Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 1049–1054
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Erratum

Erratum to: "Multiple critical points of perturbed symmetric strongly indefinite functionals" [http://dx.doi.org/10.1016/j.anihpc.2008.06.002]

Denis Bonheure [a,*,1], Miguel Ramos [b,2]

[a] Université libre de Bruxelles, Département de Mathématique, U.L.B., CP214, Boulevard du Triomphe, 1050 Bruxelles, Belgium
[b] University of Lisbon, CMAF – Faculty of Science, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

Received 21 October 2008; accepted 14 January 2009

Available online 17 March 2009

Abstract

We correct the statement and the proof of Proposition 9 in [D. Bonheure, M. Ramos, Multiple critical points of perturbed symmetric strongly indefinite functionals, http://dx.doi.org/10.1016/j.anihpc.2008.06.002].

MSC: 35J50; 35J55; 58E05

Keywords: Elliptic system; Strongly indefinite functional; Perturbation from symmetry; Lyapunov–Schmidt reduction; Genericity

The proof of [1, Proposition 9] is incorrect. We weaken the statement of this proposition and present a proof of it. The weaker statement however is enough for the purposes of [1]. We use the notation and assumptions introduced in [1].

Let $I^*: E \to \mathbb{R}$ be the functional associated to the problem

$$\begin{cases} -\Delta u = |v|^{q-2}v & \text{in } \Omega, \\ -\Delta v = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Consider the associated reduced functional

$$J^*(\alpha) := I^*\big(\alpha + \psi_\alpha^*, \alpha - \psi_\alpha^*\big) := \max_{\psi \in H_0^1(\Omega)} I^*(\alpha + \psi, \alpha - \psi). \tag{1.2}$$

DOI of original article: 10.1016/j.anihpc.2008.06.002.
* Corresponding author.
E-mail addresses: denis.bonheure@ulb.ac.be (D. Bonheure), mramos@ptmat.fc.ul.pt (M. Ramos).
1 Research supported by the F.R.S.-FNRS (Belgium).
2 Research partially supported by FCT (Fundação para a Ciência e Tecnologia), program POCI-ISFL-1-209 (Portugal/Feder-EU).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2009.01.007

Recall that if α is a critical point of J^* then

$$-2\Delta\alpha = f\big(u_\alpha^*\big) + g\big(v_\alpha^*\big), \tag{1.3}$$

where $u_\alpha^* := \alpha + \psi_\alpha^*$, $v_\alpha^* := \alpha - \psi_\alpha^*$ and ψ_α^* is the unique solution of the following equation in $H_0^1(\Omega)$:

$$-2\Delta\psi_\alpha^* = g\big(v_\alpha^*\big) - f\big(u_\alpha^*\big). \tag{1.4}$$

Denote by $m^*(\alpha)$ the *augmented Morse index* of the critical point α with respect to J^*, i.e. the number of non-positive eigenvalues of the quadratic form $(J^*)''(\alpha)$.

We will be interested in special critical points constructed via a min-max argument. To that purpose, we introduce the following notations. Let us write

$$H := H_0^1(\Omega) = E_k \oplus E_k^\perp,$$

where, for each $k \in \mathbb{N}_0$, E_k is spanned by the first k eigenfunctions of the Laplacian operator in $H_0^1(\Omega)$. Arguing as in [1, Lemma 3], we can provide a large constant $R_k > 0$ such that $J^*(\alpha) < 0$ for every $\alpha \in E_k$ satisfying $\|\alpha\| > R_k$. Let

$$G_k := \big\{\sigma \in C\big(B_{R_k}(0) \cap E_k; H\big) \;\big|\; \sigma(-\alpha) = -\sigma(\alpha)\ \forall\alpha \in H,\ \sigma_{|\partial B_{R_k}(0)\cap E_k} = \mathrm{Id}\big\}$$

and define the minimax levels

$$b_k := \inf_{\sigma\in G_k} \max\big\{J^*\big(\sigma(\alpha)\big)\colon \alpha \in B_{R_k}(0) \cap E_k\big\}. \tag{1.5}$$

We next derive a bound on b_k.

Proposition 9. *Assume $2 < p \leqslant q \leqslant 2^*$. There exist $C > 0$ and $k_0 \in \mathbb{N}_0$ such that for every $k \geqslant k_0$,*

$$k \leqslant C b_k^{(1-\frac{1}{p}-\frac{1}{q})\frac{N}{2}}.$$

Proof. For the sake of clarity we divide the proof in several steps.

Step 1. According to (1.3) and (1.4), if α is a critical point of J^*, $m^*(\alpha)$ is the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-2\Delta\varphi = \mu\big(f'\big(u_\alpha^*\big)(\varphi + \phi) + g'\big(v_\alpha^*\big)(\varphi - \phi)\big), \quad \varphi \in H_0^1(\Omega), \tag{1.6}$$

where $\phi \in H_0^1(\Omega)$ solves

$$-2\Delta\phi = g'\big(v_\alpha^*\big)(\varphi - \phi) - f'\big(u_\alpha^*\big)(\varphi + \phi). \tag{1.7}$$

By denoting $V = (f'(u_\alpha^*) + g'(v_\alpha^*))/2$ and $W = (f'(u_\alpha^*) - g'(v_\alpha^*))/2$, we can rephrase (1.6)–(1.7) by

$$-\Delta\varphi = \mu(V\varphi + W\phi) \quad \text{and} \quad (-\Delta + V)\phi = -W\varphi.$$

Hence, $m^*(\alpha)$ is the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-\Delta\varphi = \mu T\varphi, \quad \varphi \in H_0^1(\Omega),$$

where T is the compact operator

$$T := V - W(-\Delta + V)^{-1}W.$$

Since the operator $W(-\Delta + V)^{-1}W$ is positive, we have that $m^*(\alpha) \leqslant m_V^*(\alpha)$, where the latter quantity denotes the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-\Delta\varphi = \mu V(x)\varphi, \quad \varphi \in H_0^1(\Omega).$$

According to a well-known estimate obtained in [2,3,5] (see e.g. [6] for a proof), we have that

$$m_V^*(\alpha) \leqslant C \int V(x)^{N/2} \tag{1.8}$$

for some universal constant $C > 0$. Going back to the original system $-\Delta u = |v|^{q-2}v$, $-\Delta v = |u|^{p-2}u$, we observe that $\int |u_\alpha^*|^p = \int |v_\alpha^*|^q$ and

$$J^*(\alpha) = I^*\big(u_\alpha^*, v_\alpha^*\big) = \left(\frac{1}{2} - \frac{1}{p}\right)\int \big|u_\alpha^*\big|^p + \left(\frac{1}{2} - \frac{1}{q}\right)\int \big|v_\alpha^*\big|^q$$
$$= \left(1 - \frac{1}{p} - \frac{1}{q}\right)\int \big|u_\alpha^*\big|^p$$
$$= \left(1 - \frac{1}{p} - \frac{1}{q}\right)\int \big|v_\alpha^*\big|^q.$$

By using this and by applying Hölder inequality in (1.8) we get that

$$m^*(\alpha) \leqslant C\big(J^*(\alpha)^{(p-2)N/2p} + J^*(\alpha)^{(q-2)N/q}\big). \tag{1.9}$$

We will next refine this estimate by introducing a free parameter.

Step 2. For any $\lambda > 0$, define

$$J_\lambda^*(\alpha) := I^*\left(\lambda\alpha + \psi_{\alpha.\lambda}^*, \alpha - \frac{\psi_{\alpha,\lambda}^*}{\lambda}\right) := \max_{\psi\in H} I^*\left(\lambda\alpha + \psi, \alpha - \frac{\psi}{\lambda}\right),$$

so that $J_1^*(\alpha) = J^*(\alpha)$. Arguing as in step 1, we can check that if α is a critical point of J_λ^* then the corresponding Morse index $m_\lambda^*(\alpha)$ is given by the number of eigenvalues $\mu \leqslant 1$ of the problem

$$-\Delta\varphi = \mu T_\lambda \varphi, \quad \varphi \in H_0^1(\Omega),$$

where T_λ is the compact operator

$$T_\lambda := V_\lambda - W_\lambda(-\Delta + V_\lambda)^{-1} W_\lambda,$$

with $V_\lambda = \frac{1}{2}(\lambda f'(u_\alpha^*) + g'(v_\alpha^*)/\lambda)$ and $W_\lambda = \frac{1}{2}(f'(u_\alpha^*) - g'(v_\alpha^*)/\lambda^2)$. Here, of course, $u_\alpha^* := \lambda\alpha + \psi_{\alpha,\lambda}^*$, $v_\alpha^* := \alpha - \frac{\psi_{\alpha,\lambda}^*}{\lambda}$. Then, similarly to (1.8)–(1.9), we get that

$$m_\lambda^*(\alpha) \leqslant C\left(\big(\lambda J_\lambda^*(\alpha)^{(p-2)/p}\big)^{N/2} + \left(\frac{J_\lambda^*(\alpha)^{(q-2)/q}}{\lambda}\right)^{N/2}\right). \tag{1.10}$$

Step 3. As a further preliminary step in our proof, we introduce the map

$$\theta_\lambda(\alpha) := \frac{\lambda+1}{2}\alpha + \frac{\lambda-1}{2\lambda}\psi_{\alpha,\lambda}^*.$$

We claim that $\theta_\lambda : H \to H$ is an odd homeomorphism such that

$$J_\lambda^*\big(\theta_\lambda^{-1}(\alpha)\big) \leqslant J_1^*(\alpha), \quad \forall \alpha \in H. \tag{1.11}$$

We can already assume that $\lambda \neq 1$, otherwise our statement is obvious. Observe that given $\beta \in H$ it is possible to find an unique $\alpha \in H$ such that

$$\psi_{\alpha,\lambda}^* = \frac{2\lambda}{\lambda-1}\beta - \frac{\lambda(\lambda+1)}{\lambda-1}\alpha.$$

Indeed, using the definition of $\psi_{\alpha,\lambda}^*$, this means that we must solve the equation in H:

$$-2\lambda\frac{\lambda+1}{\lambda-1}\Delta\alpha = \frac{-4\lambda}{\lambda-1}\Delta\beta - g\left(\frac{2\lambda}{\lambda-1}\alpha - \frac{2}{\lambda-1}\beta\right) + \lambda f\left(\frac{-2\lambda}{\lambda-1}\alpha + \frac{2\lambda}{\lambda-1}\beta\right)$$

and, clearly, this problem has a unique solution $\alpha \in H$ for any given $\beta \in H$. As for (1.11), given $\alpha \in H$, let $\beta = \theta_\lambda^{-1}(\alpha)$. Then $\alpha = \theta_\lambda(\beta)$ and

$$J_\lambda^*(\beta) := I^*\left(\lambda\beta + \psi_{\beta,\lambda}, \beta - \frac{\psi_{\beta,\lambda}}{\lambda}\right) = I^*(\alpha + \varphi, \alpha - \varphi) \leqslant J_1^*(\alpha),$$

where $\varphi := \lambda\beta + \psi_{\beta,\lambda} - \alpha$ and we have used the definition of J_1^* in the last inequality.

Step 4. We now describe the following min-max construction. By using Fatou's lemma, we easily see that for any finite dimensional subspace Y of H, $J_\lambda^*(\alpha) \to -\infty$ as $\|\alpha\| \to \infty$, $\alpha \in Y$. From now on, we denote by Q_k^λ a large ball $Q_k^\lambda = B_{R_k^\lambda}(0) \cap E_k$ and

$$\partial Q_k^\lambda := \{\alpha \in E_k\colon \|\alpha\| = R_k^\lambda\}, \qquad S := \{\alpha \in H_{k-1}^\perp\colon \|\alpha\| = \rho\}.$$

The constant $\rho = \rho_k$ is defined in the following way: from now on we restrict ourselves to a fixed interval $\lambda \in [\lambda^*, +\infty[$ with $0 < \lambda^* < 1$; then it is possible to fix $\rho \in]0, R_k[$ in such a way that

$$\inf\left\{ I^*\left(\frac{2\lambda}{\lambda+1}\alpha, \frac{2\lambda}{\lambda+1}\alpha\right)\colon \alpha \in S\right\} > 0. \tag{1.12}$$

We stress that ρ does not depend on λ. The positive constant R_k^λ is taken large enough so that $J_\lambda^*(\alpha) < 0$ for every $\alpha \in E_k$ such that $\|\alpha\| \geqslant R_k^\lambda$ and we choose $R_k^1 = R_k$. By possibly taking a larger R_k^λ, we also require that

$$\|\theta_\lambda^{-1}(\alpha)\| > \rho, \quad \forall \alpha \in \partial Q_k^\lambda. \tag{1.13}$$

Finally, we require that $R_k^\lambda \geqslant R_k^1$ for every $\lambda \in [\lambda^*, +\infty[$.

Given $A \subset H$, we say that A and S *link* if:

(i) A is compact and symmetric;
(ii) A contains a subset B which is odd homeomorphic to ∂Q_k^λ and $\sup_B J_\lambda^* < 0$;
(iii) $\gamma(A) \cap S \neq \emptyset$ for every odd and continuous map $\gamma : A \to H$ such that $\gamma|_B = \mathrm{Id}$. Accordingly, we denote

$$\mathcal{A}_\lambda := \{A \subset H\colon A \text{ and } S \text{ link}\}$$

and

$$c_\lambda^* := \inf_{A \in \mathcal{A}_\lambda} \sup_A J_\lambda^*.$$

The class $\mathcal{A}_\lambda$ contains the set Q_k^λ, since $\rho < R_k^\lambda$. We also observe that $c_\lambda^* > 0$. Indeed, by definition, we have $c_\lambda^* \geqslant \inf_S J_\lambda^*$ while, using the very definition of J_λ^*, for every $\alpha \in H$,

$$J_\lambda^*(\alpha) \geqslant I^*\left(\lambda\alpha + \varphi, \alpha - \frac{\varphi}{\lambda}\right) = I^*\left(\frac{2\lambda}{\lambda+1}\alpha, \frac{2\lambda}{\lambda+1}\alpha\right),$$

where $\varphi := \frac{\lambda(1-\lambda)}{1+\lambda}\alpha$, and our claim follows from (1.12).

Now, it is standard that there exists a critical point α_λ of J_λ^* at level c_λ^* satisfying $m_\lambda^*(\alpha_\lambda) \geqslant k$.

It then follows from (1.9) that

$$k \leqslant C\left(\left(\lambda\left(c_\lambda^*\right)^{(p-2)/p}\right)^{N/2} + \left(\frac{(c_\lambda^*)^{(q-2)/q}}{\lambda}\right)^{N/2}\right). \tag{1.14}$$

This estimate holds uniformly in λ and in k, provided λ is bounded away from zero. In our final step below we prove that, for every λ,

$$c_\lambda^* \leqslant b_k. \tag{1.15}$$

By inserting (1.15) in the inequality (1.14) with $\lambda := b_k^{\frac{1}{p}-\frac{1}{q}}$ completes then the proof of Proposition 9. We stress that indeed $b_k \geqslant 1$ for every large $k \in \mathbb{N}$.

Step 5. In order to prove (1.15), given $\varepsilon > 0$, let $\sigma \in G_k$ be such that $\sup_{\sigma(Q_k^1)} J_1^* \leqslant b_k + \varepsilon$; we recall that $Q_k^1 = B_{R_k^1}(0) \cap E_k = B_{R_k}(0) \cap E_k$. Since $R_k^1 \leqslant R_k^\lambda$, we can extend σ to Q_k^λ by setting

$$\tilde\sigma(\alpha) := \begin{cases} \sigma(\alpha) & \text{if } \alpha \in Q_k^1, \\ \alpha & \text{if } \alpha \in Q_k^\lambda \setminus Q_k^1. \end{cases}$$

Let $A := \theta_\lambda^{-1}(\tilde\sigma(Q_k^\lambda))$. We claim that $A \in \mathcal{A}_\lambda$. Indeed, by letting $B := \theta_\lambda^{-1}(\tilde\sigma(\partial Q_k^\lambda)) = \theta_\lambda^{-1}(\partial Q_k^\lambda)$, thanks to (1.11) we see that

$$\sup_B J_\lambda^* \leqslant \sup_{\tilde\sigma(\partial Q_k^\lambda)} J_1^* = \sup_{\partial Q_k^\lambda} J_1^* < 0$$

since $R_k^\lambda \geqslant R_k^1$. On the other hand, let $\gamma : A \to H$ be any odd and continuous map such that $\gamma(\alpha) = \alpha$ for all $\alpha \in \theta_\lambda^{-1}(\partial Q_k^\lambda)$. Define

$$\mathcal{U} := \{\alpha \in Q_k^\lambda\colon\ \|\gamma(\theta_\lambda^{-1}(\tilde\sigma(\alpha)))\| < \rho\}.$$

This is a bounded, symmetric neighborhood of the origin in E_k. According to the Borsuk–Ulam theorem, there exists $\alpha \in \partial\mathcal{U}$ such that $\gamma(\beta) \in E_{k-1}^\perp$, with $\beta := \theta_\lambda^{-1}(\tilde\sigma(\alpha))$. Of course we have $\|\gamma(\beta)\| \leqslant \rho$. Now, if $\alpha \in \partial Q_k^\lambda$ then $\gamma(\beta) = \theta_\lambda^{-1}(\alpha)$ and so $\|\gamma(\beta)\| > \rho$, see (1.13). Thus $\alpha \notin \partial Q_k^\lambda$ and therefore $\|\gamma(\beta)\| = \rho$. In conclusion, $\gamma(\beta) \in \gamma(A) \cap S$. This proves the required linking property and shows that $A \in \mathcal{A}_\lambda$. As a consequence,

$$c_\lambda^* \leqslant \sup_A J_\lambda^*.$$

But, again by (1.11) and the fact that $R_k^\lambda \geqslant R_k^1$ we have that

$$\sup_A J_\lambda^* = \sup_{\theta_\lambda^{-1}(\tilde\sigma(Q_k^\lambda))} J_\lambda^* \leqslant \sup_{\tilde\sigma(Q_k^\lambda)} J_1^* = \sup_{(Q_k^\lambda\setminus Q_k^1)\cup\sigma(Q_k^1)} J_1^* = \sup_{\sigma(Q_k^1)} J_1^*.$$

In conclusion,

$$c_\lambda^* \leqslant b_k + \varepsilon, \quad \forall\varepsilon > 0,$$

and this establishes (1.15). □

The proof of [1, Theorem 1] follows easily from Proposition 9. [1, Claim 2 of Theorem 1] has to be adapted to the new statement of Proposition 9. We include the details for completeness.

We denote again by E_k the eigenspace associated to the first k eigenfunctions of the Laplacian operator in $H_0^1(\Omega)$ and we fix a large constant $\tilde R_k > 0$ such that $\tilde J(\alpha) < 0$ for every $\alpha \in E_k$ satisfying $\|\alpha\| > \tilde R_k$. Let

$$G_k := \{\sigma \in C(B_{\tilde R_k}(0) \cap E_k; H_0^1(\Omega)) \mid \sigma(-\alpha) = -\sigma(\alpha),\ \sigma|\partial B_{\tilde R_k}(0) \cap E_k = \mathrm{Id}\},$$

and define the minimax levels

$$\tilde b_k := \inf_{\sigma\in G_k} \max\{\tilde J(\sigma(\alpha))\colon\ \alpha \in B_{\tilde R_k}(0) \cap E_k\}. \tag{1.16}$$

Proof of [1, Theorem 1]. Assume by contradiction that $\tilde J$ does not admit an unbounded sequence of critical values. Let $(\tilde b_k)_k$ be the sequence of minimax levels of $\tilde J$ defined by (1.16).

Claim 1. *There exist $C, k_0 > 0$ such that for all $k \geqslant k_0$,*

$$\tilde b_k \leqslant C k^{p/(p-1)}. \tag{1.17}$$

The claim follows exactly as in [4, Prop. 10.46].

Claim 2. *There exist $C' > 0$ and $k_0' > 0$ such that for all $k \geqslant k_0'$,*

$$\tilde b_k \geqslant C' k^{2pq/N(pq-p-q)}. \tag{1.18}$$

Let us fix a small $c > 0$ in such a way that the functional

$$\hat I(u,v) = \int \big(\langle\nabla u, \nabla v\rangle - cF(u) - cG(v)\big)$$

is such that $\tilde{I} - \hat{I}$ is bounded from below in $H_0^1(\Omega) \times H_0^1(\Omega)$. We also consider the associated reduced functional $\hat{J}$ defined by

$$\hat{J}(\alpha) := \hat{I}(\alpha + \hat{\psi}_\alpha, \alpha - \hat{\psi}_\alpha) := \max_{\psi \in H_0^1(\Omega)} \hat{I}(\alpha + \psi, \alpha - \psi)$$

and the corresponding minimax numbers

$$\hat{b}_k := \inf_{\sigma \in G_k} \max\{\hat{J}(\sigma(\alpha)) \colon \alpha \in B_{\hat{R}_k}(0) \cap E_k\},$$

where taking $\hat{R}_k = \tilde{R}_k$ larger if necessary, we can assume that $\hat{J}(\alpha) < 0$ for every $\alpha \in E_k$ satisfying $\|\alpha\| > \hat{R}_k$. Clearly, the sequence $\tilde{b}_k - \hat{b}_k$ is bounded from below. According to Proposition 9, we have that $k^{2pq/N(pq-p-q)} \leqslant C\hat{b}_k$, so that the claim follows.

The conclusion easily follows. □

References

[1] D. Bonheure, M. Ramos, Multiple critical points of perturbed symmetric strongly indefinite functionals, http://dx.doi.org/10.1016/j.anihpc.2008.06.002.

[2] M. Cwickel, Weak type estimates and the number of bound states of Schrödinger operators, Ann. Math. 106 (1977) 93–102.

[3] E.H. Lieb, Bounds on the eigenvalues of the Laplace and Schrödinger operators, Bull. Amer. Math. Soc. 82 (1976) 751–753.

[4] P. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf., vol. 65, Amer. Math. Soc., Providence, RI, 1986.

[5] G. Rosenbljum, The distribution of the discrete spectrum for singular differential operators, Soviet Math. Dokl. 13 (1972) 245–249.

[6] B. Simon, Functional Integration and Quantum Physics, Academic Press, 1979.