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Conservation laws on complex networks

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Abstract

This paper considers a system described by a conservation law on a general network and deals with solutions to Cauchy problems. The main application is to vehicular traffic, for which we refer to the Lighthill–Whitham–Richards (LWR) model. Assuming to have bounds on the conserved quantity, we are able to prove existence of solutions to Cauchy problems for every initial datum in L^1_{loc}. Moreover Lipschitz continuous dependence of the solution with respect to initial data is discussed.

MSC: 90B20; 35L65

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1. Introduction

Various fluid dynamic models were developed in the literature in order to describe the evolution of vehicular traffic in roads. They treat traffic from a macroscopic point of view: just the evolution of macroscopic variables, such as density and average velocity of cars, is considered. The Lighthill–Whitham–Richards (LWR) model (see [34,37]), introduced in the 50s, is the prototype. It is based on the conservation of the number of cars and it consists of a single partial differential equation in conservation form.

From 1975 several second order models, i.e. models with two equations, were considered, see for example [1,13, 25,27,36,38–40], while a third order model was presented in [28]. An extension to multipopulation can be found; see [7]. We refer the reader to [6,24,29] for a general presentation of the various models.

More recently, a growing attention was devoted to extensions of the same models to networks; see for instance [4,11,12,21–23,30–32]. The interest was also motivated by other applications: data networks [19], supply chains [18,26], air traffic management [5], gas pipelines [2,14,15]. Here we focus on the LWR model on a network, but the results are of use to other research domains.

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The main interest is in the Cauchy problem for a complex network. In some previous papers [12,19,21,24], existence of weak entropic solutions was proved only for networks with nodes with at most two incoming and two outgoing arcs and some specific dynamics at nodes.

Our construction is based on the wave-front tracking method; see [9,17,33]. More precisely, first we consider Riemann problems at nodes, which are Cauchy problems with constant initial data on each arc. Notice that the only conservation of cars is not sufficient to determine a unique solution. Thus one has to prescribe solutions for every initial data and we call the relative map a *Riemann solver* at nodes. Then it is possible to construct approximate solutions using classical self-similar entropic solutions for Riemann problems inside arcs and an assigned Riemann solver at nodes. As usual, the approach relies on three estimates: the number of waves, the number of wave interactions and total variation of the solution. While these estimates are straightforward on a real line (see [9]), they become difficult to be proved on complex networks (see [24]). In particular one has to rely on estimates on the total variation of the flux of the solution.

We provide a general strategy to overcome the technical problems: three key properties of Riemann solvers are defined (see Definitions 8–10), which guarantee the needed bounds and thus the existence of solutions to Cauchy problems. Our approach is valid for general networks, with no limitation on the type of nodes: in particular we extend all results of the literature. The main technical novelty is to get bounds on the total variation (in space) of solution flux via bounds on the positive variation (in time) of incoming fluxes at nodes.

To prove the validity of our approach, we show that the three key properties are shared by various Riemann solvers proposed in the literature. In particular, we consider three different kind of solutions at J, which we call Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$. The Riemann solver $\mathcal{RS}_1$ was proposed for vehicular traffic in [12]. It prescribes first a fixed distribution of traffic in outgoing arcs, and then the maximization of the flux through the node. The Riemann solver $\mathcal{RS}_2$ was introduced for data networks in [19]: first one maximizes the flux through the node and then prescribes a distribution of traffic. The Riemann solver $\mathcal{RS}_3$ models car traffic at T-junctions; see [35]. Thanks to finite velocity of waves, one can reduce to treat the case of a single node, with arcs of infinite length.

The continuous dependence of solutions with respect to initial data is an open problem in the case of Riemann solver $\mathcal{RS}_1$. We remark that, in general, the Lipschitz continuous dependence with respect to initial data does not hold; see [12,24]. As regards the Riemann solver $\mathcal{RS}_2$, we prove the Lipschitz continuous dependence with respect to initial conditions, by viewing L^1 as a Finsler manifold and considering "generalized tangent vectors". This method was proposed by Bressan [8] and improved in [10].

The paper is organized as follows. Section 2 contains the main definitions and notations. Section 3 deals with Riemann problems at the node J, while the Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$ are analyzed in Section 4. In Section 5 there are the statements of the main result about existence of solutions to Cauchy problems in the network, while in Section 5.1 the wave-front method is briefly described. Sections 5.2 and 5.3 give, respectively, some bounds on the total variation of the flux for approximate solutions and the proof of the existence of a wave-front tracking approximate solution. Section 5.4 contains the proof about existence of a solution to the Cauchy problem, while Section 6 deals with the Lipschitz continuous dependence of the solution with respect to initial conditions. Finally Appendix A contains some technical results.

2. Basic definitions and notations

A complex networks is formed by a collection of arcs and nodes. However, relying on finite velocity of waves, one can reduce to consider Cauchy problems for single nodes; see Theorem 4.3.9 of [24]. Thus, from now on, for sake of simplicity, we focus on a single node with arcs of infinite length.

Consider a node J with n incoming arcs $I_1, \dots, I_n$ and m outgoing arcs $I_{n+1}, \dots, I_{n+m}$. We model each incoming arc I_i ($i \in \{1, \dots, n\}$) of the node with the real interval $I_i =]-\infty, 0]$. Similarly we model each outgoing arc I_j ($j \in \{n+1, \dots, n+m\}$) of the node with the real interval $I_j = [0, +\infty[$. On each arc I_l ($l \in \{1, \dots, n+m\}$) we consider the partial differential equation

$$(\rho_l)_t + f(\rho_l)_x = 0, \tag{1}$$

where $\rho_l = \rho_l(t, x) \in [0, \rho_{max}]$, is the *density* of cars, $v_l = v_l(\rho_l)$ is the *velocity* of cars and $f(\rho_l) = v_l(\rho_l)\rho_l$ is the *flux*. Hence the datum is given by a finite collection of functions ρ_l defined on $[0, +\infty[\times I_l$. For simplicity, we put $\rho_{max} = 1$.

On the flux f we make the following assumption

($\mathcal{F}$) $f:[0,1]\to\mathbb{R}$ is a Lipschitz continuous and concave function satisfying
1. $f(0)=f(1)=0$;
2. there exists a unique $\sigma\in\,]0,1[$ such that f is strictly increasing in $[0,\sigma[$ and strictly decreasing in $]\sigma,1]$.

The definitions of entropic solutions on arcs and weak solutions at nodes are as follows.

Definition 1. A function $\rho_l\in C([0,+\infty[;L^1_{loc}(I_l))$ is an entropy-admissible solution to (1) in the arc I_l if, for every $k\in[0,1]$ and every $\tilde{\varphi}:[0,+\infty[\,\times I_l\to\mathbb{R}$ smooth, positive with compact support in $]0,+\infty[\,\times(I_l\setminus\{0\})$, it holds

$$\int_0^{+\infty}\int_{I_l}\left(|\rho_l-k|\frac{\partial\tilde{\varphi}}{\partial t}+\operatorname{sgn}(\rho_l-k)\big(f(\rho_l)-f(k)\big)\frac{\partial\tilde{\varphi}}{\partial x}\right)dx\,dt\geqslant 0. \tag{2}$$

Definition 2. A collection of functions $\rho_l\in C([0,+\infty[;L^1_{loc}(I_l))$, where $l\in\{1,\dots,n+m\}$, is a weak solution at J if

1. for every $l\in\{1,\dots,n+m\}$, the function ρ_l is an entropy-admissible solution to (1) in the arc I_l;
2. for every $l\in\{1,\dots,n+m\}$ and for a.e. $t>0$, the function $x\mapsto\rho_l(t,x)$ has a version with bounded total variation;
3. for a.e. $t>0$, it holds
$$\sum_{i=1}^{n}f\big(\rho_i(t,0-)\big)=\sum_{j=n+1}^{n+m}f\big(\rho_j(t,0+)\big), \tag{3}$$
where ρ_l stands for the version with bounded total variation of 2.

For a collection of functions $\rho_l\in C([0,+\infty[;L^1_{loc}(I_l))$ ($l\in\{1,\dots,n+m\}$) such that, for every $l\in\{1,\dots,n+m\}$ and a.e. $t>0$ the map $x\mapsto\rho_l(t,x)$ has a version with bounded total variation, we define the functionals

$$\Gamma(t):=\sum_{i=1}^{n}f\big(\rho_i(t,0-)\big) \tag{4}$$

and

$$\text{Tot.Var.}_f(t):=\sum_{l=1}^{n+m}\text{Tot.Var.}\big(f\big(\rho_l(t,\cdot)\big)\big). \tag{5}$$

It is clear that these functionals are well defined for a.e. positive time. By definition we easily derive the bound

$$0\leqslant\Gamma(t)\leqslant nf(\sigma) \tag{6}$$

for a.e. $t\geqslant 0$.

We now define a set of matrices to describe solutions at nodes. First consider the set

$$\mathcal{A}:=\left\{A=\{a_{ji}\}_{i=1,\dots,n,\ j=n+1,\dots,n+m}:\ 0<a_{ji}<1\ \forall i,j,\ \sum_{j=n+1}^{n+m}a_{ji}=1\ \forall i\right\}. \tag{7}$$

Let $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbb{R}^n$. For every $i=1,\dots,n$, we denote $H_i=\{e_i\}^\perp$. If $A\in\mathcal{A}$, then we write, for every $j=n+1,\dots,n+m$, $a_j=(a_{j1},\dots,a_{jn})\in\mathbb{R}^n$ and $H_j=\{a_j\}^\perp$. Let $\mathcal{K}$ be the set of indices $\mathbf{k}=(k_1,\dots,k_\ell)$, $1\leqslant\ell\leqslant n-1$, such that $0\leqslant k_1<k_2<\cdots<k_\ell\leqslant n+m$ and for every $\mathbf{k}\in\mathcal{K}$ define

$$H_{\mathbf{k}}=\bigcap_{h=1}^{\ell}H_{k_h}.$$

Writing $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^n$ and following [12] we define the set

$$\mathfrak{N}:=\big\{A\in\mathcal{A}:\ \mathbf{1}\notin H_{\mathbf{k}}^\perp\text{ for every }\mathbf{k}\in\mathcal{K}\big\}. \tag{8}$$

Notice that, if $n > m$, then $\mathfrak{N} = \emptyset$. The matrices of $\mathfrak{N}$ will give rise to a unique solution to Riemann problems at J.

For later use, define also the set

$$\Theta = \left\{ \boldsymbol{\theta} = (\theta_1, \ldots, \theta_{n+m}) \in \mathbb{R}^{n+m} \colon \theta_1 > 0, \ldots, \theta_{n+m} > 0, \ \sum_{i=1}^{n} \theta_i = \sum_{j=n+1}^{n+m} \theta_j = 1 \right\}. \tag{9}$$

3. The Riemann problem

Fix $\rho_{1,0}, \ldots, \rho_{n+m,0} \in [0,1]$. Consider the Riemann problem at J

$$\begin{cases} \dfrac{\partial}{\partial t}\rho_l + \dfrac{\partial}{\partial x} f(\rho_l) = 0, \\ \rho_l(0,\cdot) = \rho_{l,0}, \end{cases} \quad l \in \{1, \ldots, n+m\}. \tag{10}$$

Remark 1. The Riemann problem (10) can be interpreted as a collection of initial–boundary value problems, one for each arc, with coupling conditions. Concerning this type of problems for conservation laws, we refer to [3] and to [20] for general theory.

Conditions 2 and 3 of Definition 4 ensure that, on each arc, an admissible solution to the corresponding initial–boundary value problem is achieved. See also Remark 2 below.

A solution to the Riemann problem at J is defined following Definition 2, i.e.

Definition 3. A solution to the Riemann problem (10) is a weak solution at J, in the sense of Definition 2, such that $\rho_l(0,x) = \rho_{l,0}$ for every $l \in \{1, \ldots, n+m\}$ and for a.e. $x \in I_l$.

We are now ready to introduce the key concept of Riemann solver at J.

Definition 4. A Riemann solver $\mathcal{RS}$ is a function

$$\begin{aligned} \mathcal{RS}\colon \quad [0,1]^{n+m} &\longrightarrow [0,1]^{n+m}, \\ (\rho_{1,0}, \ldots, \rho_{n+m,0}) &\longmapsto (\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) \end{aligned}$$

satisfying the following

1. $\sum_{i=1}^{n} f(\bar{\rho}_i) = \sum_{j=n+1}^{n+m} f(\bar{\rho}_j)$;
2. for every $i \in \{1, \ldots, n\}$, the classical Riemann problem
$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = \begin{cases} \rho_{i,0}, & \text{if } x < 0, \\ \bar{\rho}_i, & \text{if } x > 0, \end{cases} \end{cases}$$
is solved with waves with negative speed;
3. for every $j \in \{n+1, \ldots, n+m\}$, the classical Riemann problem
$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = \begin{cases} \bar{\rho}_j, & \text{if } x < 0, \\ \rho_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$
is solved with waves with positive speed.

Remark 2. By Definition 4, a Riemann solver produces a solution to the Riemann problem (10), which conserves the mass at J and which generates waves with negative speed in incoming arcs and waves with positive speed in outgoing arcs.

To effectively describe a solution to Riemann problems at J, a Riemann solver needs to satisfy the following consistency condition:

Definition 5. We say that a Riemann solver $\mathcal{RS}$ satisfies the consistency condition if

$$\mathcal{RS}\big(\mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0})\big)=\mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0})$$

for every $(\rho_{1,0},\dots,\rho_{n+m,0})\in[0,1]^{n+m}$.

Now we can state the three key properties of a Riemann solver, which will ensure the necessary bounds on approximate solutions (via wave-front tracking) and thus the existence of solutions to Cauchy problems. First we need some additional notation.

Definition 6. We say that $(\rho_{1,0},\dots,\rho_{n+m,0})$ is an equilibrium for the Riemann solver $\mathcal{RS}$ if

$$\mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0})=(\rho_{1,0},\dots,\rho_{n+m,0}).$$

Definition 7. We say that a datum $\rho_i\in[0,1]$ in an incoming arc is a good datum if $\rho_i\in[\sigma,1]$ and a bad datum otherwise.

We say that a datum $\rho_j\in[0,1]$ in an outgoing arc is a good datum if $\rho_i\in[0,\sigma]$ and a bad datum otherwise.

The first property requires that equilibria are determined only by bad data values, more precisely:

Definition 8. We say that a Riemann solver $\mathcal{RS}$ has the property (P1) if the following condition holds. Given $(\rho_{1,0},\dots,\rho_{n+m,0})$ and $(\rho'_{1,0},\dots,\rho'_{n+m,0})$ two initial data such that $\rho_{l,0}=\rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum, then

$$\mathcal{RS}(\rho_{1,0},\dots,\rho_{n+m,0})=\mathcal{RS}\big(\rho'_{1,0},\dots,\rho'_{n+m,0}\big). \tag{11}$$

The second property asks for bounds in the increase of the flux variation for waves interacting with J. More precisely the latter should be bounded in terms of the strength of the interacting wave as well as the variation in the incoming fluxes.

Definition 9. We say that a Riemann solver $\mathcal{RS}$ has the property (P2) if there exists a constant $C\geqslant 1$ such that the following condition holds. For every equilibrium $(\rho_{1,0},\dots,\rho_{n+m,0})$ of $\mathcal{RS}$ and for every wave $(\rho_{l,0},\rho_l)$ $(l\in\{1,\dots,n+m\})$ interacting with J at time $\bar t>0$ and producing waves in the arcs according to $\mathcal{RS}$, we have

$$\mathrm{Tot.Var.}_f(\bar t+)-\mathrm{Tot.Var.}_f(\bar t-)\leqslant C\min\big\{\big|f(\rho_{l,0})-f(\rho_l)\big|,\big|\Gamma(\bar t+)-\Gamma(\bar t-)\big|\big\}. \tag{12}$$

Finally, we state the third property: a wave interacting with J and provoking a flux decrease on a specific arc should also gives rise to a decrease in the incoming fluxes.

Definition 10. We say that a Riemann solver $\mathcal{RS}$ has the property (P3) if, for every equilibrium $(\rho_{1,0},\dots,\rho_{n+m,0})$ of $\mathcal{RS}$ and for every wave $(\rho_{l,0},\rho_l)$ $(l\in\{1,\dots,n+m\})$ with $f(\rho_l)<f(\rho_{l,0})$ interacting with J at time $\bar t>0$ and producing waves in the arcs according to $\mathcal{RS}$, we have

$$\Gamma(\bar t+)\leqslant\Gamma(\bar t-). \tag{13}$$

4. Riemann solvers

In this section we present some different Riemann solvers for the Riemann problem (10), proposed in recent literature. We verify for all of them the three key properties stated in the previous section.

Let us first illustrate some common facts to all Riemann solvers. Introduce the following sets and notations

1. for every $i\in\{1,\dots,n\}$ define

$$\Omega_i=\begin{cases}[0,f(\rho_{i,0})], & \text{if } 0\leqslant\rho_{i,0}\leqslant\sigma,\\ [0,f(\sigma)], & \text{if } \sigma\leqslant\rho_{i,0}\leqslant 1;\end{cases} \tag{14}$$

2. for every $j \in \{n+1, \dots, n+m\}$ define

$$\Omega_j = \begin{cases} [0, f(\sigma)], & \text{if } 0 \leqslant \rho_{j,0} \leqslant \sigma, \\ [0, f(\rho_{j,0})], & \text{if } \sigma \leqslant \rho_{j,0} \leqslant 1; \end{cases} \tag{15}$$

3. for every $l \in \{1, \dots, n+m\}$ denote

$$\gamma_l^{max} = \max \Omega_l. \tag{16}$$

For a flux satisfying $(\mathcal{F})$, we define:

Definition 11. Let $\tau : [0,1] \to [0,1]$ be the map such that:

1. $f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$;
2. $\tau(\rho) \neq \rho$ for every $\rho \in [0,1] \setminus \{\sigma\}$.

Clearly, the function τ is well defined and satisfies

$$0 \leqslant \rho \leqslant \sigma \iff \sigma \leqslant \tau(\rho) \leqslant 1, \qquad \sigma \leqslant \rho \leqslant 1 \iff 0 \leqslant \tau(\rho) \leqslant \sigma.$$

Then we can state the following:

Proposition 1. *The following statements hold.*

1. *For every $i \in \{1, \dots, n\}$, an element $\bar\gamma$ belongs to Ω_i if and only if there exists $\bar\rho_i \in [0,1]$ such that $f(\bar\rho_i) = \bar\gamma$ and point* 2 *of Definition* 4 *is satisfied.*
2. *For every $j \in \{n+1, \dots, n+m\}$, an element $\bar\gamma$ belongs to Ω_j if and only if there exists $\bar\rho_j \in [0,1]$ such that $f(\bar\rho_j) = \bar\gamma$ and point* 3 *of Definition* 4 *is satisfied.*

Proof. From 2 of Definition 4, $\bar\rho_i \in \{\rho_{i,0}\} \cup]\tau(\rho_{i,0}), 1]$ if $\rho_{i,0} < \sigma$, while $\bar\rho_i \in [\sigma, 1]$ otherwise. By definition of Ω_i, the first statement follows.

Similarly, by 3 of Definition 4, $\bar\rho_j \in \{\rho_{j,0}\} \cup [0, \tau(\rho_{j,0})[$ if $\rho_{j,0} > \sigma$, while $\bar\rho_j \in [0, \sigma]$ otherwise. By definition of Ω_j, the second statement follows. □

4.1. Riemann solver $\mathcal{RS}_1$

In this subsection, we consider the Riemann solver introduced for vehicular traffic in [12]. The construction can be summarized as follows.

1. Fix a matrix $A \in \mathfrak{N}$ and consider the closed, convex and not empty set

$$\Omega = \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i \colon A \cdot (\gamma_1, \dots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}. \tag{17}$$

2. Find the point $(\bar\gamma_1, \dots, \bar\gamma_n) \in \Omega$ which maximizes the function

$$E(\gamma_1, \dots, \gamma_n) = \gamma_1 + \dots + \gamma_n, \tag{18}$$

and define $(\bar\gamma_{n+1}, \dots, \bar\gamma_{n+m})^T := A \cdot (\bar\gamma_1, \dots, \bar\gamma_n)^T$. Since $A \in \mathfrak{N}$, the point $(\bar\gamma_1, \dots, \bar\gamma_n)$ is uniquely defined.
3. For every $i \in \{1, \dots, n\}$, set $\bar\rho_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar\gamma_i$, or by the solution to $f(\rho) = \bar\gamma_i$ such that $\bar\rho_i \geqslant \sigma$. For every $j \in \{n+1, \dots, n+m\}$, set $\bar\rho_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar\gamma_j$, or by the solution to $f(\rho) = \bar\gamma_j$ such that $\bar\rho_j \leqslant \sigma$. Finally, define $\mathcal{RS}_1 : [0,1]^{n+m} \to [0,1]^{n+m}$ by

$$\mathcal{RS}_1(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\bar\rho_1, \dots, \bar\rho_n, \bar\rho_{n+1}, \dots, \bar\rho_{n+m}). \tag{19}$$

We now verify the consistency condition as well as properties (P1)–(P3).

Lemma 1. *The function defined in* (19) *satisfies the consistency condition*

$$\mathcal{RS}_1\big(\mathcal{RS}_1(\rho_{1,0},\dots,\rho_{n+m,0})\big)=\mathcal{RS}_1(\rho_{1,0},\dots,\rho_{n+m,0}) \tag{20}$$

for every $(\rho_{1,0},\dots,\rho_{n+m,0})\in[0,1]^{n+m}$.

For a proof, see [12,24].

Proposition 2. *The Riemann solver* $\mathcal{RS}_1$ *satisfies property* (P1).

Proof. Fix two initial data $(\rho_{1,0},\dots,\rho_{n+m,0})$ and $(\rho'_{1,0},\dots,\rho'_{n+m,0})$ with the property that $\rho_{l,0}=\rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum. For every $l\in\{1,\dots,n+m\}$, consider Ω_l and Ω'_l the sets (14)–(15) respectively for the initial data $(\rho_{1,0},\dots,\rho_{n+m,0})$ and $(\rho'_{1,0},\dots,\rho'_{n+m,0})$. We easily deduce that $\Omega_l=\Omega'_l$ for every $l\in\{1,\dots,n+m\}$. Indeed if $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad data, then $\rho_{l,0}=\rho'_{l,0}$ and so $\Omega_l=\Omega'_l$. If $\rho_{l,0}$ is a good datum, then also $\rho'_{l,0}$ is a good datum (and vice versa) and so $\Omega_l=\Omega'_l=[0,f(\sigma)]$. Consequently we have the thesis, since the solution depends only on these sets and on the matrix A. $\square$

Lemma 2. *Fix an equilibrium* $(\rho_{1,0},\dots,\rho_{n+m,0})$ *for* $\mathcal{RS}_1$ *and consider, for some* $l\in\{1,\dots,n+m\}$, $\rho_l\in[0,1]$ *such that the wave* $(\rho_l,\rho_{l,0})$ *has positive speed if* $l\leqslant n$, *while the wave* $(\rho_{l,0},\rho_l)$ *has negative speed if* $l>n$. *There exists a constant* $\tilde C\geqslant 1$ *such that*

$$\sum_{\substack{h=1\\h\neq l}}^{n+m}\big|f(\hat\rho_h)-f(\rho_{h,0})\big|+\big|f(\hat\rho_l)-f(\rho_l)\big|\leqslant\tilde C\big|f(\rho_l)-f(\rho_{l,0})\big|, \tag{21}$$

where

$$(\hat\rho_1,\dots,\hat\rho_{n+m})=\mathcal{RS}_1(\rho_{1,0},\dots,\rho_{l-1,0},\rho_l,\rho_{l+1,0},\dots,\rho_{n+m,0}).$$

Proof. Denote with Ω^- and with Ω the sets, defined in (17), respectively for the initial data $(\rho_{1,0},\dots,\rho_{n+m,0})$ and $(\rho_{1,0},\dots,\rho_l,\dots,\rho_{n+m,0})$. It is easy to see that, by construction, $\Omega^-\subseteq\Omega$ or $\Omega\subseteq\Omega^-$. We have two different possibilities:

1. $\max_{(\gamma_1,\dots,\gamma_n)\in\Omega^-}E(\gamma_1,\dots,\gamma_n)=\max_{(\gamma_1,\dots,\gamma_n)\in\Omega}E(\gamma_1,\dots,\gamma_n)$,
2. $\max_{(\gamma_1,\dots,\gamma_n)\in\Omega^-}E(\gamma_1,\dots,\gamma_n)\neq\max_{(\gamma_1,\dots,\gamma_n)\in\Omega}E(\gamma_1,\dots,\gamma_n)$,

where E is the linear function defined in (18). If

$$\max_{(\gamma_1,\dots,\gamma_n)\in\Omega^-}E(\gamma_1,\dots,\gamma_n)=\max_{(\gamma_1,\dots,\gamma_n)\in\Omega}E(\gamma_1,\dots,\gamma_n),$$

then, since $A\in\mathfrak{N}$, there exists a unique

$$(\bar\gamma_1,\dots,\bar\gamma_n)=\big(f(\rho_{1,0}),\dots,f(\rho_{n,0})\big)\in\Omega\cap\Omega^-$$

such that

$$E(\bar\gamma_1,\dots,\bar\gamma_n)=\max_{(\gamma_1,\dots,\gamma_n)\in\Omega^-}E(\gamma_1,\dots,\gamma_n)=\max_{(\gamma_1,\dots,\gamma_n)\in\Omega}E(\gamma_1,\dots,\gamma_n).$$

Therefore there is only one wave, produced by $\mathcal{RS}_1$ at J, in the arc I_l. Hence

$$\sum_{\substack{h=1\\h\neq l}}^{n+m}\big|f(\hat\rho_h)-f(\rho_{h,0})\big|+\big|f(\hat\rho_l)-f(\rho_l)\big|=\big|f(\hat\rho_l)-f(\rho_l)\big|=\big|f(\rho_{l,0})-f(\rho_l)\big|$$

and the conclusion follows.

Consider the other case, i.e.

$$\max_{(\gamma_1,\dots,\gamma_n)\in\Omega^-}E(\gamma_1,\dots,\gamma_n)\neq\max_{(\gamma_1,\dots,\gamma_n)\in\Omega}E(\gamma_1,\dots,\gamma_n).$$

Denote with $(\gamma_1^-, \ldots, \gamma_n^-) \in \Omega^-$ and with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \Omega$ the points of maximum of E respectively on Ω^- and on Ω. Clearly, we have that $(\gamma_1^-, \ldots, \gamma_n^-) = (f(\rho_{1,0}), \ldots, f(\rho_{n,0}))$, $(\gamma_1^-, \ldots, \gamma_n^-) \in \partial\Omega^-$ and $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \partial\Omega$. Since the directions of the faces of Ω^- and Ω depend only on the coefficients of A and the difference between the two sets depends only by the variation of a single constraint, then there exists a constant $\bar{C}$ such that

$$\left|\left(\gamma_1^-, \ldots, \gamma_n^-\right) - (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)\right| \leqslant \bar{C}\left|f(\rho_{l,0}) - f(\rho_l)\right|.$$

Hence

$$\begin{aligned}\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| &\leqslant \sum_{h=1}^{n+m}\left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| \\ &\leqslant 2\bar{C}\sum_{i=1}^{n}|\bar{\gamma}_i - \gamma_{i,0}| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| \leqslant (2\bar{C}+1)\left|f(\rho_{l,0}) - f(\rho_l)\right|\end{aligned}$$

and the conclusion follows. □

Proposition 3. *The Riemann solver $\mathcal{RS}_1$ satisfies property* (P2).

Proof. Fix an equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ for $\mathcal{RS}_1$ and $l \in \{1, \ldots, n+m\}$. Assume $l \leqslant n$, $\rho_l \in [0,1]$ is such that the wave $(\rho_l, \rho_{l,0})$ has positive speed and interacts with J at time $\bar{t}$, the other case being similar. Define

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \ldots, \rho_{n+m,0}).$$

Lemma 2 implies that

$$\begin{aligned}\text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) &= \sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| - \left|f(\rho_{l,0}) - f(\rho_l)\right| \\ &\leqslant (\tilde{C} - 1)\left|f(\rho_{l,0}) - f(\rho_l)\right|.\end{aligned}$$

Clearly we have $\Gamma(\bar{t}-) = \sum_{i=1}^{n} f(\rho_{i,0})$ and $\Gamma(\bar{t}+) = \sum_{i=1}^{n} f(\hat{\rho}_i)$. Since the direction of the faces of the set Ω, defined in (17), depend only on the matrix $A \in \mathfrak{N}$ and the solution for the flux lies on the boundary of Ω, we have that $|\Gamma(\bar{t}-) - \Gamma(\bar{t}+)|$ is proportional to

$$\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| - \left|f(\rho_{l,0}) - f(\rho_l)\right|$$

and so the conclusion follows. □

Proposition 4. *The Riemann solver $\mathcal{RS}_1$ satisfies property* (P3).

Proof. Fix an equilibrium $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ for $\mathcal{RS}_1$ and $l \in \{1, \ldots, n+m\}$. Consider just the case $l \leqslant n$, the other case being similar. Assume that $\rho_l \in [0,1]$ is such that the wave $(\rho_l, \rho_{l,0})$ has positive speed, interacts with J at time $\bar{t}$ and $f(\rho_l) < f(\rho_{l,0})$. Define

$$(\hat{\rho}_1, \ldots, \hat{\rho}_{n+m}) = \mathcal{RS}(\rho_{1,0}, \ldots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \ldots, \rho_{n+m,0}).$$

The Rankine–Hugoniot condition implies that $\rho_l < \rho_{l,0}$ and so ρ_l is a bad datum. Call Ω^- and Ω^+ respectively the sets (17) for the initial data $(\rho_{1,0}, \ldots, \rho_{n+m,0})$ and $(\rho_{1,0}, \ldots, \rho_l, \ldots, \rho_{n+m,0})$. Since ρ_l is a bad datum and $f(\rho_l) < f(\rho_{l,0})$, then $\Omega^+ \subseteq \Omega^-$ and so

$$\Gamma(\bar{t}-) = \sum_{i=1}^{n} f(\rho_{i,0}) \geqslant \sum_{i=1}^{n} f(\hat{\rho}_i) = \Gamma(\bar{t}+).$$

The proof is finished. □

4.2. Riemann solver $\mathcal{RS}_2$

In this subsection, we consider the Riemann solver, introduced in [19] for data networks; see also [24]. The construction consists of the following steps.

1. Fix $\boldsymbol{\theta} \in \Theta$ and define
$$\Gamma_{inc} = \sum_{i=1}^{n} \gamma_i^{max}, \qquad \Gamma_{out} = \sum_{j=n+1}^{n+m} \gamma_j^{max},$$
then the maximal possible through-flow at the crossing is
$$\Gamma = \min\{\Gamma_{inc}, \Gamma_{out}\}.$$
2. Introduce the closed, convex and not empty sets
$$I = \left\{ (\gamma_1, \ldots, \gamma_n) \in \prod_{i=1}^{n} \Omega_i \colon \sum_{i=1}^{n} \gamma_i = \Gamma \right\},$$
$$J = \left\{ (\gamma_{n+1}, \ldots, \gamma_{n+m}) \in \prod_{j=n+1}^{n+m} \Omega_j \colon \sum_{j=n+1}^{n+m} \gamma_j = \Gamma \right\}.$$
3. Denote with $(\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$ the orthogonal projection on the convex set I of the point $(\Gamma\theta_1, \ldots, \Gamma\theta_n)$ and with $(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m})$ the orthogonal projection on the convex set J of the point $(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m})$.
4. For every $i \in \{1, \ldots, n\}$, define $\bar{\rho}_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or by the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geqslant \sigma$. For every $j \in \{n+1, \ldots, n+m\}$, define $\bar{\rho}_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or by the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leqslant \sigma$. Finally, define $\mathcal{RS}_2 : [0,1]^{n+m} \to [0,1]^{n+m}$ by
$$\mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0}) = (\bar{\rho}_1, \ldots, \bar{\rho}_n, \bar{\rho}_{n+1}, \ldots, \bar{\rho}_{n+m}). \tag{22}$$

The following result holds.

Lemma 3. *The function defined in* (22) *satisfies the consistency condition*
$$\mathcal{RS}_2\big(\mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0})\big) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0}) \tag{23}$$
for every $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

Proof. Consider $(\rho_{1,0}, \ldots, \rho_{n+m,0}) \in [0,1]^{n+m}$, call $\Gamma_{0,inc}$, $\Gamma_{0,out}$, Γ_0 the numbers defined in 1 of $\mathcal{RS}_2$ and call I_0 and J_0 the sets defined in 2 of $\mathcal{RS}_2$. Let
$$(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \ldots, \rho_{n+m,0})$$
and
$$(\bar{\gamma}_1, \ldots, \bar{\gamma}_{n+m}) = \big(f(\bar{\rho}_1), \ldots, f(\bar{\rho}_{n+m})\big).$$
Similarly to above, call Γ_{inc}, Γ_{out}, Γ, the numbers defined in 1 and I and J the sets defined in 2 with respect to the initial condition $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})$. In order to prove (23), we need to consider the following possibilities.

1. $\Gamma_0 = \Gamma_{0,inc} \leqslant \Gamma_{0,out}$.
2. $\Gamma_0 = \Gamma_{0,out} \leqslant \Gamma_{0,inc}$.

We restrict to the first case, since the second one is completely symmetric. For every $i \in \{1, \ldots, n\}$, $\bar{\rho}_i \in \{\rho_{i,0}, \sigma\}$. More precisely, if $\rho_{i,0} < \sigma$, then $\bar{\rho}_i = \rho_{i,0}$, while if $\rho_{i,0} \geqslant \sigma$, then $\bar{\rho}_i = \sigma$.

Applying $\mathcal{RS}_2$ to the point $(\bar{\rho}_1, \ldots, \bar{\rho}_{n+m})$, we deduce $\Gamma_{inc} = \Gamma_{0,inc}$, $\Gamma_{out} \geqslant \Gamma_{0,out}$, $\Gamma = \Gamma_0$, $I = I_0$, and $J_0 \subseteq J$. More precisely, $\Gamma_{out} > \Gamma_{0,out}$ if and only if there exists $j \in \{n+1, \ldots, n+m\}$ such that $\rho_{j,0} > \sigma$ and $\bar{\rho}_j \leqslant \sigma$. Define
$$\tilde{A} = \big\{ j \in \{n+1, \ldots, n+m\} \colon \rho_{j,0} > \sigma,\ \bar{\rho}_j \leqslant \sigma \big\}$$

and $\tilde{B} = \{n+1, \ldots, n+m\} \setminus \tilde{A}$. We easily deduce that the projection of $(\Gamma_0\theta_1, \ldots, \Gamma_0\theta_n)$ on I_0 is the same as the projection of $(\Gamma\theta_1, \ldots, \Gamma\theta_n)$ on I. We also claim that the projection of $(\Gamma_0\theta_{n+1}, \ldots, \Gamma_0\theta_{n+m})$ on J_0 is the same as the projection of $(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m})$ on J. In fact, if $J = J_0$, then the claim is obvious. Assume therefore that $J_0 \subsetneqq J$. If we denote with P_C the orthogonal projection on a closed and convex subset C of $\mathbb{R}^m$, then

$$(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) = P_{J_0}(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m}).$$

Therefore, if we choose a point $(x_{n+1}, \ldots, x_{n+m}) \in J_0$, then the scalar product

$$(\Gamma\theta_{n+1} - \bar{\gamma}_{n+1}, \ldots, \Gamma\theta_{n+m} - \bar{\gamma}_{n+m}) \cdot (x_{n+1} - \bar{\gamma}_{n+1}, \ldots, x_{n+m} - \bar{\gamma}_{n+m}) \leqslant 0.$$

Notice that $J \setminus J_0$ is given by points $(\gamma_{n+1}, \ldots, \gamma_{n+m})$ satisfying

$$f(\rho_{j,0}) < \gamma_j \leqslant f(\sigma)$$

for some $j \in \tilde{A}$. Since $\bar{\gamma}_j < f(\rho_{j,0})$ for every $j \in \tilde{A}$, then for every point $(\tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m})$ of J such that $\tilde{x}_j > \bar{\gamma}_j$ for some $j \in \tilde{A}$, there exist $\zeta > 0$ and a point $(x_{n+1}, \ldots, x_{n+m}) \in J_0$ such that $x_j > \bar{\gamma}_j$ for some $j \in \tilde{A}$ and

$$(\tilde{x}_{n+1} - \bar{\gamma}_{n+1}, \ldots, \tilde{x}_{n+m} - \bar{\gamma}_{n+m}) = \zeta(x_{n+1} - \bar{\gamma}_{n+1}, \ldots, x_{n+m} - \bar{\gamma}_{n+m}).$$

This fact permits to conclude that

$$(\Gamma\theta_{n+1} - \bar{\gamma}_{n+1}, \ldots, \Gamma\theta_{n+m} - \bar{\gamma}_{n+m}) \cdot (\tilde{x}_{n+1} - \bar{\gamma}_{n+1}, \ldots, \tilde{x}_{n+m} - \bar{\gamma}_{n+m}) \leqslant 0$$

for every $(\tilde{x}_{n+1}, \ldots, \tilde{x}_{n+m}) \in J$ and so

$$(\bar{\gamma}_{n+1}, \ldots, \bar{\gamma}_{n+m}) = P_J(\Gamma\theta_{n+1}, \ldots, \Gamma\theta_{n+m}).$$

This concludes the proof. □

Before proving (P1)–(P3), we need to prove some technical lemmas about projections.

Lemma 4. *Fix $N \in \mathbb{N} \setminus \{0\}$, a set $\mathcal{P} = \prod_{l=1}^{N}[0, a_l]$, where $a_l > 0$ for every $l \in \{1, \ldots, N\}$, and an N-dimensional vector $(\vartheta_1, \ldots, \vartheta_N)$ such that $\vartheta_l > 0$ for every $l \in \{1, \ldots, N\}$ and $\sum_{l=1}^{N} \vartheta_l = 1$. For $0 \leqslant \Lambda \leqslant \sum_{l=1}^{N} a_l$, denote with $(\zeta_1, \ldots, \zeta_N) = P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ the orthogonal projection of $(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ on the set*

$$\mathcal{I} = \left\{(\gamma_1, \ldots, \gamma_N) \in \mathcal{P}\colon \sum_{l=1}^{N} \gamma_l = \Lambda\right\}.$$

Then the value ζ_l ($l \in \{1, \ldots, N\}$) depends on Λ in a continuous way. Moreover, for all but a finite number of $0 < \Lambda < \sum_{l=1}^{N} a_l$, the derivative of ζ_l with respect to Λ exists and satisfies $\frac{\partial}{\partial\Lambda}\zeta_l \geqslant 0$.

Proof. The continuity of ζ_l w.r.t. Λ is trivial. The differentiability of ζ_l w.r.t. Λ is instead granted for all values of Λ such that locally the projection $P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ either is $(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ or lies in the same face of $\mathcal{I}$. By linearity, this happens for all but a finite number of values of Λ. Thus we are left with last statement.

The conclusion is evident if $P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$ is equal to $(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N)$. So assume that

$$P_{\mathcal{I}}(\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N) \neq (\Lambda\vartheta_1, \ldots, \Lambda\vartheta_N),$$

i.e. $(\zeta_1, \ldots, \zeta_N)$ belongs to the boundary of $\mathcal{I}$. Moreover the case $N = 1$ is trivial, so we consider $N \geqslant 2$.

Since $\vartheta_k > 0$ for every $k \in \{1, \ldots, N\}$, then $\zeta_k > 0$ for every $k \in \{1, \ldots, N\}$. Assume, for simplicity, that there exists $\bar{k} \in \{1, \ldots, N-1\}$, such that

$$\zeta_k = a_k,$$

for every $k = \bar{k}+1, \ldots, N$, and $\zeta_k < a_k$ otherwise. The vector $(\zeta_1, \ldots, \zeta_N)$ can be written in the form

$$\Lambda(\vartheta_1, \ldots, \vartheta_N) + t(v_1, \ldots, v_N),$$

where $t > 0$, $(v_1, \ldots, v_N)$ depends on Λ and on a_k and it satisfies $\sum_{l=1}^{N} v_l = 0$. Hence, for every $k = \bar{k}+1, \ldots, N$, we deduce that

$$t = \frac{a_k - \vartheta_k\Lambda}{v_k}$$

and

$$v_k = \frac{a_k - \vartheta_k \Lambda}{a_N - \vartheta_N \Lambda} v_N.$$

Since the projection minimizes the distance, in order to find $(\zeta_1, \dots, \zeta_n)$, it is sufficient to minimize t (or equivalently to maximize v_N^2) under the constraints

$$v_k = \frac{a_k - \vartheta_k \Lambda}{a_N - \vartheta_N \Lambda} v_N, \quad k = \bar{k}+1, \dots, N, \tag{24}$$

$$\left\|(v_1, \dots, v_N)\right\|^2 = 1, \tag{25}$$

$$\sum_{l=1}^{N} v_l = 0. \tag{26}$$

We apply the Lagrangian multiplier method to maximize v_N^2 under the constraints (24)–(26). For simplicity, define $f = v_N^2$, $g_k = v_k - \frac{a_k - \vartheta_k \Lambda}{a_N - \vartheta_N \Lambda} v_N$ for $k = \bar{k}+1, \dots, N$, $g_{N+1} = \sum_{l=1}^{N} v_l$ and finally $g_{N+2} = \|(v_1, \dots, v_N)\|^2 - 1$. So we deal with the critical points of the function

$$f + \sum_{k=\bar{k}+1}^{N+2} \lambda_k g_k,$$

depending on the variables $(v_1, \dots, v_N)$, where the coefficients λ_k belong to $\mathbb{R}$. Differentiating the previous function with respect to v_i $(i = 1, \dots, \bar{k})$, we find that

$$\lambda_{N+1} + 2\lambda_{N+2} v_i = 0,$$

which implies that $v_1 = \dots = v_{\bar{k}} = \bar{v}$ for some $\bar{v} \neq 0$, since λ_{N+1} and λ_{N+2} are nontrivial. Thus Eqs. (24) and (26) imply that

$$\bar{v} = -\frac{A v_N}{\bar{k}},$$

where

$$A = 1 + \sum_{k=\bar{k}+1}^{N-1} \frac{a_k - \vartheta_k \Lambda}{a_N - \vartheta_N \Lambda}.$$

Hence, for every $i = 1, \dots, \bar{k}$,

$$\begin{aligned}
\frac{\partial}{\partial \Lambda} \zeta_i &= \frac{\partial}{\partial \Lambda} (\Lambda \vartheta_i + t v_i) = \frac{\partial}{\partial \Lambda} \left(\Lambda \vartheta_i - \frac{a_N - \vartheta_N \Lambda}{v_N} \cdot \frac{A v_N}{\bar{k}} \right) \\
&= \frac{\partial}{\partial \Lambda} \left(\Lambda \vartheta_i - \frac{a_N - \vartheta_N \Lambda}{\bar{k}} \cdot A \right) \\
&= \frac{\partial}{\partial \Lambda} \left(\Lambda \vartheta_i - \frac{1}{\bar{k}} \sum_{k=\bar{k}+1}^{N} (a_k - \vartheta_k \Lambda) \right) \\
&= \vartheta_i + \frac{1}{\bar{k}} \sum_{k=\bar{k}+1}^{N} \vartheta_k > 0,
\end{aligned}$$

while, for every $i = \bar{k}+1, \dots, N$, we have $\frac{\partial}{\partial \Lambda} \zeta_i = 0$. $\square$

Lemma 5. *Under the same assumptions as Lemma* 4*, the value ζ_l, for $l \in \{1, \dots, N\}$, depends in a continuous way on a_h for $h \in \{1, \dots, N\}$. Moreover, if $l \neq h$, then for all but a finite number of a_h it is differentiable and it holds* $\frac{\partial \zeta_l}{\partial a_h} \leqslant 0$.

Proof. The proof of continuity and differentiability of ζ_l w.r.t. a_h is similar to that of Lemma 4. Thus we consider only the last statement.

The case $N = 1$ is trivial, hence we assume that $N \geqslant 2$ and $l, h \in \{1, \dots, N\}$ with $l \neq h$. If $(\zeta_1, \dots, \zeta_N)$ is equal to $\Lambda(\vartheta_1, \dots, \vartheta_N)$, then the claim is obvious. Assume therefore that

$$(\zeta_1, \dots, \zeta_N) \neq \Lambda(\vartheta_1, \dots, \vartheta_N).$$

In this case $(\zeta_1, \dots, \zeta_N)$ belongs to the topological boundary of $\mathcal{P}$ contained in the space $\sum_{i=1}^{n} \gamma_i = \Lambda$.

As in the proof of Lemma 4, we deduce $\zeta_k > 0$ for every $k \in \{1, \dots, N\}$ and assume there exists $\bar{k} \in \{1, \dots, N-1\}$, such that $\zeta_k = a_k$, for every $k = \bar{k}+1, \dots, N$, and $\zeta_k < a_k$ otherwise. Again (see the proof of Lemma 4) we write $(\zeta_1, \dots, \zeta_N) = \Lambda(\vartheta_1, \dots, \vartheta_N) + t(v_1, \dots, v_N)$, and deduce $v_1 = \dots = v_{\bar{k}} = \bar{v}$ for some $\bar{v} \neq 0$.

Now, notice that $\frac{\partial}{\partial a_h}\zeta_i = 0$ if $i \geqslant \bar{k}+1$ and $i \neq h$. While, if $i \leqslant \bar{k}$, then

$$\frac{\partial}{\partial a_h}\zeta_i = \frac{\partial}{\partial a_h}(\Lambda\vartheta_i + t v_i) = \frac{\partial}{\partial a_h}(t\bar{v}),$$

since Λ is fixed. Thus $\frac{\partial}{\partial a_h}\zeta_i$ is independent from i and, finally, the equation $\sum_{i=1}^{n} \zeta_i = \Lambda$ implies that

$$\frac{\partial}{\partial a_h}\zeta_i \leqslant 0;$$

so the proof is finished. □

Remark 3. Note that, in Lemma 4, we assume that every a_l ($l \in \{1, \dots, N\}$) is fixed and that the coefficient Λ varies.

On the contrary, in Lemma 5, we assume that Λ is fixed and that the coefficients a_l vary.

Proposition 5. *The Riemann solver* $\mathcal{RS}_2$ *satisfies property* (P1).

The proof is similar to that of Proposition 2; hence we omit it.

Lemma 6. *Fix an equilibrium* $(\rho_{1,0}, \dots, \rho_{n+m,0})$ *for* $\mathcal{RS}_2$ *and consider, for some* $l \in \{1, \dots, n+m\}$, $\rho_l \in [0,1]$ *such that the wave* $(\rho_l, \rho_{l,0})$ *has positive speed if* $l \leqslant n$, *while the wave* $(\rho_{l,0}, \rho_l)$ *has negative speed if* $l > n$. *Then*

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left|f(\hat{\rho}_h) - f(\rho_{h,0})\right| + \left|f(\hat{\rho}_l) - f(\rho_l)\right| = \left|f(\rho_l) - f(\rho_{l,0})\right|, \tag{27}$$

where

$$(\hat{\rho}_1, \dots, \hat{\rho}_{n+m}) = \mathcal{RS}_2(\rho_{1,0}, \dots, \rho_{l-1,0}, \rho_l, \rho_{l+1,0}, \dots, \rho_{n+m,0}).$$

Proof. In this proof we use the following notation.

- Γ_{inc}^-, Γ_{out}^-, Γ^-, I^- and J^- denote the numbers and the sets defined in points 1 and 2 of Section 4.2 for the initial condition $(\rho_{1,0}, \dots, \rho_{n+m,0})$.
- Γ_{inc}^+, Γ_{out}^+, Γ^+, I^+ and J^+ denote the numbers and the sets defined in points 1 and 2 of Section 4.2 for the initial condition $(\hat{\rho}_1, \dots, \hat{\rho}_{n+m})$.
- $\bar{\Gamma}_{inc}$, $\bar{\Gamma}_{out}$, $\bar{\Gamma}$, $\bar{I}$ and $\bar{J}$ denote the numbers and the sets defined in points 1 and 2 of Section 4.2 for the initial condition $(\rho_{1,0}, \dots, \rho_l, \dots, \rho_{n+m,0})$.
- $\Omega_{h,0}$ denotes the set defined in (14) or in (15) with respect to $\rho_{h,0}$ for $h \in \{1, \dots, n+m\}$.
- $\overline{\Omega}_l$ denotes the set defined in (14) or in (15) with respect to ρ_l.

Notice that $\bar{\Gamma} = \Gamma^+$. We have the following two possibilities.

1. $\Gamma_{inc}^- \leqslant \Gamma_{out}^-$.
2. $\Gamma_{inc}^- > \Gamma_{out}^-$.

We deal only with the proof of the first case, since the second one can be treated in the same way. Assume, therefore, $\Gamma_{inc}^- \leqslant \Gamma_{out}^-$. In this case $\Gamma^- = \Gamma_{inc}^-$ and $\rho_{i,0} \leqslant \sigma$ for every $i \in \{1, \ldots, n\}$. There are two different situations: $l \leqslant n$ and $l > n$.

Assume first $l \leqslant n$. We noticed that $\rho_{l,0} \leqslant \sigma$ and so $\rho_l < \sigma$, since the speed of the wave is positive. We have

$$\bar{\Gamma}_{inc} = \Gamma_{inc}^- - f(\rho_{l,0}) + f(\rho_l) \tag{28}$$

and

$$\bar{\Gamma}_{out} = \Gamma_{out}^-.$$

If $\bar{\Gamma}_{inc} \leqslant \bar{\Gamma}_{out}$, then no wave is produced in incoming arcs and at most m waves are produced in outgoing arcs. The total variation of the flux due to these waves is

$$\sum_{j=n+1}^{n+m} \left| f(\hat{\rho}_j) - f(\rho_{j,0}) \right|.$$

Therefore

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left| f(\hat{\rho}_h) - f(\rho_{h,0}) \right| + \left| f(\hat{\rho}_l) - f(\rho_l) \right| - \left| f(\rho_l) - f(\rho_{l,0}) \right|$$

$$= -\left| f(\rho_l) - f(\rho_{l,0}) \right| + \sum_{j=n+1}^{n+m} \left| f(\rho_{j,0}) - f(\hat{\rho}_j) \right|.$$

If $f(\rho_l) < f(\rho_{l,0})$, then $\bar{\Gamma} < \Gamma^-$ and the sets $\bar{J} \subseteq J^-$ differ only for the values of $\bar{\Gamma}$, Γ^-, since the wave $(\rho_l, \rho_{l,0})$ does not affect $\Omega_{j,0}$ for every $j \in \{n+1, \ldots, n+m\}$; hence we apply Lemma 4 and deduce that $f(\hat{\rho}_j) \leqslant f(\rho_{j,0})$ for every $j \in \{n+1, \ldots, n+m\}$.

If instead $f(\rho_l) > f(\rho_{l,0})$, then $\bar{\Gamma} > \Gamma^-$ and, with similar considerations as the previous ones, we have that $f(\hat{\rho}_j) \geqslant f(\rho_{j,0})$ for every $j \in \{n+1, \ldots, n+m\}$. Therefore, we have

$$\sum_{\substack{h=1\\h\neq l}}^{n+m} \left| f(\hat{\rho}_h) - f(\rho_{h,0}) \right| + \left| f(\hat{\rho}_l) - f(\rho_l) \right| - \left| f(\rho_l) - f(\rho_{l,0}) \right|$$

$$\begin{aligned}
&= \operatorname{sgn}\bigl(f(\rho_l) - f(\rho_{l,0})\bigr) \cdot \left(-f(\rho_l) + f(\rho_{l,0}) + \sum_{j=n+1}^{n+m} \bigl(f(\hat{\rho}_j) - f(\rho_{j,0}) \bigr) \right) \\
&= \operatorname{sgn}\bigl(f(\rho_l) - f(\rho_{l,0})\bigr)\bigl(-f(\rho_l) + f(\rho_{l,0}) + \Gamma^+ - \Gamma^-\bigr) \\
&= \operatorname{sgn}\bigl(f(\rho_l) - f(\rho_{l,0})\bigr)\bigl(-f(\rho_l) + f(\rho_{l,0}) + \bar{\Gamma} - \Gamma_{inc}^-\bigr) \\
&= \operatorname{sgn}\bigl(f(\rho_l) - f(\rho_{l,0})\bigr)\bigl(-f(\rho_l) + f(\rho_{l,0}) + \bar{\Gamma}_{inc} - \Gamma_{inc}^-\bigr) = 0,
\end{aligned}$$

where we used Eq. (28) and the equality $\bar{\Gamma} = \Gamma^+$. Thus the conclusion follows in the case $\bar{\Gamma}_{inc} \leqslant \bar{\Gamma}_{out}$.

If $\bar{\Gamma}_{inc} > \bar{\Gamma}_{out}$, then $\bar{\Gamma} = \Gamma^+ = \bar{\Gamma}_{out} = \Gamma_{out}^+$ and $\bar{\Gamma}_{inc} > \bar{\Gamma} \geqslant \Gamma_{inc}^-$. Moreover we deduce that $f(\rho_l) > f(\rho_{l,0})$ and so the total variation of the flux due to the interacting wave is, in this case, equal to

$$\left| f(\rho_l) - f(\rho_{l,0}) \right| = f(\rho_l) - f(\rho_{l,0}).$$

Since $\bar{\Gamma} = \bar{\Gamma}_{out}$, then in the outgoing arcs there is the formation of at most m waves and the trace of the flux of the solution at the node is the maximum possible. This implies that $f(\hat{\rho}_j) \geqslant f(\rho_{j,0})$ for every $j \in \{n+1, \ldots, n+m\}$. Therefore the total variation of the flux in outgoing arcs after the interaction produced at J is given by $\bar{\Gamma} - \Gamma_{inc}^-$.

By $\bar{\Gamma}_{inc} > \bar{\Gamma}$, in the incoming arcs there is the production of at most n waves. In this case, the trace of the solution in an incoming arc is a good datum (see Definition 7), since $\rho_{i,0} \leqslant \sigma$ for every $i \in \{1, \ldots, n\}$, $\rho_l < \sigma$ and the speed of the produced waves is negative. Then $f(\rho_{h,0}) \geqslant f(\hat{\rho}_h)$ for every $h \in \{1, \ldots, n\}$, $h \neq l$ and $f(\hat{\rho}_l) \leqslant f(\rho_l)$.

Without loss of generality, we may assume that the interacting wave is in the arc I_1, i.e. $l = 1$; hence the total variation of the flux due to the waves is

$$
\begin{aligned}
&\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_l)-f(\rho_{l,0})\right|\\
&\quad=\sum_{h=2}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_1)-f(\rho_1)\right|-\left|f(\rho_1)-f(\rho_{1,0})\right|\\
&\quad=\sum_{i=2}^{n}\left[f(\rho_{i,0})-f(\hat\rho_i)\right]+\bar\Gamma-\Gamma_{inc}^{-}+f(\rho_1)-f(\hat\rho_1)-f(\rho_1)+f(\rho_{1,0})\\
&\quad=\sum_{i=1}^{n}\left[f(\rho_{i,0})\right]-\sum_{i=1}^{n}\left[f(\hat\rho_i)\right]+\bar\Gamma-\Gamma_{inc}^{-}\\
&\quad=\sum_{i=1}^{n}\left[f(\rho_{i,0})\right]-\Gamma^{+}+\bar\Gamma-\Gamma_{inc}^{-}\\
&\quad=\Gamma_{inc}^{-}-\Gamma^{+}+\bar\Gamma-\Gamma_{inc}^{-}=0.
\end{aligned}
$$

Thus the conclusion follows provided $\bar\Gamma_{inc}>\bar\Gamma_{out}$. Therefore the case $l\leqslant n$ is completed.

Assume now $l>n$ and, without loss of generality, $l=n+1$. We consider three different situations.

If $\Gamma_{out}^{-}\leqslant\bar\Gamma_{out}$, then $\bar\Gamma=\bar\Gamma_{inc}=\Gamma_{inc}^{-}$, and so nothing happens in incoming arcs. If $\Gamma_{out}^{-}=\bar\Gamma_{out}$, then both $\rho_{n+1,0}$ and ρ_{n+1} are good data and this is not possible by the velocity of the wave. Since the wave $(\rho_{n+1,0},\rho_{n+1})$ has negative speed, then $\Gamma_{out}^{-}<\bar\Gamma_{out}$. The only possibility is that $\rho_{n+1,0}$ is a bad datum, $\rho_{n+1}\in[\sigma,\rho_{n+1,0}[$ and so $f(\rho_{n+1})>f(\rho_{n+1,0})$. Moreover, since the wave $(\hat\rho_{n+1},\rho_{n+1})$ has positive speed, then $f(\rho_{n+1})\geqslant f(\hat\rho_{n+1})$. Therefore

$$
\begin{aligned}
&\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_l)-f(\rho_{l,0})\right|\\
&\quad=\sum_{j=n+2}^{n+m}\left|f(\rho_{j,0})-f(\hat\rho_j)\right|+\left|f(\hat\rho_{n+1})-f(\rho_{n+1})\right|-\left|f(\rho_{n+1})-f(\rho_{n+1,0})\right|\\
&\quad=\sum_{j=n+2}^{n+m}\left|f(\rho_{j,0})-f(\hat\rho_j)\right|+f(\rho_{n+1})-f(\hat\rho_{n+1})-f(\rho_{n+1})+f(\rho_{n+1,0})\\
&\quad=\sum_{j=n+2}^{n+m}\left|f(\rho_{j,0})-f(\hat\rho_j)\right|-f(\hat\rho_{n+1})+f(\rho_{n+1,0}).
\end{aligned}
$$

Since $\bar\Gamma=\Gamma^{-}$ and $\Omega_{l,0}\subseteq\overline{\Omega}_l$, we may apply Lemma 5 and deduce that $f(\rho_{j,0})\geqslant f(\hat\rho_j)$ for every $j\in\{n+2,\dots,n+m\}$ and so

$$
\begin{aligned}
&\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_l)-f(\rho_{l,0})\right|\\
&\quad=\sum_{j=n+1}^{n+m}\left[f(\rho_{j,0})\right]-\sum_{j=n+1}^{n+m}\left[f(\hat\rho_j)\right]=\Gamma^{-}-\Gamma^{+}=0
\end{aligned}
$$

and so we have the thesis in the case $\Gamma_{out}^{-}\leqslant\bar\Gamma_{out}$.

If $\Gamma_{out}^{-}>\bar\Gamma_{out}\geqslant\Gamma_{inc}^{-}$, then $\rho_{n+1,0}<\rho_{n+1}$ and $f(\rho_{n+1})<f(\rho_{n+1,0})$, since the wave $(\rho_{n+1,0},\rho_{n+1})$ has negative speed. Thus $\bar\Gamma=\Gamma_{inc}^{-}$ and no wave is produced in incoming arcs and also no wave is produced in the arc I_{n+1}; i.e. $\hat\rho_{n+1}=\rho_{n+1}$. Moreover $\overline{\Omega}_l\subseteq\Omega_{l,0}$ and so, by Lemma 5, we have $f(\rho_{j,0})\leqslant f(\hat\rho_j)$ for every $j\in\{n+2,\dots,n+m\}$. Therefore

$$\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_l)-f(\rho_{l,0})\right|$$
$$=\sum_{j=n+2}^{n+m}\left|f(\hat\rho_j)-f(\rho_{j,0})\right|-\left|f(\rho_{n+1})-f(\rho_{n+1,0})\right|$$
$$=\sum_{j=n+2}^{n+m}\left[f(\hat\rho_j)-f(\rho_{j,0})\right]+f(\rho_{n+1})-f(\rho_{n+1,0})$$
$$=\sum_{j=n+1}^{n+m}\left[f(\hat\rho_j)-f(\rho_{j,0})\right]=\bar\Gamma-\Gamma^-=0$$

and so we have the thesis in the case $\Gamma_{out}^->\bar\Gamma_{out}\geqslant\Gamma_{inc}^-$.

If $\Gamma_{out}^->\bar\Gamma_{out}$ and $\bar\Gamma_{out}<\Gamma_{inc}^-$, then $\rho_{n+1,0}<\rho_{n+1}$ and $f(\rho_{n+1})<f(\rho_{n+1,0})$, since the wave $(\rho_{n+1,0},\rho_{n+1})$ has negative speed. Moreover $\bar\Gamma=\bar\Gamma_{out}$ and so no wave is produced in I_{n+1}, waves with decreasing flux are produced in incoming arcs, and waves with increasing flux are produced in outgoing arcs, i.e. $\hat\rho_{n+1}=\rho_{n+1}$, $f(\hat\rho_i)\leqslant f(\rho_{i,0})$ for every $i\in\{1,\dots,n\}$ and $f(\hat\rho_j)\geqslant f(\rho_{j,0})$ for every $j\in\{n+2,\dots,n+m\}$. Hence

$$\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_l)-f(\rho_{l,0})\right|$$
$$=\sum_{i=1}^{n}\left|f(\hat\rho_i)-f(\rho_{i,0})\right|+\sum_{j=n+2}^{n+m}\left|f(\hat\rho_j)-f(\rho_{j,0})\right|-\left|f(\rho_{n+1,0})-f(\rho_{n+1})\right|$$
$$=\sum_{i=1}^{n}\left[f(\rho_{i,0})-f(\hat\rho_i)\right]+\sum_{j=n+2}^{n+m}\left[f(\hat\rho_j)-f(\rho_{j,0})\right]+f(\rho_{n+1})-f(\rho_{n+1,0})$$
$$=\Gamma^--\Gamma^++\sum_{j=n+1}^{n+m}\left[f(\hat\rho_j)-f(\rho_{j,0})\right]=\Gamma^--\Gamma^++\Gamma^+-\Gamma^-=0,$$

and we have the thesis in the case $\Gamma_{out}^->\bar\Gamma_{out}$ and $\bar\Gamma_{out}<\Gamma_{inc}^-$.

The proof is thus finished. □

Proposition 6. *The Riemann solver $\mathcal{RS}_2$ satisfies property* (P2).

Proof. Fix an equilibrium $(\rho_{1,0},\dots,\rho_{n+m,0})$ for $\mathcal{RS}_2$ and consider, for some $l\in\{1,\dots,n+m\}$, $\rho_l\in[0,1]$ such that the wave $(\rho_l,\rho_{l,0})$ has positive speed if $l\leqslant n$, while the wave $(\rho_{l,0},\rho_l)$ has negative speed if $l>n$. Assume that the wave interacts with J at time $\bar t>0$ and define

$$(\hat\rho_1,\dots,\hat\rho_{n+m})=\mathcal{RS}_2(\rho_{1,0},\dots,\rho_{l-1,0},\rho_l,\rho_{l+1,0},\dots,\rho_{n+m,0}).$$

By Lemma 6 we have

$$\begin{aligned}\mathrm{Tot.Var.}_f(\bar t+)-\mathrm{Tot.Var.}_f(\bar t-)&=\sum_{\substack{h=1\\h\neq l}}^{n+m}\left|f(\hat\rho_h)-f(\rho_{h,0})\right|+\left|f(\hat\rho_l)-f(\rho_l)\right|-\left|f(\rho_{l,0})-f(\rho_l)\right|\\&=\left|f(\rho_{l,0})-f(\rho_l)\right|-\left|f(\rho_{l,0})-f(\rho_l)\right|=0\end{aligned}$$

and so (P2) holds. □

Proposition 7. *The Riemann solver $\mathcal{RS}_2$ satisfies property* (P3).

Proof. Fix an equilibrium $(\rho_{1,0},\dots,\rho_{n+m,0})$ for $\mathcal{RS}_2$ and $l\in\{1,\dots,n+m\}$. Consider just the case $l\leqslant n$, the other case being similar. Assume that $\rho_l\in[0,1]$ is such that the wave $(\rho_l,\rho_{l,0})$ has positive speed, interacts with J at time $\bar t>0$ and $f(\rho_l)<f(\rho_{l,0})$. Define

$$(\hat\rho_1,\dots,\hat\rho_{n+m})=\mathcal{RS}(\rho_{1,0},\dots,\rho_{l-1,0},\rho_l,\rho_{l+1,0},\dots,\rho_{n+m,0}).$$

The Rankine–Hugoniot condition implies that $\rho_l<\rho_{l,0}$ and so ρ_l is a bad datum. Call Γ^- and Γ^+ respectively the values, defined in point 1 of the procedure for $\mathcal{RS}_2$, for initial data $(\rho_{1,0},\dots,\rho_{n+m,0})$ and $(\rho_{1,0},\dots,\rho_l,\dots,\rho_{n+m,0})$. Since ρ_l is a bad datum, then $\Gamma^-\geqslant\Gamma^+$ and so

$$\Gamma(\bar t-)=\sum_{i=1}^n f(\rho_{i,0})\geqslant\sum_{i=1}^n f(\hat\rho_i)=\Gamma(\bar t+).$$

The proof is finished. □

4.3. Riemann solver $\mathcal{RS}_3$

In this subsection, we consider the Riemann solver, introduced in [35] to model T-nodes. Consider a node J with n incoming and $m=n$ outgoing arcs and fix a positive coefficient Γ_J, which is the maximum capacity of the node. The construction can be done in the following way.

1. Fix $\boldsymbol{\theta}\in\Theta$. For every $i\in\{1,\dots,n\}$, define
$$\Gamma_i=\min\{\gamma_i^{max},\gamma_{i+n}^{max}\}$$
where the numbers γ_l^{max} are defined in (16). Then the maximal possible through-flow at J is
$$\Gamma=\sum_{i=1}^n\Gamma_i.$$
2. Introduce the closed, convex and not empty set
$$I=\left\{(\gamma_1,\dots,\gamma_n)\in\prod_{i=1}^n[0,\Gamma_i]:\ \sum_{i=1}^n\gamma_i=\min\{\Gamma,\Gamma_J\}\right\}.$$
3. Denote with $(\bar\gamma_1,\dots,\bar\gamma_n)$ the orthogonal projection on the convex set I of the point $(\min\{\Gamma,\Gamma_J\}\theta_1,\dots,\min\{\Gamma,\Gamma_J\}\theta_n)$ and set $(\bar\gamma_{n+1},\dots,\bar\gamma_{2n})=(\bar\gamma_1,\dots,\bar\gamma_n)$.
4. For every $i\in\{1,\dots,n\}$, define $\bar\rho_i$ either by $\rho_{i,0}$ if $f(\rho_{i,0})=\bar\gamma_i$, or by the solution to $f(\rho)=\bar\gamma_i$ such that $\bar\rho_i\geqslant\sigma$. For every $j\in\{n+1,\dots,n+m\}$, define $\bar\rho_j$ either by $\rho_{j,0}$ if $f(\rho_{j,0})=\bar\gamma_j$, or by the solution to $f(\rho)=\bar\gamma_j$ such that $\bar\rho_j\leqslant\sigma$. Finally, define $\mathcal{RS}_3:[0,1]^{n+m}\to[0,1]^{n+m}$ by
$$\mathcal{RS}_3(\rho_{1,0},\dots,\rho_{n+m,0})=(\bar\rho_1,\dots,\bar\rho_n,\bar\rho_{n+1},\dots,\bar\rho_{n+m}).\tag{29}$$

The following result holds.

Lemma 7. *The function defined in* (29) *satisfies the consistency condition*

$$\mathcal{RS}_3\big(\mathcal{RS}_3(\rho_{1,0},\dots,\rho_{n+m,0})\big)=\mathcal{RS}_3(\rho_{1,0},\dots,\rho_{n+m,0})\tag{30}$$

for every $(\rho_{1,0},\dots,\rho_{n+m,0})\in[0,1]^{n+m}$.

For a proof, see Proposition 2.4 of [35].

Proposition 8. *The Riemann solver* $\mathcal{RS}_3$ *satisfies property* (P1).

The proof is similar to that of Proposition 2; hence we omit it.

Proposition 9. *The Riemann solver* $\mathcal{RS}_3$ *satisfies properties* (P2) *and* (P3).

The proof is completely similar to the proofs of properties (P2) and (P3) for the Riemann solver $\mathcal{RS}_2$ and so omitted.

5. The Cauchy problem

In this section, we deal with the Cauchy problem at the node J. Fix n initial data for incoming arcs $\rho_{1,0}, \ldots, \rho_{n,0} \in BV(]-\infty, 0]; [0, 1])$ and m initial data for outgoing arcs $\rho_{n+1,0}, \ldots, \rho_{n+m,0} \in BV([0, +\infty[; [0, 1])$. Consider the Cauchy problem at J:

$$\begin{cases} \frac{\partial}{\partial t}\rho_l(t,x) + \frac{\partial}{\partial x} f\big(\rho_l(t,x)\big) = 0, & x \in I_l \setminus \{0\},\ t > 0, \\ \rho_l(0,x) = \rho_{0,l}(x), & x \in I_l, \end{cases} \qquad l \in \{1, \ldots, n+m\}. \tag{31}$$

The main result is the following theorem.

Theorem 8. *Consider the Cauchy problem* (31) *and a Riemann solver* $\mathcal{RS}$ *satisfying the consistency condition and the properties* (P1)–(P3). *Then there exists a weak solution at* J $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x))$ *such that*

1. *for every* $l \in \{1, \ldots, n+m\}$, $\rho_l(0,x) = \rho_{0,l}(x)$ *for a.e.* $x \in I_l$;
2. *for a.e.* $t > 0$,
$$\mathcal{RS}\big(\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+)\big) = \big(\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+)\big). \tag{32}$$

The proof of the theorem is given in next sections. In [12,19,24,35], existence of solutions was proved for Riemann solvers $\mathcal{RS}_1$, $\mathcal{RS}_2$ and $\mathcal{RS}_3$ for a node J with at most two incoming and two outgoing arcs.

Remark 4. Notice that, under the hypotheses of the paper, every weak solution at J $(\rho_1(t,x), \ldots, \rho_{n+m}(t,x))$ admits strong traces $(\rho_1(t,0-), \ldots, \rho_{n+m}(t,0+))$ for a.e. $t > 0$; see [3, Lemma 1].

5.1. Wave-front tracking

Since solutions to Riemann problems are given, we are able to construct piecewise constant approximations via wave-front tracking algorithm; see [9] for the general theory and [24] in the case of networks.

Definition 12. Given $\varepsilon > 0$ and a Riemann solver $\mathcal{RS}$, we say that the map $\bar\rho_\varepsilon = (\bar\rho_{1,\varepsilon}, \ldots, \bar\rho_{n+m,\varepsilon})$ is an ε-approximate wave-front tracking solution to (31) with respect to $\mathcal{RS}$ if the following conditions hold.

1. For every $l \in \{1, \ldots, n+m\}$, $\bar\rho_{l,\varepsilon} \in C([0, +\infty[; L^1_{loc}(I_l; [0, 1]))$.
2. For every $l \in \{1, \ldots, n+m\}$, $\bar\rho_{l,\varepsilon}(t,x)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in the (t,x)-plane. Moreover jumps of $\bar\rho_{l,\varepsilon}(t,x)$ can be shocks or rarefactions and are indexed by $\mathcal{J}_l(t) = \mathcal{S}_l(t) \cup \mathcal{R}_l(t)$.
3. For every $l \in \{1, \ldots, n+m\}$, along each shock $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{S}_l(t)$, we have
$$\bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)-\big) < \bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)+\big).$$
Moreover
$$\left| \dot x_{l,\alpha}(t) - \frac{f(\bar\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-)) - f(\bar\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+))}{\bar\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)-) - \bar\rho_{l,\varepsilon}(t, x_{l,\alpha}(t)+)} \right| \leqslant \varepsilon.$$
4. For every $l \in \{1, \ldots, n+m\}$, along each rarefaction front $x(t) = x_{l,\alpha}(t)$, $\alpha \in \mathcal{R}_l(t)$, we have
$$\bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)+\big) < \bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)-\big) < \bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)+\big) + \varepsilon.$$
Moreover
$$\dot x_{l,\alpha}(t) \in \big[f'\big(\bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)-\big)\big), f'\big(\bar\rho_{l,\varepsilon}\big(t, x_{l,\alpha}(t)+\big)\big)\big].$$

5. For every $l \in \{1, \dots, n+m\}$,

$$\left\| \bar{\rho}_{l,\varepsilon}(0,\cdot) - \rho_{0,l}(\cdot) \right\|_{L^1(I_l)} < \varepsilon.$$

6. For a.e. $t > 0$

$$\mathcal{RS}\big(\bar{\rho}_{1,\varepsilon}(t,0-), \dots, \bar{\rho}_{n+m,\varepsilon}(t,0+)\big) = \big(\bar{\rho}_{1,\varepsilon}(t,0-), \dots, \bar{\rho}_{n+m,\varepsilon}(t,0+)\big).$$

Fix a Riemann solver $\mathcal{RS}$ satisfying the consistency condition and the properties (P1)–(P3). For every $l \in \{1, \dots, n+m\}$, consider a sequence $\rho_{0,l,\nu}$ of piecewise constant functions defined on I_l such that $\rho_{0,l,\nu}$ has a finite number of discontinuities and $\lim_{\nu \to +\infty} \rho_{0,l,\nu} = \rho_{0,l}$ in $L^1_{loc}(I_l; [0,1])$. For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve the Riemann problem at J (according to $\mathcal{RS}$) and all Riemann problems in each arc. We approximate every rarefaction wave with a rarefaction fan, formed by rarefaction shocks of strength less than $\frac{1}{\nu}$ travelling with the Rankine–Hugoniot speed. Moreover, if σ is in the range of a rarefaction shock, then its speed is zero. We repeat the previous construction at every time at which interactions between waves or of waves with J happen.

Remark 5. By slightly modifying the speed of waves, we may assume that, at every positive time t, at most one interaction happens. Moreover, at every interaction time, either two waves interact in an arc or a wave reaches the node J.

Remark 6. For interactions in arcs, we split rarefaction waves into rarefaction fans just at time $t = 0$. At the node J, instead, we allow the formation of rarefaction fans at every positive time.

Let us introduce the concepts of generation order for waves, of big shocks and of waves with increasing or decreasing flux. We need these definitions in the proof of existence of a wave-front tracking approximate solution and in the bounds for the total variation of the flux.

Definition 13. A wave of $\bar{\rho}_\varepsilon$, generated at time $t = 0$, is said an original wave or a wave with generation order 1.

If a wave with generation order $k \geqslant 1$ interacts with J, then the produced waves are said of generation $k+1$.

If a wave with generation order $k \geqslant 1$ interacts in an arc with a wave with generation order $k' \geqslant 1$, then the produced wave is said of generation $\min\{k, k'\}$.

Definition 14. We say that a wave (ρ_l, ρ_r) in an arc is a big shock if $\rho_l < \sigma < \rho_r$.

Definition 15. We say that a wave (ρ_l, ρ_r) interacting with J from an incoming arc has decreasing flux (respectively increasing flux) if $f(\rho_l) < f(\rho_r)$ (respectively $f(\rho_l) > f(\rho_r)$).

We say that a wave (ρ_l, ρ_r) interacting with J from an outgoing arc has decreasing flux (respectively increasing flux) if $f(\rho_l) > f(\rho_r)$ (respectively $f(\rho_l) < f(\rho_r)$).

5.2. Bounds on the total variation of the flux

The aim of this subsection is to give a bound to the total variation of the flux for an approximate solution. Fix a Riemann solver satisfying the properties (P1)–(P3). Let us start with some technical results.

Lemma 9. *The following statements hold.*

1. *Assume that a wave with decreasing flux, connecting ρ_l with ρ_r, reaches J from an incoming arc. Then (ρ_l, ρ_r) is a shock wave and ρ_l is a bad datum.*
2. *Assume that a wave with decreasing flux, connecting ρ_l and ρ_r, reaches J from an outgoing arc. Then (ρ_l, ρ_r) is a shock wave and ρ_r is a bad datum.*

Proof. Let us consider an incoming arc and a wave (ρ_l, ρ_r), which reaches the node J with decreasing flux. The wave has positive speed and so $\rho_l < \rho_r$. Since f is decreasing in $[\sigma, 1]$ then $\rho_l < \sigma$. It means that the wave is a shock wave and ρ_l is a bad datum.

The situation for an outgoing arc is completely symmetric. □

Corollary 1. *If a wave generated at J returns to J without interacting with waves with generation order* 1*, then it has decreasing flux and produces a decrease of Γ.*

Proof. Consider a wave generated at J, which does not interact with waves with generation order 1. Since the network is composed by a single node, then the speed of the wave can change only if the wave interacts with waves with generation order $k \geqslant 2$, i.e. with waves produced by J. Under these assumptions, the speed of the wave can change sign, only in the case the wave is a big shock or it interacts with a big shock; see Lemma 4.3.7 of [24] (see Appendix A). In any case, the wave is a big shock when it returns to J. Moreover it must have positive velocity if it is in an incoming arc, while it must have negative velocity in the other case. Therefore it is a wave with decreasing flux and the conclusion follows by property (P3). □

Lemma 10. *Assume that a wave (ρ_l, ρ_r) interacts with J at a time $\bar{t} > 0$, then*

$$\text{Tot.Var.}_f(\bar{t}+) \leqslant (C+1)\,\text{Tot.Var.}_f(\bar{t}-), \tag{33}$$

where C is given by property (P2) *of the Riemann solver $\mathcal{RS}$.*

Proof. By property (P2), we get

$$\text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) \leqslant C\big|f(\rho_l) - f(\rho_r)\big|.$$

Therefore we have

$$\begin{aligned}
\text{Tot.Var.}_f(\bar{t}+) &\leqslant \text{Tot.Var.}_f(\bar{t}-) + C\big|f(\rho_l) - f(\rho_r)\big| \\
&= \text{Tot.Var.}_f(\bar{t}-) - \big|f(\rho_l) - f(\rho_r)\big| + (C+1)\big|f(\rho_l) - f(\rho_r)\big| \\
&\leqslant \max\{C+1, 1\}\big[\text{Tot.Var.}_f(\bar{t}-) - \big|f(\rho_l) - f(\rho_r)\big|\big] + \max\{C+1, 1\}\big|f(\rho_l) - f(\rho_r)\big| \\
&= (C+1)\,\text{Tot.Var.}_f(\bar{t}-),
\end{aligned}$$

and this concludes the proof. □

Lemma 11. *Assume that a wave (ρ_l, ρ_r) interacts with J at a time $\bar{t} > 0$. Then*

$$\Gamma(\bar{t}+) \leqslant \Gamma(\bar{t}-) + (C+2)\big|f(\rho_l) - f(\rho_r)\big|, \tag{34}$$

where C is given by property (P2)*.*

Proof. The variation of Γ at $\bar{t}$ is the sum of the variation of the fluxes for the incoming arcs. Therefore

$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) \leqslant \text{Tot.Var.}_f(\bar{t}+) - \text{Tot.Var.}_f(\bar{t}-) + 2\big|f(\rho_l) - f(\rho_r)\big|.$$

Hence, by (P2), it is bounded by $(C+2)|f(\rho_l) - f(\rho_r)|$. □

The next lemma gives a bound for the positive total variation of Γ.

Lemma 12. *We have*

$$\text{Tot.Var.}^+\,\Gamma(\cdot) \leqslant (C+2)\,\text{Tot.Var.}_f(0+), \tag{35}$$

where C is given by property (P2) *and* Tot.Var.$^+\,\Gamma(\cdot)$ *denotes the positive total variation of Γ.*

Proof. By property (P3), an increment of the functional Γ can happen only when a wave with increasing flux interacts with J. Moreover a wave, generated at J, can come back at J only with a decreasing flux. Indeed, consider the case of an incoming arc, the other one being completely symmetric. Assume that a wave with increasing flux (ρ_l, ρ_r) interacts with J. Since $f(\rho_l) > f(\rho_r)$ and the velocity of the wave is positive, then we deduce that $\rho_l > \rho_r$. By contradiction, if $\rho_l > \sigma$, then clearly $\rho_r \in [0, \tau(\rho_l)[$ and so (ρ_l, ρ_r) is a rarefaction wave, whose velocity is not positive. Hence $\rho_l \leqslant \sigma$ and ρ_r is a bad datum. By [24, Lemma 4.3.6] (see Appendix A), the wave (ρ_l, ρ_r) is not a wave coming back to J. More precisely, it is a wave, which can be generated by interactions between original waves, but no one of these interacting waves is produced at J.

If two waves interact in an arc, then, by the construction of the approximate solution, the total variation of the flux diminishes after the interaction. Therefore, the previous considerations allow to conclude, thanks to Lemma 11. □

Lemma 13. *For C given by property* (P2)*, we have*

$$\text{Tot.Var.}\, \Gamma(\cdot) \leqslant 2(C+2)\, \text{Tot.Var.}_f(0+) + nf(\sigma). \tag{36}$$

Proof. It is a direct consequence of Lemma 12 and of the bound (6). □

Lemma 14. *For every $t > 0$ we have*

$$\text{Tot.Var.}_f(t) \leqslant C_1\, \text{Tot.Var.}_f(0+) + Cnf(\sigma), \tag{37}$$

where $C_1 = 1 + 2C(C+2)$ and C is given by property (P2).

Proof. Notice that the functional Tot.Var._f can increase only when a wave interacts with J and, by property (P2) of the Riemann solver $\mathcal{RS}$, produces a variation of Γ. If we denote with $g(t)$ the function $\text{Tot.Var.}_f(t)$, then the positive variation $\text{Tot.Var.}^+\, g(\cdot)$ of g is bounded by $C \cdot \text{Tot.Var.}\, \Gamma(\cdot)$. Thus, by Lemma 13,

$$\text{Tot.Var.}^+\, g(\cdot) \leqslant 2C(C+2)\, \text{Tot.Var.}_f(0+) + Cnf(\sigma)$$

and so, for every $t > 0$,

$$\begin{aligned} \text{Tot.Var.}_f(t) &\leqslant \text{Tot.Var.}_f(0+) + 2C(C+2)\, \text{Tot.Var.}_f(0+) + Cnf(\sigma) \\ &= C_1\, \text{Tot.Var.}_f(0+) + Cnf(\sigma). \end{aligned}$$

The proof is finished. □

5.3. Existence of a wave-front tracking solution

In this subsection, we prove the existence of a wave-front tracking approximate solution. We have the following proposition.

Proposition 10. *For every $\nu \in \mathbb{N} \setminus \{0\}$ the construction in Section* 5.1 *can be done for every positive time, producing a $\frac{1}{\nu}$-approximate wave-front tracking solution to* (31) *with respect to $\mathcal{RS}$.*

Proof. For every $l \in \{1, \ldots, n+m\}$ and every $\nu \in \mathbb{N} \setminus \{0\}$, call $\rho_{l,\nu}$ the function built by the previous procedure. Moreover, for every $l \in \{1, \ldots, n+m\}$, $\nu \in \mathbb{N} \setminus \{0\}$, $k \in \mathbb{N} \setminus \{0\}$ and for every time $t \geqslant 0$, define the functions $N_{l,\nu}(t)$ and $M_{l,k,\nu}(t)$, which count respectively the number of discontinuities of $\rho_{l,\nu}(t, \cdot)$ and the number of waves with generation order k of $\rho_{l,\nu}(t, \cdot)$.

Assume, by contradiction, that, there exist $\bar{\nu} \in \mathbb{N} \setminus \{0\}$ and $T > 0$ such that

$$\sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) < +\infty$$

for every $t \in [0, T[$, and

$$\limsup_{t \to T^-} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(t) = +\infty. \tag{38}$$

Note that, for every time t,

$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t) \leqslant \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) < +\infty.$$

Indeed, $\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(t)$ is locally constant and can vary only at interaction times in the following way:

1. if at $\bar{t} > 0$ a wave with generation order 1 reaches the node J, then
$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$
2. if at $\bar{t} > 0$ two waves with generation order 1 interact in an arc, then
$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-) - 1;$$
3. if at $\bar{t} > 0$ a wave with generation order k_1 interacts with a wave of order k_2 in an arc with $k_1 + k_2 \geqslant 3$, then
$$\sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}+) = \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(\bar{t}-).$$

Moreover, for every $l \in \{1,\dots,n+m\}$ and for every $k \geqslant 0$, the function $M_{l,k,\bar{\nu}}(\cdot)$ is decreasing inside the arcs. For every $k \in \mathbb{N} \setminus \{0\}$ and for every time $t > 0$, we have

$$\sum_{l=1}^{n+m} M_{l,k,\bar{\nu}}(t) \leqslant (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} M_{l,1,\bar{\nu}}(0+) = (K_{\bar{\nu}})^{k-1} \sum_{l=1}^{n+m} N_{l,\bar{\nu}}(0+) < +\infty,$$

where $K_{\bar{\nu}} = (n+m)\bar{\nu}$. This bound is due to the fact that each wave with generation order k can interact with J and produce at most $\bar{\nu}$ waves with generation order $k+1$ in each arc (in the case of rarefactions).

Now, there exists $0 < \eta < T$ such that no wave with generation order 1 interacts with J in the time interval $(T-\eta, T)$. Eq. (38) implies also that in $(T-\eta, T)$ there are an infinite number of interactions of waves with J. Since waves of generation order 1 do not interact in $(T-\eta, T)$, the only possibility is that a wave with generation order $k \geqslant 2$ comes back to J producing waves of order $k+1$, some of which come back to J producing waves of order $k+2$ and so on. Moreover by Lemma 4.3.7 of [24] (see Appendix A), if a wave of generation order $k \geqslant 2$, interacts with J from an arc in $(T-\eta, T)$, then, after the interaction, the datum in that arc is bad, since the wave cannot interact with waves of generation order 1 and come back to J. In an arc a bad datum at J can change only in the following cases:

1. an original wave interacts with J from the arc;
2. a wave, which is a big shock, is originated at J on an arc and the new datum at J is good.

Obviously in the time interval $(T-\eta, T)$ the first possibility cannot happen; so only the second possibility may happen. Assume that there exist $t_1, t_2 \in (T-\eta, T)$ with $t_1 < t_2$ such that a big shock is originated at J at time t_1 in an arc and comes back to J at time t_2. In this arc, the datum before t_1 is bad, since a big shock is originated at time t_1. Moreover the big shock comes back to J at time t_2, and so an original wave cannot interact with the big shock; hence the bad datum of the big shock does not change. Therefore, in that arc after the time t_2, the datum is bad and is the same as the datum before t_1. Thus every arc I_l may take only a precise bad value $\bar{\rho}_l$, otherwise good values. The key point is that, at every time $t \in (T-\eta, T)$, there are finitely many possible combinations of bad data at the node J (obtained choosing the arcs which present a bad datum at J, the precise value being fixed). By property (P1) (i.e. the image of $\mathcal{RS}$ depends only on the values of bad data) we deduce that, for $t \in (T-\eta, T)$, $\rho_{\bar{\nu}}(t)$ at J may take only a finite number of values, thus waves produced by J have a finite set of possible velocities.

Denote with $\mathcal{G}$ the set of all $l \in \{1,\dots,n+m\}$ such that $\rho_{\bar{\nu},l}(t,0)$ is a good datum for every time t in a left neighborhood of T.

Consider $\bar{l} \in \mathcal{G}$. We claim that there exists a constant $C_{\bar{l}} > 0$ such that $N_{\bar{l},\bar{\nu}}(t) \leqslant C_{\bar{l}}$ for every time t in a left neighborhood of T. Indeed the number of different states, which can be produced at J, is finite by the previous considerations. Since all states are good, there is a minimal size of a flux jump along a discontinuity. Then the total number of discontinuities is necessary bounded by Lemma 14.

Consider now $\bar{l} \in \{1, \ldots, n+m\} \setminus \mathcal{G}$. If $\rho_{\bar{\nu},\bar{l}}(t, 0)$ is a bad datum for every time t in a left neighborhood of T, then clearly $N_{\bar{l},\bar{\nu}}(t)$ is uniformly bounded in the same time interval. The other case is that a big shock is originated in the arc $I_{\bar{l}}$ and comes back to J infinitely many times. We claim that there exists a constant $C_{\bar{l}} > 0$ such that $N_{\bar{l},\bar{\nu}}(t) \leqslant C_{\bar{l}}$ for every time $t \in [\tilde{t}_1, \tilde{t}_2]$, where $\tilde{t}_1$ and $\tilde{t}_2$ are the times, at which a big shock respectively is originated at J in $I_{\bar{l}}$ and comes back to J. In fact, in the time interval $]\tilde{t}_1, \tilde{t}_2[$, the datum $\rho_{\bar{\nu},\bar{l}}(t, 0)$ is good and the number of possible different states, between J and the big shock, is finite. Therefore, as before, if the number of discontinuity cannot be bounded by a constant, then also the total variation of the flux cannot be bounded and this is not true, by Lemma 14.

This concludes the proof by contradiction. □

Remark 7. Notice that the proof of the previous proposition shows that the number of waves of a wave-front tracking approximate solution is uniformly bounded, while the interactions can be accumulated at time T.

In the case of Riemann solver $\mathcal{RS}_1$, it is also possible to prove that the interactions do not accumulate at T. In fact, consider a wave interacting with J from an arc I_l ($l \in \{1, \ldots, n+m\}$) at time $\bar{t} > 0$. Then, by [16, Lemma 1], there exists a constant $\tilde{C} > 0$, depending only on the matrix $A \in \mathfrak{N}$, such that

$$\left|\Gamma(\bar{t}+) - \Gamma(\bar{t}-)\right| \geqslant \tilde{C}\left|f\left(\rho_h(\bar{t}+, 0)\right) - f\left(\rho_h(\bar{t}-, 0)\right)\right|$$

for every $h \in \{1, \ldots, n+m\}$, $h \neq l$. This estimate permits to conclude in similar way as in the end of the proof of Proposition 10, by using Lemma 13.

5.4. Existence of solutions

This subsection is devoted to the proof of Theorem 8.

Proof of Theorem 8. Fix an ε-approximate wave-front tracking solution $\bar{\rho}_\varepsilon$ to (31), in the sense of Definition 12, with respect to a Riemann solver $\mathcal{RS}$ satisfying the consistency condition and the properties (P1)–(P3).

By Lemma 14, we deduce that there exists a constant $M > 0$, depending on the total variation of the flux of the initial datum, such that

$$\text{Tot.Var.}_f(\cdot) \leqslant M.$$

For every $l \in \{1, \ldots, n+m\}$ and every $\nu \in \mathbb{N}$, using the concept of generalized characteristic introduced by Dafermos [17], we construct a curve $Y_{l,\nu}$ bounding the region of influence of waves generated by the node J on the approximate solution $\rho_{l,\nu}$. More precisely, we follow the generalized characteristic emanating from 0 at time 0, sticking to the boundary of I_l each time $Y_{l,\nu}$ is at 0 and the characteristic speed is positive (respectively negative) if I_l is an incoming (respectively outgoing) arc. The curve $Y_{l,\nu} : [0, +\infty[\to I_l$ then satisfies

1. $Y_{l,\nu}(0) = 0$;
2. in $D_1^{l,\nu} = \{(t, x) \in [0, +\infty[\times I_l\colon |x| \geqslant Y_{l,\nu}(t)\}$, the function $\rho_{l,\nu}$ depends only on the initial condition;
3. in $D_2^{l,\nu} = [0, +\infty[\times I_l \setminus D_1^{l,\nu}$ the function $\rho_{l,\nu}$ depends also on the data from other arcs and on the Riemann solver $\mathcal{RS}$.

By uniform Lipschitz continuity, possibly by passing to a subsequence, the curves $Y_{l,\nu}$ converge uniformly as $\nu \to +\infty$ to some Lipschitz continuous limit curves. Thus, for every $l \in \{1, \ldots, n+m\}$, there exist two sets $D_1^l, D_2^l \subseteq [0, +\infty[\times I_l$, which are "limits" of the regions $D_1^{l,\nu}$, $D_2^{l,\nu}$, in the sense that $\text{meas}(D_1^l \Delta D_1^{l,\nu}) \to 0$ and $\text{meas}(D_2^l \Delta D_2^{l,\nu}) \to 0$, where Δ indicates the set-theoretic symmetric difference.

For every $l \in \{1, \ldots, n+m\}$, $\rho_{l,\nu}$ converges to a limit function ρ_l in L^1_{loc} on D_1^l by the theory of conservation laws on a real line; see [9].

Now recall that, for every $l \in \{1, \ldots, n+m\}$ and $\nu \in \mathbb{N}$, $\rho_{l,\nu} \in L^\infty$. Therefore, possibly up to a subsequence, on D_2^l the sequence $\rho_{l,\nu}$ weakly converges to a limit function ρ_l in L^1 and $f(\rho_{l,\nu})$ strongly converges to $\bar{f}_l$ in L^1 for some $\bar{f}_l$.

By [24, Lemma 4.3.6] (see Appendix A), for every $\bar{t}$ the set $D_2^{l,\nu} \cap \{(t,x)\colon\ t = \bar{t}\}$ contains at most one big shock. This permits to conclude that $\rho_{l,\nu}$ strongly converges to ρ_l (being f invertible possibly subdividing furtherly $D_2^{l,\nu}$). Finally, the vector $(\rho_1(t,x),\dots,\rho_{n+m}(t,x))$ is a weak solution at J satisfying 1 and 2 of Theorem 8. $\square$

Remark 8. In the case of Riemann solver $\mathcal{RS}_2$, when a wave interacts with J at time $\bar{t}$, the total variation of the flux does not change, i.e.

$$\text{Tot.Var.}_f(\bar{t}-) = \text{Tot.Var.}_f(\bar{t}+).$$

For a detailed proof of this fact, see Lemma 6. Therefore the constant M in the proof of Theorem 8 can be chosen equal to $\text{Tot.Var.}_f(0+)$.

6. Dependence of solutions on initial data

It is known that the Lipschitz continuous dependence of the solution to the Cauchy problem (31) with respect to the initial datum in general does not hold in the case of Riemann solver $\mathcal{RS}_1$. More precisely in [12,24] there is a counterexample to the Lipschitz continuous dependence property in the case of a node with two incoming and two outgoing arcs.

On the other side, the Lipschitz continuous dependence of the solution to (31) with respect to the initial datum was proved in the case of Riemann solver $\mathcal{RS}_2$ and simple nodes in [19]; see also [24]. In this section we want to prove that the property holds for every type of nodes.

Let us introduce the concept of Finsler manifold.

Definition 16. Consider a differentiable manifold M and, for every $x \in M$, a norm $\|\cdot\|_x$ on the tangent space T_xM. The manifold M is said a Finsler manifold if

1. $\|\cdot\|_x$ depends in a continuous way on x;
2. for every $x \in M$ and $v \in T_xM$ the Hessian of the function

$$\begin{aligned} L_x : T_xM &\longrightarrow \mathbb{R}, \\ w &\longmapsto \|w\|_x^2 \end{aligned}$$

   is positive definite at v.

Given a Finsler manifold M, a metric d is naturally defined by

$$d(x,y) = \inf_{\Omega(x,y)} \int_0^1 \|\dot{\gamma}(\theta)\|\,d\theta$$

where $\Omega(x,y)$ is the set of smooth curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Our main idea is to put a Finsler type structure on $L^1(\mathbb{R})$, which measures the norm of generalized tangent vectors and is not defined on the whole space, thus not ensuring the second property of Definition 16. To do this we first focus on piecewise constant functions and define "generalized" tangent vectors in terms of shift of discontinuities. Still we can define a distance among piecewise constant functions, which happens to coincide with the usual L^1 metric and thus can be naturally extended to the whole L^1. The difference is in the differential structure at the base of this new metric, which will permit to prove the Lipschitz continuous dependence.

Consider a curve $\gamma : [0,1] \to L^1$ taking values on the set of piecewise constant functions with N discontinuities, indicating by $x_1(\theta) < x_2(\theta) < \cdots < x_N(\theta)$ the discontinuity points of $\gamma(\theta)$. Then γ admits as tangent vector $(v,\xi)(\theta) \in L^1 \times \mathbb{R}^N$ if the following holds:

$$L^1 \ni v(\theta,x) \doteq \lim_{h\to 0} \frac{\gamma(\theta+h,x)-\gamma(\theta,x)}{h}, \quad \text{for a.e. } x,$$

$$\xi_i(\theta) \doteq \lim_{h\to 0} \frac{x_i(\theta+h)-x_i(\theta)}{h}, \quad i = 1,\dots,N.$$

In this case we write $\dot\gamma(\theta) = (v, \xi)(\theta)$. Notice that γ is not differentiable according to the usual differential structure of L^1, since the L^1-limit of $(\gamma(\theta + h) - \gamma(\theta))/h$ does not exist (indeed such ratio converges to a finite sum of Dirac deltas).

The norm of $(v, \xi)(\theta)$ is defined by

$$\|(v,\xi)(\theta)\| = \|v(\theta)\|_{L^1} + \sum_{i=1}^{N} |\xi_i(\theta)| |\gamma(\theta, x_i+) - \gamma(\theta, x_i-)|.$$

The norm of (v, ξ) measures precisely the infinitesimal L^1 displacement of γ.

Then, for every couple of piecewise constant functions $u, u' \in L^1$ we can define the distance:

$$d(u,u') = \inf_{\Omega(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta$$

where $\Omega(u, u')$ is the set of curves $\gamma : [0, 1] \to L^1$ admitting piecewise smooth tangent vectors (thus having a piecewise constant number of discontinuities), such that $\gamma(0) = u$ and $\gamma(1) = u'$. We easily get that d coincides with the usual L^1-distance (since we defined suitably the norm of tangent vectors). Then d can be extended to the whole L^1 using the usual L^1-distance, namely we can set

$$d(u,u') = \inf\{\|u - w\|_{L^1} + d(w,w') + \|w' - u'\|_{L^1} \colon\ w, w' \text{ piecewise constant}\}.$$

Moreover d can be recovered just using curves with tangent vectors having a zero L^1 component. More precisely:

Lemma 15. *Given $u, u' \in L^1$ piecewise constant, let us indicate by $\widetilde{\Omega}(u, u')$ the set of curves $\gamma : [0, 1] \to L^1$, $\gamma(0) = u$, $\gamma(1) = u'$, admitting piecewise smooth tangent vector (v, ξ) such that $v \equiv 0$. Then it holds*:

$$\inf_{\widetilde{\Omega}(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta = \inf_{\Omega(u,u')} \int_0^1 \|\dot\gamma(\theta)\| \, d\theta = d(u,u') = \|u - u'\|_{L^1}.$$

Proof. Consider the curve defined for $t \in]0, 1[$ by

$$\gamma(t) = u(x)\chi_{]-\infty, \tan(\pi t - \frac{\pi}{2})]} + u'(x)\chi_{]\tan(\pi t - \frac{\pi}{2}), +\infty[},$$

where χ is the indicator function, and setting, by continuity, $\gamma(0) = u$ and $\gamma(1) = u'$. Then clearly γ admits a piecewise smooth tangent vector (v, ξ) with $v \equiv 0$. Indeed, for every t such that $x(t) = \tan(\pi t - \frac{\pi}{2})$ is neither a discontinuity point of u nor of u' we get

$$\dot\gamma(t) = \left(0, \pi\left[1 + \tan^2\left(\pi t - \frac{\pi}{2}\right)\right]\right)$$

and so

$$\|\dot\gamma(t)\| = \pi\left[1 + \tan^2\left(\pi t - \frac{\pi}{2}\right)\right]\left|u'\big(x(t)\big) - u\big(x(t)\big)\right|.$$

Moreover, the norm of $\dot\gamma$ spans exactly the area contained between the graphs of u and u' so that:

$$\int_0^1 \|\dot\gamma(\theta)\| \, d\theta = \|u - u'\|_{L^1},$$

which gives the conclusion. □

Remark 9. The technique of generalized tangent vectors was used in [10] for systems. In that case one has to introduce weights in the definition of the norm of a tangent vector. Therefore the metric d happens to be equivalent but not equal to the L^1 metric. Moreover Lemma 15 does no more hold true.

Now the main idea to prove Lipschitz continuous dependence is the following. We consider the same Finsler structure on the set $L^1(\prod_{l=1}^{n+m} I_l)$. Given two initial data $\rho(0)$ and $\rho'(0)$, we focus on wave-front tracking approximate solutions $\rho_\nu(t)$, $\rho'_\nu(t)$. We fix a sampling procedure for the first step of the wave-front tracking, for instance sampling the initial datum at points j/ν, $j \in \mathbb{N}$. For every $\gamma_0 \in \Omega(\rho(0), \rho'(0))$ we can define γ_t to be the evolution of γ_0 at time t: for $t > 0$ and $\theta \in [0,1]$, $\gamma_t(\theta)$ is the wave-front tracking approximate solution to (31), evaluated at time t, starting from the initial condition $\gamma_0(\theta)$. It is easy to prove that γ_t admits, for a.e. θ, a tangent vector $(v,\xi)_t$ such that:

$$\left\|(v,\xi)_t(\theta)\right\| \leqslant \left\|(v,\xi)_0(\theta)\right\|. \tag{39}$$

Then, denoting by Ω_t the set of all the evolution curves of γ_0, which varies in $\Omega(\rho(0), \rho'(0))$, we get

$$\begin{aligned} d\big(\rho_\nu(t), \rho'_\nu(t)\big) &= \inf_{\Omega(\rho_\nu(t), \rho'_\nu(t))} \int_0^1 \left\|\dot\gamma(\theta)\right\| d\theta \\ &\leqslant \inf_{\Omega_t} \int_0^1 \left\|\dot\gamma(\theta)\right\| d\theta = \inf_{\Omega_t} \int_0^1 \left\|(v,\xi)_t(\theta)\right\| d\theta \\ &\leqslant \inf_{\Omega(\rho(0), \rho'(0))} \int_0^1 \left\|(v,\xi)_0(\theta)\right\| d\theta = d\big(\rho_\nu(0), \rho'_\nu(0)\big). \end{aligned}$$

Passing to the limit in ν and recalling that d coincides with the usual L^1 metric, we conclude the Lipschitz continuous dependence on initial data.

Let us now pass to estimates on the shift of waves along wave-front tracking approximate solutions. We start with a definition.

Definition 17. Fix $\xi \in \mathbb{R}$ and a wave (ρ_l, ρ_r) of an ε-approximate wave-front tracking solution to (31). We say that ξ forms a shift for the wave (ρ_l, ρ_r) if we consider the same ε-approximate wave-front tracking solution, except for the position of the wave (ρ_l, ρ_r), which is translated by the quantity ξ in the x-direction.

The proof of the continuous dependence is based on the following general lemma.

Lemma 16. *Fix an ε-approximate wave-front tracking solution to* (31) *$\bar\rho_\varepsilon$ with respect to a Riemann solver $\mathcal{RS}$, satisfying the consistency condition. Assume that a wave $(\hat\rho_k^-, \tilde\rho_k^-)$ in an arc I_k $(k \in \{1, \dots, n+m\})$ interacts with J producing waves $(\hat\rho_l^+, \tilde\rho_l^+)$ in (possibly) all the arcs of the node J. If the interacting wave in I_k is shifted by ξ_k^-, then all the produced waves at J are shifted by ξ_l^+ $(l \in \{1, \dots, n+m\})$, which satisfy the relations*

$$\left|\xi_k^- \frac{\hat\rho_k^- - \tilde\rho_k^-}{f(\hat\rho_k^-) - f(\tilde\rho_k^-)}\right| = \left|\xi_l^+ \frac{\hat\rho_l^+ - \tilde\rho_l^+}{f(\hat\rho_l^+) - f(\tilde\rho_l^+)}\right| \tag{40}$$

for every $l \in \{1, \dots, n+m\}$.

Proof. Note that, applying the shift ξ_k^-, the interaction of the wave $(\hat\rho_k^-, \tilde\rho_k^-)$ with J is shifted in time by

$$\xi_k^- \frac{\hat\rho_k^- - \tilde\rho_k^-}{f(\hat\rho_k^-) - f(\tilde\rho_k^-)}.$$

The shift in time of the waves generated by this interaction must be the same and so the proof easily follows. □

Theorem 17. *Fix $\boldsymbol{\theta} \in \Theta$ and consider the Cauchy problem* (31) *with the Riemann solver $\mathcal{RS}_2$. There exists a unique $(\rho_1(t,x), \dots, \rho_{n+m}(t,x))$, weak solution at J, such that*

1. *for every* $l \in \{1, \dots, n+m\}$, $\rho_l(0, x) = \rho_{0,l}(x)$ *for a.e.* $x \in I_l$;
2. *for a.e.* $t > 0$,

$$\mathcal{RS}_2\big(\rho_1(t,0), \dots, \rho_{n+m}(t,0)\big) = \big(\rho_1(t,0), \dots, \rho_{n+m}(t,0)\big). \tag{41}$$

Moreover the solution depends in a Lipschitz continuous way on the initial datum with respect to the L^1*-topology.*

Proof. As explained above, we can restrict to estimate the L^1-distance among wave-front tracking solutions. For this, it is enough to show that

$$\big\|(v,\xi)_t(\theta)\big\| \leqslant \big\|(v,\xi)_0(\theta)\big\|$$

for every $t > 0$ and $\theta \in [0, 1]$. We prove the latter estimating the evolution of the tangent vector norm at each time. Moreover, by Lemma 15, we can restrict the study to the evolution of shifts.

Fix a time $\bar{t} > 0$ and, without loss of generality, treat the following cases:

(a) no interaction of waves takes place in any arc at $\bar{t}$ and no wave interacts with J;
(b) two waves interact at $\bar{t}$ on an arc and no other interaction takes place;
(c) a wave interacts with J at $\bar{t}$ and no other interaction takes place.

In case (a) the shifts are constant, while in case (b) the norms are decreasing by Lemma 2.7.2 of [24] (see Appendix A).

Assume now that a wave $(\hat{\rho}_{\bar{l}}^-, \tilde{\rho}_{\bar{l}}^-)$ interacts with J at time $\bar{t}$ from the arc $I_{\bar{l}}$. Using Lemmas 6 and 16, we deduce

$$\begin{aligned}
\big\|(v,\xi)(\bar{t}+)\big\| - \big\|(v,\xi)(\bar{t}-)\big\| &= \sum_{l=1}^{n+m} \big|\xi_l^+\big|\big|\hat{\rho}_l^+ - \tilde{\rho}_l^+\big| - \big|\xi_{\bar{l}}^-\big|\big|\hat{\rho}_{\bar{l}}^- - \tilde{\rho}_{\bar{l}}^-\big| \\
&= \left[\sum_{l=1}^{n+m} \left|\frac{f(\hat{\rho}_l^+) - f(\tilde{\rho}_l^+)}{f(\hat{\rho}_{\bar{l}}^-) - f(\tilde{\rho}_{\bar{l}}^-)}\right| - 1\right] \big|\xi_{\bar{l}}^-\big|\big|\hat{\rho}_{\bar{l}}^- - \tilde{\rho}_{\bar{l}}^-\big| \\
&= 0,
\end{aligned}$$

where $(\hat{\rho}_l^+, \tilde{\rho}_l^+)$ is the wave produced in the arc I_l after the interaction. This estimate permits to conclude. □

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Appendix A. Technical lemmas

In this section we report the statements of Lemmas 2.7.2, 4.3.6 and 4.3.7 of [24], for readers' convenience.

Lemma 2.7.2 of [24]. *Consider two waves, with speeds* λ_1 *and* λ_2 *respectively, that interact together at* $\bar{t}$ *producing a wave with speed* λ_3*. If the first wave is shifted by* ξ_1 *and the second wave by* ξ_2*, then the shift of the resulting wave is given by*

$$\xi_3 = \frac{\lambda_3 - \lambda_2}{\lambda_1 - \lambda_2}\xi_1 + \frac{\lambda_1 - \lambda_3}{\lambda_1 - \lambda_2}\xi_2. \tag{42}$$

Moreover we have that

$$\Delta\rho_3\xi_3 = \Delta\rho_1\xi_1 + \Delta\rho_2\xi_2, \tag{43}$$

where $\Delta\rho_i$ *are the signed strengths of the corresponding waves.*

Lemma 4.3.6 of [24]. *If an arc* I_l *of a node* J *has a good datum, then it remains good after interactions with* J *of waves coming from other arcs. Moreover, no big shock can be produced in this way. If an arc* I_l *has a bad datum, then* *after any interaction with J of waves coming from other arcs, either the datum of I_l is unchanged or a big shock is produced (and the new datum is good).*

Lemma 4.3.7 of [24]. *If a wave produced from a node J on an incoming arc I_i comes back to J, interacting only with waves produced by J, then the wave connects a bad left datum to a right good datum. The converse is true for outgoing arcs.*

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