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Gelfand type elliptic problems under Steklov boundary conditions

Elvise Berchio, Filippo Gazzola *, Dario Pierotti

Dipartimento di Matematica del Politecnico, Piazza L. da Vinci 32, 20133 Milano, Italy

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Abstract

For a Gelfand type semilinear elliptic equation we extend some known results for the Dirichlet problem to the Steklov problem. This extension requires some new tools, such as non-optimal Hardy inequalities, and discovers some new phenomena, in particular a different behavior of the branch of solutions and three kinds of blow-up for large solutions in critical growth equations. We also show that small values of the boundary parameter play against strong growth of the nonlinear source.

1. Introduction

In a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$, we consider the problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u_\nu + cu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $c, \lambda > 0$ and

$$g \in C^1[0, +\infty) \text{ is a strictly positive, increasing and convex function} \tag{2}$$

such that

$$\liminf_{s\to+\infty} \frac{g'(s)s}{g(s)} > 1. \tag{3}$$

Problem (1) has wide applications to physical models. Among others, it describes problems of thermal self-ignition [15], diffusion phenomena induced by nonlinear sources [18], a ball of isothermal gas in gravitational equilibrium as proposed by lord Kelvin [8], the problem of temperature distribution in an object heated by the application of a uniform electric current [20]. We also refer to [17,23] where different models and further references may be found.

* Corresponding author.
E-mail address: filippo.gazzola@polimi.it (F. Gazzola).

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The equation in (1) has been intensively studied under *Dirichlet* boundary conditions. With no hope of being complete, let us mention the works in [3,5,9,14,17,22–24] and references therein. The main results in these papers will be recalled during the course.

Our purpose is to study problem (1). When $c = 0$ this reduces to the Neumann problem, whereas the limit case where $c \to +\infty$ may be seen as the Dirichlet problem. Steklov conditions (also called conditions of the third kind or Robin conditions when $c > 0$) are considered a more "realistic" description of the interactions at the boundary of a physical system. For example, the heat flow through the surface of a body generally depends on the value of the temperature at the surface itself.

We first prove that there exists $\lambda^* > 0$ such that problem (1) admits a solution if and only if $0 < \lambda \leqslant \lambda^*$. Then, a particular attention deserves the limiting situation where $\lambda = \lambda^*$ since, in some models, λ^* corresponds to the maximal current which can be applied to a body Ω. In this case, we show that the *extremal* solution u^* is unique. The main concern is then to establish whether it is bounded or not. This depends on the space dimension n and on the domain Ω. In general domains, partial results may be obtained by adapting the analysis in [23]. We have more precise results in the ball where, as shown in [5], one can take advantage of some *Hardy inequalities*. For the Steklov problem (1), we need to use some Hardy inequalities which do not involve the optimal Hardy constant. Moreover, the analysis of (1) when Ω is the ball shows that the solutions have new features not observable under Dirichlet boundary conditions. For instance, when $g(u) = e^u$, the solutions branch arising from $\lambda = 0$ (and $u = 0$) may bend back from λ^* until an asymptote $\lambda = \bar{\lambda} > 0$ *without* oscillating around it. Therefore, Steklov boundary conditions highlight some "further dimensions" with respect to the limit case $c = +\infty$, namely under Dirichlet conditions. We also study in some detail power-like nonlinearities g and show that small values of c play against large values of the power. In particular, for the critical growth equation, the blow-up of large solutions as $\lambda \to 0$ strongly depends on the parameter c. For small values of c blow-up occurs globally and *without* concentration as for *subcritical* problems, whereas for large values of c concentration occurs. Finally, the transition between these two situations occurs at a single value of c for which concentration is combined with global blow-up.

This paper is organized as follows. In Section 2 we state our main results which can be divided in two classes. The first kind of results (Theorems 1 and 2) are quite standard and we obtain their proofs by adapting techniques from [3,5,9,22]. The sketch of these proofs is given in Section 4. The second kind of results considers some specific nonlinearities which allow to prove more precise statements, in particular when dealing with radial solutions in the ball, see Theorems 3–5. Their proofs are postponed to Sections 5–7. Finally, Hardy inequalities with boundary terms are obtained in Section 3, see Theorem 8.

2. Main results

2.1. General nonlinearities

Throughout this paper we assume that $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ is a $C^{2,\alpha}$ bounded domain such that $0 \in \Omega$. In some cases, we restrict our attention to $\Omega = B$, the unit ball in $\mathbb{R}^n$. With $\|.\|_p$ we denote the $L^p(\Omega)$ norm $(1 \leqslant p \leqslant +\infty)$. Since problem (1) may be at supercritical growth, we cannot work within a variational framework and there is no canonical space for weak solutions to (1). Hence we set

$$X_c(\Omega) := \left\{ v \in C^2(\overline{\Omega});\ v_\nu + cv = 0 \text{ on } \partial\Omega \right\}$$

and we say that $u \in L^1(\Omega)$ is a solution of (1) if $g(u) \in L^1(\Omega)$ and

$$-\int_\Omega u \Delta v \, dx = \lambda \int_\Omega g(u) v \, dx \quad \text{for all } v \in X_c(\Omega). \tag{4}$$

Moreover, if $u \in L^\infty(\Omega)$ we say that u is *regular* while if $u \notin L^\infty(\Omega)$ we say that u is *singular*. We say that a solution u_λ of (1) is *minimal* if $u_\lambda \leqslant u$ a.e. in Ω, for any other solution u of (1). By elliptic regularity, we know that regular solutions are smooth and solve (1) in a classical sense. We have

Theorem 1. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ *be a smooth bounded domain and assume that* g *satisfies* (2) *and* (3). *Then there exists* $\lambda^* = \lambda^*(c) > 0$ *such that*:

(i) *For* $0 < \lambda < \lambda^*$ *problem* (1) *admits a minimal regular solution* u_λ *and the map* $\lambda \mapsto u_\lambda(x)$ *is strictly increasing for all* $x \in \overline{\Omega}$. *Moreover, if* u *and* v *are two distinct solutions such that* $u(x) \leqslant v(x)$ *a.e. in* Ω, *then the inequality is strict and* $u \equiv u_\lambda$.
(ii) *For* $\lambda = \lambda^*$ *problem* (1) *admits a unique solution, not necessarily regular, that belongs to the energy class* $H^1(\Omega)$.
(iii) *For* $\lambda > \lambda^*$ *problem* (1) *admits no solution.*

Furthermore, the map $c \mapsto \lambda^*(c)$ *is bounded, strictly increasing and* $\lambda^*(c) \to 0$ *as* $c \to 0$.

When $\lambda = \lambda^*$, we call *extremal solution* the unique solution u^* of (1) which, under the extra condition (3), lies in $H^1(\Omega)$. As far as we are aware, it is not clear whether it is possible to remove assumption (3) even under Dirichlet boundary conditions. When g satisfies just assumption (2), the best known results for problem (1) with $c = +\infty$, state that the extremal solution lies in $H_0^1(\Omega)$ for any $n \leqslant 5$ (see [25, Theorem 1]) while, if $\Omega = B$, the result holds for every $n \geqslant 2$ (see [6, Theorem 1.1]). The paper [10] establishes further regularity for u^* but under some additional growth condition on g. When $c \to 0$, Theorem 1 tells us that the "spectrum" $(0, \lambda^*)$ reduces to the empty set. This is related to the fact that, by the divergence theorem, there exist no positive solutions to the Neumann problem.

By means of their stability, we may characterize *singular* extremal solutions in the energy class.

Theorem 2. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ *be a smooth bounded domain and assume that* g *satisfies* (2) *and* (3). *Let* $u \in H^1(\Omega)$ *be a singular weak solution of* (1). *Then, the following facts are equivalent*:

(i) $g'(u) \in L^1(\Omega)$ *and*
$$\int_\Omega |\nabla v|^2\,dx + c\int_{\partial\Omega} v^2\,d\sigma \geqslant \lambda \int_\Omega g'(u)v^2\,dx \quad \textit{for all } v \in X_c(\Omega); \tag{5}$$
(ii) $\lambda = \lambda^*$ *and* $u = u^*$.

We stress that the assumption $u \in H^1(\Omega)$ in Theorem 2 is crucial, see Remark 15. Furthermore, note that if $u = u_\lambda$ is the minimal solution, by Theorem 1 $g'(u_\lambda) \in L^\infty(\Omega)$ and (5) can be extended to any $v \in H^1(\Omega)$. On the other hand, if $u = u^*$ and it is singular, Theorem 2(i) ensures that the right-hand side in (5) is finite and, by density arguments, one has that (5) holds for all v in the energy class $H^1(\Omega)$.

2.2. Some model nonlinearities

In order to perform a precise analysis of the regularity of the extremal solution we restrict our attention to some model nonlinearities. When $\Omega = B$, several computations can be performed explicitly.

We first consider the case where g is the exponential function.

Theorem 3. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ *be a smooth bounded domain, assume that* $g(u) = e^u$ *and let* λ^* *be the extremal parameter, then*:

(i) *if* $n = 2$ *and* $\lambda \in (0, \lambda^*)$, *there exist at least two solutions of problem* (1) *in the energy class and any solution in the energy class is regular*;
(ii) *if* $n \leqslant 9$, u^* *is regular.*

If $n \geqslant 10$ *and* $\Omega = B$, *let* $c_n := \frac{n-2-\sqrt{(n-2)(n-10)}}{2}$, *then*:

(iii) *if* $0 < c < c_n$, *then* $\lambda^* > 2(n-2)e^{-2/c}$ *and* u^* *is regular*;
(iv) *if* $c \geqslant c_n$, *then* $\lambda^* = 2(n-2)e^{-2/c}$ *and* $u^*(x) = 2(\frac{1}{c} - \log|x|)$, *so that the extremal solution of* (1) *is singular.*

In Appendix A we perform a careful analysis of radial solutions to (1) when $\Omega = B$ and $g(u) = e^u$. It turns out that the branch containing the minimal solution has the following behavior.

Since Dirichlet boundary conditions correspond to $c = +\infty$, the third displayed picture highlights a phenomenon which *is not* observable under Dirichlet boundary conditions, see [23].

Next, we consider the power case. When either $n = 2$ or $n \geqslant 3$ and $1 < p < \frac{n+2}{n-2}$, by standard boot-strap arguments for subcritical elliptic problems, any energy solution is regular. The same can be proved for $p = \frac{n+2}{n-2}$, see [4]. In these cases Theorem 1(ii) ensures that the extremal solution is regular. This is the reason why, in what follows, we just focus on the supercritical case. For $p > \frac{n+2}{n-2}$ and $c > \frac{2}{p-1}$, we put

$$\lambda_s = \lambda_s(n, p, c) = \frac{2(n(p-1) - 2p)}{(p-1)^{p+1}} \left(\frac{c(p-1) - 2}{c} \right)^{p-1} \tag{6}$$

and, for $n \geqslant 11$, we let

$$p_n := \frac{n^2 - 8n + 4 + 8\sqrt{n-1}}{(n-2)(n-10)} > \frac{n+2}{n-2}.$$

The constant p_n was originally introduced for the Dirichlet case in [5], see also [23]. Due to the Steklov boundary conditions, a further number has to be defined. For $n \geqslant 11$ and $p \geqslant p_n$, the number

$$c_{n,p} := \frac{1}{2}\left(n - 2 - \sqrt{(n-2)^2 + \frac{8p}{p-1}\left(\frac{2p}{p-1} - n \right)} \right) \tag{7}$$

is well-defined and positive. Furthermore the map $p \mapsto c_{n,p}$ is decreasing in $[p_n, \infty)$, $c_{n,p_n} = \frac{n-2}{2}$ and $c_{n,p} > \frac{2}{p-1}$ for all $p \geqslant p_n$. Then, we prove:

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) be a smooth bounded domain, assume that $g(u) = (1+u)^p$ for some $p > \frac{n+2}{n-2}$ and let λ^* be the extremal parameter, then*:

(i) *if $n \leqslant 10$ or $n \geqslant 11$ and $p < p_n$, u^* is regular.*

If $\Omega = B$, we have that:

(ii) *if $0 < c \leqslant \frac{2}{p-1}$, then any radial solution is regular*;
(iii) *if $n \geqslant 11$ and $p \geqslant p_n$, then*:
- *if $0 < c < c_{n,p}$, u^* is regular*;
- *if $c \geqslant c_{n,p}$, then $\lambda^* = \lambda_s$ and $u^*(x) = \frac{c(p-1)}{c(p-1)-2}|x|^{-2/(p-1)} - 1$, so that the extremal solution of* (1) *is singular.*

The first part of the statement of Theorems 3 and 4 tells that in low dimensions the extremal solution is regular regardless of the domain Ω and of the value of c. The second part of the statement shows that the result is sharp since in higher dimensions there exists a domain (the ball) and values of c ($\geqslant c_n$ or $c_{n,p}$) for which the extremal solution is singular. We emphasize that, again, the results under Dirichlet boundary conditions can be obtained as limit case for $c \to +\infty$.

When $\Omega = B$ and $n \geqslant 11$, the regions in the plane (c, p) describing the regularity of solutions are summarized in Fig. 2. In the grey region, where $1 < p \leqslant \max\{\frac{2+c}{c}, \frac{n+2}{n-2}\}$, any radial solution in the energy class is regular. In the striped region, where either $c \geqslant c_{n,p}$ or $p \geqslant p_n$, u^* is singular. In the remaining white part, u^* is regular. In some sense, this shows that the "lack of regularity" for large exponents p disappears in presence of small values of c. We may say that

$$\textit{small values of } c \textit{ weaken the strength of large values of } p \textit{ in radial problems.} \tag{8}$$

For the branches of radial solutions in B, we expect pictures similar to those displayed in the exponential case. For instance, in the grey region, we expect the first picture in Fig. 1.

When $n \geqslant 3$ and $p = \frac{n+2}{n-2}$ we may determine explicitly the solutions of (1) and highlight a further interesting phenomenon. For $c > 0$ and $\varepsilon > \varepsilon_0(c) := \max\{0, \frac{n-2}{c} - 1\}$, consider the function

$$\varphi(\varepsilon) := \frac{[n(n-2)]^{n-2}}{c^4} \frac{[c(1+\varepsilon) - n + 2]^4 \varepsilon^{n-2}}{(1+\varepsilon)^{2n}}. \tag{9}$$

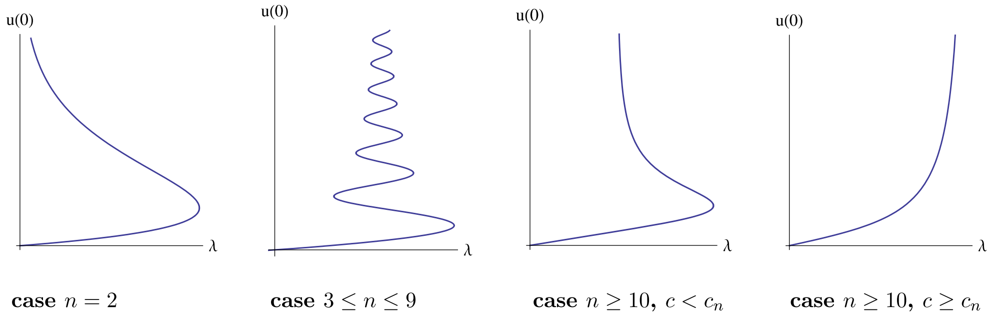

Fig. 1. Bifurcation branches in the exponential case.

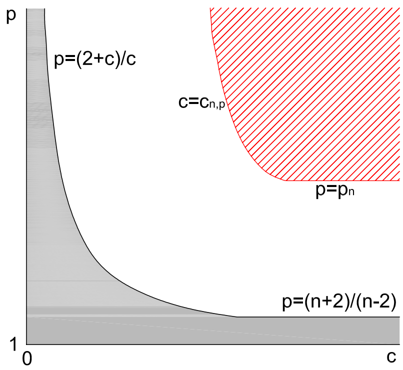

Fig. 2. Regularity of the solutions in the power case for $n \geqslant 11$.

It is readily seen that $\varphi(\varepsilon_0) = 0 = \lim_{\varepsilon \to +\infty} \varphi(\varepsilon)$, that φ attains a global maximum at

$$\bar{\varepsilon} := \frac{n + 2 + \sqrt{(n+2)^2 - 4c(n-2-c)}}{2c},$$

that φ increases on $(\varepsilon_0, \bar{\varepsilon})$ and decreases on $(\bar{\varepsilon}, +\infty)$. Hence, for any $\lambda \in (0, \lambda_n)$, where $\lambda_n(c) := (\varphi(\bar{\varepsilon}))^{1/(n-2)}$,

$$\text{there exist} \quad \varepsilon_i = \varepsilon_i(c, \lambda) \quad (i = 1, 2) \quad \text{such that} \quad \varphi(\varepsilon_i) = \lambda^{n-2}. \tag{10}$$

If $\lambda = \lambda_n$, then $\varepsilon_1 = \varepsilon_2 = \bar{\varepsilon}$. We prove:

Theorem 5. *Let $\Omega = B \subset \mathbb{R}^n$ $(n \geqslant 3)$ and assume that $g(u) = (1+u)^{\frac{n+2}{n-2}}$. Then, if $\lambda_n > 0$ and $\varepsilon_0 < \varepsilon_2 \leqslant \bar{\varepsilon} \leqslant \varepsilon_1$ are defined as in* (10)*, we have*:

(i) *for every $\lambda \in (0, \lambda_n)$, there exist two radial solutions of problem* (1)*, the minimal solution u_1 and a larger one u_2, given by*

$$u_i(x) = \left(\frac{n(n-2)\varepsilon_i}{\lambda}\right)^{(n-2)/4} \left(\varepsilon_i + |x|^2\right)^{-(n-2)/2} - 1, \quad i = 1, 2;$$

(ii) *the extremal parameter satisfies $\lambda^* = \lambda_n$ and the extremal solution u^* of* (1) *is given by*

$$u^*(x) = \left(\frac{n(n-2)\bar{\varepsilon}}{\lambda_n}\right)^{(n-2)/4} \left(\bar{\varepsilon} + |x|^2\right)^{-(n-2)/2} - 1.$$

Furthermore, as $\lambda \to 0$, the large solution u_2 blows-up as follows:

- *global blow-up*: *if* $0 < c < n-2$, *then*

$$u_2(0) \sim \left(\frac{cn(n-2)}{\lambda(n-2-c)}\right)^{(n-2)/4} \quad \text{and} \quad u_2(1) \sim \left(\frac{cn(n-2-c)}{\lambda(n-2)}\right)^{(n-2)/4};$$

- *global blow-up with concentration*: *if* $c = n-2$, *then*

$$u_2(0) \sim \left(\frac{n(n-2)}{\lambda}\right)^{n(n-2)/2(n+2)} \quad \text{and} \quad u_2(1) \sim \left(\frac{n(n-2)}{\lambda}\right)^{(n-2)/(n+2)};$$

- *concentration*: *if* $c > n-2$, *then*

$$u_2(0) \sim \frac{c-n+2}{c}\left(\frac{n(n-2)}{\lambda}\right)^{(n-2)/2} \quad \text{and} \quad u_2(1) \to \frac{n-2}{c-n+2}.$$

Remark 6. Letting $c \to +\infty$ in Theorem 5, one obtains known results under Dirichlet boundary conditions, see [14, Theorem 7]. In particular, $\bar{\varepsilon}(c) \to 1$ and $\lambda^*(c) \to \frac{n(n-2)}{4}$. If we approach the Neumann case, that is if we let $c \to 0$, then $\bar{\varepsilon}(c) \sim \frac{n+2}{c}$ and $\lambda^*(c) \sim \frac{n(n-2)2^{8/(n-2)}}{(n+2)^{(n+2)/(n-2)}}c$.

The striking difference of blow-up in the cases $c \lesseqgtr n-2$ is somehow a consequence of nonexistence of solutions to related problems, see [26, Theorem 4.2]. Further results can be found in [19]. In the subcritical case $p < \frac{n+2}{n-2}$ one can adapt to Steklov boundary conditions [14, Theorem 6] (see also [13, Theorem 2]) and obtain:

Proposition 7. *Let* $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ *be a smooth bounded domain and let* $g(u) = (1+u)^p$ *for some* $1 < p < \frac{n+2}{n-2}$. *For* $\lambda \in (0, \lambda^*)$, *with* λ^* *being the extremal parameter, let* U_λ *be a large (mountain-pass) solution of problem* (1). *Then, there exists a mountain-pass solution* V *to*

$$\begin{cases} -\Delta V = V^p & \text{in } \Omega, \\ V > 0 & \text{in } \Omega, \\ V_\nu + cV = 0 & \text{on } \partial\Omega, \end{cases}$$

such that

$$\lim_{\lambda \to 0} \lambda^{1/(p-1)} U_\lambda \to V \quad \text{in } C^{2,\alpha}(\overline{\Omega}).$$

Combined with this statement, Theorem 5 tells us that for $c < n-2$ (problem close to Neumann), the critical growth equation behaves subcritically. This is a further argument in favor of the rule (8).

3. Hardy inequalities with a boundary term

For $c > 0$ fixed, throughout this section we endow the Sobolev space $H^1(\Omega)$ with the following scalar product and corresponding norm

$$(u, v) := \int_\Omega \nabla u \nabla v \, dx + c \int_{\partial\Omega} uv \, d\sigma, \qquad \|u\|^2 := \int_\Omega |\nabla u|^2 \, dx + c \int_{\partial\Omega} u^2 \, d\sigma. \tag{11}$$

Several versions of Hardy inequality [16] are available in literature. Our starting point is the optimal inequality in $H^1(\Omega)$ involving a *boundary term*. It is shown in [1,27] that there exists a positive constant $C_n = C_n(\Omega)$ such that

$$\int_\Omega |\nabla u|^2 \, dx + C_n \int_{\partial\Omega} u^2 \, d\sigma \geqslant \frac{(n-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} \, dx \quad \text{for all } u \in H^1(\Omega). \tag{12}$$

When $\Omega = B$, the optimal (smallest) value of C_n has been determined, $C_n(B) = \frac{n-2}{2}$.

One may then wonder if when C_n is replaced by a *smaller* constant, a similar inequality remains true provided $\frac{(n-2)^2}{4}$ is also replaced by a smaller constant. In other words, for any $c \in [0, C_n]$ we wish to determine the largest $h(c) \in [0, \frac{(n-2)^2}{4}]$ such that

$$\int_\Omega |\nabla u|^2\,dx + c\int_{\partial\Omega} u^2\,d\sigma \geqslant h(c)\int_\Omega \frac{u^2}{|x|^2}\,dx \quad \text{for all } u \in H^1(\Omega). \tag{13}$$

Hence, for any $c > 0$, we define

$$h(c) := \inf_{u\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx + c\int_{\partial\Omega} u^2\,d\sigma}{\int_\Omega \frac{u^2}{|x|^2}\,dx}. \tag{14}$$

Formally, the Euler equation corresponding to the variational problem (14) is the following eigenvalue problem under Steklov boundary conditions

$$\begin{cases} -\Delta u = h(c)\dfrac{u}{|x|^2} & \text{in } \Omega, \\ u_\nu + cu = 0 & \text{on } \partial\Omega. \end{cases} \tag{15}$$

By solutions of (15) we mean weak solutions, that is functions $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u\nabla v\,dx + c\int_{\partial\Omega} uv\,d\sigma = h(c)\int_\Omega \frac{uv}{|x|^2}\,dx \quad \text{for all } v \in H^1(\Omega). \tag{16}$$

We prove:

Theorem 8. *Let $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ be a smooth bounded domain, let C_n be as in* (12), *let $c \geqslant 0$ and let $h(c)$ be as in* (14). *Then we have*:

(i) $h(0) = 0$ *and $h(c)$ is strictly increasing with respect to $c \in [0, C_n]$*;
(ii) $h(c) = \frac{(n-2)^2}{4}$ *for every $c \geqslant C_n$.*

Moreover the infimum in (14) *is achieved if and only if $0 \leqslant c < C_n$ and, up to a multiplicative constant, the minimizer $\bar u \in H^1(\Omega)$ is unique, strictly positive in Ω and it solves* (15).

Hence, for any bounded smooth domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ and for every $c > 0$, inequality (13) holds with $h(c)$ defined as in (14) and behaving as explained in Theorem 8. It is worth noting that the lower bound $c = 0$, in correspondence of which (13) becomes trivial, is the first Steklov boundary eigenvalue for $-\Delta$ and, clearly, $\bar u(x) \equiv 1$ is a corresponding eigenfunction (solution to the Neumann problem).

Proof. The properties of $h(c)$ follow directly from its definition combined with inequality (12). In particular, in view of the optimality of the constants in (12), it must be

$$h(c) < \frac{(n-2)^2}{4} \quad \text{for every } c < C_n.$$

We first show that the infimum in (14) is attained for every $c < C_n$. Let $\{u_m\} \subset H^1(\Omega)$ be a minimizing sequence for $h(c)$ such that

$$\int_\Omega \frac{u_m^2}{|x|^2}\,dx = 1. \tag{17}$$

Then,

$$\|u_m\|^2 = \int_\Omega |\nabla u_m|^2\,dx + c\int_{\partial\Omega} u_m^2\,d\sigma = h(c) + o(1) \quad \text{as } m \to +\infty, \tag{18}$$

which shows that $\{u_m\}$ is bounded in $H^1(\Omega)$. Exploiting the compactness of the trace map $H^1(\Omega) \to L^2(\partial\Omega)$, we conclude that there exists $u \in H^1(\Omega)$ such that

$$u_m \rightharpoonup u \quad \text{in } H^1(\Omega), \qquad u_m \to u \quad \text{in } L^2(\partial\Omega), \qquad \frac{u_m}{|x|} \rightharpoonup \frac{u}{|x|} \quad \text{in } L^2(\Omega), \tag{19}$$

up to a subsequence. Assume that $u_m \to 0$ in $L^2(\partial\Omega)$, then by (12) and (17)–(19) we infer that

$$h(c) + o(1) = \int_\Omega |\nabla u_m|^2\,dx + C_n\int_{\partial\Omega} u_m^2\,d\sigma + o(1) \geqslant \frac{(n-2)^2}{4} + o(1)$$

a contradiction. Hence, $u \not\equiv 0$, if we set $v_m := u_m - u$, from (19) we obtain

$$v_m \rightharpoonup 0 \quad \text{in } H^1(\Omega), \qquad v_m \to 0 \quad \text{in } L^2(\partial\Omega), \qquad \frac{v_m}{|x|} \rightharpoonup 0 \quad \text{in } L^2(\Omega). \tag{20}$$

In view of (19)–(20) we may rewrite (18) as

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla v_m|^2\,dx + c\int_{\partial\Omega} u^2\,d\sigma = h(c) + o(1). \tag{21}$$

Moreover, by (17) and (20), we have

$$1 = \int_\Omega \frac{u_m^2}{|x|^2}\,dx = \int_\Omega \frac{u^2}{|x|^2}\,dx + \int_\Omega \frac{v_m^2}{|x|^2}\,dx + o(1) \leqslant \int_\Omega \frac{u^2}{|x|^2}\,dx + \frac{4}{(n-2)^2}\int_\Omega |\nabla v_m|^2\,dx + o(1).$$

Since $h(c) \geqslant 0$, this last inequality gives

$$h(c) \leqslant h(c)\int_\Omega \frac{u^2}{|x|^2}\,dx + \frac{4h(c)}{(n-2)^2}\int_\Omega |\nabla v_m|^2\,dx + o(1).$$

By combining this with (21), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^2\,dx + c\int_{\partial\Omega} u^2\,d\sigma &\leqslant h(c)\int_\Omega \frac{u^2}{|x|^2}\,dx + \left(\frac{4h(c)}{(n-2)^2} - 1\right)\int_\Omega |\nabla v_m|^2\,dx + o(1)\\ &\leqslant h(c)\int_\Omega \frac{u^2}{|x|^2}\,dx + o(1)\end{aligned}$$

which shows that $u \not\equiv 0$ is a minimizer.

Concerning the positivity of a minimizer u, one may simply replace u with $|u|$ if necessary. Then, by the Lagrange multiplier method, u is a nonnegative solution of problem (15) with $h = h(c)$, hence a superharmonic function. By the maximum principle (Lemma 10), this implies $u > 0$ in Ω and, in turn, in $\overline{\Omega}$. Indeed, if u vanishes somewhere on $\partial\Omega$, then the boundary condition ($u = -cu_\nu$ on $\partial\Omega$) would contradict the Hopf boundary lemma.

By arguing as in Lemma 12 we see that, up to a multiplicative constant, the minimizer $\bar{u}$ is unique. In order to show that the infimum in (14) is not achieved if $c = C_n$, we may proceed as in [1, (a_4), p. 429]. Indeed the argument there is local and does not take into account the boundary conditions. □

When $\Omega = B$, $h(c)$ can be explicitly determined and we obtain as a consequence a result by Barbatis, Filippas and Tertikas [2].

Theorem 9. *(See [2].) Let $n \geqslant 3$. Then, for every $c \geqslant 0$ there holds*

$$\int_B |\nabla u|^2\,dx + c\int_{\partial B} u^2\,d\sigma \geqslant h(c)\int_B \frac{u^2}{|x|^2}\,dx \quad \textit{for all } u \in H^1(B), \tag{22}$$

where $h(c) = c(n-2-c)$ for every $0 \leqslant c < \frac{n-2}{2}$, while $h(c) = \frac{(n-2)^2}{4}$ for every $c \geqslant \frac{n-2}{2}$. Furthermore, the constants are optimal and equality in (22) *is attained if and only if $0 \leqslant c < \frac{n-2}{2}$ by real multiples of the function $\bar{u}(x) = |x|^{-c}$.*

Theorem 9 will be of crucial importance in the proofs of Theorems 3 and 4.

4. Sketch of the proofs of Theorems 1 and 2

We first need a weak maximum principle and a weak form of the super-sub-solution method.

Lemma 10. *For all $f \in L^1(\Omega)$ such that $f \geqslant 0$ a.e. in Ω and $f \not\equiv 0$ there exists a unique $u \in L^1(\Omega)$ such that*

$$-\int_\Omega u\Delta v\,dx = \int_\Omega f v\,dx \quad \textit{for all } v \in X_c(\Omega), \tag{23}$$

and $u > 0$ a.e. in Ω.

If $\bar{u} \in L^1(\Omega)$ is such that $\bar{u} \geqslant 0$ a.e. in Ω, $g(\bar{u}) \in L^1(\Omega)$ and

$$-\int_\Omega \bar{u}\Delta v\,dx \geqslant \lambda\int_\Omega g(\bar{u})v\,dx \quad \textit{for all } v \in X_c(\Omega),\ v \geqslant 0 \textit{ in } \Omega,$$

then there exists a solution u of (1) *such that $0 \leqslant u \leqslant \bar{u}$ in Ω.*

Proof. The existence of a weak solution u to (23) may be obtained by applying a suitable approximation by truncation argument, see [3, Lemma 1]. The positivity of u is deduced, arguing by contradiction, by combining the maximum principle for superharmonic functions with the Hopf boundary lemma.

With the just proved results, to get the second part of the proof, one may apply the monotone iteration argument illustrated in [3, Lemma 3]. □

Thanks to Lemma 10, we get the regularity of the minimal solution.

Lemma 11. *Assume that for some $\mu > 0$ there exists a (possibly singular) solution w of* (1) *for $\lambda = \mu$. Then, for all $0 < \lambda < \mu$ there exists a regular solution of* (1).

Proof. Our purpose is to construct a regular super-solution of problem (1) and to conclude by applying Lemma 10. To this aim, exploiting the ideas of [3, Lemmas 2 and 4], we define

$$h(u) = \int_0^u \frac{ds}{\mu g(s)}, \qquad h_\eta(u) = \eta^{-1}h(u) \quad \text{and} \quad \Phi_\eta(u) = h_\eta^{-1}\big(h(u)\big) = h^{-1}\big(\eta h(u)\big),$$

for all $u \geqslant 0$ and $\eta \in (0,1)$. One has that:

(a) $0 \leqslant \Phi_\eta(u) < u$, for all $u \geqslant 0$;
(b) Φ_η is increasing, concave and $\Phi'_\eta(u) < 1$, for all $u \geqslant 0$;
(c) if $h(\infty) < \infty$, then $\Phi_\eta(\infty) < \infty$.

Set now $f(x) := \mu g(w(x))$. Then, $f \in L^1(\Omega)$ and $f > 0$ a.e. in Ω, so that there exists a sequence $\{f_n\}_{n\geqslant 0} \subset C_c^\infty(\Omega)$, $f_n \geqslant 0$ such that $f_n \to f$ in $L^1(\Omega)$. To each f_n we associate the unique solution $w_n \in X_c(\Omega)$ (recall $\partial\Omega \in C^{2,\alpha}$) of

$$\begin{cases} -\Delta w_n = f_n & \text{in } \Omega, \\ w_n > 0 & \text{in } \Omega, \\ (w_n)_\nu + c w_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 10 implies that $w_n \to w$ in $L^1(\Omega)$. On the other hand, some computations give

$$\Delta\Phi_\eta(w_n) = \Phi'_\eta(w_n)\Delta w_n + \Phi''_\eta(w_n)|\nabla w_n|^2 \leqslant \Phi'_\eta(w_n)\Delta w_n = -\Phi'_\eta(w_n) f_n$$

and, in turn,

$$-\int_\Omega \Delta\Phi_\eta(w_n) v\,dx \geqslant \int_\Omega \Phi'_\eta(w_n) f_n v\,dx \quad \text{for all } v \in X_c(\Omega),\ v \geqslant 0 \text{ in } \Omega.$$

Integrating by parts and exploiting the boundary conditions, this gives

$$-\int_\Omega \Phi_\eta(w_n)\Delta v\,dx + c\int_{\partial\Omega} \big[\Phi'_\eta(w_n) w_n - \Phi_\eta(w_n)\big] v\,d\sigma \geqslant \int_\Omega \Phi'_\eta(w_n) f_n v\,dx \tag{24}$$

for all $v \in X_c(\Omega)$ such that $v \geqslant 0$ in Ω. Notice that

$$\Phi'_\eta(w_n) w_n - \Phi_\eta(w_n) = \eta \frac{g(\Phi_\eta(w_n))}{g(w_n)} w_n - \Phi_\eta(w_n) \leqslant 0 \quad \text{for all } 0 < \eta < 1.$$

This inequality can be checked by observing that the function

$$F_\eta(s) := \eta \frac{g(\Phi_\eta(s))}{g(s)} s - \Phi_\eta(s), \quad s \geqslant 0,$$

has strictly negative derivative (provided $0 < \eta < 1$) and $F_\eta(0) = 0$. Indeed, some computations give

$$F'_\eta(s) = \frac{\eta g(\Phi_\eta(s)) s}{g^2(s)} \big[\eta g'\big(\Phi_\eta(s)\big) - g'(s)\big] \leqslant \frac{\eta g(\Phi_\eta(s)) s}{g^2(s)} g'(s)(\eta - 1) \leqslant 0 \quad \text{for } 0 < \eta < 1,$$

where in the last step we combine (a) with the convexity of g.

By passing to the limit in (24), the above arguments yield

$$-\int_\Omega \Phi_\eta(w)\Delta v\,dx \geqslant \int_\Omega \Phi'_\eta(w) f v\,dx = \eta \int_\Omega \frac{g(\Phi_\eta(w))}{g(w)} f v\,dx = \eta\mu \int_\Omega g\big(\Phi_\eta(w)\big) v\,dx.$$

Let $\bar v := \Phi_\eta(w)$. Then, $\bar v \leqslant w \in L^1(\Omega)$ and $g(\bar v) \leqslant g(w) \in L^1(\Omega)$. Moreover, the latter inequality implies that, for $0 < \eta < 1$, $\bar v$ is a weak super-solution of (1) with $\lambda = \eta\mu$.

If $\int_0^{+\infty} \frac{ds}{g(s)} < \infty$, by (c) we conclude that $\bar v < +\infty$, which means that $\bar v$ is bounded and the thesis comes from Lemma 10.

If $\int_0^{+\infty} \frac{ds}{g(s)} = \infty$, we first observe that by the concavity of h we have $h(w) \leqslant h(\bar v) + \frac{w - \bar v}{\mu g(\bar v)}$, so that $g(\bar v) \leqslant \frac{w}{\mu h(w)} \leqslant C(1 + w)$ for some $C > 0$. On the other hand, by Lemma 10, the existence of the weak super-solution $\bar v$ implies the existence of a weak solution u_1 of (1) with $\lambda = \eta\mu$, such that $0 < u_1 \leqslant \bar v$ and $0 \leqslant g(u_1) \leqslant g(\bar v) \in L^1(\Omega)$. Hence $u_1 \in L^{p_1}(\Omega)$, for all $1 \leqslant p_1 < \frac{n}{n-2}$ (see [21, Theorem 5.4]). By iteration, the same construction allows to show the existence of a sequence of functions $u_k \in L^1(\Omega)$ which solve (1) with $\lambda = \eta^k\mu$, such that $0 < u_k \leqslant u_{k-1}$ and $g(u_k) \leqslant C(1 + u_{k-1}) \in L^{p_{k-1}}(\Omega)$, for all $1 \leqslant p_{k-1} < \frac{n}{n-2(k-1)}$. For $k > n/2$, this procedure gives a bounded super-solution of (1) with $\lambda = \eta^k\mu$. By arbitrariness of $\eta \in (0, 1)$, this concludes the proof. □

For $c > 0$ fixed, we set

$$\lambda_1(c) := \inf_{v \in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2\,dx + c\int_{\partial\Omega} v^2\,d\sigma}{\int_\Omega v^2\,dx} \tag{25}$$

and, if also $\lambda \in (0, \lambda^*)$ is fixed, we set

$$\mu_1(c,\lambda) = \inf_{v\in H^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2\,dx + c\int_{\partial\Omega} v^2\,d\sigma - \lambda\int_\Omega g'(u_\lambda)v^2\,dx}{\int_\Omega v^2\,dx}, \tag{26}$$

where u_λ is the minimal (regular) solution of problem (1). Notice that λ_1 and μ_1 are, respectively, the first eigenvalue of $-\Delta$, and of the linearized operator $-\Delta - \lambda g'(u_\lambda)$, under Steklov boundary conditions. We have:

Lemma 12. *The eigenvalue λ_1 in* (25) *is simple and any corresponding eigenfunction is strictly of one sign in Ω. Let $\lambda \in (0,\lambda^*)$ and let u_λ be the corresponding minimal regular solution to* (1)*, then the eigenvalue μ_1 in* (26) *is positive and any corresponding eigenfunction is strictly of one sign in Ω.*

Proof. From the compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$ we infer that the infimum in (25) is achieved so a minimizer ϕ_1 exists. If ϕ_1 changes sign in Ω, then $|\phi_1|$ is a minimizer that vanishes in Ω, against the maximum principle. Also the proof that λ_1 is simple can be obtained by contradiction. Let $\phi_2 \in H^1(\Omega)$ be another eigenfunction corresponding to λ_1 so that $\phi_2 > 0$ in Ω. For every $k \in \mathbb{R}$, define $\psi_k := \phi_1 + k\phi_2$. Since the problem is linear, also ψ_k is an eigenfunction. But, unless ϕ_2 is a multiple of ϕ_1, there exists some k such that ψ_k changes sign in Ω, a contradiction.

To show that $\mu_1 > 0$ one can follow [9, Proposition 2.15]. Taking into account the regularity of u_λ, the rest of the statement follows as for λ_1. □

Proof of Theorem 1. If $c > 0$, $\lambda = 0$ and we drop the requirement that $u > 0$, then (1) only admits the trivial solution. So, we put

$$\Lambda := \{\lambda \geqslant 0\colon \text{(1) admits a nonnegative solution}\} \quad \text{and} \quad \lambda^* := \sup \Lambda. \tag{27}$$

As we just remarked, $0 \in \Lambda$ so that $\Lambda \neq \emptyset$ and λ^* is well defined. For any $\varepsilon > 0$, consider the problem

$$\begin{cases} -\Delta u = \varepsilon & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u_\nu + cu = 0 & \text{on } \partial\Omega, \end{cases}$$

which admits a classical solution $u \in X_c(\Omega)$. Taking $\bar\lambda = \varepsilon/g(\|u\|_\infty)$, one has that u is a super-solution of problem (1) for any $\lambda \in (0,\bar\lambda)$. By Lemma 10, we deduce that $\lambda^* > 0$. Moreover, we also infer that for every $\lambda \in \Lambda$ minimal solutions u_λ exist and by Lemma 11 they are regular. Lemma 10 also tells that (for fixed $c > 0$) the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing for all $x \in \Omega$ and, by the Hopf boundary lemma, this holds up to the boundary. In particular, Λ is an interval. The second statement in (i) follows by combining Lemma 10 with the arguments of [11, Theorem 5].

Now we show that λ^* is finite. To this end, let $\lambda \in \Lambda$ and let u be the corresponding positive solution of (1). By (2) there exists $\alpha > 0$ such that $g(s) \geqslant \alpha s$ for every $s \in [0,+\infty)$ and $g(s) \not\equiv \alpha s$. Then, if λ_1 and ϕ_1 are as in Lemma 12, by (4) we obtain

$$\lambda_1 \int_\Omega u\phi_1\,dx = -\int_\Omega u\Delta\phi_1\,dx = \lambda\int_\Omega g(u)\phi_1\,dx > \lambda\alpha\int_\Omega u\phi_1\,dx.$$

This yields

$$\lambda^* < \frac{\lambda_1}{\alpha}. \tag{28}$$

In turn, by taking $v \equiv 1$ in (25), we readily obtain

$$\lambda_1(c) \leqslant \frac{c|\partial\Omega|}{|\Omega|} \tag{29}$$

which, combined with (28), shows that $\lambda^*(c) \to 0$, as $c \to 0$.

To study the case $\lambda = \lambda^*$ we adapt to the Steklov boundary conditions some arguments of [5,22]. We know that the map $\lambda \mapsto u_\lambda(x)$ is strictly increasing for all $x \in \overline{\Omega}$ so that we may define

$$u^*(x) := \lim_{\lambda\to\lambda^*} u_\lambda(x) \quad (x \in \Omega). \tag{30}$$

We claim that $u^* \in H^1(\Omega)$ and that it solves (1). To this end, for $\lambda \in (0, \lambda^*)$, let u_λ be the minimal regular solution of (1). Then

$$-\int_\Omega u_\lambda \Delta v\, dx = \lambda \int_\Omega g(u_\lambda) v\, dx \quad \text{for all } v \in X_c(\Omega), \tag{31}$$

so that by Lemma 12, after an integration by parts, we get

$$\lambda \int_\Omega g'(u_\lambda) u_\lambda^2\, dx \leqslant \int_\Omega |\nabla u_\lambda|^2\, dx + c \int_{\partial\Omega} u_\lambda^2\, d\sigma = -\int_\Omega u_\lambda \Delta u_\lambda\, dx = \lambda \int_\Omega g(u_\lambda) u_\lambda\, dx. \tag{32}$$

From (3) it follows that there exist $\varepsilon > 0$ and $C > 0$ such that $(1+\varepsilon) g(s)s \leqslant g'(s)s^2 + C$ for all $s \in [0, +\infty)$. This fact, combined with (32), yields the existence of $C_1 > 0$ such that:

$$\int_\Omega g(u_\lambda) u_\lambda\, dx < C_1 \quad \text{for all } \lambda \in \big(0, \lambda^*\big).$$

Therefore

$$\|u_\lambda\|^2 = \int_\Omega |\nabla u_\lambda|^2\, dx + c \int_{\partial\Omega} u_\lambda^2\, d\sigma = \lambda \int_\Omega g(u_\lambda) u_\lambda\, dx < \lambda^* C_1.$$

Hence, up to a subsequence, we have $u_\lambda \rightharpoonup u^*$ in $H^1(\Omega)$ as $\lambda \to \lambda^*$. This, together with a suitable variant of the Lebesgue Theorem, allows us to pass to the limit in (31) and to conclude that $u^* \in H^1(\Omega)$ solves (1) for $\lambda = \lambda^*$. The claim is so proved.

We prove the uniqueness of the extremal solution u^*, by showing that (1), with $\lambda = \lambda^*$, admits no weak super-solution in the sense of Lemma 10. In particular, if $u \in L^1(\Omega)$, $u \geqslant 0$, is a weak super-solution of (1) with $\lambda = \lambda^*$, then u is the minimal solution. Thanks to Lemmas 10 and 11, the proof can be obtained simply by replacing Dirichlet with Steklov boundary conditions in [22, Theorem 1.1]. Indeed, by Lemma 10, we deduce that there exists a minimal weak solution $0 \leqslant u_{\lambda^*} \leqslant u$ of (1). Then, assuming by contradiction that $u \neq u_{\lambda^*}$ and exploiting the convexity of g, one shows that $z := (u + u_{\lambda^*})/2$ is a strict weak super-solution of (1) with $\lambda = \lambda^*$. In turn, arguing as in [22, Lemma 2.2], this allows to construct a weak super-solution of (1) with $\lambda = \lambda^* + \varepsilon$, for some $\varepsilon > 0$, in contradiction with the definition of λ^*.

We finally show that the map $c \mapsto \lambda^*(c)$ is strictly increasing. The extremal solution u_1^* of problem (1) with $c = c_1$, is a solution for $\lambda = \lambda^*(c_1)$. Let $c_1 > c_2$, then testing the equation with $v \in X_{c_2}$, $v \geqslant 0$ in Ω, and integrating by parts, we deduce

$$\begin{aligned}
-\int_\Omega u_1^* \Delta v\, dx - \lambda^*(c_1) \int_\Omega g\big(u_1^*\big) v\, dx &= \int_{\partial\Omega} \big(-u_1^* v_\nu + v\big(u_1^*\big)_\nu\big)\, d\sigma \\
&= \int_{\partial\Omega} \big(c_2 u_1^* + \big(u_1^*\big)_\nu\big) v\, d\sigma > \int_{\partial\Omega} \big(c_1 u_1^* + \big(u_1^*\big)_\nu\big) v\, d\sigma = 0.
\end{aligned}$$

Therefore, u_1^* is a super-solution of problem (1) with $c = c_2$ and $\lambda = \lambda^*(c_1)$. By Lemma 10 this implies that $\lambda^*(c_1) \leqslant \lambda^*(c_2)$. The inequality is in fact strict since, as shown above, there exists no super-solution when $\lambda = \lambda^*$. □

Proof of Theorem 2. The implication (ii) ⇒ (i) follows by combining the characterization given in Lemma 12 with (30). For the converse implication, we assume (i) and, for contradiction, that $\lambda < \lambda^*$. We take $v = u - u_\lambda$ as test function in (5), where u_λ is the minimal solution. Exploiting the boundary conditions, we get

$$\lambda \int_\Omega (u - u_\lambda)\big(g(u) - g(u_\lambda)\big)\, dx = -\int_\Omega (u - u_\lambda) \Delta (u - u_\lambda)\, dx \geqslant \lambda \int_\Omega g'(u)(u - u_\lambda)^2\, dx.$$

Then, by convexity of the function g, we infer that $u = u_\lambda$. But u_λ is regular, a contradiction. Hence, $\lambda = \lambda^*$ so that Theorem 1(ii) implies $u = u^*$ and (ii) follows. □

5. Proof of Theorem 3

Proof of (i). The existence of a second solution in the energy class follows from the fact that $u \mapsto e^u$ is subcritical and hence a compact map from $H^1(\Omega)$ to $L^1(\Omega)$ when $n = 2$. More precisely, for any $\lambda \in (0, \lambda^*)$ let u_λ be the minimal solution and consider the functional $I_\lambda(w) = J_\lambda(w + u_\lambda) - J_\lambda(u_\lambda)$, where

$$J_\lambda(w) = \frac{1}{2}\left(\int_\Omega |\nabla w|^2\,dx + c\int_{\partial\Omega} w^2\right) - \lambda\int_\Omega e^w\,dx \quad \text{for all } w \in H^1(\Omega)$$

and $H^1(\Omega)$ is endowed with the norm (11). Then, the second solution U_λ can be characterized variationally as a mountain-pass critical point of I_λ, see [9, Theorem 2.1]. The fact that any energy solution is a classical solution follows from embedding arguments and elliptic regularity. □

Proof of (ii). We follow the idea developed in [23] but taking into account the presence of a boundary term. For every $\lambda \in (0, \lambda^*)$, we know that the minimal solution u_λ satisfies the equation

$$\int_\Omega \nabla u_\lambda \nabla v\,dx + c\int_{\partial\Omega} u_\lambda v\,d\sigma = \lambda\int_\Omega e^{u_\lambda} v\,dx \quad \text{for all } v \in H^1(\Omega)$$

and, by Lemma 12, the stability condition

$$\int_\Omega |\nabla w|^2\,dx + c\int_{\partial\Omega} w^2\,d\sigma \geqslant \lambda\int_\Omega e^{u_\lambda} w^2\,dx \quad \text{for all } w \in H^1(\Omega).$$

We choose as test functions, respectively, $v = e^{(q-1)u_\lambda}$ and $w = e^{\frac{q-1}{2}u_\lambda}$, where $q > 1$. We get

$$(q-1)\int_\Omega e^{(q-1)u_\lambda}|\nabla u_\lambda|^2\,dx + c\int_{\partial\Omega} u_\lambda e^{(q-1)u_\lambda}\,d\sigma = \lambda\int_\Omega e^{qu_\lambda}\,dx$$

and

$$\left(\frac{q-1}{2}\right)^2\int_\Omega e^{(q-1)u_\lambda}|\nabla u_\lambda|^2\,dx + c\int_{\partial\Omega} e^{(q-1)u_\lambda}\,d\sigma \geqslant \lambda\int_\Omega e^{qu_\lambda}\,dx.$$

By putting together these inequalities we obtain

$$\frac{4c}{q-1}\int_{\partial\Omega} e^{(q-1)u_\lambda}\,d\sigma - c\int_{\partial\Omega} u_\lambda e^{(q-1)u_\lambda}\,d\sigma \geqslant \lambda\left(\frac{4}{q-1} - 1\right)\int_\Omega e^{qu_\lambda}\,dx.$$

Assume that $1 < q < 5$ so that $\frac{4}{q-1} > 1$. As $\lambda \to \lambda^*$, the left-hand side cannot blow-up since the leading term is $u_\lambda e^{(q-1)u_\lambda}$. Therefore, the right-hand side remains bounded, this means that e^{u_λ} is uniformly bounded in $L^q(\Omega)$. Since u_λ solves the equation, by elliptic regularity this means that $\{u_\lambda\}$ is uniformly bounded in $W^{2,q}(\Omega)$ for all $1 < q < 5$. Since $n \leqslant 9$, by Sobolev embedding this shows that $\{u_\lambda\}$ is uniformly bounded in $L^\infty(\Omega)$ so that $u^* \in L^\infty(\Omega)$ in view of (30). □

Proof of (iii). For $\lambda \in (0, \lambda^*)$ the minimal solution u_λ may be obtained by an iterative method starting from $u_0 = 0$. We let u^m, m positive integer, be the unique solution to

$$\begin{cases} -\Delta u^m = \lambda e^{u^{m-1}} & \text{in } B, \\ u^m_\nu + cu^m = 0 & \text{on } \partial B. \end{cases} \tag{33}$$

Hence u^m is radially symmetric and so is u_λ and, by (30), also u^*. We restrict our attention to radial solutions of problem (1). Following an idea of Tartar (see [24]), for $r = |x| \in (0, 1)$ we set

$$s = \log r \in (-\infty, 0), \qquad v(s) = \frac{d}{ds}u\big(e^s\big), \qquad w(s) = -\lambda e^{2s} e^{u(e^s)}, \tag{34}$$

and we rewrite problem (1) as a dynamical system, namely

$$\begin{cases} v' = w - (n-2)v, \\ w' = (v+2)w, \end{cases} \tag{35}$$

whereas the Steklov condition reads

$$w(0) = -\lambda e^{-v(0)/c}. \tag{36}$$

By definition, $w(s) < 0$. For any radial solution u, the equation combined with Lemma 10 gives $u'(r) < 0$ in $(0, 1)$, so that also $v(s) < 0$. Therefore, we may study the trajectory of (35) in the region of the phase plane where both v and w are negative. System (35) admits the two equilibrium points

$$O = (0, 0) \quad \text{and} \quad P = \big(-2, -2(n-2)\big).$$

The point O is a saddle point, regardless the dimension $n \geqslant 3$, its stable manifold is the v-axis while the unstable manifold is tangent to the line $w = nv$. The point P is a sink if $n \geqslant 10$, a spiral sink if $n < 10$. Moreover, independently of the boundary conditions, from [24] (see also [12]) we know that:

Lemma 13. *(See [24].) Let $\Omega = B$, $g(u) = e^u$ and let u be a radial solution of the equation in* (1)*. Let $\Phi(s) = (v(s), w(s))$ be the corresponding trajectory of* (35)*, then*:

- *u is regular if and only if $\lim_{s\to-\infty} \Phi(s) = O$;*
- *u is singular if and only if $\Phi(s) = P$ for every $s \in (-\infty, 0)$.*

Here and below, Γ will denote the graph relative to the unstable manifold of O. Hence, Γ is the heteroclinic joining O (as $s \to -\infty$) with P (as $s \to +\infty$). Since system (35) is autonomous, if $s_1 \in (-\infty, 0)$ is such that the corresponding trajectory satisfies $w(s_1) = -\lambda e^{-v(s_1)/c}$, with no loss of generality, we can assume that $s_1 = 0$. Then, the problem of studying (35) under condition (36) corresponds to looking for the intersections of Γ with the graph of the function

$$\gamma_c : v \mapsto w = -\lambda e^{-v/c}, \quad v \in (-\infty, 0).$$

In view of Lemma 13, if u is a singular radial solution of problem (1) for some $c > 0$ and for $\lambda \in (0, \lambda^*]$, then the graph of the corresponding function γ_c contains P. This condition readily implies that, for every $c > 0$, there exists a unique value of λ giving rise to a singular radial solution of (1), $\lambda = \lambda_s := 2(n-2)e^{-2/c}$. Furthermore, since the corresponding trajectory is P, we also conclude that the unique singular radial solution is $u_s(x) := 2(\frac{1}{c} - \log|x|)$.

Let us go back to the proof of (iii), recalling that $n \geqslant 10$ and $c < c_n$. With the above choice of u_s and λ_s, if $h = h(c)$ denotes the function defined in Theorem 9, we have

$$\frac{h(c)}{|x|^2} < \frac{2(n-2)}{|x|^2} = \lambda_s e^{u_s} \quad \text{for all } x \in B \setminus \{0\}. \tag{37}$$

Since $c < c_n < \frac{n-2}{2}$, the function $\bar{u} \in H^1(B)$ in Theorem 9 achieves equality in (22), namely

$$\int_B |\nabla \bar{u}|^2 \, dx + c \int_{\partial B} \bar{u}^2 \, d\sigma = h(c) \int_B \frac{\bar{u}^2}{|x|^2} \, dx.$$

Hence, using (37) which holds for functions in $H^1(B)$, we get

$$\int_B |\nabla \bar{u}|^2 \, dx + c \int_{\partial B} \bar{u}^2 \, d\sigma < \lambda_s \int_B e^{u_s} \bar{u}^2 \, dx.$$

By Theorem 2, this tells us that $\lambda^* > \lambda_s$ and $u^* \neq u_s$, thereby completing the proof of statement (iii). $\square$

Proof of (iv). Since $c \geqslant c_n$, we have now that

$$\frac{h(c)}{|x|^2} \geqslant \frac{2(n-2)}{|x|^2} = \lambda_s e^{u_s} \quad \text{for all } x \in B \setminus \{0\}.$$

Using this inequality in (22) yields

$$\int_B |\nabla u|^2\,dx + c\int_{\partial B} u^2\,d\sigma \geqslant h(c)\int_B \frac{u^2}{|x|^2}\,dx \geqslant \lambda_s \int_B e^{u_s}u^2\,dx \quad \text{for all } u \in H^1(B).$$

Statement (iv) then follows from Theorem 2. □

6. Proof of Theorem 4

Proof of (i). We follow again [23]. For every $\lambda \in (0,\lambda^*)$, the minimal solution u_λ of problem (1) with $g(u) = (1+u)^p$ satisfies

$$\int_\Omega \nabla u_\lambda \nabla v\,dx + c\int_{\partial\Omega} u_\lambda v\,d\sigma = \lambda\int_\Omega (1+u_\lambda)^p v\,dx \quad \text{for all } v \in H^1(\Omega)$$

and

$$\int_\Omega |\nabla w|^2\,dx + c\int_{\partial\Omega} w^2\,d\sigma \geqslant \lambda p\int_\Omega (1+u_\lambda)^{p-1}w^2\,dx \quad \text{for all } w \in H^1(\Omega).$$

We choose as test functions, respectively, $v = (1+u_\lambda)^{(q-1)p}$ and $w = (1+u_\lambda)^{\frac{(q-1)p+1}{2}}$, where $q > 1$. Then we get

$$(q-1)p\int_\Omega (1+u_\lambda)^{(q-1)p-1}|\nabla u_\lambda|^2\,dx + c\int_{\partial\Omega} u_\lambda(1+u_\lambda)^{(q-1)p}\,d\sigma = \lambda\int_\Omega (1+u_\lambda)^{qp}\,dx,$$

and

$$\left(\frac{(q-1)p+1}{2}\right)^2\int_\Omega (1+u_\lambda)^{(q-1)p-1}|\nabla u_\lambda|^2\,dx + c\int_{\partial\Omega}(1+u_\lambda)^{(q-1)p+1}\,d\sigma \geqslant \lambda p\int_\Omega (1+u_\lambda)^{qp}\,dx.$$

By comparing the two expressions found, we conclude that

$$\frac{4(q-1)pc}{((q-1)p+1)^2}\int_{\partial\Omega}(1+u_\lambda)^{(q-1)p+1}\,d\sigma - c\int_{\partial\Omega} u_\lambda(1+u_\lambda)^{(q-1)p}\,d\sigma$$
$$\geqslant \lambda\left(\frac{4(q-1)p^2}{((q-1)p+1)^2} - 1\right)\int_\Omega (1+u_\lambda)^{qp}\,dx.$$

If we assume that $(1+u_\lambda)^p \notin L^q(\Omega)$ for some $q > 1$ such that

$$\frac{4(q-1)p^2}{((q-1)p+1)^2} > 1 \quad \Longleftrightarrow \quad q < \frac{3p-1+2\sqrt{p(p-1)}}{p},$$

then we get a contradiction. We now apply the same bootstrap argument of [23, Theorem 4] and we infer that $u^* \in L^\infty(\Omega)$ for $n < n_p := 6 + 4(\frac{1}{p-1} + \sqrt{1+\frac{1}{p-1}})$. Notice that the map $p \mapsto n_p$ is a decreasing function of p and tends to 10 as $p \to +\infty$, thus $n_p > 10$ for all $p > 1$ and, in particular, for all $p > \frac{n+2}{n-2}$. On the other hand, when $n \geqslant 11$ one may check that the condition $n < n_p$ is equivalent to $p < p_n$, with p_n defined as in the statement, and (ii) follows. □

Proof of (ii). Let $\Omega = B$, $g(u) = (1+u)^p$. For $r = |x| \in (0,1)$ we set

$$s = \log r \in (-\infty, 0), \qquad v(s) = \frac{d}{ds}z(s) \quad \text{and} \quad w(s) = -\lambda(p-1)e^{2s}e^{z(s)},$$

where z is such that $1+u(r)=e^{\frac{z(\log r)}{p-1}}=e^{\frac{z(s)}{p-1}}$, and we rewrite problem (1) as the dynamical system

$$\begin{cases} v'=w-(n-2)v-\dfrac{1}{p-1}v^2, \\ w'=(v+2)w, \end{cases} \tag{38}$$

with the Steklov condition

$$w(0)=-\lambda(p-1)\left(1+\frac{v(0)}{c(p-1)}\right)^{1-p}. \tag{39}$$

By definition, $w(s)<0$. Furthermore, since the equation combined with Lemma 10 gives $u'(r)<0$ in $(0,1)$, we also deduce that $v(s)<0$. Hence we study the trajectory of (38) in the region of the phase plane where both v and w are negative.

System (38) admits three stationary points: $O=(0,0)$, $P=(-2,-2(n-2-\frac{2}{p-1}))$ and $Q=(-(p-1)(n-2),0)$. Since $p>\frac{n+2}{n-2}$, the point P belongs to the region of the phase plane where both v and w are negative. The point O is a saddle point, independently of the dimension, its stable manifold is the v-axis while the unstable manifold is tangent to the line $w=nv$. The point Q, since $p>\frac{n+2}{n-2}$, is a saddle point, its stable manifold is the v-axis while the unstable manifold is tangent to the line $w=-v(p(n-2)-2)-(p-1)(n-2)(p(n-2)-2)$. For $n\geqslant 11$, the point P is a spiral sink if $p\in(\frac{n+2}{n-2},p_n)$ and a sink if $p\geqslant p_n$. If $n\leqslant 10$, P is always a stable spiral point.

Since the system (38) differs from (35) only for the negative term $-\frac{1}{p-1}v^2$ in the first equation, some minor changes allow us to argue as in [24, Lemmas 2 and 3] and prove:

Lemma 14. *Let $\Omega=B$, $g(u)=(1+u)^p$, with $p>\frac{n+2}{n-2}$, and let u be a radial solution of the equation in* (1). *Let $\Phi_p(s)=(v(s),w(s))$ be the corresponding trajectory of* (38), *then*:

- *u is regular if and only if $\lim_{s\to-\infty}\Phi_p(s)=O$;*
- *u is singular if and only if $\Phi_p(s)=P$ for every $s\in(-\infty,0)$.*

We denote with Γ_p the graph relative to the unstable manifold of O. Solutions of (38) under condition (39) correspond to intersections of Γ_p with the curve

$$\gamma_{c,p}(v):=-\lambda(p-1)\left(1+\frac{v}{c(p-1)}\right)^{1-p},\quad v\in(-\infty,0).$$

In view of Lemma 14, if $u=v_s$ is a singular radial solution of problem (1) for some $c>0$ and for $\lambda=\lambda_s$, then P belongs to the support of $\gamma_{c,p}$, that is λ_s must be as in (6). On the other hand, being λ_s well defined only for $c>2/(p-1)$, one has no singular radial solutions for $0<c\leqslant 2/(p-1)$ so that, in particular, the extremal solution is regular.

Furthermore, by invoking again Lemma 14, we conclude that the unique singular radial solution is

$$v_s(x):=\frac{c(p-1)}{c(p-1)-2}|x|^{-2/(p-1)}-1. \tag{40}$$

We conclude by noting that $v_s\in L^1(B)$, and hence it weakly solves problem (1) if and only if $p>\frac{n}{n-2}$, furthermore $v_s\in H^1(B)$ if and only if $p>\frac{n+2}{n-2}$. $\square$

Proof of (iii). Let $n\geqslant 11$ and $p\geqslant p_n$, this range is not covered by statement (i), hence the regularity of u^* is unknown. For $0<c\leqslant 2/(p-1)$, by (ii), we know that there exists no singular solution. Let $c\in(\frac{2}{p-1},c_{n,p})$, with $c_{n,p}$ as in (7) so that $c_{n,p}<\frac{n-2}{2}$. If $h=h(c)$ denotes the function defined in Theorem 9, we have

$$\frac{h(c)}{|x|^2}<\frac{p\lambda_s}{|x|^2}\left(\frac{c(p-1)}{c(p-1)-2}\right)^{p-1}=p\lambda_s(1+v_s)^{p-1}\quad\text{for all } x\in B\setminus\{0\}.$$

Repeating the same argument that follows (37), we deduce that $\lambda^*>\lambda_s$ and $u^*\neq v_s$.

If $c \geqslant c_{n,p}$, we have

$$\frac{h(c)}{|x|^2} \geqslant \frac{p\lambda_s}{|x|^2}\left(\frac{c(p-1)}{c(p-1)-2}\right)^{p-1} = p\lambda_s(1+v_s)^{p-1} \quad \text{for all } x \in B \setminus \{0\}.$$

Inserting this inequality in (22) yields

$$\int_B |\nabla u|^2\,dx + c\int_{\partial B} u^2\,d\sigma \geqslant h(c)\int_B \frac{u^2}{|x|^2}\,dx \geqslant p\lambda_s\int_B (1+v_s)^{p-1}u^2\,dx$$

for all $u \in H^1(B)$. The second part of statement (iii) then follows from Theorem 2. □

Remark 15. By arguing as in the proof of Theorem 4(iii), one may check that, when $c \geqslant \frac{n-2}{2}$, the singular solution v_s in (40) satisfies the stability condition (5) for any $\frac{n}{n-2} < p < \frac{n+2\sqrt{n-1}}{n-4+2\sqrt{n-1}} < \frac{n+2}{n-2}$ but, since it does not lie in the energy class, it cannot be the extremal solution. This strange phenomenon, that is the existence of solutions that cannot be approached by the branch of classical solutions, was already noticed under Dirichlet boundary conditions, see [5, Theorem 6.2].

7. Proof of Theorem 5

For $\lambda > 0$, let u be a radial solution to problem (1) with $g(u) = (1+u)^{\frac{n+2}{n-2}}$ and $\Omega = B$. Then $w := \lambda^{\frac{n-2}{4}}(u+1)$ solves

$$\begin{cases} -\Delta w = w^{\frac{n+2}{n-2}} & \text{in } B,\\ w > \lambda^{\frac{n-2}{4}} & \text{in } B,\\ w_\nu + cw = c\lambda^{(n-2)/4} & \text{on } \partial B. \end{cases} \tag{41}$$

Arguing as in [14, Theorem 7], one sees that u may be extended as a positive entire solution of the same equation in $\mathbb{R}^n$. But positive radial solutions of the equation

$$-\Delta w = w^{(n+2)/(n-2)} \quad \text{in } \mathbb{R}^n,$$

are explicitly given by

$$w_\varepsilon(x) := \big(n(n-2)\varepsilon\big)^{(n-2)/4}\big(\varepsilon + |x|^2\big)^{-(n-2)/2}, \quad \varepsilon > 0,$$

see [7] and references therein. The restriction to B of the functions w_ε solves problem (41) provided

$$w_\varepsilon'(1) + cw_\varepsilon(1) = c\lambda^{(n-2)/4} \quad \Longleftrightarrow \quad \varphi(\varepsilon) = \lambda^{n-2}, \tag{42}$$

with $\varphi(\varepsilon)$ as in (9). Equality (42) gives the bound $\varepsilon > \varepsilon_0(c) := \max\{0, \frac{n-2}{c} - 1\}$. The first part of the statement is then a consequence of (10).

By (42) we get

$$\lambda(\varepsilon) = \frac{n(n-2)\varepsilon[c(1+\varepsilon) - n + 2]^{4/(n-2)}}{c^{4/(n-2)}(1+\varepsilon)^{2n/(n-2)}} \to 0 \quad \Longleftrightarrow \quad \varepsilon \searrow \varepsilon_0 \quad \text{or} \quad \varepsilon \nearrow +\infty.$$

Furthermore, since $u_2(x) = (\lambda(\varepsilon_2))^{-(n-2)/4}w_{\varepsilon_2}(x) - 1$, we have

$$u_2(0) \sim \left(\frac{n(n-2)}{\lambda(\varepsilon_2)\varepsilon_2}\right)^{(n-2)/4} \quad \text{and} \quad u_2(1) \sim \left(\frac{n(n-2)\varepsilon_2}{\lambda(\varepsilon_2)(1+\varepsilon_2)^2}\right)^{(n-2)/4} \quad \text{as } \varepsilon_2 \searrow \varepsilon_0.$$

If $0 < c < n-2$, the second part of the statement follows from the fact that $\varepsilon_0(c) = \frac{n-2-c}{c}$, while if $c \geqslant n-2$, since $\varepsilon_0(c) = 0$, one has to take in account the fact that

$$\lambda^{n-2}(\varepsilon_2) \sim \begin{cases} (n(n-2))^{n-2}\varepsilon_2^{n+2} & \text{if } c = n-2,\\ (n(n-2)\varepsilon_2)^{n-2}(\frac{c-n+2}{c})^4 & \text{if } c > n-2 \end{cases} \quad \text{as } \varepsilon_2 \searrow 0.$$

Appendix A. Description of the solutions branch when $g(u) = e^u$

Assume that $\Omega = B$ and $g(u) = e^u$. In this section we describe analytically the (radial) solutions branch, thereby justifying the four pictures displayed in Fig. 1. For any $c > 0$, let

$$\lambda_s := 2(n-2)e^{-2/c}, \qquad u_s(x) := 2\left(\frac{1}{c} - \log|x|\right) \quad \text{and} \quad \gamma_c(v) := -\lambda e^{-v/c}, \quad \lambda \in (0, \lambda^*].$$

We denote by γ_c^* the curve corresponding to $\lambda = \lambda^*(c)$ and by γ_c^s the one corresponding to $\lambda = \lambda_s(c)$. We also denote by Γ the unstable manifold of O, that is the heteroclinic joining O and P. Note that $\lambda^* < \frac{nc}{e}$ in view of (28)–(29) since $\alpha = e$.

A radial solution u of problem (1) is strictly radially decreasing, hence we have $\|u\|_\infty = u(0)$. Thus, to plot the diagram, one has to study the dependence of $u(0)$ on λ. The continuity of the branch is a consequence of the implicit function theorem together with the convexity of $u \mapsto e^u$, see [9]. We first prove a result which is well known for the Dirichlet problem but which appears less obvious for the Steklov problem

Proposition 16. *Let $n \geqslant 2$, $c > 0$ and $0 < \lambda_1 < \lambda_2 < \lambda^*$. Assume that u_1 and u_2 are two regular radial solutions of problem* (1) *corresponding, respectively, to $\lambda = \lambda_1$ and $\lambda = \lambda_2$, then $u_1(0) \neq u_2(0)$.*

Proof. Any regular radial solution $u = u(r)$ of problem (1) for some $\lambda > 0$ corresponds to the part of Γ between O and its intersection $(\bar{v}, \overline{w})$ with γ_c. In the Dirichlet case, γ_c is the straight line $w = -\lambda$. Since by (34) any dilation with respect to r becomes a translation in the s-variable, there exists $\beta > 0$ such that $u(\beta r)$ solves the Dirichlet problem for $\lambda = -\overline{w}$.

By contradiction, assume that $u_1(0) = u_2(0) = \delta > 0$ and let $(\bar{v}_i, \overline{w}_i)$, $i = 1, 2$, be the corresponding points in the phase plane. Then there exist $\beta_1, \beta_2 > 0$ such that $\bar{u}_1(r) := u_1(\beta_1 r)$ and $\bar{u}_2(r) := u_2(\beta_2 r)$ vanish at $r = 1$ and solve the following Cauchy problem

$$\begin{cases} -u''(r) - \dfrac{n-1}{r}u'(r) = \lambda e^{u(r)}, & r \in (0,1), \\ u(0) = \delta > 0, \\ u'(0) = 0 \end{cases} \tag{43}$$

with $\lambda = \bar{\lambda}_1 = -\overline{w}_1$ and $\lambda = \bar{\lambda}_2 = -\overline{w}_2$, respectively. Moreover, $\bar{\lambda}_1 \neq \bar{\lambda}_2$, in view of the monotonicity of γ_c. Then, by uniqueness of the solution to the Cauchy problem,

$$\bar{u}_2(r) = \bar{u}_1\left(r\sqrt{\frac{\bar{\lambda}_2}{\bar{\lambda}_1}}\right)$$

which contradicts the fact that $\bar{u}_1(1) = \bar{u}_2(1) = 0$. □

The next result justifies the first picture in Fig. 1.

Proposition 17. *Let $n = 2$, then*:

- $\lambda^*(c) = c(\sqrt{c^2+4} - c)e^{-1-\frac{2}{c}+\frac{\sqrt{c^2+4}}{c}}$;
- *for every $\lambda \in (0, \lambda^*)$, there exist two regular radial solutions, the minimal one u_λ and a larger one U_λ. The maps $\lambda \mapsto u_\lambda(0)$ and $\lambda \mapsto U_\lambda(0)$ are, respectively, increasing and decreasing with respect to λ and $u_\lambda(0) \searrow 0$ whereas $U_\lambda(0) \nearrow +\infty$ as $\lambda \searrow 0$;*
- *for $\lambda = \lambda^*$ the extremal solution u^* is regular and the solutions branch has a turning point.*

Proof. When $n = 2$ the equilibrium points of system (35) are not isolated and lie on the v-axis while the trajectories in the phase plane are the parabolas $w = v^2/2 + 2v + C$, with $C < 2$. Hence, Γ is the parabola having equation $w(v) = v^2/2 + 2v$. Indeed, a slight modification of Lemma 13 shows that the trajectory corresponding to a regular solution of (1) starts in O and ends in $(-4,0)$ moving on Γ. On the other hand, there exist no singular radial solutions (the only candidate is $(-4,0)$) so that u^* is regular.

The statement on the number of solutions follows from the fact that we must intersect the (convex) parabola Γ with the concave graph of the function $w=\gamma_c(v)$. The minimal solution u_λ corresponds to the intersection which is closer to O, the large solution U_λ to the other intersection. The value of $\lambda^*(c)$ is explicitly determined by imposing to γ_c to be tangent to Γ.

By Lemma 10, we known that the branch of minimal solutions is strictly increasing with respect to λ, the behavior of the second branch arising from λ^* comes from Proposition 16. If $\lambda=0$, then $\gamma_c(v)\equiv 0$, hence the two intersection points with Γ are O and $(-4,0)$. To O corresponds the solution $u_0\equiv 0$. To $(-4,0)$ corresponds the solution $U_0(x)=-2\log|x|$. Hence, as $\lambda\to 0$, we get that $U_\lambda(0)\to+\infty$. □

Concerning the second picture in Fig. 1, we have:

Proposition 18. *Let* $3\leqslant n\leqslant 9$*, then*:

- $\lambda^*>\lambda_s$ *and the solutions branch has infinitely many turning points clustering on both sides of* λ_s;
- *for* $\lambda=\lambda^*$ *the extremal solution* u^* *is regular and the solutions branch has a turning point*;
- *if* $\lambda=\lambda_s$ *there exist infinitely many solutions.*

Proof. By Lemma 13, combined with the stability properties of the stationary points O and P, we know that the trajectory Φ tends to O as $s\to-\infty$ and spirals around P as $s\to+\infty$. Therefore, for some s, $\Phi(s)$ lies in the strip $-2<v(s)<0$ (on the right of P) in the phase plane. Hence, there exists a limit value $\lambda^*>\lambda_s$, such that the corresponding curve γ_c^* (not containing P) becomes tangent to Γ while γ_c^s contains P. For $\lambda>\lambda^*$ no intersections can be found and no solution exists. When $\lambda<\lambda^*$, the curve γ_c intersects at least twice Γ. The intersection point nearest to O corresponds to the minimal solution u_λ. Since we already know that the branch of minimal solutions is strictly increasing with respect to λ, Proposition 16 justifies the second part of the solutions branch displayed in Fig. 1. □

We now turn to high dimensions $n\geqslant 10$. In this case, the number

$$c_n:=\frac{n-2-\sqrt{(n-2)(n-10)}}{2}$$

is well defined. The next statement justifies the third picture in Fig. 1.

Proposition 19. *Let* $n\geqslant 10$ *and* $0<c<c_n$*, then*:

- $\lambda^*>\lambda_s$ *and for every* $\lambda\in(0,\lambda_s)$ *there exists a unique regular radial solution*;
- *for every* $\lambda\in(\lambda_s,\lambda^*)$ *there exist two regular radial solutions, the minimal one* u_λ *and a large one* U_λ*. Furthermore* $\lambda\mapsto u_\lambda(0)$ *and* $\lambda\mapsto U_\lambda(0)$ *are, respectively, increasing and decreasing with respect to* λ *and* $U_\lambda(0)\nearrow+\infty$ *as* $\lambda\searrow\lambda_s$;
- *if* $\lambda=\lambda^*$ *the extremal solution* u^* *is regular and the solutions branch has a turning point.*

Proof. By Lemma 13, we know that the trajectory Φ starts in O and ends in P. We show that, when $n\geqslant 10$, Γ lies in the region T, where

$$T:=\left\{(v,w)\colon\ -2\leqslant v\leqslant 0,\ -2(n-2)+(v+2)\frac{n-2+\sqrt{(n-2)(n-10)}}{2}\leqslant w\leqslant (n-2)v\right\},$$

and, furthermore, Γ is tangent in O to the line $w=nv$ and in P to the line

$$w=-2(n-2)+(v+2)\frac{n-2+\sqrt{(n-2)(n-10)}}{2}. \tag{44}$$

The tangent lines at the two stationary points are determined by the eigenvectors of the corresponding linearized system. More precisely, $w=nv$ is the tangent line to the unstable manifold of O, while (44) is the tangent line to the stable manifold of P corresponding to the eigenvalue having the smallest absolute value. Then, starting from O, a close look at (35) shows that v must lie in the interval $(-2,0)$ and furthermore $w \leqslant (n-2)v$. Assume by contradiction that Γ intersects the line (44) for some $s=\bar{s}$, with $v(\bar{s}) \in (-2,0)$. Some computations then give

$$\frac{dw(\bar{s})}{ds} - \frac{n-2+\sqrt{(n-2)(n-10)}}{2}\frac{dv(\bar{s})}{ds} = \frac{n-2+\sqrt{(n-2)(n-10)}}{2}\big(v(\bar{s})+2\big)^2 > 0.$$

Hence, recalling that $\frac{dv(\bar{s})}{ds} < 0$, we conclude that $\frac{dw}{dv} < \frac{n-2+\sqrt{(n-2)(n-10)}}{2}$, a contradiction.

When $\lambda = \lambda_s$, γ_c^s is tangent to Γ for $c = c_n$. When $c \in (0, c_n)$, we show that γ_c^s intersects Γ twice, in P and for some $\bar{v} \in (-2,0)$. Since, for $-2 \leqslant v \leqslant 0$, Γ is the graph of a function $w(v) = w(s(v))$, we study the sign of

$$F(v) := \gamma_c^s(v) - w(v) = -2(n-2)e^{-(v+2)/c} - w(v), \quad -2 \leqslant v \leqslant 0.$$

We have that

$$F'(v) = \gamma_c'(v) - \frac{dw(v)}{dv} = \frac{2(n-2)}{c}e^{-(v+2)/c} - \frac{(v+2)w(v)}{w(v)-(n-2)v}$$

and we observe that $F(-2)=0$, $F(0)<0$ and $F'(-2)>0$. Hence F admits at least one zero in the interval $(-2,0)$. Moreover, if $\bar{v} \in (0,2)$ is such that $F(\bar{v})=0$, we deduce

$$F'(\bar{v}) = 2e^{-(\bar{v}+2)/c}\left(\frac{n-2}{c} - \frac{\bar{v}+2}{\bar{v}+2e^{-(\bar{v}+2)/c}}\right) := 2e^{-(\bar{v}+2)/c}H(\bar{v}).$$

The sign of the function H tells us which is the position of the tangent vectors to γ_c and Γ when they intersect. Some computations give

$$H'(v) = \frac{2(c-(c+v+2)e^{-(\bar{v}+2)})}{c(v+2e^{-(\bar{v}+2)})^2} =: \frac{2h(v)}{c(v+2e^{-(\bar{v}+2)})^2},$$

but $h(-2)=0$ and $h'(v) = 2(v+2)e^{-(\bar{v}+2)}c^{-2} > 0$, so $H'(v)>0$ for every $v \in (-2,0)$. In terms of F this means that $F'(\bar{v})$ changes sign at most once in $(-2,0)$. If we assume by contradiction that there exist $-2 < v_1 < v_2 < 0$ such that $F(v_1) = 0 = F(v_2)$, the observations so far collected allow to conclude that $F'(v_1) < 0$, $F'(v_2) = 0$ and there exists $\delta > 0$ such that $F(v) < 0$, or equivalently $w(v) > \gamma_c(v)$, for $v \in (v_2+\delta, v_2+2\delta)$. Inserting this into $F'(v)$ we finally conclude that $F'(v) > 0$ for $v \in (v_2+\delta, v_2+2\delta)$, a contradiction.

As a consequence of the above discussion, we get that γ_c intersects Γ once, for every $\lambda \in (0,\lambda_s)$, and twice, for every $\lambda \in [\lambda_s, \lambda^*)$, where $\lambda^* > \lambda_s$ turns to be the value of λ in correspondence of which γ_c is tangent to Γ. To get the second statement, we repeat the arguments of Proposition 17 with minor changes. □

We conclude with the last picture in Fig. 1.

Proposition 20. *Let $n \geqslant 10$ and $c \geqslant c_n$, then:*

- *for every $\lambda \in (0,\lambda^*)$ there exists a unique regular solution;*
- *for $\lambda^* = \lambda_s$ the extremal solution u^* is singular and $u^* = u_s$.*

Proof. The behavior of the trajectory Γ is the same as described in Proposition 19. Furthermore γ_c^s is tangent to Γ at $c = c_n$ and lies below the line (44) if $c > c_n$. By this we conclude that, for any $\lambda \in (0,\lambda_s]$, γ_c intersects Γ just once and does not intersect Γ for $\lambda > \lambda_s$, hence $\lambda^* = \lambda_s$. □

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