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Global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical Hartree equation in $\mathbb{R}^d$

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Abstract

We prove the global well-posedness and scattering for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation with low regularity data in $\mathbb{R}^d$, $d \geqslant 3$. Precisely, we show that a unique and global solution exists for initial data in the Sobolev space $H^s(\mathbb{R}^d)$ with $s > 4(\gamma - 2)/(3\gamma - 4)$, which also scatters in both time directions. This improves the result in [M. Chae, S. Hong, J. Kim, C.W. Yang, Scattering theory below energy for a class of Hartree type equations, Comm. Partial Differential Equations 33 (2008) 321–348], where the global well-posedness was established for any $s > \max(1/2, 4(\gamma - 2)/(3\gamma - 4))$. The new ingredients in our proof are that we make use of an interaction Morawetz estimate for the smoothed out solution Iu, instead of an interaction Morawetz estimate for the solution u, and that we make careful analysis of the monotonicity property of the multiplier $m(\xi) \cdot \langle \xi \rangle^p$. As a byproduct of our proof, we obtain that the H^s norm of the solution obeys the uniform-in-time bounds.

MSC: 35Q40; 35Q55; 47J35

Keywords: Almost interaction Morawetz estimate; Well-posedness; Hartree equation; I-method; Uniform bound

1. Introduction

In this paper, we study the global well-posedness of the following initial value problem (IVP) for the defocusing $H^{\frac{1}{2}}$-subcritical (that is, $2 < \gamma < 3$) Hartree equation.

$$\begin{cases} iu_t + \Delta u = (|x|^{-\gamma} * |u|^2)u, \quad d \geqslant 3, \\ u(0) = u_0(x) \in H^s(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where H^s denotes the usual inhomogeneous Sobolev space of order s. It is a classical model introduced in [26].

We adopt the following standard notion of local well-posedness, that is, we say that the IVP (1.1) is locally well-posed in H^s if for any $u_0 \in H^s$, there exists a positive time $T = T(\|u_0\|_s)$ depending only on the norm of the initial data, such that a solution to the IVP exists on the time interval $[0, T]$, is unique in a certain Banach space of functional

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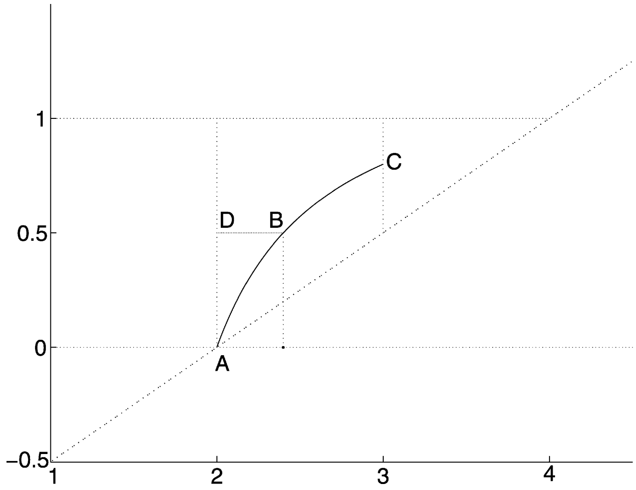

Fig. 1. The curve "ABC" is described by "$s = \frac{4(\gamma-2)}{3\gamma-4}$".

$X \subset C([0,T], H^s)$, and the solution map from H_x^s to $C([0,T], H^s)$ depends continuously. If T can be taken arbitrarily large, we say that the IVP (1.1) is globally well-posed.

Local well-posedness for the IVP (1.1) in H^s for any $s > \frac{\gamma}{2} - 1$ was established in [18]. A local solution also exists for $H^{\frac{\gamma}{2}-1}$ initial data, but the time of existence depends not only on the $H^{\frac{\gamma}{2}-1}$ norm of u_0, but also on the profile of u_0. For more details on local well-posedness see [18].

L^2 solutions of (1.1) enjoy mass conservation

$$\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R}^d)} = \left\|u_0(\cdot)\right\|_{L^2(\mathbb{R}^d)}.$$

Moreover, H^1 solutions enjoy energy conservation

$$E(u)(t) = \frac{1}{2}\left\|\nabla u(t)\right\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{4}\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{|x-y|^\gamma}\left|u(t,x)\right|^2\left|u(t,y)\right|^2 dx\,dy = E(u)(0),$$

which together with mass conservation and the local theory immediately yields global well-posedness for (1.1) with initial data in H^1. A large amount of work has been devoted to global well-posedness and scattering for the Hartree equation, see [7–11,13,15,17–23].

Existence of global solutions in $\mathbb{R}^3$ to (1.1) corresponding to initial data below the energy threshold was recently obtained in [5] by using the method of "almost conservation laws" or "I-method" (for a detailed description of this method, see [25] or Section 3 below) and the interaction Morawetz estimate for the solution u, where global well-posedness was obtained in $H^s(\mathbb{R}^3)$ with $s > \max(1/2, 4(\gamma-2)/(3\gamma-4))$ (see Fig. 1). Since authors in [5] used the interaction Morawetz estimate, which involves $\dot{H}^{1/2}$ norm of the solution, the restriction condition $s \geqslant \frac{1}{2}$ is prerequisite. In order to resolve IVP (1.1) in H^s, $s < \frac{1}{2}$ by still using the interaction Morawetz estimate, we need return to the interaction Morawetz estimate for the smoothed out version Iu of the solution, which is used simultaneously in [2] and [6].

In this paper, we consider the case $d \geqslant 3$ and we prove the following result:

Theorem 1.1. *Let $2 < \gamma < 3 \leqslant d$, the initial value problem* (1.1) *is globally well-posedness in $H^s(\mathbb{R}^d)$ for any $s > \frac{4(\gamma-2)}{3\gamma-4}$. Moreover the solution satisfies*

$$\sup_{t\in[0,\infty)} \left\|u(t)\right\|_{H^s(\mathbb{R}^d)} \leqslant C\left(\|u_0\|_{H^s}\right),$$

and there is scattering for these solutions, that is, the wave operators exist and there is asymptotic completeness on all of $H^s(\mathbb{R}^d)$.

Remark 1.1. As for the case $3 \leqslant \gamma < 4 \leqslant d$, local well-posedness for the IVP (1.1) in H^s holds for any $s > \frac{\gamma}{2} - 1$. Note that in this case, we have

$$\frac{\gamma}{2} - 1 \geqslant \frac{1}{2},$$

which satisfies the need of the regularity of the interaction Morawetz estimate. Hence for the case $3 \leqslant \gamma < 4 \leqslant d$, we only need to combine "I-method" with the interaction Morawetz estimate for the solution, instead of the interaction Morawetz estimate for the smoothed out version Iu of the solution, to obtain the low regularity global solution of the IVP (1.1), just as in [3] .

Our method follows closely the recent developments in [2,4] and [6], where the main two ingredients are the "I-method" and an almost Morawetz type estimate following the work of Lin and Strauss [16]. In order to obtain the low regularity global solution of the IVP (1.1), we combine I-method with an interaction Morawetz estimate for the smoothed out version Iu of the solution. By comparison with the interaction Morawetz estimate for the solution in [5], such a Morawetz estimate for an almost solution is the main novelty of this paper, which helps us to lower the need on the regularity of the initial data. In addition, we do not use the monotonicity property of the multiplier $m(\xi) \cdot \langle \xi \rangle^p$ in the proof of the almost conservation law.

Last, we organize this paper as following: In Section 2, we introduce some notation and state some important propositions that we will used throughout this paper. In Section 3, we review the I-method, prove the local well-posedness theory for Iu and obtain an upper bound on the increment of the modified energy. In Section 4, we prove the "almost interaction Morawetz estimate" for the smoothed out version Iu of the solution. Finally in Section 5, we give the details of the proof of the global well-posedness stated in Theorem 1.1.

2. Notation and preliminaries

2.1. Notation

In what follows, we use $A \lesssim B$ to denote an estimate of the form $A \leqslant CB$ for some constant C. If $A \lesssim B$ and $B \lesssim A$, we say that $A \approx B$. We write $A \ll B$ to denote an estimate of the form $A \leqslant cB$ for some small constant $c > 0$. In addition $\langle a \rangle := 1 + |a|$ and $a\pm := a \pm \epsilon$ with $0 < \epsilon \ll 1$. The reader also has to be alert that we sometimes do not explicitly write down constants that depend on the L^2 norm of the solution. This is justified by the conservation of the L^2 norm.

2.2. Definition of spaces

We use $L_x^r(\mathbb{R}^d)$ to denote the Lebesgue space of functions $f : \mathbb{R}^d \to \mathbb{C}$ whose norm

$$\|f\|_{L_x^r} := \bigg(\int\limits_{\mathbb{R}^d} \big|f(x)\big|^r \, dx \bigg)^{\frac{1}{r}}$$

is finite, with the usual modification in the case $r = \infty$. We also use the space–time Lebesgue spaces $L_t^q L_x^r$ which are equipped with the norm

$$\|u\|_{L_t^q L_x^r} := \bigg(\int\limits_J \big\|u(t,x)\big\|_{L_x^r}^q \, dt \bigg)^{\frac{1}{q}}$$

for any space–time slab $J \times \mathbb{R}$, with the usual modification when either q or r are infinity. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

As usual, we define the Fourier transform of $f(x) \in L_x^1$ by

$$\hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int\limits_{\mathbb{R}^d} e^{-ix\xi} f(x) \, dx.$$

We define the fractional differentiation operator $|\nabla_x|^\alpha$ for any real α by

$$\widehat{|\nabla|^\alpha u}(\xi) := |\xi|^\alpha \hat{u}(\xi),$$

and analogously

$$\widehat{\langle\nabla\rangle^\alpha u}(\xi) := \langle\xi\rangle^\alpha \hat{u}(\xi).$$

The (in)homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$ ($H^s(\mathbb{R}^d)$) is given via

$$\|u\|_{\dot{H}^s} := \big\||\nabla|^s u\big\|_{L^2(\mathbb{R}^d)}, \qquad \|u\|_{H^s} := \big\|\langle\nabla\rangle^s u\big\|_{L^2(\mathbb{R}^d)},$$

and more general (in)homogeneous Sobolev spaces are given via

$$\|u\|_{\dot{H}^{s,p}} := \big\||\nabla|^s u\big\|_{L^p(\mathbb{R}^d)}, \qquad \|u\|_{H^{s,p}} := \big\|\langle\nabla\rangle^s u\big\|_{L^p(\mathbb{R}^d)}.$$

Let $S(t)$ denote the solution operator to the linear Schrödinger equation

$$iu_t + \Delta u = 0, \quad x \in \mathbb{R}^d.$$

We denote by $X^{s,b}(\mathbb{R}\times\mathbb{R}^d)$ the completion of $\mathcal{S}(\mathbb{R}\times\mathbb{R}^d)$ with respect to the following norm

$$\|u\|_{X^{s,b}} = \big\|S(-t)u\big\|_{H^s_x H^b_t} = \big\|\langle\tau + |\xi|^2\rangle^b \langle\xi\rangle^s \tilde{u}(\tau,\xi)\big\|_{L^2_\tau L^2_\xi(\mathbb{R}\times\mathbb{R}^d)},$$

where $\tilde{u}$ is the space–time Fourier transform

$$\tilde{u}(\tau,\xi) = (2\pi)^{-\frac{d+1}{2}} \iint_{\mathbb{R}\times\mathbb{R}^d} e^{-i(x\cdot\xi + t\tau)} u(t,x)\,dt\,dx.$$

Furthermore for a given time interval J, we define

$$\|u\|_{X^{s,b}(J)} = \inf\big\{\|v\|_{X^{s,b}};\ v = u \text{ on } J\big\}.$$

2.3. Some known estimates

Now we recall a few known estimates that we shall need. First we state the following Strichartz estimate [1,14]. Let $d \geqslant 3$, we recall that a pair of exponents (q,r) is called admissible if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) \triangleq \delta(r), \quad 2 \leqslant q, r \leqslant \infty.$$

Proposition 2.1. *Let $d \geqslant 3$, (q,r) and $(\tilde{q},\tilde{r})$ be any two admissible pairs. Suppose that u is a solution to*

$$iu_t + \Delta u = F(t,x), \quad t \in J,\ x \in \mathbb{R}^d,$$
$$u(0) = u_0(x).$$

Then we have the estimate

$$\|u\|_{L^q_t L^r_x(J\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(J\times\mathbb{R}^d)},$$

where the prime exponents denote Hölder dual exponents.

From the above Strichartz estimate and Lemma 2.1 in [12], we have

$$\|u\|_{L^q_t L^r_x} \lesssim \|u\|_{X^{0,\frac{1}{2}+}}$$

for any admissible (q,r). Then by Sobolev embedding theorem, we have

Proposition 2.2. *Let $d \geqslant 3$. For $r < \infty$, $0 \leqslant \frac{2}{q} \leqslant \min(\delta(r), 1)$, we have*

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{X^{\delta(r)-\frac{2}{q}, \frac{1}{2}+}}.$$

While for $2 \leqslant q \leqslant \infty$, $r = \infty$, we have

$$\|u\|_{L_t^q L_x^\infty} \lesssim \|u\|_{X^{\frac{d}{2}-\frac{2}{q}+, \frac{1}{2}+}}.$$

We will also need the Littlewood–Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leqslant 2$ which equals 1 on the ball $|\xi| \leqslant 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood–Paley operators

$$\widehat{P_{\leqslant N} f}(\xi) := \varphi\left(\frac{\xi}{N}\right)\hat{f}(\xi), \qquad \widehat{P_{>N} f}(\xi) := \left(1 - \varphi\left(\frac{\xi}{N}\right)\right)\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \left(\varphi\left(\frac{\xi}{N}\right) - \varphi\left(\frac{2\xi}{N}\right)\right)\hat{f}(\xi).$$

Similarly we can define $P_{<N}$, $P_{\geqslant N}$, and $P_{M<\cdot\leqslant N} = P_{\leqslant N} - P_{\leqslant M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\leqslant N}$ for $P_{\leqslant N} f$ and similarly for the other operators.

The Littlewood–Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every L_x^p and $\dot{H}_x^s$ space for $1 \leqslant p \leqslant \infty$ and $s \geqslant 0$. They also obey the following Sobolev and Bernstein estimates

$$\|P_{\geqslant N} f\|_{L^p} \lesssim N^{-s} \big\| |\nabla|^s P_{\geqslant N} f \big\|_{L^p},$$
$$\big\| |\nabla|^s P_{\leqslant N} f \big\|_{L^q} \lesssim N^{s+\frac{n}{p}-\frac{n}{q}} \|P_{\leqslant N} f\|_{L^p},$$
$$\big\| |\nabla|^{\pm s} P_N f \big\|_{L^q} \lesssim N^{\pm s+\frac{n}{p}-\frac{n}{q}} \|P_N f\|_{L^p},$$

whenever $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$.

3. The I-method and the modified local well-posedness

3.1. The I-operator and the hierarchy of energies

Let us define the operator I. For $s < 1$ and a parameter $N \gg 1$, let $m(\xi)$ be the following smooth monotone multiplier:

$$m(\xi) := \begin{cases} 1, & \text{if } |\xi| < N, \\ (\frac{N}{|\xi|})^{1-s}, & \text{if } |\xi| > 2N. \end{cases}$$

We define the multiplier operator $I : H^s \to H^1$ by

$$\widehat{Iu}(\xi) = m(\xi)\hat{u}(\xi).$$

The operator I is smoothing of order $1 - s$ and we have from Bernstein's estimates that

$$\|u\|_{H^{s_0}} \lesssim \|Iu\|_{H^{s_0+1-s}} \lesssim N^{1-s} \|u\|_{H^{s_0}},$$
$$\|u\|_{X^{s_0,b_0}} \lesssim \|Iu\|_{X^{s_0+1-s,b_0}} \lesssim N^{1-s} \|u\|_{X^{s_0,b_0}}$$

for any $s_0, b_0 \in \mathbb{R}$.

We set

$$\widetilde{E}(u) = E(Iu), \tag{3.1}$$

where

$$E(u)(t) = \frac{1}{2}\big\|\nabla u(t)\big\|_{L^2}^2 + \frac{1}{4}\iint \frac{1}{|x-y|^\gamma}\big|u(t,x)\big|^2\big|u(t,y)\big|^2\,dx\,dy.$$

We call $\widetilde{E}(u)$ the modified energy. Since we will focus on the analysis of the modified energy, we collect some facts concerning the calculus of multilinear forms used to define the modified energy.

If $k \geqslant 2$ is an even integer, we define a spatial multiplier of order k to be the function $M_k(\xi_1, \xi_2, \dots, \xi_k)$ on

$$\Gamma_k = \left\{ (\xi_1, \xi_2, \dots, \xi_k) \in (\mathbb{R}^d)^k \colon \sum_{j=1}^{k} \xi_j = 0 \right\},$$

which we endow with the standard measure $\delta(\xi_1 + \xi_2 + \cdots + \xi_k)$. If M_k is a multiplier of order k, $1 \leqslant j \leqslant k$ is an index and $l \geqslant 1$ is an even integer, the elongation $X_j^l(M_k)$ of M_k is defined to be the multiplier of order $k+l$ given by

$$X_j^l(M_k)(\xi_1, \xi_2, \dots, \xi_{k+l}) = M_k(\xi_1, \dots, \xi_{j-1}, \xi_j + \cdots + \xi_{j+l}, \xi_{j+l+1}, \dots, \xi_{k+l}).$$

Also if M_k is a multiplier of order k and $u_1, u_2, \dots, u_k$ are functions on $\mathbb{R}^d$, we define the k-linear functional

$$\Lambda_k(M_k; u_1, u_2, \dots, u_k) = \operatorname{Re} \int_{\Gamma_k} M_k(\xi_1, \xi_2, \dots, \xi_k) \prod_{j=1}^{k} \hat{u}_j(\xi_j)$$

and we adopt the notation $\Lambda_k(M_k; u) = \Lambda_k(M_k; u, \bar{u}, \dots, u, \bar{u})$. We observe that the quantity $\Lambda_k(M_k; u)$ is invariant

(1) if one permutes the even arguments $\xi_2, \xi_4, \dots, \xi_k$ of M_k;
(2) if one permutes the odd arguments $\xi_1, \xi_3, \dots, \xi_{k-1}$ of M_k;
(3) if one makes the change of

$$M_k(\xi_1, \xi_2, \dots, \xi_{k-1}, \xi_k) \mapsto M_k(-\xi_2, -\xi_1, \dots, -\xi_k, -\xi_{k-1}).$$

If u is a solution of (1.1), the following differentiation law holds for the multiplier forms $\Lambda_k(M_k; u)$

$$\partial_t \Lambda_k(M_k; u) = \Lambda_k\left(i M_k \sum_{j=1}^{k} (-1)^j |\xi_j|^2; u \right) + \Lambda_{k+2}\left(i \sum_{j=1}^{k} (-1)^j |\xi_{j+1,j+2}|^{-(d-\gamma)} X_j^2(M_k); u \right) \tag{3.2}$$

where we used the notational convention $\xi_{a,b} = \xi_a + \xi_b$, $\xi_{a,b,c} = \xi_a + \xi_b + \xi_c$, etc. Indeed, note that

$$i\partial_t u_j + \Delta u_j = \left(|x|^{-\gamma} * |u_j|^2\right) u_j, \quad \text{for } 0 \leqslant j \leqslant k,$$

we take Fourier transformation in spatial variable and deduce

$$\partial_t \hat{u}_j(\xi_j) = -i|\xi_j|^2 \hat{u}_j(\xi_j) - i \int_{\xi_j = \tilde{\xi}_{j,j+1,j+2}} |\tilde{\xi}_{j+1,j+2}|^{-(d-\gamma)} \hat{u}_j(\tilde{\xi}_j) \hat{\bar{u}}_j(\tilde{\xi}_{j+1}) \hat{u}_j(\tilde{\xi}_{j+2})\, d\tilde{\xi}_{j+1}\, d\tilde{\xi}_{j+2},$$

$$\partial_t \hat{\bar{u}}_j(\xi_j) = +i|\xi_j|^2 \hat{\bar{u}}_j(\xi_j) + i \int_{\xi_j = \tilde{\xi}_{j,j+1,j+2}} |\tilde{\xi}_{j+1,j+2}|^{-(d-\gamma)} \hat{\bar{u}}_j(\tilde{\xi}_j) \hat{u}_j(\tilde{\xi}_{j+1}) \hat{\bar{u}}_j(\tilde{\xi}_{j+2})\, d\tilde{\xi}_{j+1}\, d\tilde{\xi}_{j+2}.$$

From the above identities, we can obtain (3.2).

Using the above notation, the modified energy (3.1) can be written as follows:

$$\widetilde{E}(u) = \Lambda_2\left(-\frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u \right) + \Lambda_4\left(\frac{1}{4} |\xi_{2,3}|^{-(d-\gamma)} m_1 m_2 m_3 m_4; u \right)$$

where we abbreviate $m(\xi_j)$ as m_j.

Together with the differentiation rules (3.2) and the symmetry properties of k-linear functional $\Lambda_k(M_k; u)$, we obtain

$$\partial_t \Lambda_2\left(-\frac{1}{2} \xi_1 m_1 \cdot \xi_2 m_2; u \right) = \Lambda_2\left(-\frac{i}{2} \xi_1 m_1 \cdot \xi_2 m_2 \sum_{j=1}^{2} (-1)^j |\xi_j|^2; u \right)$$

$$+\Lambda_4\left(-\frac{i}{2}\sum_{j=1}^{2}(-1)^j|\xi_{j+1,j+2}|^{-(d-\gamma)}X_j^2(\xi_1 m_1\cdot\xi_2 m_2);u\right)$$
$$=\Lambda_4\big(i|\xi_{2,3}|^{-(d-\gamma)}m_1^2|\xi_1|^2;u\big),$$

where we used the fact that $\sum_{j=1}^2(-1)^j|\xi_j|^2=0$ on the hyperplane Γ_2. At the same time, we have

$$\begin{aligned}
&\partial_t\Lambda_4\left(\frac14|\xi_{2,3}|^{-(d-\gamma)}m_1m_2m_3m_4;u\right)\\
&=\Lambda_4\left(\frac{i}{4}|\xi_{2,3}|^{-(d-\gamma)}m_1m_2m_3m_4\sum_{j=1}^4(-1)^j|\xi_j|^2;u\right)\\
&\quad+\Lambda_6\left(\frac{i}{4}\sum_{j=1}^4(-1)^j|\xi_{j+1,j+2}|^{-(d-\gamma)}X_j^2\big(|\xi_{2,3}|^{-(d-\gamma)}m_1m_2m_3m_4\big);u\right)\\
&=-\Lambda_4\big(i|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1m_2m_3m_4;u\big)-\Lambda_6\big(i|\xi_{2,3}|^{-(d-\gamma)}|\xi_{4,5}|^{-(d-\gamma)}m_{1,2,3}m_4m_5m_6;u\big)\\
&=-\Lambda_4\big(i|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1m_2m_3m_4;u\big)+\Lambda_6\big(i|\xi_{2,3}|^{-(d-\gamma)}|\xi_{4,5}|^{-(d-\gamma)}m_{1,2,3}(m_{1,2,3}-m_4m_5m_6);u\big).
\end{aligned}$$

The fundamental theorem of calculus together with these estimates implies the following proposition, which will be used to prove that $\widetilde{E}$ is almost conserved.

Proposition 3.1. *Let u be an H^1 solution to* (1.1)*. Then for any $T\in\mathbb{R}$ and $\delta>0$, we have*

$$\widetilde{E}(u)(T+\delta)-\widetilde{E}(u)(T)=\int_T^{T+\delta}\Lambda_4(M_4;u)\,dt+\int_T^{T+\delta}\Lambda_6(M_6;u)\,dt$$

with

$$M_4=i|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1(m_1-m_2m_3m_4);$$
$$M_6=i|\xi_{2,3}|^{-(d-\gamma)}|\xi_{4,5}|^{-(d-\gamma)}m_{1,2,3}(m_{1,2,3}-m_4m_5m_6).$$

Furthermore if $|\xi_j|\ll N$ for all j, then the multipliers M_4 and M_6 vanish on Γ_4 and Γ_6, respectively.

3.2. Modified local well-posedness

In this subsection, we shall prove a local well-posedness result for the modified solution Iu and some a priori estimates for it.

Let $J=[t_0,t_1]$ be an interval of time. We denote by $Z_I(J)$ the following space:

$$Z_I(J)=S_I(J)\cap X_I^{1,\frac12+}(J)$$

where

$$S_I(J)=\Big\{u;\ \sup_{(q,r)\text{ admissible}}\big\|\langle\nabla\rangle Iu\big\|_{L_t^qL_x^r(J\times\mathbb{R}^d)}<\infty\Big\},$$
$$X_I^{1,b}(J)=\big\{u;\ \|Iu\|_{X^{1,\frac12+}(J\times\mathbb{R}^d)}<\infty\big\}.$$

Proposition 3.2. *Let $2<\gamma<3\leqslant d$, $s>\frac{\gamma}{2}-1$, and consider the IVP*

$$\begin{aligned}
&iIu_t+\Delta Iu=I\big(|x|^{-\gamma}*|u|^2u\big),\quad x\in\mathbb{R}^d,\ t\in\mathbb{R},\\
&Iu(t_0,x)=Iu_0(x)\in H^1\big(\mathbb{R}^d\big).
\end{aligned}\tag{3.3}$$

Then for any $u_0\in H^s$, there exists a time interval $J=[t_0,t_0+\delta]$, $\delta=\delta(\|Iu_0\|_{H^1})$ and there exists a unique $u\in Z_I(J)$ solution to (3.3)*. Moreover it is continuity with respect to the initial data.*

Proof. The proof of this proposition proceeds by the usual fixed point method on the space $Z_I(J)$. Since the estimates are very similar to the ones we provide in the proof of Proposition 3.3 below, in particular (3.9) and (3.10) , we omit the details. □

Proposition 3.3. *Let $2 < \gamma < 3 \leqslant d$ and $s > \frac{\gamma}{2} - 1$. If u is a solution to the IVP* (3.3) *on the interval $J = [t_0, t_1]$, which satisfies the following a priori bound*

$$\|Iu\|^4_{L^4_t \dot{H}_x^{-\frac{d-3}{4},4}(J\times\mathbb{R}^d)} < \mu,$$

where μ is a small universal constant, then

$$\|u\|_{Z_I(J)} \lesssim \|Iu_0\|_{H^1}.$$

Proof. We start by obtaining a control of the Strichartz norms. Applying $\langle\nabla\rangle$ to (3.3) and using the Strichartz estimate in Proposition 2.1. For any pair of admissible exponents (q, r), we obtain

$$\big\|\langle\nabla\rangle Iu\big\|_{L^q_t L^r_x} \lesssim \|Iu_0\|_{H^1} + \big\|\langle\nabla\rangle I\big(\big(|x|^{-\gamma} * |u|^2\big)u\big)\big\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d+4}}}. \tag{3.4}$$

Now we notice that the multiplier $\langle\nabla\rangle I$ has symbol which is increasing as a function of $|\xi|$ for any $s \geqslant \frac{\gamma}{2} - 1$. Using this fact one can modify the proof of the Leibnitz rule for fractional derivatives and prove its validity for $\langle\nabla\rangle I$. See also Principle A.5 in the appendix of [25]. This remark combined with (3.4) implies that

$$\begin{aligned}
\big\|\langle\nabla\rangle Iu\big\|_{L^q_t L^r_x} &\lesssim \|Iu_0\|_{H^1} + \big\|\langle\nabla\rangle I\big(\big(|x|^{-\gamma} * |u|^2\big)u\big)\big\|_{L_t^{\frac{3}{2}} L_x^{\frac{6d}{3d+4}}} \\
&\lesssim \|Iu_0\|_{H^1} + \big\||x|^{-\gamma} * \langle\nabla\rangle I\big(|u|^2\big)\big\|_{L^2_t L_x^{\frac{2d}{\gamma}}} \|u\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} \\
&\quad + \big\||x|^{-\gamma} * |u|^2\big\|_{L^3_t L_x^{\frac{3d}{4}}} \big\|\langle\nabla\rangle Iu\big\|_{L^3_t L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \|Iu_0\|_{H^1} + \big\|\langle\nabla\rangle Iu\big\|_{L^3_t L_x^{\frac{6d}{3d-4}}} \|u\|^2_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}}
\end{aligned} \tag{3.5}$$

where we used Hölder's inequality and Hardy–Littlewood–Sobolev's inequality.

In order to obtain an upper bound on $\|u\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}}$, we perform a Littlewood–Paley decomposition along the following lines. We note that a similar approach was used in [3]. We write

$$u = u_{N_0} + \sum_{j=1}^{\infty} u_{N_j}, \tag{3.6}$$

where u_{N_0} has spatial frequency support for $\langle\xi\rangle \leqslant N$, while u_{N_j} is such that its spatial Fourier support transform is supported for $\langle\xi\rangle \approx N_j = 2^{h_j}$ with $h_j \gtrsim \log N$ and $j = 1, 2, \ldots$. By the triangle inequality and Hölder's inequality, we have

$$\begin{aligned}
\|u\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} &\lesssim \|u_{N_0}\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} + \sum_{j=1}^{\infty} \|u_{N_j}\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} \\
&\lesssim \|u_{N_0}\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} + \sum_{j=1}^{\infty} \|u_{N_j}\|^{2-\frac{\gamma}{2}}_{L^6_t L_x^{\frac{6d}{3d-2}}} \|u_{N_j}\|^{\frac{\gamma}{2}-1}_{L^6_t L_x^{\frac{6d}{3d-8}}}.
\end{aligned} \tag{3.7}$$

On the other hand, by using the definition of the operator I, the definition of the u_{N_j}'s and the Marcinkiewicz multiplier theorem, we observe that for some $0 < \theta_i < 1$, $i = 1, \ldots, 4$, $\sum_{j=1}^4 \theta_i = 1$

$$\begin{aligned}
\|u_{N_0}\|_{L^6_t L_x^{\frac{6d}{3d+4-3\gamma}}} &\lesssim \|u_{N_0}\|^{\theta_1}_{L^4_t \dot{H}_x^{-\frac{d-3}{4},4}} \|u_{N_0}\|^{\theta_2}_{L^6_t L_x^{\frac{6d}{3d-8}}} \|u_{N_0}\|^{\theta_3}_{L^6_t L_x^{\frac{6d}{3d-2}}} \|u_{N_0}\|^{\theta_4}_{L^\infty_t L^2_x} \\
&\lesssim \|Iu_{N_0}\|^{\theta_1}_{L^4_t \dot{H}_x^{-\frac{d-3}{4},4}} \|u_{N_0}\|^{1-\theta_1}_{Z_I(J)},
\end{aligned}$$

$$\big\|\langle\nabla\rangle I u_{N_j}\big\|_{L_t^6 L_x^{\frac{6d}{3d-2}}} \approx N_j\left(\frac{N}{N_j}\right)^{1-s}\|u_{N_j}\|_{L_t^6 L_x^{\frac{6d}{3d-2}}}, \quad j=1,2,\ldots,$$

$$\|I u_{N_j}\|_{L_t^6 L_x^{\frac{6d}{3d-8}}} \approx \left(\frac{N}{N_j}\right)^{1-s}\|u_{N_j}\|_{L_t^6 L_x^{\frac{6d}{3d-8}}}, \quad j=1,2,\ldots.$$

Now we use these estimates to obtain the following upper bound on (3.7)

$$\begin{aligned}\|u\|_{L_t^6 L_x^{\frac{6d}{3d+4-3\gamma}}} &\lesssim \|I u_{N_0}\|^{\theta_1}_{L_t^4 \dot H_x^{-\frac{d-3}{4},4}}\|u_{N_0}\|^{1-\theta_1}_{Z_I(J)}\\ &\quad+\sum_{j=1}^{\infty}\left(\frac{1}{N_j}\left(\frac{N_j}{N}\right)^{1-s}\|u_{N_j}\|_{Z_I(J)}\right)^{2-\frac{\gamma}{2}}\left(\left(\frac{N_j}{N}\right)^{1-s}\|u_{N_j}\|_{Z_I(J)}\right)^{\frac{\gamma}{2}-1}\\ &\lesssim \mu^{\frac{\theta_1}{4}}\|u\|^{1-\theta_1}_{Z_I(J)}+N^{-(2-\frac{\gamma}{2})}\|u\|_{Z_I(J)}, \qquad (3.8)\end{aligned}$$

which together with (3.5) implies that

$$\big\|\langle\nabla\rangle I u\big\|_{L_t^q L_x^r} \lesssim \|I u_0\|_{H^1}+\mu^{\frac{\theta_1}{2}}\|u\|^{3-2\theta_1}_{Z_I(J)}+N^{-(4-\gamma)}\|u\|^3_{Z_I(J)}. \qquad (3.9)$$

Now we shall obtain a control of the $X^{s,b}$ norm. We use Duhamel's formula and the theory of $X^{s,b}$ spaces [12,25] to obtain

$$\begin{aligned}\|I u\|_{X^{1,\frac{1}{2}+}} &\lesssim \|I u_0\|_{H^1}+\big\|\langle\nabla\rangle I\big(\big(|x|^{-\gamma}*|u|^2\big)u\big)\big\|_{X^{0,-\frac{1}{2}+}}\\ &\lesssim \|I u_0\|_{H^1}+\big\|\langle\nabla\rangle I\big(\big(|x|^{-\gamma}*|u|^2\big)u\big)\big\|_{L_t^{\frac{3}{2}+} L_x^{\frac{6d}{3d+4}+}}\\ &\lesssim \|I u_0\|_{H^1}+\big\|\langle\nabla\rangle I u\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}\|u\|_{L_t^6 L_x^{\frac{6d}{3d+4-3\gamma}}}\|u\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}. \qquad (3.10)\end{aligned}$$

An upper bound on $\|u\|_{L_t^6 L_x^{\frac{6d}{3d+4-3\gamma}}}$ is given by (3.8). In order to obtain an upper bound on $\|u\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}$, we proceed as follows. First we perform a dyadic decomposition and write u as in (3.6). The triangle inequality applied on (3.6) gives for any $0<\delta<\frac{\gamma}{2}-1$

$$\begin{aligned}\|u\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}} &\lesssim \|u_{N_0}\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}+\sum_{j=1}^{\infty}\|u_{N_j}\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}\\ &= \|I u_{N_0}\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}+\sum_{j=1}^{\infty}N_j^{\delta-s}N^{s-1}\big\|\langle\nabla\rangle^{1-\delta} I u_{N_j}\big\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}\\ &\lesssim \|I u\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}}+\big\|\langle\nabla\rangle^{1-\delta} I u\big\|_{L_t^{6+} L_x^{\frac{6d}{3d+4-3\gamma}+}} \lesssim \|I u\|_{X^{1,\frac{1}{2}+}}, \qquad (3.11)\end{aligned}$$

where we used Proposition 2.2. By applying the inequalities (3.8) and (3.11) to bound the right-hand side of (3.10), we obtain

$$\|I u\|_{X^{1,\frac{1}{2}+}} \lesssim \|I u_0\|_{H^1}+\mu^{\frac{\theta_1}{4}}\|u\|^{3-\theta_1}_{Z_I(J)}+N^{-(2-\frac{\gamma}{2})}\|u\|^3_{Z_I(J)}. \qquad (3.12)$$

The desired bound follows from (3.9) and (3.12) by choosing N sufficiently large. □

3.3. An upper bound on the increment of $\widetilde{E}(u)$

Decomposition remark. Our approach to prove a decay for the increment of the modified energy is based on obtaining certain multilinear estimates in appropriate functional spaces which are L^2-based (more details, see [24]). Hence, whenever we perform a Littlewood–Paley decomposition of a function we shall assume that the Fourier transforms of the Littlewood–Paley pieces are positive. Moreover, we will ignore the presence of conjugates. At the end we will always keep a decay factor $C(N_1, N_2, \ldots)$ in order to perform the summations.

Now we proceed to prove the almost conservation law of the modified energy. In Proposition 3.1, we prove that an increment of the modified energy can be expressed as

$$\widetilde{E}(u)(T+\delta)-\widetilde{E}(u)(T)=\int_T^{T+\delta}\Lambda_4(M_4;u)\,dt+\int_T^{T+\delta}\Lambda_6(M_6;u)\,dt$$

with

$$M_4=i|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1(m_1-m_2m_3m_4);$$
$$M_6=i|\xi_{2,3}|^{-(d-\gamma)}|\xi_{4,5}|^{-(d-\gamma)}m_{1,2,3}(m_{1,2,3}-m_4m_5m_6).$$

Hence in order to control the increment of the modified energy, we shall find an upper bound on the $\Lambda_4(M_4;u)$ and $\Lambda_6(M_6;u)$ forms, which we do in the following propositions. First we give the quadrilinear estimate.

Proposition 3.4. *For any Schwartz function u, and any $\delta\approx 1$ just as in Proposition 3.2, we have that*

$$\left|\int_T^{T+\delta}\Lambda_4(M_4;u)\,dt\right|\lesssim N^{-1+}\|Iu\|^4_{X^{1,\frac12+}},\tag{3.13}$$

for $s>\frac{\gamma}{2}-1$.

Proof. By Plancherel's theorem, we aim to prove that

$$\left|\int_T^{T+\delta}\int_{\Gamma_4}|\xi_{2,3}|^{-(d-\gamma)}|\xi_1|^2m_1(m_1-m_2m_3m_4)\hat{u}_1(t,\xi_1)\hat{\bar{u}}_2(t,\xi_2)\hat{u}_3(t,\xi_3)\hat{\bar{u}}_4(t,\xi_4)\right|$$
$$\lesssim N^{-1+}C(N_1,N_2,N_3,N_4)\prod_{j=1}^4\|Iu_j\|_{X^{1,\frac12+}},\tag{3.14}$$

where $C(N_1,N_2,N_3,N_4)$ is a decay just as the remark above, and it allows us to sum over all dyadic shells. The analysis which follows will not rely on the complex conjugate structure in $\Lambda_4(M_4;u)$. Thus, by symmetry, we may assume that $N_2\geqslant N_3\geqslant N_4$.

Case 1. $N\gg N_2$. According to the definition of $m(\xi)$, the multiplier

$$|\xi_{2,3}|^{-(d-\gamma)}m_1(m_1-m_2m_3m_4)$$

is identically 0, the bound (3.13) holds trivially.

Case 2. $N_2\gtrsim N\gg N_3\geqslant N_4$. Since $\sum_{j=1}^4\xi_j=0$, we have $N_1\approx N_2$. We aim for (3.14) with a decay factor

$$C(N_1,N_2,N_3,N_4)=N_2^{0-}.$$

By the mean value theorem, we have the following pointwise bound

$$\begin{aligned}\left|m_1(m_1-m_2m_3m_4)\right|&=\left|m_1(m_{2,3,4}-m_2m_3m_4)\right|\\&\lesssim m_1\left|\nabla m(\xi)\cdot(\xi_3+\xi_4)\right|\quad\text{where } |\xi|\approx|\xi_2|\\&\lesssim m_1m_2\frac{N_3}{N_2}.\end{aligned}$$

Hence by Hölder's inequality and Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we obtain

$$\begin{aligned}\text{LHS of (3.14)} &\lesssim N_1^2 m_1 m_2 \frac{N_3}{N_2}\left|\int_T^{T+\delta}\int_{\Gamma_4} |\xi_{2,3}|^{-(d-\gamma)}\hat{u}_1(t,\xi_1)\hat{\bar{u}}_2(t,\xi_2)\hat{u}_3(t,\xi_3)\hat{\bar{u}}_4(t,\xi_4)\right| \\ &\lesssim N_1^2 m_1 m_2 \frac{N_3}{N_2}\|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}\|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}\|u_3\|_{L_t^6 L_x^{\frac{6d}{3d-2}}}\|u_4\|_{L_t^6 L_x^{\frac{6d}{3d+10-6\gamma}}} \\ &\lesssim N_1^2 m_1 m_2 \frac{N_3}{N_2} N_4^{\gamma-2}\prod_{j=1}^4 \|u_j\|_{X^{0,\frac{1}{2}+}},\end{aligned}$$

where we used the fact that

$$\begin{aligned}&\frac{1}{3}+\frac{1}{3}+\frac{1}{6}+\frac{1}{6}=1, \qquad \frac{3d-4}{6d}+\frac{3d-4}{6d}+\frac{3d-2}{6d}+\frac{3d+10-6\gamma}{6d}+\frac{\gamma}{d}=2,\\ &\delta\left(\frac{6d}{3d-4}\right)-\frac{2}{3}=0, \qquad \delta\left(\frac{6d}{3d-2}\right)-\frac{2}{6}=0, \qquad \delta\left(\frac{6d}{3d+10-6\gamma}\right)-\frac{2}{6}=\gamma-2.\end{aligned}$$

It suffices to show that

$$N_1^2 m_1 m_2 \frac{N_3}{N_2} N_4^{\gamma-2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_2 \langle N_3\rangle\langle N_4\rangle.$$

We reduce to show that

$$N^{1-} N_2^{0+} \lesssim N_2\, \langle N_3\rangle N_3^{-1}\, \langle N_4\rangle N_4^{2-\gamma}.$$

This is true since

$$N_2 \gtrsim N^{1-} N_2^{0+}; \qquad \langle N_3\rangle N_3^{-1} \gtrsim 1; \qquad \langle N_4\rangle N_4^{2-\gamma} \gtrsim 1.$$

Case 3. $N_2 \geqslant N_3 \gtrsim N$. In this case, we use the trivial pointwise bound

$$\left|m_1(m_1 - m_2 m_3 m_4)\right| \lesssim m_1^2.$$

The frequency interactions fall into two subcategories, depending on which frequency is comparable to N_2.

Case 3a. $N_1 \approx N_2 \geqslant N_3 \gtrsim N$. In this case, we prove the decay factor

$$C(N_1, N_2, N_3, N_4) = N_3^{0-}$$

in (3.14). This allows us to directly sum in N_3 and N_4, and sum in N_1 and N_2 after applying Cauchy–Schwarz to those factors.

By Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we obtain

$$\begin{aligned}\text{LHS of (3.14)} &\lesssim N_1^2 m_1^2\left|\int_T^{T+\delta}\int_{\Gamma_4} |\xi_{2,3}|^{-(d-\gamma)}\hat{u}_1(t,\xi_1)\hat{\bar{u}}_2(t,\xi_2)\hat{u}_3(t,\xi_3)\hat{\bar{u}}_4(t,\xi_4)\right| \\ &\lesssim N_1^2 m_1^2 \|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}\|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}\|u_3\|_{L_t^6 L_x^{\frac{6d}{3d-2}}}\|u_4\|_{L_t^6 L_x^{\frac{6d}{3d+10-6\gamma}}} \\ &\lesssim N_1^2 m_1^2 N_4^{\gamma-2}\prod_{j=1}^4 \|u_j\|_{X^{0,\frac{1}{2}+}}.\end{aligned}$$

It suffices to show that

$$N_1^2 m_1^2 N_4^{\gamma-2} \lesssim N^{-1+} N_3^{0-} m_1 N_1 m_2 N_2 m_3 N_3 \langle N_4\rangle.$$

We reduce to show that

$$N^{1-}N_3^{0+} \lesssim m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma}.$$

This is true since for $s \geqslant \gamma - 2$, we have

$$m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma} \geqslant m_3 N_3 m_4 \langle N_4\rangle^{3-\gamma} \gtrsim m_3 N_3 \gtrsim N^{1-}N_3^{0+},$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-decreasing if $s + p \geqslant 1$. While for $\frac{\gamma}{2} - 1 < s < \gamma - 2$, we have

$$m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma} \gtrsim m_3 N_3 m_3 N_3^{3-\gamma} \gtrsim N^{4-\gamma-}N_3^{0+} \gtrsim N^{1-}N_3^{0+},$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-increasing if $s + p < 1$.

Case 3b. $N_2 \approx N_3 \gtrsim N$, $N_2 \gtrsim N_1$. In this case, we prove the decay factor

$$C(N_1, N_2, N_3, N_4) = N_2^{0-}$$

in (3.14). This will allow us to directly sum in all the N_i.

By Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate once again, we obtain

$$\begin{aligned}
\text{LHS of (3.14)} &\lesssim N_1^2 m_1^2 \left| \int_T^{T+\delta} \int_{\Gamma_4} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}_1(t,\xi_1)\hat{\bar{u}}_2(t,\xi_2)\hat{u}_3(t,\xi_3)\hat{\bar{u}}_4(t,\xi_4) \right| \\
&\lesssim N_1^2 m_1^2 \|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_3\|_{L_t^6 L_x^{\frac{6d}{3d-2}}} \|u_4\|_{L_t^6 L_x^{\frac{6d}{3d+10-6\gamma}}} \\
&\lesssim N_1^2 m_1^2 N_4^{\gamma-2} \prod_{j=1}^4 \|u_j\|_{X^{0,\frac{1}{2}+}}.
\end{aligned}$$

It suffices to show that

$$N_1^2 m_1^2 N_4^{\gamma-2} \lesssim N^{-1+}N_2^{0-} m_1 N_1 m_2 N_2 m_3 N_3 \langle N_4\rangle.$$

Note that

$$N_1^2 m_1^2 \lesssim m_1 N_1 m_2 N_2.$$

We reduce to show that

$$N^{1-}N_2^{0+} \lesssim m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma}.$$

This is true since for $s \geqslant \gamma - 2$, we have

$$m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma} \geqslant m_3 N_3 m_4 \langle N_4\rangle^{3-\gamma} \gtrsim m_3 N_3 \approx m_2 N_2 \gtrsim N^{1-}N_2^{0+},$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-decreasing if $s + p \geqslant 1$. While for $\frac{\gamma}{2} - 1 < s < \gamma - 2$, we have

$$m_3 N_3 m_4 \langle N_4\rangle N_4^{2-\gamma} \gtrsim m_3 N_3 m_3 N_3^{3-\gamma} \approx m_2^2 N_2^{4-\gamma} \gtrsim N^{4-\gamma-}N_3^{0+} \gtrsim N^{1-}N_3^{0+},$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-increasing if $s + p < 1$. This completes the proof. □

In order to make use of quadrilinear estimate (Proposition 3.4) to obtain sextilinear estimate, we first give a lemma

Lemma 3.1. *Assume u, δ are as in Proposition* 3.2*, and $P_{N_{1,2,3}}$ the Littlewood–Paley projection onto the $N_{1,2,3}$ frequency shell. Then*

$$\left\| P_{N_{1,2,3}}\left(I\left(u|\nabla|^{-(d-\gamma)}|u|^2\right)\right)\right\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \lesssim N_{1,2,3}\|Iu\|^3_{X^{1,\frac{1}{2}+}}.$$

Proof. We write $u = u_L + u_H$ where

$$\operatorname{supp} \hat{u}_l(t,\xi) \subseteq \{|\xi| < 2\},$$
$$\operatorname{supp} \hat{u}_H(t,\xi) \subseteq \{|\xi| > 1\}.$$

Hence,

$$\begin{aligned}
&\big\| P_{N_{1,2,3}}\big(I\big(u|\nabla|^{-(d-\gamma)}|u|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \big\| P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}|u_L|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} + \big\| P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\quad + \big\| P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} + \big\| P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_L|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\quad + \big\| P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}\bar{u}_L u_H\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} + \big\| P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}u_L \bar{u}_H\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\quad + \big\| P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}\bar{u}_L u_H\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} + \big\| P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}u_L \bar{u}_H\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}.
\end{aligned}$$

Consider the first term. By Hölder's inequality, Hardy–Littlewood–Sobolev's inequality and Proposition 2.2, we have

$$\begin{aligned}
\big\| P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}|u_L|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} &\lesssim \big\| u_L|\nabla|^{-(d-\gamma)}|u_L|^2\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \|u_L\|^3_{L_t^9 L_x^{\frac{18d}{9d-6\gamma-4}}} = \|Iu_L\|^3_{L_t^9 L_x^{\frac{18d}{9d-6\gamma-4}}} \\
&\lesssim \|Iu_L\|^3_{X^{1,\frac{1}{2}+}} \leqslant N_{1,2,3}\|Iu\|^3_{X^{1,\frac{1}{2}+}},
\end{aligned}$$

since

$$N_{1,2,3} \geqslant 1, \quad \text{and} \quad 0 \leqslant d \times \left(\frac{1}{2} - \frac{9d-6\gamma-4}{18d}\right) - \frac{2}{9} = \frac{\gamma}{3} \leqslant 1.$$

We estimate the second term. By Sobolev's inequality and using the Leibniz rule for the operator $|\nabla|^{2-\frac{\gamma}{2}} I$ (Principle A.5 in [25]) and Proposition 2.2, we have

$$\begin{aligned}
\left\| \frac{1}{N_{1,2,3}} P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\right\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} &\lesssim \big\| |\nabla|^{-1} P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \big\| |\nabla|^{2-\frac{\gamma}{2}} I\big(u_H|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-3\gamma+14}}} \\
&\lesssim \big\| |\nabla|^{2-\frac{\gamma}{2}} Iu_H\big\|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \|u_H\|^2_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \\
&\lesssim \big\| |\nabla|^{2-\frac{\gamma}{2}} Iu_H\big\|^3_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \lesssim \big\| |\nabla| Iu_H\big\|^3_{L_t^9 L_x^{\frac{18d}{9d-4}}} \\
&\leqslant \|Iu\|^3_{X^{1,\frac{1}{2}+}}.
\end{aligned}$$

As for the third term. By Sobolev's inequality and using the Leibniz rule for the operator $|\nabla|^{2-\frac{\gamma}{2}} I$ and Proposition 2.2 again, we have

$$\begin{aligned}
&\left\| \frac{1}{N_{1,2,3}} P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\right\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \big\| |\nabla|^{-1} P_{N_{1,2,3}}\big(I\big(u_L|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\
&\lesssim \big\| |\nabla|^{2-\frac{\gamma}{2}} I\big(u_L|\nabla|^{-(d-\gamma)}|u_H|^2\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-3\gamma+14}}}
\end{aligned}$$

$$\lesssim \big\||\nabla|^{2-\frac{\gamma}{2}} Iu_H\big\|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \|u_H\|_{L_t^9 L_x^{\frac{18d}{9d-4}}} \|u_L\|_{L_t^9 L_x^{\frac{18d}{9d-18\gamma+32}}} + \|u_H\|^2_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}} \big\||\nabla|^{2-\frac{\gamma}{2}} Iu_L\big\|_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+14}}}$$
$$\lesssim \big\||\nabla| Iu_H\big\|_{L_t^9 L_x^{\frac{18d}{9d-4}}} \big\||\nabla| Iu_H\big\|_{L_t^9 L_x^{\frac{18d}{9d-4}}} \big\||\nabla|^{\gamma-2} Iu_L\big\|_{L_t^9 L_x^{\frac{18d}{9d-4}}} + \big\||\nabla| Iu_H\big\|^2_{L_t^9 L_x^{\frac{18d}{9d-4}}} \big\||\nabla| Iu_L\big\|_{L_t^9 L_x^{\frac{18d}{9d-4}}}$$
$$\lesssim \|Iu\|^3_{X^{1,\frac{1}{2}+}}.$$

Now we estimate the fourth term. By Sobolev's inequality and Hölder's inequality, we obtain

$$\begin{aligned}\left\|\frac{1}{N_{1,2,3}} P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_L|^2\big)\big)\right\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} &\lesssim \big\||\nabla|^{-1} P_{N_{1,2,3}}\big(I\big(u_H|\nabla|^{-(d-\gamma)}|u_L|^2\big)\big)\big\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \\ &\lesssim \big\|u_H|\nabla|^{-(d-\gamma)}|u_L|^2\big\|_{L_t^3 L_x^{\frac{6d}{3d+2}}} \\ &\lesssim \|u_H\|_{L_t^9 L_x^{\frac{18d}{9d-4}}} \|u_L\|^2_{L_t^9 L_x^{\frac{18d}{9d-9\gamma+5}}} \lesssim \|Iu\|^3_{X^{1,\frac{1}{2}+}}.\end{aligned}$$

The remainder terms are similar to the third and fourth terms because we can ignore the complex conjugates. This completes the proof. □

Now we proceed to prove the sextilinear estimate. Indeed, after performing the Littlewood–Paley decomposition of a function, we know that the factor $P_{N_{1,2,3}}(I(u|\nabla|^{-(d-\gamma)}|u|^2))$ in Λ_6 takes the same role as the factor ΔIu_1 in Λ_4. At the same time, we always take the ΔIu_1 factor in $L_t^3 L_x^{6d/(3d-4)}$ in the proof of Proposition 3.4, and estimate this by $N_1\|Iu_1\|_{X^{1,\frac{1}{2}+}}$. Combining with Lemma 3.1, we can obtain the following estimate.

Proposition 3.5. *For any Schwartz function u, and any $\delta \approx 1$ as in Proposition 3.2, we have that*

$$\left|\int_T^{T+\delta} \Lambda_6(M_6; u)\,dt\right| \lesssim N^{-1+}\|Iu\|^6_{X^{1,\frac{1}{2}+}}$$

for $s > \frac{\gamma}{2} - 1$.

4. Almost interaction Morawetz estimate

In this section, we aim to prove the interaction Morawetz estimate for the smoothed out solution Iu, that is, "almost Morawetz estimate". For this, we consider $a(x_1, x_2) = |x_1 - x_2| : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, a convex and locally integrable function of polynomial growth. In all of our arguments, we will work with the Schwartz solutions. This will simplify the calculations and will enable us to justify the steps in the subsequent proofs, since we can approximate the H^s solutions by the Schwartz solutions.

Theorem 4.1. *Let u be a Schwarz solution to*

$$iu_t + \Delta u = \widetilde{\mathcal{N}}(u), \quad (x,t) \in \mathbb{R}^d \times [0,T],$$

where $\widetilde{\mathcal{N}}(u) = (|x|^{-\gamma} * |u|^2)u$. Let Iu be a solution to

$$iIu_t + \Delta Iu = I\big(\widetilde{\mathcal{N}}(u)\big), \quad (x,t) \in \mathbb{R}^d \times [0,T]. \tag{4.1}$$

Then

$$\big\||\nabla|^{-\frac{d-3}{4}} Iu\big\|^4_{L_T^4 L_x^4} \lesssim \|Iu\|_{L_T^\infty \dot H_x^1} \|Iu\|^3_{L_T^\infty L_x^2} + \int_0^T \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\mathrm{bad}}, Iu(t,x_1)Iu(t,x_2)\big\}_p\,dx_1\,dx_2\,dt \tag{4.2}$$

with $\{\cdot,\cdot\}_p$ is the momentum bracket defined by

$$\{f,g\}_p = \mathrm{Re}(f\nabla\bar g - g\nabla\bar f),$$

and

$$\widetilde{\mathcal{N}}_{\mathrm{bad}} = \sum_{i=1}^{2}\big(I\widetilde{\mathcal{N}}_i(u_i) - \widetilde{\mathcal{N}}_i(Iu_i)\big)\prod_{j=1,\, j\neq i}^{2} Iu_j,$$

where u_i is a solution to

$$iu_t + \Delta u = \widetilde{\mathcal{N}}(u), \quad (x_i, t) \in \mathbb{R}^d \times \mathbb{R},\ d \geqslant 3, \tag{4.3}$$

here $x_i \in \mathbb{R}^d$, not a coordinate. In particular, on a time interval J_k where the local well-posedness Proposition 3.2 *holds, we have that*

$$\int_{J_k}\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\mathrm{bad}}, Iu(t,x_1)Iu(t,x_2)\big\}_p\, dx_1\, dx_2\, dt \lesssim \frac{1}{N^{1-}}\|u\|_{Z_I(J_k)}^6.$$

Toward this goal, we recall the idea of the proof of the interaction Morawetz estimate for the defocusing nonlinear cubic Schrödinger equation in three space dimensions [3]. We present the result using a tensor of Schrödinger solutions that emerged in [2,6]. We first recall the generalized Virial identity [2,16].

Proposition 4.1. *Let $a : \mathbb{R}^d \to \mathbb{R}$ be convex and u be a smooth solution to the solution*

$$iu_t + \Delta u = \widetilde{\mathcal{N}}(u), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{4.4}$$

Then the following inequality holds

$$\int_0^T\int_{\mathbb{R}^d} (-\Delta\Delta a)\big|u(t,x)\big|^2\, dx\, dt + 2\int_0^T\int_{\mathbb{R}^d} \nabla a \cdot \{\widetilde{\mathcal{N}}, u\}_p\, dx\, dt \lesssim \big|M_a(T) - M_a(0)\big|$$

where $M_a(t)$ is the Morawetz action corresponding to u and is given by

$$M_a(t) = 2\int_{\mathbb{R}^d} \nabla a(x) \cdot \mathrm{Im}\big(\bar{u}(x)\nabla u(x)\big)\, dx.$$

Proof of Theorem 4.1. Now we rewrite Eq. (4.1) as

$$iIu_t + \Delta Iu = \widetilde{\mathcal{N}}(Iu) + \big(I\big(\widetilde{\mathcal{N}}(u)\big) - \widetilde{\mathcal{N}}(Iu)\big).$$

By Proposition 4.1, we have

$$\begin{aligned}
&\int_0^T\int (-\Delta\Delta a)\big|Iu(t,x)\big|^2\, dx\, dt - \int_0^T\iiint \big|Iu(t,y)\big|^2 \frac{x-y}{|x-y|}\big(\nabla|\nabla|^{-(n-4)}|Iu|^2|Iu|^2\big)(t,x)\, dx\, dy\, dt \\
&\quad \lesssim \sup_{t\in[0,T]}\left|\int \nabla a(x)\cdot \mathrm{Im}\big(\overline{Iu}(x)\nabla Iu(x)\big)\, dx\right| + \left|\int_0^T\int \nabla a \cdot \big\{I\widetilde{\mathcal{N}}(u) - \widetilde{\mathcal{N}}(Iu), Iu\big\}_p\, dx\, dt\right|,
\end{aligned}$$

where the second term on the left-hand side

$$-\int_0^T\iiint \big|Iu(t,y)\big|^2 \frac{x-y}{|x-y|}\big(\nabla|\nabla|^{-(n-4)}|Iu|^2|Iu|^2\big)(t,x)\, dx\, dy\, dt \tag{4.5}$$

is created by the term $\widetilde{\mathcal{N}}(Iu)$ and the commutator $I(\widetilde{\mathcal{N}}(u)) - \widetilde{\mathcal{N}}(Iu)$ creates the second term on the right-hand side

$$\left|\int_0^T\int \nabla a \cdot \big\{I\widetilde{\mathcal{N}}(u) - \widetilde{\mathcal{N}}(Iu), Iu\big\}_p\, dx\, dt\right|. \tag{4.6}$$

By symmetry, we know that the term (4.5) is a positive term that we can ignore, which is analogue to the case in [19,20]. Next we turn to the estimate of the error term (4.6). In addition, the conjugation will play no crucial role in the forthcoming argument.

Now define the tensor product $u := (u_1 \otimes u_2)(t,x)$ for x in

$$\mathbb{R}^{d+d} = \{x = (x_1, x_2):\ x_1 \in \mathbb{R}^d,\ x_2 \in \mathbb{R}^d\}$$

by the formula

$$(u_1 \otimes u_2)(t,x) = u_1(t,x_1)u_2(t,x_2),$$

let us set

$$IU(t,x) = \prod_{j=1}^{2} Iu(t,x_j).$$

If u solves (4.4) for d dimensions, then IU solves (4.4) for $2d$ dimensions, with right-hand side $\widetilde{\mathcal{N}}_I$ given by

$$\widetilde{\mathcal{N}}_I = \sum_{i=1}^{2}\left(I\big(\widetilde{\mathcal{N}}_i(u_i)\big) \prod_{j=1, j\neq i}^{2} Iu_j\right).$$

Now let us decompose

$$\begin{aligned}\widetilde{\mathcal{N}}_I &= \widetilde{\mathcal{N}}_{\text{good}} + \widetilde{\mathcal{N}}_{\text{bad}}\\ &\triangleq \sum_{i=1}^{2}\left(\widetilde{\mathcal{N}}_i(Iu_i) \prod_{j=1,\, j\neq i}^{2} Iu_j\right) + \sum_{i=1}^{2}\left(\big(I\big(\widetilde{\mathcal{N}}_i(u_i)\big) - \widetilde{\mathcal{N}}_i(Iu_i)\big) \prod_{j=1, j\neq i}^{2} Iu_j\right).\end{aligned}$$

The first term summand creates a positive term that we can ignore again. The term we call $\widetilde{\mathcal{N}}_{\text{bad}}$ produces the error term. Now we pick $a(x) = a(x_1,x_2) = |x_1 - x_2|$ where $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Hence we have

$$\big\||\nabla|^{-\frac{d-3}{4}} Iu\big\|^4_{L^4_T L^4_x} \lesssim \|Iu\|_{L^\infty_T \dot H^1_x} \|Iu\|^3_{L^\infty_T L^2_x} + \left|\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\text{bad}}, Iu(t,x_1)Iu(t,x_2)\big\}_p \, dx_1\, dx_2\, dt\right|.$$

Note that the second term of the right-hand side comes from the momentum bracket term in the proof of Proposition 4.1. Following with the same calculations in [2], we deduce that

$$\begin{aligned}\mathcal{E} &:= \left|\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\text{bad}}, Iu(t,x_1)Iu(t,x_2)\big\}_p \, dx_1\, dx_2\, dt\right|\\ &\lesssim \big(\big\|I\big(\widetilde{\mathcal{N}}(u)\big) - \widetilde{\mathcal{N}}(Iu)\big\|_{L^1_t L^2_x} + \big\|\nabla_x\big(I\big(\widetilde{\mathcal{N}}(u)\big) - \widetilde{\mathcal{N}}(Iu)\big)\big\|_{L^1_t L^2_x}\big)\|u\|^3_{Z_I(J)}.\end{aligned} \tag{4.7}$$

Now we proceed to estimate $\|\nabla_x(I(\widetilde{\mathcal{N}}(u)) - \widetilde{\mathcal{N}}(Iu))\|_{L^1_t L^2_x}$, which is the harder term. The term $\|I(\widetilde{\mathcal{N}}(u)) - \widetilde{\mathcal{N}}(Iu)\|_{L^1_t L^2_x}$ can be estimated in the same way. Note that

$$\widetilde{\mathcal{N}}(u) = \big(|x|^{-\gamma} * |u|^2\big)u,$$

we have

$$\begin{aligned}&\mathcal{F}_x\big(\nabla_x\big(I\big(\widetilde{\mathcal{N}}(u)\big) - \widetilde{\mathcal{N}}(Iu)\big)\big)(\xi)\\ &\quad = \int_{\xi=\sum_{j=1}^3 \xi_j} i\xi|\xi_{2,3}|^{-(d-\gamma)}\big(m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3)\big)\hat u(\xi_1)\hat{\bar u}(\xi_2)\hat u(\xi_3)\, d\xi_1\, d\xi_2\, d\xi_3.\end{aligned}$$

We decompose u into a sum of dyadic pieces u_j localized around N_j, then

$$\begin{aligned}
&\big\|\nabla_x\big(I\big(\widetilde{\mathcal{N}}(u)\big)-\widetilde{\mathcal{N}}(Iu)\big)\big\|_{L_t^1L_x^2}\\
&\quad=\big\|\mathcal{F}_x\big(\nabla_x\big(I\big(\widetilde{\mathcal{N}}(u)\big)-\widetilde{\mathcal{N}}(Iu)\big)\big)(\xi)\big\|_{L_t^1L_\xi^2}\\
&\quad\lesssim\sum_{N_1,N_2,N_3}\bigg\|\int\limits_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3\xi_j}}|\xi||\xi_{2,3}|^{-(d-\gamma)}\big|m(\xi)-m(\xi_1)m(\xi_2)m(\xi_3)\big|\hat u(\xi_1)\hat{\bar{u}}(\xi_2)\hat u(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3\bigg\|_{L_t^1L_\xi^2}.
\end{aligned}$$

Since the conjugation plays no crucial role here, without loss of generality, we assume that

$$N_1\geqslant N_2\geqslant N_3.$$

Set

$$\sigma(\xi_1,\xi_2,\xi_3)=|\xi_1+\xi_2+\xi_3|\big|m(\xi_1+\xi_2+\xi_3)-m(\xi_1)m(\xi_2)m(\xi_3)\big|,$$

then

$$\sigma(\xi_1,\xi_2,\xi_3)=\sum_{j=1}^4\chi_j(\xi_1,\xi_2,\xi_3)\sigma(\xi_1,\xi_2,\xi_3):=\sum_{j=1}^4\sigma_j(\xi_1,\xi_2,\xi_3),$$

where $\chi_j(\xi_1,\xi_2,\xi_3)$ is a smooth characteristic function of the set Ω_j defined as follows:

- $\Omega_1=\{|\xi_i|\approx N_i,\ i=1,2,3;\ N\gg N_1\}$;
- $\Omega_2=\{|\xi_i|\approx N_i,\ i=1,2,3;\ N_1\gtrsim N\gg N_2\}$;
- $\Omega_3=\{|\xi_i|\approx N_i,\ i=1,2,3;\ N_1\geqslant N_2\gtrsim N\gg N_3\}$;
- $\Omega_4=\{|\xi_i|\approx N_i,\ i=1,2,3;\ N_1\geqslant N_2\geqslant N_3\gtrsim N\}$.

Hence, we have

$$\begin{aligned}
&\big\|\nabla_x\big(I\big(\widetilde{\mathcal{N}}(u)\big)-\widetilde{\mathcal{N}}(Iu)\big)\big\|_{L_t^1L_x^2}\\
&\quad\lesssim\sum_{N_1,N_2,N_3}\sum_{j=1}^4\bigg\|\int\limits_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3\xi_j}}|\xi_{2,3}|^{-(d-\gamma)}\sigma_j(\xi_1,\xi_2,\xi_3)\hat u(\xi_1)\hat{\bar{u}}(\xi_2)\hat u(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3\bigg\|_{L_t^1L_\xi^2}\\
&\quad:=\sum_{N_1,N_2,N_3}\sum_{j=1}^4L_j.
\end{aligned}$$

Contribution of L_1. Since σ_1 is identically zero when $N\geqslant 4N_1$, L_1 gives no contribution to the sum above.

Contribution of L_2. By the mean value theorem, we have the pointwise bound

$$\sigma_2(\xi_1,\xi_2,\xi_3)\lesssim N_1\cdot m_1\frac{N_2}{N_1}=m_1N_2.$$

Hence, by Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we obtain

$$\begin{aligned}
L_2&=\bigg\|\int\limits_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3\xi_j}}|\xi_{2,3}|^{-(d-\gamma)}\sigma_2(\xi_1,\xi_2,\xi_3)\hat u(\xi_1)\hat u(\xi_2)\hat u(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3\bigg\|_{L_t^1L_\xi^2}\\
&\lesssim m_1N_2\bigg\|\int\limits_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3\xi_j}}|\xi_{2,3}|^{-(d-\gamma)}\hat u(\xi_1)\hat u(\xi_2)\hat u(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3\bigg\|_{L_t^1L_\xi^2}
\end{aligned}$$

$$\lesssim m_1 N_2 \|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_3\|_{L_t^3 L_x^{\frac{6d}{3d-6\gamma+8}}}$$

$$\lesssim m_1 N_2 N_3^{\gamma-2} \prod_{J=1}^{3} \|u_j\|_{X^{0,\frac{1}{2}+}},$$

where we used the fact that

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1, \qquad \frac{3d-4}{6d}+\frac{3d-4}{6d}+\frac{3d-6\gamma+8}{6d}+\frac{\gamma}{d}=\frac{1}{2}+1,$$
$$\delta\left(\frac{6d}{3d-4}\right)-\frac{2}{3}=0, \qquad \delta\left(\frac{6d}{3d-6\gamma+8}\right)-\frac{2}{3}=\gamma-2.$$

It suffices to show that

$$m_1 N_2 N_3^{\gamma-2} \lesssim N^{-1+} N_1^{0-} m_1 N_1 \langle N_2\rangle \langle N_3\rangle.$$

We reduce to show that

$$N^{1-} N_1^{0+} \lesssim N_1 \langle N_2\rangle N_2^{-1} \langle N_3\rangle N_3^{2-\gamma}.$$

This is true since

$$N_1 \gtrsim N^{1-} N_1^{0+}; \qquad \langle N_2\rangle N_2^{-1} \gtrsim 1; \qquad \langle N_3\rangle N_3^{2-\gamma} \gtrsim 1.$$

Contribution of L_3. Note that

$$\sigma_3(\xi_1,\xi_2,\xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1.$$

Hence, by Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we have

$$L_3 = \left\| \int_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3 \xi_j}} |\xi_{2,3}|^{-(d-\gamma)} \sigma_3(\xi_1,\xi_2,\xi_3) \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 \right\|_{L_t^1 L_\xi^2}$$

$$\lesssim m_1 N_1 \left\| \int_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3 \xi_j}} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 \right\|_{L_t^1 L_\xi^2}$$

$$\lesssim m_1 N_1 \|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_3\|_{L_t^3 L_x^{\frac{6d}{3d-6\gamma+8}}}$$

$$\lesssim m_1 N_1 N_3^{\gamma-2} \prod_{J=1}^{3} \|u_j\|_{X^{0,\frac{1}{2}+}}.$$

It suffices to show that

$$m_1 N_1 N_3^{\gamma-2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_2 \langle N_3\rangle.$$

We reduce to show that

$$N^{1-} N_2^{0+} \lesssim m_2 N_2 \langle N_3\rangle N_3^{2-\gamma}.$$

This is true since

$$m_2 N_2 \gtrsim N^{1-} N_2^{0+}; \qquad \langle N_3\rangle N_3^{2-\gamma} \gtrsim 1.$$

Contribution of L_4. Note that

$$\sigma_4(\xi_1,\xi_2,\xi_3) \lesssim N_1 m_1 + N_1 m_1 m_2 \lesssim N_1 m_1.$$

Hence, by Hölder's inequality, Hardy–Littlewood–Sobolev's inequality, Proposition 2.2 and Bernstein's estimate, we obtain

$$\begin{aligned}
L_4 &= \Bigg\| \int_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3 \xi_j}} |\xi_{2,3}|^{-(d-\gamma)} \sigma_4(\xi_1,\xi_2,\xi_3)\hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3 \Bigg\|_{L_t^1 L_\xi^2} \\
&\lesssim m_1 N_1 \Bigg\| \int_{\substack{|\xi_j|\approx N_j,\\ \xi=\sum_{j=1}^3 \xi_j}} |\xi_{2,3}|^{-(d-\gamma)} \hat{u}(\xi_1)\hat{u}(\xi_2)\hat{u}(\xi_3)\,d\xi_1\,d\xi_2\,d\xi_3 \Bigg\|_{L_t^1 L_\xi^2} \\
&\lesssim m_1 N_1 \|u_1\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_2\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} \|u_3\|_{L_t^3 L_x^{\frac{6d}{3d-6\gamma+8}}} \\
&\lesssim m_1 N_1 N_3^{\gamma-2} \prod_{J=1}^3 \|u_j\|_{X^{0,\frac12+}}.
\end{aligned}$$

It suffices to show that

$$m_1 N_1 N_3^{\gamma-2} \lesssim N^{-1+} N_2^{0-} m_1 N_1 m_2 N_2 m_3 N_3.$$

We reduce to show that

$$N^{1-} N_2^{0+} \lesssim m_2 N_2 m_3 N_3 N_3^{2-\gamma}.$$

This is true since for $s \geqslant \gamma - 2$, we have

$$m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim m_2 N_2 \gtrsim N^{1-} N_2^{0+}$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-decreasing if $s+p \geqslant 1$. While for $\frac{\gamma}{2} - 1 < s < \gamma - 2$, we have

$$m_2 N_2 m_3 N_3^{3-\gamma} \gtrsim m_2 N_2 m_2 N_2^{3-\gamma} \gtrsim N^{4-\gamma-} N_2^{0+} \gtrsim N^{1-} N_2^{0+}$$

where we used the fact that $m(\xi)\langle\xi\rangle^p$ is monotone non-increasing if $s+p<1$. $\square$

5. Proof of Theorem 1.1

We first scale the solution. Suppose that $u(t,x)$ is a global in time solution to (1.1) with initial data $u_0 \in C_0^\infty(\mathbb{R}^d)$. Setting

$$u^\lambda(t,x) = \lambda^{-\frac{n+2-\gamma}{2}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right),$$

we choose a parameter λ so that $\|Iu_0^\lambda\|_{H^1} = O(1)$, that is

$$\lambda \approx N^{\frac{1-s}{s-\gamma/2+1}}. \tag{5.1}$$

Next, let us define

$$S := \left\{0 \leqslant t < \infty:\ \left\|Iu^\lambda\right\|_{L^4 \dot{H}^{-\frac{d-3}{4},4}([0,t]\times\mathbb{R}^d)} \leqslant K\lambda^{\frac34(\frac{\gamma}{2}-1)}\right\},$$

with K a constant to be chosen later. We claim that S is the whole interval $[0,\infty)$. Indeed, assume by contradiction that it is not so, then since

$$\left\|Iu^\lambda\right\|_{L^4 \dot{H}^{-\frac{d-3}{4},4}([0,t]\times\mathbb{R}^d)}$$

is a continuous function of time, there exists a time $T \in [0, \infty)$ such that

$$\|Iu^\lambda\|_{L^4 \dot H^{-\frac{d-3}{4},4}([0,T]\times\mathbb{R}^d)} > K\lambda^{\frac{3}{4}(\frac{\gamma}{2}-1)}, \tag{5.2}$$

$$\|Iu^\lambda\|_{L^4 \dot H^{-\frac{d-3}{4},4}([0,T]\times\mathbb{R}^d)} \leqslant 2K\lambda^{\frac{3}{4}(\frac{\gamma}{2}-1)}. \tag{5.3}$$

We now split the interval $[0, T]$ into subintervals J_k, $k = 1, \ldots, L$, in such a way that

$$\|Iu^\lambda\|^4_{L^4 \dot H^{-\frac{d-3}{4},4}(J_k\times\mathbb{R}^d)} \leqslant \mu,$$

with μ as in Proposition 3.3. This is possible because of (5.3). Then, the number L of possible subintervals must satisfy

$$L \approx \frac{(2K\lambda^{\frac{3}{4}(\frac{\gamma}{2}-1)})^4}{\mu} \approx \frac{(2K)^4\lambda^{3(\frac{\gamma}{2}-1)}}{\mu}. \tag{5.4}$$

From Propositions 3.2 and 3.3, we know that

$$\sup_{t\in[0,T]} E\big(Iu^\lambda(t)\big) \lesssim E(Iu_0^\lambda) + \frac{L}{N^{1-}}$$

and by our choice (5.1) of λ, $E(Iu_0^\lambda) \lesssim 1$. Hence, in order to guarantee that

$$E\big(Iu^\lambda(t)\big) \lesssim 1$$

holds for all $t \in [0, T]$, we need to require that

$$L \lesssim N^{1-}.$$

According to (5.4), this is fulfilled as long as

$$\frac{(2K)^4\lambda^{3(\frac{\gamma}{2}-1)}}{\mu} \lesssim N^{1-}. \tag{5.5}$$

From our choice of λ, the expression (5.5) implies that

$$\frac{(2K)^4}{\mu} \lesssim N^{1-\frac{1-s}{s-\gamma/2+1}3(\frac{\gamma}{2}-1)-}.$$

Thus this is possible for $s > \frac{4(\gamma-2)}{3\gamma-4}$ and a large number N.

Now recall the a priori estimate (4.2)

$$\big\||\nabla|^{-\frac{d-3}{4}} Iu^\lambda\big\|^4_{L_T^4 L_x^4} \lesssim \|Iu^\lambda\|_{L_T^\infty \dot H_x^1}\|Iu^\lambda\|^3_{L_T^\infty L_x^2} + \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\mathrm{bad}}, Iu^\lambda(t,x_1)Iu^\lambda(t,x_2)\big\}_p \, dx_1\, dx_2\, dt.$$

Set

$$Error(t) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla a \cdot \big\{\widetilde{\mathcal{N}}_{\mathrm{bad}}, Iu^\lambda(t,x_1)Iu^\lambda(t,x_2)\big\}_p \, dx_1\, dx_2.$$

By Theorem 4.1 and Proposition 3.3 on each interval J_k, we have that

$$\left|\int_{J_k} Error(t)\,dt\right| \lesssim \frac{1}{N^{1-}}\|u^\lambda\|^6_{Z_I(J_k)} \lesssim \frac{1}{N^{1-}}.$$

Summing all the J_k's, we have that

$$\left|\int_0^T Error(t)\,dt\right| \lesssim \frac{L}{N^{1-}} \lesssim N^{0+}.$$

Therefore, by our choice (5.1) of λ, we obtain

$$\big\||\nabla|^{-\frac{d-3}{4}} Iu^\lambda\big\|^4_{L^4_T L^4_x} \lesssim \big\|Iu^\lambda\big\|_{L^\infty_T \dot H^1_x}\big\|Iu^\lambda\big\|^3_{L^\infty_T L^2_x} + N^{0+} \lesssim C\lambda^{3(\frac{\gamma}{2}-1)}.$$

This estimate contradicts (5.2) for an appropriate choice of K. Hence $S=[0,\infty)$. In addition, let T_0 be chosen arbitrarily, we have also proved that for $s > \frac{4(\gamma-2)}{3\gamma-4}$,

$$\big\|Iu^\lambda\big(\lambda^2 T_0\big)\big\|_{H^1_x} = O(1).$$

Then

$$\begin{aligned}\big\|u(T_0)\big\|_{H^s} &= \big\|u(T_0)\big\|_{L^2} + \big\|u(T_0)\big\|_{\dot H^s}\\ &= \|u_0\|_{L^2} + \lambda^{s-\frac{\gamma}{2}+1}\big\|u^\lambda\big(\lambda^2 T_0\big)\big\|_{\dot H^s}\\ &\lesssim \lambda^{s-\frac{\gamma}{2}+1}\big\|Iu^\lambda\big(\lambda^2 T_0\big)\big\|_{H^1} \lesssim \lambda^{s-\frac{\gamma}{2}+1} \approx N^{1-s}.\end{aligned}$$

Since T_0 is arbitrarily large, the a priori bound on the H^s norm concludes the global well-posedness of the Cauchy problem (1.1).

Note that we have obtained that

$$\|Iu\|_{L^4 \dot H^{-\frac{d-3}{4},4}([0,+\infty)\times\mathbb{R}^d)} \leqslant C\big(\|u_0\|_{H^s}\big),$$

this together with Propositions 2.2, 3.3 and the property of the operator I imply that

$$\sup_{(q,r)\ \text{admissible}} \big\|\langle\nabla\rangle^s u\big\|_{L^q L^r([0,+\infty)\times\mathbb{R}^d)} \leqslant \|u\|_{Z_I([0,+\infty))} \lesssim C\big(\|Iu_0\|_{H^1}\big) \lesssim C\big(\|u_0\|_{H^s}\big).$$

The scattering result can be obtained from the well-known standard argument [1,3]. This completes the proof.

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