(\Omega)$. By density we obtain $\limsup_\delta M_\delta \leqslant M$, therefore $M_\delta \to M$ as $\delta \to 0$.

If μ_τ is a weak limit point of μ_τ^δ along some sequence $\delta_i \to 0$, the lower semi-continuity of Φ_λ gives, since $\Phi_\lambda^{\delta_i} \geqslant \Phi_\lambda$ for any i,

$$\Phi_\lambda(\mu_\tau) + \frac{1}{2\tau}\mathcal{W}_2^2(\mu_\tau,\mu) \leqslant \liminf_{i\to\infty} \Phi_\lambda\big(\mu_\tau^{\delta_i}\big) + \frac{1}{2\tau}\mathcal{W}_2^2\big(\mu_\tau^{\delta_i},\mu\big) \leqslant \liminf_{i\to\infty} M_{\delta_i} = M,$$

therefore μ_τ is a solution of (3.4). As a consequence

$$\lim_{i\to\infty} \Phi_\lambda\big(\mu_\tau^{\delta_i}\big) + \delta_i \int_\Omega \big|\hat{\mu}_\tau^{\delta_i}\big|^4 + \frac{1}{2\tau}\mathcal{W}_2^2\big(\mu_\tau^{\delta_i},\mu\big) = \Phi_\lambda(\mu_\tau) + \frac{1}{2\tau}\mathcal{W}_2^2(\mu_\tau,\mu).$$

By the lower semi-continuity of Φ_λ and $\nu \mapsto \mathcal{W}_2^2(\nu,\mu)$ it follows that $\Phi_\lambda(\mu_\tau^{\delta_i}) \to \Phi_\lambda(\mu_\tau)$, $\mathcal{W}_2^2(\mu_\tau^{\delta_i},\mu) \to \mathcal{W}_2^2(\mu_\tau,\mu)$ and $\delta_i \int_\Omega (\hat{\mu}_\tau^{\delta_i})^4 \to 0$. Now, since $\Phi_\lambda(\nu)$ is itself the sum of two lower semi-continuous terms, namely $\Phi_0(\nu)$ and $\lambda|\nu|(\Omega)/2$, we obtain

$$\lim_{i\to\infty} \lambda\big|\mu_\tau^{\delta_i}\big|(\Omega) = \lambda|\mu_\tau|(\Omega) \quad \text{and} \quad \lim_{i\to\infty} \int_\Omega |\nabla h_{\mu_\tau^{\delta_i}}|^2 + (h_{\mu_\tau^{\delta_i}} - 1)^2 = \int_\Omega |\nabla h_{\mu_\tau}|^2 + (h_{\mu_\tau} - 1)^2.$$

In particular $\hat{\mu}_\tau^{\delta_i} \to \hat{\mu}_\tau$ strongly in $H^{-1}(\Omega)$. □

Next we derive an Euler equation for problem (3.6), which will give a characterization of the discrete velocity of the scheme. It is useful to begin with the analysis of the corresponding minimization problem on the whole plane. This way, we can deal with competitors of the form $\mathbf{t}_\#\nu$, which can have some mass outside $\overline{\Omega}$.

Lemma 3.2. *Any minimizer ν of*

$$\min\left\{\Phi_\lambda^\delta(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu):\ \nu \in \mathcal{M}_{\kappa,M}\big(\mathbb{R}^2\big),\ |\nu|\big(\mathbb{R}^2\big) \leqslant |\mu|(\overline{\Omega}),\ \int_{\mathbb{R}^2} |x|^2\, d|\nu| < +\infty\right\} \tag{3.7}$$

satisfies

$$-3\delta\nabla\big((\hat\nu)^4\big)-\nabla h_\nu\hat\nu=\frac{1}{\tau}\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^+\big)+\frac{1}{\tau}\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^-\big)\quad in\ \mathcal{D}'\big(\mathbb{R}^2\big),\tag{3.8}$$

where γ_0^+ and γ_0^- are the optimal transport plans from ν to μ given by splitting, with the notation of (2.16)*: $\gamma_0^+\in\Gamma_0(\nu^+,\mu_0^+)$ and $\gamma_0^-\in\Gamma_0(\nu^-,\mu_0^-)$, where μ_0^+ and μ_0^- are suitable submeasures of μ^+ and μ^- respectively.*

Proof. We perform a variation of the internal part of the optimal measure ν along a smooth vector field $\boldsymbol{\xi}:\mathbb{R}^2\to\mathbb{R}^2$.

Let ϑ^+,ϑ^- be the optimal couple for $\mathcal{W}_2(\nu,\mu)$, such that (2.14) holds. If $\gamma^+\in\Gamma_0(\vartheta^+,\mu^+)$ and $\gamma^-\in\Gamma_0(\vartheta^-,\mu^-)$, we can consider a splitting as (2.16). Accordingly, ϑ_1 denotes the common part of ϑ^+ and ϑ^- so $\vartheta^+=\nu^++\vartheta_1$, $\vartheta^-=\nu^-+\vartheta_1$ and

$$\mathcal{W}_2^2(\nu,\mu)=\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\big(\gamma_0^++\gamma_0^-+\gamma_1^++\gamma_1^-\big)(x,y).\tag{3.9}$$

Let

$$\nu_\varepsilon=\tilde\nu+(\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\hat\nu\tag{3.10}$$

and

$$\Omega_\varepsilon=\big\{x\in\Omega\colon\ x+\varepsilon\boldsymbol{\xi}(x)\in\Omega\big\}.\tag{3.11}$$

For small ε, $\mathbf{I}+\varepsilon\boldsymbol{\xi}$ is injective, and it is clear that $\nu_\varepsilon(\mathbb{R}^2)=\nu(\mathbb{R}^2)$ and $|\nu_\varepsilon|(\mathbb{R}^2)=|\nu|(\mathbb{R}^2)$. Let moreover

$$\begin{aligned}\gamma_\varepsilon^+&=(\mathbf{I}+\varepsilon\boldsymbol{\xi},\mathbf{I})_\#\big(\chi_{\Omega\times\overline\Omega}\gamma_0^+\big)+\chi_{\partial\Omega\times\overline\Omega}\gamma_0^++\gamma_1^+,\\ \gamma_\varepsilon^-&=(\mathbf{I}+\varepsilon\boldsymbol{\xi},\mathbf{I})_\#\big(\chi_{\Omega\times\overline\Omega}\gamma_0^-\big)+\chi_{\partial\Omega\times\overline\Omega}\gamma_0^-+\gamma_1^-.\end{aligned}\tag{3.12}$$

We have

$$\mathcal{W}_2^2(\nu_\varepsilon,\mu)\leqslant W_2^2\big(\tilde\vartheta^++\hat\vartheta_1+(\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\hat\nu^+,\mu^+\big)+W_2^2\big(\tilde\vartheta^-+\hat\vartheta_1+(\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\hat\nu^-,\mu^-\big),$$

but it is clear from (3.12) that

$$\gamma_\varepsilon^+\in\Gamma\big(\tilde\vartheta^++\hat\vartheta_1+(\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\hat\nu^+,\mu^+\big)\quad\text{and}\quad\gamma_\varepsilon^-\in\Gamma\big(\tilde\vartheta^-+\hat\vartheta_1+(\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\hat\nu^-,\mu^-\big),$$

hence

$$\mathcal{W}_2^2(\nu_\varepsilon,\mu)\leqslant\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\big(\gamma_\varepsilon^++\gamma_\varepsilon^-\big).$$

We write the last integral as

$$\begin{aligned}\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\big(\gamma_\varepsilon^++\gamma_\varepsilon^-\big)&=\int\limits_{\Omega\times\overline\Omega}\big|x+\varepsilon\boldsymbol{\xi}(x)-y\big|^2\,d\gamma_0^++\int\limits_{\partial\Omega\times\overline\Omega}|x-y|^2\,d\gamma_0^+\\&\quad+\int\limits_{\Omega\times\overline\Omega}\big|x+\varepsilon\boldsymbol{\xi}(x)-y\big|^2\,d\gamma_0^-+\int\limits_{\partial\Omega\times\overline\Omega}|x-y|^2\,d\gamma_0^-\\&\quad+\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\big(\gamma_1^++\gamma_1^-\big)\\&=\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\gamma_0^++2\varepsilon\int\limits_{\Omega\times\overline\Omega}\boldsymbol{\xi}(x)\cdot(x-y)\,d\gamma_0^+\\&\quad+\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\gamma_0^-+2\varepsilon\int\limits_{\Omega\times\overline\Omega}\boldsymbol{\xi}(x)\cdot(x-y)\,d\gamma_0^-+o(\varepsilon)\\&\quad+\int\limits_{\overline\Omega\times\overline\Omega}|x-y|^2\,d\big(\gamma_1^++\gamma_1^-\big).\end{aligned}$$

Then, recalling also (3.9), we find

$$\mathcal{W}_2^2(\nu_\varepsilon,\mu)-\mathcal{W}_2^2(\nu,\mu)\leqslant 2\varepsilon\int_{\Omega\times\overline{\Omega}}\boldsymbol{\xi}(x)\cdot(x-y)\,d\gamma_0^+ + 2\varepsilon\int_{\Omega\times\overline{\Omega}}\boldsymbol{\xi}(x)\cdot(x-y)\,d\gamma_0^- + o(\varepsilon). \tag{3.13}$$

So we obtain

$$\limsup_{\varepsilon\to 0}\frac{(\mathcal{W}_2^2(\nu_\varepsilon,\mu)-\mathcal{W}_2^2(\nu,\mu))}{2\varepsilon}\leqslant\int_{\overline{\Omega}}\boldsymbol{\xi}(z)\cdot d\big[\pi^1_\#\big(\chi_\Omega(x)(x-y)\big(\gamma_0^++\gamma_0^-\big)\big)\big](z). \tag{3.14}$$

For the derivative of $\Phi_\lambda^\delta(\nu_\varepsilon)$, we take advantage of the $L^4(\Omega)$ convergence of $\hat{\nu}_\varepsilon$ to $\hat{\nu}$ as $\varepsilon\to 0$, which gives the $W^{2,4}(\Omega)$ convergence of h_{ν_ε} to h_ν and, by smoothness of $\partial\Omega$, the $C^1(\overline{\Omega})$ convergence as well. We begin by making use of the equality (3.3) about the functional Φ_0:

$$\begin{aligned}
\Phi_0(\nu_\varepsilon)-\Phi_0(\nu) &= \nu(\Omega)-\nu_\varepsilon(\Omega)+\frac{1}{2}\int_\Omega (h_{\nu_\varepsilon}+h_\nu)\,d(\nu_\varepsilon-\nu)\\
&= \nu(\Omega)-\nu_\varepsilon(\Omega)+\frac{1}{2}\int_{\Omega_\varepsilon}\big(h_{\nu_\varepsilon}\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})+h_\nu\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})\big)\,d\nu-\frac{1}{2}\int_\Omega (h_{\nu_\varepsilon}+h_\nu)\,d\nu\\
&= \nu(\Omega)-\nu_\varepsilon(\Omega)+\frac{1}{2}\int_\Omega\big(h_{\nu_\varepsilon}\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})-h_{\nu_\varepsilon}+h_\nu\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})-h_\nu\big)\,d\hat{\nu}\\
&\quad -\frac{1}{2}\int_{\Omega\setminus\Omega_\varepsilon}\big(h_{\nu_\varepsilon}\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})+h_\nu\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})\big)\,d\nu.
\end{aligned}$$

But the $C^1(\overline{\Omega})$ regularity, the fact that $h_\nu=1$ on $\partial\Omega$ yields

$$\int_{\Omega\setminus\Omega_\varepsilon}\big(h_{\nu_\varepsilon}\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})+h_\nu\circ(\mathbf{I}+\varepsilon\boldsymbol{\xi})\big)\,d\nu=2\nu(\Omega\setminus\Omega_\varepsilon)+O(\varepsilon)\hat{\nu}(\Omega\setminus\Omega_\varepsilon)=2\big(\nu(\Omega)-\nu_\varepsilon(\Omega)\big)+o(\varepsilon),$$

using $\hat{\nu}\in L^4$. As a consequence

$$\Phi_0(\nu_\varepsilon)-\Phi_0(\nu)=\varepsilon\int_\Omega\nabla h_\nu\cdot\boldsymbol{\xi}\,d\hat{\nu}+o(\varepsilon).$$

Since $|\nu_\varepsilon|(\Omega)\leqslant|\nu|(\Omega)$ we also have

$$\Phi_\lambda(\nu_\varepsilon)-\Phi_\lambda(\nu)\leqslant\varepsilon\int_\Omega\nabla h_\nu\cdot\boldsymbol{\xi}\,d\hat{\nu}+o(\varepsilon). \tag{3.15}$$

For the regularizing term, we make use of the change of variables formula for the push forward (see for instance [3, Section 5.5]). Since $\det(J(\mathbf{I}+\varepsilon\boldsymbol{\xi}))=1+\varepsilon\nabla\cdot\boldsymbol{\xi}+o(\varepsilon)$, we get

$$\begin{aligned}
\frac{\delta}{\varepsilon}\bigg[\int_\Omega|\hat{\nu}_\varepsilon|^4-\int_\Omega|\hat{\nu}|^4\bigg] &= \frac{\delta}{\varepsilon}\bigg[\int_{\Omega_\varepsilon}\frac{\hat{\nu}^4}{\det^3(J(\mathbf{I}+\varepsilon\boldsymbol{\xi}))}-\int_\Omega\hat{\nu}^4\bigg]\\
&\leqslant -3\delta\int_\Omega\hat{\nu}^4\nabla\cdot\boldsymbol{\xi}+o(1).
\end{aligned} \tag{3.16}$$

As in the proof of [4, Proposition 5.1], we combine (3.14), (3.15) and (3.16). By the minimality of ν, and considering that we can change the sign of the arbitrary vector $\boldsymbol{\xi}$, we find the equality

$$-3\delta\int_{\Omega}\hat{\nu}^4\nabla\cdot\boldsymbol{\xi}+\int_{\Omega}\nabla h_\nu\cdot\boldsymbol{\xi}\,d\nu+\frac{1}{\tau}\int_{\overline{\Omega}}\boldsymbol{\xi}\cdot d\big[\pi^1_\#\big(\chi_\Omega(x)(x-y)\big(\gamma_0^++\gamma_0^-\big)\big)\big]=0,$$

for any $\boldsymbol{\xi}\in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$. The result follows. □

Corollary 3.3. *Let $\nu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$ be a minimizer of* (3.6). *Then* (3.8) *holds.*

Proof. Let ν_P be the minimizer of (3.7). Since any element of $\mathcal{M}_{\kappa,M}(\overline{\Omega})$ is admissible for this problem, there holds

$$\Phi^\delta_\lambda(\nu_P)+\frac{1}{2\tau}\mathcal{W}_2^2(\nu_P,\mu)\leqslant\Phi^\delta_\lambda(\sigma)+\frac{1}{2\tau}(\sigma,\mu),\quad\forall\sigma\in\mathcal{M}_{\kappa,M}(\overline{\Omega}).\tag{3.17}$$

Let ϑ_P^+ and ϑ_P^- be the optimal couple corresponding to ν_P, such that the infimum in the definition of $\mathcal{W}_2$ is attained and $\mathcal{W}_2^2(\nu_P,\mu)=W_2^2(\vartheta_P^+,\mu^+)+W_2^2(\vartheta_P^-,\mu^-)$. Denote by γ_P^+ and γ_P^- the corresponding optimal transport plans. Consider the map $\Psi(x,y)=(x,y')$, where y' is equal to y if $y\in\Omega$, and is equal to the first point on the segment from x to y hitting $\partial\Omega$ otherwise; let θ^+ and θ^- be the first marginals of $\Psi_\#\gamma_P^+$ and $\Psi_\#\gamma_P^-$ respectively, and let $\nu=\theta^+-\theta^-$. It is clear that $\nu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$. We claim that ν is the minimum for (3.6). For the proof, notice that $\hat{\nu}=\hat{\nu}_P$ (so that $\Phi^\delta_\lambda(\nu_P)=\Phi^\delta_\lambda(\nu)$). Moreover

$$W_2^2\big(\theta^+,\mu^+\big)\leqslant W_2^2\big(\theta_P^+,\mu^+\big)\quad\text{and}\quad W_2^2\big(\theta^-,\mu^-\big)\leqslant W_2^2\big(\theta_P^-,\mu^-\big),$$

since μ is supported in $\overline{\Omega}$ and the projection decreases distances. Since $\theta^+-\theta^-=\nu$, we get

$$\mathcal{W}_2^2(\nu,\mu)\leqslant\mathcal{W}_2^2(\nu_P,\mu).$$

Combining this information with (3.17), the claim is readily seen to follow. In order to conclude, it is sufficient to notice that (3.8) depends only on the interior part of the minimizer. □

4. The entropy argument

One of the key points in this paper consists in showing that the regularity of the initial datum is kept by the discrete minimizers. This way, we will establish that the analogous result for positive measures (in [4]) actually extends to the general real measure framework. For this, we need the regularity of the reference measure μ in (3.6). Hence, in this section we will let $\mu=\hat{\mu}$.

From now on, we will say that $\varphi:[0,+\infty)\to\mathbb{R}$ is an entropy function if it is nondecreasing, C^2 and there holds

$$\begin{aligned}&x\varphi'(x)=\varphi(x)\quad\text{in }[0,1],\\&2x^2\varphi''(x)\geqslant x\varphi'(x)-\varphi(x)\quad\big(\text{McCann [15] displacement convexity}\big).\end{aligned}\tag{4.1}$$

Given an entropy φ, we let it be extended by oddness to $(-\infty,0)$; we will also consider an even convex function ψ on $\mathbb{R}$ such that $\psi'(x)=x\varphi'(x)-\varphi(x)$ for all $x\geqslant 0$.

Lemma 4.1. *Let φ be an entropy and let $\mu\in\mathcal{M}_{\kappa,M}(\overline{\Omega})$ be such that $\mu=\hat{\mu}\in L^4(\Omega)$ and $\int_\Omega\varphi(|\hat{\mu}|)<\infty$. Then, for any minimizer μ^δ_τ of* (3.6)*, we have*

$$\int_\Omega\varphi\big(\big|\hat{\mu}^\delta_\tau\big|\big)\leqslant\int_\Omega\varphi\big(|\hat{\mu}|\big).$$

Proof. We know that $\hat{\nu}:=\hat{\mu}^\delta_\tau$ has $L^4(\Omega)$ regularity. But in view of the Euler equation (3.8) we can find even more regularity. In fact, since $\hat{\nu}\ll\mathcal{L}^2$, we know by Brenier's theorem that $\chi_{\Omega\times\overline{\Omega}}\gamma_0^+$ and $\chi_{\Omega\times\overline{\Omega}}\gamma_0^-$ are plans induced by optimal transport maps r_1 and r_2, which are bounded since Ω is. These maps correspond to the gradients of two convex Lipschitz functions (defined on $\mathbb{R}^2$). Therefore we have $r_1,r_2\in BV_{loc}(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and

$$\begin{aligned}&\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^+\big)=(I-r_1)\nu^+,\\&\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^-\big)=(I-r_2)\nu^-.\end{aligned}$$

This way, (3.8) becomes

$$-3\delta\nabla\big((\hat\nu)^4\big)-\nabla h_\nu\hat\nu=\frac{1}{\tau}(I-r_1)\nu^++\frac{1}{\tau}(I-r_2)\nu^-\quad\text{in }\mathcal{D}'\big(\mathbb{R}^2\big).\tag{4.2}$$

Since $r_1, r_2\in L^\infty(\Omega)$, the right-hand side is in $L^4(\Omega)$. But since $\nabla h_\nu\in C^0(\overline{\Omega})$, we have $\nabla h_\nu\hat\nu\in L^4(\Omega)$, so that by comparison in (4.2) we find $\hat\nu^4\in W^{1,4}(\Omega)$, and by Sobolev embedding $\hat\nu\in C^0(\overline{\Omega})$. Let us now define

$$r=r_1\chi_{\{\hat\nu>0\}}+r_2\chi_{\{\hat\nu<0\}}.$$

Dividing (4.2) by $|\hat\nu|$, we obtain that $\hat\nu$-a.e. in Ω

$$3\delta\,\mathrm{sgn}(\hat\nu)\frac{\nabla((\hat\nu)^4)}{\hat\nu}+\nabla h_\nu\,\mathrm{sgn}(\hat\nu)=\frac{1}{\tau}(r_1-I)\chi_{\{\hat\nu>0\}}+\frac{1}{\tau}(r_2-I)\chi_{\{\hat\nu<0\}},$$

which, by definition of r, corresponds to

$$3\delta\,\mathrm{sgn}(\hat\nu)\frac{\nabla((\hat\nu)^4)}{\hat\nu}+\nabla h_\nu\,\mathrm{sgn}(\hat\nu)=\frac{1}{\tau}(r-I)\quad\hat\nu\text{-a.e.}\tag{4.3}$$

Mind that r_1 transports $\hat\nu^+$ to a submeasure of $\mu^+=\hat\mu^+\in L^4(\Omega)$ (and similarly for r_2), so $r_{1\#}\hat\nu^+\leqslant\hat\mu^+$ and $r_{2\#}\hat\nu^-\leqslant\hat\mu^-$. Since φ is nondecreasing on $(0,+\infty)$, and since the relations (4.1) hold, we have (see [3, Lemma 10.4.4])

$$\int_{\mathbb{R}^2}\varphi\big(\hat\mu^+\big)-\varphi\big(\hat\nu^+\big)\geqslant\int_{\mathbb{R}^2}\varphi\big(r_{1\#}\hat\nu^+\big)-\varphi\big(\hat\nu^+\big)\geqslant-\int_{\mathbb{R}^2}\psi'\big(\hat\nu^+\big)\,\mathrm{tr}\big(\nabla(r_1-I)\big)$$

and

$$\int_{\mathbb{R}^2}\varphi\big(\hat\mu^-\big)-\varphi\big(\hat\nu^-\big)\geqslant\int_{\mathbb{R}^2}\varphi\big(r_{2\#}\hat\nu^-\big)-\varphi\big(\hat\nu^-\big)\geqslant-\int_{\mathbb{R}^2}\psi'\big(\hat\nu^-\big)\,\mathrm{tr}\big(\nabla(r_2-I)\big).$$

We sum the last two inequalities using the fact that $\varphi(0)=0$, and deduce

$$\int_{\mathbb{R}^2}\varphi\big(|\hat\mu|\big)-\varphi\big(|\hat\nu|\big)\geqslant-\int_{\mathbb{R}^2}\psi'\big(\hat\nu^+\big)\,\mathrm{tr}\big(\nabla(r_1-I)\big)-\int_{\mathbb{R}^2}\psi'\big(\hat\nu^-\big)\,\mathrm{tr}\big(\nabla(r_2-I)\big).\tag{4.4}$$

But, r_1 and r_2 are gradients of convex functions, so that we have $\mathrm{tr}(\nabla(r_1-I))\leqslant\mathrm{div}(r_1-I)$ and $\mathrm{tr}(\nabla(r_2-I))\leqslant\mathrm{div}(r_2-I)$ (these divergences are to be understood in the distributional sense, and are measures since r_1, r_2 are BV). Now consider the quantity $\mathrm{div}((r_1-I)\chi_{\{\hat\nu>0\}})$. Formally by the Volpert formula for BV functions (see [2]) we have

$$\mathrm{div}\big((r_1-I)\chi_{\{\hat\nu>0\}}\big)=\mathrm{div}(r_1-I)\chi_{\{\hat\nu>0\}}+\langle r_1-I,n_{\{\hat\nu=0\}}\rangle\,d\mathcal{H}^1\llcorner\big(\{\hat\nu=0\}\big),\tag{4.5}$$

where n denotes the normal. The computation is formal because the level set $\{\hat\nu=0\}$ need not be $\mathcal{H}^1$-rectifiable. But for almost any $\varepsilon>0$ the boundaries of the sublevels $\{|\hat\nu|<\varepsilon\}$ are, by the BV regularity of $\hat\nu$. Since we are dealing with integrals of the form

$$\int_{\partial\{|\hat\nu|<\varepsilon\}}\psi'(\hat\nu)\,d\mathcal{H}^1=0,$$

where ψ' vanishes in a whole interval containing 0, we can take ε small enough and use the formula above. As a consequence,

$$\int_{\mathbb{R}^2}\psi'\big(\hat\nu^+\big)\,\mathrm{div}\big((r_1-I)\chi_{\{\hat\nu>0\}}\big)=\int_{\mathbb{R}^2}\psi'\big(\hat\nu^+\big)\,\mathrm{div}(r_1-I)\chi_{\{\hat\nu>0\}}.$$

The same holds for $\hat{\nu}^-$ on $\{\hat{\nu} < 0\}$. This way, from (4.4) we deduce

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(|\hat{\mu}|) - \varphi(|\hat{\nu}|) &\geqslant -\int_{\mathbb{R}^2} \psi'(\hat{\nu}^+)\,\mathrm{tr}(\nabla(r_1 - I)) - \int_{\mathbb{R}^2} \psi'(\hat{\nu}^-)\,\mathrm{tr}(\nabla(r_2 - I)) \\ &\geqslant -\int_{\{\hat{\nu}>0\}} \psi'(|\hat{\nu}|)\,\mathrm{div}(r_1 - I) - \int_{\{\hat{\nu}<0\}} \psi'(|\hat{\nu}|)\,\mathrm{div}(r_2 - I) \\ &= -\int_{\mathbb{R}^2} \psi'(|\hat{\nu}|)\,\mathrm{div}((r_1 - I)\chi_{\{\hat{\nu}>0\}}) - \int_{\mathbb{R}^2} \psi'(|\hat{\nu}|)\,\mathrm{div}((r_2 - I)\chi_{\{\hat{\nu}<0\}}) \\ &= -\int_{\mathbb{R}^2} \psi'(|\hat{\nu}|)\,\mathrm{div}(r - I) \\ &= -\int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}(r - I).\end{aligned}$$

We make use of (4.3) to estimate the last integral, that is, by means of (4.3) (valid $\hat{\nu}$-a.e. and since $\psi' = 0$ in a neighborhood of 0), from the latter inequality we have

$$\int_{\mathbb{R}^2} \varphi(|\hat{\mu}|) - \varphi(|\hat{\nu}|) \geqslant -\tau \int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}\left[3\delta\,\mathrm{sgn}(\hat{\nu})\frac{\nabla((\hat{\nu})^4)}{\hat{\nu}} + \mathrm{sgn}(\hat{\nu})\nabla h_\nu\right]. \tag{4.6}$$

Since ψ' is odd and $\hat{\nu}$ vanishes on $\mathbb{R}^2 \setminus \Omega$, arguing as above (using Volpert's formula and $\psi' = 0$ in $[-1, 1]$), we find

$$\int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}(\mathrm{sgn}(\hat{\nu})\nabla h_\nu) = \int_{\Omega} \psi'(\hat{\nu})\Delta h_\nu.$$

Moreover, by convexity of ψ we obtain

$$\int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}(\mathrm{sgn}(\hat{\nu})\nabla h_\nu) = \int_{\Omega} \psi'(\hat{\nu})(h_\nu - \hat{\nu}) \leqslant \int_{\Omega} \psi(h_\nu) - \psi(\hat{\nu}). \tag{4.7}$$

Now consider the equation $-\Delta h_\nu + h_\nu = \nu$ in Ω. Multiplying it by $\psi'(h_\nu)$, and integrating by parts yields

$$\int_{\Omega} \psi''(h_\nu)|\nabla h_\nu|^2 + \psi'(h_\nu)(h_\nu - \hat{\nu}) = 0,$$

where we used $\psi'(h_\nu) = \psi'(1) = 0$ on $\partial\Omega$ by continuity of ψ'; so, with the convexity of ψ on $\mathbb{R}$, we obtain

$$\int_{\Omega} \psi''(h_\nu)|\nabla h_\nu|^2 \leqslant \int_{\Omega} \psi(\hat{\nu}) - \psi(h_\nu).$$

Inserting this inequality into (4.7) we get

$$\int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}(\mathrm{sgn}(\hat{\nu})\nabla h_\nu) \leqslant -\int_{\Omega} \psi''(h_\nu)|\nabla h_\nu|^2. \tag{4.8}$$

On the other hand, the same type of argument also yields

$$\begin{aligned}\int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}\left(\mathrm{sgn}(\hat{\nu})\frac{\nabla\hat{\nu}^4}{\hat{\nu}}\right) &= \int_{\Omega} \psi'(|\hat{\nu}|)\,\mathrm{div}\left(\frac{\nabla\hat{\nu}^4}{|\hat{\nu}|}\right) \\ &= -\int_{\Omega} \nabla\psi'(|\hat{\nu}|)\cdot\frac{\nabla\hat{\nu}^4}{|\hat{\nu}|}\end{aligned}$$

$$= -\int_\Omega g'\big(\hat\nu^4\big)\frac{|\nabla\hat\nu^4|^2}{|\hat\nu|},$$

where $g(x) = \psi'(x^{1/4})$ (hence $g' \geqslant 0$), and so

$$\int_\Omega \psi'\big(|\hat\nu|\big)\,\mathrm{div}\bigg(\mathrm{sgn}(\hat\nu)\frac{\nabla\hat\nu^4}{\hat\nu}\bigg) \leqslant 0. \tag{4.9}$$

Inserting (4.8) and (4.9) into (4.6), we find

$$\int_\Omega \varphi\big(|\hat\mu|\big) - \varphi\big(|\hat\nu|\big) \geqslant \tau\int_\Omega \psi''(h_\nu)|\nabla h_\nu|^2. \tag{4.10}$$

Since $\psi'' \geqslant 0$ (by convexity of ψ) we conclude. □

Corollary 4.2. *Let $\mu = \hat\mu \in L^p(\Omega)$, $p \geqslant 4$. Then there exists a minimizer μ_τ of* (3.4) *such that $\hat\mu_\tau \in L^p(\Omega)$. In particular there holds*

$$\int_\Omega \varphi\big(|\hat\mu_\tau|\big) \leqslant \int_\Omega \varphi\big(|\hat\mu|\big) < +\infty$$

for suitable entropies φ, enjoying a p-growth at infinity.

Proof. Let us consider a p-growing entropy:

$$\varphi(x) := \begin{cases} x & \text{for } 0 \leqslant x \leqslant 1,\\ x^p + (p-1)(1 + (p-1)x - \frac{1}{2}p(1+x^2)) & \text{for } x > 1, \end{cases} \tag{4.11}$$

extended by oddness to $(-\infty, 0)$ (one may check it is indeed C^2). By Lemma 4.1, we have

$$\int_\Omega \varphi\big(|\hat\mu_\tau^\delta|\big) \leqslant \int_\Omega \varphi\big(|\hat\mu|\big).$$

On the other hand, by Lemma 3.1 we know we have a limit point μ_τ of μ_τ^δ, as $\delta \to 0$, which minimizes (3.4). But the above inequality gives a weak compactness in $L^p(\Omega)$ for the sequence $(\hat\mu_\tau^\delta)$. The weak lower semi-continuity in $L^p(\Omega)$ of $\nu \mapsto \int_\Omega \varphi(|\nu|)$ (which holds because φ, being an entropy, is convex) allows to conclude. □

The limiting case as $p \uparrow \infty$ of the previous corollary gives:

Corollary 4.3. *Let $\mu = \hat\mu \in L^\infty(\Omega)$ and $K = \max\{1, \|\hat\mu\|_\infty\}$. There exists a minimizer μ_τ of* (3.4) *such that*

$$\|\hat\mu_\tau\|_\infty \leqslant K, \qquad |h_{\mu_\tau}| \leqslant K. \tag{4.12}$$

Proof. Since $K \geqslant 1$ we can construct a sequence (φ_n) of p-growing entropies converging monotonically on $\mathbb{R}^+$ to

$$\tilde\varphi(x) := \begin{cases} x & \text{for } 0 \leqslant x \leqslant K,\\ +\infty & \text{for } x > K. \end{cases} \tag{4.13}$$

Let also ψ_n be such that $\psi_n'(x) = x\varphi_n'(x) - \varphi_n(x)$, with ψ_n'' converging monotonically to $+\infty$ if $|x| > K$. By (4.10) we have

$$\int_\Omega \varphi_n\big(|\hat\mu_\tau^\delta|\big) + \tau\int_\Omega \psi_n''(h_{\mu_\tau^\delta})|\nabla h_{\mu_\tau^\delta}|^2 \leqslant \int_\Omega \varphi_n\big(|\hat\mu|\big).$$

Now we apply Lemma 3.1 to obtain a limit point μ_τ of μ_τ^δ, as $\delta \to 0$, such that μ_τ is a minimizer of (3.4). Then, the weak lower semi-continuity of $|\mu| \mapsto \int_\Omega \varphi_n(|\mu|)$ in L^p, the continuity of ψ_n'' and the convergence of $h_{\mu_\tau^\delta}$ yield

$$\int_\Omega \varphi_n\big(|\hat\mu_\tau|\big) + \tau \int_\Omega \psi_n''(h_{\mu_\tau})|\nabla h_{\mu_\tau}|^2 \leqslant \int_\Omega \varphi_n\big(|\hat\mu|\big) \leqslant \int_\Omega \tilde\varphi\big(|\hat\mu|\big).$$

From the convergence properties of φ_n and ψ_n we get $|\hat\mu_\tau| \leqslant K$ a.e. in Ω and $|h_{\mu_\tau}| \leqslant K$. □

Finally, the L^p regularity of a minimizer enables us to use the first variation argument used for the perturbed problem. We are led to:

Lemma 4.4. *Let $p \geqslant 4$ and $\mu = \hat\mu \in L^p(\Omega)$. There exists a minimizer μ_τ of* (3.4) *satisfying*

$$-\nabla h_{\mu_\tau}\hat\mu_\tau = \frac{1}{\tau}\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^+\big) + \frac{1}{\tau}\pi^1_\#\big(\chi_\Omega(x)(x-y)\gamma_0^-\big) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big), \tag{4.14}$$

where $\gamma_0^+ \in \Gamma_0(\mu_\tau^+, \mu_0^+)$ and $\gamma_0^- \in \Gamma_0(\mu_\tau^-, \mu_0^-)$, with respect to Notation 2.5.

Proof. By the previous corollaries we know that (3.4) possesses a minimizer with $L^4(\Omega)$ interior part. Then, we can perform the same variational argument in Lemma 3.2 (with some simplifications, since the term $\delta \int \hat\mu^4\,dx$ is now absent) and Corollary 3.3 to deduce (4.14). □

5. Back to the continuous model

Let us fix the initial datum $\mu^0 \in \mathcal{M}_{\kappa,M}(\overline{\Omega}) \cap H^{-1}(\Omega)$ and let $\mu_\tau^0 = \mu^0$. We define a sequence of discrete solutions μ_τ^k. At each step, we minimize starting from the interior part of the previous point, and then we simply add its boundary part. This way, more and more mass is accumulated on the boundary at each step, and never returns to the interior of the domain. This is reminiscent of the analysis of [4], in the framework of probability measures. Indeed, in such context it is proven that no mass enters from the boundary, by means of energy comparison. So, the recursive scheme will be the following. Given a time step $\tau > 0$ and $\mu_\tau^k \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$, define ν_τ^{k+1} as a minimizer of the discrete problem

$$\min_{\nu \in \mathcal{M}_{\kappa',M}(\overline{\Omega}),\, |\nu|(\overline{\Omega}) \leqslant |\hat\mu_\tau^k|(\Omega)} \Phi_\lambda(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2\big(\nu, \hat\mu_\tau^k\big), \quad k \in \mathbb{N}, \tag{5.1}$$

where $\kappa' = \hat\mu_\tau^k(\Omega)$. Since we minimize starting from the internal part of μ_τ^k, we can choose ν_τ^{k+1} satisfying the regularity properties obtained by virtue of the entropy argument in the previous section. Then we let

$$\mathcal{M}_{\kappa,M}(\overline{\Omega}) \ni \mu_\tau^{k+1} = \nu_\tau^{k+1} + \tilde\mu_\tau^k, \quad k \in \mathbb{N}. \tag{5.2}$$

Also, we define the piecewise constant interpolation $\overline{\mu}_\tau(t) := \mu_\tau^{\lceil t/\tau \rceil}$ for any $t \geqslant 0$. The following result shows that a minimizing movement does exist, as the pointwise limit of $\overline{\mu}_\tau(t)$.

Proposition 5.1 *(Existence of a limit curve). There exists an infinitesimal sequence (τ_n) such that $\overline{\mu}_{\tau_n}(t)$ converge to some $\mu(t) \in \mathcal{M}_{\kappa,M}(\overline{\Omega})$, weakly in the sense of measures, for any $t \geqslant 0$. Furthermore, $|\hat\mu|$ is uniformly bounded in $L^4(\Omega)$ if the internal part of μ^0 belongs to $L^4(\Omega)$.*

Proof. Since ν_τ^{k+1} is a minimizer of the one-step minimization starting from $\hat\mu_\tau^k$, we have, for any k,

$$\mathcal{W}_2^2\big(\nu_\tau^{k+1}, \hat\mu_\tau^k\big) \leqslant 2\tau\Phi_\lambda\big(\hat\mu_\tau^k\big) - 2\tau\Phi_\lambda\big(\nu_\tau^{k+1}\big).$$

Since $\hat\mu_\tau^{k+1} = \hat\nu_\tau^{k+1}$, and since Φ_λ depends only on the interior part of measures, we find

$$\mathcal{W}_2^2\big(\nu_\tau^{k+1}, \hat\mu_\tau^k\big) \leqslant 2\tau\Phi_\lambda\big(\hat\mu_\tau^k\big) - 2\tau\Phi_\lambda\big(\hat\mu_\tau^{k+1}\big). \tag{5.3}$$

Let us insert (2.15) and take (2.7) into account, so

$$\mathbb{W}_1^2\big(\mu_\tau^{k+1},\mu_\tau^k\big)=\mathbb{W}_1^2\big(\nu_\tau^{k+1},\hat{\mu}_\tau^k\big)\leqslant 4M\tau\big(\Phi_\lambda\big(\hat{\mu}_\tau^k\big)-\Phi_\lambda\big(\hat{\mu}_\tau^{k+1}\big)\big). \tag{5.4}$$

Of course this also implies

$$\Phi_\lambda\big(\mu_\tau^k\big)\leqslant\Phi_\lambda\big(\mu^0\big),\quad \forall k>0. \tag{5.5}$$

Let $t\in(k_1\tau,(k_1+1)\tau]$ and $s\in(k_2\tau,(k_2+1)\tau]$, for some $k_1,k_2>0$, with $k_2>k_1$. Using the interpolation of minimizers $\overline{\mu}_\tau(t)$, summing the relations (5.4) and making use of the triangle inequality (mind that $\mathbb{W}_1$ is a distance), along with (5.5) and the positiveness of Φ_λ, we have

$$\begin{aligned}\mathbb{W}_1^2\big(\overline{\mu}_\tau(t),\overline{\mu}_\tau(s)\big)=\mathbb{W}_1^2\big(\mu_\tau^{k_1+1},\mu_\tau^{k_2+1}\big)&\leqslant(k_2-k_1)\sum_{k=k_1+1}^{k_2}\mathbb{W}_1^2\big(\mu_\tau^k,\mu_\tau^{k+1}\big)\\&\leqslant 2\tau(k_2-k_1)\sum_{k=k_1+1}^{k_2}\big(\Phi_\lambda\big(\mu_\tau^{k_1+1}\big)-\Phi_\lambda\big(\mu_\tau^{k_2+1}\big)\big)\\&\leqslant 2\tau(k_2-k_1)\Phi_\lambda\big(\mu^0\big).\end{aligned}$$

Hence

$$\mathbb{W}_1\big(\overline{\mu}_\tau(t),\overline{\mu}_\tau(s)\big)\leqslant\sqrt{2\Phi_\lambda\big(\mu^0\big)}\sqrt{|t-s|+\tau},\quad \forall s,t\in[0,+\infty).$$

The discrete $C^{0,1/2}$ estimate allows to find (for this see [4], or also [3, Chapter 11] for the precise argument) a subsequence $\tau_n\to 0$ such that in the sense of measures

$$\lim_{n\to\infty}\overline{\mu}_{\tau_n}(t)=\mu(t),\quad \forall t\geqslant 0. \tag{5.6}$$

This concludes the proof. □

Notation 5.2. The transportation is described by the cost $\mathcal{W}_2(\nu_\tau^{k+1},\hat{\mu}_\tau^k)$, corresponding to an optimal couple of measures $(\vartheta^+)_\tau^{k+1}$, $(\vartheta^-)_\tau^{k+1}$ as in (2.14). That is,

$$\mathcal{W}_2^2\big(\nu_\tau^{k+1},\hat{\mu}_\tau^k\big)=W_2^2\big((\vartheta^+)_\tau^{k+1},(\hat{\mu}^+)_\tau^k\big)+W_2^2\big((\vartheta^-)_\tau^{k+1},(\hat{\mu}^-)_\tau^k\big). \tag{5.7}$$

With reference to Notation 2.5, we let $(\vartheta_1)_\tau^{k+1}$ be the common part of $(\vartheta^+)_\tau^{k+1}$ and $(\vartheta^-)_\tau^{k+1}$. The two terms in the right-hand side of (5.7) correspond to optimal plans $(\gamma^+)_\tau^{k+1}$ and $(\gamma^-)_\tau^{k+1}$ and can be split as

$$\big(\gamma^+\big)_\tau^{k+1}=\big(\gamma_0^+\big)_\tau^{k+1}+\big(\gamma_1^+\big)_\tau^{k+1}\quad\text{and}\quad\big(\gamma^-\big)_\tau^{k+1}=\big(\gamma_0^-\big)_\tau^{k+1}+\big(\gamma_1^-\big)_\tau^{k+1}. \tag{5.8}$$

Here

$$\big(\gamma_0^+\big)_\tau^{k+1}\in\Gamma_0\big(\big(\nu^+\big)_\tau^{k+1},\big(\hat{\mu}_0^+\big)_\tau^k\big)\quad\text{and}\quad\big(\gamma_1^+\big)_\tau^{k+1}\in\Gamma_0\big((\vartheta_1)_\tau^{k+1},\big(\hat{\mu}_1^+\big)_\tau^k\big),$$

where $(\hat{\mu}_0^+)_\tau^k$ and $(\hat{\mu}_1^+)_\tau^k$ are suitable positive submeasures of $(\hat{\mu}^+)_\tau^k$, which is their sum. Similarly for the negative parts.

The discrete velocity of the scheme (3.4) (neglecting the common parts) could be defined by $(x-y)/\tau$ with $(x,y)\in(\mathrm{supp}(\gamma_0^+)_\tau^{k+1})\cup(\mathrm{supp}(\gamma_0^-)_\tau^{k+1})$. The characterization of the discrete velocity is crucial to interpret our recursive scheme as the discrete version of a differential equation. But we do not have a standard continuity equation for the signed case, and therefore we cannot proceed as in [4] (see Section 6 therein). Instead, we will see how to obtain a partial result by constructing the limiting differential equation "by hand".

In [16], the authors are able to produce solutions for a similar model (see Section 6 below) by means of an explicit, rather than implicit, discrete scheme. They take advantage of strong regularity hypotheses on the initial datum, which are preserved during the evolution, guaranteeing good compactness properties. Here we would like to address the case of mere L^p initial data.

We start by introducing a basic estimate. With the notation above, we have shown that

$$\mathcal{W}_2^2\big(\nu_\tau^{k+1},\hat{\mu}_\tau^k\big)=\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|^2\,d\big(\big(\gamma_0^+\big)_\tau^{k+1}+\big(\gamma_0^-\big)_\tau^{k+1}+\big(\gamma_1^+\big)_\tau^{k+1}+\big(\gamma_1^-\big)_\tau^{k+1}\big)(x,y). \tag{5.9}$$

From (5.3), summing the telescopic series, we immediately see that

$$\sum_{k=0}^{\infty}\mathcal{W}_2^2\big(\nu_\tau^{k+1},\hat{\mu}_\tau^k\big)\leqslant 2\tau\Phi_\lambda\big(\mu^0\big). \tag{5.10}$$

Hence each of the four terms in the right-hand side of (5.9) satisfies the same bound.

The next proposition shows that there is no contribution from the transport plans γ_1^+ and γ_1^-, which can be thought as accounting for self-annihilation of mass, in the subsequent limit process.

Proposition 5.3. *Let $\phi\in C_b^1(\overline{\Omega})$. Then*

$$\lim_{\tau\to0}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big|\phi(y)-\phi(x)\big|\,d\big(\gamma_1^+\big)_\tau^{k+1}(x,y)=0,$$

$$\lim_{\tau\to0}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big|\phi(y)-\phi(x)\big|\,d\big(\gamma_1^-\big)_\tau^{k+1}(x,y)=0.$$

Proof. By definition of $\mathcal{W}_2$, and taking into account the constraint in the discrete minimization problem $|\nu_\tau^{k+1}|(\overline{\Omega})\leqslant|\mu_\tau^k|(\Omega)$, we see that for any k there holds $(\vartheta^+)_\tau^{k+1}(\overline{\Omega})+(\vartheta^-)_\tau^{k+1}(\overline{\Omega})=|\mu_\tau^k|(\Omega)$. Then, recalling that $(\vartheta_1)_\tau^k$ is the common part of $(\vartheta^+)_\tau^k$ and $(\vartheta^-)_\tau^k$, it is clear that $\sum_{k=0}^{\infty}(\vartheta_1)_\tau^k\leqslant|\mu^0|(\overline{\Omega})\leqslant M$, hence

$$\sum_{k=0}^{\infty}\big(\gamma_1^+\big)_\tau^{k+1}(\overline{\Omega}\times\overline{\Omega})+\big(\gamma_1^-\big)_\tau^{k+1}(\overline{\Omega}\times\overline{\Omega})\leqslant M. \tag{5.11}$$

Now we compute

$$\begin{aligned}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big|\phi(y)-\phi(x)\big|\,d\big(\gamma_1^+\big)_\tau^{k+1}(x,y)&\leqslant\|\phi\|_{\mathrm{Lip}}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,d\big(\gamma_1^+\big)_\tau^{k+1}(x,y)\\&=\|\phi\|_{\mathrm{Lip}}\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,d\sum_{k=0}^{\infty}\big(\gamma_1^+\big)_\tau^{k+1}(x,y).\end{aligned}$$

With Hölder's inequality we see that the last term is controlled by

$$\|\phi\|_{\mathrm{Lip}}\Bigg(\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|^2\,d\sum_{k=0}^{\infty}\big(\gamma_1^+\big)_\tau^{k+1}(x,y)\Bigg)^{1/2}\Bigg(\int\limits_{\overline{\Omega}\times\overline{\Omega}}d\sum_{k=0}^{\infty}\big(\gamma_1^+\big)_\tau^{k+1}(x,y)\Bigg)^{1/2}.$$

But making use of (5.10) we see that the first factor is bounded by $\sqrt{2\tau\Phi_\lambda(\mu^0)}$, while by (5.11) the second is less than or equal to $\sqrt{M}$. Hence

$$\limsup_{\tau\to0}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big|\phi(y)-\phi(x)\big|\,d\big(\gamma_1^+\big)_\tau^{k+1}(x,y)\leqslant\lim_{\tau\to0}M\|\phi\|_{\mathrm{Lip}}\sqrt{2\tau\Phi_\lambda\big(\mu^0\big)}=0.$$

Similarly one shows that

$$\limsup_{\tau\to 0}\sum_{k=0}^{\infty}\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big|\phi(y)-\phi(x)\big|\,d\big(\gamma_1^-\big)_\tau^{k+1}(x,y)=0,$$

obtaining the thesis. □

We will also need the following similar result.

Proposition 5.4. *Let $\phi\in C_b^1(\overline{\Omega})$. There holds*

$$\lim_{\tau\to 0}\sum_{k=0}^{\infty}\int\limits_{\partial\Omega\times\overline{\Omega}}\big|\phi(x)-\phi(y)\big|\,d\big(\gamma_0^+\big)_\tau^{k+1}(x,y)=0,$$

and the same for the analogous sum involving $(\gamma_0^-)_\tau^k$.

Proof. Reasoning as in the previous proposition, one estimates the sum above by

$$\|\phi\|_{\mathrm{Lip}}\Bigg(\sum_{k=0}^{\infty}\int\limits_{\partial\Omega\times\overline{\Omega}}|x-y|^2\,d\big(\gamma_0^+\big)_\tau^{k+1}(x,y)\Bigg)^{1/2}\Bigg(\sum_{k=0}^{\infty}\big(\gamma_0^+\big)_\tau^{k+1}(\partial\Omega\times\overline{\Omega})\Bigg)^{1/2}. \tag{5.12}$$

Since no mass on the boundary returns to the interior of the domain during the discrete steps, we have

$$\sum_{k=1}^{\infty}\big(\gamma_0^+\big)_\tau^{k}(\partial\Omega\times\overline{\Omega})=\sum_{k=1}^{\infty}\big(\nu_\tau^k\big)^+(\partial\Omega)\leqslant\sum_{k=1}^{\infty}\big|\nu_\tau^k\big|(\partial\Omega)\leqslant\big|\mu^0\big|(\overline{\Omega})\leqslant M.$$

This shows that the second factor in (5.12) is uniformly bounded. The first one is controlled again by $\|\phi\|_{\mathrm{Lip}}\sqrt{2\tau\Phi_\lambda(\mu^0)}$, as a consequence of (5.9) and (5.10). The same argument gives the thesis if $(\gamma_0^+)_\tau^k$ are replaced by $(\gamma_0^-)_\tau^k$. □

Lemma 5.5 *(Convergence of the total variations). Let (τ_n) be given by Proposition* 5.1. *Then there exist positive measures $\varrho^+(t)$, $\varrho^-(t)$ such that, possibly on a subsequence, there hold*

$$\overline{\mu}_{\tau_n}^+(t)\rightharpoonup\varrho^+(t),\qquad\overline{\mu}_{\tau_n}^-(t)\rightharpoonup\varrho^-(t),\qquad\big|\overline{\mu}_{\tau_n}(t)\big|\rightharpoonup\varrho^+(t)+\varrho^-(t). \tag{5.13}$$

Proof. We prove the convergence of the positive parts. By difference the result follows for the negative parts. Let φ be a bounded Lipschitz function over $\overline{\Omega}$. Possibly adding a constant, we can assume that φ is nonnegative. Let

$$a_\tau^k:=\int\limits_{\overline{\Omega}}\varphi\,d\big(\nu^+\big)_\tau^{k+1}-\int\limits_{\overline{\Omega}}\varphi\,d\big(\hat{\mu}_0^+\big)_\tau^k,\qquad b_\tau^k:=\int\limits_{\overline{\Omega}}\varphi\,d\big(\hat{\mu}_1^+\big)_\tau^k.$$

We have, by (5.9),

$$\begin{aligned}
a_\tau^k&=\int\limits_{\overline{\Omega}\times\overline{\Omega}}\big(\varphi(x)-\varphi(y)\big)\,d\big(\gamma_0^+\big)_\tau^{k+1}(x,y)\\
&\leqslant\|\varphi\|_{\mathrm{Lip}}\Bigg(\int\limits_{\overline{\Omega}\times\overline{\Omega}}d\big(\gamma_0^+\big)_\tau^{k+1}\Bigg)^{1/2}\Bigg(\int\limits_{\overline{\Omega}\times\overline{\Omega}}|x-y|^2\,d\big(\gamma_0^+\big)_\tau^{k+1}(x,y)\Bigg)^{1/2}\\
&\leqslant\|\varphi\|_{\mathrm{Lip}}\sqrt{M}\,\mathcal{W}_2\big(\nu_\tau^{k+1},\hat{\mu}_\tau^k\big),
\end{aligned}$$

which gives, making use of (5.10),

$$\sum_{k=0}^{\infty}\frac{(a_\tau^k)^2}{\tau}\leqslant\frac{M}{\tau}\|\varphi\|_{\mathrm{Lip}}^2\sum_{k=0}^{\infty}\mathcal{W}_2^2\big(\mu_\tau^{k+1},\mu_\tau^k\big)\leqslant 2M\|\varphi\|_{\mathrm{Lip}}^2\Phi_\lambda\big(\mu^0\big).$$

As

$$\sum_{k=0}^{\infty}\frac{(a_\tau^k)^2}{\tau}=\int_0^{+\infty}\left|\frac{(a_\tau^{\lceil t/\tau\rceil})}{\tau}\right|^2dt,$$

we infer the $L^2(0,+\infty)$ weak compactness of the sequence $(a_\tau^{\lceil t/\tau\rceil}/\tau)$. We can assume, possibly extracting from (τ_n) a subsequence, that $(a_{\tau_n}^{\lceil t/\tau_n\rceil}/\tau_n)$ weakly converge to some $f\in L^2(0,+\infty)$. In particular we have

$$A_{\tau_n}:=\sum_{k=0}^{\infty}a_\tau^k\delta_{\{k\tau_n\}}\rightharpoonup f\mathcal{L}^1.$$

Indeed, letting $\zeta\in C_c(0,+\infty)$, we have

$$\int_0^{\infty}\zeta(t)\,d\left(\sum_{k=0}^{\infty}a_{\tau_n}^k\delta_{\{\tau_n k\}}(t)\right)=\sum_{k=0}^{\infty}a_{\tau_n}^k\zeta(\tau_n k)=\int_0^{\infty}\frac{a_{\tau_n}^{\lceil t/\tau_n\rceil}}{\tau_n}\zeta\big(\tau_n\lceil t/\tau_n\rceil\big)\,dt\to\int_0^{\infty}f(t)\zeta(t)\,dt.$$

Hence, for any t there holds

$$\lim_{n\to\infty}\sum_{k=0}^{\infty}a_{\tau_n}^k\delta_{\{k\tau_n\}}\big([0,t]\big)\to\int_0^t f(y)\,dy.$$

Next, notice that, by (5.2),

$$\frac{d}{dt}\int_{\overline{\Omega}}\varphi\,d\overline{\mu}_{\tau_n}^+(t)\leqslant\sum_{k=0}^{\infty}\big(a_{\tau_n}^k-b_{\tau_n}^k\big)\delta_{\{k\tau_n\}}.$$

Since $b_{\tau_n}^k\geqslant 0$, we see that

$$\frac{d}{dt}\left(\int_{\overline{\Omega}}\varphi\,d\overline{\mu}_{\tau_n}^+(t)-A_{\tau_n}\big([0,t]\big)\right)\leqslant 0.$$

We have a family of monotone functions. We can apply Helly's pointwise compactness theorem (see for instance [3, Lemma 3.3.3]) to obtain that, possibly extracting one more subsequence, the pointwise, nonincreasing limit of this family of functions exists. The convergence of A_{τ_n} now yields

$$\int_{\overline{\Omega}}\varphi\,d\overline{\mu}_\tau^+(t)\to L_\varphi(t),\quad\forall t\geqslant 0.$$

The convergence holds for any positive Lipschitz φ, hence for any Lipschitz φ. By a diagonal argument we can find an infinitesimal sequence, that we still denote by τ_n, such that

$$\int_{\overline{\Omega}}\varphi\,d\overline{\mu}_{\tau_n}^+(t)\to L_\varphi(t),\quad\forall t\geqslant 0\text{ and }\forall\varphi\in\mathcal{D},$$

where $\mathcal{D}$ is a countable dense subset of $C_b^0(\overline{\Omega})$ made of Lipschitz functions. Then, for any t, $\varphi\mapsto L_\varphi(t)$ can be extended uniquely to a weakly continuous linear functional on $C_b^0(\overline{\Omega})$. By the Riesz representation theorem we conclude that $L_\varphi(t)=\int_{\overline{\Omega}}\varphi\,d\varrho^+(t)$, for some $\varrho^+\in\mathcal{M}^+(\overline{\Omega})$, and for any t there holds $\overline{\mu}_\tau^+(t)\rightharpoonup\varrho^+(t)$.

Letting $\varrho^-(t)$ be the pointwise weak limit of $\overline{\mu}_\tau^-(t)$ obtained by the same reasoning, we also infer the convergence of the total variations: $|\overline{\mu}_\tau(t)| \rightharpoonup \varrho(t)$, where $\varrho(t) = \varrho^+(t) + \varrho^-(t)$. □

Eventually, we are able to produce a limiting equation.

Theorem 5.6 *(Equation in the limit). Let $\mu^0 \in L^4(\Omega)$. The minimizing movement $\mu(t)$ given by Proposition* 5.1 *satisfies*

$$\frac{d}{dt}\mu(t) - \operatorname{div}\big(\chi_\Omega \nabla h_{\mu(t)} \varrho(t)\big) = 0 \quad \text{in } \mathcal{D}'\big((0,+\infty) \times \mathbb{R}^2\big), \tag{5.14}$$

where $\varrho(t)$ is a suitable positive measure satisfying $\hat{\varrho}(t) \geqslant |\hat{\mu}(t)|$.

Proof. Let (τ_n) be the sequence given by Proposition 5.1 and Lemma 5.5. Just for simplicity of notation, in the sequel we shall write τ instead of τ_n.

Let $\phi \in C^2(\overline{\Omega})$. Let us compute the derivative of the time interpolated measure $\overline{\mu}_\tau(t)$. We have, in the sense of distributions,

$$\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\overline{\mu}_\tau(t) = \sum_{k=0}^{\infty}\bigg(\int_{\overline{\Omega}} \phi \, d\mu_\tau^{k+1} - \int_{\overline{\Omega}} \phi \, d\mu_\tau^k\bigg)\delta_{\{k\tau\}}.$$

But

$$\int_{\overline{\Omega}} \phi \, d\mu_\tau^{k+1} - \int_{\overline{\Omega}} \phi \, d\mu_\tau^k = \int_{\overline{\Omega}} \phi \, d\nu_\tau^{k+1} - \int_{\overline{\Omega}} \phi \, d\hat{\mu}_\tau^k = \int_{\overline{\Omega}\times\overline{\Omega}} \big(\phi(y) - \phi(x)\big)\, d\gamma_\tau^{k+1},$$

where, using the notation introduced in (5.8),

$$\gamma_\tau^{k+1} := \big(\gamma_0^+\big)_\tau^{k+1} - \big(\gamma_0^-\big)_\tau^{k+1} + \big(\gamma_1^+\big)_\tau^{k+1} - \big(\gamma_1^-\big)_\tau^{k+1}.$$

So we may write

$$\begin{aligned}\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\overline{\mu}_\tau(t) &= \sum_{k=0}^{\infty}\delta_{\{k\tau\}} \int_{\overline{\Omega}\times\overline{\Omega}} \big(\phi(x) - \phi(y)\big)\, d\gamma_\tau^{k+1}(x,y) \\ &= \sum_{k=0}^{\infty}\delta_{\{k\tau\}}\bigg(\int_{\overline{\Omega}\times\overline{\Omega}} \big\langle \nabla\phi(x), x-y\big\rangle\, d\gamma_\tau^{k+1}(x,y) + \mathcal{R}_\tau^k\bigg).\end{aligned} \tag{5.15}$$

Let us estimate the remainder $\mathcal{R}_\tau^k$ by writing it in integral form. We have

$$\begin{aligned}\mathcal{R}_\tau^k &= \frac{1}{2}\int_0^1 \int_{\overline{\Omega}\times\overline{\Omega}} \big|\big\langle \nabla^2\phi\big((1-\theta)x + \theta y\big)(y-x), y-x\big\rangle\big|\, d\gamma_\tau^{k+1}(x,y)\, d\theta \\ &\leqslant \frac{1}{2}\big\|\nabla^2\phi\big\|_\infty \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|^2\, d\gamma_\tau^{k+1}(x,y) \\ &\leqslant \frac{1}{2}\big\|\nabla^2\phi\big\|_\infty \mathcal{W}_2^2\big(\nu_\tau^{k+1}, \hat{\mu}_\tau^k\big).\end{aligned} \tag{5.16}$$

By (5.10) we see that

$$\lim_{\tau\downarrow 0}\sum_{k=0}^{\infty}\mathcal{R}_\tau^k = 0.$$

Together with Propositions 5.3 and 5.4, this shows that (5.15) can be written, for $\tau \to 0$, as

$$\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\overline{\mu}_\tau(t) = \sum_{k=0}^{\infty} \delta_{\{k\tau\}} \biggl(\int_{\overline{\Omega}\times\overline{\Omega}} \langle \nabla\phi(x), x-y \rangle \chi_\Omega(x) \, d\bigl(\bigl(\gamma_0^+\bigr)_\tau^{k+1} - \bigl(\gamma_0^-\bigr)_\tau^{k+1}\bigr)(x,y) \biggr) + o(1). \tag{5.17}$$

The Euler equation for discrete minimizers ν_τ^k of (3.4), since $\hat{\nu}_\tau^k = \hat{\mu}_\tau^k$, reads (see Lemma 4.4)

$$-\nabla h_{\mu_\tau^k}\bigl(\bigl(\hat{\mu}_\tau^k\bigr)^+ - \bigl(\hat{\mu}_\tau^k\bigr)^-\bigr) = \frac{1}{\tau}\bigl(\pi_\#^1\bigl(\chi_\Omega(x)(x-y)\bigl(\gamma_0^+\bigr)_\tau^k\bigr) + \pi_\#^1\bigl(\chi_\Omega(x)(x-y)\bigl(\gamma_0^-\bigr)_\tau^k\bigr)\bigr),$$

but notice that the first term in the right-hand side can be different from zero only on $\operatorname{supp}(\hat{\mu}_\tau^k)^+$. Similarly for the second term. Hence we can split the equation in

$$-\nabla h_{\mu_\tau^k}\bigl(\hat{\mu}_\tau^k\bigr)^+ = \frac{1}{\tau}\pi_\#^1\bigl(\chi_\Omega(x)(x-y)\bigl(\gamma_0^+\bigr)_\tau^k\bigr),$$
$$\nabla h_{\mu_\tau^k}\bigl(\hat{\mu}_\tau^k\bigr)^- = \frac{1}{\tau}\pi_\#^1\bigl(\chi_\Omega(x)(x-y)\bigl(\gamma_0^-\bigr)_\tau^k\bigr).$$

Substituting in (5.17), we find

$$\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\overline{\mu}_\tau(t) = -\sum_{k=0}^{\infty} \tau\delta_{\{k\tau\}} \int_{\Omega} \bigl\langle \nabla\phi(x), \nabla h_{\mu_\tau^k}(x) \bigr\rangle \, d\bigl|\hat{\mu}_\tau^k\bigr|(x) + o(1).$$

Passing to the limit as τ goes to zero (more precisely, along the sequence τ_n) we get

$$\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\mu(t) + \int_{\Omega} \langle \nabla\phi, \nabla h_{\mu(t)} \rangle \, d\varrho(t) = 0,$$

where $\varrho(t)$ is given by Lemma 5.5, hence satisfying $\hat{\varrho}(t) \geqslant |\hat{\mu}(t)|$ and $\varrho(t)(\overline{\Omega}) \leqslant |\mu^0|(\overline{\Omega})$. □

Remark 5.7 *(The positive case).* In the case of positive (or negative) measures, it is immediately seen that both (2.3) and (2.12) reduce to the standard 2-Wasserstein distance. In particular, if μ is positive and ν is not, by (2.13) we get $\mathcal{W}_2(\nu,\mu) = \inf\emptyset = +\infty$. Then, it is clear that, if the initial datum μ^0 is positive, the discrete minimizers are positive as well. As a consequence, passing to the limit in the discrete scheme, as shown in [4] (the schemes coincide in the positive case), produces a positive solution of

$$\frac{d}{dt}\mu(t) - \operatorname{div}\bigl(\chi_\Omega \nabla h_{\mu(t)}\mu(t)\bigr) = 0 \quad \text{in } \mathcal{D}'\bigl((0,+\infty)\times\mathbb{R}^2\bigr).$$

If $\mu^0 \in L^\infty(\Omega)$ is compactly supported such solution is locally unique in time, and also globally unique if suitable boundary conditions hold (see [13]).

Remark 5.8 *(System formulation).* We could also derive separately the equations, in the limit as $\tau \to 0$, for the positive and negative parts of $\overline{\mu}_\tau(t)$. Let $\phi \in C^2(\overline{\Omega})$. Regarding the positive part, we reason as in Theorem 5.6, taking advantage in particular of Proposition 5.4, so that as $\tau \to 0$

$$\begin{aligned}
\frac{d}{dt}\int_{\overline{\Omega}} \phi \, d\overline{\mu}_\tau^+(t) &= \sum_{k=0}^{\infty} \delta_{\{k\tau\}} \biggl(\int_{\overline{\Omega}} \phi \, d\bigl(\mu^+\bigr)_\tau^{k+1} - \int_{\overline{\Omega}} \phi \, d\bigl(\mu^+\bigr)_\tau^k \biggr) \\
&= \sum_{k=0}^{\infty} \delta_{\{k\tau\}} \biggl(\int_{\overline{\Omega}} \phi \, d\bigl(\nu^+\bigr)_\tau^{k+1} - \int_{\overline{\Omega}} \phi \, d\bigl(\hat{\mu}_0^+\bigr)_\tau^k - \int_{\overline{\Omega}} \phi \, d\bigl(\hat{\mu}_1^+\bigr)_\tau^k \biggr) + o(1) \\
&= \sum_{k=0}^{\infty} \delta_{\{k\tau\}} \biggl(\int_{\Omega\times\overline{\Omega}} \bigl\langle \nabla\phi(x), x-y \bigr\rangle \, d(\gamma_0)_\tau^{k+1}(x,y) - \int_{\overline{\Omega}} \phi \, d\bigl(\hat{\mu}_1^+\bigr)_\tau^k \biggr) + o(1),
\end{aligned} \tag{5.18}$$

where $(\hat{\mu}_0^+)_\tau^k$ is the part of $(\hat{\mu}^+)_\tau^k$ that comes from $(\nu^+)_\tau^{k+1}$ and $(\hat{\mu}_1^+)_\tau^k$ is the part of $(\hat{\mu}^+)_\tau^k$ which gets transported by $(\gamma_1^+)_\tau^{k+1}$, as in Notation 5.2. A term that was not present in (5.17) appears. Let us analyze it. First, notice that, since $(\hat{\mu}_1^+)_\tau^k$ accounts for mass cancellation at every step, we have as usual

$$\sum_{k=0}^{\infty} (\hat{\mu}_1^+)_\tau^k(\overline{\Omega}) \leqslant |\mu^0|(\overline{\Omega}) \leqslant M.$$

This entails

$$\int_0^\infty \int_\Omega \frac{(\hat{\mu}_1^+)_\tau^{\lceil t/\tau \rceil}}{\tau}\, dx\, dt \leqslant M,$$

so that the sequence $(\hat{\mu}_1^+)_\tau^{\lceil t/\tau \rceil}/\tau$ is bounded in $L^1(\Omega \times (0, +\infty))$ and hence possesses a subsequence converging weakly in measure. We denote by $\sigma(x,t)$ a suitable space–time weak limit measure, which is of course positive. Notice that, if $\zeta(t) \in C_c(0, +\infty)$,

$$\int_0^\infty \int_\Omega \zeta(t)\phi(x)\, d\left(\sum_{k=0}^{\infty} (\hat{\mu}_1^+)_\tau^k \delta_{\{k\tau\}}\right) = \int_0^\infty \int_\Omega \zeta\left(\tau \lceil t/\tau \rceil\right)\phi(x)\, \frac{1}{\tau} (\hat{\mu}_1^+)_\tau^k(x,t),$$

so that this quantity converges to $\int_0^\infty \int_\Omega \zeta(t)\phi(x)\, d\sigma(x,t)$ for $\tau \to 0$ on a suitable sequence. This fact, together with the arguments of Theorem 5.6, shows that passing to the limit in (5.18) we get, in the sense of distributions,

$$\frac{d}{dt} \int_{\overline{\Omega}} \phi(x)\, d\varrho^+(t) = -\int_\Omega \langle \nabla\phi, \nabla h_{\mu(t)} \rangle\, d\varrho^+(t) - \int_\Omega \phi(x)\, d\sigma(x,t),$$

where $\varrho^+(t)$ is a suitable limit of $(\overline{\mu}_\tau^+)_\tau^{\lceil t/\tau \rceil}$ as $\tau \to 0$ (given by Lemma 5.5). The similar argument applies for negative parts, so that one ends up with the system (1.8). The system is coupled because $\mu(t) = \varrho^+(t) - \varrho^-(t)$. This formulation is probably more meaningful, since we can see the structure of two continuity equations with equal and opposite vector fields, with a negative term in the right-hand side (the same for the two equations) which has the meaning of a mass sink. The actual solution is then recovered as the difference of ϱ^+ and ϱ^-.

Remark 5.9 *(L^∞ bounds on the boundary part).* We see from (5.1) and (5.2) that some mass accumulated on the boundary at each step and then does not play a role in the subsequent discrete minimizations. Of course there can also be cancellations on the boundary. Hence, we can say that $t \mapsto |\hat{\mu}(t)|(\Omega)$ and $t \mapsto |\mu(t)|(\overline{\Omega})$ are nonincreasing, but not that $t \mapsto |\tilde{\mu}(t)|(\partial\Omega)$ is nondecreasing. All these quantities, as well as $\rho_t^+(\overline{\Omega})$ and $\rho_t^-(\overline{\Omega})$, are uniformly bounded by the initial mass. Looking at the equation in the limit, one can also obtain a mass dissipation estimate on the boundary. Let $\varrho^+(0) = \mu^+(0)$ and $\varrho^-(0) = \mu^-(0)$ belong to $L^\infty(\Omega)$, so that (1.8) admits a solution $(\varrho^+(t), \varrho^-(t))$ with $(\hat{\varrho}^+(t), \hat{\varrho}^-(t)) \in L^\infty((0,T), L^\infty(\Omega)^2)$. Let $\Omega_\varepsilon \subset \Omega$ be the set of all points with distance from $\partial\Omega$ greater than ε, and let C be such that $\|\nabla h_{\mu(t)} \hat{\varrho}(t)\|_\infty \leqslant C$ in $[0,T]$, where $\varrho(t) = \varrho^+(t) + \varrho^-(t)$. Using the weak formulation of (5.14) with a test function $\phi \in C_c^1(\mathbb{R}^2)$, we find

$$\frac{d}{dt} \int_{\overline{\Omega} \setminus \Omega_\varepsilon} \phi\, d\mu(t) = -\int_{\Omega \setminus \Omega_\varepsilon} \nabla h_{\mu(t)} \cdot \nabla\phi\, d\varrho(t) - \int_{\partial\Omega_\varepsilon} \phi \frac{\partial h_{\mu(t)}}{\partial\nu} \hat{\varrho}(t)\, d\mathcal{H}^1 \tag{5.19}$$

for a.e. $\varepsilon > 0$. Now we can bound the last integral with $C \int_{\partial\Omega_\varepsilon} |\phi|\, d\mathcal{H}^1 \llcorner \partial\Omega_\varepsilon$, and passing to the limit as $\varepsilon \downarrow 0$ we see that the first term in the right-hand side vanishes. We obtain

$$\left| \frac{d}{dt} \int_{\partial\Omega} \phi\, d\mu(t) \right| \leqslant C \int_{\partial\Omega} |\phi|\, d\mathcal{H}^1 \llcorner \partial\Omega.$$

Integrating in time we get

$$\left|\int_{\partial\Omega} \phi\, d\mu(t)\right| \leqslant Ct \int_{\partial\Omega} |\phi|\, d\mathcal{H}^1 \llcorner \partial\Omega.$$

Since ϕ is arbitrary, this shows that $\tilde{\mu}(t) = f(t,\cdot)\mathcal{H}^1 \llcorner \partial\Omega$ for a suitable density f satisfying $\|f(t,\cdot)\|_\infty \leqslant Ct$.

6. The equation in the whole plane

In this section we are going to analyze a different model, where the constitutive coupling density–velocity relation (1.2) is replaced by

$$-\Delta h_\mu = \mu \tag{6.1}$$

in the whole plane and $\mu \in \mathcal{M}_{\kappa,M}(\mathbb{R}^2)$. Hence, the evolution equation takes the form

$$\frac{d}{dt}\mu(t) - \operatorname{div}\big(\nabla\Delta^{-1}\mu(t)\big|\mu(t)\big|\big) = 0 \quad \text{in } \mathcal{D}'\big((0,+\infty)\times\mathbb{R}^2\big). \tag{6.2}$$

This vortex model was investigated in [12,16]. In particular, in [16] the authors are able to prove the existence and uniqueness by means of an approximation scheme in the case of Sobolev initial data, as already mentioned.

Rather, we would like again to obtain a Wasserstein gradient flow for the corresponding energy functional. We have to pay attention to the definition of the energy. Indeed, since the fundamental solution of the Laplace equation in two dimensions is $-\frac{1}{2\pi}\log|x|$, the solution to (6.1) we are considering is the one given by

$$h_\mu(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, d\mu(y),$$

and then

$$\nabla h_\mu(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, d\mu(y).$$

We stress that h_μ does not decay to zero at infinity and $|\nabla h_\mu|^2$ is not integrable over the whole plane, unless $\mu(\mathbb{R}^2) = 0$. Therefore, $\int_{\mathbb{R}^2} |\nabla h_\mu|^2$ is not in general a well-defined quantity. In order to overcome this problem, we introduce for the case $\kappa \neq 0$ an auxiliary measure μ_0, whose density can be any smooth compactly supported function, and such that $\mu_0(\mathbb{R}^2) = \kappa$. Next we define h_0 as $(-\Delta^{-1})\mu_0$. As a consequence, if we set $w_\mu := h_\mu - h_0$, by linearity we see that

$$w_\mu = (-\Delta)^{-1}(\mu - \mu_0), \tag{6.3}$$

and this time $\mu - \mu_0$ has zero integral over $\mathbb{R}^2$ so w_μ and ∇w_μ decay sufficiently fast at infinity. We may then introduce the energy functional

$$\Psi(\mu) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_\mu|^2 + \int_{\mathbb{R}^2} w_\mu \mu_0, \tag{6.4}$$

which is now well defined.

Remark 6.1. This procedure can be seen as a "renormalized" way of defining the energy, and is inspired from a similar procedure for Ginzburg–Landau in the plane, see [18,5]. Indeed, an alternative way of dealing with the non-integrability of $|\nabla h_\mu|^2$ would be to define the energy functional by

$$\lim_{R\to\infty}\left(\frac{1}{2}\int_{B_R} |\nabla h_\mu|^2 - \frac{\kappa^2}{4\pi}\log R\right), \tag{6.5}$$

where B_R is the ball of radius R, centered in the origin. The limit is finite because of the logarithmic behavior at infinity of $\int_{B_R} |\nabla h_\mu|^2$, which comes from the structure of the fundamental solution in two dimensions. Writing h_μ as $h_0 + w_\mu$, expanding, integrating by parts, and taking the limit $R \to +\infty$, one can see (see [5] for example) that the definition (6.5) is equivalent, up to a constant depending on the choice of μ_0, to that of Ψ.

We proceed with the gradient flow of the functional (6.4). We give a brief discussion, omitting some details of the proofs, since the arguments are similar to the ones in the previous sections. Note that here we could also have added a $\lambda|\mu|(\mathbb{R}^2)$ term to the energy functional, which would again probably influence the mass cancellation rate in the (changing sign) solution. As a result, different λ's could lead to different solutions.

Let us consider the perturbed discrete scheme associated to Ψ, in the space of measures with finite second moment $\mathcal{M}^2_{\kappa,M}(\mathbb{R}^2) := \{\mu \in \mathcal{M}_{\kappa,M}(\mathbb{R}^2)\colon \int_{\mathbb{R}^2} |x|^2\, d|\mu| < +\infty\}$, that is, for $\delta > 0$,

$$\min_{\nu\in\mathcal{M}^2_{\kappa,M}(\mathbb{R}^2),\ |\nu|(\mathbb{R}^2)\leqslant|\mu|(\mathbb{R}^2)} \Psi(\nu) + \delta\int_{\mathbb{R}^2} |\nu|^4 + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu). \tag{6.6}$$

Remark 6.2. Since Ψ is strictly convex, we can also infer that the solution to (6.6) is unique (for any $\delta \geqslant 0$). Notice that we could not conclude the same in problems (3.4) and (3.6), as the strictly convex part of the functional therein, namely $\Phi_0^\delta(\mu)$, depends only on the interior part $\hat{\mu}$ of the measure. The absence of boundary mass for the problem on the whole plane allows us to retrieve uniqueness.

We have again

Proposition 6.3. *Let ν be solution to* (6.6). *Then $\nu \in L^4(\mathbb{R}^2)$ and*

$$-3\delta\nabla\big((\nu)^4\big) - \nabla h_\nu \nu = \frac{1}{\tau}\pi^1_\#\big((x-y)\big(\gamma_0^+ + \gamma_1^+\big)\big) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big), \tag{6.7}$$

where $\gamma_0^+ \in \Gamma_0(\nu^+, \mu_0^+)$ and $\gamma_0^- \in \Gamma_0(\nu^-, \mu_0^-)$ are plans corresponding to an optimal splitting of the cost $\mathcal{W}_2(\nu,\mu)$, as in Notation 2.5.

Proof. Let $\nu_\varepsilon = (\mathbf{I} + \varepsilon\boldsymbol{\xi})_\#\nu$, where $\boldsymbol{\xi}$ is a smooth, compactly supported vector field. Let moreover

$$\gamma_\varepsilon^+ = (\mathbf{I} + \varepsilon\boldsymbol{\xi}, \mathbf{I})_\#\gamma_0^+ + \gamma_1^+,$$
$$\gamma_\varepsilon^- = (\mathbf{I} + \varepsilon\boldsymbol{\xi}, \mathbf{I})_\#\gamma_0^- + \gamma_1^-.$$

Then

$$\mathcal{W}_2^2(\nu_\varepsilon,\mu) \leqslant W_2^2\big((\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\nu^+ + \vartheta_1, \mu^+\big) + W_2^2\big((\mathbf{I}+\varepsilon\boldsymbol{\xi})_\#\nu^- + \vartheta_1 + \mu^-\big)$$
$$\leqslant \int_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^2\, d\big(\gamma_\varepsilon^+ + \gamma_\varepsilon^-\big).$$

We find

$$\mathcal{W}_2^2(\nu_\varepsilon,\mu) - \mathcal{W}_2^2(\nu,\mu) \leqslant 2\varepsilon \int_{\mathbb{R}^2\times\mathbb{R}^2} \boldsymbol{\xi}(x)(x-y)\, d\big(\gamma_0^+ + \gamma_0^-\big)(x,y) + o(\varepsilon).$$

The differentiation of the term $\delta\int_{\mathbb{R}^2} |\nu_\varepsilon|^4$ is done exactly as in Lemma 3.2. Coming to the term $\Psi(\nu_\varepsilon)$, integrating by parts, we have

$$\Psi(\nu_\varepsilon) - \Psi(\nu) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla w_{\nu_\varepsilon}|^2 - |\nabla w_\nu|^2 + \int_{\mathbb{R}^2} (w_{\nu_\varepsilon} - w_\nu)\,\mu_0$$
$$= \frac{1}{2}\int_{\mathbb{R}^2} (w_{\nu_\varepsilon} + w_\nu)\, d(\nu_\varepsilon - \nu) + \int_{\mathbb{R}^2} h_0\, d(\nu_\varepsilon - \nu)$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} \big(w_{\nu_\varepsilon} \circ (\mathbf{I} + \varepsilon \boldsymbol{\xi}) - w_{\nu_\varepsilon} + w_\nu \circ (\mathbf{I} + \varepsilon \boldsymbol{\xi}) - w_\nu\big)\, d\nu + \int_{\mathbb{R}^2} \big(h_0 \circ (\mathbf{I} + \varepsilon \boldsymbol{\xi}) - h_0\big)\, d\nu.$$

All the integrals above are well defined, since w_ν, w_{ν_ε}, h_0 are continuous and $\nu_\varepsilon - \nu$ is compactly supported. Since $\boldsymbol{\xi}$ is compactly supported, making use of the C^1 convergence of w_{ν_ε} to w_ν we find

$$\Psi(\nu_\varepsilon) - \Psi(\nu) = \varepsilon \int_{\mathbb{R}^2} \nabla w_\nu \cdot \boldsymbol{\xi}\, d\nu + \varepsilon \int_{\mathbb{R}^2} \nabla h_0 \cdot \boldsymbol{\xi}\, d\nu + o(\varepsilon) = \varepsilon \int_{\mathbb{R}^2} \nabla h_\nu \cdot \boldsymbol{\xi}\, d\nu + o(\varepsilon).$$

Then from the minimality of ν and the arbitrariness of $\boldsymbol{\xi}$, one gets the thesis, the details being as in the proof of Lemma 3.2. □

In order to get regularity and pass to the limit as $\delta \to 0$, one has to establish the corresponding entropy argument. We have the following.

Lemma 6.4. *Let $\mu \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$. Then the minimizer ν of* (6.6) *satisfies*

$$\int_{\mathbb{R}^2} \varphi\big(|\nu|\big) \leqslant \int_{\mathbb{R}^2} \varphi\big(|\mu|\big) < +\infty, \tag{6.8}$$

for any C^2 entropy φ with the following properties: $\varphi(x) = x$ *for* $0 \leqslant x \leqslant 1$ *and* $\varphi(x) \leqslant C|x|^4$ *for* $x > 1$, *for some constant* $C > 0$.

Proof. Step 1. Assume first that μ has compact support. This step repeats the argument of Lemma 4.1. Since $\nu \ll \mathcal{L}^2$, again the plans γ_0^+ and γ_0^- are induced by optimal transport maps r_1 and r_2 respectively. Notice that, since we are assuming that the respective targets μ_0^+ and μ_0^- are compactly supported, these maps are bounded. Therefore, the Euler–Lagrange equation (6.7) can be written as

$$-3\delta \nabla\big((\nu)^4\big) - \nabla h_\nu \nu = \frac{1}{\tau}(I - r_1)\nu^+ + \frac{1}{\tau}(I - r_2)\nu^- \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big),$$

and since $\nu \in L^4(\mathbb{R}^2)$ the right-hand side is in $L^4(\mathbb{R}^2)$ as well. Now we can repeat the same argument of the proof of Lemma 4.1 to obtain the regularity of ν. Then we divide again by $|\nu|$ to obtain

$$3\delta\, \mathrm{sgn}(\nu) \frac{\nabla((\nu)^4)}{\nu} + \nabla h_\nu\, \mathrm{sgn}(\nu) = \frac{1}{\tau}(r - I) \quad \nu\text{-a.e.} \tag{6.9}$$

where $r = r_1 \chi_{\{\nu > 0\}} + r_2 \chi_{\{\nu < 0\}}$. In order to use the displacement convexity inequality [3, Lemma 10.4.4] we need the finiteness of the integrals $\int_{\mathbb{R}^2} \varphi(|\mu|)$ and $\int_{\mathbb{R}^2} \varphi(|\nu|)$. This property is ensured by the bound $|\nu|(\mathbb{R}^2) \leqslant |\mu|(\mathbb{R}^2)$ and by the choice of φ as an entropy with at most quartic growth. Indeed, by assumption on φ we have

$$\int_{\mathbb{R}^2} \varphi\big(|\nu|\big) \leqslant \int_{\{|\nu| \leqslant 1\}} |\nu| + C \int_{\{|\nu| > 1\}} |\nu|^4 < +\infty,$$

since $\nu \in L^4(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. Thus, we get the equivalent of (4.6), that is

$$\int_{\mathbb{R}^2} \varphi\big(|\mu|\big) - \varphi\big(|\nu|\big) \geqslant -\tau \int_{\mathbb{R}^2} \psi'\big(|\nu|\big)\, \mathrm{div}\left[3\delta\, \mathrm{sgn}(\nu) \frac{\nabla((\nu)^4)}{\nu} + \mathrm{sgn}(\nu) \nabla h_\nu\right]. \tag{6.10}$$

Here ψ is again an even convex function on $\mathbb{R}$ such that $\psi'(x) = x\varphi'(x) - \varphi(x)$. We saw the left-hand side in (6.10) is finite, the right-hand side is also finite because it is nonnegative. Indeed, since ψ' is odd and increasing and vanishes at 0, we have in particular

$$\int_{\mathbb{R}^2} \psi'(|\nu|)\operatorname{div}\big(\operatorname{sgn}(\nu)\nabla h_\nu\big) = \int_{\mathbb{R}^2} \psi'(\nu)\Delta h_\nu = -\int_{\mathbb{R}^2} \psi'(\nu)\nu \leqslant 0.$$

On the other hand,

$$\int_{\mathbb{R}^2} \psi'(|\nu|)\operatorname{div}\left(\operatorname{sgn}(\nu)\frac{\nabla((\nu)^4)}{\nu}\right) \leqslant 0.$$

This is found as in the last part of the proof of Lemma 4.1. The estimate (6.8) follows.

Step 2. We establish the result for the generic initial measure $\mu \in L^4(\mathbb{R}^2)$, by approximation with compactly supported measures. Let us introduce the following truncation of a measure $\sigma \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2)$:

$$\breve{\sigma}^+_R := \sigma^+_R\big(\mathbb{R}^2\big)\frac{\chi_{B_R}\sigma^+}{\sigma^+(B_R)}, \qquad \breve{\sigma}^-_R := \sigma^-\big(\mathbb{R}^2\big)\frac{\chi_{B_R}\sigma^-}{\sigma^-(B_R)}, \quad R > 0.$$

This way, $\breve{\sigma}^+_R$ and $\breve{\sigma}^-_R$ have the same integral as σ^+ and σ^- (the positive and negative parts of σ) respectively. Letting $\breve{\sigma}_R := \breve{\sigma}^+_R - \breve{\sigma}^-_R$, we see that $\breve{\sigma}_R$ is still in $\mathcal{M}^2_{\kappa,M}(\mathbb{R}^2)$ and $|\breve{\sigma}_R|(\mathbb{R}^2) = |\sigma|(\mathbb{R}^2)$. Moreover, if $R \to \infty$ we clearly have the weak convergence $\breve{\sigma}^+_R \rightharpoonup \sigma^+$, and the convergence of second moments, and the same for negative parts. If σ is also in $L^p(\mathbb{R}^2)$, then $\breve{\sigma}_R \to \sigma$ in the strong $L^p(\mathbb{R}^2)$ topology.

We would like to show that, if ν_n is solution to (6.6) starting from $\breve{\mu}_n$, $n \in \mathbb{N}$, then weak limits of the sequence (ν_n) are solutions to the same problem starting from μ. For this, we need a Γ-convergence argument.

First notice that the minimality of ν_n for problem (6.6) implies

$$\mathcal{W}_2^2(\nu_n, \breve{\mu}_n) \leqslant \tau\left(\Psi(\breve{\mu}_n) + \delta\int_{\mathbb{R}^2} |\breve{\mu}_n|^4\right) \to \tau\left(\Psi(\mu) + \delta\int_{\mathbb{R}^2} |\mu|^4\right)$$

where the convergence follows by elliptic regularity. Therefore the second moments of (ν_n) are uniformly bounded, giving the tightness of the sequence. On the other hand, by the estimate found in Step 1 in the compactly supported case, for a 4-growing entropy we have

$$\int_{\mathbb{R}^2} \varphi\big(|\nu_n|\big) \leqslant \int_{\mathbb{R}^2} \varphi\big(|\breve{\mu}_n|\big)$$

where the right-hand side is uniformly bounded in n, by the definition of $\breve{\mu}_n$. By appropriate choice of φ, we can deduce that (ν_n) is also bounded in $L^4(\mathbb{R}^2)$. In particular, (ν_n) has limits, up to subsequences, both in the weak topology of measures and in the weak $L^4(\mathbb{R}^2)$ topology.

For a general sequence (ν_n) converging to ν weakly in measures and in $L^4(\mathbb{R}^2)$, let us make use of the following notation. Let $\vartheta^1_n(\breve{\mu}_n)$, $\vartheta^2_n(\breve{\mu}_n)$, with $\vartheta^1_n(\breve{\mu}_n) - \vartheta^2_n(\breve{\mu}_n) = \nu_n$, be a couple realizing the infimum in the definition of $\mathcal{W}_2(\nu_n, \breve{\mu}_n)$ (as seen in (2.14), the infimum is indeed attained), so that

$$\mathcal{W}_2^2(\nu_n, \breve{\mu}_n) = W_2^2\big(\vartheta^1_n(\breve{\mu}_n), \breve{\mu}^+_n\big) + W_2^2\big(\vartheta^2_n(\breve{\mu}_n), \breve{\mu}^-_n\big). \tag{6.11}$$

Notice that $\vartheta^1_n(\breve{\mu}_n)$ and $\breve{\mu}^+_n$ have the same integral over $\mathbb{R}^2$ (the same as μ^+ of course), and similarly for negative parts. As a consequence we have for any n

$$\mathcal{W}_2^2(\nu_n, \mu) \leqslant W_2^2\big(\vartheta^1_n(\breve{\mu}_n), \mu^+\big) + W_2^2\big(\vartheta^2_n(\breve{\mu}_n), \mu^-\big).$$

Then, by the weak lower semi-continuity of $\nu \mapsto \mathcal{W}_2(\nu, \mu)$, (6.11) and the triangle inequality, we get

$$\begin{aligned}
\mathcal{W}_2^2(\nu, \mu) &\leqslant \liminf_{n\to\infty} \mathcal{W}_2^2(\nu_n, \mu) \\
&\leqslant \liminf_{n\to\infty} W_2^2\big(\vartheta^1_n(\breve{\mu}_n), \mu^+\big) + W_2^2\big(\vartheta^2_n(\breve{\mu}_n), \mu^-\big) \\
&\leqslant \liminf_{n\to\infty}\big(W_2\big(\vartheta^1_n(\breve{\mu}_n), \breve{\mu}^+_n\big) + W_2\big(\breve{\mu}^+_n, \mu^+\big)\big)^2 + \big(W_2\big(\vartheta^2_n(\breve{\mu}_n), \breve{\mu}^-_n\big) + W_2\big(\breve{\mu}^-_n, \mu^-\big)\big)^2 \\
&= \liminf_{n\to\infty} W_2^2\big(\vartheta^1_n(\breve{\mu}_n), \breve{\mu}^+_n\big) + W_2^2\big(\vartheta^2_n(\breve{\mu}_n), \breve{\mu}^-_n\big) = \liminf_{n\to\infty} \mathcal{W}_2^2(\nu_n, \breve{\mu}_n),
\end{aligned} \tag{6.12}$$

where we have used the weak convergence, plus convergence of second moments, of $\breve{\mu}_n^+$ and $\breve{\mu}_n^-$. On the other hand, let $\vartheta^1(\mu)$, $\vartheta^2(\mu)$ be an optimal couple corresponding to $\mathcal{W}_2(\nu,\mu)$. As before,

$$\mathcal{W}_2^2(\nu,\breve{\mu}_n) \leqslant W_2^2\big(\vartheta^1(\mu),\breve{\mu}_n^+\big) + W_2^2\big(\vartheta^2(\mu),\breve{\mu}_n^-\big),$$

hence again the weak convergence plus convergence of second moments of $\breve{\mu}_n^+$ and $\breve{\mu}_n^-$ entails

$$\begin{aligned}\limsup_{n\to\infty}\mathcal{W}_2^2(\nu,\breve{\mu}_n) &\leqslant \limsup_{n\to\infty} W_2^2\big(\vartheta^1(\mu),\breve{\mu}_n^+\big) + W_2^2\big(\vartheta_2(\mu),\breve{\mu}_n^-\big)\\ &= \limsup_{n\to\infty} W_2^2\big(\vartheta^1(\mu),\mu^+\big) + W_2^2\big(\vartheta^2(\mu),\mu^-\big) = \mathcal{W}_2^2(\mu,\nu).\end{aligned} \tag{6.13}$$

Since $\Psi(\cdot) + \delta\int_{\mathbb{R}^2}|\cdot|^4$ is weakly lower semi-continuous, these two inequalities, concerning generic sequences (ν_n), prove by definition the Γ-convergence of

$$\nu \mapsto \Psi(\nu) + \delta\int_{\mathbb{R}^2}|\nu|^4 + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\breve{\mu}_n) + \mathbf{1}_{\{|\nu|(\mathbb{R}^2)\leqslant|\mu|(\mathbb{R}^2)\}}(\nu)$$

to

$$\nu \mapsto \Psi(\nu) + \delta\int_{\mathbb{R}^2}|\nu|^4 + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu) + \mathbf{1}_{\{|\nu|(\mathbb{R}^2)\leqslant|\mu|(\mathbb{R}^2)\}}(\nu),$$

as $n\to\infty$, where $\mathbf{1}_A(\nu)$ is the function with value 0 if $\nu\in A$ and $+\infty$ if $\nu\notin A$. The Γ-convergence implies, as usual, that a weak limit ν of a sequence (ν_n) of minimizers for the first functional is a minimizer for the second functional above. Since that functional is strictly convex, it has a unique minimizer, and thus the whole sequence of minimizers (ν_n) converges to the solution ν of (6.6).

Eventually, the proof is concluded invoking the weak $L^4(\mathbb{R}^2)$ lower semi-continuity of $\nu\mapsto\int_{\mathbb{R}^2}\varphi(|\nu|)$ (since any entropy φ is convex) and the inequality proved in Step 1 for compactly supported initial measures:

$$\int_{\mathbb{R}^2}\varphi\big(|\nu|\big) \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^2}\varphi\big(|\nu_n|\big) \leqslant \liminf_{n\to\infty}\int_{\mathbb{R}^2}\varphi\big(|\breve{\mu}_n|\big) \leqslant \int_{\mathbb{R}^2}\varphi\big(|\mu|\big),$$

as desired. □

Corollary 6.5. *Let $\mu\in\mathcal{M}_{\kappa,M}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$, $p\geqslant 4$. Then the unique solution μ_τ of the unperturbed problem*

$$\min_{\nu\in\mathcal{M}^2_{\kappa,M}(\mathbb{R}^2),\,|\nu|(\mathbb{R}^2)\leqslant|\mu|(\mathbb{R}^2)} \Psi(\nu) + \frac{1}{2\tau}\mathcal{W}_2^2(\nu,\mu) \tag{6.14}$$

belongs to $L^p(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2}\varphi\big(|\mu_\tau|\big) \leqslant \int_{\mathbb{R}^2}\varphi\big(|\mu|\big),$$

where φ is the entropy defined in (4.11)*, if $p<+\infty$. On the other hand, $\|\mu_\tau\|_p\leqslant\max\{1,\|\mu\|_p\}$ if $p=+\infty$. Moreover, μ_τ satisfies*

$$-\nabla h_{\mu_\tau}\mu_\tau = \frac{1}{\tau}\pi^1_\#\big((x-y)\big(\gamma_0^+ + \gamma_0^-\big)\big). \tag{6.15}$$

Proof. Of course Lemma 3.1 applies also to the functional Ψ. Then there exists a sequence δ_i such that $\mu_\tau^{\delta_i}$, minimizers of (6.6), converge narrowly to μ_τ, where μ_τ is the minimizer of (6.14). Let φ be the entropy defined by (4.11) for $p=4$. In this case Lemma 6.4 directly applies, giving

$$\int_{\mathbb{R}^2}\varphi\big(|\mu_\tau^{\delta_i}|\big) \leqslant \int_{\mathbb{R}^2}\varphi\big(|\mu|\big),$$

hence we deduce weak $L^4(\mathbb{R}^2)$ compactness, and the weak lower semi-continuity of $\theta \mapsto \int_{\mathbb{R}^2} \varphi(|\theta|)$ in $L^4(\mathbb{R}^2)$ yields the desired $L^4(\mathbb{R}^2)$ bound.

Next we take advantage of the following fact: any entropy can be approximated monotonically from below on $[0,+\infty)$ by a sequence of entropies with at most quadratic growth at infinity. To see this, it is enough, given the generic entropy φ, to construct the sequence φ_n in the following way:

$$\varphi_n(x) = \begin{cases} \varphi(x) & \text{for } 0 \leqslant x \leqslant n, \\ ax^2 + bx + c & \text{for } x > n, \end{cases}$$

where the coefficients a, b, c have to be suitably chosen in order to make φ_n enjoy a C^2 regularity. It is straightforward to check that indeed the functions φ_n are displacement convex, for any n.

Now let φ be defined by (4.11) if $4 < p < +\infty$. Otherwise, if $p = +\infty$ we let φ be given by (4.13) (which is itself an entropy), with $K := \max\{1, \|\mu\|_\infty\}$. For both cases, using Lemma 6.4, for any n we have

$$\int_{\mathbb{R}^2} \varphi_n\big(\big|\mu_\tau^{\delta_i}\big|\big) \leqslant \int_{\mathbb{R}^2} \varphi_n\big(|\mu|\big) \leqslant \int_{\mathbb{R}^2} \varphi\big(|\mu|\big) < +\infty.$$

By monotone convergence we get

$$\int_{\mathbb{R}^2} \varphi\big(\big|\mu_\tau^{\delta_i}\big|\big) \leqslant \int_{\mathbb{R}^2} \varphi\big(|\mu|\big). \tag{6.16}$$

If $p < \infty$ we find weak $L^p(\mathbb{R}^2)$ compactness, and again we conclude by weak lower semi-continuity of $\theta \mapsto \int_{\mathbb{R}^2} \varphi(|\theta|)$ in $L^p(\mathbb{R}^2)$. If $p = +\infty$, (6.16) shows that $|\mu_\tau^{\delta_i}| \leqslant K$, then $|\mu_\tau| \leqslant K$.

Finally, (6.15) is proven as in Lemma 4.4. □

The construction of the minimizing movement is done in the usual way: given $\mu_\tau^k \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2)$, μ_τ^{k+1} is the minimizer of (6.14), satisfying the properties of Corollary 6.5. Moreover, we don't need to split the minimizers as in (5.2). Then, starting from the basic inequality

$$\mathcal{W}_2^2\big(\mu_\tau^{k+1}, \mu_\tau^k\big) \leqslant 2\tau\Psi\big(\mu_\tau^k\big) - 2\tau\Psi\big(\mu_\tau^{k+1}\big),$$

the argument of Proposition 5.1 repeats, giving the limiting curve $t \mapsto \mu(t) \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2)$. Finally one reasons as in Lemma 5.5, Theorem 5.6 and Remark 5.8 (the proofs being simplified by the absence of boundary) and gets the limiting equation. Note that for the particular case of positive measures, the analogue of Remark 5.7 holds, i.e. we produce a true solution of (6.2). We summarize these results in the following

Theorem 6.6. *Let $\mu^0 \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2) \cap L^4(\mathbb{R}^2)$. There exists an infinitesimal sequence (τ_n) such that $\bar{\mu}_{\tau_n}(t)$ converge to some $\mu(t) \in \mathcal{M}^2_{\kappa,M}(\mathbb{R}^2)$ weakly in the sense of measures, for any $t \geqslant 0$, with $\mu(0) = \mu^0$. Also, for any $t \geqslant 0$, $\bar{\mu}^+_{\tau_n}(t)$ and $\bar{\mu}^-_{\tau_n}(t)$ converge weakly to some positive measures $\varrho^+(t)$ and $\varrho^-(t)$ with finite second moment and total mass bounded by M. $\varrho^+(t)$ and $\varrho^-(t)$ are uniformly bounded in $L^4(\mathbb{R}^2)$ and satisfy $\varrho^+(t) - \varrho^-(t) = \mu(t)$. Moreover, letting $\varrho(t) = \varrho^+(t) + \varrho^-(t)$, there holds*

$$\frac{d}{dt}\mu(t) - \operatorname{div}\big(\big(\nabla\Delta^{-1}\mu(t)\big)\varrho(t)\big) = 0 \quad \text{in } \mathcal{D}'\big((0,+\infty) \times \mathbb{R}^2\big).$$

Finally, there exists a positive space–time measure $\sigma \in \mathcal{M}((0,+\infty) \times \mathbb{R}^2)$ such that, in $\mathcal{D}'((0,+\infty) \times \mathbb{R}^2)$,

$$\begin{cases} \dfrac{d}{dt}\varrho^+(t) - \operatorname{div}\big(\nabla\Delta^{-1}\mu(t)\varrho^+(t)\big) = -\sigma(t), \\ \dfrac{d}{dt}\varrho^-(t) + \operatorname{div}\big(\nabla\Delta^{-1}\mu(t)\varrho^-(t)\big) = -\sigma(t). \end{cases}$$

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