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Ergodicity of Hamilton–Jacobi equations with a noncoercive nonconvex Hamiltonian in $\mathbb{R}^2/\mathbb{Z}^2$

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Abstract

The paper investigates the long time average of the solutions of Hamilton–Jacobi equations with a noncoercive, nonconvex Hamiltonian in the torus $\mathbb{R}^2/\mathbb{Z}^2$. We give nonresonance conditions under which the long-time average converges to a constant. In the resonant case, we show that the limit still exists, although it is nonconstant in general. We compute the limit at points where it is not locally constant.

Résumé

Nous considérons le comportement en temps grand de la moyenne temporelle de solutions d'équations de Hamilton–Jacobi pour un hamiltonien non convexe et non coercif dans le tore $\mathbb{R}^2/\mathbb{Z}^2$. Nous mettons en évidence des conditions de non-résonnance sous lesquelles cette moyenne converge vers une constante. Dans le cas où il y a résonnance, nous montrons que la limite existe, bien qu'étant non constante en général. Nous calculons la limite aux points où celle-ci est non localement constante.

1. Introduction

Since the pioneering work of Lions, Papanicolaou and Varadhan [12], the ergodic problem for Hamilton–Jacobi equations has attracted considerable attention. For equations of evolutionary type:

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ u(x, 0) = 0 & \text{in } \mathbb{R}^N \end{cases} \tag{1}$$

where $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous, one is interested in the long time behavior of the time average $u(x, t)/t$. For equations of stationary type:

$$\lambda v_\lambda + H(x, Dv_\lambda) = 0 \quad \text{in } \mathbb{R}^N, \tag{2}$$

the object of investigation is the limiting behavior, when the discount factor λ vanishes, of the quantity λv_λ.

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A typical result in this framework is the following: If $H(\cdot, p)$ is $\mathbb{Z}^N$-periodic and $H(x,\cdot)$ is coercive:

$$\lim_{|p|\to+\infty} H(x,p) = +\infty \quad \text{uniformly with respect to } x, \tag{3}$$

the following ergodicity property holds:

$$\lim_{t\to+\infty} \frac{u(x,t)}{t} = \lim_{\lambda\to 0} \lambda v_\lambda(x) = c \quad \text{uniformly with respect to } x, \tag{4}$$

where c is the unique constant for which equation

$$H(x, D\chi) = -c \quad \text{in } \mathbb{R}^N \tag{5}$$

has a continuous, periodic solution χ. The periodicity condition has been relaxed in many situations (compact manifolds, almost periodic setting, stochastic homogenization, ...). However, for first order Hamilton–Jacobi equations, the coercivity of $H(x,\cdot)$ plays a central role. Indeed, as can readily be seen from the equation, this condition ensures the family of functions $\{v_\lambda\}$ to be equicontinuous, which in turn implies the existence of a corrector (or approximate corrector in more general frameworks), i.e., a solution of (5).

When the Hamiltonian is not coercive, this crucial equicontinuity property fails and few results are available. Most of them rely on some partial coercivity or on some reduction property, which, somehow, compensates the lack of coercivity: let us quote Alvarez and Bardi [1,2], Alvarez and Ishii [3], Artstein and Gaitsgory [5], Bardi [6], Barles [7], Birindelli and Wigniolle [8], Gomes [10] and Imbert and Monneau [11]. We follow here a completely different approach, based on *nonresonance conditions*, initiated for Hamilton–Jacobi equations by Arisawa and Lions [4]. In [4] the authors investigate—among other problems—Eqs. (1) and (2) for Hamiltonians of the form

$$H(x,p) = H(p) - \ell(x), \quad \forall (x,p) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\ell : \mathbb{R}^N \to \mathbb{R}$ is continuous, $\mathbb{Z}^N$-periodic and $H : \mathbb{R}^N \to \mathbb{R}$ is positively homogeneous and convex. Under these assumptions, (4) holds as soon as

$$\forall k \in \mathbb{Z}^N\backslash\{0\}, \quad \exists a \in \partial H(0) \quad \text{with } \langle k, a\rangle \neq 0. \tag{6}$$

This is the nonresonance condition.

The first aim of our paper is to investigate nonresonance conditions for Hamilton–Jacobi equations with *nonconvex* Hamiltonians. Since this is a very delicate issue, we concentrate on plane problems and on equations of the form

$$\begin{cases} u_t + H(Du) - \ell(x) = 0 & \text{in } \mathbb{R}^2 \times (0,+\infty), \\ u(x,0) = 0 \end{cases} \tag{7}$$

and of the form

$$\lambda v_\lambda + H(Dv_\lambda) - \ell(x) = 0 \quad \text{in } \mathbb{R}^2, \tag{8}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous and $\ell : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $\mathbb{Z}^2$-periodic. We now present our main result. Let us assume that there is some $k \geqslant 1$ and some k-positively homogeneous, locally Lipschitz continuous Hamiltonian $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{s\to 0^+} s^k H(p/s) = H_\infty(p) \quad \text{locally uniformly in } p$$

and such that

$$\forall \bar p \in \mathbb{R}^2\backslash\{0\} \text{ with } H_\infty(\bar p) = 0,\ DH_\infty(\bar p) \text{ exists and is nonzero.}$$

Let us set

$$\mathcal{P} = \big\{p = (p_1,p_2) \in \mathbb{R}^2 \;\big|\; |p| = 1,\ H_\infty(p) = H_\infty(-p) = 0,\ [p_2 = 0 \text{ or } p_1/p_2 \in \mathbb{Q}]\big\}. \tag{9}$$

Then we show (Theorem 4.1) that if either $\mathcal{P} = \emptyset$ or

$$\forall p \in \mathcal{P}, \quad \lim_{s\to 0^+} \left| H\left(\frac{p}{s}\right)\right| = +\infty \tag{10}$$

then ergodicity (4) holds for any $\ell : \mathbb{R}^2 \to \mathbb{R}$. For instance condition $\mathcal{P} = \emptyset$ holds for $H(p_1, p_2) = -|p_1| + \alpha|p_2|$ if and only if $\alpha > 0$ is irrational. For $H(p_1, p_2) = -(p_1 + a)^2 + (p_2 + b)^2$, we have $\mathcal{P} = \{(\pm 1, \pm 1)\}$ and condition (10) holds if and only if $|a| \neq |b|$.

In order to underline the difference between our result and the nonresonance condition of [4] described above, let us explain the main ideas of the proofs. As pointed out in [4], the interesting feature of Eqs. (7) and (8) is that they provide uniform continuity of $u(\cdot, t)/t$ in $\mathbb{R}^2$ for any $t \geqslant 1$ and of λv_λ in $\mathbb{R}^2$ for any $\lambda \in (0, 1]$ (this holds true in any space dimension). Let now w be a uniform limit of some subsequence $(\lambda_n v_{\lambda_n})$. Then classical arguments of viscosity solutions show that w is a Lipschitz continuous, periodic solution of

$$H_\infty(Dw) = 0 \quad \text{in } \mathbb{R}^2. \tag{11}$$

The main issue amounts to establish some rigidity properties for the solutions of this equation. When H_∞ is convex and 1-positively homogeneous, then it is proved in [4] that any continuous, periodic solution w of (11) is also a solution of

$$\langle a, Dw \rangle = 0 \quad \text{in } \mathbb{R}^2 \tag{12}$$

for any $a \in \partial H_\infty(0)$. Indeed, w is a viscosity subsolution of (12), hence a subsolution in the sense of distributions, and, integrating (12) over $[0, 1]^2$ readily gives that equality holds by periodicity. Note that Eq. (12) means that w is constant along the lines $t \to x + ta$ for any $a \in \partial H_\infty(0)$. In particular, if (6) holds, one can cover the torus by such lines and any continuous periodic solutions of (11) has to be constant.

If now H is nonconvex, then the reduction to linear equations of the form (12) does not hold. However we are able to show *in the plane* that, if (9) holds and if w is a Lipschitz continuous solution of Eq. (11), then at any point $\bar{x}$ at which w has a nonzero derivative, the map $t \to w(\bar{x} - tDH(Dw(\bar{x})))$ is constant on $[0, +\infty)$ (see Lemma 3.1). This result is somewhat surprising since the map $t \to w(\bar{x} + tDH(Dw(\bar{x})))$ need not be constant on $[0, +\infty)$. As a consequence we prove in Theorem 3.3 that any nonconstant, periodic solution of (11) is of the form $w(x) = \bar{w}(\langle \bar{p}, x \rangle)$ for some map $\bar{w} : \mathbb{R} \to \mathbb{R}$ and some $\bar{p} \in \mathcal{P}$. In particular, if $\mathcal{P} = \emptyset$, any limit of (λv_λ) is constant, which implies (4). Ergodicity in the case (10) is more subtle and relies on the fact that any cluster point w of the (λv_λ) for the uniform topology has to be of the form $w(x) = \bar{w}(\langle \bar{p}, x \rangle)$ for some $\bar{p} \in \mathcal{P}$ and some $\bar{w} : \mathbb{R} \to \mathbb{R}$.

The second aim of our paper is to analyse the behavior of the solutions of (7) and (8) in the resonant case ($\mathcal{P} \neq \emptyset$). In this case, we cannot expect ergodicity to hold in general: for instance, if $H(p_1, p_2) = -|p_1| + |p_2|$ and $\ell(p_1, p_2) = \bar{\ell}(p_1 - p_2)$ for some continuous periodic function $\bar{\ell} : \mathbb{R} \to \mathbb{R}$, then one easily checks that $v_\lambda = \ell/\lambda$, so that (4) does not hold. In this resonant case, very few is known. The only work we are aware of is due to Quincampoix and Renault who show in [13] that, if H is convex with respect to the gradient variable and has a "weak dependence" with respect to x, then the limit of $(u(\cdot, t)/t)$ as $t \to +\infty$ exists. We give a similar result for nonconvex H in dimension 2: under suitable assumptions on H, the limits of $u(\cdot, t)/t$ and of λv_λ as $t \to +\infty$ and $\lambda \to 0^+$ exist and are equal (Theorem 5.1). Of course this common limit w need not be constant in general. However, since w is a Lipschitz continuous, periodic solution of (11), our rigidity result implies that w has to be of the form $w(x) = \bar{w}(\langle \bar{p}, x \rangle)$ for some function $\bar{w} : \mathbb{R} \to \mathbb{R}$ and some $\bar{p} \in \mathcal{P}$. In fact we can compute explicitely $\bar{w}$ at any point $s \in \mathbb{R}$ at which $\bar{w}$ is not locally constant. The quantity $\bar{w}(s)$ is the sum of two terms: one is the average of ℓ over the line $\langle \bar{p}, x \rangle = s$; the other is related to the behavior of $H(p)$ when $|p| \to +\infty$. The proof of these results relies on the existence of "correctors" of a linearized equation along the lines $\langle \bar{p}, x \rangle = s$. We use these correctors in order to build sub-solutions with state constraints on sets of the form $\{w \geqslant \theta\}$.

The paper is organized in the following way. We first recall in Section 2 some well-known results on Eqs. (7) and (8). Then we establish in Section 3 the rigidity property of Eq. (11). The proof of ergodicity is given in Section 4, while Section 5 is devoted to resonant case. We complete the paper by a discussion on some open problems.

2. Generalities

In this section, we recall some known results on the Hamilton–Jacobi equations (7) and (8) (see also in particular [4]). We always work in the framework of viscosity solutions [9]. Let us first focus on the stationary equation:

$$\lambda v_\lambda + H(Dv_\lambda) = \ell(x) \quad \text{in } \mathbb{R}^N.$$

In the above equation, $H:\mathbb{R}^N\to\mathbb{R}$ is continuous and $\ell:\mathbb{R}^N\to\mathbb{R}$ is continuous and $\mathbb{Z}^N$-periodic. Under these assumptions, (8) has a unique viscosity solution v_λ. This solution is continuous and $\mathbb{Z}^N$-periodic. We set $w_\lambda=\lambda v_\lambda$.

Lemma 2.1. *The map w_λ is continuous, uniformly with respect to λ. Moreover, if ℓ is Lipschitz continuous, then so is w_λ, with a Lipschitz constant independent of λ.*

Proof. Let ω be a modulus of continuity of ℓ. For any $z\in\mathbb{R}^N$, $v_\lambda(\cdot+z)+\omega(|z|)/\lambda$ (resp. $v_\lambda(\cdot+z)-\omega(|z|)/\lambda$) is a supersolution (resp. a subsolution) of (8). By comparison, we get

$$v_\lambda(x+z)-\omega\big(|z|\big)/\lambda\leqslant v_\lambda(x)\leqslant v_\lambda(x+z)+\omega\big(|z|\big)/\lambda\quad\forall x\in\mathbb{R}^N,$$

which implies that ω is a modulus of continuity for w_λ for any $\lambda>0$. Note that if ℓ is Lipschitz continuous with a Lipschitz constant L, then we can take $\omega(t)=Lt$, so that the w_λ are all L-Lipschitz continuous. □

From now on we assume that H has a recession function: there exists a continuous function $H_\infty:\mathbb{R}^N\to\mathbb{R}$ and $k\geqslant 1$ such that

$$\lim_{s\to 0^+} s^k H(p/s)=H_\infty(p)\quad\text{locally uniformly in } p. \tag{13}$$

Note that H_∞ is k-positively homogeneous.

Lemma 2.2. *Under assumption* (13)*, if w is any limit of a uniformly converging subsequence of (w_λ) as $\lambda\to 0^+$, then w satisfies*

$$H_\infty(Dw)=0. \tag{14}$$

Proof. Indeed w_λ solves

$$\lambda^k w_\lambda+\lambda^k H(Dw_\lambda/\lambda)=\lambda^k\ell\quad\text{in } \mathbb{R}^N.$$

Letting $\lambda\to 0$ gives the result. □

Lemma 2.3. *Let w be any limit of a uniformly converging subsequence of (w_λ) as $\lambda\to 0^+$. Then*

$$\limsup_{\lambda\to 0}\max_{x\in\mathbb{R}^N} w_\lambda(x)=\max_{x\in\mathbb{R}^N} w(x)\quad\textit{and}\quad \liminf_{\lambda\to 0}\min_{x\in\mathbb{R}^N} w_\lambda(x)=\min_{x\in\mathbb{R}^N} w(x).$$

In particular, if w is constant, then (w_λ) uniformly converges to the constant w as $\lambda\to 0^+$.

Proof. For any λ, n, let $x_{\lambda,n}$ be a maximum point of $v_\lambda-v_{\lambda_n}$. We have (formally)

$$Dv_\lambda(x_{\lambda,n})=Dv_{\lambda_n}(x_{\lambda,n}).$$

From the equations satisfied by v_λ and v_{λ_n} we also have

$$\lambda v_\lambda(x_{\lambda,n})-\lambda_n v_{\lambda_n}(x_{\lambda,n})\leqslant -H\big(Dv_\lambda(x_{\lambda,n})\big)+H\big(Dv_{\lambda_n}(x_{\lambda,n})\big)=0.$$

Hence

$$\begin{aligned}\max\lambda v_\lambda&\leqslant\lambda\max_x(v_\lambda-\lambda v_{\lambda_n})+\lambda\max_x v_{\lambda_n}\\&\leqslant\lambda v_\lambda(x_{\lambda,n})-\lambda v_{\lambda_n}(x_{\lambda,n})+\lambda\max v_{\lambda_n}\\&\leqslant\lambda_n v_{\lambda_n}(x_{\lambda,n})+2\lambda|v_{\lambda_n}|_\infty.\end{aligned}$$

When $\lambda\to 0$, $x_{\lambda,n}\to x_n$ (up to some subsequence) and we get

$$\limsup_{\lambda\to 0^+}\max\lambda v_\lambda\leqslant\lambda_n v_{\lambda_n}(x_n).$$

Letting finally $n\to+\infty$ we obtain

$$\limsup_{\lambda\to 0^+} \max \lambda v_\lambda \leqslant \max_x w.$$

The above argument can be made rigorous by using standard technique of doubling variables. The reverse inequality

$$\liminf_{\lambda\to 0^+} \min \lambda v_\lambda \geqslant \min_x w$$

can be proved in the same way by minimizing $v_\lambda - v_{\lambda_n}$. □

We now turn to the analysis of the solutions of the evolution equation

$$\begin{cases} u_t + H(Du) - \ell(x) = 0 & \text{in } \mathbb{R}^N \times [0, +\infty[, \\ u(x,0) = 0 & \text{in } \mathbb{R}^N \end{cases}$$

where, as before, $H : \mathbb{R}^N \to \mathbb{R}$ is continuous, $\ell : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\mathbb{Z}^N$-periodic.

Lemma 2.4. *Under the above assumptions, the map $u(\cdot,t)/t$ is $\mathbb{Z}^N$-periodic, bounded and continuous in x uniformly with respect to $t \geqslant 1$.*

Proof. By comparison principle, $|u(x,t)| \leqslant t\|\ell\|_\infty$. So $u(\cdot,t)/t$ is bounded by $\|\ell\|_\infty$ for $t \geqslant 1$. Let ω be a modulus of continuity of ℓ. Then, for any $z \in \mathbb{R}^N$, $(x,t) \to u(x+z,t) + \omega(z)(1+t)$ is a super-solution of (7). So $u(x,t) \leqslant u(x+z,t) + \omega(z)(1+t)$, which proves that $u(\cdot,t)/t$ has ω as a modulus of continuity with respect to x for any $t \geqslant 1$. □

3. Rigidity of equation $H_\infty(Dw) = 0$

From now on we work in the plane. We denote by (x_1, x_2) or (p_1, p_2) a generic element of $\mathbb{R}^2$. The aim of this section is to investigate the continuous periodic solutions of equation $H_\infty(Dw) = 0$. In order to simplify the notations, we denote by H the Hamiltonian of this section. Our assumptions on $H : \mathbb{R}^2 \to \mathbb{R}$ are:

$$H \text{ is locally Lipschitz continuous, } k\text{-positively homogeneous, for some } k \geqslant 1. \tag{15}$$

Lemma 3.1. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ satisfying* (15). *Let w be a locally Lipschitz continuous viscosity solution of $H(Dw) = 0$ in $\mathbb{R}^2$ and $\bar{x}$ be a point of differentiability of w with $Dw(\bar{x}) \neq (0,0)$. If $DH(Dw(\bar{x}))$ exists, then*

$$w\big(\bar{x} - tDH\big(Dw(\bar{x})\big)\big) = w(\bar{x}) \quad \forall t \geqslant 0.$$

Remarks.

1. Equality $w(\bar{x} + tDH(Dw(\bar{x}))) = w(\bar{x})$ $\forall t \geqslant 0$ does not hold in general. For instance, if $H(p_1, p_2) = -|p_1| + |p_2|$, then $w(x_1, x_2) = \max\{x_1 + x_2, x_2 - x_1\}$ is a viscosity solution to $H(Dw) = 0$ in $\mathbb{R}^2$ and $(1,1)$ is a point of differentiability of w with $Dw(1,1) = (1,1)$ and $DH(Dw(\bar{x})) = (-1,1)$. However
$$w\big((1,1) + tDH\big(Dw(\bar{x})\big)\big) = \max\{2, 2t\} > 2 = w(1,1) \quad \forall t > 1.$$
2. Our result also holds if w is a solution of $H(Dw) = 0$ in an open set $\mathcal{O} \subset \mathbb{R}^2$. In this case, we have $w(\bar{x} - tDH(Dw(\bar{x}))) = w(\bar{x})$ for any $t > 0$ such that $\bar{x} - sDH(Dw(\bar{x})) \in \mathcal{O}$ for all $s \in [0,t]$. The proof is exactly the same.

Proof of Lemma 3.1. Replacing H by $\tilde{H}(p) = |p|H(p/|p|)$ if necessary, we can assume that H is 1-positively homogeneous. Moreover, H is then globally Lipschitz continuous. Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$ and let us set $\bar{p} = Du(\bar{x})$ and $\xi = DH(Dw(\bar{x}))$. Without loss of generality we can also suppose that $u(\bar{x}) = 0$ and $\bar{p} = e_1$. Since H is positively homogeneous, we note for later use that $\langle \bar{p}, \xi \rangle = H(\bar{p}) = 0$.

Step 1. We claim that, for any $\varepsilon > 0$ small enough there is some convex and positively homogeneous map $H_\varepsilon^+ : \mathbb{R}^2 \to \mathbb{R}$ such that

$$H \leqslant H_\varepsilon^+, \quad H_\varepsilon^+(\bar p)=0, \quad \partial H_\varepsilon^+(\bar p)\subset B(\xi,\varepsilon) \quad \text{and} \quad 0\notin \partial H_\varepsilon^+(0), \tag{16}$$

where $B(\xi,\varepsilon)$ denotes the closed ball centered at ξ and of radius ε.

Proof of Step 1. Since H is differentiable at $\bar p$, for any $\varepsilon>0$ we can find $\eta>0$ such that

$$\bigl|H(p)-\bigl\langle \xi,(p-\bar p)\bigr\rangle\bigr| \leqslant \varepsilon|p-\bar p| \quad \forall p\in B(\bar p,\eta). \tag{17}$$

Let us denote by C_η the convex cone $\{p=(p_1,p_2)\in\mathbb{R}^2 \mid |p_2|\leqslant \eta p_1\}$. By homogeneity of H and using the fact that $\langle \xi,\bar p\rangle=0$ and $\bar p=e_1$, (17) leads to

$$H(p)\leqslant \langle \xi,p\rangle+\varepsilon|p_2| \quad \forall p=(p_1,p_2)\in C_\eta.$$

Then, since H is Lipschitz continuous, we can choose a constant M such that $H\leqslant H_\varepsilon^+$ where

$$H_\varepsilon^+(p)=\langle \xi,p\rangle+\varepsilon|p_2|+Md_{C_\eta}(p) \quad \forall p\in\mathbb{R}^2,$$

where $d_{C_\eta}(p)$ denotes the distance from p to C_η. We note that H_ε^+ is convex and positively homogeneous. Moreover, $H_\varepsilon^+(\bar p)=0$ and $\partial H_\varepsilon^+(\bar p)\subset B(\xi,\varepsilon)$ by construction. Finally, we note that, if we had $0\in\partial H_\varepsilon^+(0)$, then this would imply that, for any $h>0$ sufficiently small,

$$0=\bigl\langle 0,(\bar p-h\xi)\bigr\rangle\leqslant H_\varepsilon^+(\bar p-h\xi)=-h|\xi|^2+\varepsilon h|\xi|<0,$$

a contradiction if we choose $\varepsilon>0$ small. So $0\notin\partial H_\varepsilon^+(0)$. □

Step 2. Let us fix $\delta>0$ small and let us denote by $\bar y_\delta$ and $\bar z_\delta$ a projection of $\bar x$ onto the sets $\{w\leqslant-\delta\}$ and $\{w\geqslant\delta\}$. Since w is differentiable at $\bar x$ with $|\bar p|=|Dw(\bar x)|=1$, we have

$$\lim_{\delta\to 0^+}\frac{\bar x-\bar y_\delta}{\delta}=\lim_{\delta\to 0^+}\frac{\bar z_\delta-\bar x}{\delta}=\frac{\bar p}{|\bar p|^2}=\bar p. \tag{18}$$

In particular, for $\delta>0$ sufficiently small, we have

$$-\delta<w(z)<\delta \quad \forall z\in\,]\bar y_\delta,\bar z_\delta[. \tag{19}$$

We claim that

$$\bigl\langle(\bar y_\delta-\bar x),\xi\bigr\rangle\leqslant 0 \quad \text{while } \bigl\langle(\bar z_\delta-\bar x),\xi\bigr\rangle\geqslant 0. \tag{20}$$

Proof of claim (20). Let $\mathbf{1}_{\{w>-\delta\}}$ denote the indicatrix function of the set $\{w>-\delta\}$. From classical stability result the map $\mathbf{1}_{\{w>-\delta\}}$ is a super-solution of $H(Dw)=0$. For $\delta>0$ small, the map

$$z(x)=-\delta+\bigl(\delta^2+|\bar x-\bar y_\delta|^2-|\bar x-x|^2\bigr)^{\frac12}$$

satisfies:

$$z(x)\leqslant|\bar x-\bar y_\delta|\leqslant 1=\mathbf{1}_{\{w>-\delta\}}(x) \quad \text{if } |\bar x-x|<|\bar x-\bar y_\delta|$$

by definition of y_δ and

$$z(x)\leqslant 0\leqslant\mathbf{1}_{\{w>-\delta\}}(x) \quad \text{if } |\bar x-x|\geqslant|\bar x-\bar y_\delta|.$$

Since moreover $z(\bar y_\delta)=0=\mathbf{1}_{\{w>-\delta\}}(\bar y_\delta)$, we have by definition of supersolutions:

$$H\bigl(Dz(\bar y_\delta)\bigr)=H\biggl(\frac{\bar x-\bar y_\delta}{\delta}\biggr)\geqslant 0.$$

Now we use the fact that we are in dimension 2, that $H(\bar p)=0$, $\xi=DH(\bar p)\neq 0$, $(\bar x-\bar y_\delta)/\delta\to\bar p$ and that H is homogeneous, to get that $\langle(\bar y_\delta-\bar x),\xi\rangle\leqslant 0$. The other inequality of (20) can be proved in a symmetric way. □

To proceed further we need the following lemma, the proof of which is postponed:

Lemma 3.2. *Let $G:\mathbb{R}^N\to\mathbb{R}$ be a convex positively homogeneous map and w be a continuous super-solution of $G(Dw)=0$. Then, for any $x_0\in\mathbb{R}^N$, there is an absolutely continuous map $x:[0,+\infty)\to\mathbb{R}^N$ such that $x(0)=x_0$,*

$$x'(t)\in-\partial G(0) \text{ for almost all } t\geqslant 0 \text{ and } t\to w\big(x(t)\big) \text{ is nonincreasing on } [0,+\infty).$$

Step 3. Let $H_\varepsilon^+:\mathbb{R}^2\to\mathbb{R}^2$ be the convex, positively homogeneous map defined in Step 1. Since $H\leqslant H_\varepsilon^+$, w is a supersolution of $H_\varepsilon^+(Dw)=0$. From Lemma 3.2 there is some absolutely continuous map $y_\delta:[0,+\infty)\to\mathbb{R}^2$ starting from $\bar y_\delta$ with

$$y_\delta'(t)\in-\partial H_\varepsilon^+(0) \text{ for almost all } t\geqslant 0 \text{ and } t\to w\big(y_\delta(t)\big) \text{ is nonincreasing on } [0,+\infty).$$

Let us fix $T>0$ and let Q_δ be the closed quadrilateral with vertices $\bar y_\delta$, $\bar z_\delta$, $\bar z_\delta-T\xi$, $\bar y_\delta-T\xi$. We set

$$\theta_\delta^-=\inf\big\{t\geqslant 0 \bigm| y_\delta(t)\notin Q_\delta\big\}.$$

Since $0\notin\partial H_\varepsilon^+(0)$, the separation theorem states that there is some $\eta>0$ and there is some direction $\zeta\in\mathbb{R}^2\backslash\{0\}$ such that $\langle y_\delta'(t),\zeta\rangle\geqslant\eta$ for almost all $t\geqslant 0$. Thus θ_δ^- is well defined and finite because Q_δ is bounded. We claim that $y_\delta(\theta_\delta^-)\in[\bar z_\delta,\bar z_\delta-T\xi]\cup[\bar z_\delta-T\xi,\bar y_\delta-T\xi]$.

Proof of the claim. By construction we have $w(y_\delta(t))\leqslant-\delta$ for any $t\geqslant 0$. From (19) this implies that $y_\delta(\theta_\delta^-)\notin\,]\bar y_\delta,\bar z_\delta]$. Since $y_\delta'(t)\in-\partial H_\varepsilon^+(0)$ a.e., we have

$$\big\langle y_\delta'(t),\bar p\big\rangle\geqslant-H_\varepsilon^+(\bar p)=0 \quad \text{a.e.},$$

so that $\langle(y_\delta(t)-\bar y_\delta),\bar p\rangle\geqslant 0$ for any $t\geqslant 0$. Thus $y_\delta(\theta_\delta^-)\notin\,]\bar y_\delta-T\xi,\bar y_\delta[$.

It remains to show that $y_\delta(\theta_\delta^-)\neq\bar y_\delta$. For this it is enough to prove that, if $y_\delta(\theta)=\bar y_\delta$ for some $\theta\geqslant 0$, then there is some $\sigma>0$ such that $y_\delta(\theta+s)\in Q_\delta$ for any $s\in[0,\sigma]$. Let us fix $\theta\geqslant 0$ such that $y_\delta(\theta)=\bar y_\delta$. Since the open ball $\mathring{B}(\bar x,|\bar y_\delta-\bar x|)$ is contained in $\{w>-\delta\}$, we have $y_\delta(t)\notin\mathring{B}(\bar x,|\bar y_\delta-\bar x|)$ for any $t\geqslant 0$. We now consider two cases. If $\langle(\bar y_\delta-\bar x),\xi\rangle<0$, then there is some $\eta>0$ such that

$$\big\{z\in B(\bar y_\delta,\eta)\bigm| z\notin\mathring{B}\big(\bar x,|\bar y_\delta-\bar x|\big) \text{ and } \big\langle(z-\bar y_\delta),\bar p\big\rangle\geqslant 0\big\}\subset Q_\delta.$$

Since, for $s>0$ small, the point $y_\delta(\theta+s)$ belongs to the set in the left-hand side, we have $y_\delta(\theta+s)\in Q_\delta$ for any $s\in[0,\sigma]$ for some $\sigma>0$. Let us now suppose that $\langle(\bar y_\delta-\bar x),\xi\rangle=0$. This implies that $\bar y_\delta-\bar x=-|\bar y_\delta-\bar x|\bar p$. Let z be an accumulation point of $(y_\delta(t)-\bar y_\delta)/(t-\theta)$ as $t\to\theta^+$. Then $z\in-\partial H_\varepsilon^+(0)$. In particular $\langle z,\bar p\rangle\geqslant-H_\delta^+(\bar p)=0$. Since $y_\delta(t)\notin\mathring{B}(\bar x,|\bar y_\delta-\bar x|)$ for any $t\geqslant 0$, we also have

$$0\leqslant\big\langle(\bar y_\delta-\bar x),z\big\rangle\geqslant-|\bar y_\delta-\bar x|\langle\bar p,z\rangle.$$

Hence $\langle z,\bar p\rangle=0$ and $z=\lambda\xi$ for some $\lambda\in\mathbb{R}$. Since $\partial H_\varepsilon^+(\bar p)\subset B(\xi,\varepsilon)$ we can find $\eta>0$ such that $\partial H_\varepsilon^+(\bar p-h\xi)\subset B(\xi,2\varepsilon)$ for any $h\in[0,\eta]$. So, for any $h\in(0,\eta]$, we have

$$\big\langle z,(\bar p-h\xi)\big\rangle\geqslant-H_\varepsilon^+(\bar p-h\xi)=-H_\varepsilon^+(\bar p)+\big\langle\big(\bar p-(\bar p-h\xi)\big),q_h\big\rangle$$

for some $q_h\in\partial H_\varepsilon^+(\bar p+h\xi)\subset B(\xi,2\varepsilon)$. Thus

$$-\lambda h|\xi|^2=\big\langle z,(\bar p-h\xi)\big\rangle\geqslant h\langle\xi,q_h\rangle \quad \forall h\in(0,\eta],$$

which entails that $\lambda\leqslant-|\xi|^2+2\varepsilon|\xi|<0$. Therefore we have proved that any limit point z of $(y_\delta(t)-\bar y_\delta)/(t-\theta)$ as $t\to\theta^+$ is of the form $z=\lambda\xi$ where $\lambda<0$. The definition of Q_δ then easily implies that there is some $\sigma>0$ such that $y_\delta(\theta+s)\in Q_\delta$ for any $s\in[0,\sigma]$. □

Step 4. We now note that $-w$ is a solution to $-H(-Dw)=0$. Arguing as above with $-w$, $-H(-\cdot)$ and $\bar z_\delta$ instead of w, H and $\bar y_\delta$, we can find some absolutely continuous arc $z_\delta:[0,+\infty)\to\mathbb{R}^2$ starting from $\bar z_\delta$ such that $w(z_\delta(t))\geqslant\delta$ for any $t\geqslant 0$ and such that, if we set

$$\theta_\delta^+ = \inf\{t \geqslant 0 \mid z_\delta(t) \notin Q_\delta\},$$

then θ_δ^+ is finite and $z_\delta(\theta_\delta^+) \in [\bar{y}_\delta, \bar{y}_\delta - T\xi] \cup [\bar{z}_\delta - T\xi, \bar{y}_\delta - T\xi]$. Since $w(y_\delta(t)) \leqslant -\delta$ and $w(z_\delta(t)) \geqslant \delta$ for any $t \geqslant 0$, $y_\delta([0, \theta_\delta^-]) \cap z_\delta([0, \theta_\delta^+]) = \emptyset$. Since we are in the plane, this implies that $y_\delta(\theta_\delta^-) \in [\bar{z}_\delta - T\xi, \bar{y}_\delta - T\xi]$ and $z_\delta(\theta_\delta^+) \in [\bar{z}_\delta - T\xi, \bar{y}_\delta - T\xi]$. Letting $\delta \to 0^+$, the maps y_δ and z_δ converge, up to subsequence, to some absolutely continuous maps y and z, while $\theta_\delta^- \to \theta^-$ and $\theta_\delta^+ \to \theta^+$ with $y(t) \in [\bar{x}, \bar{x} - T\xi]$ for any $t \in [0, \theta^-]$ and $y(\theta^-) = \bar{x} - T\xi$, while $z(t) \in [\bar{x}, \bar{x} - T\xi]$ for any $t \in [0, \theta^+]$ and $z(\theta^+) = \bar{x} - T\xi$. Moreover, $w(y(t)) \leqslant w(\bar{x})$ for any $t \in [0, \theta^-]$ while $w(z(t)) \geqslant w(\bar{x})$ for any $t \in [0, \theta^+]$. Therefore $w(\bar{x} - t\xi) = w(\bar{x})$ for any $t \in [0, T]$. This completes the proof since T is arbitrary. □

Proof of Lemma 3.2. Since G is defined on $\mathbb{R}^N$, convex and positively homogeneous, $\partial G(0)$ is a convex compact subset of $\mathbb{R}^N$. Let $z : \mathbb{R}^N \to \mathbb{R}$ be the solution to

$$\begin{cases} z_t + G(Dz) = 0 & \text{in } \mathbb{R}^N \times (0, +\infty), \\ z(\cdot, 0) = w & \text{in } \mathbb{R}^N. \end{cases}$$

Since w satisfies $G(Dw) \geqslant 0$, we have $w(x) \geqslant z(x,t)$ for any (x,t). From Lax representation formula we get

$$w(x) \geqslant z(x,t) = \min_{(x-y) \in t\partial G(0)} w(y) \quad \forall (x,t) \in \mathbb{R}^N \times (0, +\infty).$$

In particular we have proved that, for any $x \in \mathbb{R}^N$ and any $\tau > 0$, there is some $y \in \mathbb{R}^N$ such that $(x - y)/\tau \in \partial G(0)$ and $w(x) \geqslant w(y)$.

By induction, we can then show that there is a sequence $(y_n)_n$ such that $y_0 = x_0$, $y_{n+1} \in y_n - \tau\partial G(0)$ and $w(y_{n+1}) \leqslant w(y_n)$ for any $n \in \mathbb{N}$. Let $x_\tau : [0, +\infty)$ be an affine interpolation of (y_n) such that $x_\tau(n\tau) = y_n$ for any $n \in \mathbb{N}$. Note that $x'_\tau(t) \in -\partial G(0)$ for almost all $t \geqslant 0$. In particular the family x_τ is equi-Lipschitz continuous and a subsequence converge to some $x : [0, +\infty) \to \mathbb{R}^N$ such that $x(0) = x_0$, $x'(t) \in -\partial G(0)$ for almost every $t \geqslant 0$ and $w(x(t)) \leqslant w(x(s))$ whenever $t \geqslant s$. □

As a consequence of Lemma 3.1, we get the following rigidity result:

Theorem 3.3. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ satisfy* (15) *and such that*

$$\forall \bar{p} \in \mathbb{R}^2 \backslash \{0\} \text{ with } H(\bar{p}) = 0,\ DH(\bar{p}) \text{ exists and is nonzero.} \tag{21}$$

Then equation

$$H(Dw) = 0 \quad \text{in } \mathbb{R}^2 \tag{22}$$

admits a nonconstant, Lipschitz continuous, $\mathbb{Z}^2$-periodic solution if and only if there is some $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathbb{R}^2 \backslash \{0\}$ such that $H(\bar{p}) = H(-\bar{p}) = 0$ and such that either $\bar{p}_2 = 0$ or $\bar{p}_1/\bar{p}_2 \in \mathbb{Q}$.

In this case, any continuous, periodic solution w of (22) *is one-dimensional*: *namely, there is some map $\bar{w} : \mathbb{R} \to \mathbb{R}$ and some $\bar{p} \in \mathbb{R}^2 \backslash \{0\}$ with $H(\bar{p}) = H(-\bar{p}) = 0$, and either $\bar{p}_2 = 0$ or $\bar{p}_1/\bar{p}_2 \in \mathbb{Q}$, such that*

$$w(x) = \bar{w}(\langle x, \bar{p} \rangle) \quad \forall x \in \mathbb{R}^2.$$

For instance, if $H(p) = -|p_1| + \alpha|p_2|$ for some $\alpha > 0$, then Eq. (22) admits a nonconstant, $\mathbb{Z}^2$-periodic Lipschitz continuous solution if and only if $\alpha \in \mathbb{Q}$.

Proof of Theorem 3.3. Let w be a nonconstant, Lipschitz continuous and periodic solution of (22). Then there is some point of differentiability $\bar{x} \in \mathbb{R}^2$ of w such that $\bar{p} := Dw(\bar{x}) \neq 0$. From Lemma 3.1, we have

$$w(\bar{x} - t\xi) = w(\bar{x}) \quad \forall t \geqslant 0 \tag{23}$$

where $\xi = DH(Dw(\bar{x}))$. Let us consider the set

$$E = \{x \in \mathbb{R}^2/\mathbb{Z}^2 \mid \exists t_n \to +\infty,\ \bar{x} - t_n\xi \to x \text{ in } \mathbb{R}^2/\mathbb{Z}^2\}.$$

Then E is a closed subset of $\mathbb{R}^2/\mathbb{Z}^2$ and $w = w(\bar{x})$ on E. Moreover E satisfies

$$x - t\xi \in E \quad \forall x \in E,\ \forall t \in \mathbb{R}.$$

It is known that a set of the form $\{x - t\xi \mid t \in \mathbb{R}\}$ is dense in $\mathbb{R}^2/\mathbb{Z}^2$ if and only if ξ has rationally independent coordinates. In this case, w has to be constant, which contradicts of assumption. So the pair (ξ_1, ξ_2) is not rationally independent, which amounts to saying that either $\xi_2 = 0$ or $\xi_1/\xi_2 \in \mathbb{Q}$. Since $\langle \bar{p}, \xi\rangle = 0$ and $\bar{p} \neq 0$, this also implies that either $\bar{p}_2 = 0$ or $\bar{p}_1/\bar{p}_2 \in \mathbb{Q}$.

We now claim that w is one-dimensional. Indeed let $\bar{x}'$ be another point of differentiability of w such that $w(\bar{x}') \neq w(\bar{x})$ and $\bar{p}' = Dw(\bar{x}') \neq 0$. Let $\xi' = DH(\bar{p}')$. According to the previous discussion, we have either $\xi_2' = 0$ or $\xi_1'/\xi_2' \in \mathbb{Q}$. Let

$$E' = \{x \in \mathbb{R}^2/\mathbb{Z}^2 \mid \exists t_n \to +\infty,\ \bar{x}' - t_n\xi' \to x \text{ in } \mathbb{R}^2/\mathbb{Z}^2\}.$$

As before we have $w = w(\bar{x}')$ in E'. From the particular form of ξ and ξ' we also have

$$E = \bar{x} + \mathbb{R}\xi \quad \text{and} \quad E' = \bar{x}' + \mathbb{R}\xi'.$$

Note that $E \cap E' = \emptyset$ because $w = w(\bar{x})$ on E and $w = w(\bar{x}')$ on E' and $w(\bar{x}) \neq w(\bar{x}')$. So ξ and ξ' are parallel, which implies that $\bar{p}$ and $\bar{p}'$ are also parallel: indeed we have $\langle \xi, \bar{p}\rangle = 0$ and $\langle \xi', \bar{p}'\rangle = 0$ and we are in dimension 2. This shows that any level-set of w is invariant by the flow $x \to x + t\xi$, which implies that w is one-dimensional: there is some $\bar{w} : \mathbb{R} \to \mathbb{R}$ such that $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ for any x.

We now check that $H(-\bar{p}) = 0$. Indeed, otherwise, one has $H(-\lambda\bar{p}) \neq 0$ for any $\lambda > 0$ because H is homogeneous. Since $H(Dw) = H(\bar{w}'\bar{p}) = 0$, this implies that, for almost all $s \in \mathbb{R}$, $\bar{w}'(s) \geqslant 0$. Hence $\bar{w}$ is non-nondecreasing, which contradicts the assumption that w is periodic and nonconstant.

Conversely, let us assume that there exists $\bar{p} = (\bar{p}_1, \bar{p}_2) \in \mathbb{R}^2\backslash\{0\}$ such that $H(\bar{p}) = H(-\bar{p}) = 0$, and either $\bar{p}_2 \neq 0$ or $\bar{p}_1/\bar{p}_2 \in \mathbb{Q}$. Let $(a, b) \in \mathbb{Z} \times \mathbb{N}$ with either $p_2 = b = 0$ or $a/b = \bar{p}_1/\bar{p}_2$. Then

$$w(x) = \sin(\langle (a, b), x\rangle)$$

is a periodic, nonconstant Lipschitz continuous solution of $H(Dw) = 0$ because w is smooth and

$$H(Dw(x)) = \left(\frac{b}{\bar{p}_2}\right)^k H(\cos(\langle (a, b), x\rangle)\bar{p}) = 0 \quad \forall x \in \mathbb{R}^2. \qquad \square$$

4. Ergodicity

In this section we investigate conditions under which Eqs. (7) and (8) have an ergodic behavior. Recall that $u = u(x, t)$ is the solution of the evolution equation (7) while, for any $\lambda > 0$, $v_\lambda = v_\lambda(x)$ is the solution of (8).

Let $H : \mathbb{R}^2 \to \mathbb{R}$ be locally Lipschitz continuous such that there is some $k \geqslant 1$ and some k-positively homogeneous, locally Lipschitz continuous Hamiltonian $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ with

$$\lim_{s \to 0^+} s^k H(p/s) = H_\infty(p) \quad \text{locally uniformly in } p.$$

We also assume that H_∞ satisfies

$$\forall \bar{p} \in \mathbb{R}^2\backslash\{0\} \text{ with } H_\infty(\bar{p}) = 0,\ DH_\infty(\bar{p}) \text{ exists and is nonzero.} \tag{24}$$

In view of Theorem 3.3 we introduce the notation:

$$\mathcal{P} = \{p = (p_1, p_2) \in \mathbb{R}^2 \mid |p| = 1,\ H_\infty(p) = H_\infty(-p) = 0,\ [p_2 = 0 \text{ or } p_1/p_2 \in \mathbb{Q}]\}. \tag{25}$$

Then equation $H_\infty(Dw) = 0$ admits a nonconstant, Lipschitz continuous, $\mathbb{Z}^2$-periodic solution if and only if $\mathcal{P} \neq \emptyset$. In particular, if $\mathcal{P} = \emptyset$, then combining Lemma 2.3 and Theorem 3.3 readily entails the convergence of the (λv_λ) towards a constant as $\lambda \to 0$. Hence condition $\mathcal{P} = \emptyset$ can be understood as a nonresonance condition.

Very surprizingly, ergodicity actually holds under a much weaker assumption. Namely:

Theorem 4.1. *Assume that either $\mathcal{P} = \emptyset$ or that*

$$\forall p \in \mathcal{P}, \quad \lim_{s \to 0^+} \left| H\left(\frac{p}{s}\right)\right| = +\infty. \tag{26}$$

Then the (λv_λ) and the $u(t,\cdot)/t$ converge to the same constant as $\lambda \to 0$ and $T \to +\infty$.

For instance, if H_∞ is some k-positively homogeneous Hamiltonian (for some $k > 1$) satisfying (24) and $H(p) = H_\infty(p + a)$ where $\langle DH_\infty(p), a\rangle \neq 0$ for any $p \in \mathcal{P}$, then (26) holds, because

$$\begin{aligned}\left| H\left(\frac{p}{s}\right)\right| &= (1/s)^k \left|H_\infty(p + sa)\right| = (1/s)^k \left|H_\infty(p) + s\langle DH_\infty(p), a\rangle + o(s)\right| \\ &= (1/s)^{k-1}\left|\langle DH_\infty(p), a\rangle + o(1)\right| \to +\infty \quad \text{as } s \to 0^+.\end{aligned}$$

Proof of Theorem 4.1. We first analyse the behaviour of the (λv_λ). For this we assume that $\ell : \mathbb{R}^2 \to \mathbb{R}$ is Lipschitz continuous. This assumption is removed later.

Let w be the uniform limit of some sequence $(\lambda_n v_{\lambda_n})$ where $\lambda_n \to 0$. Let us assume that w is not constant. Since the λv_λ are uniformly Lipschitz continuous and $\mathbb{Z}^2$-periodic, so is w. Since, from Lemma 2.2, w is a solution of $H_\infty(Dw) = 0$, Theorem 3.3 states that there is some $\bar{p} \in \mathcal{P}$ and some continuous map $\bar{w} : \mathbb{R} \to \mathbb{R}$ such that $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ for any $x \in \mathbb{R}^2$. To fix the ideas, let us assume for instance that

$$\lim_{s \to 0^+} H\left(\frac{\bar{p}}{s}\right) = +\infty.$$

We claim that, for any smooth test function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\bar{w} - \varphi$ has a strict local maximum at some point $\bar{z}$, we have $\varphi'(\bar{z}) \leqslant 0$. Indeed assume on the contrary that $\varphi'(\bar{z}) > 0$. From standard perturbation arguments, there is a sequence (x_n) such that $\lambda_n v_{\lambda_n} - \varphi(\langle \bar{p}, \cdot\rangle)$ has a maximum at x_n and $\langle x_n, \bar{p}\rangle \to \bar{z}$. Then, setting $z_n = \langle x_n, \bar{p}\rangle$, we have

$$\varphi(z_n) + H\left(\frac{1}{\lambda_n}\varphi'(z_n)\bar{p}\right) - \ell(x_n) \leqslant 0.$$

Since $\varphi'(z_n) \to \varphi'(\bar{z}) > 0$, our assumption states that

$$\lim_n H\left(\frac{1}{\lambda_n}\varphi'(z_n)\bar{p}\right) = +\infty,$$

which leads to a contradiction since $\varphi(z_n) \to \varphi(\bar{z})$ and ℓ is bounded.

So for any smooth test function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\bar{w} - \varphi$ has a strict local maximum at some point $\bar{z}$, we have $\varphi'(\bar{z}) \leqslant 0$. This implies that $\bar{w}$ is nonincreasing. But the function $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ is $\mathbb{Z}^2$-periodic. Hence w is constant. Thanks to Lemma 2.3 we can now complete the proof of the convergence of the (λv_λ) towards a constant in the case where ℓ is Lipschitz continuous.

If ℓ is only continuous, we proceed by approximation: Let (ℓ_k) be a sequence of smooth periodic functions converging to ℓ as $k \to +\infty$. Let v_λ^k be the unique bounded solution to

$$\lambda v_\lambda^k + H\left(Dw_\lambda^k\right) - \ell_k = 0 \quad \text{in } \mathbb{R}^2.$$

Then from comparison principle we have:

$$\left\|\lambda v_\lambda^k - \lambda v_\lambda\right\|_\infty \leqslant \|\ell_k - \ell\|_\infty \quad \forall k \geqslant 0,\ \forall \lambda > 0. \tag{27}$$

Since ℓ_k is smooth, we already know that the (λv_λ^k) converge to a constant c_k as $\lambda \to 0^+$. From (27) we easily see that (c_k) is a Cauchy sequence and hence converges to some $c \in \mathbb{R}$. Since

$$\|c - \lambda v_\lambda\|_\infty \leqslant |c - c_k| + \left\|c_k - \lambda v_\lambda^k\right\|_\infty + \left\|\lambda v_\lambda^k - \lambda v_\lambda\right\|_\infty,$$

inequality (27) shows that the (λv_λ) converge to c as $\lambda \to 0^+$.

We now consider the convergence of the $u(t,\cdot)/t$. Let us denote by $c \in \mathbb{R}$ the limit of the λv_λ. The proof is then standard: let us fix $\varepsilon > 0$ and let λ sufficiently small so that $\|\lambda v_\lambda - c\|_\infty \leqslant \varepsilon$. Let

$$Z(x,t) = v_\lambda + (c-\varepsilon)t - \|v_\lambda\|_\infty.$$

Then Z is a sub-solution of (7) because $Z(x,0) \leqslant 0$ and

$$Z_t + H(DZ) - \ell = c - \varepsilon + H(Dv_\lambda) - \ell = c - \varepsilon - \lambda v_\lambda \leqslant 0.$$

By comparison, we have $u(x,t) \geqslant Z(x,t)$ for any $(x,t) \in \mathbb{R}^2 \times [0,+\infty)$. Thus

$$\liminf_{t\to+\infty} \min_{x\in\mathbb{R}^2} \frac{u(x,t)}{t} \geqslant \liminf_{t\to+\infty} \min_{x\in\mathbb{R}^2} \frac{Z(x,t)}{t} = c - \varepsilon.$$

In the same way one can show that

$$\limsup_{t\to+\infty} \max_{x\in\mathbb{R}^2} \frac{u(x,t)}{t} \leqslant c + \varepsilon,$$

which completes the proof since ε is arbitrary. □

Remark 4.2. In fact we have proved the following result: if there is a sequence $\lambda_n \to 0^+$, $\bar{p} \in \mathcal{P}$ and a nonconstant, periodic map $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ such that the sequence $(\lambda_n v_{\lambda_n})$ uniformly converges to w, then

$$\liminf_{s\to0^+} \left| H\left(\frac{\bar{p}}{s}\right)\right| < +\infty \quad \text{and} \quad \liminf_{s\to0^+} \left| H\left(\frac{-\bar{p}}{s}\right)\right| < +\infty.$$

We use this remark in the next section.

Application to homogenization. Theorem 4.1 can be applied to the homogenization of HJ equations of the form

$$\begin{cases} z_t^\varepsilon + H\big(Dz^\varepsilon(x)\big) - \ell(x/\varepsilon) = 0 & \text{in } \mathbb{R}^2 \times (0,T), \\ z^\varepsilon(x,0) = z_0(x) & \text{in } \mathbb{R}^2 \end{cases} \tag{28}$$

where H and ℓ are as above.

Corollary 4.3. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a Lipschitz continuous map. Let us assume that* (24) *holds with $k = 1$ and that $\mathcal{P} = \emptyset$, where $\mathcal{P}$ is defined by* (25). *Then there is a Lipschitz continuous Hamiltonian $h : \mathbb{R}^2 \to \mathbb{R}$ such that, for any bounded uniformly continuous map $z_0 : \mathbb{R}^2 \to \mathbb{R}$, the solution z^ε to* (28) *uniformly converges to the solution z of*

$$\begin{cases} z_t + h\big(Dz(x)\big) = 0 & \textit{in } \mathbb{R}^2 \times (0,T), \\ z(x,0) = z_0(x) & \textit{in } \mathbb{R}^2. \end{cases}$$

Remark. We do not known if the result holds true when H is only locally Lipschitz continuous.

Proof of Corollary 4.3. For any $p \in \mathbb{R}^2$, let v_λ^p be the unique continuous $\mathbb{Z}^2$-periodic solution of

$$\lambda v_\lambda^p + H\big(Dv_\lambda^p + p\big) - \ell = 0 \quad \text{in } \mathbb{R}^2.$$

From Theorem 4.1, we know that λv_λ^p uniformly converges to some constant that we denote $-h(p)$. Since H is Lipschitz continuous, then so is h because, for any $p, p' \in \mathbb{R}^2$, we have by comparison principle:

$$\big\|\lambda v_\lambda^p - \lambda v_\lambda^{p'}\big\|_\infty \leqslant \mathrm{Lip}(H)\big|p - p'\big|.$$

The rest of the proof is standard. □

5. Case of resonance

Let again $u = u(x,t)$ and $v_\lambda = v_\lambda(x)$ denote the solution to (7) and (8) respectively. In this section we investigate the existence of a limit for u/t and λv_λ in case of resonance.

For this we assume that $H : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous and that there is some 1-positively homogeneous, Lipschitz continuous Hamiltonian $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{s\to 0^+} sH(p/s) = H_\infty(p) \quad \text{locally uniformly in } p.$$

We also assume that H_∞ satisfies

$$\forall \bar p \in \mathbb{R}^2\backslash\{0\} \text{ with } H_\infty(\bar p) = 0,\ DH_\infty(\bar p) \text{ exists and is nonzero.} \tag{29}$$

We still use the notation:

$$\mathcal{P} = \bigl\{p = (p_1, p_2) \in \mathbb{R}^2 \bigm| |p| = 1,\ H_\infty(p) = H_\infty(-p) = 0,\ [p_2 = 0 \text{ or } p_1/p_2 \in \mathbb{Q}]\bigr\}$$

and we denote by $\mathcal{P}_0$ the subset of $p \in \mathcal{P}$ such that

$$\liminf_{s\to 0^+} \left| H\left(\frac{p}{s}\right)\right| < +\infty \quad \text{and} \quad \liminf_{s\to 0^+} \left| H\left(\frac{-p}{s}\right)\right| < +\infty.$$

We have seen in Theorem 4.1 that, if $\mathcal{P}_0 = \emptyset$, then the (u/t) and (λv_λ) converge to a constant. In order to investigate the resonant case, we assume that $\mathcal{P}_0 \neq \emptyset$ and that, for any $\bar p \in \mathcal{P}_0$, there are $\alpha(\bar p) \in \mathbb{R}^2\backslash\{0\}$ and $\beta(\bar p) \in \mathbb{R}$ such that, for any $M > 0$, the convergence

$$\lim_{s\to 0^+} H\left(\frac{\theta \bar p}{s} + b\right) = \bigl\langle \alpha(\bar p), b\bigr\rangle + \beta(\bar p) \tag{30}$$

holds uniformly with respect to $\theta \geqslant 1/M$ and $b \in \mathbb{R}^2$ with $|b| \leqslant M$.

Example. Let us assume that $H_\infty : \mathbb{R}^2 \to \mathbb{R}$ is positively homogeneous, satisfies (29) and that $\mathcal{P} \neq \emptyset$. Let $H(p) = H_\infty(p + a)$ for some $a \in \mathbb{R}^2$. Then (30) holds because $\mathcal{P}_0 = \mathcal{P}$ and

$$\begin{aligned} H\left(\frac{\theta \bar p}{s} + b\right) &= \frac{\theta}{s} H_\infty\left(\bar p + \frac{s}{\theta}(a+b)\right) = \frac{\theta}{s}\left(H_\infty(\bar p) + \frac{s}{\theta}\bigl\langle DH_\infty(\bar p), (a+b)\bigr\rangle + o\left(\frac{s}{\theta}\right)\right) \\ &= \bigl\langle \alpha(\bar p), b\bigr\rangle + \beta(\bar p) + o(1) \end{aligned}$$

where $\alpha(\bar p) = DH_\infty(\bar p) \neq 0$ and $\beta(\bar p) = \langle DH_\infty(\bar p), a\rangle$.

Our main result in the case of resonance is the following:

Theorem 5.1. *Under the above assumptions, there is a continuous $\mathbb{Z}^2$-periodic function $w : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$w(x) = \lim_{t\to+\infty} \frac{u(x,t)}{t} = \lim_{\lambda\to 0^+} v_\lambda(x) \quad \textit{uniformly w.r.t. } x \in \mathbb{R}^2. \tag{31}$$

In order to describe the limit function w, we need a remark and some terminology.

Remark 5.2. From assumption (30), we have $\langle \alpha(\bar p), \bar p\rangle = 0$. In particular, since $\bar p \in \mathcal{P}$ and $\alpha(\bar p) \neq 0$, we have $\alpha(\bar p) = (\alpha_1, \alpha_2)$ with either $\alpha_2 = 0$ or $\alpha_1/\alpha_2 \in \mathbb{Q}$. Hence there is some $T(\bar p) > 0$ such that $T(\bar p)\alpha(\bar p) \in \mathbb{Z}^2$.

We will say that a map $\bar w : \mathbb{R} \to \mathbb{R}$ has a point of increase at $\bar s \in \mathbb{R}$ if there is a sequence $s_n \to \bar s$ and a sequence $h_n \to 0^+$ such that $\bar w(s_n + h_n) > \bar w(s_n)$. The map $\bar w$ has a point of decrease at $\bar s$ if $-\bar w$ has a point of increase at $\bar s$.

Proposition 5.3. *Let w be defined by* (31). *There is some $\bar p \in \mathcal{P}_0$ and some continuous map $\bar w : \mathbb{R} \to \mathbb{R}$ such that*

$$w(x) = \bar w\big(\langle \bar p, x\rangle\big) \quad \forall x \in \mathbb{R}^2.$$

Moreover, we have at any point of increase s of $\bar w$:

$$\bar w(s) = \frac{1}{T(\bar p)} \int_0^{T(\bar p)} \ell\big(s\bar p + t\alpha(\bar p)\big)\, ds - \beta(\bar p),$$

while at any point of decrease s we have:

$$\bar w(s) = \frac{1}{T(\bar p)} \int_0^{T(\bar p)} \ell\big(s\bar p + t\alpha(\bar p)\big)\, ds - \beta(-\bar p).$$

The proofs of Theorem 5.1 and of Proposition 5.3 require several steps. From now on we assume that ℓ is a smooth function. This restriction is removed at the end of the proof. Recall that $w_\lambda = \lambda v_\lambda$. Let $\mathcal{W}$ be the set of cluster points in the uniform topology of w_λ as $\lambda \to 0$. The key step in the proof amounts to show that $\mathcal{W}$ consists in a singleton. From Lemma 2.2 we know that $H_\infty(Dw) = 0$ for any $w \in \mathcal{W}$.

Lemma 5.4. *There is some $\bar p \in \mathcal{P}_0$ such that, for any $w \in \mathcal{W}$, $w(x) = \bar w(\langle \bar p, x\rangle)$ for some $\bar w : \mathbb{R} \to \mathbb{R}$.*

Proof. If $\mathcal{W}$ contains a constant function $\bar w$, then Lemma 2.3 states that $\mathcal{W} = \{\bar w\}$, and the result is obvious. Otherwise, for any $\bar p \in \mathcal{P}_0$, let $\mathcal{W}_{\bar p}$ be the set of $w \in \mathcal{W}$ such that $w(x) = \bar w(\langle \bar p, x\rangle)$ for some $\bar w : \mathbb{R} \to \mathbb{R}$. Combining Theorem 3.3 and Remark 4.2, we have

$$\bigcup_{\bar p \in \mathcal{P}_0} \mathcal{W}_{\bar p} = \mathcal{W}.$$

Moreover the $\mathcal{W}_{\bar p}$ are closed in $\mathcal{W}$. Since we work in the plane, for any $\bar p \neq \bar p' \in \mathcal{P}_0$, either $\bar p = -\bar p'$, in which case $\mathcal{W}_{\bar p} = \mathcal{W}_{\bar p'}$, or the set $\mathcal{W}_{\bar p} \cap \mathcal{W}_{\bar p'}$ only consists of constant functions. So we actually have either $\mathcal{W}_{\bar p} = \mathcal{W}_{\bar p'}$ or $\mathcal{W}_{\bar p} \cap \mathcal{W}_{\bar p'} = \emptyset$. Since the set $\mathcal{W}$ is connected, this implies that $\mathcal{W}_{\bar p} = \mathcal{W}$ for some $\bar p \in \mathcal{P}_0$. □

From now on we fix $\bar p$ as in Lemma 5.4. In order to simplify the notations, we set $\alpha^\pm = \alpha(\pm \bar p)$, $T^\pm = T(\pm\bar p)$ and $\beta^\pm = \beta(\pm\bar p)$. We note that, since $\langle \alpha^+, \bar p\rangle = \langle \alpha^-, \bar p\rangle = 0$, the vectors α^+ and α^- are in fact proportional. Finally we set $\hat\alpha^\pm = \alpha^\pm/|\alpha^\pm|$. Recall that $|\bar p| = 1$.

The first step of the proof consists in building special sub- and super-solutions of the Hamilton–Jacobi equation (8). Let us set

$$\bar c(s) = \frac{1}{T^+} \int_0^{T^+} \ell\big(s\bar p + t\alpha^+\big)\, dt \quad \forall s \in \mathbb{R}, \tag{32}$$

$$\chi^\pm(x) = \int_0^{\langle \hat\alpha^\pm, x\rangle/|\alpha^\pm|} \ell\big(\big(\langle \bar p, x\rangle\big)\bar p + t\alpha^\pm\big)\, dt - \frac{\langle \hat\alpha^\pm, x\rangle}{|\alpha^\pm|} \bar c\big(\langle \bar p, x\rangle\big) \quad \forall x \in \mathbb{R}^2. \tag{33}$$

We note that $\bar c(s)$ is the average of ℓ on the set $\langle \bar p, x\rangle = s$.

Lemma 5.5. *The $\chi^\pm$ are smooth and the maps $x \to \bar c(\langle \bar p, x\rangle)$ and $\chi^\pm$ are $m\mathbb{Z}^2$-periodic for some $m \in \mathbb{N}^*$. Moreover,*

$$\big\langle \alpha^\pm, D\chi^\pm(x)\big\rangle = \ell(x) - \bar c\big(\langle \bar p, x\rangle\big) \quad \forall x \in \mathbb{R}^2. \tag{34}$$

Proof. By definition of T^+, $(a,b) := T^+\alpha^+ \in \mathbb{Z}^2\backslash\{0\}$. Let us assume to fix the ideas that $b \neq 0$ (the proof being easier if $b = 0$). Since $\langle\alpha^+, \bar{p}\rangle = 0$, we have $\bar{p}_2 = -\bar{p}_1 a/b$, and, since $|\bar{p}| = 1$, $\bar{p}_1 = \pm(1 + (a/b)^2)^{-\frac{1}{2}}$. We also note that $\hat{\alpha}^+/|\alpha^+| = T^+(a,b)/(a^2+b^2)$. Setting $m = (a^2+b^2) \in \mathbb{N}^*$ we get

$$\big(\langle\bar{p}, k\rangle\big)\bar{p} \in \mathbb{Z}^2 \quad \text{and} \quad \frac{\langle\hat{\alpha}^+, k\rangle}{|\alpha^+|} \in T^+\mathbb{Z} \quad \forall k \in m\mathbb{Z}^2.$$

This proves the $m\mathbb{Z}^2$-periodicity of $x \to \bar{c}(\langle\bar{p}, x\rangle)$ and of χ^+. The periodicity of χ^- can be established with similar arguments (changing m if necessary), and assertion (34) is straightforward. □

Lemma 5.6. *Let $\varepsilon \in \{+,-\}$ and $\varphi : [c,d] \to \mathbb{R}$ be a smooth function such that $\varepsilon\varphi' > 0$ and $\varphi \leqslant \bar{c} - \beta^\varepsilon$ in $[c,d]$. Then, for any $\eta > 0$ there is some $\bar{\lambda} > 0$ such that, for any $\lambda \in (0, \bar{\lambda})$, the map*

$$x \to \frac{1}{\lambda}\big(\varphi\big(\langle\bar{p}, x\rangle\big) - \eta\big) + \chi^\varepsilon(x) - \|\chi^\varepsilon\|_\infty$$

is a sub-solution with state-constraints of (8) *in the set $\{x \in \mathbb{R}^2 \mid c \leqslant \langle\bar{p}, x\rangle < d\}$ if $\varepsilon = +$ and in the set $\{x \in \mathbb{R}^2 \mid c < \langle\bar{p}, x\rangle \leqslant d\}$ if $\varepsilon = -$.*

Proof. Let us assume for instance that $\varepsilon = -$ and that $\varphi : [c,d] \to \mathbb{R}$ is a smooth function such that $\varphi' < 0$ and $\varphi \leqslant \bar{c} - \beta^-$ in $[c,d]$. For any $\lambda > 0$, let us set

$$\zeta_\lambda(x) = \frac{1}{\lambda}\big(\varphi\big(\langle\bar{p}, x\rangle\big) - \eta\big) + \chi^-(x) - \|\chi^-\|_\infty \quad \forall x \in \mathbb{R}^2.$$

Let $M > 0$ be such that

$$\|\chi^-(x)\|_\infty \leqslant M \quad \text{and} \quad \varphi'(t) \leqslant -\frac{1}{M} \quad \forall t \in [c,d].$$

From assumption (30) we can find $\bar{\lambda} > 0$ such that

$$H\left(-\frac{\theta\bar{p}}{\lambda} + b\right) \leqslant \langle\alpha^-, b\rangle + \beta^- + \eta \quad \forall |b| \leqslant M,\ \theta \geqslant \frac{1}{M},\ \lambda \in (0, \bar{\lambda}).$$

Then for any $\lambda \in (0, \bar{\lambda})$ and at any point $x \in \mathbb{R}^2$ with $c < \langle\bar{p}, x\rangle < d$, we have

$$\begin{aligned}
\lambda\zeta_\lambda + H(D\zeta_\lambda) - \ell &= \varphi - \eta + \lambda\big(\chi^- - \|\chi^-\|_\infty\big) + H\left(\frac{\phi'}{\lambda}\bar{p} + D\chi^-\right) - \ell \\
&\leqslant \varphi - \eta + \big(\langle\alpha^-, D\chi^-\rangle + \beta^- + \eta\big) - \ell \\
&\leqslant \varphi - \eta + \big(\ell - \bar{c} + \beta^- + \eta\big) - \ell \leqslant 0
\end{aligned}$$

where we have used the definition of $\bar{\lambda}$, (34) and the fact that $\bar{\phi} \leqslant \bar{c} - \beta^-$ respectively.

Let now $\lambda \in (0, \bar{\lambda})$, $\bar{x} \in \mathbb{R}^2$ such that $\langle\bar{p}, \bar{x}\rangle = d$ and ψ be a smooth test function such that $\zeta_\lambda - \psi$ has a local maximum at $\bar{x}$ on the set $\{c < \langle\bar{p}, x\rangle \leqslant d\}$. Then there is some $\theta_\lambda \geqslant 0$ such that

$$D(\zeta_\lambda - \psi)(\bar{x}) = \theta_\lambda\bar{p}.$$

Arguing as above, we get

$$\begin{aligned}
\lambda\zeta_\lambda + H(D\psi) - \ell &= \varphi - \beta^- - \eta + \lambda\big(\chi^- - \|\chi^-\|_\infty\big) + H\left(\left(\frac{\phi'}{\lambda} - \theta_\lambda\right)\bar{p} + D\chi^-\right) - \ell \\
&\leqslant \varphi - \beta^- - \eta + \big(\langle\alpha^-, D\chi^-\rangle + \beta^- + \eta\big) - \ell \\
&\leqslant \varphi - \beta^- - \eta + \big(\ell - \bar{c} + \beta^- + \eta\big) - \ell \leqslant 0. \qquad \square
\end{aligned}$$

Lemma 5.7. *Let $\varepsilon \in \{+,-\}$, $w \in \mathcal{W}$, $\bar{w} : \mathbb{R} \to \mathbb{R}$ be such that $w(x) = \bar{w}(\langle\bar{p}, x\rangle)$ and $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth test function such that $\psi \leqslant \bar{w}$ (resp. $\psi \geqslant \bar{w}$) with an equality at $\bar{s} \in \mathbb{R}$. If $\varepsilon\psi'(\bar{s}) > 0$, then*

$$\bar{w}(\bar{s}) \geqslant \bar{c}(\bar{s}) - \beta^\varepsilon \quad \big(\text{resp. } \bar{w}(\bar{s}) \leqslant \bar{c}(\bar{s}) - \beta^\varepsilon\big),$$

where $\bar{c}$ is defined by (32).

Proof. To fix the ideas we work in the case $\varepsilon = +$ and suppose that $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth test function such that $\psi \leqslant \bar{w}$ with an equality at $\bar{s} \in \mathbb{R}$ and that $\psi'(\bar{s}) > 0$. Without loss of generality, we also assume that there is some $\delta \in (0, 1/2)$ such that $\psi(t) < \bar{w}(t)$ for $t \in [\bar{s} - \delta, \bar{s} + \delta] \setminus \{\bar{s}\}$, $\psi(\bar{s}) = \bar{w}(\bar{s})$ and $\psi'(t) > 0$ for $t \in [\bar{s} - \delta, \bar{s} + \delta]$. Let $\lambda_n \to 0$ be such that $(\lambda_n v_n := \lambda_n v_{\lambda_n})$ uniformly converges to w and let χ^+ be defined by (33). For n large enough, the function

$$\zeta_n(x) = v_n(x) - \left(\frac{1}{\lambda_n}\psi\big(\langle \bar{p}, x\rangle\big) + \chi^+(x)\right)$$

is $m\mathbb{Z}^2$-periodic for some $m \in \mathbb{N}^*$ in the set $\{|\langle \bar{p}, x\rangle - \bar{s}| \leqslant \delta\}$. So ζ_n has a minimum point in $\{|\langle \bar{p}, x\rangle - \bar{s}| \leqslant \delta\}$ at a point x_n such that $|\langle \alpha^+, x_n\rangle| \leqslant m$. In particular the sequence (x_n) is bounded. Since the function χ^+ is also bounded, (x_n) converges, up to a subsequence, to a maximum point of $x \to w(x) - \psi(\langle \bar{p}, x\rangle)$ in $\{|\langle \bar{p}, x\rangle - \bar{s}| \leqslant \delta\}$. Hence $(\langle \bar{p}, x_n\rangle)$ converges to $\bar{s}$ and x_n is an interior maximum point of ζ_n for any n large enough.

Since v_n is the solution of (8) we have, for n large enough,

$$\lambda_n v_n(x_n) + H\left(\frac{\psi'(\langle \bar{p}, x_n\rangle)}{\lambda_n}\bar{p} + D\chi^+(x_n)\right) - \ell(x_n) \geqslant 0.$$

Arguing as in the proof of Lemma 5.6, for any $\eta > 0$ we have, for n large enough:

$$\lambda_n v_n(x_n) - \bar{c}\big(\langle \bar{p}, x_n\rangle\big) + \beta^+ - \eta \geqslant 0.$$

Letting $n \to +\infty$ and then $\eta \to 0^+$ gives the desired result. □

Corollary 5.8. *Let $w \in \mathcal{W}$ and $\bar{w}$ be such that $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$. If $\bar{w}$ has a point of increase (resp. decrease) at $\bar{s}$, then*

$$\bar{w}(\bar{s}) = \bar{c}(\bar{s}) - \beta^+ \quad \big(\textit{resp. } \bar{w}(\bar{s}) = \bar{c}(\bar{s}) - \beta^-\big),$$

where $\bar{c}$ is defined by (32). *In particular, the range of $\bar{w}$ is contained in the intersection of the range of $s \to \bar{c}(s) - \beta^+$ and of the range of $s \to \bar{c}(s) - \beta^-$.*

Proof. Since $\bar{w}$ has a point of increase at $\bar{s}$, there are $s_n \to \bar{s}$ and $h_n \to 0^+$ such that $\bar{w}(s_n + h_n) > \bar{w}(s_n)$. Hence one can find some smooth functions ϕ_n and ψ_n and points $a_n, b_n \in (s_n, s_n + h_n)$ such that

- $\phi_n \geqslant \bar{w}$ on $(s_n, s_n + h_n)$ with an equality at a_n and $\phi_n'(a_n) > 0$,
- $\psi_n \leqslant \bar{w}$ on $(s_n, s_n + h_n)$ with an equality at b_n and $\psi_n'(b_n) > 0$.

From Lemma 5.7 we have

$$\bar{w}(a_n) \leqslant \bar{c}(a_n) - \beta^+ \quad \text{and} \quad \bar{w}(b_n) \geqslant \bar{c}(b_n) - \beta^+.$$

Letting $n \to +\infty$ gives the result. □

Recall that the integer $m \in \mathbb{N}^*$ is given by Lemma 5.5.

Lemma 5.9. *Let $w \in \mathcal{W}$ and $\lambda_n \to 0$ be such that $(\lambda_n v_{\lambda_n})$ uniformly converges to w as $n \to +\infty$. Then for any $\theta \in (\min w, \max w)$, $\eta > 0$ sufficiently small and n large enough, there is a Lipschitz continuous, $m\mathbb{Z}^2$-periodic function $\tilde{v}_{\lambda_n}$ which is a state-constraint viscosity sub-solution of* (8) *in some closed, periodic, neighbourhood K of $\{w \geqslant \theta\}$ and such that $\lambda_n \tilde{v}_{\lambda_n} \geqslant \theta - \eta$ in K.*

Proof. Let $\bar{w} : \mathbb{R} \to \mathbb{R}$ be such that $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ and let $\theta \in (\min w, \max w)$. From Corollary 5.8 θ lies in the intersection of the ranges of $\bar{c} - \beta^+$ and of $\bar{c} - \beta^-$. Hence, perturbing slightly θ if necessary, we can assume without loss of generality that θ is a noncritical value of $\bar{c} - \beta^+$ and of $\bar{c} - \beta^-$. Let $E^+ = \{w \geqslant \theta\}$. Since $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ is periodic and $\bar{p} \in \mathcal{P}$, the set E^+ is of the form

$$E^+ = \big\{x \in \mathbb{R}^2 \bigm| \langle \bar{p}, x\rangle \in I^+\big\}$$

where I^+ is a closed, periodic subset of $\mathbb{R}$ (the period is not 1 in general). From Corollary 5.8, $w(x) = \theta \in \{\bar{c}(\langle \bar{p}, x\rangle) - \beta^+, \bar{c}(\langle \bar{p}, x\rangle) - \beta^-\}$ for any $x \in \partial E^+$. The value θ being noncritical for $\bar{c} - \beta^+$ and for $\bar{c} - \beta^-$, the sets $(\bar{c} - \beta^+)^{-1}(\theta)$ and $(\bar{c} - \beta^-)^{-1}(\theta)$ are locally finite and thus the set I^+ consists locally in a finite number of closed, disjoint intervals. Let $[a, b]$ be such an interval. Then $\bar{c}'(a) \neq 0$ and $\bar{c}'(b) \neq 0$. From Corollary 5.8 again, we can find $\eta \in (0, 1)$ such that

- $\bar{w}(s) = \bar{c}(s) - \beta^+$ and $\bar{c}'(s) > 0$ if $s \in [a - 2\eta, a]$,
- $\bar{w}(s) = \bar{c}(s) - \beta^-$ and $\bar{c}'(s) < 0$ if $s \in [b, b + 2\eta]$.

We also choose $\sigma > 0$ so small that, if we set

$$\varphi_b(s) := \bar{c}(s) - \beta^- - \sigma(b + \eta - s)^2 \quad \text{and} \quad \varphi_a(s) = \bar{c}(s) - \beta^+ - \sigma(s - a + \eta)^2,$$

then $\varphi_b' < 0$ on $[b, b + 2\eta]$ and $\varphi_a' > 0$ on $[a - 2\eta, a]$. Then Lemma 5.6 states that, for any n sufficiently large,

$$\xi_b^n(x) := \frac{1}{\lambda_n}\bigl(\varphi_b\bigl(\langle \bar{p}, x\rangle\bigr) - \eta\bigr) + \chi^-(x) - \|\chi^-\|_\infty$$

is a state-constraint sub-solution in $\{b < x - y \leqslant b + \eta\}$ while

$$\xi_a^n(x) = \frac{1}{\lambda_n}\bigl(\varphi_a\bigl(\langle \bar{p}, x\rangle\bigr) - \eta\bigr) + \chi^+(x) - \|\chi^+\|_\infty$$

is a state-constraint sub-solution in $\{a - \eta \leqslant x - y < a\}$. Let us finally set

$$\tilde{v}_{\lambda_n}(x) = \begin{cases} \max\{v_{\lambda_n}(x) - \frac{\eta + \sigma\eta^4}{\lambda_n}, \xi_b^n(x)\} & \text{if } b < \langle \bar{p}, x\rangle \leqslant b + \eta, \\ \max\{v_{\lambda_n}(x) - \frac{\eta + \sigma\eta^4}{\lambda_n}, \xi_a^n(x)\} & \text{if } a - \eta \leqslant \langle \bar{p}, x\rangle < a, \\ v_{\lambda_n}(x) - \frac{\eta + \sigma\eta^4}{\lambda_n} & \text{if } a \leqslant \langle \bar{p}, x\rangle \leqslant b. \end{cases}$$

We note that, if $\langle \bar{p}, x\rangle = b + \eta$, then

$$\lim_{n\to+\infty} \lambda_n\left(v_{\lambda_n}(x) - \frac{\eta + \sigma\eta^4}{\lambda_n}\right) = \bar{c}(b + \eta) - \beta^- - \eta - \sigma\eta^4 \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n\xi_b^n(x) = \bar{c}(b + \eta) - \beta^- - \eta,$$

while, if $\langle \bar{p}, x\rangle = b$, then

$$\lim_{n\to+\infty} \lambda_n\left(v_{\lambda_n}(x) - \frac{\eta + \sigma\eta^4}{\lambda_n}\right) = \bar{c}(b) - \beta^- - \eta - \sigma\eta^4 \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n\xi_b^n(x) = \bar{c}(b) - \beta^+ - \eta - \sigma\eta^2.$$

Since $\eta \in (0, 1)$, for n large enough we have

$$\tilde{v}_{\lambda_n}(x) = \xi_b^n(x) \quad \text{for } \langle \bar{p}, x\rangle \text{ close to } b + \eta$$

and

$$\tilde{v}_{\lambda_n}(x) = v_{\lambda_n}(x) - \bigl(\eta + \sigma\eta^4\bigr)/\lambda_n \quad \text{for } \langle \bar{p}, x\rangle \text{ close to } b.$$

Therefore $\tilde{v}_{\lambda_n}$ is a Lipschitz continuous state-constraint sub-solution of (8) in $\{a < \langle \bar{p}, x\rangle \leqslant b + \eta\}$ if n is large enough. Arguing in the same way we can show that $\tilde{v}_{\lambda_n}$ is a sub-solution of (8) with state-constraints in $\{a - \eta \leqslant \langle \bar{p}, x\rangle < b\}$ if n is large enough. Doing the same construction on each connected component of I^+ then completes the proof thanks to the periodicity of I^+. □

Proof of the existence of a limit of the (v_λ) for smooth ℓ's. Let us fix $w \in \mathcal{W}$ and let $\lambda_n \to 0$ be such that $\lambda_n v_{\lambda_n} \to \bar{w}$ uniformly. Let $m \in \mathbb{N}^*$ be defined by Lemma 5.5, $\theta \in (\min \bar{w}, \max \bar{w})$ and let $\eta > 0$ be sufficiently small. From Lemma 5.9, for n large enough one can find some Lipschitz continuous, $m\mathbb{Z}^2$-periodic function $\tilde{v}_{\lambda_n}$ which is a sub-solution with state-constraints of (8) in a closed, periodic, neighbourhood K of $\{w \geqslant \theta\}$ and which is such that $\tilde{v}_{\lambda_n} \geqslant \theta - \eta$ in K. Let us fix such a n.

For $\sigma, \mu > 0$ let us consider a minimum point $(x_{\mu,\sigma}, x'_{\mu,\sigma})$ on the set $\mathbb{R}^2 \times K$ of the function

$$\Phi_\sigma(x, x') = v_\mu(x) - \tilde{v}_{\lambda_n}(x') - \frac{1}{2\sigma}|x - x'|^2.$$

Since v_μ and $\tilde{v}_{\lambda_n}$ are $m\mathbb{Z}^2$-periodic, we can assume that $(x_{\mu,\sigma})$ and $(x'_{\mu,\sigma})$ are bounded and converge up to some subsequence to some x_μ as $\sigma \to 0$. Then x_μ is a minimum point of $v_\mu - \tilde{v}_{\lambda_n}$ on K:

$$\min_{x\in K}(v_\mu - \tilde{v}_{\lambda_n})(x) = (v_\mu - \tilde{v}_{\lambda_n})(x_\mu). \tag{35}$$

Since v_μ is a solution of (8) and $\tilde{v}_{\lambda_n}$ is a subsolution with state constraints in K of (8) we have

$$\mu v_\mu(x_{\mu,\sigma}) + H\left(\frac{1}{\sigma}(x_{\mu,\sigma} - x'_{\mu,\sigma})\right) - \ell(x_{\mu,\sigma}) \geqslant 0$$

while

$$\lambda_n \tilde{v}_{\lambda_n}(x'_{\mu,\sigma}) + H\left(\frac{1}{\sigma}(x_{\mu,\sigma} - x'_{\mu,\sigma})\right) - \ell(x'_{\mu,\sigma}) \leqslant 0,$$

where

$$|x_{\mu,\sigma} - x'_{\mu,\sigma}| \leqslant 2\left(\|v_\mu\|_\infty + \|\tilde{v}_{\lambda_n}\|_\infty\right)^{\frac{1}{2}}\sqrt{\sigma}.$$

This implies that

$$\mu v_\mu(x_{\mu,\sigma}) - \lambda_n \tilde{v}_{\lambda_n}(x'_{\mu,\sigma}) \geqslant -2\,\mathrm{Lip}(\ell)\left(\|v_\mu\|_\infty + \|\tilde{v}_{\lambda_n}\|_\infty\right)^{\frac{1}{2}}\sqrt{\sigma}.$$

Letting $\eta \to 0$ and using the definition $\tilde{v}_{\lambda_n}$ leads to

$$\mu v_\mu(x_\mu) \geqslant \lambda_n \tilde{v}_{\lambda_n}(x_\mu) \geqslant \theta - \eta. \tag{36}$$

Let now $\hat{w} \in \mathcal{W}$ and $\mu_p \to 0$ be such that $\mu_p v_{\mu_p} \to \hat{w}$ uniformly. Then

$$\min_K \hat{w} = \lim_p \min_K \mu_p v_{\mu_p} \geqslant \liminf_p \min_K \mu_p(v_{\mu_p} - \tilde{v}_{\lambda_n}) = \liminf_p \mu_p(v_{\mu_p} - \tilde{v}_{\lambda_n})(x_{\mu_p}),$$

where the last equality comes from (35). Note that, from (36), we have

$$\liminf_p \mu_p(v_{\mu_p} - \tilde{v}_{\lambda_n})(x_{\mu_p}) \geqslant \theta - \eta - \limsup_p \mu_p \tilde{v}_{\lambda_n}(x_{\mu_p}) = \theta - \eta.$$

So $\min_K \hat{w} \geqslant \theta - \eta$. Letting $\varepsilon \to 0$ then gives

$$\hat{w} \geqslant \theta \quad \text{in } \{w \geqslant \theta\}\ \forall \theta \in (\min w, \max w). \tag{37}$$

Combining the above inequality with Lemma 2.3 we get $\hat{w} \geqslant w$. Reversing the roles of w and $\hat{w}$ finally shows $w = \hat{w}$, i.e., that $\mathcal{W}$ is a singleton. □

Proof of the existence of a limit of the (v_λ) for general ℓ's. We proceed by approximation. Let (ℓ_k) be a sequence of smooth periodic functions converging to ℓ as $k \to +\infty$. Let w_λ^k be the unique bounded solution to

$$\lambda w_\lambda^k + H(Dw_\lambda^k) - \ell_k = 0 \quad \text{in } \mathbb{R}^2.$$

Then from the comparison principle we have:

$$\|w_\lambda^k - w_\lambda\|_\infty \leqslant \|\ell_k - \ell\|_\infty \quad \forall k \geqslant 0,\ \forall \lambda > 0. \tag{38}$$

Since ℓ_k is smooth, we already know that w_λ^k converges to a limit w^k as $\lambda \to 0^+$. From (38) we easily see that (w^k) is a Cauchy sequence. Hence (w^k) uniformly converges to some continuous periodic function w. Since

$$\|w - w_\lambda\|_\infty \leqslant \|w - w^k\|_\infty + \|w^k - w_\lambda^k\|_\infty + \|w_\lambda^k - w_\lambda\|_\infty,$$

inequality (38) shows that w_λ converges to w as $\lambda \to 0^+$.

Let us now assume that w is not constant. Then w^k is not constant for k large enough. Since the ℓ_k are smooth there is some $\bar{p}_k \in \mathcal{P}_0$ and some $\bar{w}_k : \mathbb{R} \to \mathbb{R}$ such that $w^k(x) = \bar{w}_k(\langle \bar{p}_k, x\rangle)$. From assumption (29) the set $\mathcal{P}_0$ is finite. So we can as well assume that $\bar{p}_k$ is constant: $\bar{p}_k = \bar{p}$ for all $k \geqslant 0$ where $\bar{p} \in \mathcal{P}_0$. We note that $\bar{w}_k$ uniformly converges to $\bar{w}$.

Let now $\bar{s}$ be a point of increase of $\bar{w}$: there exist a sequence $s_n \to s$ and a sequence $h_n \to 0^+$ such that $\bar{w}(s_n + h_n) > \bar{w}(s_n)$. Let us fix n. Then for k large enough, $\bar{w}_k(s_n + h_n) > \bar{w}_k(s_n)$. This means that there is a point of increase $t_{nk} \in (s_n, s_n + h_n)$ for $\bar{w}_k$. Hence, from Corollary 5.8

$$\bar{w}_k(t_{nk}) = \frac{1}{T(\bar{p})} \int_0^{T(\bar{p})} \ell_k\big(t_{nk}\bar{p} + t\alpha(\bar{p})\big)\, dt - \beta(\bar{p}).$$

(Indeed the quantities $T(\bar{p})$, $\alpha(\bar{p})$, $\beta(\bar{p})$ only depend on H and $\bar{p}$, which are fixed here.) Letting first $t_{nk} \to t_n \in [s_n, s_n + h_n]$ up to a subsequence as $k \to +\infty$, and then $t_n \to \bar{s}$ gives the desired equality:

$$\bar{w}(\bar{s}) = \frac{1}{T(\bar{p})} \int_0^{T(\bar{p})} \ell\big(\bar{s}\bar{p} + t\alpha(\bar{p})\big)\, dt - \beta(\bar{p}).$$

The proof of the symmetric equality in the case of decrease can be obtained in the same way. □

Proof of the existence of a limit of the $(u(\cdot,t)/t)$. We again assume that ℓ is a smooth function: this restriction can be removed exactly as for the (v_λ). Let w be the limit of the (λv_λ) and $\bar{p} \in \mathcal{P}_0$ and $\bar{w} : \mathbb{R} \to \mathbb{R}$ be such that $w(x) = \bar{w}(\langle \bar{p}, x\rangle)$ for any $x \in \mathbb{R}^2$.

We first note that

$$\min_x w(x) \leqslant \liminf_{t\to+\infty} \min_x \frac{u(x,t)}{t} \leqslant \limsup_{t\to+\infty} \min_x \frac{u(x,t)}{t} \leqslant \max_x w(x). \tag{39}$$

Indeed, let $\eta > 0$ and λ small enough such that $\lambda v_\lambda \leqslant \max w + \eta$. Then the map

$$Z(x,t) = t(\max w + \eta) + v_\lambda + \|v_\lambda\|_\infty$$

is a supersolution of (8). By comparison we get

$$\limsup_{t\to+\infty} \max_x \frac{u(x,t)}{t} \leqslant \limsup_{t\to+\infty} \max_x \frac{Z(x,t)}{t} = \max w + \eta.$$

Whence the right-hand side of (39) since η is arbitrary. The left-hand side can be proved by symmetric arguments.

Let now $\theta \in (\min w, \max w)$. From Lemma 5.9, for any $\eta > 0$ and for any $\lambda > 0$ small enough, there is some Lipschitz continuous, periodic function $\tilde{v}_\lambda$ which is state-constraint sub-solution of (8) in a closed, periodic, neighbourhood K of $\{w \geqslant \theta\}$ and which is such that $\tilde{v}_\lambda \geqslant \theta - \eta$ in K.

As in the proof of Theorem 4.1 one can check that

$$Z(x,t) = t(\theta - \eta) + \tilde{v}_\lambda(x) - \|\tilde{v}_\lambda\|_\infty$$

is a state-constraint sub-solution of (7) in $K \times (0, +\infty)$. By comparison we get

$$u(x,t) \geqslant Z(x,t) \quad \forall (x,t) \in K \times [0, +\infty).$$

This implies that

$$\liminf_{t\to+\infty} \frac{u(x,t)}{t} \geqslant \theta - \eta \quad \text{in } K.$$

Since η is arbitrary, we have

$$\liminf_{t\to+\infty} \frac{u(x,t)}{t} \geqslant \theta \quad \text{in } \{w \geqslant \theta\}\ \forall \theta \in (\min w, \max w). \tag{40}$$

In the same way, working with $-u$ instead of u, we can prove that

$$\limsup_{t\to+\infty} \frac{u(x,t)}{t} \leqslant \theta \quad \text{in } \{w \leqslant \theta\}\ \forall \theta \in (\min w, \max w). \tag{41}$$

Combining (39), (40) and (41) finally gives the equality

$$\lim_{t\to+\infty} \frac{u(x,t)}{t} = w(x) \quad \forall x \in \mathbb{R}^2. \qquad \square$$

6. Conclusion and open problems

In this paper we have addressed two questions: the first one is the existence of an ergodic limit for Hamilton–Jacobi equations with nonconvex Hamiltonians; the second one is related to the existence of a (possibly nonconstant) limit for the quantities $\frac{u(x,t)}{t}$ and $\lambda v_\lambda(x)$ as $t \to +\infty$ and $\lambda \to 0$. Although our results shed some new light on these problems in the plane, we are very far from having a complete picture.

First the case of higher dimension problems is completely open: we suspect that our rigidity result (Lemma 3.1) still holds for $N \geqslant 3$. However the consequence of such a result on the ergodic problem is not clear.

Second, even in dimension 2, our analysis is not complete:

- When $\mathcal{P} = \emptyset$, we have no characterization of the limit constant c. In particular, we do not know if one can associate to the constant c a cell-problem of the form (5).
- We are able to treat the resonant case only under a global Lipschitz continuity assumption on H: for instance we cannot deal with $H(p_1, p_2) = -p_1^2 + p_2^2$.
- Even when we know in the resonant case that there exist a limit and that this limit is of the form $w = \bar{w}(\langle \bar{p}, x\rangle)$ for some $\bar{p} \in \mathcal{P}$, we do not know how this vector $\bar{p}$ is related to the function ℓ. For instance, when $H(p_1, p_2) = -|p_1| + |p_2|$, then $\mathcal{P} = \{\pm 1, \pm 1\}$; is there a criterium on ℓ to explain that the limit is of the form $\bar{w}(x - y)$ or of the form $\bar{w}(x + y)$?
- Finally, although we can compute explicitely the limit function $\bar{w}$ at points s where $\bar{w}$ has a point of increase or of decrease, we do not know how to compute $\bar{w}$ at points where $\bar{w}$ is locally constant. In particular what are the maxima and minima of $\bar{w}$?

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