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On the uniqueness of the second bound state solution of a semilinear equation ☆

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Abstract

We establish the uniqueness of the second radial bound state solution of

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^n. \tag{P}$$

We assume that the nonlinearity $f \in C(-\infty, \infty)$ is an odd function satisfying some convexity and growth conditions of superlinear type, and either has one zero at $b > 0$, is nonpositive and not identically 0 in $(0, b)$, and is differentiable and positive $[b, \infty)$, or is positive and differentiable in $[0, \infty)$.

Keywords: Bound state; Uniqueness; Separation lemmas

1. Introduction and main results

In this paper we establish the uniqueness of the second bound state solution to

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^n \tag{P}$$

in the radially symmetric case. That is, we will prove that the problem

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0, \quad r > 0,\ n > 2, \\ &u'(0) = 0, \qquad \lim_{r\to\infty} u(r) = 0, \end{aligned} \tag{1}$$

has at most one solution $u \in C^2[0, \infty)$ such that

$$\begin{aligned} &\text{there exists } R > 0 \text{ such that} \quad u(r) > 0, \quad r \in (0, R), \quad u(R) = 0, \\ &u(r) < 0, \quad \text{for } r > R. \end{aligned} \tag{2}$$

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Any nonconstant solution to (1) is called a bound state solution. Bound state solutions such that $u(r) > 0$ for all $r > 0$, are referred to as a first bound state solution, or a ground state solution.

The function $f : \mathbb{R} \to \mathbb{R}$ is supposed to be continuous and odd, and we set $F(s) = \int_0^s f(t)\,dt$.

We will assume that f satisfies (f_1)–(f_5), where

(f_1) $f(0) = 0$, and there exists $b \geqslant 0$ such that $f(s) > 0$ for $s > b$, and if $b > 0$, $f(s) \leqslant 0$, $f(s) \not\equiv 0$ for $s \in [0, b]$.

(f_2) f is continuous in $[0, \infty)$ and differentiable in $[b, \infty)$.

(f_3) $f(s) \leqslant f'(s)(s - b)$, for all $s \geqslant b$. Furthermore, if $b = 0$, we also assume that

$$1 < \lim_{s \to 0} \frac{sf'(s)}{f(s)}. \tag{3}$$

(f_4) The function $s \to \frac{sf'(s)}{f(s)}$ is nonincreasing in (b, ∞).

(f_5) If $b > 0$ we assume that

$$\frac{\beta f'(\beta)}{f(\beta)} \leqslant \frac{n}{n-2},$$

where $\beta > b$ is the unique positive solution of $F(x) = 0$.

If $b = 0$, we also assume

$$\frac{sf'(s)}{f(s)} \leqslant \frac{n}{n-2} \quad \text{for all } s > 0.$$

We have imposed (3) in the case $b = 0$, because otherwise, from the assumptions $f(s) \leqslant sf'(s)$ and (f_4), f would be linear, and in this case we obviously do not have uniqueness.

In addition, we will prove the existence of at most one solution to the Dirichlet–Newmann free boundary problem

$$\begin{aligned}
&u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0, \quad r > 0,\ n > 2,\\
&u'(0) = 0, \quad \text{there exist } 0 < R < \bar{R} \text{ such that} \quad u(r) > 0, \quad r \in (0, R), \quad u(R) = 0,\\
&u(r) < 0, \quad \text{for } r \in (R, \bar{R}), \qquad u(\bar{R}) = u'(\bar{R}) = 0.
\end{aligned} \tag{4}$$

In order to prove our results, we will study the behavior of the solutions to the initial value problem

$$\begin{aligned}
&u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0 \quad r > 0,\ n > 2,\\
&u(0) = \alpha, \quad u'(0) = 0.
\end{aligned} \tag{5}$$

for $\alpha \in (0, \infty)$. As usual, we will denote by $u(r, \alpha)$ a C^2 solution of (5).

We state now our main result.

Theorem 1.1. *Assume that f satisfies (f_1)–(f_5) with $b > 0$. Then problems* (1)–(2) *and* (4) *have at most one nontrivial solution.*

This result will follow from the following monotonicity theorem:

Theorem 1.2. *Assume that f satisfies (f_1)–(f_5) and let $0 < \alpha_1 < \alpha_2$, $u_1(r) := u(r, \alpha_1)$, $u_2(r) := u(r, \alpha_2)$.*

(i) *If $b = 0$, then the problems* (1)–(2) *and* (4) *have no nontrivial solutions.*

(ii) *If u_1 has two zeros $0 < R_1 < \bar{R}_1$ with $u_1(r) > 0$ in $(0, R_1)$ and $u_1(r) < 0$ in $(R_1, \bar{R}_1)$, then u_2 has at least two zeros in $(0, \bar{R}_1)$, $0 < R_2 < \bar{R}_2$ with $R_1 > R_2$, $\bar{R}_1 > \bar{R}_2$, $u_2(r) > 0$ in $(0, R_2)$ and $u_2(r) < 0$ in $(R_2, \bar{R}_2)$. Furthermore,*

$$u_1'(\bar{R}_1) < u_2'(\bar{R}_2).$$

(iii) *Let $b > 0$. If u_1 is a solution of* (1)–(2)*, then u_2 has at least two zeros $0 < R_2 < \bar{R}_2$ with $R > R_2$, $u_2(r) > 0$ in $(0, R_2)$ and $u_2(r) < 0$ in $(R_2, \bar{R}_2)$.*

As a consequence of Theorem 1.2, we have the following uniqueness result for the Dirichlet problem in a given ball.

Theorem 1.3. *Assume that f satisfies (f_1)–(f_5). Given $\rho > 0$, the problem*

$$\begin{aligned} &u''(r) + \frac{n-1}{r}u'(r) + f(u) = 0, \quad r > 0,\ n > 2, \\ &u'(0) = 0, \qquad u(\rho) = 0, \end{aligned} \tag{6}$$

has at most one solution u such that

$$\begin{aligned} &\text{there exists } R \in (0, \rho) \text{ such that} \quad u(r) > 0, \quad r \in (0, R), \quad u(R) = 0, \\ &u(r) < 0, \quad \text{for } R < r < \rho. \end{aligned} \tag{7}$$

The uniqueness of the first bound state solution of (1) or for the quasilinear situation involving the m-Laplacian operator $\nabla \cdot (|\nabla u|^{m-2}\nabla u)$, $m > 1$, has been exhaustively studied during the last thirty years, see for example the works [3,4,6,7,10,12–14,16–19]. The study was initiated by Coffman, see [4], where the case $f(s) = s^p - s^q$ is treated for $p = 3$ and $q = 1$ in dimension $n = 3$. Berestycki and Lions proposed in [1], as an open problem, the uniqueness for other nonlinearities than the very special one considered by Coffman. In this direction, McLeod and Serrin improved the results of Coffmann to more general nonlinearities f, including in particular $f(s) = s^p - s$ for $1 < p \leqslant n/(n-2)$. Kwong [12] extended the range for p to $1 < p < (n+2)/(n-2)$. All these works still assumed differentiability of f in $[0,\infty)$. Peletier and Serrin, see [16,17], proved a crucial Monotone Separation result and established uniqueness of positive bound states for f locally Lipschitz in $(0,\infty)$ and under a sublinear type of assumption. Chen and Lin, see [3], proved uniqueness of positive bound states under a superlinear and subcritical type of assumptions for $f \in C^1[0,\infty)$, $f(0) = 0$. Using a combination of the arguments given in [4,12] and [16,17], Cortázar, Felmer and Elgueta, see [6,7], extended their result for f continuous in $[0,\infty)$, $f(0) = 0$ and locally Lipschitz in (b,∞), under the superlinear assumption (f_3) and (f_4). Pucci and Serrin in [18], dealt with a very general operator (including the m-Laplacian case), and proved the uniqueness of the first bound state under the assumptions $f(s) < 0$ in $(0,b)$, $f(s) > 0$ in (b,∞), $f \in L^1_{\mathrm{loc}}(0,\delta) \cap C^1(0,\infty)$ and the subcritical condition $(F/f)'(s) \geqslant (n-2)/2n$ for all $s > 0$, $s \neq b$. Finally, we mention the work of Serrin and Tang, see [19], to our knowledge the most complete for the m-Laplacian operator, where the authors established the uniqueness of the first bound state solution assuming only $f \in C(0,\infty) \cap C^1(b,\infty)$ and (f_4).

In [19] the authors also conjectured that their methods could be adapted to the study of the uniqueness of positive solutions to the Dirichlet problem (6) for $f(s) = s^p - s^q$ and in the quasilinear case of the m-Laplacian. This conjecture was proved true in [11, Theorem 1.2] in the superlinear situation for a certain range of the parameters involved. This was done using the methods in [7,8]. In the case of the semilinear problem, the result in [11] applies to the canonical nonlinearity $f(s) = s^p - s^q$, with $0 < q < p$, $p \geqslant 1$. Note that Theorem 1.3 gives a uniqueness result for the second bound state of the Dirichlet problem in a ball.

To the best of our knowledge, there is only one work concerning the uniqueness of higher bound states: Troy, see [20, Theorems 1.1, 1.3] studied the existence and uniqueness of the solution to (1)–(2) in dimension $n = 3$ for

$$f(s) = \begin{cases} s+1, & s \leqslant -1/2, \\ -s, & s \in (-1/2, 1/2), \\ s-1, & s \geqslant 1/2. \end{cases}$$

Note that in this case $b = 1$, $\beta = 1 + \sqrt{2}/2$, and for $s > b$,

$$\frac{sf'(s)}{f(s)} = 1 + \frac{1}{s-1}.$$

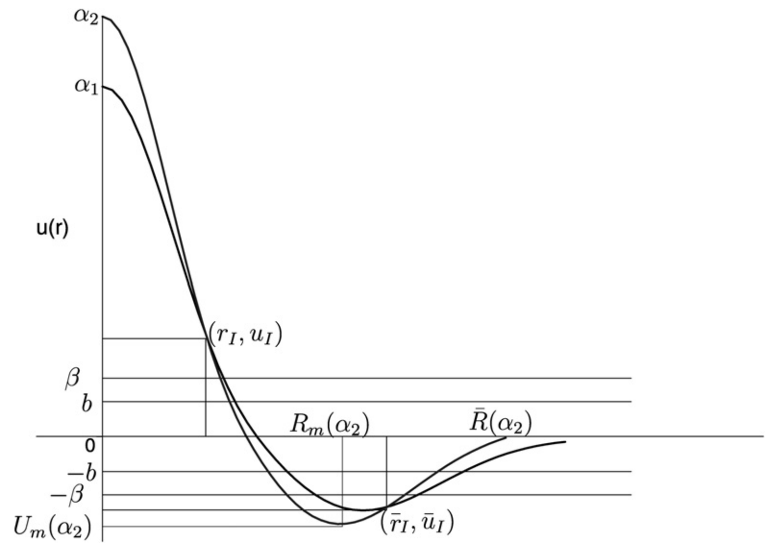

Fig. 1. The graph of two solutions in the neighborhood V illustrating their intersection points.

Hence all assumptions (f_1)–(f_5) are satisfied since

$$\frac{\beta f'(\beta)}{f(\beta)} = 1 + \sqrt{2} < 3 = \frac{n}{n-2}.$$

His study is carried out through a careful analysis of the behavior of the function $\partial u/\partial\alpha$.

Classical examples of a function f satisfying (f_1)–(f_5) are given by $f(s) = s^p - s^q$ in dimension $n = 3$ for $0 < q < 1 \leqslant p$ with $p + q \leqslant 2$ in the case $b > 0$, and $f(s) = s^p$, with $1 < p \leqslant \frac{n}{n-2}$ in the case $b = 0$, see Section 7.

The existence of sign changing bound state solutions of (1)–(2) has been established by Coffman in [5] and McLeod, Troy and Weissler in [15], where $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous and satisfies appropriate sign conditions and is of subcritical growth. Their proof uses shooting techniques and a scaling argument.

Finally we describe our approach. Our theorems will follow after a series of comparison results between two solutions to (5) with initial value in some small neighborhood of α^*, where $u(\cdot, \alpha^*)$ is either a second bound state, or $u(r, \alpha^*)$ has at least two zeros. We will show (see Propositions 2.2 and 4.1.1), that there exists a neighborhood V of α^* such that any solution to (5) with $\alpha \in V$ has a first minimum value $U_m(\alpha)$ at $r = R_m(\alpha)$ satisfying $U_m(\alpha) < -\beta$ (see Fig. 1). Then, we divide our study according to the interval where $u(r, \alpha)$ belongs: $[b, \alpha]$, $[-\beta, b]$ and $[U_m, -\beta]$ before the minimum, and then $[U_m, -b]$ and $[-\beta, 0)$ after the minimum. In Section 3 we follow the ideas of Coffman, see [4], and use the function $\varphi(r, \alpha) = \frac{\partial}{\partial\alpha} u(r, \alpha)$ to study the behavior of the solutions in $[b, \alpha]$. It is here that assumptions (f_4)–(f_5) play a fundamental role allowing us to determine the "good" comparison between solutions at $u = b$. In Section 4 we study the case $b > 0$. In Subsection 4.1, using the comparison of the solutions at $u = b$, and following ideas in [16,17], we study the behavior of the solutions in the interval $[-\beta, b]$, which implies the corresponding good comparison of the solutions when they cross the value $-\beta$. This is done by considering a modified functional $\tilde{W}$ defined by

$$\tilde{W}(s, \alpha) = r^{n-1}(s, \alpha)\sqrt{\big(u'\big(r(s, \alpha), \alpha\big)\big)^2 + 2F(s)}, \quad s \in \big[U_m(\alpha), \alpha\big]$$

where $r(s, \alpha)$ denotes the inverse of u before the minimum. Then in Subsection 4.2, we use ideas of Pucci, Serrin and Tang in [18,19] to study the behavior of the solutions in the interval $[U_m, -\beta]$ before the minimum. We do so by considering the celebrated functional introduced first by Erbe and Tang in [9]:

$$P(s, \alpha) = -2n\frac{F}{f}(s)\frac{r^{n-1}(s, \alpha)}{r'(s, \alpha)} - \frac{r^n(s, \alpha)}{(r'(s, \alpha))^2} - 2r^n(s, \alpha)F(s), \quad s \neq b.$$

Again, the good behavior at $-\beta$ will yield a good comparison of $P(U_m(\alpha), \alpha)$. In Subsection 4.3 we deal with the behavior of the solutions in $[U_m, -b]$ after the minimum using the same ideas. In particular we derive the behavior of the solutions at their second intersection point. This in turn allows us to determine the behavior of the solutions in the interval $[-\beta, 0)$ after the minimum. We do this in Subsection 4.4, and the main tool we use is the functional W defined by

$$W(s,\alpha) = \bar{r}(s,\alpha)\sqrt{\big(u'\big(\bar{r}(s,\alpha),\alpha\big)\big)^2 + 2F(s)}, \quad s \in \big[U_m(\alpha), \alpha\big],$$

introduced by Peletier and Serrin in [16,17]. Here $\bar{r}(s,\alpha)$ denotes the inverse of u after the minimum. Section 5 is devoted to derive the corresponding separation results for the case $b = 0$. Our main results are proved in Section 6. Finally in Section 7 we give some examples and we make some remarks to show that our results remain valid for the m-Laplacian.

The study of uniqueness under weaker assumptions on the function f remains open, see the remark at the end of Section 3.

2. Preliminaries

The aim of this section is to establish several properties of the solutions to the initial value problem (5).

The functional

$$I(r,\alpha) = \big(u'(r,\alpha)\big)^2 + 2F\big(u(r,\alpha)\big) \tag{8}$$

will play a fundamental role. A simple calculation yields

$$I'(r,\alpha) = -\frac{2(n-1)}{r}\big(u'(r)\big)^2, \tag{9}$$

and therefore, as $n > 2$, we have that I is decreasing in r. It can be seen that for $\alpha \in (b,\infty)$, one has $u(r,\alpha) > 0$ and $u'(r,\alpha) < 0$ for r small enough, and thus we can define

$$R(\alpha) := \sup\big\{r > 0 \,\big|\, u(s,\alpha) > 0 \text{ and } u'(s,\alpha) < 0 \text{ for all } s \in (0,r)\big\}.$$

Following [16,17] we set

$$\mathcal{N} = \big\{\alpha\colon\ u(R(\alpha),\alpha) = 0 \text{ and } u'\big(R(\alpha),\alpha\big) < 0\big\},$$
$$\mathcal{G} = \big\{\alpha\colon\ u\big(R(\alpha),\alpha\big) = 0 \text{ and } u'\big(R(\alpha),\alpha\big) = 0\big\},$$
$$\mathcal{P} = \big\{\alpha\colon\ u\big(R(\alpha),\alpha\big) > 0\big\}.$$

As in [6], the sets $\mathcal{N}$ and $\mathcal{P}$ are open intervals, and moreover, if $\mathcal{N} \neq \emptyset$, then $\mathcal{N} = (a,\infty)$ for some $a > 0$. If our problems (1)–(2) or (4) have a solution, then $\mathcal{N} \neq \emptyset$. Let

$$\mathcal{F}_1 = \big\{\alpha \in \mathcal{N}\colon\ u'(r,\alpha) < 0 \text{ for all } r > 0\big\}.$$

For $\alpha \notin \mathcal{F}_1$ we define

$$R_m(\alpha) := \inf\big\{r > R(\alpha)\colon\ u'(r,\alpha) = 0\big\}, \qquad U_m(\alpha) = u\big(R_m(\alpha),\alpha\big),$$

and if $\alpha \in \mathcal{F}_1$, we set $R_m(\alpha) = \infty$. Also, for $\alpha \in \mathcal{N} \setminus \mathcal{F}_1$ we set

$$\bar{R}(\alpha) := \sup\big\{r > R_m(\alpha) \,\big|\, u(s,\alpha) < 0 \text{ and } u'(s,\alpha) > 0 \text{ for all } s \in (R_m, r)\big\},$$

and $\bar{U}(\alpha) := u(\bar{R}(\alpha),\alpha)$. Let now

$$\mathcal{F}_2 = \big\{\alpha \in \mathcal{N} \setminus \mathcal{F}_1\colon\ u\big(\bar{R}(\alpha),\alpha\big) < 0\big\},$$
$$\mathcal{N}_2 = \big\{\alpha \in \mathcal{N} \setminus \mathcal{F}_1\colon\ u\big(\bar{R}(\alpha),\alpha\big) = 0 \text{ and } u'\big(\bar{R}(\alpha),\alpha\big) > 0\big\},$$
$$\mathcal{G}_2 = \big\{\alpha \in \mathcal{N} \setminus \mathcal{F}_1\colon\ u\big(\bar{R}(\alpha),\alpha\big) = 0 \text{ and } u'\big(\bar{R}(\alpha),\alpha\big) = 0\big\},$$
$$\mathcal{P}_2 = \mathcal{F}_1 \cup \mathcal{F}_2.$$

Concerning the sets $\mathcal{N}_2$ and $\mathcal{P}_2$ we have:

Proposition 2.1. *The sets $\mathcal{N}_2$ and $\mathcal{P}_2$ are open.*

Proof. The proof that $\mathcal{N}_2$ is open is by continuity and follows as in [7] with obvious modifications, so we omit it.

The proof that $\mathcal{P}_2$ is open is based on the fact that the functional I defined in (8) is decreasing in r, and $\alpha \in \mathcal{P}_2$ if and only if $\alpha \in \mathcal{N}$ and $I(r_1, \alpha) < 0$ for some $r_1 \in (0, T(\alpha))$, where $T(\alpha) = \bar{R}(\alpha)$ if $\alpha \in \mathcal{F}_2$ and $T(\alpha) = \infty$ if $\alpha \in \mathcal{F}_1$.

Let $\alpha \in \mathcal{P}_2$ and assume first that $\bar{R}(\alpha) = \infty$. We claim that

$$\lim_{r\to\infty} u(r,\alpha) = -b, \qquad \lim_{r\to\infty} u'(r,\alpha) = 0.$$

Since $u(\cdot,\alpha)$ is monotone (decreasing for all $r > 0$ if $\alpha \in \mathcal{F}_1$ and increasing in $(R_m(\alpha), \infty)$ if $\alpha \in \mathcal{F}_2$), there exists L such that $\lim_{r\to\infty} u(r,\alpha) = L$. Furthermore, since $I(\cdot,\alpha)$ is decreasing and bounded and $F(s) \to \infty$ as $s \to \pm\infty$, we have that L is finite and $\lim_{r\to\infty} u'(r,\alpha) = 0$. Moreover, from the equation and applying L'Hôpital's rule twice, we conclude that

$$0 = \lim_{r\to\infty} \frac{u(r,\alpha) - L}{r^2} = \lim_{r\to\infty} \frac{r^{n-1}u'(r,\alpha)}{2r^n} = -\frac{f(L)}{2n}.$$

Thus, $L = -b$ as we claimed, implying that

$$\lim_{r\to\bar{R}(\alpha)} I(r,\alpha) = 2F(-b) < 0.$$

Assume next $\bar{R}(\alpha) < \infty$ and hence $\alpha \in \mathcal{F}_2$. Then $\bar{R}(\alpha)$ is a maximum point of $u(\cdot,\alpha)$ implying that

$$0 \leqslant -u''\big(\bar{R}(\alpha),\alpha\big) = f\big(u\big(\bar{R}(\alpha),\alpha\big)\big)$$

and thus $-b < u(\bar{R}(\alpha),\alpha) < 0$ ($u(\bar{R}(\alpha),\alpha) \neq -b$ from the uniqueness of the solutions and since $u(0,\alpha) = \alpha$). Hence

$$I\big(\bar{R}(\alpha),\alpha\big) = 2F\big(u\big(\bar{R}(\alpha),\alpha\big)\big) < 0.$$

Conversely, if $\alpha \notin \mathcal{P}_2$ and $\alpha \in \mathcal{N}$, then $\alpha \in \mathcal{G}_2 \cup \mathcal{N}_2$, and thus the claim follows from the fact that $I(r,\alpha) \geqslant I(\bar{R}(\alpha),\alpha) \geqslant 0$ for all $r \in (0, \bar{R}(\alpha))$. Hence the openness of $\mathcal{P}_2$ follows from the continuous dependence of solutions to (5) in the initial value α and from the openness of $\mathcal{N}$. □

Finally in this section we establish the existence of a neighborhood of α^* so that solutions with initial value in this interval cannot be decreasing for all $r > 0$ (see Fig. 2).

Proposition 2.2. *Let $\alpha^* \in \mathcal{G}_2 \cup \mathcal{N}_2$. Then there exists $\delta_0 > 0$ such that $(\alpha^* - \delta_0, \alpha^* + \delta_0) \subseteq \mathcal{N} \setminus \mathcal{F}_1$.*

Proof. Since $\alpha^* \in \mathcal{G}_2 \cup \mathcal{N}_2$, there exists $\tau > R_m(\alpha^*)$ such that $u'(\tau,\alpha^*) > 0$. By continuity, there exists $\delta_0 > 0$ such that

$$u'(\tau,\alpha) > 0 \quad \text{for all } \alpha \in \big(\alpha^* - \delta_0, \alpha^* + \delta_0\big),$$

implying that

$$R_m(\alpha) < \tau \quad \text{for all } \alpha \in \big(\alpha^* - \delta_0, \alpha^* + \delta_0\big),$$

and thus

$$\big(\alpha^* - \delta_0, \alpha^* + \delta_0\big) \subset \mathcal{N} \setminus \mathcal{F}_1. \qquad \square$$

We will study the behavior of the solutions to the initial value problem (5). To this end, $\alpha^* \in \mathcal{G}_2 \cup \mathcal{N}_2$ is fixed and $\alpha \in (\alpha^* - \delta_0, \alpha^* + \delta_0)$, where $\delta_0 > 0$ is given in Proposition 2.2. Then

$$u(\cdot,\alpha) : \big[0, R_m(\alpha)\big] \longrightarrow \big[U_m(\alpha), \alpha\big]$$

is invertible with inverse $r(\cdot,\alpha)$, and

$$u(\cdot,\alpha) : \big[R_m(\alpha), \bar{R}(\alpha)\big) \longrightarrow \big[U_m(\alpha), \bar{U}(\alpha)\big),$$

is also invertible with inverse $\bar{r}(\cdot,\alpha)$.

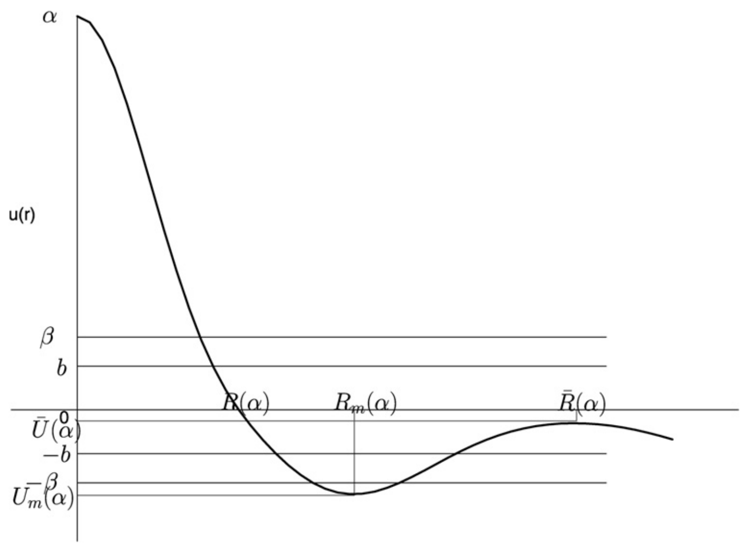

Fig. 2. The graph of the solution for $\alpha \in \mathcal{P}_2$ in the neighborhood of $\alpha^* \in \mathcal{G}_2$.

3. Behavior of the solutions in $[b, \alpha]$

Under assumptions (f_1) and (f_2), and for every $\alpha \in (b, \infty)$, the functions $u(r, \alpha)$ and $u'(r, \alpha)$ are of class C^1 in

$$\mathcal{O} = \{(r, \alpha)\colon\ \alpha \in (b, \infty) \text{ and } r \in [0, r(b, \alpha))\},$$

hence for $(r, \alpha) \in \mathcal{O}$, we set

$$\varphi(r, \alpha) = \frac{\partial u}{\partial \alpha}(r, \alpha), \qquad ' = \frac{\partial}{\partial r}.$$

Then, φ satisfies the linear differential equation

$$\begin{aligned} &\varphi''(r) + \frac{N-1}{r}\varphi'(r) + f'(u)\varphi = 0, \quad n > 2, \\ &\varphi(0, \alpha) = 1, \quad \varphi'(0, \alpha) = 0. \end{aligned} \tag{10}$$

It can be proven just as in [7] that both $\varphi(r, \alpha)$ and $\varphi'(r, \alpha)$ are continuous in

$$\underline{\mathcal{O}} = \{(r, \alpha)\colon\ \alpha \in (b, \infty) \text{ and } r \in [0, r(b, \alpha)]\},$$

and the extension of $\varphi'(r, \alpha)$ is the left derivative with respect to r of $\varphi(r, \alpha)$.

Set

$$\varphi(r) = \varphi(r, \alpha), \quad \text{and} \quad r(b, \alpha) = R_b.$$

The following result appears in [3], we include its proof for the sake of completeness.

Proposition 3.1. *Let f satisfy (f_1)–(f_3). Then φ has a first zero at $z \in (0, r(b, \alpha)]$. Moreover, if $b = 0$, then $z \in (0, R_b)$ (and thus $u(z, \alpha) > 0$).*

Proof. If $\varphi(r) > 0$ in $(0, R_b)$ we will prove that $\varphi(R_b) = 0$ and thus the lemma follows.

Multiplying the equation in (10) by $r^{n-1}(u - b)$ and integrating by parts over $(0, R_b)$, we have that

$$-\int_0^{R_b} r^{n-1} u'(r)\varphi'(r)\, dr + \int_0^{R_b} f'(u(r))\varphi(r)(u(r) - b) r^{n-1}\, dr = 0,$$

and a second integration by parts yields

$$\int_0^{R_b} \big(f'\big(u(r)\big)\big(u(r)-b\big)-f\big(u(r)\big)\big)\varphi(r)r^{n-1}\,dr=-R_b^{n-1}\varphi(R_b)\big|u'(R_b)\big|. \tag{11}$$

Using now that from (f_3), $f'(u(r))(u(r)-b)-f(u(r))\geqslant 0$ for $r\in(0,R_b)$, and that $u'(R_b)<0$, we deduce from (11) that $\varphi(R_b)=0$. If $b=0$, then by our assumption (f_3), we have that the integral in (11) is strictly positive and thus $\varphi(R_b)<0$, a contradiction. □

We set $U_z=u(z,\alpha)$, where z is as in Proposition 3.1. The first part of the next lemma is also contained in [3,6]. We include its proof for the sake of completeness.

Lemma 3.1. *Let f satisfy (f_1)–(f_4). Then z is the only zero of φ in $(0,R_b]$. Assume that f satisfies (f_5). If either $b=0$ or $b>0$ and $U_z\geqslant\beta$, then $\varphi'(R_b)\leqslant 0$.*

Proof. If $U_z=b$, then $z=R_b$ and $\varphi'(R_b)\leqslant 0$.

Assume next that $U_z>b$. We will show that

$$\frac{U_z f'(U_z)}{f(U_z)}>1.$$

From (f_3), this is always the case when $b>0$. If $b=0$, assume there exists a first value $s_0\in(0,\alpha)$ at which $s_0 f'(s_0)=f(s_0)$ for some $s_0\in(b,\alpha)$. Then from (f_3) and (f_4) it must be that $f(s)=\frac{f(s_0)}{s_0}s$ for all $s\geqslant s_0$ and hence $\alpha\varphi(r)=u(r)$ for $r\in(0,r_0)$, where $u(r_0)=s_0$. This implies that $\varphi(r)>0$ in $(0,r_0]$ and thus $U_z<s_0$ implying that there exists $c>0$ such that

$$\frac{U_z f'(U_z)}{f(U_z)}=1+\frac{2}{c}.$$

Then, since by (f_4), the function

$$r\to c\frac{u(r)f'(u(r))}{f(u(r))}-c-2$$

is increasing in $(0,R_b)$, we have that

$$\phi(r):=f\big(u(r)\big)\bigg(c\frac{u(r)f'(u(r))}{f(u(r))}-c-2\bigg)$$

is nonpositive in $(0,z)$ and nonnegative in (z,R_b).

Let us set $v(r)=ru'(r)+cu(r)$. Then v satisfies

$$v''+\frac{n-1}{r}v'+f'\big(u(r)\big)v=\phi(r),$$

and, as long as $\varphi(r)$ does not change sign in (z,r), with $r\in(z,R_b)$, we have

$$\begin{aligned}0\geqslant\int_0^r t^{n-1}\varphi(t)\phi(t)\,dt&=\int_0^r t^{n-1}(\varphi\Delta v-v\Delta\varphi)\,dt\\&=r^{n-1}\big(\varphi(r)v'(r)-\varphi'(r)v(r)\big),\end{aligned} \tag{12}$$

and therefore

$$\varphi(r)v'(r)-\varphi'(r)v(r)\leqslant 0, \tag{13}$$

implying in particular that $v(z)\leqslant 0$. Now we can prove that z is the only zero of φ in $(0,R_b]$. Indeed, if φ has a second zero at $z_1\in(z,R_b)$, then from (13), it must be that $v(z_1)\geqslant 0$. On the other hand, since $ru'(r)/u(r)$ is decreasing in

$(0, R_b)$ (see [3] and the proofs of [7, Lemma 3.1] or [11, Lemma 4.1], which apply to both cases $b = 0$ or $b > 0$), we find that

$$v(z_1) = u(z_1)\left(\frac{z_1 u'(z_1)}{u(z_1)} + c\right) < u(z_1)\left(\frac{z u'(z)}{u(z)} + c\right) = \frac{u(z_1)}{u(z)} v(z) \leqslant 0$$

implying that $v(z_1) < 0$, a contradiction. Hence φ has exactly one zero in $(0, R_b]$.

Assume now that f also satisfies (f_5) and suppose that either $b = 0$ or $b > 0$ and $U_z \geqslant \beta$. Then by (f_4) and (f_5), we have that $c \geqslant n - 2$ and thus

$$v'(r) = r u''(r) + (c+1)u'(r) \leqslant r u''(r) + (n-1)u'(r) = -r f\big(u(r)\big) < 0$$

for all $r \in (0, R_b)$, hence

Evaluating (13) at $r = R_b$, we find that

$$\varphi(R_b)v'(R_b) - \varphi'(R_b)v(R_b) \leqslant 0,$$

implying $\varphi'(R_b) \leqslant 0$. □

Remark. As it will become clear in the next sections, if we a priori know that $\varphi'(R_b) \leqslant 0$, then we can replace assumptions (f_4) and (f_5) with the much weaker subcritical assumption

$$\left(\frac{F}{f}\right)'(s) \geqslant \frac{n-2}{2n} \quad \text{for all } s > b.$$

4. The case $b > 0$

In this section we will study the behavior of the solutions to (5) in the neighborhood of $\alpha^* \in \mathcal{G}_2 \cup \mathcal{N}_2$. We will do so by analyzing the solutions in the intervals $[-\beta, b]$ before the minimum, $[U_m(\alpha), -\beta]$ (before and after the minimum) and $[-\beta, 0]$ after the minimum.

4.1. Behavior in $[-\beta, b]$ before the minimum

We start by showing that there exists a neighborhood V of α^* such that any solution to (5) with $\alpha \in V$ has a first minimum value $U_m(\alpha)$ at $r = R_m(\alpha)$ satisfying $U_m(\alpha) < -\beta$.

Proposition 4.1.1. *Let f satisfy (f_1)–(f_4) and let α^* and $\delta_0 > 0$ be as in Proposition 2.2. Then there exists $\delta_1 \in (0, \delta_0]$ such that*

$$U_m(\alpha) = u\big(R_m(\alpha), \alpha\big) < -\beta \quad \text{for all } \alpha \in \big(\alpha^* - \delta_1, \alpha^* + \delta_1\big). \tag{14}$$

Proof. The assumption $\alpha^* \in \mathcal{G}_2 \cup \mathcal{N}_2$ implies that the functional defined in (8) satisfies

$$I\big(\bar{R}(\alpha^*), \alpha^*\big) \geqslant 0,$$

and thus $I(r, \alpha^*) > 0$ for all $r \in (0, \bar{R}(\alpha^*))$. Also, from the continuity of $R_m(\alpha)$ for $\alpha \in (\alpha^* - \delta_0, \alpha^* + \delta_0)$, and the fact that

$$2F\big(u\big(R_m(\alpha^*), \alpha^*\big)\big) = I\big(R_m(\alpha^*), \alpha^*\big) > 0,$$

we conclude that there exists $\delta_1 \leqslant \delta_0$ such that (14) holds. □

Let $\delta_1 > 0$ be as given by Proposition 4.1.1. Given $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$, we will denote

$$u_1(r) = u(r, \alpha_1), \qquad u_2(r) = u(r, \alpha_2),$$

and

$$r_1(s) = r(s, \alpha_1), \qquad r_2(s) = r(s, \alpha_2).$$

For $\alpha \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$, consider the energy-like functional

$$\tilde{W}(s,\alpha) = r^{n-1}(s,\alpha)\sqrt{\big(u'\big(r(s,\alpha),\alpha\big)\big)^2 + 2F(s)}, \quad s \in \big[U_m(\alpha), \alpha\big],$$

and set

$$\tilde{W}_1(s) = \tilde{W}(s,\alpha_1), \qquad \tilde{W}_2(s) = \tilde{W}(s,\alpha_2).$$

We note from (9) and (14) that

$$\big(u'\big(r(s,\alpha),\alpha\big)\big)^2 + 2F(s) \geqslant 2F\big(U_m(\alpha)\big) > 0, \quad s \in \big[U_m(\alpha), \alpha\big],$$

and thus $\tilde{W}(s,\alpha)$ is well defined for all $s \in [U_m(\alpha), \alpha]$. We have the following separation lemma.

Lemma 4.1.1. *Let f satisfy (f_1)–(f_4). Let α_1, $\alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$ with $\alpha_1 < \alpha_2$. Assume that there exists $U \in [0, \beta]$ such that*

$$r_1(U) \geqslant r_2(U) \quad \text{and} \quad \tilde{W}_1(U) < \tilde{W}_2(U), \tag{15}$$

then

$$r_1(s) > r_2(s) \quad \text{and} \quad \tilde{W}_1(s) < \tilde{W}_2(s), \quad \text{for all } s \in [-\beta, U).$$

Proof. From (15) we easily obtain that $|r_1'(U)| > |r_2'(U)|$, and thus $r_1 > r_2$ in some small left neighborhood of U. Hence, there exists $c \in [-\beta, U)$ such that

$$\tilde{W}_1 \leqslant \tilde{W}_2, \quad r_1 > r_2, \quad \text{and} \quad r_1' < r_2' \quad \text{in } [c, U).$$

Next, we will show that $\tilde{W}_1 - \tilde{W}_2$ is increasing in $[c, U)$, and this will imply that the infimum of such c is $-\beta$, and thus the conclusion of the theorem will follow.

Indeed, from the definition of $\tilde{W}(s,\alpha)$ we have

$$\frac{\partial \tilde{W}}{\partial s}(s,\alpha) = \frac{2(n-1)r^{n-2}(s,\alpha)F(s)}{u'(r(s,\alpha),\alpha)\sqrt{(u'(r(s,\alpha),\alpha))^2 + 2F(s)}},$$

and thus for $s \in [c, U)$

$$\begin{aligned}
&\frac{1}{2(n-1)}\left(\frac{\partial \tilde{W}_1}{\partial s}(s) - \frac{\partial \tilde{W}_2}{\partial s}(s)\right) \\
&\quad = F(s)\left(\frac{r_1^{n-2}(s)}{u_1'(r_1(s))\sqrt{(u_1'(r_1(s)))^2 + 2F(s)}} - \frac{r_2^{n-2}(s)}{u_2'(r_2(s))\sqrt{(u_2'(r_2(s)))^2 + 2F(s)}}\right) \\
&\quad \geqslant r_2^{n-2}(s)\big|F(s)\big|\left(\frac{1}{|u_1'(r_1(s))|\sqrt{(u_1'(r_1(s)))^2 + 2F(s)}} - \frac{1}{|u_2'(r_2(s))|\sqrt{(u_2'(r_2(s)))^2 + 2F(s)}}\right) \\
&\quad \geqslant 0. \qquad \square
\end{aligned}$$

Since $(\alpha^* - \delta_1, \alpha^* + \delta_1) \subset \mathcal{N}$, we have, by [11, Theorem 4.4], that if $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$, then $R(\alpha_2) < R(\alpha_1)$, and therefore there exists a first point $r_I \in (0, R(\alpha_2))$ such that $u_1(r_I) = u_2(r_I)$. We denote by U_I this common value.

Proposition 4.1.2. *Let f satisfy (f_1)–(f_5). Then there exists $\delta \in (0, \delta_1]$ such that for all $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$ it holds that*

$$r_1(s) > r_2(s) \quad \text{and} \quad \tilde{W}_1(s) < \tilde{W}_2(s), \quad \text{for all } s \in [-\beta, U_{bI}),$$

where $U_{bI} = \min\{b, U_I\}$.

Proof. Let $U_z = u(z, \alpha^*)$, where z is given in Proposition 3.1 and assume first that $U_z < \beta$. By continuity, there exists $\delta \leqslant \delta_1$ such that for $\alpha_1, \ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$, it holds that $U_I < \beta$. Since $|r_1'(U_I)| > |r_2'(U_I)|$, it must be that $\tilde{W}_1(U_I) < \tilde{W}_2(U_I)$. The result follows now from Lemma 4.1.1 with $U = U_I$.

Assume next that $U_z \geqslant \beta$. By direct computation we have that

$$\frac{\partial r}{\partial \alpha}(s, \alpha) = -\frac{\varphi(r(s,\alpha), \alpha)}{u'(r(s,\alpha), \alpha)}$$

and

$$\frac{\partial \tilde{W}}{\partial \alpha}(s, \alpha) = \frac{r(s,\alpha) f(s) \varphi(r(s,\alpha), \alpha) u'(r(s,\alpha))}{u'(r(s,\alpha))\sqrt{(u'(r(s,\alpha),\alpha))^2 + 2F(s)}} r^{n-2}(s,\alpha) + \frac{r(s,\alpha)(u'(r(s,\alpha)))^2 \varphi'(r(s,\alpha),\alpha) - 2(n-1)F(s)\varphi(r(s,\alpha),\alpha)}{u'(r(s,\alpha))\sqrt{(u'(r(s,\alpha),\alpha))^2 + 2F(s)}} r^{n-2}(s,\alpha).$$

Hence, evaluating at $s = b$ and using Lemma 3.1 we find that

$$\frac{\partial r}{\partial \alpha}(b, \alpha) = -\frac{\varphi(R_b)}{u'(R_b, \alpha)} \leqslant 0$$

and

$$\frac{\partial \tilde{W}}{\partial \alpha}(b, \alpha) = \frac{R_b (u'(R_b,\alpha))^2 \varphi'(R_b) - 2(n-1)F(b)\varphi(R_b)}{u'(R_b,\alpha)\sqrt{(u'(R_b,\alpha))^2 + 2F(b)}} R_b^{n-2} > 0.$$

Therefore, if $\alpha_1, \ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$ it holds that

$$r_1(b) \geqslant r_2(b) \quad \text{and} \quad \tilde{W}_1(b) < \tilde{W}_2(b),$$

and thus the result follows from Lemma 4.1.1 with $U = b$. □

4.2. Behavior in $[U_m, -\beta]$ before the minimum

From now on we assume that f satisfies (f_1)–(f_5) and that $\delta > 0$ is fixed by Proposition 4.1.2.

In this and the next section we follow the ideas of Pucci, Serrin and Tang in [18,19]. To this end, for $s \in (U_m(\alpha), -\beta]$ we set

$$P(s, \alpha) = -2n \frac{F}{f}(s) \frac{r^{n-1}(s,\alpha)}{r'(s,\alpha)} - \frac{r^n(s,\alpha)}{(r'(s,\alpha))^2} - 2r^n(s,\alpha)F(s).$$

Then,

$$P'(s, \alpha) = \frac{\partial P}{\partial s}(s, \alpha) = \left(n - 2 - 2n\left(\frac{F}{f}\right)'(s)\right) \frac{r^{n-1}(s,\alpha)}{r'(s,\alpha)}. \tag{16}$$

We will need the following technical proposition.

Proposition 4.2.1. *Assumptions (f_3), (f_4) and (f_5) imply*

(f_6) $(\frac{F}{f})'(s) \geqslant \frac{n-2}{2n}$ *for all $s > b$.*

Proof. By (f_3), $f'(s) > 0$ for $s \geqslant \beta$, and from (f_4),

$$F(s) = \int_\beta^s f(t)\,dt = \int_\beta^s \frac{f(t)}{tf'(t)} tf'(t)\,dt \leqslant \frac{f(s)}{sf'(s)} \int_\beta^s tf'(t)\,dt$$

$$= \frac{f(s)}{sf'(s)}\big(sf(s) - \beta f(\beta) - F(s)\big).$$

Using now that $f(\beta) \geqslant 0$, we obtain

$$F(s)\left(1 + \frac{f(s)}{sf'(s)}\right) \leqslant \frac{f^2(s)}{f'(s)},$$

implying that

$$s \to \frac{sf(s)}{F(s)} \quad \text{is decreasing in } (\beta, \infty). \tag{17}$$

On the other hand (f_4) and (f_5) imply that $\frac{sf'(s)}{f(s)} \leqslant \frac{n+2}{n-2}$ for all $s \in (\beta, \infty)$, and thus

$$s \to \frac{(f(s))^{\frac{n-2}{n+2}}}{s} \quad \text{is decreasing in } (\beta, \infty). \tag{18}$$

(Note that we have used a much weaker assumption than (f_5).) From (17), (18), and since both functions involved are positive, by multiplication we obtain that

$$s \to \frac{(f(s))^{\frac{2n}{n+2}}}{F(s)} \quad \text{is decreasing in } (\beta, \infty)$$

and thus, differentiating we obtain $\frac{2n}{n+2} f^{\frac{2n}{n+2}-1}(s) f'(s) F(s) \leqslant f^{\frac{2n}{n+2}+1}$ which is equivalent to

$$\frac{n-2}{2n} \leqslant 1 - \frac{f'(s)F(s)}{f^2(s)} = \left(\frac{F}{f}\right)'(s).$$

Since this last inequality holds trivially for $s \in (b, \beta]$, (f_6) follows. □

Hence under the assumptions of our main theorems, it holds that $P'(s, \alpha) \geqslant 0$ for all $s \in (U_m(\alpha), -\beta]$.

Let now $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$, with $\alpha_1 < \alpha_2$, and set

$$P_1(s) = P(s, \alpha_1), \qquad P_2(s) = P(s, \alpha_2),$$
$$U_{1m} = u_1\big(R_m(\alpha_1)\big), \qquad U_{2m} = u_2\big(R_m(\alpha_2)\big).$$

As in [9,19], we set

$$S_{12}(s) = \frac{r_1^{n-1} r_2'}{r_2^{n-1} r_1'}(s).$$

Then

$$S_{12}'(s) = S_{12}(s) f(s)\big(\big(r_2'(s)\big)^2 - \big(r_1'(s)\big)^2\big). \tag{19}$$

Proposition 4.2.2. *For any $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$, with $\alpha_1 < \alpha_2$, we have that $U_{1m} > U_{2m}$ and*

$$r_1 > r_2 \quad \textit{and} \quad P_1 > P_2 \quad \textit{in } [U_{1m}, -\beta].$$

Moreover,

$$P_1(U_{1m}) > P_2(U_{2m}).$$

Proof. We will prove first that $U_{1m} > U_{2m}$ and that for all $s \in [U_{1m}, -\beta]$ we have

$$S_{12}(s) < 1, \qquad \big|r_1'(s)\big| > \big|r_2'(s)\big|, \qquad r_1(s) > r_2(s). \tag{20}$$

From Proposition 4.1.2 and since $F(-\beta) = 0$, we have that $S_{12}(-\beta) \leqslant 1$ and $r_1(-\beta) > r_2(-\beta)$, and thus, $r_1'(-\beta) < r_2'(-\beta)$. Moreover, from (19) we have that S_{12}(s) is increasing as long as $|r_1'(s)| > |r_2'(s)|$, for $s < -\beta$. If (20) does not hold for all $s \in (\max\{U_{1m}, U_{2m}\}, -\beta)$, then at the largest point s_0 where it fails, it must be that $|r_1'(s_0)| = |r_2'(s_0)|$

and $r_1(s_0) > r_2(s_0)$ implying that $S_{12}(s_0) > 1$, a contradiction. Thus (20) holds in $(\max\{U_{1m}, U_{2m}\}, -\beta)$, implying that $U_{1m} = \max\{U_{1m}, U_{2m}\}$.

Next we prove that $P_1 > P_2$, in $[U_{1m}, -\beta]$. From the definition of P_1 and P_2 we have

$$(P_1 - P_2)(-\beta) = \left(\frac{r_2^n}{(r_2')^2} - \frac{r_1^n}{(r_1')^2}\right)(-\beta)$$
$$= \left(\frac{r_2^n}{(r_2')^2}\left[1 - S_{12}^2 \frac{r_2^{n-2}}{r_1^{n-2}}\right]\right)(-\beta) > 0.$$

On the other hand, from (f_6) and (20),

$$(P_1 - P_2)'(s) = \big(S_{12}(s) - 1\big)\left(n - 2 - 2n\left(\frac{F}{f}\right)'(s)\right)\frac{r_2^{n-1}}{r_2'}(s) < 0,$$

implying that $P_1 > P_2$ in $[U_{1m}, -\beta]$. In particular, $P_1(U_{1m}) > P_2(U_{1m})$. Now, since $P_2' > 0$, we have that $P_2(U_{1m}) > P_2(U_{2m})$, and thus $P_1(U_{1m}) > P_2(U_{2m})$, ending the proof of the theorem. □

4.3. Behavior in $[U_m, -b]$ after the minimum

We recall that for $\alpha \in (\alpha^* - \delta, \alpha^* + \delta)$, $u(r, \alpha)$ is strictly increasing in $[R_m(\alpha), \bar{R}(\alpha))$, and we have denoted its inverse by $\bar{r}(s, \alpha)$.

Lemma 4.3.1. *Let $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. If $\bar{r}_1(U_{1m}) < \bar{r}_2(U_{1m})$, then there exists $\bar{U}_I \in (U_{1m}, -b)$ such that $\bar{r}_1(\bar{U}_I) = \bar{r}_2(\bar{U}_I)$ and $\bar{r}_1(s) < \bar{r}_2(s)$ for all $s \in (U_{1m}, \bar{U}_I)$.*

Proof. Assume by contradiction that $u_1 > u_2$ for all $r \in (\bar{r}_1(U_{1m}), \bar{r}_1(-b))$, then we will have that $u_1 > u_2$ for all $r \in (r_1(-b), \bar{r}_1(-b))$.

Since f is an odd function from (f_3) we obtain that $\frac{f(s)}{s+b}$ is decreasing for all $s < -b$, and thus

$$\begin{aligned}
0 &< \int_{r_1(-b)}^{\bar{r}_1(-b)} r^{n-1}\left(\frac{f(u_2)}{u_2 + b} - \frac{f(u_1)}{u_1 + b}\right)(u_2 + b)(u_1 + b)\,dr \\
&= \int_{r_1(-b)}^{\bar{r}_1(-b)} r^{n-1}\big(f(u_2)(u_1 + b) - f(u_1)(u_2 + b)\big)\,dr \\
&= \int_{r_1(-b)}^{\bar{r}_1(-b)} r^{n-1} f(u_1)(u_1 - u_2) - \int_{r_1(-b)}^{\bar{r}_1(-b)} r^{n-1}\big(f(u_1) - f(u_2)\big)(u_1 + b)\,dr \\
&= \int_{r_1(-b)}^{\bar{r}_1(-b)} r^{n-1} f(u_1)(u_1 - u_2) + \int_{r_1(-b)}^{\bar{r}_1(-b)} \big(r^{n-1}(u_1 - u_2)'\big)'(u_1 + b)\,dr \\
&= I_1 + I_2.
\end{aligned} \tag{21}$$

Integrating I_2 twice by parts we obtain that

$$I_2 = 0 - r^{n-1}(u_1 - u_2)(u_1 + b)'\Big|_{r_1(-b)}^{\bar{r}_1(-b)} + \int_{r_1(-b)}^{\bar{r}_1(-b)} (u_1 - u_2)\big(r^{n-1}u_1'\big)'\,dr$$
$$< -I_1,$$

a contradiction with (21). □

As in the previous subsection we define

$$\begin{aligned}
&\bar P(s,\alpha) = -2n\frac{F}{f}(s)\frac{\bar r^{n-1}(s,\alpha)}{\bar r'(s,\alpha)} - \frac{\bar r^n(s,\alpha)}{(\bar r'(s,\alpha))^2} - 2\bar r^n(s,\alpha)F(s),\\
&\bar P'(s,\alpha) = \left(n-2-2n\left(\frac{F}{f}\right)'(s)\right)\frac{\bar r^{n-1}(s,\alpha)}{\bar r'(s,\alpha)},\\
&\bar S_{12}(s) = \frac{\bar r_1^{n-1}\bar r_2'}{\bar r_2^{n-1}\bar r_1'}(s),\\
&\bar S_{12}'(s) = \bar S_{12}(s) f(s)\big(\big(\bar r_2'(s)\big)^2 - \big(\bar r_1'(s)\big)^2\big).
\end{aligned} \tag{22}$$

Lemma 4.3.2. *For any $\alpha_1,\ \alpha_2 \in (\alpha^*-\delta, \alpha^*+\delta)$, with $\alpha_1 < \alpha_2$, it holds that*

$$\bar P_1(U_{1m}) > \bar P_2(U_{1m}).$$

Proof. Note that $\bar P_1(U_{1m}) = P_1(U_{1m})$ and $\bar P_2(U_{2m}) = P_2(U_{2m})$. From Proposition 4.2.2 we have that $P_1(U_{1m}) > P_2(U_{2m})$. Hence $\bar P_1(U_{1m}) > \bar P_2(U_{2m})$. From (f_6), $\bar P_2'(s) \leqslant 0$, hence since $U_{1m} > U_{2m}$ we obtain $\bar P_1(U_{1m}) > \bar P_2(U_{2m}) > \bar P_2(U_{1m})$. □

Lemma 4.3.3. *Let $\alpha_1,\ \alpha_2 \in (\alpha^*-\delta, \alpha^*+\delta)$ with $\alpha_1 < \alpha_2$. If $\bar r_1(U_{1m}) < \bar r_2(U_{1m})$, then*

$$\frac{\bar r_1^{n-1}}{\bar r_1'}(s) < \frac{\bar r_2^{n-1}}{\bar r_2'}(s) \quad \text{and} \quad \bar P_1(s) > \bar P_2(s) \quad \text{for all } s \in [U_{1m}, \bar U_I],$$

with $\bar U_I$ as in Lemma 4.3.1.

Proof. Observe first that $\bar S_{12}(U_{1m}) = 0$ and $\bar S_{12}(\bar U_I) < 1$. If there exists a point $s \in (U_{1m}, \bar U_I)$ such that $\bar S_{12}'(s) = 0$, then $\bar r_1'(s) = \bar r_2'(s)$ and hence

$$\bar S_{12}(s) = \frac{\bar r_1^{n-1}}{\bar r_2^{n-1}}(s) < 1,$$

implying $\bar S_{12}(s) < 1$ for $s \in [U_{1m}, \bar U_I]$.

On the other hand, from the second in (22), using that $\bar S_{12}(s) < 1$ and (f_6), we obtain

$$(\bar P_1 - \bar P_2)'(s) = \left((\bar S_{12} - 1)\left(n-2-2n\left(\frac{F}{f}\right)'\right)\frac{\bar r_2^{n-1}}{\bar r_2'}\right)(s) > 0.$$

Hence, for all $s \in (U_{1m}, \bar U_I)$, $\bar P_1(s) - \bar P_2(s) > \bar P_1(U_{1m}) - \bar P_2(U_{1m}) > 0$, from Lemma 4.3.2. □

Let us define

$$U_{II} = \begin{cases} U_{1m} & \text{if } \bar r_1(U_{1m}) \geqslant \bar r_2(U_{1m}),\\ \bar U_I & \text{if } \bar r_1(U_{1m}) < \bar r_2(U_{1m}).\end{cases}$$

Proposition 4.3.1. *Let $\alpha_1, \alpha_2 \in (\alpha^*-\delta, \alpha^*+\delta)$ with $\alpha_1 < \alpha_2$. Then,*

$$\frac{\bar r_1}{\bar r_1'} < \frac{\bar r_2}{\bar r_2'} \quad \text{at } \bar U_I \quad \text{if } U_{II} \geqslant -\beta\ ,$$

and

$$\bar r_1 > \bar r_2, \quad \text{and} \quad \frac{\bar r_1}{\bar r_1'} < \frac{\bar r_2}{\bar r_2'} \quad \text{in } (U_{II}, -\beta] \ \text{ if } U_{II} < -\beta. \tag{23}$$

Proof. If $U_{II} \geqslant -\beta$, we obtain from the fact that $U_{1m} < -\beta$, that $U_{II} = \bar{U}_I$ and thus the result follows from the definition of $\bar{U}_I$.

We assume next that $U_{II} < -\beta$. As $\bar{r}_1(U_{II}) \geqslant \bar{r}_2(U_{II})$ and $\bar{S}_{12}(U_{II}) < 1$, then

$$\frac{\bar{r}_1}{\bar{r}_1'}(U_{II}) < \frac{\bar{r}_2}{\bar{r}_2'}(U_{II}).$$

Assume by contradiction that (23) does not occur. Then, there exists a first point $t \in (U_{II}, -\beta)$ such that

$$\frac{\bar{r}_1}{\bar{r}_1'}(t) = \frac{\bar{r}_2}{\bar{r}_2'}(t) \quad \text{and} \quad \bar{r}_1(s) > \bar{r}_2(s), \quad \text{for all } s \in (U_{II}, t],$$

implying

$$\bar{S}_{12}(t) = \left(\frac{\bar{r}_1(t)}{\bar{r}_2(t)}\right)^{n-2} = D > 1.$$

Also, from the definition of $\bar{P}_1$ and $\bar{P}_2$, we have that

$$(\bar{P}_1 - D\bar{P}_2)(t) = 2\big(D\bar{r}_2^n - \bar{r}_1^n\big)F(t) = 2\bar{r}_1^{n-2}\big(\bar{r}_2^2 - \bar{r}_1^2\big)F(t) < 0.$$

If $U_{II} = \bar{U}_I$, then by Lemma 4.3.3, we have that $(\bar{P}_1 - \bar{P}_2)(U_{II}) > 0$, while if $U_{II} = U_{1m}$, then by Lemma 4.3.2, $(\bar{P}_1 - \bar{P}_2)(U_{II}) > 0$. From (22) and (f_6), we have that $\bar{P}_1'(s,\alpha) \leqslant 0$ for $s \geqslant \bar{U}_{1m}$ and thus $\bar{P}_1(\bar{U}_I) \leqslant \bar{P}_1(\bar{U}_{1m}) < 0$, (see Proposition 4.1.1) implying that $\bar{P}_2(U_{II}) < \bar{P}_1(U_{II}) < 0$. Since $D > 1$ we conclude that

$$(\bar{P}_1 - D\bar{P}_2)(U_{II}) > 0.$$

From the last in (22) we obtain that $\bar{S}_{12}$ is increasing in (U_{II}, t) implying that $\bar{S}_{12}(s) < D$. Finally, using (f_6) we deduce

$$\big(\bar{P}_1 - D\bar{P}_2\big)'(s) = \left((\bar{S}_{12} - D)\left(n - 2 - 2n\left(\frac{F}{f}\right)'\right)\frac{\bar{r}_2^{n-1}}{\bar{r}_2'}\right)(s) > 0,$$

and thus

$$(\bar{P}_1 - D\bar{P}_2)(t) > 0,$$

a contradiction. □

4.4. *Behavior in* $[-\beta, 0)$ *after the minimum*

In this section we will examine the behavior of the solutions for $u \in [-\beta, 0)$ after the minimum. To do this, we will use the functional W defined below, first introduced by Peletier and Serrin, see [16,17]:

$$W(s,\alpha) = \bar{r}(s,\alpha)\sqrt{\big(u'\big(\bar{r}(s,\alpha),\alpha\big)\big)^2 + 2F(s)}, \quad s \in \big[U_m(\alpha), Z(\alpha)\big),$$

where

$$Z(\alpha) := \sup\big\{s \in \big(U_m(\alpha), \bar{U}(\alpha)\big):\ u'\big(\bar{r}(s,\alpha),\alpha\big)^2 + 2F(s) > 0\big\}.$$

The functional W is well defined in this interval, since

$$\frac{d}{ds}\big[\big(u'\big(\bar{r}(s,\alpha),\alpha\big)\big)^2 + 2F(s)\big] = -2(n-1)\frac{u'(\bar{r}(s,\alpha),\alpha)}{\bar{r}(s,\alpha)} < 0,$$

and

$$u'\big(\bar{r}\big(U_m(\alpha),\alpha\big),\alpha\big)^2 + 2F\big(U_m(\alpha)\big) = 2F\big(U_m(\alpha)\big) > 0.$$

We note that $Z(\alpha) = 0$ if and only if $\alpha \in \mathcal{G}_2 \cup \mathcal{N}_2$. Indeed, if $\alpha \in \mathcal{G}_2 \cup \mathcal{N}_2$, then

$$\big(u'\big(\bar{r}(s,\alpha),\alpha\big)\big)^2 + 2F(s) \geqslant 0 \quad \text{at } s = \bar{U}(\alpha),$$

and if $\alpha \in \mathcal{F}_2$, then as in the proof of Proposition 2.1, we can prove that $I(\bar{R}(\alpha),\alpha) < 0$ which implies that $Z(\alpha) < \bar{U}(\alpha) < 0$.

Lemma 4.4.1. *Let $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. Assume that there exists $U \in [-\beta, 0]$ such that*

$$\bar r_1(U) \geqslant \bar r_2(U) \quad \textit{and} \quad W_1(U) < W_2(U), \tag{24}$$

then

$$Z(\alpha_1) \leqslant Z(\alpha_2)$$

and

$$\bar r_1(s) > \bar r_2(s), \qquad W_1(s) < W_2(s), \quad \textit{and} \quad u_1'\big(r_1(s)\big) < u_2'\big(r_2(s)\big), \quad s \in \big(U, Z(\alpha_1)\big].$$

Moreover, if $\alpha_2 \in \mathcal{G}_2$, then $\alpha_1 \in \mathcal{F}_2$, and if $\alpha_1 \in \mathcal{G}_2 \cup \mathcal{N}_2$, then $\alpha_2 \in \mathcal{N}_2$ and

$$\bar R(\alpha_1) > \bar R(\alpha_2) \quad \textit{and} \quad u_1'\big(\bar R(\alpha_1)\big) < u_2'\big(\bar R(\alpha_2)\big).$$

Proof. If we set $l = \min\{Z(\alpha_1), Z(\alpha_2)\} \geqslant U$, then W_1 and W_2 are well defined in $[-\beta, l]$. From (24) we deduce that $u_1'(\bar r_1(U)) < u_2'(\bar r_2(U))$, and thus $\bar r_1 > \bar r_2$ in some small right neighborhood of U. Observe that as long as $W_1 \leqslant W_2$, $\bar r_1$ and $\bar r_2$ do not intersect and hence neither do $u_1'(r_1(s))$ and $u_2'(r_2(s))$.

Let $d \in (U, l]$ such that

$$W_1(s) \leqslant W_2(s), \qquad \bar r_1(s) > \bar r_2(s), \quad \text{and} \quad u_1'\big(\bar r_1(s)\big) < u_2'\big(\bar r_2(s)\big) \quad \text{in } (U, d]. \tag{25}$$

We will show that $W_1 - W_2$ is strictly decreasing in $(U, d]$, and this will imply that the supremum of such d is l. Moreover, from (25) it will follow that $u_1'(\bar r_1(l)) < u_2'(\bar r_2(l))$ implying that $Z(\alpha_1) \leqslant Z(\alpha_2)$. Indeed, if $Z(\alpha_1) > Z(\alpha_2)$, then $Z(\alpha_2) < 0$, and $W_2(Z(\alpha_2)) = 0$, and from (25), $W_1(Z(\alpha_2)) = 0$, a contradiction with $Z(\alpha_1) > Z(\alpha_2)$. Hence $l = Z(\alpha_1)$.

From the definition of $W(s, \alpha)$ we have

$$\frac{\partial W}{\partial s}(s, \alpha) = \frac{2F(s) - (n-2)(u'(\bar r(s, \alpha), \alpha))^2}{u'(\bar r(s, \alpha), \alpha)\sqrt{u'^2(\bar r(s, \alpha), \alpha) + 2F(s)}}. \tag{26}$$

For any fixed $s \in (-\beta, 0)$ let us define

$$h(p) = \frac{2F(s) - (n-2)p^2}{p\sqrt{p^2 + 2F(s)}}, \quad p > 0.$$

As $F(s) \leqslant 0$, h is strictly increasing. Thus, from (26)

$$\frac{\partial W_1}{\partial s}(s) - \frac{\partial W_2}{\partial s}(s) = h\big(u_1'\big(\bar r_1(s)\big)\big) - h\big(u_2'\big(\bar r_2(s)\big)\big) < 0,$$

implying that $W_1 - W_2$ is strictly decreasing in $(U, d]$ and the first part of our result follows.

Now, if $\alpha_1 \in \mathcal{G}_2 \cup \mathcal{N}_2$, then $Z(\alpha_1) = 0$ and thus $Z(\alpha_2) = 0$, implying that $\alpha_2 \in \mathcal{G}_2 \cup \mathcal{N}_2$, and thus

$$\bar R(\alpha_1) = \bar r_1(0) > \bar r_2(0) = \bar R(\alpha_2) \quad \text{and} \quad u_1'\big(\bar R(\alpha_1)\big) < u_2'\big(\bar R(\alpha_2)\big)$$

and hence $\alpha_2 \in \mathcal{N}_2$.

Let now $\alpha_2 \in \mathcal{G}_2$. If $Z(\alpha_1) = 0$, then $u_1'(\bar R(\alpha_1)) < u_2'(\bar R(\alpha_2)) = 0$, a contradiction, implying $Z(\alpha_1) < 0$ and thus $\alpha_1 \in \mathcal{F}_2$. □

Proposition 4.4.1. *Let $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. Then*

$$Z(\alpha_1) \leqslant Z(\alpha_2)$$

and there exists $U \in [-\beta, 0]$ such that

$$\bar r_1(s) > \bar r_2(s), \qquad W_1(s) < W_2(s), \quad \textit{and} \quad u_1'\big(\bar r_1(s)\big) < u_2'\big(\bar r_2(s)\big), \quad s \in \big(U, Z(\alpha_1)\big].$$

Moreover, if $\alpha_1 \in \mathcal{G}_2 \cup \mathcal{N}_2$, then $\alpha_2 \in \mathcal{N}_2$ and hence

$$\bar R(\alpha_1) > \bar R(\alpha_2) \quad \textit{and} \quad u_1'\big(\bar R(\alpha_1)\big) < u_2'\big(\bar R(\alpha_2)\big), \tag{27}$$

and if $\alpha_2 \in \mathcal{G}_2$, then $\alpha_1 \in \mathcal{F}_2$.

Proof. Let

$$U := \begin{cases} -\beta & \text{if } U_{II} < -\beta, \\ \bar{U}_I & \text{if } U_{II} \geqslant -\beta. \end{cases}$$

From Proposition 4.3.1 we have

$$\bar{r}_1(U) \geqslant \bar{r}_2(U), \quad \text{and} \quad \bar{r}_1(U)u_1'\big(\bar{r}_1(U)\big) < \bar{r}_2(U)u_2'\big(\bar{r}_2(U)\big),$$

and since $F(U) \leqslant 0$, we obtain that $W_1(U) < W_2(U)$. Hence the result follows from Lemma 4.4.1. □

5. The case $b = 0$

In this case we need to examine the behavior of the solutions in the intervals $[0, \alpha]$ and $[U_m(\alpha), 0]$ before and after the minimum. We start with the analogue of Proposition 4.1.2.

Proposition 5.1. *Let f satisfy (f_1)–(f_5). Then for all $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$ with $\alpha_1 < \alpha_2$ it holds that*

$$r_1(0) > r_2(0) \quad \text{and} \quad r_1^{n-1}(0)\big|u_1'\big(r_1(0)\big)\big| \leqslant r_2^{n-1}(0)\big|u_2'\big(r_2(0)\big)\big|.$$

Proof. By direct computation, using Lemma 3.1 and Proposition 3.1, we have that

$$\frac{\partial r}{\partial \alpha}(s, \alpha)\Big|_{s=0} = -\frac{\varphi(r(0, \alpha), \alpha)}{u'(r(0, \alpha), \alpha)} < 0$$

and

$$\frac{\partial}{\partial \alpha} r^{n-1}(s, \alpha)u'\big(r(s, \alpha), \alpha\big)\Big|_{s=0} = r^{n-1}(0, \alpha)\varphi'\big(r(0, \alpha), \alpha\big),$$

and thus the result follows. □

We have the following analogues of Proposition 4.2.2 and Lemmas 4.3.1, 4.3.1 and Proposition 4.3.1. Their proofs follow by a step by step modification of the ones given in Section 4, so we omit them.

Proposition 5.2. *For any $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$, with $\alpha_1 < \alpha_2$, we have that $U_{1m} > U_{2m}$ and*

$$r_1 > r_2 \quad \text{and} \quad P_1 > P_2 \quad \text{in } [U_{1m}, 0].$$

Moreover,

$$P_1(U_{1m}) > P_2(U_{2m}).$$

Lemma 5.1. *Let $\alpha_1,\ \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. If $\bar{r}_1(U_{1m}) < \bar{r}_2(U_{1m})$, then there exists $\bar{U}_I \in (U_{1m}, 0)$ such that $\bar{r}_2(\bar{U}_I) = \bar{r}_1(\bar{U}_I)$ and $\bar{r}_2(s) > \bar{r}_1(s)$ for all $s \in (U_{1m}, \bar{U}_I)$.*

Lemma 5.2. *Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta, \alpha^* + \delta)$ with $\alpha_1 < \alpha_2$. If $\bar{r}_1(U_{1m}) < \bar{r}_2(U_{1m})$, then*

$$\frac{\bar{r}_1^{n-1}}{\bar{r}_1'}(s) < \frac{\bar{r}_2^{n-1}}{\bar{r}_2'}(s) \quad \text{and} \quad \bar{P}_1(s) > \bar{P}_2(s) \quad \text{for all } s \in [U_{1m}, \bar{U}_I],$$

with $\bar{U}_I$ as in Lemma 5.1.

We recall the definition of U_{II} from the previous section:

$$U_{II} = \begin{cases} U_{1m} & \text{if } \bar{r}_1(U_{1m}) \geqslant \bar{r}_2(U_{1m}), \\ \bar{U}_I & \text{if } \bar{r}_1(U_{1m}) < \bar{r}_2(U_{1m}). \end{cases}$$

Proposition 5.3. *Let $\alpha_1, \alpha_2 \in (\alpha^* - \delta_1, \alpha^* + \delta_1)$ with $\alpha_1 < \alpha_2$. Then*

$$\bar{r}_1 > \bar{r}_2, \quad \text{and} \quad \frac{\bar{r}_1}{\bar{r}_1'} < \frac{\bar{r}_2}{\bar{r}_2'} \quad \text{in } (U_{II}, 0].$$

In particular,

$$\bar{R}(\alpha_1) > \bar{R}(\alpha_2), \quad \text{and} \quad u_1'\big(\bar{R}(\alpha_1)\big) < u_2'\big(\bar{R}(\alpha_2)\big). \tag{28}$$

6. Proof of the main results

We start by proving Theorem 1.2.

Proof of Theorem 1.2(i). This is a well-known result, see for example [2], where much stronger nonexistence results are proved. We give its proof for the sake of completeness. From $(f_4)-(f_5)$, we obtain that there exist $C_0>0$ and $s_0>0$ such that

$$f(s)\geqslant C_0 s^{\frac{n}{n-2}}\quad \text{for all } s\in(0,s_0).$$

Assume now by contradiction that there exists $\alpha\in\mathcal{G}_2$ and set $v(r)=-u(r,\alpha)$. Let $r_0>R(\alpha)$ be such that $v\leqslant s_0$ and $v'(r)<0$ for $r\geqslant r_0$. Integrating the equation in (1) over $(r/2,r)$, $r\geqslant 2r_0$, we find that

$$-r^{n-1}v'(r)=\int\limits_{r/2}^{r} t^{n-1}f\big(v(t)\big)\,dt\geqslant \text{Const.} f\big(v(r)\big)r^n\geqslant \text{Const.}\big(v(r)\big)^{\frac{n}{n-2}}r^n, \tag{29}$$

and thus

$$-v'(r)v^{-\frac{n}{n-2}}(r)\geqslant \text{Const.} r\quad \text{for all } r\geqslant 2r_0$$

implying that

$$v^{-\frac{2}{n-2}}(2r_0)-v^{-\frac{2}{n-2}}(r)\geqslant \text{Const.}\big(r^2-4r_0^2\big)$$

in contradiction with $\lim_{r\to\infty}v(r)=0$.

Hence $\mathcal{G}_2=\emptyset$ and (i) follows. □

Proof of Theorem 1.2(ii). Assume first $b=0$. From (i), $\mathcal{G}_2=\emptyset$, and thus, as under the assumptions of (ii), $\mathcal{N}_2\neq\emptyset$, therefore from Proposition 2.1, we have $\mathcal{P}_2=\emptyset$. Hence $\mathcal{N}_2=\mathcal{N}=(a,\infty)$ for some $a>0$. Also, $\bar{R}(\alpha)$ and $u'(\bar{R}(\alpha),\alpha)$ are continuous in $[\alpha_1,\infty)$, and from (28) in Proposition 5.3, $\bar{R}(\alpha)$ is locally strictly decreasing and $u'(\bar{R}(\alpha),\alpha)$ is locally strictly increasing in (α_1,∞), hence the result will follow.

Assume next $b>0$. From the assumptions, $\alpha_1\in\mathcal{G}_2\cup\mathcal{N}_2$ hence by Proposition 4.4.1, $(\alpha_1,\alpha_1+\delta)\subset\mathcal{N}_2$. Let

$$\bar{\alpha}=\sup\{\alpha>\alpha_1\colon\ \alpha\in\mathcal{N}_2\}.$$

Assume $\bar{\alpha}<\infty$. Since $\mathcal{P}_2$ and $\mathcal{N}_2$ are open, we deduce that $\bar{\alpha}\in\mathcal{G}_2$. By Proposition 4.4.1, $(\bar{\alpha}-\delta,\bar{\alpha})\subset\mathcal{F}_2$, a contradiction, and thus $(\alpha_1,\infty)\subset\mathcal{N}_2$. As $\bar{R}(\alpha)$ and $u'(\bar{R}(\alpha),\alpha)$ are continuous in $[\alpha_1,\infty)$, and, also by (27) in Proposition 4.4.1, $\bar{R}(\alpha)$ is locally strictly decreasing and $u'(\bar{R}(\alpha),\alpha)$ is locally strictly increasing in (α_1,∞), the result follows. □

Proof of Theorem 1.2(iii). In this case $\alpha_1\in\mathcal{G}_2$, hence the same argument as above yields $(\alpha_1,\infty)\subset\mathcal{N}_2$ implying the desired result. □

Proof of Theorem 1.1. Follows directly from Theorem 1.2(iii). □

Proof of Theorem 1.3. Follows directly from Theorem 1.2(ii). □

7. Concluding remarks and examples

First we observe that our method can be used to establish the uniqueness of the radial solutions of the corresponding problems when we consider the more general equation

$$-\Delta u=|x|^\theta f(u),\quad \theta>-2.$$

Indeed, by making the change of variables

$$t=\frac{2}{2+\theta}r^{\frac{\theta+2}{2}},\qquad v(t)=u(r),$$

we obtain that v is a solution of

$$-\left(t^{N-1}v'(t)\right)' = t^{N-1}f(v),$$

where

$$N = \frac{2(n+\theta)}{\theta+2}.$$

Uniqueness follows provided we change (f_5) to

$$(f_5^\theta) \qquad \frac{\beta f'(\beta)}{f(\beta)} \leqslant \frac{n+\theta}{n-2} \quad \text{if } b>0, \qquad \frac{sf'(s)}{f(s)} < \frac{n+\theta}{n-2} \quad \text{for all } s>0 \text{ if } b=0.$$

We also remark that we have assumed f odd and worked the problem with the Laplacian operator instead of the m-Laplacian operator for the sake of simplicity. In particular, the results remain valid if instead of f odd we assume

- (f_1) $f(0)=0$, and there exist $b^+>0>b^-$ such that $f(u)>0$ for $u>b^+$, $f(u)<0$ for $u<b^-$, and $f(u)\leqslant 0$, $f(u)\not\equiv 0$, for $u\in(0,b^+)$ and $f(u)\geqslant 0$, $f(u)\not\equiv 0$, for $u\in(b^-,0)$.
- (f_2) f is continuous in $\mathbb{R}$ and differentiable in $(-\infty,b^-]\cup[b^+,\infty)$,
- (f_3) $f(u)\leqslant f'(u)(u-b^+)$, for all $u\geqslant b$, and $f(u)\geqslant f'(u)(u-b^-)$ for all $u\leqslant b^-$,
- (f_4) the function $u\to\frac{uf'(u)}{f(u)}$ is decreasing in (b^+,∞) and increasing in $(-\infty,b^-)$,
- (f_5) $\frac{\beta^+f'(\beta^+)}{f(\beta^+)}\leqslant\frac{n}{n-2}$, and $\frac{\beta^-f'(\beta^-)}{f(\beta^-)}\leqslant\frac{n}{n-2}$

where $\beta^+>0>\beta^-$ are the nonzero solutions of $F(x)=0$.

On the other hand, consider the m-Laplacian operator with $n>m>1$, instead of the Laplacian in problems (1)–(2), (4) and (6)–(7), and assume

- (f_3^m) $(m-1)f(u)\leqslant f'(u)(u-b)$, for all $u\geqslant b$,
- (f_5^m) $\frac{\beta f'(\beta)}{f(\beta)}\leqslant\frac{n(m-1)}{n-m}$,

instead of (f_3) and (f_5) (or the appropriate assumptions for the case $f\geqslant 0$). All the proofs in the previous sections can be carried out, with the obvious modifications, provided that we redefine the functionals I, $\tilde{W}$ and W as follows:

$$I_m(r,\alpha) = \left(u'(r,\alpha)\right)^m + m'F\left(u(r,\alpha)\right),$$

$$\tilde{W}_m(s,\alpha) = r^{\frac{n-1}{m-1}}(s,\alpha)\sqrt{\left(u'\left(r(s,\alpha),\alpha\right)\right)^m + m'F(s)}$$

and

$$W_m(s,\alpha) = r(s,\alpha)\sqrt{\left(u'(r(s,\alpha),\alpha)\right)^m + m'F(s)},$$

where as usual $m'=m/(m-1)$. Hence Theorems 1.1–1.3 are also valid in this case.

Finally we give some examples. As we mentioned in the introduction, our results apply to the canonical example $f(s)=s^p-s^q$ in dimension $n=3$ for $0<q<1\leqslant p$ with $p+q\leqslant 2$. In this case,

$$b=1, \qquad \beta=\left(\frac{p+1}{q+1}\right)^{\frac{1}{p-q}}, \qquad \frac{sf'(s)}{f(s)} = p+\frac{p-q}{s^{p-q}-1}.$$

Thus the function f satisfies (f_1)–(f_4) for any $p\geqslant 1$ and $0<q<p$, and (f_5) if $p+q\leqslant\frac{2}{n-2}$. Other examples can be constructed as follows: Let $b>0$ be fixed. If $f\in C^1[b,\infty)\cap C[0,\delta]$ for some $\delta\in(0,b)$, satisfies (f_3)–(f_4), $f(0)=0$, $f(s)<0$ in $(0,\delta)$ and

$$\lim_{s\to\infty}\frac{sf'(s)}{f(s)} < \frac{n}{n-2}, \tag{30}$$

then we can extend f continuously to $[\delta,b]$ in order that (f_5) is satisfied. Hence we can construct f with any prescribed behavior near zero and near infinity to which our results apply, provided (30) is satisfied.

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