# Erratum to: "The Schrödinger-Maxwell system with Dirac mass" [Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) 773-793] औै 

G.M. Coclite ${ }^{\mathrm{a}, \mathrm{b}}$, H. Holden ${ }^{\text {a,c, },}$<br>${ }^{\text {a }}$ Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, No-0316 Oslo, Norway<br>${ }^{\text {b }}$ Department of Mathematics, University of Bari, Via E. Orabona 4, I-70125 Bari, Italy<br>${ }^{c}$ Department of Mathematical Sciences, Norwegian University of Science and Technology, No-7491 Trondheim, Norway

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#### Abstract

We correct the proof of [G.M. Coclite, H. Holden, The Schrödinger-Maxwell system with Dirac mass, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) 773-793, Lemma 4.1]. © 2006 Elsevier Masson SAS. All rights reserved.


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## 1. Introduction

The proof of [1, Lemma 4.1] is incorrect. We here present a new proof. We employ the notation and assumptions of [1].

Lemma 4.1. Assume $\alpha$ and $\beta$ are positive constants. There exists a subsequence $\left\{v_{n_{k}}\right\}_{k \in \mathbb{N}}$ and a map $v \in H^{2}(\Omega) \cap$ $H_{0}^{1}(\Omega)$ such that

$$
\begin{equation*}
v_{n_{k}} \rightharpoonup v \quad \text { weakly in } H^{2}(\Omega) \cap H_{0}^{1}(\Omega) . \tag{1}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
v_{n_{k}} \longrightarrow v \quad \text { uniformly in } \Omega . \tag{2}
\end{equation*}
$$

Proof. We split the proof in two steps.

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Step 1. We claim that the following inequality

$$
\begin{equation*}
J_{n}\left(v_{n}\right)=\min _{v \in H_{0}^{1}(\Omega)} J_{n}(v) \leqslant 0 \tag{3}
\end{equation*}
$$

holds for infinitely many $n \in \mathbb{N}$.
Let $\varphi_{0}$ be the normalized positive first eigenfunction of $-\Delta$ on $\Omega$, namely $\varphi_{0}$ is the unique smooth map satisfying the following conditions

$$
\begin{cases}-\Delta \varphi_{0}=\omega_{0} \varphi_{0}, & \text { in } \Omega,  \tag{4}\\ \varphi_{0}>0, & \text { in } \Omega, \\ \varphi_{0}=0, & \text { on } \partial \Omega \\ \left\|\varphi_{0}\right\|_{L^{2}(\Omega)}=1 . & \end{cases}
$$

Since

$$
\int_{\Omega}\left|\nabla \varphi_{0}(x)\right|^{2} \mathrm{~d} x=\omega_{0} \int_{\Omega} \varphi_{0}^{2}(x) \mathrm{d} x=\omega_{0},
$$

we find, by evaluating $J_{n}$ in ${ }^{1} \lambda \operatorname{sign}\left(v_{n-1}\left(x_{0}\right)\right) \varphi_{0}, \lambda>0$, that (writing $\left.v_{n-1}=\operatorname{sign}\left(v_{n-1}\left(x_{0}\right)\right)\right)$

$$
\begin{aligned}
J_{n}\left(\lambda v_{n-1} \varphi_{0}\right)= & \frac{\lambda^{2}}{2} \int_{\Omega}\left|\nabla \varphi_{0}\right|^{2} \mathrm{~d} x+\lambda^{4} \frac{\alpha}{4} \int_{\Omega \times \Omega} G(x, y) \varphi_{0}^{2}(y) \varphi_{0}^{2}(x) \mathrm{d} x \mathrm{~d} y \\
& -\left|v_{n-1}\left(x_{0}\right)\right| \lambda^{3} \frac{\alpha}{\beta} \int_{\Omega \times \Omega} G(x, y) G\left(y, x_{0}\right) \varphi_{0}(y) \varphi_{0}^{2}(x) \mathrm{d} x \mathrm{~d} y \\
& +v_{n-1}^{2}\left(x_{0}\right) \lambda^{2} \frac{\alpha}{\beta^{2}} \int_{\Omega \times \Omega} G(x, y) G\left(y, x_{0}\right) G\left(x, x_{0}\right) \varphi_{0}(y) \varphi_{0}(x) \mathrm{d} x \mathrm{~d} y \\
& +v_{n-1}^{2}\left(x_{0}\right) \lambda^{2} \frac{\alpha}{2 \beta^{2}} \int_{\Omega \times \Omega} G(x, y) G^{2}\left(y, x_{0}\right) \varphi_{0}^{2}(x) \mathrm{d} x \mathrm{~d} y \\
& -\left|v_{n-1}^{3}\left(x_{0}\right)\right| \lambda \frac{\alpha}{\beta^{3}} \int_{\Omega \times \Omega} G(x, y) G^{2}\left(y, x_{0}\right) G\left(x, x_{0}\right) \varphi_{0}(x) \mathrm{d} x \mathrm{~d} y \\
& -\frac{\omega}{2} \lambda^{2} \int_{\Omega} \varphi_{0}^{2}(x) \mathrm{d} x-\left|v_{n-1}\left(x_{0}\right)\right| \lambda \frac{\omega}{\beta} \int_{\Omega} G\left(x, x_{0}\right) \varphi_{0}(x) \mathrm{d} x \\
= & \lambda^{2} \frac{\omega_{0}-\omega}{2}+\lambda^{4} \kappa_{1}-\left|v_{n-1}\left(x_{0}\right)\right| \lambda^{3} \kappa_{2}+v_{n-1}^{2}\left(x_{0}\right) \lambda^{2} \kappa_{3}-\left|v_{n-1}^{3}\left(x_{0}\right)\right| \lambda \kappa_{4}-\left|v_{n-1}\left(x_{0}\right)\right| \lambda \kappa_{5} .
\end{aligned}
$$

Due to the positivity and the boundedness of $\varphi_{0}$, we have $\kappa_{1}, \ldots, \kappa_{5}>0$, hence

$$
\begin{equation*}
J_{n}\left(\lambda v_{n-1} \varphi_{0}\right) \leqslant \lambda^{2} \frac{\omega_{0}-\omega}{2}+\lambda^{4} \kappa_{1}+v_{n-1}^{2}\left(x_{0}\right) \lambda^{2} \kappa_{3}-\left|v_{n-1}\left(x_{0}\right)\right| \lambda \kappa_{5} . \tag{5}
\end{equation*}
$$

We have the following two cases.
(i) If

$$
\liminf _{n}\left|v_{n-1}\left(x_{0}\right)\right|=0
$$

then there exists an $n_{0}$ such that, by passing to a subsequence and using, e.g., [1, (2.6)], we find

$$
\begin{equation*}
J_{n}\left(v_{n}\right)=\min _{v \in H_{0}^{1}(\Omega)} J_{n}(v) \leqslant \min _{\lambda>0} J_{n}\left(\lambda v_{n-1} \varphi_{0}\right) \leqslant \min _{\lambda>0}\left(\lambda^{2} \frac{\omega_{0}-\omega}{2}+\lambda^{4} \kappa_{1}\right)<0, \quad n>n_{0} \tag{6}
\end{equation*}
$$

[^1](ii) If
$$
0<\liminf _{n}\left|v_{n-1}\left(x_{0}\right)\right| \leqslant \infty,
$$
then there exists $n_{0}$ and $c_{1} \in\left(0,\left|v_{n-1}\left(x_{0}\right)\right|\right)$ for $n>n_{0}$ such that
\[

$$
\begin{align*}
J_{n}\left(v_{n}\right) & =\min _{v \in H_{0}^{1}(\Omega)} J_{n}(v) \leqslant \min _{\lambda>0} J_{n}\left(\lambda v_{n-1} \varphi_{0}\right) \\
& \leqslant \min _{\lambda>0}\left(\lambda^{2} \frac{\omega_{0}-\omega}{2}+\lambda^{4} \kappa_{1}+\left|v_{n-1}\left(x_{0}\right)\right|^{2} \lambda^{2} \kappa_{3}-c_{1} \lambda \kappa_{5}\right)<0, \quad n>n_{0} \tag{7}
\end{align*}
$$
\]

Clearly, (6) and (7) prove (3). So Step 1 is concluded.
Step 2. We prove (1) and (2). Clearly it suffices to prove that
the sequence $\left\{v_{n}\right\}_{n \in \mathbb{N}}$ is bounded in $H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$.
Due to [1, Lemmas 2.1 and 3.5] we only have to show that
the sequence $\left\{v_{n}\right\}_{n \in \mathbb{N}}$ is bounded in $H_{0}^{1}(\Omega)$,
the sequence $\left\{v_{n}\left(x_{0}\right)\right\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}$.
If, by contradiction, (9) does not hold, we have that

$$
\begin{equation*}
\limsup _{n}\left\|v_{n}\right\|_{H_{0}^{1}(\Omega)}=\infty \tag{11}
\end{equation*}
$$

Therefore, [1, Lemma 3.3] and a diagonal argument guarantee

$$
\begin{equation*}
\limsup _{n} J_{n}\left(v_{n}\right)=\infty \tag{12}
\end{equation*}
$$

Since, by construction, $v_{n}$ is a minimizer for $J_{n}$, Eq. (3) says

$$
\begin{equation*}
J_{n}\left(v_{n}\right)=\min _{f \in H_{0}^{1}(\Omega)} J_{n}(f) \leqslant 0, \quad n \in \mathbb{N} \tag{13}
\end{equation*}
$$

which contradicts (12). This proves (9).
We conclude by proving (10). Assume by contradiction that $\left\{v_{n}\left(x_{0}\right)\right\}_{n \in \mathbb{N}}$ is not bounded, namely (passing to a subsequence)

$$
\begin{equation*}
\lim _{n}\left|v_{n}\left(x_{0}\right)\right|=\infty \tag{14}
\end{equation*}
$$

Multiplying [1, (2.21)] by $u_{n}$, which is defined by [1, (2.20)], and integrating over $\Omega$ we get

$$
\int_{\Omega}\left|\nabla v_{n}(x)\right|^{2} \mathrm{~d} x+\frac{v_{n-1}\left(x_{0}\right)}{\beta} \int_{\Omega} G\left(x, x_{0}\right) \Delta v_{n}(x) \mathrm{d} x+\alpha \int_{\Omega \times \Omega} G(x, y) u_{n}^{2}(y) u_{n}^{2}(x) \mathrm{d} x \mathrm{~d} y=\omega \int_{\Omega} u_{n}^{2}(x) \mathrm{d} x .
$$

Integration by parts gives

$$
-\frac{v_{n-1}\left(x_{0}\right)}{\beta} \int_{\Omega} G\left(x, x_{0}\right) \Delta v_{n}(x) \mathrm{d} x=-\frac{v_{n-1}\left(x_{0}\right)}{\beta} \int_{\Omega} v_{n}(x) \Delta G\left(x, x_{0}\right) \mathrm{d} x=\frac{v_{n-1}\left(x_{0}\right) v_{n}\left(x_{0}\right)}{\beta}
$$

thus

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{n}(x)\right|^{2} \mathrm{~d} x+\alpha \int_{\Omega \times \Omega} G(x, y) u_{n}^{2}(y) u_{n}^{2}(x) \mathrm{d} x \mathrm{~d} y=\omega \int_{\Omega} u_{n}^{2}(x) \mathrm{d} x+\frac{v_{n}\left(x_{0}\right) v_{n-1}\left(x_{0}\right)}{\beta} . \tag{15}
\end{equation*}
$$

Introducing the notation

$$
\begin{aligned}
& a_{n}=\alpha \int_{\Omega \times \Omega} G(x, y)\left(\frac{v_{n}(y)}{v_{n-1}\left(x_{0}\right)}-\frac{G\left(y, x_{0}\right)}{\beta}\right)^{2}\left(\frac{v_{n}(x)}{v_{n-1}\left(x_{0}\right)}-\frac{G\left(x, x_{0}\right)}{\beta}\right)^{2} \mathrm{~d} x \mathrm{~d} y \\
& b_{n}=\omega \int_{\Omega}\left(\frac{v_{n}(x)}{v_{n-1}\left(x_{0}\right)}-\frac{G\left(x, x_{0}\right)}{\beta}\right)^{2} \mathrm{~d} x
\end{aligned}
$$

Eq. (15) reads (cf. [1, (2.20)])

$$
\begin{equation*}
\int_{\Omega}\left|\nabla v_{n}\right|^{2} \mathrm{~d} x+v_{n-1}^{4}\left(x_{0}\right) a_{n}=v_{n-1}^{2}\left(x_{0}\right) b_{n}+\frac{v_{n}\left(x_{0}\right) v_{n-1}\left(x_{0}\right)}{\beta} . \tag{16}
\end{equation*}
$$

Due to (9) and (14)

$$
\begin{equation*}
a_{n} \rightarrow \alpha \int_{\Omega \times \Omega} G(x, y) \frac{G^{2}\left(y, x_{0}\right)}{\beta^{2}} \frac{G^{2}\left(x, x_{0}\right)}{\beta^{2}} \mathrm{~d} x \mathrm{~d} y, \quad b_{n} \rightarrow \omega \int_{\Omega} \frac{G^{2}\left(x, x_{0}\right)}{\beta^{2}} \mathrm{~d} x . \tag{17}
\end{equation*}
$$

Passing to a subsequence we can assume that the sequence $\left\{v_{n}\left(x_{0}\right) / v_{n-1}^{3}\left(x_{0}\right)\right\}$ has a limit as $n \rightarrow \infty$. We have the following three cases.
(i) If

$$
\begin{equation*}
\lim _{n} \frac{v_{n}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right)}=\infty \tag{18}
\end{equation*}
$$

we divide (16) by $v_{n-1}^{4}\left(x_{0}\right)$

$$
\begin{equation*}
\int_{\Omega}\left|\frac{\nabla v_{n}}{v_{n-1}^{2}\left(x_{0}\right)}\right|^{2} \mathrm{~d} x+a_{n}=\frac{b_{n}}{v_{n-1}^{2}\left(x_{0}\right)}+\frac{v_{n}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right) \beta} \tag{19}
\end{equation*}
$$

Using (9), (14), and (17) in (19), we have

$$
\alpha \int_{\Omega \times \Omega} G(x, y) \frac{G^{2}\left(y, x_{0}\right)}{\beta^{2}} \frac{G^{2}\left(x, x_{0}\right)}{\beta^{2}} \mathrm{~d} x \mathrm{~d} y=\infty,
$$

which is a contradiction.
(ii) If

$$
\begin{equation*}
\lim _{n} \frac{v_{n}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right)}=0, \tag{20}
\end{equation*}
$$

we divide (16) by $v_{n-1}\left(x_{0}\right) v_{n}\left(x_{0}\right)$

$$
\begin{equation*}
\int_{\Omega} \frac{\left|\nabla v_{n}\right|^{2}}{v_{n-1}\left(x_{0}\right) v_{n}\left(x_{0}\right)} \mathrm{d} x+\frac{a_{n} v_{n-1}^{3}\left(x_{0}\right)-b_{n} v_{n-1}\left(x_{0}\right)}{v_{n}\left(x_{0}\right)}=\frac{1}{\beta} \tag{21}
\end{equation*}
$$

Since by (9), (14), and (17),

$$
\lim _{n} \frac{a_{n} v_{n-1}^{3}\left(x_{0}\right)-b_{n} v_{n-1}\left(x_{0}\right)}{v_{n}\left(x_{0}\right)}=\lim _{n} \frac{v_{n-1}^{3}\left(x_{0}\right)}{v_{n}\left(x_{0}\right)}\left(a_{n}-\frac{b_{n}}{v_{n-1}^{2}\left(x_{0}\right)}\right)=\infty,
$$

which implies, using (21), that $\infty=\frac{1}{\beta}$, which is a contradiction.
(iii) Finally, if

$$
\lim _{n} \frac{v_{n}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right)}=\ell \in(0, \infty)
$$

we observe that (see (14))

$$
\lim _{n} \frac{v_{n+1}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right)}=\lim _{n} \frac{v_{n+1}\left(x_{0}\right)}{v_{n}^{3}\left(x_{0}\right)} \frac{v_{n}\left(x_{0}\right)}{v_{n-1}^{3}\left(x_{0}\right)} v_{n}^{2}\left(x_{0}\right)=\ell^{2} \lim _{n} v_{n}^{2}\left(x_{0}\right)=\infty,
$$

therefore the subsequence $\left\{v_{2 n}\left(x_{0}\right)\right\}_{n \in \mathbb{N}}$ satisfies (18) and we get a contradiction.
Therefore (10) is proved.

## References

[1] G.M. Coclite, H. Holden, The Schrödinger-Maxwell system with Dirac mass, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) $773-793$.


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    * Corresponding author.

    E-mail addresses: coclitegm@dm.uniba.it (G.M. Coclite), holden@math.ntnu.no (H. Holden).
    URLs: http://www.dm.uniba.it/Members/coclitegm, http://www.math.ntnu.no/~holden/.

[^1]:    ${ }^{1}$ The point $x_{0}$ is where the point interaction is located, cf. [1].

