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# Asymptotic analysis of the $p$-Laplacian flow in an exterior domain ☆

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**Abstract**

We consider the Dirichlet problem for the $p$-Laplacian evolution equation, $u_t = \Delta_p u$, where $p > 2$, posed in an exterior domain in $\mathbf{R}^N$, with zero Dirichlet boundary condition and with integrable and nonnegative initial data. We are interested in describing the influence of the holes of the domain on the large time behaviour of the solutions. Such behaviour varies depending on the relative values of $N$ and $p$. We must distinguish between the behaviour near infinity of space (outer analysis), and near the holes (inner analysis). We obtain that the outer analysis is given in all cases by certain self-similar solutions and the inner analysis is given by quasi-stationary states. Logarithmic corrections to exact self-similarity appear in the critical case $N = p$, which is mathematically more interesting. In this first paper we treat only the cases $N > p$ and $N = p$, the case $N < p$ will be considered in a companion work.


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*Keywords:* $p$-Laplacian equation; Exterior domain; Asymptotic behaviour; Domain with holes; Matched asymptotics

## 1. Introduction and description of results

In this work we study the asymptotic behaviour of the solutions of the parabolic $p$-Laplacian equation in an exterior domain with Dirichlet boundary conditions. More precisely, let $G \subset \mathbf{R}^N$ be a bounded open set with smooth boundary (of class $C^{2,\alpha}$) and let $\Omega = \mathbf{R}^N \setminus G$. We think of $G$ as the "holes". We assume moreover that $\Omega$ is connected, which implies no essential loss of generality for our purposes. Thus, we consider the following problem:

$$\begin{cases} u_t = \Delta_p u, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where $p > 2$. On the initial data we make the assumptions that $u_0 \in L^1(\Omega)$ and it is nonnegative in $\Omega$. For most of the paper we also assume that $u_0$ has compact support in $\overline{\Omega}$. This $p$-Laplacian equation and its variants have been intensively studied in the last decades, on the one hand because it is a relevant example of nonlinear parabolic equation of degenerate type which leads to a very interesting and highly nontrivial theory, like the well-posedness and regularity questions described in DiBenedetto's book [4]; on the other hand, because it is one of the basic models on nonlinear diffusion appearing in applications that range from non-Newtonian fluids to image processing, see references in Chapter 11 of [18]. As it is well known, the $p$-Laplacian equation has the property of finite propagation speed for $p > 2$, hence for any time $t > 0$ the solutions with such initial data remain compactly supported in $\overline{\Omega}$ and this behaviour gives rise to a free boundary, which is the surface of separation between the region where $u > 0$ and the region where $u = 0$.

We are interested in describing the influence of the holes of the domain on the large time behaviour of the solutions. For a complete study of the asymptotic behaviour in an exterior domain, one has to perform two different steps in the analysis, typical of the technique of matched asymptotics. First, the *outer analysis* gives the asymptotic rates and profiles of the solutions in the far field near infinity. Afterwards, one has to perform the *inner analysis* of the problem, which means studying what happens in the region near the holes (more precisely in bounded subdomains).

In this paper we will analyze the asymptotic behaviour of general solutions of the $p$-Laplacian equations only in dimension $N \geqslant p$, the remaining case $1 \leqslant N < p$ being considered in a companion paper, [12], since it has special features that take space to develop. As we will show, there exists a difference too between the cases $N > p$ and the limit case $N = p$, where the results and the techniques used in the proofs are more involved and mathematically more interesting. No such division into ranges occurs for the $p$-Laplacian equation posed in the whole space or in a bounded domain, beyond the basic requirement that $p > 2$ that implies finite propagation speed. Therefore, the present division reflects the varying influence of the holes.

### 1.1. Preliminaries

In order to describe the asymptotic behaviour, we need to introduce some preliminary facts and results concerning the parabolic $p$-Laplacian equation. A very important class of solutions of the $p$-Laplacian equation consists of the so-called source-type solutions, which are very similar to the ZKB solutions of the porous medium equation. They have the form

$$B_C(x,t) = t^{-\alpha} F_C(y), \tag{1.2}$$

where $y = xt^{-\beta}$ and

$$F_C(y) = \left(C - k|y|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}. \tag{1.3}$$

The function $F_C$ is called the profile of the source-type solution, and the exponents $\alpha$ and $\beta$ are the self-similarity exponents. In our case they have the values

$$\alpha = \frac{N}{N(p-2)+p}, \qquad \beta = \frac{1}{N(p-2)+p} \tag{1.4}$$

and the parameter $k$ is also known, $k = ((p-2)/p)\beta^{\frac{1}{p-1}}$. The constant $C$ is a free parameter, that gives the height of the solution. We thus have a whole one-parameter family of solutions of the same type. We remark that the source-type solutions we have introduced satisfy the conservation law

$$\int_{\mathbf{R}^N} B_C(x,t)\,dx = \text{constant}$$

for all times. For convenience we call this integral the total mass of the solution, $M_C$. This is justified when we think of the equation as nonlinear diffusion of a substance with density $u$. It is also easy to see that the source-type solutions have as initial trace a Dirac mass, which is $M_C\delta(x)$. Moreover, the connection between the free parameter $C$ and the mass $M_C$ is given by

$$M_C = dC^{\gamma}, \tag{1.5}$$

where $\gamma = \frac{(p-1)N}{p(p-2)\alpha}$ and

$$d = N\omega_N \int_0^\infty \left(1 - ky^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} y^{N-1}\,dy.$$

For more details about the properties of the source-type solutions, the reader may consult the book [18]. On the other hand, it is proved in [11] that there exist many other self-similar solutions of the form

$$U(x,t) = t^{-\alpha}F(\xi), \quad \xi = xt^{-\beta} \tag{1.6}$$

having a compactly supported profile $F$.

Let us examine next some scaling invariance properties of the solutions of the $p$-Laplacian equation. Starting from an arbitrary solution $u$ of the $p$-Laplacian equation, we consider the family of scaled versions of $u$:

$$u_\lambda(x,t) = \lambda^\alpha u\left(\lambda^\beta x, \lambda t\right). \tag{1.7}$$

Then, by a straightforward calculation, one can check that starting from a solution $u$ of the $p$-Laplacian equation, we produce an entire family of solutions of the same equation that are zoomed versions of the initial one. We remark that all the self-similar solutions of the form (1.6) enjoy the nice property of being invariant to the scaling above.

We now introduce the weak formulation of the $p$-Laplace equation. Let $Q_T = \Omega \times (0,T]$.

**Definition 1.1.** A function $u \in C((0,T]: W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ is a weak solution of problem (1.1) on [0,T] if for any test function $\Phi \in C^{2,1}(\overline{Q_T})$ with compact support in $\overline{Q_T}$ and $\Phi = 0$ on $\partial\Omega \times (0,T]$, it satisfies the integral identity

$$\int_\Omega u(x,t)\Phi(x,t)\,dx = \int_0^t\int_\Omega \left(u(x,s)\Phi_s(x,s) - |\nabla u|^{p-2}\nabla u(x,s)\cdot\nabla\Phi(x,s)\right)dx\,ds + \int_\Omega u_0(x)\Phi(x,0)\,dx \tag{1.8}$$

for any $t \in [0,T]$. We say that $u$ is a weak solution of (1.1) on $[0,\infty)$ if there is a weak solution in the sense above on $[0,T]$ for any $T > 0$.

The definitions of weak sub- and supersolution follow as usual, by replacing in Definition 1.1 the equality by the corresponding inequalities $\leqslant$ or $\geqslant$ and considering only nonnegative test functions. We will also introduce the local weak solutions, i.e. weak solutions referred only to the equation, without considering the initial and boundary condition.

**Definition 1.2.** A function $u \in C((0,T]: W_0^{1,p}(\Omega)) \cap L^\infty(Q_T)$ is a local weak solution of problem (1.1) on $[0,T]$ if for any test function $\Phi \in C^{2,1}(Q_T)$ with compact support in $Q_T$, it satisfies the integral identity

$$\int_0^T\int_\Omega \left(u(x,t)\Phi_t(x,t) - |\nabla u|^{p-2}\nabla u(x,t)\cdot\nabla\Phi(x,t)\right)dx\,dt = 0. \tag{1.9}$$

The existence and uniqueness of solutions of the $p$-Laplacian equation has been widely investigated; a good reference is the book [4], where also the optimal regularity is studied. It can be showed that the nonnegative bounded weak solutions of the $p$-Laplacian equation are such that $u$, $|\nabla u| \in C^\alpha(Q')$ for some $\alpha > 0$ and $Q' \subset Q$, where $Q = \Omega \times (0,\infty)$.

We will often use in the text the notation $u(t)$ for the function $u(t)(x) = u(x,t)$. We will denote by $P_u$ the positivity set of $u$ and by $\Gamma(t) = \partial P_u(t) \setminus \partial\Omega$ the free boundary of $u$ at time $t$.

### *1.2. Outline of results*

We will describe in few words the main results of this work, in the two different cases $N > p$ and $N = p$.

*Case $N > p$.* We prove by a scaling argument that the outer analysis is given by the profile of a particular source-type solution, of the form (1.6) and with the exponents given by (1.4). We calculate the constant $C_0$ that identifies the profile inside the family $F_C$ and prove that the rescaled function $v(x,t) = t^{\alpha}u(x,t)$ converges to $F_{C_0}$ uniformly in outer sets of the form $|x| \geqslant \delta$. We point out that there seems to be no conservation law from which the asymptotic constant $C_0$ may be derived a priori

For the inner analysis, we make a different scaling and we prove that $v(x,t)$ converges to a stationary state, which is related to the unique solution of the following exterior Dirichlet problem:

$$\begin{cases} \Delta_p H = 0 & \text{in } \Omega, \\ H = 0 & \text{on } \partial\Omega, \\ H \to 1 & \text{uniformly as } |x| \to \infty, \end{cases} \tag{1.10}$$

by multiplying it by a constant $C > 0$. To find this constant we use the technique of matched asymptotics and this will lead us also to a global formulation of the result. The study is performed in Section 2 and the main results are Theorem 2.1 for the outer analysis and Theorem 2.2 for the inner analysis. The global formulation is expressed by Theorem 2.3.

*Case $N = p$.* The analysis of this borderline case is more involved, and we use further techniques from dynamical systems. This makes it mathematically more interesting. The asymptotic profile will be similar to the one of a source-type solution, but we have to introduce a logarithmic correction in order to insure that the total mass disappears in the end. We will get a profile of the form

$$U(x,t) = t^{-\alpha}\left(C(t) - k\left(\frac{|x|}{t^{\beta}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{1.11}$$

where the dependence of the "free parameter" in time is given by

$$C(t) = C_0(\log t)^{-\frac{p-2}{(p-1)^2}}. \tag{1.12}$$

Here the self-similarity exponents become

$$\alpha = \frac{1}{p-1}, \qquad \beta = \frac{1}{p(p-1)}.$$

We deduce that in the critical case, the solution decays in time like $C_1(t\log t)^{-1/(p-1)}$ and its support expands like $|x| \sim C_2 t^{\beta}(\log t)^{-(p-2)/p(p-1)}$. Using (1.5), this gives a mass variable in time with the law $M(t) = C/\log(t)$. We prove that the outer asymptotic behaviour of general solutions is given by a profile of this type. The inner analysis is relatively similar to that of the preceding case and uses also the general idea of matched asymptotics. The study is performed in Section 3 and the main results are Theorem 3.1 for the outer analysis and Theorem 3.2 for the inner analysis. The global formulation is expressed by Theorem 3.3.

### *1.3. Precedents*

A complete study of the asymptotic behaviour of the $p$-Laplacian equation posed in the whole space was done by Kamin and Vázquez in [15]. There are a number of works on the problem of evolution in a domain with holes when the equation is the linear heat equation or the porous medium equation. In the case of the linear heat equation the analysis is made easier by the possibility of using integral representation of the solutions, cf. Ishige [13] and [14]. In the case of the porous medium equation, the asymptotic behaviour in the whole space is well known cf. [16,20], while the asymptotic behaviour for the Dirichlet problem with zero boundary condition in domains with holes was treated by Brandle at al. [2] and by Gilding and Gonzerkiewicz in [9] and [10]. In comparison with these works, the absence of a conservation law makes the asymptotic analysis in the $p$-Laplacian case more involved. On the other hand, Quirós and Vazquez [17] had treated the case of non-homogeneous boundary conditions and showed that the asymptotical results are quite different.

## 2. Case of large dimensions, $N > p$

The analysis is divided into outer and inner analysis. The former follows the outline of the proof of paper [2] for the porous medium equation, hence we will be rather sketchy. We will devote more effort to the inner analysis and the critical case $N = p$.

### 2.1. Sub- and supersolutions. Outer analysis

In this subsection we describe some appropriate sub- and supersolution that will have the same decay in time as the general solution. The construction is based on the source-type solutions presented above, but with some necessary changes.

#### 2.1.1. Supersolutions

As supersolutions, we will consider the Barenblatt functions $B_C$ already defined, with a certain delay in time, $U_{C,\tau}(x,t) = B_C(x, t+\tau)$, $\tau > 0$. It is well known that they are weak solutions of the $p$-Laplace equation and they become supersolutions for the problem (1.1), since they are positive on the boundary of the hole $\partial G$. Moreover, by well-known comparison arguments, for any compactly supported solution $u$ of the $p$-Laplacian equation, there exist constants $C, \tau > 0$ such that $u(x,t) \leqslant B_C(x, t+\tau)$ at any time. We recall that the parameter $C > 0$ is related to the constant mass $M_C$ of the Barenblatt function by

$$C = c(p, N) M^{\frac{p(p-2)}{p-1}\beta}.$$

#### 2.1.2. Subsolutions

Defining subsolutions is more involved, and we will follow a general idea of construction that has been used in the paper [2] for the porous medium equation. We remark that the Barenblatt functions, although they have a good behaviour at infinity, cannot be used as subsolutions of the boundary value problem, since they are positive on $\partial G$. The idea we follow is to consider another local subsolution, which is good near the hole, and then to combine them. A good starting point is to consider a function with separated variables, whose $x$-part is the fundamental solution of the $p$-Laplace operator, and the part in $t$ has the expected decay. We define:

$$U(x,t) = Ct^{-\alpha}\left(1 - \left(\frac{R}{|x|}\right)^{\frac{N-p}{p-1}}\right)_+. \tag{2.1}$$

By choosing $R$ such that $G \subset B(0,R)$, we get the desired behaviour of $H$ near $\partial G$. To combine these functions, we assume a delay in time $\tau > 0$ in order to avoid problems at $t = 0$ and we change $H$ in order to be dominated by the Barenblatt function far from the hole. We set:

$$U_\tau(x,t) = C(t)(t+\tau)^{-\alpha}\left(1 - \left(\frac{R}{|x|}\right)^{\frac{N-p}{p-1}} - a\frac{(|x|-r)_+^4}{(t+\tau)^l}\right)_+, \tag{2.2}$$

$$B_{C_0,\tau}(x,t) = (t+\tau)^{-\alpha}\left(C_0 - k\left(\frac{|x|}{(t+\tau)^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}, \tag{2.3}$$

where $R$, $r$, $a$, $C_0$ and $l$ are positive parameters, which are free for the moment. We observe that both subsolutions have free boundaries and we denote by $R_1(t)$ and $R_2(t)$ the radii of their free boundaries. We choose $C(t) = K(1 + (t+\tau)^{-\sigma})$, where $\sigma > 0$. We remark that

$$\max B_{C_0,\tau} = C_0^{(p-1)/(p-2)}(t+\tau)^{-\alpha}, \quad \forall t > 0,$$

and obviously

$$\max U_\tau \geqslant K\big(1 + (t+\tau)^{-\sigma}\big)(t+\tau)^{-\alpha}\left(1 - \left(\frac{R}{r}\right)^{(N-p)/(p-1)}\right).$$

We choose $r > R$ and we can insure that $\max B_{C_0,\tau} \leqslant \max U_\tau$ at $|x| = r$ by choosing $K$ sufficiently large. In this way the two subsolutions will intersect each other in a point $r^*(t)$ depending on time and after that intersection we insure that the Barenblatt subsolution dominates and $r^*(t) \leqslant R_1(t)$. Now we can finally define our family of subsolutions:

$$V_{C_0,\tau}(x,t) = \begin{cases} 0, & \text{if } |x| < R \text{ or } |x| > R_2(t), \\ U_\tau(x,t) & \text{if } R \leqslant |x| \leqslant r^*(t), \\ B_{C_0,\tau} & \text{if } r^*(t) \leqslant |x| \leqslant R_2(t). \end{cases} \tag{2.4}$$

It is easy to check that $V_{C_0,\tau}$ is a subsolution for sufficiently large times $t > t_0 > 0$, provided $0 < \sigma < l - 1$. The next technical result, whose proof follows exactly the same lines as the proof of Lemma 3.1 in [2], shows that this rather complicated construction is good for our purposes.

**Proposition 2.1.** *For any solution $u(x,t)$ of* (1.1)*, there exists a choice of the parameters $C_0$, $\tau$, $a$, $R$, $r$ and a time $t_0 > 0$ such that for any time $t > t_0$ and $x \in \Omega$ we have $V_{C_0,\tau}(x,t) \leqslant u(x,t)$.*

With these constructions, we can pass to the study of the outer analysis.

**Theorem 2.1.** *For $N > p$, if $u$ is a weak solution of the problem* (1.1)*, there exists a constant $C_0 > 0$ such that*

$$\lim_{t\to\infty} t^{\alpha}\left|u(x,t) - B_{C_0}(x,t)\right| = 0 \tag{2.5}$$

*uniformly far from the hole, i.e. on sets of the form $\{|x| \geqslant \delta t^{\beta}\}$, where $\delta > 0$ is sufficiently small.*

**Proof.** We follow the general program proposed by one of the authors in [19] for studying the asymptotic behaviour of the nonlinear diffusion problems. This program has four different steps: in the first step we consider the family of rescaled solutions $u_\lambda$ and we obtain compactness estimate for it. As a consequence, there exists a limit point $u_\infty$ of $u_\lambda$. In the second step we prove that any limit point is a Barenblatt function. In the third step we prove that the convergence along subsequences is uniform on compact sets.

The proof of the first three steps is very similar to that of Theorem 2.1 in [2]. Let us remark only here that, since the hole shrinks to the point $x = 0$ after rescaling and passing to the limit $\lambda \to \infty$, a possible bounded singularity may develop at $x = 0$ for the limit solution, but when $N > p$ this singularity is removable, which allows us to work as in the whole space. This fact makes also an important difference between the cases $N > p$ and $N \leqslant p$, where the hole, even reduced to only one point, has its influence. We dare ask the reader to verify these assertions by consulting the calculations of [2].

Summing up, we obtain from the compactness estimates that the sequence of rescaled versions of $u$ converges along a sequence of times $t_k \to \infty$ to a weak solution $U$ of the $p$-Laplace equation, defined in $\mathbf{R}^N \times (0,\infty)$, which lies between two Barenblatt functions with zero delay. This means that the initial trace has to be a nonnegative measure supported at $x = 0$, hence a multiple of the Dirac delta. By the uniqueness theorem of [15], $U$ equals $B_{C_0}(x,t)$ for some $C_0 > 0$.

We still have to prove the independence of the limit w.r.t. the subsequence of times. Since the conservation law used in [2] does not hold in our case, we will prove differently this last step. From the previous steps, we already have that $u(x,t_k) \sim B_{C_0}(x,t_k)$ as $k \to \infty$ on some subsequence of time. Suppose that there exists two subsequences $t_{k,1}$ and $t_{k,2}$ such that

$$\lim_{k\to\infty} t_{k,1}^{\alpha}\left|u(x,t_{k,1}) - B_{C_1}(x,t_{k,1})\right| = 0 \tag{2.6}$$

and

$$\lim_{k\to\infty} t_{k,2}^{\alpha}\left|u(x,t_{k,2}) - B_{C_2}(x,t_{k,2})\right| = 0 \tag{2.7}$$

uniformly on sets of type $\{x \in \Omega\colon\ |x| \geqslant \delta t_{k,1}^{\beta}\}$, resp. $\{x \in \Omega\colon\ |x| \geqslant \delta t_{k,2}^{\beta}\}$, $\delta > 0$, where $C_1$, $C_2$ are positive constants.

Let $M(t) = \int_\Omega u(x,t)\,dx$ be the mass at time $t$. It is well known that, since we have homogeneous Dirichlet boundary conditions, the mass is decreasing in time. Hence, there exists $M = \lim_{t\to\infty} M(t)$. The explicit lower bound (subsolution) implies that this *asymptotic mass* $M > 0$. With this information we can identify the limit $U$. Indeed, we pass to the limit in the relations (2.6) and (2.7), but in the renormalized variable $y = xt^{\beta}$. In this variable, these two relations are written as

$$\left|u_{t_{k,1}}(y,1) - B_{C_1}(y,1)\right| \to 0, \qquad \left|u_{t_{k,2}}(y,1) - B_{C_2}(y,1)\right| \to 0, \tag{2.8}$$

with pointwise convergence in $\mathbf{R}^N$ and uniform convergence in sets of the form $\{|y| \geqslant \delta\}$ with $\delta > 0$ small. By integrating in $y$ in (2.8) and using the dominated convergence theorem, we obtain that the mass of the Barenblatt solutions $B_{C_1}(\cdot,1)$ and $B_{C_2}(\cdot,1)$ is the same, i.e. $M = M_{C_1} = M_{C_2}$. This implies $C_1 = C_2$, hence we have a unique limit independent on the subsequence. □

We remark that this argument does not allow for a quantitative estimates concerning the mass lost in the evolution. We only obtain the correct decay in time and the profile.

### 2.2. An elliptic a priori bound

In this part, we present an a priori bound for the solutions of the inhomogeneous Dirichlet problem for the $p$-Laplace operator. This result will be useful in the study of the inner behaviour of the general solutions of the evolution equation.

**Lemma 2.1.** *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain, $f \in C(\Omega) \cap L^\infty(\Omega)$ and $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be the solution of the Dirichlet problem*:

$$\begin{cases} \Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

*Then there exists a constant $C > 0$, independent on the diameter of $\Omega$, such that*

$$|u| \leqslant C d^{\frac{p}{p-1}} \Big(\sup_{\Omega} |f|\Big)^{\frac{1}{p-1}} \quad \text{in } \Omega, \tag{2.10}$$

*where $d = \operatorname{diam}(\Omega)$.*

**Proof.** Suppose for example that $\Omega$ lie in the slab $0 < x_1 < d$, otherwise we arrive to this situation by a translation and a rotation. We rescale in order to pass to a domain with diameter one, by setting

$$\hat{u}(y) = u(dy), \quad y \in \Omega_1, \tag{2.11}$$

where $\Omega_1 = \frac{1}{d}\Omega$, hence obviously $\operatorname{diam}(\Omega_1) = 1$ and $\Omega_1$ lies in the slab $0 < x_1 < 1$. The Dirichlet problem for $u$ transforms into

$$\begin{cases} \Delta_p \hat{u} = d^p f & \text{in } \Omega_1, \\ \hat{u} = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{2.12}$$

Denote by $\hat{f} = d^p f$. We will obtain in what follows an a priori bound for the problem (2.12). In order to prove this estimate, we use the following comparison principle which is a consequence of a Picone-type inequality and can be found in detail in [1] (see also [3]):

**Lemma 2.2.** *Let $g$ be a nonnegative continuous function such that $g(u)/u^{p-1}$ is a decreasing function. If $u, v \in C^1(\Omega) \cap C(\overline{\Omega})$ are such that*

$$\begin{cases} -\Delta_p v \geqslant g(v), & v > 0 \text{ in } \Omega,\ v \geqslant 0 \text{ on } \partial\Omega, \\ -\Delta_p u \leqslant g(u), & u \geqslant 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.13}$$

*then $u \leqslant v$ in $\Omega$.*

We apply the previous lemma for our function $\hat{u}$ and the function $v$ defined by

$$v(x) = \big(e^K - e^{Kx_1}\big)\Big(\sup_{\Omega_1} |\hat{f}|\Big)^{\frac{1}{p-1}}.$$

We remark that $v$ satisfies the assumptions of the lemma, and that

$$-\Delta_p v = K^p e^{K(p-1)x_1} \sup_{\Omega_1} |\hat{f}|.$$

On the other hand, we have $-\Delta_p \hat{u} = -\hat{f} \leqslant \sup_{\Omega_1} |\hat{f}|$, hence we take the constant $\sup_{\Omega_1} |\hat{f}|$ as the function $g$ in Lemma 2.2. Since $\hat{u}$ does not satisfy the nonnegativity assumption, we define

$$\Omega_{1,+} = \big\{x \in \Omega_1:\ \hat{u}(x) > 0\big\}, \qquad \Omega_{1,-} = \big\{x \in \Omega_1:\ \hat{u}(x) < 0\big\}$$

and we apply Lemma 2.2 twice, for $\hat{u}_+ = \max\{\hat{u}, 0\}$ in $\Omega_{1,+}$, and for $\hat{u}_- = -\min\{0, \hat{u}\}$ in $\Omega_{1,-}$. Here the conditions of the lemma are all satisfied and it is enough to choose the number $K$ in the definition of $v$ such that $K^p e^{K(p-1)x_1} > 2$, for all $x \in \Omega_1$. From Lemma 2.2 we obtain the estimate

$$|\hat{u}| \leqslant C\Big(\sup_{\Omega_1}|\hat{f}|\Big)^{\frac{1}{p-1}}$$

in $\Omega_1$. Rephrasing this result, we obtain (2.10), with a constant $C$ independent of the diameter, as stated. □

**Remark.** For $p = 2$, we obtain an improvement of the result for the Poisson equation, presented as Theorem 3.7 in the book [8].

We are ready to prove the result that is needed in the sequel.

**Proposition 2.2.** *If $u \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$\begin{cases} |\Delta_p u| \leqslant \varepsilon & \text{in } \Omega, \\ |u| \leqslant \varepsilon & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

*then $|u| \leqslant Cd^{p/(p-1)}\varepsilon^{1/(p-1)} + \varepsilon$ in $\Omega$, where $d$ is the diameter of $\Omega$ and $C > 0$ is a constant independent on the diameter of $\Omega$.*

**Proof.** We rescale as before and, using the same notations, we choose in the proof of Lemma 2.1 the function $v = v_0 + \sup_{\partial\Omega_1}|\hat{u}|$, where $v_0(x) = (e^K - e^{Kx_1})(\sup_{\Omega_1}|\hat{f}|)^{\frac{1}{p-1}}$. After rephrasing, we obtain the general a priori bound

$$|u| \leqslant Cd^{\frac{p}{p-1}}\Big(\sup_{\Omega}|f|\Big)^{\frac{1}{p-1}} + \sup_{\partial\Omega}|u|. \tag{2.15}$$

where $C > 0$ does not depend on $d$. Using this estimate, the corollary follows easily. □

### *2.3. Inner analysis*

In this section we will study the asymptotic behaviour of the solutions of the $p$-Laplacian equation near the holes. We start with some formal calculations in order to guess the correct asymptotic behaviour. Set $v = t^\alpha u$. Then, the function $v$ satisfies the equation

$$\Delta_p v = t^{-p\beta}(tv_t - \alpha v). \tag{2.16}$$

If we suppose for the moment that the terms in the right-hand side converge to 0 as $t \to \infty$, then the limit of $v$ is expected to be a solution of the following problem:

$$\begin{cases} \Delta_p v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

Let $\Phi_p$ be the solution of the stationary problem:

$$\begin{cases} \Delta_p \Phi_p = 0 & \text{in } \Omega, \\ \Phi_p = 1 & \text{in } \partial\Omega, \\ \Phi_p \to 0 & \text{as } |x| \to \infty \text{ uniformly}, \end{cases} \tag{2.18}$$

and set $H_p = 1 - \Phi_p$. The existence and uniqueness of $\Phi_p$ can be easily established, since in a radial domain we have the explicit expression, and for a general exterior domain it can be proved by approximation with an expanding sequence of bounded domains and comparison with radial sub- and supersolutions. Moreover, all the solutions of the problem (2.17) are of the form $CH_p$ for some constant $C$.

In order to find the correct constant $C$ we will use the idea of matched asymptotics, taking into account the result that we have for the outer behaviour. The general principle of this technique is that the outer limit of the inner expansion should coincide with the inner limit of the outer expansion. Since

$$\lim_{t\to\infty}\big|t^\alpha u(x,t) - t^\alpha B_{C_0}(x,t)\big| = 0, \quad \forall x \in \Omega,$$

and

$$\lim_{t\to\infty}\left|t^{\alpha}u(x,t)-CH_p(x)\right|=0 \quad \forall x\in\Omega,$$

and the convergence in both cases is uniform on compact sets, it formally follows that the unique possibility is $C=C_0^{(p-1)/(p-2)}$.

In what follows we prove rigourously that the inner behaviour is given by the stationary state $C_0^{(p-1)/(p-2)}H_p(x)$, where $C_0$ is the constant giving the outer behaviour of the solution $u$. For this, we first change the scale of time, by setting $\tau=\log t$ and $w(x,\tau)=v(x,t)=e^{\alpha\tau}u(x,e^{\tau})$. Consider the time averages:

$$W_T(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}w(x,s)\,ds.$$

**Proposition 2.3.** *For any $\varepsilon>0$ and $T>0$, there exists a constant $\delta=\delta(\varepsilon,T)>0$ and a large time $\tau_{in}=\tau_{in}(\varepsilon,\delta,T)$ such that for any $\tau\geqslant\tau_{in}$ we have*

$$\left|W_T(x,\tau)-C_0^{\frac{p-1}{p-2}}H_p(x)\right|\leqslant\varepsilon, \tag{2.19}$$

*for all $x$ with $|x|\leqslant\delta e^{\beta\tau}$.*

**Proof.** From the outer limit result, we deduce that for any $\delta>0$ and $\varepsilon>0$, there exists a time $\tau_0=\tau_0(\delta,\varepsilon)$ such that

$$F_{C_0}(\delta)-\varepsilon\leqslant w(x,\tau)\leqslant F_{C_0}(\delta)+\varepsilon, \tag{2.20}$$

for all $\tau\geqslant\tau_0$ and $|x|=R(\tau)$, where $R(\tau)=\delta e^{\tau\beta}$. Recall the notation $F_C$ from (1.3).

Set $\Omega_\tau=\Omega\cap B(0,R(\tau))$ and $g=w|_{\partial B(0,R(\tau))}$. Replacing the change of variable $\tau=\log t$ in (2.16), one obtains:

$$\Delta_p w(x,\tau)=e^{-\tau p\beta}(w_\tau-\alpha w). \tag{2.21}$$

We also remark that the solution $w(x,\tau)$ of (2.21) is positive for $\tau\geqslant\tau_0$ sufficiently large, hence it is in $C^{1,\alpha}(\Omega_\tau)$ at any time $\tau\geqslant\tau_0$. This regularity allows us to make all the calculations above in a rigourous manner and to use Proposition 2.2.

The key idea is to consider the function

$$\Phi(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}w(x,s)\,ds-C_0^{\frac{p-1}{p-2}}H_p(x) \tag{2.22}$$

and to derive estimates on $\Phi$ in order to arrive to the conditions of Proposition 2.2. For the beginning we remark that, in weak sense, it holds:

$$\begin{cases}\Delta_p\Phi=\frac{1}{T}\int_\tau^{\tau+T}e^{-sp\beta}(w_s-\alpha w)\,ds & \text{in } \Omega_\tau,\\ \Phi=0 & \text{on } \partial\Omega\cap\partial\Omega_\tau,\\ \Phi=h & \text{on } \partial\Omega_\tau\cap\partial B(0,R(\tau)),\end{cases} \tag{2.23}$$

where we regard $\tau$ as a frozen coefficient for the moment and we use (2.21). Here

$$h(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}g(x,s)\,ds-C_0^{\frac{p-1}{p-2}}H_p(x),$$

for $|x|=R(\tau)$. It follows that $F_{C_0}(\delta)-\varepsilon\leqslant\frac{1}{T}\int_\tau^{\tau+T}g(x,s)\,ds\leqslant F_{C_0}(\delta)+\varepsilon$ or, equivalently,

$$F_{C_0}(\delta)-C_0^{\frac{p-1}{p-2}}H_p(x)-\varepsilon\leqslant h(x,\tau)\leqslant F_{C_0}(\delta)-C_0^{\frac{p-1}{p-2}}H_p(x)+\varepsilon,$$

hence $|h(x,\tau)|\leqslant|F_{C_0}(\delta)-C_0^{(p-1)/(p-2)}H_p(x)|+\varepsilon$, for all $x$ with $|x|=R(\tau)$. We use next the following estimate:

$$\left|F_{C_0}(\delta)-C_0^{\frac{p-1}{p-2}}H_p(x)\right|\leqslant\left|\left(C_0-k\delta^{p/(p-1)}\right)_+^{\frac{p-1}{p-2}}-C_0^{\frac{p-1}{p-2}}\right|+\left|C_0^{\frac{p-1}{p-2}}-C_0^{\frac{p-1}{p-2}}H_p(x)\right|. \tag{2.24}$$

Since $H_p(x) \to 1$ uniformly as $|x| \to \infty$, for any $\varepsilon > 0$ and $\delta > 0$ one can choose $\tau_{in} = \tau_{in}(\varepsilon, \delta, T)$ sufficiently large such that $|C_0^{\frac{p-1}{p-2}}(1 - H_p(x))| \leqslant \frac{\varepsilon}{2}$ for all $x$ with $|x| = \delta e^{\beta\tau}$. On the other hand, there exists $\delta = \delta(\varepsilon, T) > 0$ such that $|(C_0 - k\delta^{p/(p-1)})^{(p-1)/(p-2)} - C_0^{(p-1)/(p-2)}| \leqslant \frac{\varepsilon}{2}$. From these it follows that $|h(x,\tau)| \leqslant 2\varepsilon$, for all $x$ such that $|x| = \delta e^{\beta\tau}$. Consequently, we obtain the estimate on the boundary:

$$\left|\Phi(x,\tau)\right| \leqslant 2\varepsilon, \quad \forall x \in \partial\Omega_\tau. \tag{2.25}$$

We remark from the previous estimates that the dependence of $\delta$ on $\varepsilon$ satisfies $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$. Hence, by enlarging $\tau_{in}$ we can obtain $\delta$ as small as we want.

Next we want to estimate $|\Delta_p\Phi|$ in $\Omega_\tau$. We start by an integration by parts:

$$\frac{1}{T}\int_\tau^{\tau+T} e^{-sp\beta} w_s\,ds = \frac{1}{T}e^{-(\tau+T)p\beta}w(x,\tau+T) - \frac{1}{T}e^{-\tau p\beta}w(x,\tau) + \frac{1}{T}\int_\tau^{\tau+T} p\beta e^{-sp\beta} w\,ds.$$

It follows that

$$\begin{aligned} \Delta_p\Phi(x,\tau) &= \frac{1}{T}e^{-\tau p\beta}\big(e^{-Tp\beta}w(x,T+\tau) - w(x,\tau)\big) + \frac{1}{T}(p-N)\beta\int_\tau^{\tau+T} e^{-sp\beta}w\,ds \\ &\leqslant \frac{1}{T}e^{-\tau p\beta}\big(e^{-Tp\beta}w(x,T+\tau) - w(x,\tau)\big), \end{aligned} \tag{2.26}$$

since $N > p$. Using the uniform boundedness of $w$ on $\partial\Omega_\tau$ with bounds which are independent of time, and Eq. (2.21) satisfied by $w$, we deduce that $w$ is uniformly bounded in $\Omega_\tau$. This implies

$$\Delta_p\Phi(x,\tau) \leqslant e^{-\tau p\beta}C \tag{2.27}$$

for all $\tau \geqslant \tau_{in}$ sufficiently large and $x \in \Omega_\tau$, where $C > 0$ is a constant independent on $\tau$. On the other hand, we need an estimate from below, which in view of the last inequalities, will follow from an estimate of the integral term $\Psi(x,\tau) = \frac{1}{T}(p-N)\beta\int_\tau^{\tau+T} e^{-sp\beta}w\,ds$. But this estimate follows easily since $w \geqslant 0$ and $w$ is uniformly bounded near the hole. Enlarging $C$, we finally obtain:

$$\left|\Delta_p(\Phi)\right| \leqslant e^{-\tau p\beta}C \tag{2.28}$$

in $\Omega_\tau$. From (2.28), (2.25) and Proposition 2.2, we obtain that

$$\left|\Phi(x,\tau)\right| \leqslant C\delta^{p/(p-1)}C^{1/(p-1)} + 2\varepsilon, \quad \forall x \in \Omega_\tau, \tag{2.29}$$

where $C > 0$ is a constant which does not depend on $\tau$. Hence, by choosing $\delta$ small, for $\tau \geqslant \tau_{in}$ very large, we obtain the estimate (2.19). □

We now have to pass from the convergence of the time averages to the convergence of the functions $w$ to the stationary state. The following theorem completes the result:

**Theorem 2.2.** *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a sufficiently large time $t_{in} = t_{in}(\varepsilon, \delta)$ such that*

$$\left|t^\alpha u(x,t) - C_0^{\frac{p-1}{p-2}}H_p(x)\right| \leqslant \varepsilon, \tag{2.30}$$

*for all $t \geqslant t_{in}$ and for all $x \in \Omega$ with $|x| \leqslant \delta t^\beta$.*

**Proof.** We argue by contradiction and suppose that there exists $(x_n, \tau_n)_n$ (using the previous notations) enjoying the properties:

$$\tau_n \to \infty, \quad \frac{|x_n|}{\delta e^{\beta\tau_n}} \to 0, \quad w(x_n,\tau_n) \geqslant C_0^{\frac{p-1}{p-2}}H_p(x_n) + \varepsilon.$$

Using the essential fact that $w_\tau \geqslant -Cw$ for some positive $C$, proved in [5], and integrating, we obtain

$$w(x_n, \tau_n + h) \geqslant w(x_n,\tau_n)e^{-Ch} \geqslant \big(C_0^{\frac{p-1}{p-2}}H_p(x_n) + \varepsilon\big)e^{-Ch}.$$

Integrating in $h$ in the previous inequality and performing the same calculations as in [2], we arrive to the inequality

$$W_T(x_n,\tau_n) \geqslant \left(C_0^{\frac{p-1}{p-2}} H_p(x_n) + \varepsilon\right)\frac{1-e^{-CT}}{CT} \geqslant C_0^{\frac{p-1}{p-2}} H_p(x_n) + \frac{\varepsilon}{2},$$

for $T$ sufficiently small. This is a contradiction with Proposition 2.3. □

### 2.4. Global formulation

In this paragraph we gather the results of Theorems 2.1 and 2.2 in a global approximation result. The global approximant should be such that the Barenblatt solution dominates near infinity.

**Theorem 2.3.** *Let $u$ be the solution of problem* (1.1) *and let*

$$U(x,t) = \left(B_{C_0}(x,t) - t^{-\alpha} C_0^{\frac{p-1}{p-2}}\left(1 - H_p(x)\right)\right)_+, \tag{2.31}$$

*where $C_0$ is the constant that appears in the previous sections. Then,*

$$\lim_{t\to\infty} t^\alpha \left|u(x,t) - U(x,t)\right| = 0 \tag{2.32}$$

*uniformly for $x \in \Omega$. Moreover, we have:*

$$\lim_{t\to\infty} \left\|u(x,t) - U(x,t)\right\|_{L^1(\Omega)} = 0. \tag{2.33}$$

*Both* (2.32) *and* (2.33) *can be extended to the whole class of solutions with initial data $u_0 \in L^1(\Omega)$.*

**Proof.** (i) We work first with compactly supported data. Using the obvious inequality:

$$t^\alpha \left|u(x,t) - U(x,t)\right| \leqslant \left|t^\alpha u(x,t) - C_0^{\frac{p-1}{p-2}} H_p(x)\right| + \left|C_0^{\frac{p-1}{p-2}} - t^\alpha B_{C_0}(x,t)\right|$$

and Theorem 2.2, we obtain the desired uniform convergence in sets of the form $\{|x| \leqslant \delta t^\beta\}$. On the other hand, we can write similarly:

$$t^\alpha \left|u(x,t) - U(x,t)\right| \leqslant t^\alpha \left|u(x,t) - B_{C_0}(x,t)\right| + \left|C_0^{\frac{p-1}{p-2}}\left(1 - H_p(x)\right)\right|.$$

Using Theorem 2.1 and the uniform convergence of $H_p$ to 1 at infinity, we obtain the uniform convergence in the complementary sets, $\{|x| \leqslant \delta t^\beta\}$.

We remark that there is a constant $C > 0$ such that $u(x,t) = 0$ for all $x \in \Omega$ with $|x| \geqslant Ct^\beta$ because it is bounded above by a Barenblatt solution. The same happens to $U$ of course. The convergence in $L^1$ follows easily by integration on the set $\{x \in \Omega\colon\ |x| \leqslant Ct^\beta\}$.

(ii) Let us consider next initial data in $L^1$. In order to obtain the uniform convergence result, we perform an analysis by approximation from below following the ideas in [20]. Consider a compactly supported approximation $\tilde{u}_0$ of $u_0$, such that $\|u_0 - \tilde{u}_0\|_{L^1(\Omega)} \leqslant \varepsilon$, $\varepsilon > 0$, sufficiently small. Fix a radius $r$ larger than the radius of the support of $B_{C_0}(x,1)$. For $1/2 < t < 1$, the solution $\tilde{u}_\lambda$ converges uniformly to $B_{C_0}(x,t)$ for $|x| \leqslant r$. Using the $L^1$–$L^\infty$ smoothing effect described in Chapter 11 of [18], we deduce that $u_\lambda$ converges uniformly to $B_{C_0}$ in the region $|x| \leqslant r$ and for $1/2 < t < 1$. In the outer region, the mass of $u_\lambda$ is very small. The proof ends by applying to the function $u_\lambda$ in the set $|x| \geqslant r$ the same $L^1$–$L^\infty$ smoothing effect and deducing that $\|u_\lambda\|_\infty$ is as small as we want at later times in the outer region. □

**Remark.** We have:

$$t^\alpha \left|B_{C_0}(x,t) - t^{-\alpha} C_0^{\frac{p-1}{p-2}} H_p(x)\right| \leqslant \left|t^\alpha B_{C_0}(x,t) - C_0^{\frac{p-1}{p-2}}\right| + \left|C_0^{\frac{p-1}{p-2}}\left(1 - H_p(x)\right)\right| \leqslant \varepsilon$$

for $\delta = \delta(\varepsilon) > 0$ small and $1/\delta \leqslant |x| \leqslant \delta t^\beta$ with $t$ sufficiently large. It appears in this way an overlapping region. Indeed, in the region of $\Omega$ where $1/\delta \leqslant |x| \leqslant \delta t^\beta$ for $t \geqslant t_{in} > 0$, the outer and the inner behaviour hold at the same time.

We end this section with a consequence of the previous analysis concerning the evolution of the supports for compactly supported solutions. We introduce the following notations

$$r_+(t) = \max_{x\in\Gamma(t)} |x|, \qquad r_-(t) = \min_{x\in\Gamma(t)} |x|, \qquad R(t) = C^* t^{\beta}, \tag{2.34}$$

where $C^* = (C_0/k)^{(p-1)/p}$. Here $\Gamma(t)$ is the free boundary of $u$ at time $t$ and $R(t)$ is the free boundary of the Barenblatt profile $B_{C_0}$ at time $t$.

**Corollary.** *In the conditions of Theorem* 2.3, *we have*

$$\lim_{t\to\infty} \frac{r_\pm(t)}{R(t)} = 1.$$

**Proof.** The proof is very similar to that of Corollary 4.3 in [2] and we present it briefly. The lower part of the estimate follows easily from the uniform convergence in the renormalized variable $y$ in sets of the form $|y| \geqslant \delta > 0$ and the uniform positivity of the Barenblatt profile $F_{C_0}$ in sets of the form $|y| \leqslant C^* - \varepsilon$, for all $\varepsilon > 0$. We obtain the estimate $r_-(t)/R(t) \geqslant (1-\varepsilon)$ for $t > t(\varepsilon)$ sufficiently large.

For the assertion about $r_+$, we use the mass analysis. Since the limit mass of $u$ is $M_{C_0}$, for any $\varepsilon > 0$, there exists $t_0 > 0$ such that $M(t) < M_{C_0+\varepsilon}$, for all $t > t_0$. We compare $u$ with the solution $\bar{u}$ of the $p$-Laplacian equation posed in the whole space, with initial data

$$\bar{u}(x, t_0) = \begin{cases} u(x, t_0), & x \in \Omega, \\ 0, & x \in G. \end{cases}$$

We obtain that $r_+(t) \leqslant \bar{r}_+(t)$. Using the asymptotic behaviour in the whole space [15], we derive that $r_+(t)/R(t) \leqslant 1+\varepsilon$, for all $t > t_\varepsilon$ sufficiently large. The proof ends with the trivial inequality $r_-(t) \leqslant r_+(t)$ and the estimate of $r_-(t)/R(t)$ already proved. □

## 3. Critical case $N = p$

The case $N = p$ provides an important difference with the previous case in the general theory of the $p$-Laplacian, since the fundamental solution of the equation is $C|x|^{-(N-p)/(p-1)}$ for $N > p$ and $\log|x|$ for $N = p$. In this way, the dimension $N = p$ corresponds to the case $N = 2$ for the usual Laplacian. On the other hand, the hole starts to play an important role. Indeed, by performing the rescaling as before, we arrive at a solution with a singularity at $x = 0$, but this singularity is no more removable. In the proofs, we will suppose that $p$ is an integer and the problem has physical sense. In radial variables any dimension makes sense theoretically, but the proofs are perfectly similar, since all the profiles that we use for comparison are radial.

### *3.1. Formal derivation of the logarithmic correction*

In this part we follow an idea of [10] based on some formal calculations using a weighted integral. The rigourous proof will be different, but this calculation helps us to conjecture the correct asymptotic profile. For this calculation, we need to pass to the radial variables and consider the problem (in any dimension $N \geqslant p$):

$$\begin{cases} u_t = (p-1)|u_r|^{p-2} r^{\frac{1-N}{p-1}} \frac{\partial}{\partial r}\left(r^{\frac{N-1}{p-1}} u_r\right), & \text{if } (r,t) \in (1,\infty)\times[0,\infty), \\ u(1,t) = 0, \quad \forall t > 0, \\ u(r,0) = u_0(r), \quad \forall r \in (0,\infty), \end{cases} \tag{3.1}$$

where $u_0$ is compactly supported and bounded. Similarly to the usual convolution with the Green kernel, we define the following weighted integral for the Barenblatt solutions of (3.1):

$$Z : [1,\infty)\times(0,\infty) \to \mathbf{R}, \qquad Z(r,t) = \int_r^\infty k(x,r) B_C(x,t)\, dx \tag{3.2}$$

where the kernel $k$ is given by the fundamental solution:

$$k(x,r)=\begin{cases} x^{p-1}r^{p-N}(x^{N-p}-r^{N-p})/(N-p), & \text{if } N>p, \\ x\log(x/r), & \text{if } N=p. \end{cases} \tag{3.3}$$

Our goal in this subsection is to calculate the behaviour when $t\to\infty$ of $Z(r,t)$ and to remark what are the differences that appears when passing from $N>p$ to $N=p$. We first calculate it for $N>p$. We have:

$$\begin{aligned} Z(r,t) &= \frac{1}{N-p}\int_r^\infty x^{p-1}r^{p-N}\big(x^{N-p}-r^{N-p}\big)t^{-\alpha}\big(C-k(x/t^\beta)^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}\,dx \\ &= \frac{1}{N-p}\int_{r/t^\beta}^\infty \big(y^{N-1}r^{p-N}-y^{p-1}t^{p\beta-\alpha}\big)\big(C-ky^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}\,dy. \end{aligned}$$

Since $\alpha=N\beta$ and $N>p$, it follows that $p\beta-\alpha<0$, hence

$$\lim_{t\to\infty} Z(r,t)=\frac{r^{p-N}}{N-p}\int_0^\infty y^{N-1}\big(C-ky^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}\,dy<\infty.$$

This follows from the fact that the asymptotic profile is the Barenblatt and the weighted integral should have a finite limit as $t\to\infty$, obtained by cancelling the time from the independent integral in $y$. This also should pass in the case $N=p$ if we want to obtain the correct profile. Let us pass to the case $N=p$ and calculate the same integral:

$$\begin{aligned} Z(r,t) &= \int_r^\infty x^{p-1}\log\frac{x}{r}t^{-\alpha}\big(C-k\big(x/t^\beta\big)^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}\,dx \\ &= \int_{r/t^\beta}^\infty y^{p-1}t^{(p-1)\beta}t^{-\alpha}\log\frac{yt^\beta}{r}\big(C-ky^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}t^\beta\,dy \\ &= \int_{r/t^\beta}^\infty y^{p-1}\log\frac{yt^\beta}{r}\big(C-ky^{\frac{p}{p-1}}\big)_+^{\frac{p-1}{p-2}}\,dy. \end{aligned}$$

We remark that $\lim_{t\to\infty} Z(r,t)=\infty$, for any $r>1$, with logarithmic rate. For the divergence we introduce in the calculation a correction of logarithmic type, in order to compensate and obtain a finite limit. It is convenient to insert this correction into the form of the Barenblatt solution. Let us consider

$$\overline{B}_C(x,t)=t^{-\alpha}\left(C(\log t)^\gamma-k\left(\frac{x}{t^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}.$$

Analyzing the previous calculation of $Z(r,t)$, in order to compensate, we need to set $\gamma=-\frac{p-2}{(p-1)^2}$. In order to avoid the problems with the singularity of the logarithmic part, we will permit a delay in time $T>0$. Hence, we conjecture that the outer asymptotic behaviour of solutions in the case $N=p$ is given by a function from the family:

$$\begin{aligned} U_T(x,t;C) &= (t+T)^{-\alpha}\left(C\big(\log(t+T)\big)^{-\frac{p-2}{(p-1)^2}}-k\left(\frac{|x|}{(t+T)^\beta}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}} \\ &= \big[(t+T)\log(t+T)\big]^{-\frac{1}{p-1}}\left(C-k\left(\frac{|x|}{(t+T)^\beta}\right)^{\frac{p}{p-1}}\log(t+T)^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}, \end{aligned} \tag{3.4}$$

where in this case

$$\alpha=\frac{1}{p-1},\qquad \beta=\frac{1}{p(p-1)},\qquad k=\frac{p-2}{p}\left(\frac{1}{p(p-1)}\right)^{\frac{1}{p-1}},$$

see [18]. In the following sections we will prove this claim.

A convenient way of visualizing the corrected asymptotic behaviour is in terms of the mass. In our case

$$M(t)=\int_{\Omega} U_T(x,t;C)\,dx=\frac{C}{\log t}. \tag{3.5}$$

In fact, we see that the logarithmic correction we introduce is exactly the inverse of the number $\gamma$ which connects the "free parameter" of a general Barenblatt solution and its mass, see (1.5). A similar expression of mass decay happens in other critical cases, see [7] and [10]. Contrary to the latter case, here we do not have an exact conservation law to obtain the precise decay and the constant. Hence, we have to use a different technique.

### 3.2. Subsolutions

The construction of subsolutions of (1.1) is technical and follows the same idea as in the first section. Our profiles $U_T$ are indeed subsolutions of the equation, but not for the boundary-value problem (1.1), since they do not vanish on the boundary of the hole. We have to combine these profiles with other profiles that are compactly supported, vanish on the boundary of $\Omega$ and dominate near the hole. We consider the family of profiles

$$H_T(x,t)=A(t+T)\big((T+t)\log(T+t)\big)^{-\frac{1}{p-1}}\left(\log\big(|x|-r_0\big)-\frac{a(|x|-r_1)_+}{(T+t)^l}\right)_+, \tag{3.6}$$

where $A(t+T)=K(1+(t+T)^{-\sigma})$ and the parameters $a$, $r_0$, $r_1$, $K$, $l$ and $\sigma$ are free to choose.

The idea is to intersect $U_T$ and $H_T$ with the correct angle, i.e. such that the profile $U_T$ dominates far from the hole. We ask that $\max U_T \leqslant \max H_T$. But

$$\max U_T \leqslant \big((T+t)\log(T+t)\big)^{-\frac{1}{p-1}} C^{\frac{p-1}{p-2}}$$

and

$$\max H_T \geqslant H_T(r_1)=\big((T+t)\log(T+t)\big)^{-\frac{1}{p-1}} K\big(1+(t+T)^{-\sigma}\big)\log(r_1-r_0),$$

hence it is enough to choose $K$ such that $K\log(r_1-r_0)=2C^{(p-1)/(p-2)}$ and $r_1>r_0$.

Denote by $R_1(t)$ the radius of the interface of $H_T$ and by $R_2(t)$ the radius of the interface of $U_T$. Then $R_1(t)$ is the unique solution of the equation

$$a(r-r_1)=(T+t)^l\log(r-r_0) \tag{3.7}$$

with $r>r_1$ and

$$R_2(t)=\left(\frac{C}{k}\right)^{\frac{p-1}{p}}(t+T)^{\beta}\log(t+T)^{-\frac{p-2}{p(p-1)}}. \tag{3.8}$$

We choose $C$ and $T$ such that $R_2(t)>R_1(t)$, for all $t>0$. Then there exists $r^*(t)$ such that $1<r^*(t)<R_1(t)<R_2(t)$, for all $t>0$, such that $H_T$ and $U_T$ intersect at a distance $r^*(t)$. We define

$$V_T(x,t;C)=\begin{cases}0 & \text{if } |x|<1+r_0 \text{ or } |x|>R_2(t),\\ H_T(x,t) & \text{if } 1+r_0\leqslant |x|\leqslant r^*(t),\\ U_T(x,t;C) & \text{if } r^*(t)\leqslant |x|\leqslant R_2(t).\end{cases} \tag{3.9}$$

It follows by direct calculation, taking into account that $\Delta_p \log|x|=0$, that $V_T(x,t;C)$ is indeed a subsolution. The next technical result shows that these subsolutions are good.

**Proposition 3.1.** *For any solution $u$ of* (1.1)*, there exists a time $t_0>0$ large and a choice of the parameters $C$, $T$, $a$, $r_0$, $r_1$, $l$ such that $V_T(x,t;C)\leqslant u(x,t)$, for all $x\in\Omega$ and $t\geqslant t_0$.*

**Proof.** *Step 1:* Show that there exists a time $t_0>0$ and a choice of the parameters such that $V_T(x,t_0;C)\leqslant u(x,t_0)$, $\forall x\in\Omega$.

Take $t_0>0$ such that $\mathrm{Int}(\operatorname{supp} u(\cdot,t_0))$ is large enough (by well-known results, it enlarges as time passes). Choose $r_0$, $r_1$ such that the annulus $\overline{W_{r_0,r_1}(0)}\subset \mathrm{Int}(\operatorname{supp} u(\cdot,t_0))$. Then choose the constant $C$ measuring the height of the subsolution $V_T$ such that $V_T$ lies below $u$ at time $t_0$. To choose the delay $T$, we ask that $\operatorname{supp} V_T(\cdot, t_0; C) = \overline{W_{r_0+1, R_2(t_0)}(0)} \subset \operatorname{supp} u(\cdot, t_0)$. Choose $R_2(t_0)$ and $r_1$ such that $r_1 < R_2(t_0) = \xi_+(t_0) - \varepsilon < \xi_+(t_0)$ and $r_1 < \xi_+(t_0) - 2\varepsilon$, where $\xi_+(t_0) = \sup\{r > 0\colon\ B(0, r) \subset \operatorname{supp} u(\cdot, t_0)\}$. From this choice, we find $T$ as the solution of the equation

$$\left(\frac{C}{k}\right)^{\frac{p-1}{p}} (T + t_0)^{\frac{1}{p(p-1)}} \log(T + t_0)^{-\frac{p-2}{p(p-1)}} = \xi_+(t_0) - \varepsilon. \tag{3.10}$$

In order to have a unique solution of (3.10), we have to increase again the time $t_0$ such that $\log t_0 \geqslant (p - 2)$. Then the function $h(T) = (T + t_0) \log(T + t_0)^{2-p}$ is increasing and the uniqueness of the solution is obvious. The choice of $a$ comes from the condition $R_1(t_0) < R_2(t_0)$.

*Step 2:* For any $t \geqslant t_0$, $V_T(x, t; C) \leqslant u(x, t)$ for all $x \in \Omega$. To do this, we use well-known arguments of comparison, starting from $t = t_0$ as initial time. Since the subsolution and the solution are separate, the only thing that we have to prove is that the above construction can be done, i.e. $R_1(t) < R_2(t)$, for all $t > t_0$. We use the standard procedure: let $g(t) = R_2(t) - R_1(t)$. Then $g(t_0) > 0$ and suppose there exists a first time $t_1 > t_0$ such that $g(t_1) = 0$. Then $R_2(t_1) = R_1(t_1)$ and $g'(t_1) \leqslant 0$.

On the other hand, $g'(t_1) = R_2'(t_1) - R_1'(t_1)$. By differentiating in (3.8), we obtain

$$R_2'(t_1) = \frac{\beta}{t_1 + T} \frac{\log(t_1 + T) - (p - 2)}{\log(t_1 + T)} R_2(t_1).$$

To obtain $R_1'(t_1)$, we differentiate in Eq. (3.7) and from a straightforward calculation and taking into account that $R_1(t_1) = R_2(t_1)$, we have:

$$\begin{aligned} g'(t_1) &= \frac{1}{t_1 + T}\left(\beta R_2(t_1) \frac{\log(t_1 + T) - (p - 2)}{\log(t_1 + T)} - \frac{la(R_2(t_1) - r_1)(R_2(t_1) - r_0)}{a(R_2(t_1) - r_1) - (t_1 + T)^l}\right) \\ &= \frac{1}{(t_1 + T)[a(R_2(t_1) - r_1) - (t_1 + T)^l]}\left[a\left(\beta \frac{\log(t_1 + T) - (p - 2)}{\log(t_1 + T)} - l\right) R_2(t_1)^2 \right. \\ &\quad \left. + \left((t_1 + T)^l + la(r_1 + r_0) - a\beta r_1 \frac{\log(t_1 + T) - (p - 2)}{\log(t_1 + T)}\right) R_2(t_1) - r_1 r_0 la\right]. \end{aligned}$$

By enlarging the initial time $t_0$ (hence at the same time $t_1$) and choosing $l < \beta$, we obtain that $g'(t_1) > 0$, in contradiction with the assumptions on $t_1$. Hence $R_1(t) < R_2(t)$, for all $t > t_0$. $\square$

### *3.3. Continuous rescaling and supersolutions*

In this section we will prove that indeed the functions $U_T(x, t; C)$ obtained formally are the correct asymptotic profiles of the general nonnegative solutions of the $p$-Laplacian equation in dimension $N = p$. The main difficulty is that the profile $U_T(x, t; C)$ is not a self-similar solution of the equation, but a subsolution, hence we cannot use the classical comparison techniques that hold only for solutions.

Justified by the previous comments, we replace the comparison technique by the technique of continuous rescaling, see [6] or [20]. We set:

$$\eta = x(t + T)^{-\beta} \log(t + T)^{\frac{p-2}{p(p-1)}}, \qquad \tau = \log(t + T), \quad v(\eta, \tau) = \big((t + T) \log(t + T)\big)^{\frac{1}{p-1}} u(x, t). \tag{3.11}$$

The main difference between this scaling and the one of the first section is that the zoom factor change continuously with time. This justifies the name of continuous rescaling. The generality of this technique comes from the fact that the zoom factors may be changed from problem to problem and in this way the method is very flexible. Moreover, in general after a good time-dependent rescaling the resulting equation is simpler than the initial one.

In our case, we obtain the new equation satisfied by $v$:

$$v_\tau = \Delta_p v + \beta\eta \cdot \nabla v + \alpha v - \frac{p - 2}{p(p - 1)\tau} \eta \cdot \nabla v + \frac{1}{p - 1} \tau^{-\frac{p-2}{p-1}} v, \tag{3.12}$$

that we will call in the sequel as the perturbed equation. We associate its autonomous counterpart, which is:

$$v_\tau = \Delta_p v + \beta\eta \cdot \nabla v + \alpha v, \tag{3.13}$$

and will be called the *limit equation*. By these transformations, the profiles $U_T(x,t;C)$ transforms into the family

$$F_C(\eta) = \bigl(C - k|\eta|^{\frac{p}{p-1}}\bigr)^{\frac{p-1}{p-2}}, \tag{3.14}$$

which are stationary solutions of the limit equation (3.13).

On the other hand, we need some a priori estimates on the general solutions of the $p$-Laplacian equation. Since we already have enough subsolutions, we need to construct supersolutions that can be compared with any solution, in order to bound the solutions also from above. The construction of a supersolution is rather technical and is given in the next

**Proposition 3.2.** *For any $C > 0$ sufficiently large, there exists a choice of the free parameters $\gamma$, $d$, $b$ and $q < 0$ such that the following profile*:

$$\overline{U}_T(x,t;C) = \bigl((T+t)\log(T+t)\bigr)^{-\frac{1}{p-1}}\Biggl(C - k\biggl(\frac{|x|}{(T+t)^{\beta}}\log(t+T)^{\frac{p-2}{p(p-1)}} + \frac{d}{\log(t+T)^{\gamma}}\biggr)^{\frac{p}{p-1}} \\ \times\biggl(1 + \frac{b}{\log(t+T)^{\gamma}}\biggr)^{\frac{pq}{p-1}}\Biggr)_+^{\frac{p-1}{p-2}} \tag{3.15}$$

*is a supersolution for the $p$-Laplacian equation in $\Omega$.*

**Proof.** The proof consists on a very long calculation. In any case, it seems much easier checking it on the rescaled Eq. (3.12), since the profile $\overline{U}$ changes into the following simpler form:

$$v_+(\eta,\tau) = \biggl(C - k\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{p}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}}\biggr)_+^{\frac{p-1}{p-2}}. \tag{3.16}$$

The supersolution condition becomes

$$\begin{aligned}
0 \leqslant{}& \frac{kp}{p-2}\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{1}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}-1}\frac{1}{\tau^{\gamma+1}}\biggl(gd + b|\eta| + \frac{2bd}{\tau^{\gamma}}\biggr) \\
&+ C\biggl[p\beta\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{pq} + \beta\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{pq}\frac{(p-1)d}{|\eta|\tau^{\gamma}} - \alpha - \frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}\biggr] \\
&+ \frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}k\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{p}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}} \\
&+ pk\beta\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{p}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}}\biggl(1 - \biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{pq}\biggr) \\
&- k\beta\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{p}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}+pq}\frac{(p-1)d}{|\eta|\tau^{\gamma}} \\
&+ \biggl(\beta - \frac{p-2}{p(p-1)\tau}\biggr)\beta^{\frac{1}{p-1}}\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{1}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}}|\eta| \\
&- \beta^{\frac{p}{p-1}}\biggl(|\eta| + \frac{d}{\tau^{\gamma}}\biggr)^{\frac{1}{p-1}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{\frac{pq}{p-1}+pq}.
\end{aligned}$$

We will prove this complicated inequality by separating it into two parts:

(a) This is the part with the free parameter $C$ characterizing the profile. The inequality that we prove is:

$$C\biggl[p\beta\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{pq} + \beta\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{pq}\frac{(p-1)d}{|\eta|\tau^{\gamma}} - \alpha - \frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}\biggr] \geqslant 0. \tag{3.17}$$

The main fact is that all these inequalities make sense in the positive part of the profile (3.16), in the rest being trivial. For that, we have to consider that $\eta$ is bounded, more precisely

$$|\eta| \leqslant \biggl(\frac{C}{k}\biggr)^{\frac{p-1}{p}}\biggl(1 + \frac{b}{\tau^{\gamma}}\biggr)^{-q},$$

hence fixing $C > 0$, one can choose any number $\gamma < (p-2)/(p-1)$, $q < 0$ and $d$ sufficiently large in order to hold the inequality from (a) at any time $\tau > \tau_0 > 0$ fixed. This is rather easy to achieve.

(b) The inequality formed with the rest of the terms can be a little simplified and written on the form

$$\begin{aligned}
0 \leqslant{}& \frac{kp}{(p-1)\tau^{1+\gamma}}\left(d+\beta|\eta|+\frac{2bd}{\tau^\gamma}\right)+\frac{1}{p-1}\tau^{-\frac{p-2}{p-1}}k\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)\\
&-k\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\frac{(p-1)d}{|\eta|\tau^\gamma}+pk\beta\left(|\eta|+\frac{d}{\tau^\gamma}\right)\left(1+\frac{b}{\tau^\gamma}\right)\left(1-\left(1+\frac{b}{\tau^\gamma}\right)^{pq}\right)\\
&+\left(\beta-\frac{p-2}{p(p-1)\tau}\right)\beta^{\frac{1}{p-1}}|\eta|\left(1+\frac{b}{\tau^\gamma}\right)-\beta^{\frac{1}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{pq+1}.
\end{aligned}$$

We compare one by one the terms with plus and the terms with minus in the preceding inequality. We have to make a different analysis depending on the values of $|\eta|$. If $|\eta|$ is very small, then we encounter a difficulty in compensating the term with minus where we divide by $|\eta|$. But in this case we will not separate the two inequalities. We remark that this term comes from the term

$$-\beta\left(C-k\left(|\eta|+\frac{d}{\tau^\gamma}\right)^{\frac{p}{p-1}}\left(1+\frac{b}{\tau^\gamma}\right)^{\frac{pq}{p-1}}\right)\left(1+\frac{b}{\tau^\gamma}\right)\frac{(p-1)d}{|\eta|\tau}$$

obtained after calculating $\Delta_p v_+$ and simplifying. Since $|\eta|$ is small (for example $|\eta| < 1$), for $C$ sufficiently large we can make a decomposition of this terms by dividing $C$ and letting $C/2$ in the expression above and introducing only $C/2$ in the corresponding part of the inequality (3.17). In this way, the term in part (b) is compensated directly, by the remaining term with $C/2$, and in (a) we have only to replace $d$ by $2d$.

If $|\eta| > 1$, by comparing one by one the terms in the inequality in part (b), we find that it is enough to choose the parameters $b$ and $q$ such that $|q|p^2b > (p-1)d$, where $d$ is already chosen from (a), and $q < 0$. With this, the proposition is proven. □

The usefulness of this construction, which is very general (we show in fact that for any $C > 0$ we can construct such a supersolution) is illustrated in the following result:

**Proposition 3.3.** *For any solution $u$ of the $p$-Laplacian equation in $\Omega$, there exists a constant $C > 0$ and a delay $T > 0$ sufficiently large such that*

$$u(x,t) \leqslant U_T(x,t;C) \tag{3.18}$$

*in the notations introduced above.*

**Proof.** Let $u_0(x) = u(x,0)$ be the initial value of the solution. There exist a delay $T > 0$ and a constant $C > 0$ such that the function $\overline{U}_T(x,t;C)$ has the following two properties:

(I) $\operatorname{supp} u_0 \subset \operatorname{supp} \overline{U}_T(x,0;C)$;
(II) On $\operatorname{supp} u_0$, we have: $u_0(x) \leqslant \overline{U}_T(x,0;C)$.

We say in this case that $u$ and $\overline{U}_T$ are separated at time $t = 0$. Since $u$ is a solution of the equation and $\overline{U}_T$ is a supersolution and they are separated at the initial time, a well-known comparison result says that $u(x,t) \leqslant \overline{U}_T(x,t;C)$ for all $x \in \Omega$ and for any time $t > 0$. On the other hand, we have:

$$\begin{aligned}
\overline{U}_T(x,t;C) &\leqslant \big((T+t)\log(T+t)\big)^{-\frac{1}{p-1}}\left(C-k|x|^{\frac{p}{p-1}}\big(\log(t+T)\big)^{\frac{p-2}{(p-1)^2}}\left(1+\frac{b}{(\log T)^\gamma}\right)^{\frac{pq}{p-1}}\right)_+^{\frac{p-1}{p-2}}\\
&\leqslant \big((T+t)\log(T+t)\big)^{-\frac{1}{p-1}}\left(C_T-k|x|^{\frac{p}{p-1}}\big(\log(t+T)\big)^{\frac{p-2}{(p-1)^2}}\right)_+^{\frac{p-1}{p-2}}\\
&= U_T(x,t;C_T),
\end{aligned}$$

where $C_T = C(1+\frac{b}{(\log T)^\gamma})^{-pq/(p-1)}$. □

This result, together with the one about subsolutions, shows that the family $U_T$ is sufficient to control all the solutions $u$.

### 3.4. Outer analysis

It will always be given by some profile of the form (3.4), depending on the initial data of the problem. The convergence will be uniform away from the hole and with a specified rate. But let us first state precisely the main result of this subsection.

**Theorem 3.1.** *Let $u(x,t)$ be the unique weak solution of (1.1) with initial data $u_0 \in L^1(\Omega)$, nonnegative and compactly supported, in dimension $N = p$. Then there exists a constant $C_0$ depending on $u_0$ and a delay in time $T$ such that*

$$\lim_{t\to\infty} (t\log t)^{\frac{1}{p-1}} \left|u(t) - U_T(\cdot, t; C)\right| = 0, \tag{3.19}$$

*with uniform convergence in any set of the form $\{|x| \geqslant \delta\lambda(t)\}$, where $\delta > 0$ is sufficiently small and*

$$\lambda(t) = t^{\beta}(\log t)^{-\frac{p-2}{p(p-1)}}, \quad \beta = \frac{1}{p(p-1)}.$$

**Proof.** The idea of this proof is to apply the S-theorem (see [6] or [7]) in order to identify all the possible asymptotic limits of the solutions of (3.12) as solutions of (3.13). In order to do this we need some uniform boundedness and compactness estimates for the orbits $(v(\tau))_{\tau\in\mathbf{R}}$. To obtain that, the decisive argument is Proposition 3.3, since it translates the estimates from the orbit $v(\tau)$ to some particular orbits with explicit formulas, more precisely, we have

$$v(\tau) \leqslant \left(C - k|\eta|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}. \tag{3.20}$$

From (3.20) and the fact that the profiles $(C - k|\eta|^{p/(p-1)})_+^{(p-1)/(p-2)}$ are stationary and bounded uniformly in $L^q(\mathbf{R}^N)$, for all $q \in [1,\infty]$, we deduce similar uniform boundedness estimates for the orbits $v(\tau)$. The compactness estimates are easily obtained, by introducing in the energy estimates the dominating profiles.

By introducing (3.20) in the definition of a weak solution for Eq. (3.12), we also remark that, due to the stationary character of the dominating profiles, the terms in the perturbation go to 0 and in the end we remain with the weak formulation of (3.13).

On the other hand, the stability assumption of the S-theorem is clearly satisfied by the family $(F_C)_{C>0}$, since it is stationary. Hence, we may form the $\omega$-limit of the orbit of $v$, which is

$$\omega(v) = \left\{f \in L^1\left(\mathbf{R}^N\right):\ \text{there exists } \tau_j \to \infty,\ v(\eta, \tau_j) \to f(\eta)\right\},$$

where the convergence is taken in $L^q$-norm, for any $q \in [1,\infty)$. The assumption $f \in L^1(\mathbf{R}^N)$ make sense, since from the change of variables we remark that $\lim_{\tau\to\infty} \overline{\Omega}(\tau) = \mathbf{R}^N \setminus \{0\}$. Then we may consider, by passing to a subsequence, that the convergence in the definition of $\omega(v)$ holds also weakly in $W^{1,p}(\mathbf{R}^N)$.

We multiply Eq. (3.12) by any test function $\Phi \in C_0^\infty(\Omega)$ and we integrate in space and in time in $(\tau_1, \tau_2)$, where $\tau_2 = \tau_1 + T$, $T > 0$ fixed. We obtain the weak formulation of Eq. (3.12):

$$\int_{\overline{\Omega}(\tau_2)} \left(v(\tau_2) - v(\tau_1)\right)\Phi\,dx = -\int_{\tau_1}^{\tau_2}\int_{\overline{\Omega}(\tau)} |\nabla v|^{p-2}\nabla v \cdot \nabla\Phi\,dx\,d\tau + \int_{\tau_1}^{\tau_2}\int_{\overline{\Omega}(\tau)} \left[\left(\beta - \frac{p-2}{p(p-1)\tau}\right)\eta\cdot\nabla v\Phi + \frac{1}{p-1}\left(1 - \tau^{-\frac{p-2}{p-1}}\right)v\Phi\right]dx\,d\tau.$$

We pass to the limit in this weak formulation with $\tau_1 \to \infty$ and $T$ fixed. Hence $\tau_2 \to \infty$ and from the boundedness estimates, the terms with $\tau$ disappear in the limit. On the other hand, the left-hand side goes to 0. From the right-hand side it remains:

$$T\int_{\mathbf{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla\Phi\,dx + \beta T\int_{\mathbf{R}^N} \eta\cdot\nabla v\Phi\,dx + \alpha T\int_{\mathbf{R}^N} v\Phi\,dx = 0,$$

which is the weak formulation of the elliptic counterpart of Eq. (3.13). Passing to the limit is rigourously justified by the uniform estimates we have, in particular that in $W^{1,p}$. We deduce that any $f \in \omega(v)$ is a weak stationary solution of Eq. (3.13).

We have now to identify the limit as one of the profiles $F_C$ given by (3.14). For the beginning, we have to study the elliptic equation

$$\Delta_p v + \beta\eta \cdot \nabla v + \alpha v = 0 \tag{3.21}$$

satisfied by the elements of $\omega(v)$. We start with the problem of the asymptotic behaviour of solutions of the evolution $p$-Laplacian equation posed in the whole $\mathbf{R}^N$. It is well known that the asymptotic profiles are given by the Barenblatt solutions given by (1.2)–(1.3) (see for example [15]). On the other hand, if we use in this case the time-adapted rescaling, by setting

$$\eta = xt^{-\beta}, \qquad \tau = \log t, \qquad v(\eta, \tau) = t^{\alpha} u(x, t), \tag{3.22}$$

after the transformation we arrive to Eq. (3.13). Following the ideas of the similar result in [6] (where it is presented for the Porous Medium Equation), we have the following

**Lemma 3.1.** *The profiles $F_C$ can be characterized as the unique nonnegative stationary solutions of Eq.* (3.13) *such that $f \in L^1(\mathbf{R}^N)$ and $f \in W^{1,p}(\mathbf{R}^N)$.*

**Proof.** Any other stationary solution $f$ can be taken as initial data for Eq. (3.13). From the general asymptotic behaviour result (see [15]) the corresponding solution of (3.13) converges to the unique Barenblatt solution $B_C$ with the same mass as $f$. On the other hand, by defining $u(x,t) = t^{-\alpha} f(xt^{-\beta})$, we obtain a solution of the $p$-Laplacian equation converging to $f$ in the time-adapted rescaling. By uniqueness of the limit we obtain $f = F_C$. $\square$

Using the previous lemma, we obtain that all the elements of $\omega(v)$ are of type $F_C$ for some $C > 0$, and the range of constants $C$ is bounded above and below, using the corresponding sub- and supersolutions we have constructed. We still have to prove that there exists in the limit only one profile of this type, i.e. a unique constant $C$. We postpone this part of the proof after the study of regularity.

We prove now the uniform convergence to every profile in the $\omega$-limit. In order to do this, we go again to Eq. (3.12) and we study the regularity of its solutions. We know that the orbits $v(\tau)$ are uniformly bounded in all the spaces $L^q$ with $q \in [1, \infty]$. On the other hand, we can write (3.12) in the following way:

$$v_\tau - \operatorname{div}\big(a(\eta, \tau, v, \nabla v)\big) = b(\eta, \tau, v, \nabla v), \tag{3.23}$$

where

$$a(\eta, \tau, v, \nabla v) = |\nabla v|^{p-2}\nabla v + \left(\beta - \frac{p-2}{p(p-1)\tau}\right)\eta v \tag{3.24}$$

and

$$b(\eta, \tau, v, \nabla v) = \frac{1}{p-1}\left(\tau^{-\frac{p-2}{p-1}} + \frac{p-2}{\tau}\right)v. \tag{3.25}$$

Using the uniform estimates that we have on $v(\tau)$, it is very easy to show that the functions $a$ and $b$ satisfy the structure conditions required in Theorem 1.1 from Chapter 3 of the book [4] and that the estimates are uniform at infinity, since $v(\tau)$ are uniformly bounded by compactly supported stationary profiles. This, together with the Arzela–Ascoli Theorem, implies that all the elements of $\omega(v)$ are achieved as uniform limits on subsequences.

*3.4.1. Mass analysis*

In order to finish, we prove now the uniqueness of the limit profile. We already know that $\omega(v) = \{F_C\colon C_- \leqslant C \leqslant C_+\}$, these bounds coming from comparison with the subsolutions and the supersolutions constructed above. Define

$$m(\tau) = \int_{\mathbf{R}^N} v(\eta, \tau)\, d\eta. \tag{3.26}$$

Then $m(F_C)$ is increasing in $C$ and $v(\tau_j) \to F_C$ uniformly for some subsequence $\tau_j \to \infty$ if and only if $m(\tau_j) \to m(F_C)$. We argue by contradiction and suppose that there exist $C_1 < C_2$ such that $m(\tau_j) \to m(F_{C_1})$ and $m(\tau_j') \to m(F_{C_2})$ on two subsequences $\tau_j \to \infty$ and $\tau_j' \to \infty$ as $j \to \infty$. Then, for any $C \in (C_1, C_2)$, since $m(\tau)$ is bounded and continuous, there exists a subsequence $\tilde{\tau}_j \to \infty$ such that $m(\tilde{\tau}_j) \to C$ and $v(\tilde{\tau}_j) \to F_C$ uniformly in $\mathbf{R}^N$. Hence, all the points in $(C_1, C_2)$ are limit points of $m(\tau)$ and this function has a very oscillatory character as $\tau \to \infty$. By a simple calculus fact, for any $C \in (C_1, C_2)$ we may suppose not only that $v(\tilde{\tau}_j) \to F_C$ uniformly, but also that

$$\frac{\partial}{\partial \tau} m(\tilde{\tau}_j) \leqslant 0, \tag{3.27}$$

by passing to a subsequence if necessary.

On the other hand, using (3.12), we calculate:

$$\begin{aligned} \frac{\partial}{\partial \tau} m(\tau) &= \int\limits_{\overline{\Omega}(\tau)} \Delta_p v \, d\eta + \tau^{-\frac{p-2}{p-1}} \int\limits_{\overline{\Omega}(\tau)} \left( \frac{1}{p-1} v - \frac{p-2}{(p-1)\tau^{1/(p-1)}} v \right) d\eta \\ &= \frac{1}{p-1} \tau^{-\frac{p-2}{p-1}} \left[ \left( 1 - \frac{p-2}{\tau^{1/(p-1)}} \right) m(\tau) + (p-1) \tau^{\frac{p-2}{p-1}} \int\limits_{\partial \overline{\Omega}(\tau)} |\nabla v|^{p-2} \nabla v \cdot \nu \, d\sigma(\eta) \right], \end{aligned} \tag{3.28}$$

where $\nu$ is the outward normal vector to the boundary of $\overline{\Omega}(\tau)$. Since the uniform limits of $v(\tau)$ along subsequences are only profiles from the family $F_C$, it is easy to see that

$$(p-1) \tau^{\frac{p-2}{p-1}} \int\limits_{\partial \overline{\Omega}(\tau)} |\nabla v|^{p-2} \nabla v \cdot \nu \, d\sigma(\eta) \to 0 \quad \text{as } \tau \to \infty,$$

hence

$$\lim_{j \to \infty} \tilde{\tau}_j^{\frac{p-2}{p-1}} \frac{\partial}{\partial \tau} m(\tilde{\tau}_j) = \frac{1}{p-1} m(F_C) > 0 \tag{3.29}$$

in contradiction with (3.27). This contradiction shows that $\omega(v)$ contains only one element, i.e. $\omega(v) = \{F_C\}$ for some $C$ depending only on the initial data $u_0$ and the domain $\Omega$.

Since $\omega(v)$ has only one element $F_C$, we find that $v(\tau) \to F_C$ uniformly as $\tau \to \infty$, far from 0, i.e. in sets of the form $\{|\eta| \geqslant \delta\}$. Rephrasing the result in the initial variables, we obtain (3.19) far from the hole $G$, more precisely in sets of the form $\{|x| \geqslant \delta t^{\beta} (\log t)^{-(p-2)/p(p-1)}\}$, as stated. $\square$

*3.5. Inner analysis*

In this subsection we study the inner behaviour of the solutions of (1.1) posed in dimension $N = p$. We will try to use the same ideas as in the case $N < p$: deduction of the profile by formal calculations, then the rigourous proof. It appears an important difference in the rescaling we do: since we have a logarithmic correction of the type $M(t)/\log t$ in the expression of the mass, and at the same time we expect to obtain in the limit a stationary solution, we have to compensate it with the logarithmic part.

Following the previous comments, we set $\bar{w}(x,t) = t^{1/(p-1)} (\log t)^{p/(p-1)} u(x,t)$. We deduce the equation satisfied by $\bar{w}$:

$$\Delta_p \bar{w} = t^{-\frac{1}{p-1}} (\log t)^{\frac{p(p-2)}{p-1}} \left( t \bar{w}_t - \frac{p + \log t}{(p-1) \log t} \bar{w} \right). \tag{3.30}$$

Suppose for a moment that the terms in the right-hand side go to 0, i.e.

$$\lim_{t \to \infty} t^{-\frac{1}{p-1}} (\log t)^{\frac{p(p-2)}{p-1}} \left( t \bar{w}_t - \frac{p + \log t}{(p-1) \log t} \bar{w} \right) = 0.$$

Then the asymptotic limit of $\bar{w}$ is expected to be a solution of the problem (2.17). As it is well known, all the solutions of this problem tend to infinity with a logarithmic rate as $|x| \to \infty$ and they have the general form $C H_p$, where $H_p$ is the unique solution of (2.17) having in addition the property that $\lim_{|x| \to \infty} H_p(x)/(\log |x|) = 1$ uniformly.

We have to use again the method of matched asymptotics in order to find the precise constant $C$. For this, we use the outer analysis result

$$\lim_{t\to\infty}(t\log t)^{\frac{1}{p-1}}\left|u(x,t)-U_T(x,t;C_0)\right|=0,$$

uniformly for all $x\in\Omega$ with $|x|\geqslant \delta t^{\beta}(\log t)^{-(p-2)/p(p-1)}$, where we denote in this section by $C_0$ the constant corresponding to $u$. On the other hand, by our formal calculation, we expect

$$\lim_{t\to\infty}\left|\frac{\bar{w}(x,t)}{\log t}-\frac{CH_p(x)}{\log t}\right|=0$$

uniformly in sets of the form $\{|x|\leqslant \delta t^{\beta}(\log t)^{-(p-2)/(p-1)p}\}$, with $\delta>0$ small. By comparing the two limits in the curve $|x|=\delta t^{\beta}(\log t)^{-(p-2)/p(p-1)}$, we obtain that

$$C=\frac{1}{\beta}C_0^{\frac{p-1}{p-2}}. \tag{3.31}$$

In what follows we will prove rigourously that the inner behaviour is given by the quasi-stationary state $C_0^{(p-1)/(p-2)}H_p(x)/\beta\log t$. For this, we use the same strategy as in Section 1 and change the scale of time by setting $\tau=\log t$. Define $w(x,t)=\bar{w}(x,t)/\log t$, which in the new variables is $w(x,\tau)=e^{\tau/(p-1)}\tau^{1/(p-1)}u(x,e^{\tau})$. Consider the time average:

$$W_T(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}w(x,s)\,ds.$$

**Proposition 3.4.** *For any $\varepsilon>0$ and $T>0$, there exists a constant $\delta=\delta(\varepsilon,T)>0$ and a large time $\tau_{in}=\tau_{in}(\varepsilon,\delta,T)$ such that for any $\tau\geqslant\tau_{in}$ we have*

$$\left|W_T(x,\tau)-\frac{C_0^{(p-1)/(p-2)}H_p(x)}{\beta\tau}\right|\leqslant\varepsilon, \tag{3.32}$$

*for all $x$ with $|x|\leqslant\delta e^{\tau\beta}\tau^{-(p-2)/p(p-1)}$.*

**Proof.** The proof is very similar to that of Proposition 2.3. The main idea is to consider the function

$$\Phi(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}w(x,s)\,ds-\frac{C_0^{(p-1)/(p-2)}H_p(x)}{\beta\tau} \tag{3.33}$$

and to derive estimates on it in order to apply Proposition 2.2. We obtain that

$$\Delta_p\Phi=\frac{1}{T}\int_{\tau}^{\tau+T}\left(se^{-s}\right)^{-1/(p-1)}\left(sw_s-\frac{1+s}{p-1}w\right)\quad\text{in }\Omega_\tau \tag{3.34}$$

and the boundary conditions

$$\Phi=0\quad\text{on }\partial\Omega\cap\partial\Omega_\tau,\qquad \Phi=h\quad\text{on }\partial\Omega\cap\partial B\big(0,R(\tau)\big), \tag{3.35}$$

where $\Omega_\tau=\Omega\cap B(0,R(\tau))$, $R(\tau)=\delta e^{\tau\beta}\tau^{-(p-2)/p(p-1)}$ and

$$h(x,\tau)=\frac{1}{T}\int_{\tau}^{\tau+T}g(x,s)\,ds-\frac{C_0^{(p-1)/(p-2)}H_p(x)}{\beta\tau}.$$

By performing a similar analysis as in the proof of Proposition 2.3, we find that for any $\varepsilon>0$ and $\delta>0$, there exists a delay $\tau_0=\tau_0(\varepsilon,\delta)$ such that

$$\left|\Phi(x,\tau)\right|\leqslant C\delta^{p/(p-1)}+3\varepsilon,\quad\forall\tau>\tau_0,$$

where $C>0$ is a constant independent of time. By choosing $\delta$ small and $\tau>\tau_0$, we obtain the estimate (3.32). □

Passing from the convergence of time averages to the convergence of the function itself is very similar to that in the case $N > p$. We obtain the main inner behaviour result:

**Theorem 3.2.** *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a time $t_{in} = t_{in}(\varepsilon, \delta)$ sufficiently large, such that*

$$\left|(t\log t)^{\frac{1}{p-1}}u(x,t) - \frac{C_0^{(p-1)/(p-2)}H_p(x)}{\beta\log t}\right| \leqslant \varepsilon, \tag{3.36}$$

*for all $t \geqslant t_{in}$ and for all $x \in \Omega$ with $|x| \leqslant \delta t^\beta(\log t)^{-(p-2)/p(p-1)}$.*

### 3.6. Global formulation

In this last subsection we group the results of Theorems 3.1 and 3.2 in a global approximation result, as we have done in Section 2. Before stating the result, we have to modify the profile $H_p$ in the outer region, by defining

$$\Psi(x,t) = \begin{cases} H_p(x)/\beta\log t & \text{if } |x| \leqslant \delta t^\beta(\log t)^{-(p-2)/p(p-1)}, \\ H_p(\delta t^\beta(\log t)^{-(p-2)/p(p-1)}x/|x|)/\beta\log t & \text{if } |x| \leqslant \delta t^\beta(\log t)^{-(p-2)/p(p-1)}. \end{cases}$$

In this way, we obtain that in the outer region $\{|x| \geqslant \delta t^\beta(\log t)^{-(p-2)/p(p-1)}\}$ the modified function $\Psi$ converges uniformly to 1. We are ready to state the global approximation theorem.

**Theorem 3.3.** *Let $u$ be the unique solution of the problem* (1.1) *in dimension $N = p$ and*

$$V(x,t) = \left(U_T(x,t;C_0) - (t\log t)^{-\frac{1}{p-1}}C_0^{\frac{p-1}{p-2}}\left(1-\Psi(x,t)\right)\right)_+, \tag{3.37}$$

*where $C_0$ and $T$ are the constants that appear in the outer analysis. Then*

$$\lim_{t\to\infty}(t\log t)^{\frac{1}{p-1}}\left|u(x,t) - V(x,t)\right| = 0, \tag{3.38}$$

*uniformly for $x \in \Omega$. Moreover, we have*:

$$\lim_{t\to\infty}\log t\left\|u(x,t) - V(x,t)\right\|_{L^1(\Omega)} = 0. \tag{3.39}$$

*Both* (3.38) *and* (3.39) *can be extended to the whole class of solutions with initial data $u_0 \in L^1(\Omega)$.*

**Proof.** (i) Similarly as in the proof of Theorem 2.3, for compactly supported data, we obtain the uniform convergence result (3.38) from the inequalities:

$$(t\log t)^{\frac{1}{p-1}}\left|u(x,t) - V(x,t)\right| \leqslant \left|(t\log t)^{\frac{1}{p-1}}u(x,t) - C_0^{\frac{p-1}{p-2}}\Psi(x,t)\right| + \left|C_0^{\frac{p-1}{p-2}} - (t\log t)^{\frac{1}{p-1}}U_T(x,t;C_0)\right|$$

and

$$(t\log t)^{\frac{1}{p-1}}\left|u(x,t) - V(x,t)\right| \leqslant (t\log t)^{\frac{1}{p-1}}\left|u(x,t) - U_T(x,t;C_0)\right| + \left|C_0^{\frac{p-1}{p-2}}\left(1-\Psi(x,t)\right)\right|$$

and the results of Theorem 3.2, resp. Theorem 3.1.

We remark that there is a constant $C$ such that $u(x,t) = 0$ for all $x \in \Omega$ with $|x| \geqslant C\delta t^\beta(\log t)^{-(p-2)/p(p-1)}$, since $u$ is bounded above by the supersolution we construct. The same happens to the global approximant $V$. Using the previous convergence result, for $\varepsilon > 0$ and $t \geqslant t(\varepsilon)$ large enough, we have:

$$\int_\Omega \left|u(x,t) - V(x,t)\right|\log t\,dx \leqslant \log t\,(t\log t)^{-\frac{1}{p-1}}\varepsilon\left(C\delta t^\beta(\log t)^{-(p-2)/p(p-1)}\right)^p = C^p\varepsilon,$$

giving the convergence in $L^1$-norm. This extends by standard density arguments to the whole class of solutions with initial data in $L^1(\Omega)$.

(ii) We consider initial data in $L^1$ and we perform an analysis by approximation as in Section 2. Choose a compactly supported initial data $\tilde{u}_0$ such that the corresponding solution satisfies:

$$\lim_{t\to\infty}\log t\left\|u(t) - \tilde{u}(t)\right\|_{L^1} = 0.$$

We use this approximation and a very similar analysis to that in Subsection 2.4 to complete the result. Let us remark that the time decay from the smoothing effect (see [18, Theorem 11.3]) is sufficient due to the logarithmic extra-term which appears in (3.39), which compensates $(\log t)^{1/(p-1)}$ from the time scale in this case. □

**Remark.** It appears an overlapping region, where the outer and the inner behaviour hold at the same time. In strong contrast with the result in Section 2, in the critical case this overlapping region is very thin and consists in the points $x \in \Omega$ such that

$$\lim_{|x|\to\infty} \frac{H_p(x)}{\beta \log t} = 1,$$

hence it is a thin layer around the curve $|x| = \delta t^{\beta}$, with a narrowness of type $o(\log t)$.

We end this section with the study of the evolution of the supports for compactly supported solutions. Recall the notations $r_+(t)$, $r_-(t)$ and $R(t)$ introduced in (2.34). We have the following result:

**Corollary.** *In the conditions of Theorem* 3.3*,*

$$\lim_{t\to\infty} \frac{r_\pm(t)}{R(t)} = 1.$$

**Proof.** The lower estimate follows from the uniform convergence of $u$ in the rescaled variable $\eta$ in sets of the form $|\eta| \geqslant \delta > 0$ and the uniform positivity of the profile $F_{C_0}$ in sets of the form $|y| \leqslant C^* - \varepsilon$. For the assertion on $r_+(t)$, we use a similar analysis in the renormalized variable $\eta$ as in the proof of the similar result in Section 2. There is a difference that we explain next. Consider the solution $\bar{u}$ of the $p$-Laplacian equation posed in the whole space, with initial data

$$\bar{u}(x, t_0) = \begin{cases} u(x, t_0), & x \in \Omega, \\ 0, & x \in G, \end{cases}$$

for a sufficiently large time $t_0$. We compare $u$ and $\bar{u}$, but in the renormalized variables $\eta$ and $\tau$, where we transform $u$ using the rescaling (3.11) and $\bar{u}$ using the rescaling (3.22). We deduce that in the new time $\tau$ holds $r_+(\tau) \leqslant \bar{r}_+(\tau)$. We rephrase the result using the rescaling (3.11) for both solutions and, using the asymptotic result in the whole space [15], we obtain the upper estimate for $r_+(t)$. □

## 4. Comments and open problems

**1. Nonconnected domains.** The assumption of connectedness is not an essential restriction. If the domain is not connected, every connected component is treated separately. The bounded connected components follow the behaviour of bounded domains with size $u = O(t^{-1/(p-2)})$ as $t \to \infty$, as described in [20], Chapter 20. Assuming that there is a unique unbounded component containing a neighbourhood of infinity, the behaviour in that component is described by the results of the present paper.

**2. Critical case of porous medium flow.** Our method for proving the inner behaviour settles also the similar case for the porous medium equation (there critical means $N = 2$ for every $m > 1$). The outer behaviour for the critical case of the porous medium equation was done in [10], where it is proved that the outer asymptotic profile is again a logarithmic correction of the Barenblatt solution:

$$U_T(x, t; C_0) = \big((t+T)\log(t+T)\big)^{-\frac{1}{m}} \left(C_0 - k\left(\frac{|x|}{(t+T)^{\beta}}\right)^2 \big(\log(t+T)\big)^{\frac{m-1}{m}}\right)_+^{\frac{1}{m-1}}, \quad \beta = \frac{1}{2m}.$$

Using the technique of matched asymptotics and estimates which are very similar to those of Theorem 3.3, we deduce an analogous result for the inner behaviour in the porous medium equation case:

**Theorem.** *For any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and a time $t_{in} = t_{in}(\varepsilon, \delta)$ sufficiently large, such that*:

$$\left| (t\log t)^{\frac{1}{m}} u(x,t) - \frac{C_0^{1/(m-1)} H(x)^m}{\beta \log t} \right| \leqslant \varepsilon,$$

*for all $t \geqslant t_{in}$ and for all $x \in \Omega$ such that $|x| \leqslant \delta t^{\beta} (\log t)^{-(m-1)/m}$.*

Here, $H(x)$ is the solution of the corresponding stationary problem

$$\begin{cases} \Delta(H^m) = 0 & \text{in } \Omega, \\ H = 0 & \text{on } \partial\Omega, \end{cases}$$

with the additional property that $\lim_{|x|\to\infty} H(x)^m/\log|x| = 1$ uniformly in $\Omega$.

**3. Application to the linear case, $p = 2$.** In the linear case $p = 2$, a partial inner behaviour in the exterior of a ball is described in [14]. In this particular case, our results coincide with those of that paper in $N \geqslant 3$. Since [14] treats only asymptotic behaviour in fixed compact sets, in dimension $N = 2$ a logarithmic improvement of the decay rate is proved. This improvement follows also easily in our case in a fixed compact set, but not in the general inner behaviour. Moreover, by treating both inner and outer behaviour in sets with time-dependent boundaries, our methods and results are more general.

**4. Open problem: quantitative estimates for the mass.** A precise estimate of the mass $M(t)$ at any moment of time is still missing. In the porous medium case, due to a conservation law, a precise estimate was obtained in [2], Corollary 4.2. In our case, there seems to be no conservation law, and this makes more difficult to obtain a relation between the initial mass of the solution and $M(t)$.

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