Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 25 (2008) 231–266

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Asymptotic solutions for large time of Hamilton–Jacobi equations in Euclidean n space

Solutions asymptotiques en temps grand d'équations de Hamilton–Jacobi dans $\mathbf{R}^n$

Hitoshi Ishii [1]

Department of Mathematics, Faculty of Education and Integrated Arts and Sciences, Waseda University, Tokyo, 169-8050 Japan

Received 6 January 2006; received in revised form 10 August 2006; accepted 29 September 2006

Available online 8 January 2007

Abstract

We study the large time behavior of solutions of the Cauchy problem for the Hamilton–Jacobi equation $u_t + H(x, Du) = 0$ in $\mathbf{R}^n \times (0, \infty)$, where $H(x, p)$ is continuous on $\mathbf{R}^n \times \mathbf{R}^n$ and convex in p. We establish a general convergence result for viscosity solutions $u(x, t)$ of the Cauchy problem as $t \to \infty$.

Résumé

Nous étudions le comportement en temps grand des solutions du problème de Cauchy pour l'équation de Hamilton–Jacobi $u_t + H(x, Du) = 0$ dans $\mathbf{R}^n \times (0, \infty)$, où $H(x, p)$ est continu dans $\mathbf{R}^n \times \mathbf{R}^n$ et convexe en p. Nous établissons un résultat de convergence général pour les solutions de viscosité $u(x, t)$ du problème de Cauchy quand $t \to \infty$.

MSC: 35B40; 35F25; 37J99; 49L25

Keywords: Asymptotic solutions; Hamilton–Jacobi equations; Large time behavior; Weak KAM theory

1. Introduction and the main results

In recent years, there has been much interest on the asymptotic behavior of viscosity solutions of the Cauchy problem for Hamilton–Jacobi equations or viscous Hamilton–Jacobi equations. Namah and Roquejoffre [25] and Fathi [14] were the first those who established fairly general convergence results for the Hamilton–Jacobi equation $u_t(x, t) + H(x, Du(x, t)) = 0$ on a compact manifold M with smooth strictly convex Hamiltonian H. Fathi's approach to this large time asymptotic problem is based on weak KAM theory [13,15,16] which is concerned with the Hamilton–

E-mail address: ishii@edu.waseda.ac.jp.

[1] Supported in part by the Grant-in-Aids for Scientific Research, No. 18204009, JSPS.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.09.002

Jacobi equation as well as with the Lagrangian or Hamiltonian dynamical structures behind it. Barles and Souganidis [6,7] took another approach, based on PDE techniques, to the same asymptotic problem. The weak KAM approach due to Fathi to the asymptotic problem has been developed and further improved by Roquejoffre [27] and Davini and Siconolfi [12]. Motivated by these developments the author jointly with Y. Fujita and P. Loreti (see [18,19]) has recently investigated the asymptotic problem for viscous Hamilton–Jacobi equations with Ornstein–Uhlenbeck operator

$$u_t - \Delta u + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

and the corresponding Hamilton–Jacobi equations

$$u_t + \alpha x \cdot Du + H(Du) = f(x) \quad \text{in } \mathbf{R}^n \times (0, \infty),$$

where H is a convex function on $\mathbf{R}^n$, Δ denotes the Laplace operator, and α is a positive constant, and has established a convergence result similar to those obtained by [6,7,14,27,12].

In this paper we investigate the Cauchy problem

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1.1}$$

$$u(\cdot, 0) = u_0, \tag{1.2}$$

where H is a scalar function on $\mathbf{R}^n \times \mathbf{R}^n$, $u \equiv u(x,t)$ is the unknown scalar function on $\mathbf{R}^n \times [0, \infty)$, $u_t = \partial u/\partial t$, $Du = (\partial u/\partial x_1, \ldots, \partial u/\partial x_n)$, and u_0 is a given function on $\mathbf{R}^n$ describing the initial data. The function $H(x,p)$ is assumed here to be convex in p, and we call H the *Hamiltonian* and then the function L, defined by

$$L(x, \xi) = \sup_{p \in \mathbf{R}^n} \big(\xi \cdot p - H(x,p)\big),$$

the *Lagrangian*. We refer to [26] for general properties of convex functions.

We are also concerned with the *additive eigenvalue* problem:

$$H(x, Dv) = c \quad \text{in } \mathbf{R}^n, \tag{1.3}$$

where the unknown is a pair $(c, v) \in \mathbf{R} \times C(\mathbf{R}^n)$ for which v is a viscosity solution of (1.3). This problem is also called the *ergodic control* problem due to the fact that PDE (1.3) appears as the dynamic programming equation in ergodic control of deterministic optimal control theory. We remark that the additive eigenvalue problem (1.3) appears in the homogenization of Hamilton–Jacobi equations. See for this [24].

For notational simplicity, given $\phi \in C^1(\mathbf{R}^n)$, we will write $H[\phi](x)$ for $H(x, D\phi(x))$ or $H[\phi]$ for the function: $x \mapsto H(x, D\phi(x))$ on $\mathbf{R}^n$. For instance, (1.3) may be written as $H[v] = c$ in $\mathbf{R}^n$.

We make the following assumptions on the Hamiltonian H.

(A1) $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$.

(A2) H is *coercive*, that is, for any $R > 0$,

$$\lim_{r \to \infty} \inf\big\{H(x,p) \mid x \in B(0,R),\ p \in \mathbf{R}^n \setminus B(0,r)\big\} = \infty.$$

(A3) For any $x \in \mathbf{R}^n$, the function: $p \mapsto H(x,p)$ is strictly convex in $\mathbf{R}^n$.

(A4) There are functions $\phi_i \in C^{0+1}(\mathbf{R}^n)$ and $\sigma_i \in C(\mathbf{R}^n)$, with $i = 0, 1$, such that for $i = 0, 1$,

$$H\big(x, D\phi_i(x)\big) \leqslant -\sigma_i(x) \quad \text{almost every } x \in \mathbf{R}^n,$$

$$\lim_{|x| \to \infty} \sigma_i(x) = \infty, \qquad \lim_{|x| \to \infty} (\phi_0 - \phi_1)(x) = \infty.$$

By adding a constant to the function ϕ_0, we assume henceforth that

$$\phi_0(x) \geqslant \phi_1(x) \quad \text{for } x \in \mathbf{R}^n.$$

We introduce the class Φ_0 of functions by

$$\Phi_0 = \Big\{u \in C\big(\mathbf{R}^n\big) \;\Big|\; \inf_{\mathbf{R}^n}(u - \phi_0) > -\infty\Big\}.$$

We call a *modulus* a function $m:[0,\infty)\to[0,\infty)$ if it is continuous and nondecreasing on $[0,\infty)$ and if $m(0)=0$. The space of all absolutely continuous functions $\gamma:[S,T]\to\mathbf{R}^n$ will be denoted by $\mathrm{AC}([S,T],\mathbf{R}^n)$. For $x,y\in\mathbf{R}^n$ and $t>0$, $\mathcal{C}(x,t)$ (resp., $\mathcal{C}(x,t;y,0)$) will denote the spaces of all curves $\gamma\in\mathrm{AC}([0,t],\mathbf{R}^n)$ satisfying $\gamma(t)=x$ (resp., $\gamma(t)=x$ and $\gamma(0)=y$). For any interval $I\subset\mathbf{R}$ and $\gamma:I\to\mathbf{R}^n$, we call γ a curve if it is absolutely continuous on any compact subinterval of I.

We will establish the following theorems.

Theorem 1.1. *Let $u_0\in\Phi_0$ and assume that* (A1)–(A4) *hold. Then there is a unique viscosity solution $u\in C(\mathbf{R}^n\times[0,\infty))$ of* (1.1) *and* (1.2) *satisfying*

$$\inf\bigl\{u(x,t)-\phi_0(x)\mid(x,t)\in\mathbf{R}^n\times[0,T]\bigr\}>-\infty \tag{1.4}$$

for any $T\in(0,\infty)$. Moreover the function u is represented as

$$u(x,t)=\inf\left\{\int_0^t L\bigl(\gamma(s),\dot\gamma(s)\bigr)\,\mathrm{d}s+u_0\bigl(\gamma(0)\bigr)\,\Big|\,\gamma\in\mathcal{C}(x,t)\right\} \tag{1.5}$$

for $(x,t)\in\mathbf{R}^n\times(0,\infty)$.

Note that $L(x,\xi)\geqslant -H(x,0)$ for all $x\in\mathbf{R}^n$ and hence $\inf\{L(x,\xi)\mid(x,\xi)\in B(0,R)\times\mathbf{R}^n\}>-\infty$ for all $R>0$. Note as well that for any $(x,t)\in\mathbf{R}^n\times(0,\infty)$ and $\gamma\in\mathcal{C}(x,t)$ the function: $s\mapsto L(\gamma(s),\dot\gamma(s))$ is measurable. Therefore it is natural and standard to set

$$\int_0^t L\bigl(\gamma(s),\dot\gamma(s)\bigr)\,\mathrm{d}s=\infty,$$

with $\gamma\in\mathcal{C}(x,t)$, if the function: $s\mapsto L(\gamma(s),\dot\gamma(s))$ on $[0,t]$ is not integrable. In this sense the integral in formula (1.5) *always makes sense.*

Theorem 1.2. *Let* (A1)–(A4) *hold. Then there is a solution $(c,v)\in\mathbf{R}\times\Phi_0$ of* (1.3). *Moreover the constant c is unique in the sense that if $(d,w)\in\mathbf{R}\times\Phi_0$ is another solution of* (1.3)*, then $d=c$.*

The above theorem determines uniquely a constant c, which we will denote by c_H, for which (1.3) has a viscosity solution in the class Φ_0. The constant c_H is called the *additive eigenvalue* (or simply *eigenvalue*) or *critical value* for the Hamiltonian H. This definition may suggest that c depends on the choice of (ϕ_0,ϕ_1). Actually, it depends only on H, but not on the choice of (ϕ_0,ϕ_1), as the characterization of c_H in Proposition 3.4 below shows. It is clear that if (c,v) is a solution of (1.3), then $(c,v+K)$ is a solution of (1.3) for any $K\in\mathbf{R}$. As is well-known (see [24]), the structure of solutions of (1.3) is, in general, much more complicated than this one-dimensional structure.

After completing the first version of this paper the author learned that Barles and Roquejoffre [5] had studied the large time behavior of solutions of (1.1) and (1.2) and obtained, among other results, a generalization of the main result in [25] to unbounded solutions.

Theorem 1.3. *Let* (A1)–(A4) *hold and $u_0\in\Phi_0$. Let $u\in C(\mathbf{R}^n\times[0,\infty))$ be the viscosity solution of* (1.1) *and* (1.2) *satisfying* (1.4). *Then there is a viscosity solution $v_0\in\Phi_0$ of* (1.3)*, with $c=c_H$, such that as $t\to\infty$,*

$$u(x,t)+ct-v_0(x)\to 0\quad\textit{uniformly on compact subsets of }\mathbf{R}^n.$$

We call the function $v_0(x)-ct$ obtained in the above theorem the *asymptotic solution* of (1.1) and (1.2). See Theorem 8.1 for a representation formula for the function v_0.

In order to prove Theorem 1.3, we take an approach close to and inspired by the generalized dynamical approach introduced by Davini and Siconolfi [12]. However our approach does not depend on the Aubry set for the Lagrangian L and is much simpler than the generalized dynamical approach by [12].

In the following we *always assume* that (A1)–(A4) hold.

The paper is organized as follows: in Section 2 we collect some basic observations needed in the following sections. Section 3 is devoted to the additive eigenvalue problem and to establishing Theorem 1.2. In Section 4 we establish a comparison theorem for (1.1) and (1.2), from which the uniqueness part of Theorem 1.1 follows. Section 5 deals with the existence of a viscosity solution u of the Cauchy problem (1.1)–(1.2) together with an estimate on the modulus of continuity of u. In Section 6 we prove the existence of extremal curves for variational problems associated with the Lagrangian L. Section 7 combines the results in the preceding sections, to prove Theorem 1.3. In Section 8 we show a representation formula for the asymptotic solution for large time of (1.1) and (1.2) and introduce and study the Aubry set for the Hamiltonian H (or more appropriately for Lagrangian L). In Section 9 we give two sufficient conditions for H to satisfy (A4) and a two-dimensional example in which the Aubry set contains a nonempty disk consisting of nonequilibrium points. In Appendix A we show in a general setting that value functions (or in other words the action functional) associated with Hamiltonian H are viscosity solutions of the Hamilton–Jacobi equation $H = 0$. A proposition concerning the Aubry set is presented in Appendix A.

2. Preliminaries

In this section we collect some basic observations which will be needed in the following sections.

We will be concerned with functions f on $\mathbf{R}^n \times \mathbf{R}^n$. We write $D_1 f$ and $D_2 f$ for the gradients of f, respectively, in the first n variables and in the last n variables. Similarly, we use the symbols $D_1^{\pm} f$ and $D_2^{\pm} f$ to denote the sub- and superdifferentials of f in the first or last n variables.

We remark that, since $H(x,\cdot)$ is convex for any $x \in \mathbf{R}^n$, for any $u \in C^{0+1}(\Omega)$, where $\Omega \subset \mathbf{R}^n \times (0,\infty)$ is open, it is a viscosity subsolution of (1.1) in Ω if and only if it satisfies (1.1) almost everywhere (a.e. for short) in Ω. A similar remark holds true for the stationary problem (1.3).

Also, as is well known, the coercivity assumption (A2) on H guarantees that if $v \in C(\Omega)$, where Ω is an open subset of $\mathbf{R}^n$, is a viscosity subsolution of (1.3) in Ω, then it is locally Lipschitz in Ω.

Another remark related to the convexity of H is that given nonempty, uniformly bounded, family $\mathcal{S}$ of subsolutions of (1.3) in Ω, where Ω is an open subset of $\mathbf{R}^n$, the pointwise infimum $u(x) := \inf\{v(x) \mid v \in \mathcal{S}\}$ gives a viscosity subsolution u of (1.3) in Ω. For instance, this can be checked by invoking the notion of semicontinuous viscosity solutions due to Barron and Jensen [8,9]. Indeed, due to this theory (see also [3,4,21]), $v \in C^{0+1}(\Omega)$ is a viscosity subsolution of (1.3) if and only if $H(x,p) \leqslant c$ for all $p \in D^- v(x)$ and all $x \in \Omega$. It is standard to see that if $p \in D^- u(x)$ for some $x \in \Omega$, then there are sequences $\{x_k\}_{k\in\mathbf{N}} \subset \Omega$, $\{v_k\}_{k\in\mathbf{N}} \subset \mathcal{S}$, and $\{p_k\}_{k\in\mathbf{N}} \subset \mathbf{R}^n$ such that $p_k \in D^- v_k(x_k)$ for all $k \in \mathbf{N}$ and $(x_k, p_k, v_k(x_k)) \to (x, p, u(x))$ as $k \to \infty$. Here, we have $H(x_k, p_k) \leqslant c$ for all $k \in \mathbf{N}$ and conclude that $H(x,p) \leqslant c$ for all $p \in D^- u(x)$ and all $x \in \Omega$. If, instead, $\mathcal{S}$ is a family of viscosity supersolutions of (1.3) in Ω, then a classical result in viscosity solutions theory assures that u, defined as the pointwise infimum of all functions $v \in \mathcal{S}$, is a viscosity supersolution of (1.3) in Ω. In particular, if $\mathcal{S}$ is a family of viscosity solutions of (1.3) in Ω, then the function u, defined as the pointwise infimum of $v \in \mathcal{S}$, is a viscosity solution of (1.3) in Ω. We refer the reader to [3,4,11] for the general theory of viscosity solutions.

Proposition 2.1. *For each $R > 0$ there exist constants $\delta_R > 0$ and $C_R > 0$ such that $L(x,\xi) \leqslant C_R$ for all $(x,\xi) \in B(0,R) \times B(0,\delta_R)$.*

Proof. Fix any $R > 0$. By the continuity of H, there exists a constant $M_R > 0$ such that $H(x,0) \leqslant M_R$ for all $x \in B(0,R)$. Also, by the coercivity of H, there exists a constant $\rho_R > 0$ such that $H(x,p) > M_R + 1$ for all $(x,p) \in B(0,R) \times \partial B(0,\rho_R)$. We set $\delta_R = \rho_R^{-1}$. Let $\xi \in B(0,\delta_R)$ and $x \in B(0,R)$. Let $q \in B(0,\rho_R)$ be the minimum point of the function: $f(p) := H(x,p) - \xi \cdot p$ on $B(0,\rho_R)$. Noting that $f(0) = H(x,0) \leqslant M_R$ and $f(p) > M_R + 1 - \delta_R\rho_R = M_R$ for all $p \in \partial B(0,\rho_R)$, we see that $q \in \operatorname{int} B(0,\rho_R)$ and hence $\xi \in D_2^- H(x,q)$, which implies that $L(x,\xi) = \xi \cdot q - H(x,q)$. Consequently, we get

$$L(x,\xi) \leqslant \delta_R\rho_R - \min_{p\in\mathbf{R}^n} H(x,p) = 1 - \min_{B(0,R)\times\mathbf{R}^n} H.$$

Now, choosing $C_R > 0$ so that $1 - \min_{B(0,R)\times\mathbf{R}^n} H \leqslant C_R$, we obtain

$$L(x,\xi) \leqslant C_R \quad \text{for all } (x,\xi) \in B(0,R) \times B(0,\delta_R). \qquad \square$$

Proposition 2.2. *Let $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$. Then $(x,\xi) \in \operatorname{int}\operatorname{dom} L$ if and only if $\xi \in D_2^- H(x,p)$ for some $p \in \mathbf{R}^n$.*

Proof. Fix $\hat{x}, \hat{\xi} \in \mathbf{R}^n$. Suppose first that $\hat{\xi} \in D_2^- H(\hat{x}, \hat{p})$ for some $\hat{p} \in \mathbf{R}^n$. Define the function f on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$f(x,p) = H(x,p) - \hat{\xi} \cdot p + L(\hat{x}, \hat{\xi}).$$

Note that the function $f(\hat{x}, \cdot)$ attains the minimum value 0 at $\hat{p}$ and it is strictly convex on $\mathbf{R}^n$. Fix $r > 0$ and set

$$m = \min_{p \in \partial B(\hat{p}, r)} f(\hat{x}, p),$$

and note, because of the strict convexity of $f(\hat{x}, \cdot)$, that $m > 0$. Note also that the function: $x \mapsto \min_{p \in \partial B(\hat{p},r)} f(x,p)$ is continuous on $\mathbf{R}^n$. Hence there is a constant $\delta > 0$ such that

$$\min\bigl\{f(x,p) \mid x \in B(\hat{x}, \delta),\ p \in \partial B(\hat{p}, r)\bigr\} > \frac{m}{2}, \tag{2.1}$$

$$\max\bigl\{f(x,\hat{p}) \mid x \in B(\hat{x}, \delta)\bigr\} < \frac{m}{4}. \tag{2.2}$$

Fix any $(x,\xi) \in B(\hat{x}, \delta) \times B(0, \frac{m}{4})$ and consider the affine function $g(p) := r^{-1}\xi(p - \hat{p}) + \frac{m}{4}$. We show that

$$f(x,p) > g(p) \quad \text{for all } p \in \mathbf{R}^n \setminus B(\hat{p}, r). \tag{2.3}$$

To see this, we fix any $p \in \mathbf{R}^n \setminus B(\hat{p}, r)$ and set $q = \hat{p} + r(p - \hat{p})/|p - \hat{p}| \in \partial B(\hat{p}, r)$. Then, by (2.1), we have

$$f(x,q) > \frac{m}{2}.$$

Using the convexity of $f(x,\cdot)$ and noting that $q = (1 - r/|p - \hat{p}|)\hat{p} + (r/|p - \hat{p}|)p$, we get

$$f(x,q) \leqslant \Bigl(1 - \frac{r}{|p - \hat{p}|}\Bigr) f(x,\hat{p}) + \frac{r}{|p - \hat{p}|} f(x,p)$$

and hence, by using (2.2), we get

$$\begin{aligned} f(x,p) &\geqslant r^{-1}|p-\hat{p}| f(x,q) + \bigl(1 - r^{-1}|p - \hat{p}|\bigr) f(x,\hat{p}) \\ &> r^{-1}|p - \hat{p}|\frac{m}{2} + \bigl(1 - r^{-1}|p - \hat{p}|\bigr)\frac{m}{4} = \frac{m}{4}\bigl(1 + r^{-1}|p - \hat{p}|\bigr). \end{aligned} \tag{2.4}$$

On the other hand, we have

$$g(p) \leqslant \frac{m}{4}\bigl(r^{-1}|p - \hat{p}| + 1\bigr).$$

This combined with (2.4) shows that (2.3) is valid.

Next, observing that $f(x,\hat{p}) - g(\hat{p}) < \frac{m}{4} - g(\hat{p}) = 0$ by (2.2) and using (2.3), we see that the function: $p \mapsto f(x,p) - g(p)$ attains its global minimum at a point in $B(\hat{p}, r)$. Fix such a minimum point $p_{x,\xi} \in B(\hat{p}, r)$, which is indeed uniquely determined by the strict convexity of $f(x,\cdot)$. We have

$$0 \in D_2^- f(x, p_{x,\xi}) - Dg(p_{x,\xi}) = D_2^- H(x, p_{x,\xi}) - xi - r^{-1}\xi.$$

That is,

$$\hat{\xi} + r^{-1}\xi \in D_2^- H(x, p_{x,\xi}),$$

which is equivalent to saying that

$$p_{x,\xi} \in D_2^- L\bigl(x, \hat{\xi} + r^{-1}\xi\bigr).$$

In particular, we have $(x, \hat{\xi} + r^{-1}\xi) \in \operatorname{dom} L$ and $(\hat{x}, \hat{\xi}) \in \operatorname{int}\operatorname{dom} L$.

Next, we suppose that $(\hat{x}, \hat{\xi}) \in \operatorname{int}\operatorname{dom} L$. Then it is an easy consequence of the Hahn–Banach theorem that there is a $\hat{p} \in \mathbf{R}^n$ such that $xi \in D_2^- H(\hat{x}, \hat{p})$. □

Remark. Let $(x,\xi) \in \operatorname{int}\operatorname{dom} L$. According to the above theorem (and its proof), there is a unique $p(x,\xi) \in D_2^- L(x,\xi)$. That is, on the set $\operatorname{int}\operatorname{dom} L$, the multi-valued map $D_2^- L$ can be identified with the single-valued function: $(x,\xi) \mapsto p(x,\xi)$. By the above proof, we see moreover that for each $r > 0$ there is a constant $\delta > 0$ such that $p(y,\eta) \in B(p(x,\xi), r)$ for all $(y,\eta) \in B(x,\delta) \times B(\xi,\delta)$. From this observation, we easily see that the function: $(x,\xi) \mapsto p(x,\xi)$ is continuous on $\operatorname{int}\operatorname{dom} L$. Indeed, one can show that L is differentiable in the last n variables and $D_2 L$ is continuous on $\operatorname{int}\operatorname{dom} L$.

Proposition 2.3. *Let $K \subset \mathbf{R}^n \times \mathbf{R}^n$ be a compact set. Set*

$$S = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n \mid \xi \in D_2^- H(x,p) \text{ for some } p \in \mathbf{R}^n \text{ such that } (x,p) \in K\}.$$

Then S is a compact subset of $\mathbf{R}^n \times \mathbf{R}^n$ and $S \subset \operatorname{int}\operatorname{dom} L$.

Proof. We choose a constant $R > 0$ so that $K = B(0,R) \times B(0,R)$.

To see that S is compact, we first check that $S \subset \mathbf{R}^{2n}$ is a closed set. Let $\{(x_k, \xi_k)\}_{k\in\mathbf{N}} \subset S$ be a sequence converging to $(x_0, \xi_0) \in \mathbf{R}^{2n}$. For each $k \in \mathbf{N}$ there corresponds a point $p_k \in B(0,R)$ such that

$$\xi_k \in D_2^- H(x_k, p_k).$$

This is equivalent to saying that

$$\xi_k \cdot p_k = L(x_k, \xi_k) + H(x_k, p_k). \tag{2.5}$$

We may assume by replacing the sequence $\{(x_k, \xi_k, p_k)\}$ by one of its subsequences if necessary that $\{p_k\}$ is convergent. Let $p_0 \in B(0,R)$ be the limit of the sequence $\{p_k\}$. Since L is lower semicontinuous, we get from (2.5) in the limit as $k \to \infty$,

$$\xi_0 \cdot p_0 \geqslant L(x_0, \xi_0) + H(x_0, p_0),$$

which implies that $\xi_0 \in D_2^- H(x_0, p_0)$. Hence, we have $(x_0, \xi_0) \in S$ and see that S is closed.

Next we show that S is bounded. Since $H \in C(\mathbf{R}^{2n})$ and the function: $p \mapsto H(x,p)$ is convex for any $x \in \mathbf{R}^n$, we see that there is a constant $M > 0$ such that the functions: $p \mapsto H(x,p)$, with $x \in B(0,R)$, is equi-Lipschitz continuous on $B(0,R)$ with a Lipschitz bound M. This implies that

$$|\xi| \leqslant M \quad \text{for all } (x,\xi) \in S,$$

since if $(x,\xi) \in S$, then $\xi \in D_2^- H(x,p)$ for some $p \in B(0,R)$ and $|\xi| \leqslant M$. Thus we have seen that $S \subset B(0,R) \times B(0,M)$. The set S is bounded and closed in $\mathbf{R}^{2n}$ and therefore it is compact.

Finally, we apply Proposition 2.2 to $(x,\xi) \in S$, to see that $(x,\xi) \in \operatorname{int}\operatorname{dom} L$. □

Proposition 2.4. *Let Ω be an open subset of $\mathbf{R}^n$, $\phi \in C^{0+1}(\Omega)$, and $\gamma \in \mathrm{AC}([a,b], \mathbf{R}^n)$, where $a, b \in \mathbf{R}$ satisfy $a < b$. Assume that $\gamma([a,b]) \subset \Omega$. Then there is a function $q \in L^\infty(a,b,\mathbf{R}^n)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi \circ \gamma(t) = q(t) \cdot \dot\gamma(t) \quad \text{a.e. } t \in (a,b),$$
$$q(t) \in \partial_c \phi\big(\gamma(t)\big) \quad \text{a.e. } t \in (a,b).$$

Here $\partial_c\phi$ denotes the Clarke differential of ϕ (see [10]), that is,

$$\partial_c\phi(x) = \bigcap_{r>0} \overline{\mathrm{co}}\{D\phi(y) \mid y \in B(x,r),\ \phi \text{ is differentiable at } y\} \quad \text{for } x \in \Omega.$$

Proof. We may assume without loss of generality that $\Omega = \mathbf{R}^n$. Let $\rho \in C^\infty(\mathbf{R}^n)$ be a standard mollification kernel, i.e., $\rho \geqslant 0$, $\operatorname{stp}\rho \subset B(0,1)$, and $\int_{\mathbf{R}^n} \rho(x)\,\mathrm{d}x = 1$.

Set $\rho_k(x) := k^n \rho(kx)$ and $\phi_k(x) := \rho_k * \phi(x)$ for $x \in \mathbf{R}^n$ and $k \in \mathbf{N}$. Here the symbol "$*$" indicates the usual convolution of two functions. Set

$$\psi(t) = \phi \circ \gamma(t), \quad \psi_k(t) = \phi_k \circ \gamma(t), \quad \text{and} \quad q_k(t) = D\phi_k \circ \gamma(t) \quad \text{for } t \in [a,b],\ k \in \mathbf{N}.$$

We have $\dot\psi_k(t) = q_k(t) \cdot \dot\gamma(t)$ a.e. $t \in (a,b)$, and, by integration,

$$\psi_k(t) - \psi_k(a) = \int_a^t q_k(s) \cdot \dot\gamma(s)\,\mathrm{d}s \quad \text{for all } t \in [a,b]. \tag{2.6}$$

Passing to a subsequence if necessary, we may assume that for some $q \in L^\infty(a,b,\mathbf{R}^n)$,

$$q_k \to q \quad \text{weakly star in } L^\infty\big(a,b,\mathbf{R}^n\big) \text{ as } k \to \infty.$$

Therefore, from (2.6) we get in the limit as $k \to \infty$,

$$\psi(t) - \psi(a) = \int_a^t q(s) \cdot \dot\gamma(s)\,\mathrm{d}s \quad \text{for all } t \in [a, b].$$

This shows that

$$\dot\psi(t) = q(t) \cdot \dot\gamma(t) \quad \text{a.e. } t \in (a, b).$$

Noting that $\{q_k\}$ is weakly convergent to q in $L^2(a, b, \mathbf{R}^n)$, by Mazur's theorem, we may assume that there is a sequence $\{p_k\}$ such that

$$p_k \to q \quad \text{strongly in } L^2\big(a, b, \mathbf{R}^n\big) \text{ as } k \to \infty,$$
$$p_k \in \mathrm{co}\{q_j \mid j \geqslant k\} \quad \text{for all } k \in \mathbf{N}.$$

We may further assume that

$$p_k(t) \to q(t) \quad \text{a.e. } t \in (a, b) \text{ as } k \to \infty.$$

We fix a set $I \subset (a, b)$ of full measure so that

$$p_k(t) \to q(t) \quad \text{for all } t \in I \text{ as } k \to \infty. \tag{2.7}$$

Now, for any $x \in \mathbf{R}^n$ and any $k \in \mathbf{N}$, noting that

$$D\phi_k(x) = \int_{\mathbf{R}^n} \rho_k(x - y) D\phi(y)\,\mathrm{d}y,$$

we find that

$$D\phi_k(x) \in \overline{\mathrm{co}}\big\{D\phi(y) \mid y \in B\big(x, k^{-1}\big),\ \phi \text{ is differentiable at } y\big\}.$$

From this, we get

$$q_k(t) \in \overline{\mathrm{co}}\big\{D\phi(x) \mid x \in B\big(\gamma(t), k^{-1}\big),\ \phi \text{ is differentiable at } x\big\} \quad \text{for all } t \in [a, b],$$

and therefore

$$p_k(t) \in \overline{\mathrm{co}}\big\{D\phi(x) \mid x \in B\big(\gamma(t), k^{-1}\big),\ \phi \text{ is differentiable at } x\big\} \quad \text{for all } t \in [a, b]. \tag{2.8}$$

Combining (2.7) and (2.8), we get

$$q(t) \in \bigcap_{r>0} \overline{\mathrm{co}}\big\{D\phi(x) \mid x \in B\big(\gamma(t), r\big),\ \phi \text{ is differentiable at } x\big\} \quad \text{for all } t \in I.$$

That is, we have

$$q(t) \in \partial_c\phi\big(\gamma(t)\big) \quad \text{a.e. } t \in (a, b). \qquad \square$$

Proposition 2.5. *Let Ω be an open subset of $\mathbf{R}^n$ and $w \in C^{0+1}(\mathbf{R}^n)$ be such that $H(x, Dw(x)) \leqslant f(x)$ in Ω in the viscosity sense, where $f \in C(\Omega)$. Let $a, b \in \mathbf{R}$ be such that $a < b$ and let $\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n)$. Assume that $\gamma([a, b]) \subset \Omega$. Then*

$$w\big(\gamma(b)\big) - w\big(\gamma(a)\big) \leqslant \int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s + \int_a^b f\big(\gamma(s)\big)\,\mathrm{d}s.$$

Proof. By Proposition 2.4, there is a function $q \in L^\infty(a, b, \mathbf{R}^n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s} w\big(\gamma(s)\big) = q(s) \cdot \dot\gamma(s) \quad \text{and} \quad q(s) \in \partial_c w\big(\gamma(s)\big) \quad \text{a.e. } s \in (a, b).$$

Noting that $H(x, p) \leqslant f(x)$ for all $p \in \partial_c w(x)$ and all $x \in \Omega$, we calculate that

$$w\big(\gamma(b)\big)-w\big(\gamma(a)\big)=\int_a^b \frac{\mathrm{d}}{\mathrm{d}s}w\big(\gamma(s)\big)\,\mathrm{d}s=\int_a^b q(s)\dot\gamma(s)\,\mathrm{d}s\leqslant\int_a^b\big[L\big(\gamma(s),\dot\gamma(s)\big)+H\big(\gamma(s),q(s)\big)\big]\,\mathrm{d}s$$
$$\leqslant\int_a^b\big[L\big(\gamma(s),\dot\gamma(s)\big)+f\big(\gamma(s)\big)\big]\,\mathrm{d}s. \qquad \square$$

3. Additive eigenvalue problem

In this section we prove Theorem 1.2. Our proof below is parallel to that in [24].

Lemma 3.1. *There is a function $\psi_0\in C^1(\mathbf{R}^n)$ such that*

$$H\big(x,D\psi_0(x)\big)\geqslant -C_0 \quad \textit{for all } x\in\mathbf{R}^n, \tag{3.1}$$
$$\psi_0(x)\geqslant\phi_0(x) \quad \textit{for all } x\in\mathbf{R}^n \tag{3.2}$$

for some constant $C_0>0$.

Proof. We choose a modulus ρ so that

$$H(x,p)\geqslant 0 \quad \text{for all } (x,p)\in B(0,r)\times\big[\mathbf{R}^n\setminus B\big(0,\rho(r)\big)\big] \text{ and all } r\geqslant 1,$$
$$\|D\phi_0\|_{L^\infty(B(0,r))}\leqslant\rho(r) \quad \text{for all } r\geqslant 1.$$

Because of this choice, we have

$$\phi_0(x)-\phi_0\big(|x|^{-1}x\big)\leqslant\int_1^{|x|}\rho(r)\,\mathrm{d}r \quad \text{for all } x\in\mathbf{R}^n\setminus B(0,1).$$

We define the function $\psi_0\in C^1(\mathbf{R}^n)$ by

$$\psi_0(x)=\max_{B(0,1)}\phi_0+\int_0^{|x|}\rho(r)\,\mathrm{d}r.$$

It is now easily seen that

$$\phi_0(x)\leqslant\psi_0(x) \quad \text{for all } x\in\mathbf{R}^n, \tag{3.3}$$
$$\big|D\psi_0(x)\big|=\rho\big(|x|\big) \quad \text{for all } x\in\mathbf{R}^n,$$
$$H\big(x,D\psi_0(x)\big)\geqslant 0 \quad \text{for all } x\in\mathbf{R}^n\setminus B(0,1).$$

Choosing a constant $C_0>0$ so that

$$C_0\geqslant\max_{x\in B(0,1)}\big|H\big(x,D\psi_0(x)\big)\big|,$$

we have

$$H\big(x,D\psi_0(x)\big)\geqslant -C_0 \quad \text{for all } x\in\mathbf{R}^n.$$

This together with (3.3) completes the proof. $\square$

We need the following comparison theorem.

Theorem 3.2. *Let Ω be an open subset of $\mathbf{R}^n$. Let $\varepsilon>0$. Let $u,v:\overline{\Omega}\to\mathbf{R}$ be, respectively, an upper semicontinuous viscosity subsolution of*

$$H[u]\leqslant-\varepsilon \quad \textit{in } \Omega, \tag{3.4}$$

and a lower semicontinuous viscosity supersolution of

$$H[v] \geqslant 0 \quad \text{in } \Omega. \tag{3.5}$$

Assume that $v \in \Phi_0$ and $u \leqslant v$ on $\partial\Omega$. Then $u \leqslant v$ on Ω.

The main idea in the following proof how to use the convexity property of H is similar to that in [1].

Proof. We may choose an $R > 0$ so that $H(x, D\phi_1(x)) \leqslant -\varepsilon$ a.e. in $\Omega \setminus B(0, R)$ and then a constant $A_0 > 0$ so that $\phi_1(x) + A_0 > u(x)$ for all $x \in B(0, R)$.

Fix any $A \geqslant A_0$ and define $u_A \in C(\overline{\Omega})$ by $u_A(x) = \min\{\phi_1(x) + A, u(x)\}$. For almost all $x \in \Omega$, we have

$$Du_A(x) = \begin{cases} Du(x) & \text{if } u(x) \leqslant \phi_1(x) + A, \\ D\phi_1(x) & \text{if } u(x) \geqslant \phi_1(x) + A. \end{cases}$$

Hence, for almost all $x \in \Omega$, if $u(x) \leqslant \phi_1(x) + A$, then $H(x, Du_A(x)) = H(x, Du(x)) \leqslant -\varepsilon$, and if $u(x) \geqslant \phi_1(x) + A$, then $|x| > R$ and hence $H(x, Du_A(x)) = H(x, D\phi_1(x)) \leqslant -\varepsilon$. Therefore, u_A is a viscosity subsolution of (3.4).

Since $v \in \Phi_0$ and $u_A(x) \leqslant \phi_1(x) + A$ for all $x \in \mathbf{R}^n$, we have

$$\lim_{|x|\to\infty} \big(v(x) - u_A(x)\big) = \infty,$$

and we see that there is a constant $M > 0$ such that

$$u_A(x) \leqslant v(x) \quad \text{for all } x \in \overline{\Omega} \setminus B(0, M).$$

By a standard comparison theorem applied in $\overline{\Omega} \cap B(0, 2M)$, we obtain $u_A(x) \leqslant v(x)$ for all $x \in \overline{\Omega} \cap B(0, 2M)$, from which we get $u_A(x) \leqslant v(x)$ for all $x \in \overline{\Omega}$. Noting that, for each $x \in \overline{\Omega}$, we have $u_A(x) = u(x)$ if A is sufficiently large, we conclude that $u(x) \leqslant v(x)$ for all $x \in \Omega$. □

Theorem 3.3. (1) *There is a solution $(c, v) \in \mathbf{R} \times \Phi_0$ of* (1.3). (2) *If $(c, v), (d, w) \in \mathbf{R} \times \Phi_0$ are solutions of* (1.3), *then $c = d$.*

Proof. We start by showing assertion (2). Let $(c, v), (d, w) \in \mathbf{R} \times \Phi_0$ be solutions of (1.3). Suppose that $c \neq d$. We may assume that $c < d$. Also, we may assume by adding a constant to v that $v(x_0) > w(x_0)$ at some point $x_0 \in \mathbf{R}^n$. On the other hand, by Theorem 3.2, we have $v \leqslant w$ for all $x \in \mathbf{R}^n$, which is a contradiction. Thus we must have $c = d$.

In order to show existence of a solution of (1.3), we let $\lambda > 0$ and consider the problem

$$\lambda v_\lambda(x) + H\big(x, Dv_\lambda(x)\big) = \lambda\phi_0(x) \quad \text{in } \mathbf{R}^n. \tag{3.6}$$

Let $\psi_0 \in C^1(\mathbf{R}^n)$ and $C_0 > 0$ be from Lemma 3.1. We may assume by replacing C_0 by a larger number if necessary that $\sigma_0(x) \geqslant -C_0$ for all $x \in \mathbf{R}^n$. Note that $H[\phi_0] \leqslant C_0$ in $\mathbf{R}^n$ in the viscosity sense.

We define the functions $v_\lambda^\pm$ on $\mathbf{R}^n$ by

$$v_\lambda^+(x) = \psi_0(x) + \lambda^{-1}C_0 \quad \text{and} \quad v_\lambda^-(x) = \phi_0(x) - \lambda^{-1}C_0.$$

It is easily seen that v_λ^+ and v_λ^- are viscosity supersolution and a viscosity subsolution of (3.6). In view of (3.2), we have $v_\lambda^-(x) < v_\lambda^+(x)$ for all $x \in \mathbf{R}^n$. By the Perron method in viscosity solutions theory, we find that the function v_λ on $\mathbf{R}^n$ given by

$$v_\lambda(x) = \sup\big\{w(x) \mid v_\lambda^- \leqslant w \leqslant v_\lambda^+ \text{ in } \mathbf{R}^n, \lambda w + H[w] \leqslant \lambda\phi_0 \text{ in } \mathbf{R}^n \text{ in the viscosity sense}\big\} \tag{3.7}$$

is a viscosity solution of (3.6). Because of the definition of v_λ, we have

$$\phi_0(x) - \lambda^{-1}C_0 \leqslant v_\lambda(x) \leqslant \psi_0(x) + \lambda^{-1}C_0 \quad \text{for all } x \in \mathbf{R}^n. \tag{3.8}$$

Using the left-hand side inequality of (3.7), we formally calculate that

$$\lambda\phi_0(x) = \lambda v_\lambda(x) + H\big(x, Dv_\lambda(x)\big) \geqslant \lambda\phi_0(x) - C_0 + H\big(x, Dv_\lambda(x)\big),$$

and therefore, $H(x, Dv_\lambda(x)) \leqslant C_0$. Indeed, this last inequality holds in the sense of viscosity solutions. This together with the coercivity of H yields the local equi-Lipschitz continuity of the family $\{v_\lambda\}_{\lambda>0}$. As a consequence, the family $\{v_\lambda - v_\lambda(0)\}_{\lambda>0} \subset C(\mathbf{R}^n)$ is locally uniformly bounded and locally equi-Lipschitz continuous on $\mathbf{R}^n$.

Going back to (3.7), we see that

$$\lambda\phi_0(x) - C_0 \leqslant \lambda v_\lambda(x) \leqslant \lambda\psi_0(x) + C_0 \quad \text{for all } x \in \mathbf{R}^n.$$

In particular, the set $\{\lambda v_\lambda(0)\}_{\lambda\in(0,1)} \subset \mathbf{R}$ is bounded. Thus we may choose a sequence $\{\lambda_j\}_{j\in\mathbf{N}} \subset (0,1)$ such that, as $j \to \infty$,

$$\lambda_j \to 0, \qquad -\lambda_j v_{\lambda_j}(0) \to c,$$
$$v_{\lambda_j}(x) - v_{\lambda_j}(0) \to v(x) \quad \text{uniformly on bounded sets } \subset \mathbf{R}^n$$

for some real number c and some function $v \in C^{0+1}(\mathbf{R}^n)$. Since

$$\bigl|\lambda\bigl(v_\lambda(x) - v_\lambda(0)\bigr)\bigr| \leqslant \lambda L_R|x| \quad \text{for all } x \in B(0,R),$$

all $R > 0$, and some constants $L_R > 0$, we find that

$$-\lambda_j v_{\lambda_j}(x) \to c \quad \text{uniformly on bounded sets } \subset \mathbf{R}^n \text{ as } j \to \infty.$$

By a stability property of viscosity solutions, we deduce that v is a viscosity solution of (1.3) with c in hand.

Now, we show that $v \in \Phi_0$. Fix any $\lambda \in (0,1)$. As we have observed above, there is a constant $C_1 > 0$, independent of λ, such that $|\lambda v_\lambda(0)| \leqslant C_1$. Set $w_\lambda(x) = v_\lambda(x) - v_\lambda(0)$ for $x \in \mathbf{R}^n$. Note that w_λ is a viscosity solution of

$$H(x, Dw_\lambda) \geqslant \lambda(\phi_0 - w_\lambda) - C_1 \quad \text{in } \mathbf{R}^n. \tag{3.9}$$

We may choose a constant $R > 0$ so that $H(x, D\phi_0(x)) \leqslant -C_1 - 1$ a.e. in $\mathbf{R}^n \setminus B(0,R)$, and also a constant $C_2 > 0$, independent of $\lambda \in (0,1)$, so that $\max\{|\phi_0(x)|, |w_\lambda(x)|\} \leqslant C_2$ for all $x \in B(0,R)$. Set $w = \phi_0 - 2C_2$. Obviously we have $w \leqslant w_\lambda$ in $B(0,R)$, and $H(x, Dw(x)) = H(x, D\phi_0(x)) \leqslant -C_1 - 1$ a.e. $x \in \mathbf{R}^n \setminus B(0,R)$. We set $\Omega = \{x \in \mathbf{R}^n \mid w(x) > w_\lambda(x)\}$ and observe that $\Omega \subset \mathbf{R}^n \setminus B(0,R)$. We have $\phi_0(x) - w_\lambda(x) = w(x) + 2C_2 - w_\lambda(x) > 2C_2 > 0$ for all $x \in \Omega$. Hence we see from (3.8) that w_λ is a viscosity solution of $H(x, Dw_\lambda(x)) \geqslant -C_1$ in Ω. It is clear that $w(x) = w_\lambda(x)$ for all $x \in \partial\Omega$. Noting that $w_\lambda \in \Phi_0$, we may apply Theorem 3.2, to obtain $w \leqslant w_\lambda$ in Ω, which shows that $\Omega = \emptyset$, i.e., $w \leqslant w_\lambda$ on $\mathbf{R}^n$. Sending $\lambda \to 0$, we get $\phi_0 - 2C_2 \leqslant v$ in $\mathbf{R}^n$, which shows that $v \in \Phi_0$, completing the proof. □

Proposition 3.4. *The additive eigenvalue c_H is characterized as*

$$c_H = \inf\bigl\{a \in \mathbf{R} \mid \textit{there exists a viscosity solution } v \in C\bigl(\mathbf{R}^n\bigr) \textit{ of } H[v] \leqslant a \textit{ in } \mathbf{R}^n\bigr\}.$$

Proof. We write d for the right-hand side of the above formula. Let $\phi \in \Phi_0$ be a viscosity solution of $H[\phi] = c_H$ in $\mathbf{R}^n$. If $a \geqslant c_H$, then $H[\phi] \leqslant a$ in $\mathbf{R}^n$ in the viscosity sense. Thus we have $d \leqslant c_H$. Suppose that $d < c_H$. Then there is a constant $e \in (d, c_H)$ and a viscosity solution of $H[\psi] \leqslant e$ in $\mathbf{R}^n$. By Theorem 3.2, we see that $\psi + C \leqslant \phi$ in $\mathbf{R}^n$ for any $C \in \mathbf{R}$, which is clearly a contradiction. Thus we have $d = c_H$. □

4. A comparison theorem for the Cauchy problem

In this section we establish the following comparison theorem. Let $T \in (0, \infty)$.

Theorem 4.1. *Let Ω be an open subset of $\mathbf{R}^n$. Let $u, v : \overline{\Omega} \times [0,T) \to \mathbf{R}$. Assume that $u, -v$ are upper semicontinuous on $\overline{\Omega} \times [0,T)$ and that u and v are, respectively, a viscosity subsolution and a viscosity supersolution of*

$$u_t + H(x, Du) = 0 \quad \textit{in } \Omega \times (0,T). \tag{4.1}$$

Moreover, assume that

$$\lim_{r\to\infty} \inf\bigl\{v(x,t) - \phi_1(x) \mid (x,t) \in \bigl(\Omega \setminus B(0,r)\bigr) \times [0,T)\bigr\} = \infty, \tag{4.2}$$

and that $u \leqslant v$ on $(\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Then $u \leqslant v$ in $\overline{\Omega} \times [0,T)$.

Proof. We choose a constant $C > 0$ so that

$$H\big(x, D\phi_1(x)\big) \leqslant C \quad \text{a.e. } x \in \mathbf{R}^n,$$

and define the function $w \in C(\mathbf{R}^n \times \mathbf{R})$ by

$$w(x,t) := \phi_1(x) - Ct.$$

Observe that $w_t + H(x, Dw(x,t)) \leqslant 0$ a.e. $(x,t) \in \mathbf{R}^{n+1}$.

We need only to show that for all $(x,t) \in \overline{\Omega}$ and all $A > 0$,

$$\min\big\{u(x,t), w(x,t) + A\big\} \leqslant v(x,t). \tag{4.3}$$

Fix any $A > 0$. We set $w_A(x,t) = w(x,t) + A$ for $(x,t) \in \mathbf{R}^{n+1}$. The function w_A is a viscosity subsolution of (4.1). By the convexity of $H(x,p)$ in p, the function $\bar{u}$ defined by $\bar{u}(x,t) := \min\{u(x,t), w_A(x,t)\}$ is a viscosity subsolution of (4.1). Because of assumption (4.2), we see that there is a constant $R > 0$ such that $\bar{u}(x,t) \leqslant v(x,t)$ for all $(x,t) \in (\overline{\Omega} \setminus B(0,R)) \times [0,T)$. We set $\Omega_R := \Omega \cap \operatorname{int} B(0,2R)$, so that $\bar{u}(x,t) \leqslant v(x,t)$ for all $x \in \partial\Omega_R \times [0,T)$. Also, we have $\bar{u}(x,0) \leqslant u(x,0) \leqslant v(x,0)$ for all $x \in \Omega_R$.

Next we wish to use standard comparison results. However, H does not satisfy the usual assumptions for comparison. We thus take the sup-convolution of $\bar{u}$ in the variable t and take advantage of the coercivity of H. That is, for each $\varepsilon \in (0,1)$ we set

$$u^\varepsilon(x,t) := \sup_{s \in [0,T)} \left(\bar{u}(x,s) - \frac{(t-s)^2}{2\varepsilon}\right) \quad \text{for all } (x,t) \in \overline{\Omega}_R \times \mathbf{R}.$$

For each $\delta > 0$, there is a $\gamma \in (0, \min\{\delta, T/2\})$ such that $\bar{u}(x,t) - \delta \leqslant v(x,t)$ for all $(x,t) \in \overline{\Omega}_R \times [0,\gamma]$. As is well known, there is an $\varepsilon \in (0,\delta)$ such that u^ε is a viscosity subsolution of (4.1) in $\Omega_R \times (\gamma, T-\gamma)$ and $u^\varepsilon(x,t) - 2\delta \leqslant v(x,t)$ for all $(x,t) \in (\overline{\Omega}_R \times [0,\gamma]) \cup (\partial\Omega_R \times [\gamma, T-\gamma])$. Observe that the family of functions: $t \mapsto u^\varepsilon(x,t)$ on $[\gamma, T-\gamma]$, with $x \in \overline{\Omega}_R$, is equi-Lipschitz continuous, with a Lipschitz bound $C_\varepsilon > 0$, and therefore that for each $t \in [\gamma, T-\gamma]$, the function $z : x \mapsto u^\varepsilon(x,t)$ in Ω_R satisfies $H(x, Dz(x)) \leqslant C_\varepsilon$ a.e., which implies that the family of functions: $x \mapsto u^\varepsilon(x,t)$, with $t \in [\gamma, T-\gamma]$, is equi-Lipschitz continuous in Ω_R.

Now, we may apply a standard comparison theorem, to get $u^\varepsilon(x,t) \leqslant v(x,t)$ for all $(x,t) \in \Omega_R \times [\gamma, T-\gamma]$, from which we get $\bar{u}(x,t) \leqslant v(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T)$. This completes the proof. □

5. Cauchy problem

Let $c \equiv c_H$ be the (additive) eigenvalue for H. In this and the following sections we assume without loss of generality that $c = 0$. Indeed, if we set $H_c(x,y) = H(x,y) - c$ and $L_c(x,y) = L(x,y) + c$ for $(x,y) \in \mathbf{R}^{2n}$, then the stationary Hamilton–Jacobi equation $H[v] = c$ for v is exactly $H_c[v] = 0$ for v and the evolution equation $u_t + H[u] = 0$ for u is the equation $w_t + H_c[w] = 0$ for $w(x,t) := u(x,t) + ct$. Note moreover that L_c is the Lagrangian of the Hamiltonian H_c, i.e., $L_c(x,\xi) = \sup\{\xi \cdot p - H_c(x,p) \mid p \in \mathbf{R}^n\}$ for all $x, \xi \in \mathbf{R}^n$. With these relations in mind, by replacing H and L by H_c and L_c, respectively, we may assume that $c = 0$.

We make another normalization. We fix a viscosity solution $\phi \in \Phi_0$ of $H[\phi] = 0$ in $\mathbf{R}^n$. We choose a constant $r > 0$ so that $\sigma_i(x) \geqslant 0$ for all $x \in \mathbf{R}^n \setminus B(0,r)$. There is a constant $M > 0$ such that $\phi(x) - M \leqslant \phi_1(x)$ for all $x \in B(0,r)$. We set $\zeta_1(x) = \min\{\phi(x) - M, \phi_1(x)\}$ for $x \in \mathbf{R}^n$. Since $\lim_{|x|\to\infty}(\phi - \phi_1)(x) = \infty$, we have $\zeta_1(x) = \phi_1(x)$ for all $x \in \mathbf{R}^n \setminus B(0,R)$ and some $R > r$. Note that $H(x, D\zeta_1(x)) = H(x, D\phi(x)) = 0$ a.e. in $B(0,r)$, $H(x, D\zeta_1(x)) \leqslant \max\{H(x, D\phi(x)), H(x, D\phi_1(x))\} \leqslant 0$ a.e. in $B(0,R) \setminus B(0,r)$, and $H(x, D\zeta_1(x)) = H(x, D\phi_1(x)) = -\sigma_1(x)$ a.e. in $\mathbf{R}^n \setminus B(0,R)$. Therefore, by replacing ϕ_1 and σ_1 by ζ_1 and $\max\{\sigma_1, 0\}$, respectively, we may assume that $\sigma_1 \geqslant 0$ in $\mathbf{R}^n$. Similarly, we define the function $\zeta_0 \in C^{0+1}(\mathbf{R}^n)$ by setting $\zeta_0(x) = \min\{\phi(x) - M, \phi_0(x)\}$ and observe that $H[\zeta_0] \leqslant 0$ in $\mathbf{R}^n$ in the viscosity sense and that $\sup_{\mathbf{R}^n} |\zeta_0 - \phi_0| < \infty$, which implies that $u \in \Phi_0$ if and only if $\inf_{\mathbf{R}^n}(u - \zeta_0) > -\infty$. Henceforth we write ϕ_0 for ζ_0. A warning is that the function $\sigma_0 = 0$ corresponds to the current ϕ_0 and does not have the property: $\lim_{|x|\to\infty} \sigma_0(x) = \infty$.

In this section we prove Theorem 1.1 together with some estimates on the continuity of the solution of (1.1) and (1.2) which satisfies (1.4).

Our strategy here for proving the existence of a viscosity solution of (1.1) and (1.2) which satisfies (1.4) is to prove that the function u on $\mathbf{R}^n \times (0,\infty)$ given by

$$u(x,t) = \inf\left\{ \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s + u_0\big(\gamma(0)\big) \;\Big|\; \gamma \in \mathcal{C}(x,t) \right\} \tag{5.1}$$

is a viscosity solution of (1.1) by using the dynamic programming principle.

In this section u always denotes the function on $\mathbf{R}^n \times [0,\infty)$ whose value $u(x,t)$ given by (5.1) for $t>0$ and by $u_0(x)$ for $t=0$.

Lemma 5.1. *There exists a constant $C_0 > 0$ such that*

$$u(x,t) \geqslant \phi_0(x) - C_0 \quad \text{for all } (x,t) \in \mathbf{R}^n \times [0,\infty).$$

Proof. We choose $C_0 > 0$ so that $u_0(x) \geqslant \phi_0(x) - C_0$ for all $x \in \mathbf{R}^n$. Fix any $(x,t) \in \mathbf{R}^n \times (0,\infty)$. For each $\varepsilon > 0$ there is a curve $\gamma \in \mathcal{C}(x,t)$ such that

$$u(x,t) + \varepsilon > \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s + u_0\big(\gamma(0)\big).$$

By Proposition 2.5, since $H[\phi_0] \leqslant 0$ a.e., we have

$$u(x,t) + \varepsilon > \phi_0\big(\gamma(t)\big) - \phi_0\big(\gamma(0)\big) + u_0\big(\gamma(0)\big) \geqslant \phi_0(x) - C_0,$$

which shows that $u(x,t) \geqslant \phi_0(x) - C_0$. $\square$

Lemma 5.2. *We have*

$$u(x,t) \leqslant u_0(x) + L(x,0)t \quad \text{for all } (x,t) \in \mathbf{R}^n \times (0,\infty).$$

Proof. Fix any $(x,t) \in \mathbf{R}^n \times (0,\infty)$. By choosing the curve $\gamma_x(t) \equiv x$ in formula (5.1), we find that

$$\begin{aligned} u(x,t) &\leqslant \int_0^t L\big(\gamma_x(s), \dot\gamma_x(s)\big)\,\mathrm{d}s + u_0\big(\gamma_x(0)\big) \\ &= \int_0^t L(x,0)\,\mathrm{d}s + u_0(x) = u_0(x) + L(x,0)t. \qquad \square \end{aligned}$$

Proposition 5.3 *(Dynamic Programming Principle). For $t > 0$, $s > 0$, and $x \in \mathbf{R}^n$, we have*

$$u(x, s+t) = \inf\left\{ \int_0^t L\big(\gamma(r), \dot\gamma(r)\big)\,\mathrm{d}r + u\big(\gamma(0), s\big) \;\Big|\; \gamma \in \mathcal{C}(x,t) \right\}. \tag{5.2}$$

We omit giving the proof of this proposition and we refer to [23] for a proof in a standard case.

Lemma 5.4. *For each $R > 0$ there exists a modulus m_R such that*

$$u(x,t) \geqslant u_0(x) - m_R(t) \quad \text{for all } (x,t) \in B(0,R) \times (0,\infty).$$

Proof. Fix any $R > 0$. We choose $C > 0$ and then $\rho > R$ so that $\phi_1(x) + C \geqslant u_0(x) + 1$ for all $x \in B(0,R)$ and $\phi_1(x) + C \leqslant u_0(x) - 1$ for all $x \in \mathbf{R}^n \setminus B(0,\rho)$. Fix any $\varepsilon \in (0,1)$ and choose a function $u_\varepsilon \in C^1(\mathbf{R}^n)$ so that $|u_\varepsilon(x) - u_0(x)| \leqslant \varepsilon$ for all $x \in \mathbf{R}^n$.

We set

$$\phi_\varepsilon(x) = \min\{\phi_1(x) + C, u_\varepsilon(x)\} \quad \text{for } x \in \mathbf{R}^n,$$

and note that $\phi_\varepsilon(x) = u_\varepsilon(x)$ for $x \in B(0, R)$ and $\phi_\varepsilon(x) = \phi_1(x) - C$ for $x \in \mathbf{R}^n \setminus B(0, \rho)$. Next we choose an $M > 0$ so that $|H(x, D\phi_\varepsilon(x))| \leqslant M$ for all $x \in B(0, \rho)$ and observe that $H(x, D\phi_\varepsilon(x)) \leqslant M$ a.e. $x \in \mathbf{R}^n$.

Fix any $(x, t) \in \mathbf{R}^n \times (0, \infty)$ and select a curve $\gamma \in \mathcal{C}(x, t)$ so that

$$u(x, t) + \varepsilon > \int_0^t L(\gamma(s), \dot{\gamma}(s))\, \mathrm{d}s + u_0(\gamma(0)).$$

Using Proposition 2.5, we get

$$\begin{aligned} u(x, t) + \varepsilon &> \phi_\varepsilon(\gamma(t)) - \phi_\varepsilon(\gamma(0)) - Mt + u_0(\gamma(0)) \\ &\geqslant \phi_\varepsilon(x) - Mt - u_\varepsilon(\gamma(0)) + u_0(\gamma(0)), \end{aligned}$$

which shows that $u(x, t) \geqslant u_0(x) - Mt - 2\varepsilon$ for all $(x, t) \in B(0, R) \times [0, \infty)$. Writing M_ε for M in view of its dependence on ε and setting $m_R(t) = \inf\{2\varepsilon + M_\varepsilon t \mid \varepsilon \in (0, 1)\}$ for $t \geqslant 0$, we find a modulus m_R for which $u(x, t) \geqslant u_0(x) - m_R(t)$ for all $(x, t) \in B(0, R) \times [0, \infty)$. □

Theorem 5.5. *The function u is continuous in $\mathbf{R}^n \times [0, \infty)$ and is a viscosity solution of* (1.1).

This theorem together with Lemma 5.1 and Theorem 4.1 completes the proof of Theorem 1.1.

Proof. We define the upper and lower semicontinuous envelopes u^* and u_* of u, respectively, by

$$\begin{aligned} u^*(x, t) &= \lim_{r \to +0} \sup\{u(y, s) \mid (y, s) \in \mathbf{R}^n \times [0, \infty), |y - x| + |s - t| < r\}, \\ u_*(x, t) &= \lim_{r \to +0} \inf\{u(y, s) \mid (y, s) \in \mathbf{R}^n \times [0, \infty), |y - x| + |s - t| < r\}. \end{aligned}$$

We now invoke some results established in Appendix A. That is, we apply Theorems A.1 and A.2 together with remark after these theorems, to conclude that u^* and u_* are a viscosity subsolution and a viscosity supersolution of (1.1), respectively.

We observe by Lemmas 5.2 and 5.4 that $u^*(x, 0) = u_*(x, 0) = u_0(x)$ for all $x \in \mathbf{R}^n$ and by Lemma 5.1 that $u_*(x, t) \geqslant \phi_0(x) - C_0$ for all $(x, t) \in \mathbf{R}^n \times [0, \infty)$ and some constant $C_0 > 0$. We apply Theorem 4.1, to conclude that $u_* \leqslant u^*$ in $\mathbf{R}^n \times [0, \infty)$, which implies that $u = u^* = u_* \in C(\mathbf{R}^n \times [0, \infty))$, completing the proof. □

Lemma 5.6. *For each $R > 0$ there exists a constant $C_R > 0$ such that $u(x, t) \leqslant C_R$ for all $(x, t) \in B(0, R) \times [0, \infty)$.*

Proof. Fix a viscosity solution $\phi \in \Phi_0$ of (1.3). Fix any $R > 0$. We choose a constant $C > 0$ and then a constant $\rho > R$ so that $\phi_1(x) + C > \phi(x)$ for all $x \in B(0, R)$ and $\phi_1(x) + C \leqslant \phi(x)$ for all $x \in \mathbf{R}^n \setminus B(0, \rho)$.

Next we choose a constant $K > 0$ so that $\min\{\phi(x), \phi_1(x) + C\} + K \geqslant u_0(x)$ for all $x \in B(0, \rho)$ and set $v(x, t) = \min\{u(x, t), \phi_1(x) + C + K\}$ for $(x, t) \in \mathbf{R}^n \times [0, \infty)$. Observe that v is a viscosity subsolution of (1.1) and that $v(x, 0) \leqslant u_0(x) \leqslant \phi(x) + K$ for $x \in B(0, \rho)$ and $v(x, 0) \leqslant \phi_1(x) + C + K \leqslant \phi(x) + K$ for $x \in \mathbf{R}^n \setminus B(0, \rho)$. Therefore, since $w(x, t) := \phi(x) + K$ is a viscosity solution of (1.1), by Theorem 4.1 we obtain $v(x, t) \leqslant \phi(x) + K$ for all $(x, t) \in \mathbf{R}^n \times [0, \infty)$. In particular, since $\phi_1(x) + C + K > \phi(x) + K$ for all $x \in B(0, R)$, we get $u(x, t) \leqslant \phi(x) + K$ for all $(x, t) \in B(0, R) \times [0, \infty)$, from which we conclude that $u(x, t) \leqslant C_R$ for all $(x, t) \in B(0, R) \times [0, \infty)$, with $C_R := \max_{B(0,R)} \phi + K$. □

Lemma 5.7. *For each $R > 0$ there exists a modulus l_R such that $|u(x, t) - u(y, s)| \leqslant l_R(|x - y| + |t - s|)$ for all $(x, t), (y, s) \in B(0, R) \times [0, \infty)$.*

Proof. Fix any $\varepsilon \in (0, 1)$ and choose a function $v_0 \in C^1(\mathbf{R}^n)$ so that $|v_0(x) - u_0(x)| \leqslant \varepsilon$ for all $x \in \mathbf{R}^n$. Let $v \in C(\mathbf{R}^n \times [0, \infty))$ be the unique solution of (1.1)–(1.2) satisfying (1.4), with v in place of u. Existence and uniqueness of such a solution is guaranteed by Theorem 1.1. By Theorem 4.1, we have $|u(x,t)-v(x,t)| \leqslant \varepsilon$ for all $(x,t) \in \mathbf{R}^n \times [0,\infty)$.

We wish to show that for each $R>0$ the function v is Lipschitz continuous on $B(0,R)\times[0,\infty)$.

For each $\rho>0$ we choose a constant $A_\rho>0$ so that

$$\bigl|H\bigl(x, Dv_0(x)\bigr)\bigr| \leqslant A_\rho \quad \text{for all } x \in B(0,\rho). \tag{5.3}$$

In view of Lemma 5.6, for each $R>0$ we may choose a constant $C_R>0$ so that $\phi_1(x)+C_R > u(x,t)+1$ for all $(x,t)\in B(0,R)\times[0,\infty)$. In view of Lemma 5.1, we may choose a constant $C_0>0$ so that $u(x,t)\geqslant \phi_0(x)-C_0$ for all $(x,t)\in\mathbf{R}^n\times[0,\infty)$.

Fix any $R>0$ and then $\rho>R$ so that

$$\phi_0(x)-2-C_0 \geqslant \phi_1(x)+C_R \quad \text{for all } x\in\mathbf{R}^n\setminus B(0,\rho). \tag{5.4}$$

We define $w\in C(\mathbf{R}^n\times[0,\infty))$ by $w(x,t)=\min\{v_0(x)-A_\rho t, \phi_1(x)+C_R\}$.

Note that for any $(x,t)\in(\mathbf{R}^n\setminus B(0,\rho))\times[0,A_\rho^{-1})$,

$$v_0(x)-A_\rho t \geqslant u_0(x)-2 \geqslant \phi_0(x)-2-C_0 \geqslant \phi_1(x)+C_R,$$

and therefore $w(x,t)=\phi_1(x)+C_R$. Consequently, we have

$$w_t(x,t)+H\bigl(x,Dw(x,t)\bigr)\leqslant 0 \quad \text{a.e. } (x,t)\in\mathbf{R}^n\times\bigl(0,A_\rho^{-1}\bigr).$$

That is, w is a viscosity subsolution of (1.1) in $\mathbf{R}^n\times(0,A_\rho^{-1})$. Observe as well that $w(x,0)\leqslant v_0(x)$ for all $x\in\mathbf{R}^n$. We may now apply Theorem 3.1, to conclude that $w(x,t)\leqslant v(x,t)$ for all $(x,t)\in\mathbf{R}^n\times[0,A_\rho^{-1})$. Since $\phi_1(x)+C_R > v_0(x)$ for all $x\in B(0,R)$ by our choice of C_R, we see that $w(x,t)=v_0(x)-A_\rho t$ for all $(x,t)\in B(0,R)\times[0,\infty)$. Thus, setting $K_R=A_\rho$, we see that for any $R>0$,

$$v_0(x)-K_R t\leqslant v(x,t) \quad \text{for all } (x,t)\in B(0,R)\times\bigl[0,K_R^{-1}\bigr]. \tag{5.5}$$

Next we fix any $R>0$ and $0<h<K_\rho^{-1}$, where K_ρ is a constant for which (5.5) holds with ρ in place of R, and define $z\in C(\mathbf{R}^n\times[0,\infty))$ by $z(x,t)=\min\{v(x,t)-K_\rho h, \phi_1(x)+C_R\}$. Observe that z is a viscosity subsolution of (1.1), that $z(x,0)\leqslant v(x,0)-K_\rho h\leqslant v(x,h)$ for $x\in B(0,\rho)$ by (5.5), that if $x\in\mathbf{R}^n\setminus B(0,\rho)$, then

$$z(x,0)\leqslant\phi_1(x)+C_R<\phi_0(x)-2-C_0\leqslant u(x,h)-2<v(x,h).$$

Now, by comparison, we get $z(x,t)\leqslant v(x,t+h)$ for all $(x,t)\in\mathbf{R}^n\times[0,\infty)$. Noting that if $x\in B(0,R)$, then $v(x,t)-K_\rho h\leqslant u(x,t)+1<\phi_1(x)+C_R$, we find that $v(x,t)-K_\rho h=z(x,t)\leqslant v(x,t+h)$ for all $(x,t)\in B(0,R)\times[0,\infty)$. Setting $M_R=K_\rho$, we thus obtain

$$v(x,t)+M_R t\leqslant v(x,t+h)+M_R(t+h)$$

for all $(x,t)\in B(0,R)\times[0,\infty)$ and $h\in[0,M_R^{-1}]$. We now conclude that for any $R>0$ and $x\in B(0,R)$, the function: $t\mapsto v(x,t)+M_R t$ is nondecreasing on $[0,\infty)$.

Fix any $R>0$ and observe that $H(x,Dv(x,t))\leqslant M_R$ in $\operatorname{int}B(0,R)$ in the viscosity sense, which implies together with (A2) the Lipschitz continuity of $v(x,t)$ in $x\in B(0,R)$ uniformly in $t\geqslant 0$, that is, there exists a constant $L_R>0$ such that $|v(x,t)-v(y,t)|\leqslant L_R|x-y|$ for all $x,y\in B(0,R)$ and $t\geqslant 0$.

Now, we note that $\inf_{B(0,R)\times\mathbf{R}^n} H>-\infty$. We may assume by replacing L_R by a larger constant if necessary that $M_R\leqslant L_R$ and $H(x,p)\geqslant -L_R$ for all $(x,p)\in B(0,R)\times\mathbf{R}^n$. Noting that v is a viscosity solution of $v_t\leqslant L_R$ in $\operatorname{int}B(0,R)\times(0,\infty)$, we see that for any $x\in B(0,R)$ the function: $t\mapsto v(x,t)-L_R t$ is nonincreasing on $[0,\infty)$. In conclusion, we find that $|v(x,t)-v(y,s)|\leqslant L_R(|x-y|+|t-s|)$ for all $(x,t),(y,s)\in B(0,R)\times[0,\infty)$ and moreover that $|u(x,t)-u(y,s)|\leqslant 2\varepsilon+L_R(|x-y|+|t-s|)$ for all $(x,t),(y,s)\in B(0,R)\times[0,\infty)$. This ensures the existence of a modulus l_R such that $|u(x,t)-u(y,s)|\leqslant l_R(|x-y|+|t-s|)$ for all $(x,t),(y,s)\in B(0,R)\times[0,\infty)$. □

Theorem 5.8. *For each $R>0$ the function u is bounded and uniformly continuous on $B(0,R)\times[0,\infty)$.*

Proof. The required boundedness of u follows from Lemmas 5.1 and 5.6, and hence Lemma 5.7 concludes the proof. □

6. Extremal curves

We are assuming as before that $c_H = 0$. Eq. (1.3) reads

$$H\big(x, Du(x)\big) = 0 \quad \text{in } \mathbf{R}^n. \tag{6.1}$$

Henceforth $\mathcal{S}_H^-$, $\mathcal{S}_H^+$, and $\mathcal{S}_H$ denote the sets of continuous viscosity subsolutions, of continuous viscosity supersolutions, and of continuous viscosity solutions of (6.1), respectively.

Let $\phi \in \mathcal{S}_H^-$ and $I \subset \mathbf{R}$ be an interval. Note by Proposition 2.5 that if $[a, b] \subset I$ and $\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n)$, then

$$\phi\big(\gamma(b)\big) - \phi\big(\gamma(a)\big) \leqslant \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t.$$

We call any $\gamma \in C(I, \mathbf{R}^n)$ an *extremal curve* for ϕ on I if for any interval $[a, b] \subset I$, we have $\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n)$ and

$$\int_a^b L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s = \phi\big(\gamma(b)\big) - \phi\big(\gamma(a)\big).$$

Let $\mathcal{E}(I, \phi)$ denote the set of all extremal curves for ϕ on I.

In this section we are concerned with existence of extremal curves.

Theorem 6.1. *Let $S, T \in \mathbf{R}$ satisfy $S < T$. Let $x \in \mathbf{R}^n$ and $\phi \in \mathcal{S}_H \cap \Phi_0$. Then there exists a curve $\gamma \in \mathcal{E}([S, T], \phi)$ such that $\gamma(T) = x$.*

Theorem 6.1 has the following consequence.

Corollary 6.2. *Let $x \in \mathbf{R}^n$ and $\phi \in \mathcal{S}_H \cap \Phi_0$. Then there exists a curve $\gamma \in \mathcal{E}((-\infty, 0], \phi)$ such that $\gamma(0) = x$.*

Proof. Due to Theorem 6.1, for each $y \in \mathbf{R}^n$ we may choose a curve $\gamma_y \in \mathcal{E}([-1, 0], \phi)$ such that $\gamma_y(0) = y$. We define the sequence $\{\xi_j\}_{j \in \mathbf{N}} \subset \mathbf{R}^n$ inductively as $\xi_1 = \gamma_x(-1)$, $\xi_2 = \gamma_{\xi_1}(-1)$, $\xi_3 = \gamma_{\xi_2}(-1), \ldots,$ and the curve $\gamma \in C((-\infty, 0], \mathbf{R}^n)$ by

$$\gamma(t) = \begin{cases} \gamma_x(t) & \text{for } t \in (-1, 0], \\ \gamma_{\xi_1}(t+1) & \text{for } t \in (-2, -1], \\ \gamma_{\xi_2}(t+2) & \text{for } t \in (-3, -2], \\ \vdots & \end{cases}$$

It is not hard to check that $\gamma \in \mathcal{E}((-\infty, 0], \phi)$. Also, it is obvious that $\gamma(0) = x$. $\square$

We need the following lemmas for the proof of Theorem 6.1.

Lemma 6.3. *Let $T > 0$ and let $\{\gamma_k\}_{k \in \mathbf{N}} \subset \mathrm{AC}([0, T], \mathbf{R}^n)$ be a sequence converging to a function $\gamma \in C([0, T], \mathbf{R}^n)$ in the topology of uniform convergence. Assume that*

$$\liminf_{k \to \infty} \int_0^T L\big(\gamma_k(t), \dot\gamma_k(t)\big)\,\mathrm{d}t < \infty.$$

Then $\gamma \in \mathrm{AC}([0, T], \mathbf{R}^n)$ and

$$\int_0^T L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t \leqslant \liminf_{k \to \infty} \int_0^T L\big(\gamma_k(t), \dot\gamma_k(t)\big)\,\mathrm{d}t. \tag{6.2}$$

The following lemma will be used in the proof of Lemma 6.3.

Lemma 6.4. *Let $T > 0$, $C > 0$, and $R > 0$. Let $\gamma \in \mathrm{AC}([0,T], \mathbf{R}^n)$ be such that*

$$\int_0^T L\bigl(\gamma(t), \dot\gamma(t)\bigr)\,\mathrm{d}t \leqslant C \quad \text{and} \quad \gamma(t) \in B(0,R) \quad \text{for all } t \in [0,T].$$

Then for each $\varepsilon > 0$ there exits a constant $M_\varepsilon > 0$ depending only on ε, T, C, R, and H, such that for all measurable $B \subset [0,T]$,

$$\int_B \bigl|\dot\gamma(t)\bigr|\,\mathrm{d}t \leqslant \varepsilon + M_\varepsilon |B|, \tag{6.3}$$

where $|B|$ denotes the Lebesgue measure of $B \subset \mathbf{R}$.

Proof. We choose a constant $C_1 > 0$ so that $H(x,0) \leqslant C_1$ for all $x \in B(0,R)$, which guarantees that $L(x,\xi) \geqslant -C_1$ for all $(x,\xi) \in B(0,R) \times \mathbf{R}^n$. For each $\varepsilon > 0$ we set

$$M(\varepsilon) = \max\bigl\{\bigl|H(x,p)\bigr| \bigm| (x,p) \in B(0,R) \times B\bigl(0, \varepsilon^{-1}\bigr)\bigr\},$$

so that for $(x,\xi) \in B(0,R) \times \mathbf{R}^n$,

$$L(x,\xi) \geqslant \max\bigl\{\xi \cdot p - H(x,p) \mid p \in B\bigl(0,\varepsilon^{-1}\bigr)\bigr\} \geqslant \varepsilon^{-1}|\xi| - M(\varepsilon).$$

Now, let $B \subset [0,T]$ be a measurable set, and observe that

$$\int_B \bigl(L\bigl(\gamma(t), \dot\gamma(t)\bigr) + C_1\bigr)\,\mathrm{d}t \leqslant \int_0^T \bigl(L\bigl(\gamma(t), \dot\gamma(t)\bigr) + C_1\bigr)\,\mathrm{d}t \leqslant C + C_1 T,$$

from which we get

$$\int_B \bigl(\varepsilon^{-1}\bigl|\dot\gamma(t)\bigr| + C_1 - M(\varepsilon)\bigr)\,\mathrm{d}t \leqslant C + C_1 T.$$

Hence we have

$$\int_B \bigl|\dot\gamma(t)\bigr|\,\mathrm{d}t \leqslant \varepsilon(C + C_1 T) + \varepsilon M(\varepsilon)|B|,$$

which shows that (6.3) holds with $M_\varepsilon = \delta M(\delta)$, where $\delta = \varepsilon(C + C_1 T)^{-1}$. $\square$

Proof of Lemma 6.3. We choose a constant $R > 0$ so that $|\gamma_k(t)| \leqslant R$ for all $t \in [0,T]$ and all $k \in \mathbf{N}$. Passing to a subsequence of $\{\gamma_k\}_{k\in\mathbf{N}}$ if necessary, we may assume that there is a constant $C > 0$ such that

$$\int_0^T L\bigl(\gamma_k(t), \dot\gamma_k(t)\bigr)\,\mathrm{d}t \leqslant C \quad \text{for all } k \in \mathbf{N}.$$

Now, by Lemma 6.4, for each $\varepsilon > 0$ we may choose a constant $M(\varepsilon) > 0$ so that for any measurable $B \subset [0,T]$ and for all $k \in \mathbf{N}$,

$$\int_B \bigl|\dot\gamma_k(t)\bigr|\,\mathrm{d}t \leqslant \varepsilon + M(\varepsilon)|B|. \tag{6.4}$$

We deduce from (6.4) that for any $\varepsilon > 0$ and any mutually disjoint intervals $[a_i, b_i] \subset [0,T]$, with $i = 1, 2, \ldots, m$,

$$\sum_{i=1}^m \bigl|\gamma(b_i) - \gamma(a_i)\bigr| \leqslant \varepsilon + M(\varepsilon)\sum_{i=1}^m (b_i - a_i),$$

which shows that $\gamma \in \mathrm{AC}([0,T],\mathbf{R}^n)$ and

$$\int_B \left|\dot\gamma(t)\right| \mathrm{d}t \leqslant \varepsilon + M(\varepsilon)|B| \tag{6.5}$$

for any measurable subset B of $[0,T]$.

Next let $f \in \mathrm{AC}([0,T],\mathbf{R}^n)$ and observe by using integration by parts that as $k\to\infty$

$$\begin{aligned}\int_0^T f(t)\cdot\dot\gamma_k(t)\,\mathrm{d}t &= (f\cdot\gamma_k)(T)-(f\cdot\gamma_k)(0)-\int_0^T \dot f(t)\cdot\gamma_k(t)\,\mathrm{d}t\\ &\to (f\cdot\gamma)(T)-(f\cdot\gamma)(0)-\int_0^T \dot f(t)\cdot\gamma(t)\,\mathrm{d}t\\ &= \int_0^T f(t)\cdot\dot\gamma(t)\,\mathrm{d}t.\end{aligned}$$

Now we introduce the Lagrangian L_α, with $\alpha>0$, as follows. Fix $\alpha>0$ and define the function $H_\alpha : \mathbf{R}^{2n}\to(0,\infty]$ by

$$H_\alpha(x,p)=H(x,p)+\frac{|p|^2}{\alpha}+\delta_{B(0,\alpha)}(p),$$

where δ_C denotes the indicator function of $C\subset\mathbf{R}^n$ defined by $\delta_C(p)=0$ if $p\in C$ and $=\infty$ otherwise. Next define the function $L_\alpha : \mathbf{R}^{2n}\to\mathbf{R}$ as the Lagrangian of H_α, that is, $L_\alpha(x,\xi)=\sup\{\xi\cdot p-H_\alpha(x,p)\mid p\in\mathbf{R}^n\}$ for $(x,\xi)\in\mathbf{R}^{2n}$. It is easy to see that, for all $(x,\xi)\in\mathbf{R}^{2n}$, $L_\alpha(x,\xi)\leqslant L_\beta(x,\xi)\leqslant L(x,\xi)$ if $\alpha<\beta$, that $\lim_{\alpha\to\infty}L_\alpha(x,\xi)=L(x,\xi)$ for all $(x,\xi)\in\mathbf{R}^{2n}$, and that for any $(x,\xi)\in\mathbf{R}^{2n}$, if $p\in D_2^-L_\alpha(x,\xi)$, then $|p|\leqslant\alpha$. Also, as is well known, for any $\alpha>0$, L_α is differentiable in the last n variables everywhere and L_α and D_2L_α are continuous on $\mathbf{R}^{2n}$. In view of the monotone convergence theorem, in order to prove (6.2), we need only to show that for any $\alpha>0$,

$$\int_0^T L_\alpha\bigl(\gamma(t),\dot\gamma(t)\bigr)\,\mathrm{d}t\leqslant\liminf_{k\to\infty}\int_0^T L\bigl(\gamma_k(t),\dot\gamma_k(t)\bigr)\,\mathrm{d}t. \tag{6.6}$$

To show (6.6), we fix $\alpha>0$ and note by convexity that for a.e. $t\in(0,T)$ and any $k\in\mathbf{N}$,

$$L_\alpha\bigl(\gamma_k(t),\dot\gamma_k(t)\bigr)\geqslant L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)+D_2L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\cdot\bigl(\dot\gamma_k(t)-\dot\gamma(t)\bigr).$$

Since

$$\left|L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\right|\leqslant\left|L_\alpha\bigl(\gamma_k(t),0\bigr)\right|+\alpha\left|\dot\gamma(t)\right|\leqslant\max_{x\in B(0,R)}\left|L_\alpha(x,0)\right|+\alpha\left|\dot\gamma(t)\right|\in L^1(0,T),$$

by the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty}\int_0^T L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\,\mathrm{d}t=\int_0^T L_\alpha\bigl(\gamma(t),\dot\gamma(t)\bigr)\,\mathrm{d}t.$$

Next, we set $f_k(t)=D_2L_\alpha(\gamma_k(t),\dot\gamma(t))$ and $f(t)=D_2L_\alpha(\gamma(t),\dot\gamma(t))$ for $t\in[0,T]$ and $k\in\mathbf{N}$. Then $f_k, f\in L^\infty(0,T,\mathbf{R}^n)$ for all $k\in\mathbf{N}$, and $|f_k(t)|\leqslant\alpha$ and $|f(t)|\leqslant\alpha$ a.e. $t\in(0,T)$ for all $k\in\mathbf{N}$. We may choose a sequence $\{g_j\}_{j\in\mathbf{N}}\subset\mathrm{AC}([0,T],\mathbf{R}^n)$ so that $g_j(t)\to f(t)$ a.e. $t\in(0,T)$ as $j\to\infty$ and $|g_j(t)|\leqslant\alpha$ for all $t\in[0,T]$, $j\in\mathbf{N}$. Note that $f_k(t)\to f(t)$ a.e. $t\in(0,T)$ as $k\to\infty$ and recall that the almost everywhere convergence implies the convergence in measure. For each $\varepsilon>0$ we set

$$\begin{aligned}\mu(\varepsilon,k)&=\left|\bigl\{t\in(0,T)\bigm|\left|(f_k-f)(t)\right|>\varepsilon\bigr\}\right|\quad\text{for } k\in\mathbf{N},\\ \nu(\varepsilon,j)&=\left|\bigl\{t\in(0,T)\bigm|\left|(g_j-f)(t)\right|>\varepsilon\bigr\}\right|\quad\text{for } j\in\mathbf{N},\end{aligned}$$

and observe that $\lim_{k\to\infty}\mu(\varepsilon,k)=\lim_{j\to\infty}\nu(\varepsilon,j)=0$ for any $\varepsilon>0$.

Fix any $\varepsilon>0$, $\delta>0$, and $k,j\in\mathbf{N}$. Observing that

$$\bigl|\bigl\{t\in(0,T)\bigm|\bigl|(f_k-g_j)(t)\bigr|>2\varepsilon\bigr\}\bigr|\leqslant\mu(\varepsilon,k)+\nu(\varepsilon,j)$$

and using (6.5) with ε replaced by δ or 1, we get

$$\begin{aligned}\left|\int_0^T(f_k-g_j)(t)\cdot\dot\gamma_k(t)\,\mathrm{d}t\right|&\leqslant\int_{|f_k-g_j|>2\varepsilon}2\alpha\bigl|\dot\gamma_k(t)\bigr|\,\mathrm{d}t+\int_{|f_k-g_j|\leqslant2\varepsilon}2\varepsilon\bigl|\dot\gamma_k(t)\bigr|\,\mathrm{d}t\\&\leqslant2\alpha\bigl[\delta+M(\delta)\bigl(\mu(\varepsilon,k)+\nu(\varepsilon,j)\bigr)\bigr]+2\varepsilon\bigl(1+M(1)T\bigr).\end{aligned}$$

Similarly we get

$$\begin{aligned}\left|\int_0^T(g_j-f)(t)\cdot\dot\gamma(t)\,\mathrm{d}t\right|&\leqslant\int_{|g_j-f|>\varepsilon}2\alpha\bigl|\dot\gamma(t)\bigr|\,\mathrm{d}t+\int_{|g_j-f|\leqslant\varepsilon}\varepsilon\bigl|\dot\gamma(t)\bigr|\,\mathrm{d}t\\&\leqslant2\alpha\bigl(\delta+M(\delta)\nu(\varepsilon,j)\bigr)+\varepsilon\bigl(1+M(1)T\bigr).\end{aligned}$$

Hence we have

$$\begin{aligned}\left|\int_0^T(f_k\cdot\dot\gamma_k-f\cdot\dot\gamma)\,\mathrm{d}t\right|&\leqslant4\alpha\bigl(\delta+M(\delta)\bigl(\mu(\varepsilon,k)+\nu(\varepsilon,j)\bigr)\bigr)+3\varepsilon\bigl(1+M(1)T\bigr)\\&\quad+\left|\int_0^T g_j\cdot(\dot\gamma_k-\dot\gamma)\,\mathrm{d}t\right|.\end{aligned}$$

Now, since $g_j\in\mathrm{AC}([0,T],\mathbf{R}^n)$, we have

$$\lim_{k\to\infty}\int_0^T g_j\cdot(\dot\gamma_k-\dot\gamma)\,\mathrm{d}t=0,$$

and hence

$$\limsup_{k\to\infty}\left|\int_0^T(f_k\cdot\dot\gamma_k-f\cdot\dot\gamma)\,\mathrm{d}t\right|\leqslant4\alpha\bigl(\delta+M(\delta)\nu(\varepsilon,j)\bigr)+3\varepsilon\bigl(1+M(1)T\bigr)$$

for any $\varepsilon>0$, $\delta>0$, and $j\in\mathbf{N}$. Sending $j\to\infty$ and then $\varepsilon,\delta\to0$, we see that

$$\lim_{k\to\infty}\int_0^T D_2L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\cdot\dot\gamma_k(t)\,\mathrm{d}t=\int_0^T D_2L_\alpha\bigl(\gamma(t),\dot\gamma(t)\bigr)\cdot\dot\gamma(t)\,\mathrm{d}t.$$

Finally, noting by the Lebesgue dominated convergence theorem that

$$\lim_{k\to\infty}\int_0^T D_2L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\cdot\dot\gamma(t)\,\mathrm{d}t=\int_0^T D_2L_\alpha\bigl(\gamma(t),\dot\gamma(t)\bigr)\cdot\dot\gamma(t)\,\mathrm{d}t,$$

we obtain

$$\lim_{k\to\infty}\int_0^T\bigl(L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)+D_2L_\alpha\bigl(\gamma_k(t),\dot\gamma(t)\bigr)\cdot\bigl(\dot\gamma_k(t)-\dot\gamma(t)\bigr)\bigr)\,\mathrm{d}t=\int_0^T L_\alpha\bigl(\gamma(t),\dot\gamma(t)\bigr)\,\mathrm{d}t$$

and moreover

$$\int_0^T L_\alpha\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t \leqslant \liminf_{k\to\infty}\int_0^T L_\alpha\big(\gamma_k(t),\dot\gamma_k(t)\big)\,\mathrm{d}t \leqslant \liminf_{k\to\infty}\int_0^T L\big(\gamma_k(t),\dot\gamma_k(t)\big)\,\mathrm{d}t,$$

completing the proof. □

Lemma 6.5. *Let $\phi\in\mathcal{S}_H^-\cap\Phi_0$. Let $S<T$, $R>0$, and $C\geqslant 0$. Let $\gamma\in\mathrm{AC}([S,T],\mathbf{R}^n)$ satisfy $\gamma(T)\in B(0,R)$ and*

$$\phi\big(\gamma(T)\big)-\phi\big(\gamma(S)\big)+C>\int_S^T L\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t.$$

Then there exists a constant $M>0$ depending only on ϕ, ϕ_1, R, and C such that $\gamma(t)\in B(0,M)$ for all $t\in[S,T]$.

Proof. Fix any $t\in[S,T)$. By Proposition 2.5, we have

$$\phi\big(\gamma(t)\big)-\phi\big(\gamma(S)\big)\leqslant\int_S^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s.$$

Hence we get

$$\begin{aligned}\phi\big(\gamma(T)\big)-\phi\big(\gamma(t)\big)+C&\geqslant\int_S^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s-\phi\big(\gamma(t)\big)+\phi\big(\gamma(S)\big)\\&\geqslant\int_S^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s-\int_S^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s=\int_t^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s.\end{aligned}$$

Recall by our normalization that $\phi_1\in\mathcal{S}_H^-$. Using Proposition 2.5 again, we get

$$\phi_1\big(\gamma(T)\big)-\phi_1\big(\gamma(t)\big)\leqslant\int_t^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s.$$

Therefore we get

$$(\phi-\phi_1)\big(\gamma(t)\big)\leqslant(\phi-\phi_1)\big(\gamma(T)\big)+C. \tag{6.7}$$

Set $C_1=\max_{B(0,R)}(\phi-\phi_1)$. Since $\lim_{|x|\to\infty}(\phi-\phi_1)(x)=\infty$, there exists a constant $M>R$ such that $\inf_{\mathbf{R}^n\setminus B(0,M)}(\phi-\phi_1)>C_1+C$. Fix such a constant M, and observe by (6.7) that $\gamma(t)\in B(0,M)$. □

Proof of Theorem 6.1. Fix any $\phi\in\mathcal{S}_H\cap\Phi_0$ and $T>S$. We may assume without loss of generality that $S=0$.

Note that the function $u(x,t):=\phi(x)$ on $\mathbf{R}^n\times[0,\infty)$ is a viscosity solution of (1.1). By formula (5.1), we have for any $(x,T)\in\mathbf{R}^n\times(0,\infty)$,

$$\phi(x)=\inf\left\{\int_0^T L\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t+\phi\big(\gamma(0)\big)\;\Big|\;\gamma\in\mathcal{C}(x,T)\right\}. \tag{6.8}$$

Fix any $x\in\mathbf{R}^n$. According to the above identity, for each $k\in\mathbf{N}$ we may choose a curve $\gamma_k\in\mathcal{C}(x,T)$ so that

$$\phi(x)+\frac{1}{k}>\int_0^T L\big(\gamma_k(t),\dot\gamma_k(t)\big)\,\mathrm{d}t+\phi\big(\gamma_k(0)\big). \tag{6.9}$$

We use Lemma 6.5 to see that there is a constant $R > 0$ such that $\gamma_k(t) \in B(0, R)$ for all $t \in [0, T]$ and all $k \in \mathbf{N}$. It now follows from (6.9) that there exists a constant $C > 0$ such that

$$\int_0^T L\big(\gamma_k(t), \dot\gamma_k(t)\big)\,\mathrm{d}t \leqslant C \quad \text{for all } k \in \mathbf{N}.$$

Applying Lemma 6.4, we find that $\|\dot\gamma_k\|_{L^1(0,T)} \leqslant M$ for all $k \in \mathbf{N}$ and for some $M > 0$.

From these observations, we see that the sequence $\{\gamma_k\}_{k\in\mathbf{N}}$ is uniformly bounded and equi-continuous on $[0, T]$. By the Ascoli–Arzela theorem, we may assume by passing to a subsequence if necessary that the sequence $\{\gamma_k\}_{k\in\mathbf{N}}$ is convergent to a function $\gamma \in C([0, T], \mathbf{R}^n)$ in the topology of uniform convergence. Lemma 6.3 together with (6.9) now guarantees that $\gamma \in \mathcal{C}(x, T)$ and that

$$\int_0^T L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t \leqslant \phi(x) - \phi\big(\gamma(0)\big). \tag{6.10}$$

Fix any $a, b \in \mathbf{R}$ so that $0 \leqslant a < b \leqslant T$. Using (6.7) or Proposition 2.5, we have

$$\phi\big(\gamma(a)\big) - \phi\big(\gamma(0)\big) \leqslant \int_0^a L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t,$$

$$\phi\big(\gamma(b)\big) - \phi\big(\gamma(a)\big) \leqslant \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t,$$

$$\phi\big(\gamma(T)\big) - \phi\big(\gamma(b)\big) \leqslant \int_b^T L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t.$$

These together with (6.10) yield

$$\phi\big(\gamma(b)\big) - \phi\big(\gamma(a)\big) = \int_a^b L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t,$$

which shows that $\gamma \in \mathcal{E}([0, T], \phi)$. The proof is now complete. □

We give a useful property of extremal curves in the following proposition.

Proposition 6.6. *Let $T > 0$, $\phi \in \mathcal{S}_H^-$, and $\gamma \in \mathcal{E}([0, T], \phi)$. Then there exists a function $q \in L^\infty(0, T, \mathbf{R}^n)$ such that*

$$L\big(\gamma(t), \dot\gamma(t)\big) = q(t) \cdot \dot\gamma(t) \quad \textit{a.e. } t \in (0, T), \tag{6.11}$$

$$H\big(\gamma(t), q(t)\big) = 0 \quad \textit{a.e. } t \in (0, T), \tag{6.12}$$

$$q(t) \in \partial_c \phi\big(\gamma(t)\big) \quad \textit{a.e. } t \in (0, T). \tag{6.13}$$

Proof. Fix $T > 0$, $\phi \in \mathcal{S}_H^-$, and $\gamma \in \mathcal{E}([0, T], \phi)$. By Proposition 2.4, there is a function $q \in L^\infty(0, T, \mathbf{R}^n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\gamma(t)\big) = q(t) \cdot \dot\gamma(t) \quad \text{a.e. } t \in (0, T), \tag{6.14}$$

$$q(t) \in \partial_c \phi\big(\gamma(t)\big) \quad \text{a.e. } t \in (0, T). \tag{6.15}$$

Hence we get

$$H\big(\gamma(t), q(t)\big) \leqslant 0 \quad \text{a.e. } t \in (0, T). \tag{6.16}$$

Integrating (6.14) over $(0, T)$ and using (6.16), we compute that

$$\phi\big(\gamma(T)\big) - \phi\big(\gamma(0)\big) = \int_0^T q(t)\cdot\dot\gamma(t)\,\mathrm{d}t \leqslant \int_0^T \big[L\big(\gamma(t),\dot\gamma(t)\big) + H\big(\gamma(t),q(t)\big)\big]\,\mathrm{d}t$$
$$= \int_0^T L\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t = \phi\big(\gamma(T)\big) - \phi\big(\gamma(0)\big),$$

which shows that

$$\int_0^T q(t)\cdot\dot\gamma(t)\,\mathrm{d}t = \int_0^T \big[L\big(\gamma(t),\dot\gamma(t)\big) + H\big(\gamma(t),q(t)\big)\big]\,\mathrm{d}t = \int_0^T L\big(\gamma(t),\dot\gamma(t)\big)\,\mathrm{d}t. \tag{6.17}$$

In particular, we get

$$\int_0^T H\big(\gamma(t),q(t)\big)\,\mathrm{d}t = 0,$$

which together with (6.16) yields

$$H\big(\gamma(t),q(t)\big) = 0 \quad \text{a.e. } t\in(0,T).$$

Similarly, since

$$q(t)\cdot\dot\gamma(t) \leqslant L\big(\gamma(t),\dot\gamma(t)\big) + H\big(\gamma(t),q(t)\big) = L\big(\gamma(t),\dot\gamma(t)\big) \quad \text{a.e. } t\in(0,T),$$

we see from (6.17) that

$$q(t)\cdot\dot\gamma(t) = L\big(\gamma(t),\dot\gamma(t)\big) \quad \text{a.e. } t\in(0,T).$$

Thus the function q satisfies conditions (6.11), (6.12), and (6.13). □

7. Proof of Theorem 1.3

This section will be devoted to proving Theorem 1.3. As before, the eigenvalue c_H is assumed to be zero in this section.

Let $\{S_t\}_{t\geqslant 0}$ be the semi-group of mappings on Φ_0 defined by $S_t u_0 = u(\cdot,t)$, where u is the unique viscosity solution of (1.1) and (1.2) satisfying (1.4).

The following proposition is a variant of [12, Lemma 5.2].

Proposition 7.1. *Let K be a compact subset of $\mathbf{R}^n$. Then there exist a constant $\delta\in(0,1)$ and a modulus ω for which if $u_0\in\Phi_0$, $\phi\in\mathcal{S}_H^-$, $\gamma\in\mathcal{E}([0,T],\phi)$, $\gamma([0,T])\subset K$, $T>\tau\geqslant 0$ and $\tau/(T-\tau)\leqslant\delta$, then*

$$S_T u_0\big(\gamma(T)\big) - S_\tau u_0\big(\gamma(0)\big) \leqslant \phi\big(\gamma(T)\big) - \phi\big(\gamma(0)\big) + \frac{\tau T}{T-\tau}\,\omega\left(\frac{\tau}{T-\tau}\right).$$

We need the following lemma for the proof of Proposition 7.1.

Lemma 7.2. *Let K be a compact subset of $\mathbf{R}^n$. Then there exist a constant $\delta\in(0,1)$ and a modulus ω such that for any $T>0$, $\phi\in\mathcal{S}_H^-$, $\gamma\in\mathcal{E}([0,T],\phi)$ satisfying $\gamma([0,T])\subset K$, and $\lambda\in[-\delta,\delta]$,*

$$(1+\lambda)^{-1}L\big(\gamma(t),(1+\lambda)\dot\gamma(t)\big) \leqslant L\big(\gamma(t),\dot\gamma(t)\big) + |\lambda|\omega\big(|\lambda|\big) \quad \textit{a.e. } t\in(0,T).$$

Proof. Set $Q = \{(x,p)\in K\times\mathbf{R}^n \mid H(x,p)\leqslant 0\}$. It is clear that Q is a compact subset of $\mathbf{R}^{2n}$. Define the set $S\subset\mathbf{R}^n\times\mathbf{R}^n$ by

$$S := \big\{(x,\xi)\in Q \mid \xi\in D_2^- H(x,p) \text{ for some } p\in\mathbf{R}^n \text{ such that } (x,p)\in Q\big\}.$$

By Proposition 2.3, the set S is a compact subset of $\operatorname{int}\operatorname{dom} L$. Thus we may choose a constant $\varepsilon > 0$ so that

$$S_\varepsilon := \{(x,\xi) \in \mathbf{R}^{2n} \mid \operatorname{dist}((x,\xi), S) \leqslant \varepsilon\} \subset \operatorname{int}\operatorname{dom} L.$$

We choose an $R > 0$ so that $S \subset B(0,R)$ (the ball on the right-hand side is a ball in $\mathbf{R}^{2n}$) and set $\delta = \min\{1/2, \varepsilon/R\}$, so that for any $(x,\xi) \in S$ and any $\lambda \in (-\delta,\delta)$, $(x,(1+\lambda)\xi) \in S_\varepsilon$. Let ω_0 be a modulus of continuity of the uniformly continuous function D_2L on S_ε.

Fix $T > 0$, $\phi \in \mathcal{S}_H^-$, $\gamma \in \mathcal{E}([0,T],\phi)$ such that $\gamma([0,T]) \subset K$, and $\lambda \in [-\delta,\delta]$. According to Proposition 6.6, there is a function $q \in L^\infty(0,T,\mathbf{R}^n)$ such that

$$H(\gamma(t), q(t)) = 0 \quad \text{and} \quad \dot\gamma(t) \in D_2^- H(\gamma(t), q(t)) \quad \text{a.e. } t \in (0,T). \tag{7.1}$$

Therefore we have $(\gamma(t), \dot\gamma(t)) \in S$ a.e. $t \in (0,T)$. Hence, for any $\mu \in (-\delta,\delta)$, we have

$$(\gamma(t), (1+\mu)\dot\gamma(t)) \in S_\varepsilon \quad \text{a.e. } t \in (0,T).$$

Consequently, for any $\mu \in (-\delta,\delta)$, we have

$$\left|D_2L(\gamma(t),\dot\gamma(t)) - D_2L(\gamma(t),(1+\mu)\dot\gamma(t))\right| \leqslant \omega_0(|\mu||\dot\gamma|) \quad \text{a.e. } t \in (0,T).$$

In view of (7.1), we have

$$L(\gamma(t),\dot\gamma(t)) = \dot\gamma(t)\cdot q(t) - H(\gamma(t),q(t)) = \dot\gamma(t)\cdot D_2L(\gamma(t),\dot\gamma(t)) \quad \text{a.e. } t \in (0,T).$$

Now we compute that for a.e. $t \in (0,T)$,

$$L(\gamma(t),(1+\lambda)\dot\gamma(t)) = L(\gamma(t),\dot\gamma(t)) + \lambda D_2L(\gamma(t),(1+\theta_t\lambda)\dot\gamma(t))\cdot\dot\gamma(t) \tag{7.2}$$

(for some $\theta_t \in (0,1)$, and furthermore)

$$\begin{aligned}&\leqslant L(\gamma(t),\dot\gamma(t)) + \lambda D_2L(\gamma(t),\dot\gamma(t))\cdot\dot\gamma(t) + |\lambda||\dot\gamma(t)|\omega_0(|\lambda||\dot\gamma(t)|)\\ &= (1+\lambda)L(\gamma(t),\dot\gamma(t)) + |\lambda||\dot\gamma(t)|\omega_0(|\lambda||\dot\gamma(t)|).\end{aligned}$$

Setting $\omega(r) = 2R\omega_0(Rr)$, for a.e. $t \in (0,T)$, we have

$$(1+\lambda)^{-1}L(\gamma(t),(1+\lambda)\dot\gamma(t)) \leqslant L(\gamma(t),\dot\gamma(t)) + |\lambda|\omega(|\lambda|). \qquad \square$$

Proof of Proposition 7.1. Let $\delta \in (0,1)$ and ω be those from Lemma 7.2. Fix any $u_0 \in \Phi_0$, $\phi \in \mathcal{S}_H^-$, $\gamma \in \mathcal{E}([0,T],\phi)$ such that $\gamma([0,T]) \subset K$, and $T > \tau \geqslant 0$ such that $\tau(T-\tau)^{-1} \leqslant \delta$. Set $u(x,t) = S_t u_0(x)$ for $(x,t) \in \mathbf{R}^n \times [0,\infty)$. Set $\varepsilon = \tau/(T-\tau) \in [0,\delta]$.

Setting $T_\varepsilon = (1+\varepsilon)^{-1}T$, observing that

$$\begin{aligned}u(\gamma(T),T) &= u(\gamma(T), T_\varepsilon + \varepsilon T_\varepsilon)\\ &= \inf\left\{\int_0^{T_\varepsilon} L(\eta(t),\dot\eta(t))\,\mathrm{d}t + u(\eta(0),\varepsilon T_\varepsilon) \;\middle|\; \eta \in \mathcal{C}(\gamma(T),T_\varepsilon)\right\},\end{aligned}$$

and choosing $\eta(t) := \gamma((1+\varepsilon)t)$ in the above formula, we get

$$u(\gamma(T),T) \leqslant \int_0^{T_\varepsilon} L(\gamma((1+\varepsilon)t),(1+\varepsilon)\dot\gamma((1+\varepsilon)t))\,\mathrm{d}t + u(\gamma(0),\varepsilon T_\varepsilon).$$

By making the change of variables $s = (1+\varepsilon)t$ in the above inequality, we get

$$u(\gamma(T),T) \leqslant \int_0^T (1+\varepsilon)^{-1} L(\gamma(s),(1+\varepsilon)\dot\gamma(s))\,\mathrm{d}s + u(\gamma(0),\varepsilon T_\varepsilon).$$

Using Lemma 7.2, we see immediately that

$$u\big(\gamma(T),T\big) \leqslant \int_0^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s + u\big(\gamma(0),\varepsilon T_\varepsilon\big) + \varepsilon\omega(\varepsilon)T.$$

Observing that $\varepsilon T_\varepsilon = \tau$, we get

$$u\big(\gamma(T),T\big) \leqslant \int_0^T L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s + u\big(\gamma(0),\tau\big) + \frac{\tau T}{T-\tau}\omega\bigg(\frac{\tau}{T-\tau}\bigg).$$

Recalling that $\gamma \in \mathcal{E}([0,T],\phi)$, we obtain

$$u\big(\gamma(T),T\big) - u\big(\gamma(0),\tau\big) \leqslant \phi\big(\gamma(T)\big) - \phi\big(\gamma(0)\big) + \frac{\tau T}{T-\tau}\omega\bigg(\frac{\tau}{T-\tau}\bigg). \qquad \square$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. We fix any $u_0 \in \Phi_0$ and define the functions $u^{\pm} : \mathbf{R}^n \to \mathbf{R}$ by

$$u^+(x) = \limsup_{t\to\infty} S_t u_0(x), \qquad u^-(x) = \liminf_{t\to\infty} S_t u_0(x).$$

Since the function $u(x,t) := S_t u_0(x)$ is bounded and uniformly continuous on $B(0,R)\times[0,\infty)$ for any $R>0$ by Theorem 5.8, we see that $u^{\pm} \in C(\mathbf{R}^n)$ and that $u^+(x) = \limsup^*_{t\to\infty} u(x,t)$ and $u^-(x) = \liminf_{*t\to\infty} u(x,t)$ for all $x\in\mathbf{R}^n$. As is standard in viscosity solutions theory, we have $u^+ \in \mathcal{S}_H^-$ and $u^- \in \mathcal{S}_H^+$. Moreover, by the convexity of $H(x,\cdot)$, we have $u^- \in \mathcal{S}_H^-$. Also, from Lemma 5.1 we see that $u^{\pm} \in \Phi_0$.

To conclude the proof, it is enough to show that $u^+(x) = u^-(x)$ for all $x \in \mathbf{R}^n$.

We fix any $x\in\mathbf{R}^n$. By Corollary 6.2, we find an extremal curve $\gamma \in \mathcal{E}((-\infty,0],u^-)$ such that $\gamma(0)=x$. By Lemma 6.5, we may choose a constant $R>0$ so that $\gamma(t)\in B(0,R)$ for all $t\in(-\infty,0]$. By the definition of u^+, we may choose a divergent sequence $\{t_j\}\subset(0,\infty)$ such that $\lim_{j\to\infty} u(x,t_j) = u^+(x)$. Noting that the sequence $\{\gamma(-t_j)\}\subset B(0,R)$, we may assume by replacing $\{t_j\}$ by one of its subsequences if necessary that $\gamma(-t_j)\to y$ as $j\to\infty$ for some $y\in B(0,R)$.

Fix any $\varepsilon>0$, and choose a $\tau>0$ so that $u^-(y)+\varepsilon > u(y,\tau)$. Let $\delta\in(0,1)$ and ω be those from Proposition 7.1. Let $j\in\mathbf{N}$ be so large that $\tau(t_j-\tau)^{-1}\leqslant\delta$. We now apply Proposition 7.1, to get

$$u(x,t_j) = u\big(\gamma(0),t_j\big) \leqslant u\big(\gamma(-t_j),\tau\big) + u^-\big(\gamma(0)\big) - u^-\big(\gamma(-t_j)\big) + \frac{\tau t_j}{t_j-\tau}\omega\bigg(\frac{\tau}{t_j-\tau}\bigg).$$

Sending $j\to\infty$ yields

$$u^+(x) \leqslant u(y,\tau) + u^-(x) - u^-(y) < u^-(y) + \varepsilon + u^-(x) - u^-(y) = u^-(x)+\varepsilon,$$

from which we conclude that $u^+(x)\leqslant u^-(x)$. This completes the proof. $\square$

8. A formula for asymptotic solutions and Aubry sets

In the previous section we have proved Theorem 1.3 which states that the viscosity solution u of (1.1) and (1.2) satisfying (1.4) approaches to $v_0(x)-ct$ in $C(\mathbf{R}^n)$ as $t\to\infty$, where $(c,v_0)\in\mathbf{R}\times\Phi_0$ is a solution of (1.3). In this section we give a formula for the function v_0.

Let $c=c_H$. Following [17] with small variations in the presentation, we introduce the Aubry set for $H[u]=c$. First of all, we define the function $d_H\in C(\mathbf{R}^n\times\mathbf{R}^n)$ by

$$d_H(x,y) = \sup\big\{v(x) \mid v\in C\big(\mathbf{R}^n\big),\ H[v]\leqslant c \text{ in } \mathbf{R}^n,\ v(y)=0\big\} \tag{8.1}$$

where the inequality $H[v]\leqslant c$ should be understood in the viscosity sense, and $\mathcal{A}_H$ as the set of those $y\in\mathbf{R}^n$ for which the function $d_H(\cdot,y)$ is a viscosity solution of $H[u]=c$ in $\mathbf{R}^n$. We call $\mathcal{A}_H$ the *Aubry set* for H or for $H[u]=c$.

Theorem 8.1. *For any $x \in \mathbf{R}^n$,*

$$v_0(x) = \inf\{d_H(x, y) + d_H(y, z) + u_0(z) \mid y \in \mathcal{A}_H,\ z \in \mathbf{R}^n\}. \tag{8.2}$$

We need several properties of the function d_H and the Aubry set $\mathcal{A}_H$ for the proof of Theorem 8.1 and present them here.

Henceforth we assume as usual that $c = 0$ and that $\phi_0, \phi_1 \in \mathcal{S}_H^-$.

Since the equation, $H[v] = 0$ in $\mathbf{R}^n$, has a viscosity solution in the class Φ_0 by Theorem 3.3 (or 1.2), the set

$$\{v \in \mathcal{S}_H^- \mid v(y) = 0\}$$

is nonempty and, because of the coercivity assumption on H, it is locally equi-Lipschitz continuous. Therefore, the function $d_H(\cdot, y)$ defined by (8.1) is locally Lipschitz continuous on $\mathbf{R}^n$ and vanishes at $x = y$ for any $y \in \mathbf{R}^n$. Since the pointwise supremum of a family of viscosity subsolutions of (8.1) defines a function which is a viscosity subsolution of (8.1), for any $y \in \mathbf{R}^n$, we have $d_H(\cdot, y) \in \mathcal{S}_H^-$. In view of the Perron method, we deduce that, for any $y \in \mathbf{R}^n$, the function $d_H(\cdot, y)$ is a viscosity solution of (8.1) in $\mathbf{R}^n \setminus \{y\}$. Thus we see that

$$y \in \mathbf{R}^n \setminus \mathcal{A}_H \quad \text{if and only if} \quad \exists p \in D_1^- d_H(y, y) \text{ such that } H(y, p) < 0. \tag{8.3}$$

For any $y, z \in \mathbf{R}^n$, the function $w(x) := d_H(x, y) - d_H(z, y)$ is a viscosity subsolution of (8.1) and satisfies $w(z) = 0$. Therefore we have $w(x) \leqslant d_H(x, z)$. That is, we have the triangle inequality for d_H:

$$d_H(x, y) \leqslant d_H(x, z) + d_H(z, y) \quad \text{for all } x, y, z \in \mathbf{R}^n.$$

Also, we see by the definition of d_H that $v(x) - v(y) \leqslant d_H(x, y)$ for any $v \in \mathcal{S}_H^-$ and $x, y \in \mathbf{R}^n$

Proposition 8.2. *The following formula is valid for all $x, y \in \mathbf{R}^n$:*

$$d_H(x, y) = \inf\left\{\int_0^t L(\gamma(s), \dot\gamma(s))\, ds \,\middle|\, t > 0,\ \gamma \in \mathcal{C}(x, t; y, 0)\right\}. \tag{8.4}$$

Proof. We write $\rho(x, y)$ for the right-hand side of (8.4) in this proof.

Let $x, y \in \mathbf{R}^n$, $t > 0$, and $\gamma \in \mathcal{C}(x, t; y, 0)$. Since $d_H(\cdot, y) \in \mathcal{S}_H^-$, by Proposition 2.5, we have

$$d_H(x, y) = d_H(\gamma(t), y) - d_H(\gamma(0), y) \leqslant \int_0^t L(\gamma(s), \dot\gamma(s))\, ds,$$

from which we get $d_H(x, y) \leqslant \rho(x, y)$ for all $x, y \in \mathbf{R}^n$.

Next we show that for each $y \in \mathbf{R}^n$ the function $\rho(\cdot, y)$ is locally Lipschitz continuous on $\mathbf{R}^n$.

Fix any $R > 0$. By Proposition 2.1, there are constants $\varepsilon_R > 0$ and $C_R > 0$ such that $L(x, \xi) \leqslant C_R$ for all $(x, \xi) \in B(0, R) \times B(0, \varepsilon_R)$. Fix any $x, y \in B(0, R)$ and $\delta > 0$, and set $T := (\delta + |x - y|)/\varepsilon_R$ and $\xi = \varepsilon_R(x - y)/(\delta + |x - y|)$. Define the curve $\gamma \in \mathcal{C}(x, T; y, 0)$ by $\gamma(s) = y + s\xi$. Noting that $\xi \in B(0, \varepsilon_R)$, we get

$$\rho(x, y) \leqslant \int_0^T L(\gamma(s), \dot\gamma(s))\, ds = \int_0^T L(y + s\xi, \xi)\, ds \leqslant C_R T = \varepsilon_R^{-1} C_R(\delta + |x - y|).$$

Letting $\delta \to 0$ yields $\rho(x, y) \leqslant \varepsilon_R^{-1} C_R |x - y|$, which, in particular, shows that $\rho(x, x) \leqslant 0$. It is easy to see that for any $x, y, z \in \mathbf{R}^n$, $\rho(x, y) \leqslant \rho(x, z) + \rho(z, y)$. Therefore, for any $x, y, z \in B(0, R)$, we have $|\rho(x, y) - \rho(z, y)| \leqslant \varepsilon_R^{-1} C_R |x - z|$.

In order to prove that $\rho(x, y) \leqslant d_H(x, y)$ for all $x, y \in \mathbf{R}^n$, it is sufficient to show that for any $y \in \mathbf{R}^n$, the function $v := \rho(\cdot, y)$ is a viscosity subsolution of $H[v] = 0$ in $\mathbf{R}^n$. This is a consequence of a well-known observation on value functions like v. Indeed, Theorem A.1 in Appendix A applied to the current v, with $S = \{y\}$ and $\Omega = \mathbf{R}^n$, assures that $v \in \mathcal{S}_H^-$. □

Proposition 8.3. *The Aubry set $\mathcal{A}_H$ is a nonempty compact subset of $\mathbf{R}^n$.*

We need two lemmas to show Proposition 8.3.

Lemma 8.4. *For any compact $K \subset \mathbf{R}^n \setminus \mathcal{A}_H$ there are a function $\phi_K \in \mathcal{S}_H^- \cap \Phi_0$ and a constant $\delta > 0$ such that $H[\phi_K] \leqslant -\delta$ in a neighborhood of K in the viscosity sense.*

Proof. Let $y \in \mathbf{R}^n \setminus \mathcal{A}_H$. There is a function $\varphi \in C^1(\mathbf{R}^n)$ such that $\varphi(y) = 0$, $\varphi(x) < d_H(x, y)$ for all $x \in \mathbf{R}^n \setminus \{y\}$, and $H(y, D\varphi(y)) < 0$. With a sufficiently small constant $\delta > 0$, we set

$$\psi(x) = \max\{\varphi(x) + \delta, d_H(x, y)\} \quad \text{for all } x \in \mathbf{R}^n,$$

to get a function having the properties: (i) $H[\psi] \leqslant 0$ in $\mathbf{R}^n$ in the viscosity sense, (ii) $H[\psi] \leqslant -\varepsilon$ in $\operatorname{int} B(y, \varepsilon)$ in the viscosity sense, and (iii) $\psi \in \Phi_0$. Thus we see that for each $y \in \mathbf{R}^n \setminus \mathcal{A}_H$ there is a pair $(\psi_y, \varepsilon_y) \in \Phi_0 \times (0, \infty)$ such that $H[\psi_y] \leqslant 0$ in $\mathbf{R}^n$ in the viscosity sense and $H[\psi_y] \leqslant -\varepsilon_y$ in $\operatorname{int} B(y, \varepsilon_y)$ in the viscosity sense. By a compactness argument, we find a finite sequence $\{y_j\}_{j=1}^m$ such that $K \subset \bigcup_{j=1}^m \operatorname{int} B(y_j, \varepsilon_j)$, where $\varepsilon_j := \varepsilon_{y_j}$. We set $\varepsilon = \min\{\varepsilon_j \mid j = 1, 2, \ldots, m\}$ and

$$\phi_K(x) = \frac{1}{m} \sum_{j=1}^m \psi_j(x) \quad \text{for all } x \in \mathbf{R}^n, \text{ where } \psi_j := \psi_{y_j}.$$

It is easily seen that $\phi_K \in \Phi_0 \cap \mathcal{S}_H^-$ and $H[\phi_K] \leqslant -\varepsilon/m$ in a neighborhood of K in the viscosity sense. □

Lemma 8.5. *Let $\phi \in C^{0+1}(\mathbf{R}^n)$ be a viscosity solution of $H[\phi] \leqslant 0$ in $\mathbf{R}^n$, y a point in $\mathbf{R}^n$, and $\varepsilon > 0$ a constant. Assume that $H[\phi] \leqslant -\varepsilon$ a.e. in $B(y, \varepsilon)$. Then $y \notin \mathcal{A}_H$.*

Proof. Let ϕ, y, and ε be as above. We argue by contradiction and suppose that $y \in \mathcal{A}_H$. Set $u = d_H(\cdot, y)$. By continuity, there is a constant $\delta > 0$ such that the function $v \in C^{0+1}(\mathbf{R}^n)$, defined by $v(x) = \phi(x) + \delta \min\{|x - y|, \varepsilon\}$, satisfies $H[v] \leqslant 0$ a.e. in $\mathbf{R}^n$. By the definition of d_H, we have $u(x) \geqslant v(x) - v(y)$ for all $x \in \mathbf{R}^n$, which shows that $u(x) > \phi(x) - \phi(y)$ for all $x \in \partial B(y, \varepsilon/2)$ and $u(y) = \phi(y) - \phi(y) = 0$. We approximate ϕ by a sequence of functions $\phi_k \in C^1(\mathbf{R}^n)$, with $k \in \mathbf{N}$, obtained by mollifying ϕ. Here, of course, the uniform convergence $\phi_k(x) \to \phi(x)$ is assumed on any compact subsets of $\mathbf{R}^n$ as $k \to \infty$. We may assume as well that $H[\phi_k] \leqslant -\varepsilon/2$ on $B(y, \varepsilon/2)$. Noting that as $k \to \infty$,

$$\lim_{k \to \infty} \min_{x \in \partial B(y, \varepsilon/2)} \big(u(x) - \phi_k(x) - \phi_k(y)\big) \to \min_{x \in \partial B(y, \varepsilon/2)} \big(u(x) - \phi(x) - \phi(y)\big) > u(y) = 0,$$

we deduce that if k is sufficiently large, then $u - \phi_k$ attains a local minimum at a point $x_k \in B(y, \varepsilon/2)$. For such a k, since $H[u] \geqslant 0$ in $\mathbf{R}^n$ in the viscosity sense, we get

$$H\big(x_k, D\phi_k(x_k)\big) \geqslant 0.$$

On the other hand, by our choice of ϕ_k, we have

$$H\big(x, D\phi_k(x)\big) \leqslant -\frac{\varepsilon}{2} \quad \text{for all } x \in B\left(y, \frac{\varepsilon}{2}\right),$$

and, in particular, $H(x_k, D\phi_k(x_k)) \leqslant -\varepsilon/2$. Thus we get a contradiction, which proves that $y \notin \mathcal{A}_H$. □

Proof of Proposition 8.3. We begin by showing that $\mathcal{A}_H \neq \emptyset$. For this, we suppose that $\mathcal{A}_H = \emptyset$ and will get a contradiction. There is a constant $R > 0$ such that $H[\phi_1] \leqslant -1$ in $\mathbf{R}^n \setminus B(0, R)$ in the viscosity sense. By Lemma 8.4, there are a function $\psi \in \Phi_0$ and a constant $\varepsilon \in (0, 1)$ such that $H[\psi] \leqslant 0$ a.e. in $\mathbf{R}^n$ and $H[\psi] \leqslant -\varepsilon$ a.e. in $B(0, R)$. By setting $v = \frac{1}{2}(\psi + \phi_1)$, we get a function $v \in C^{0+1}(\mathbf{R}^n)$ which satisfies $H[v] \leqslant -\varepsilon/2$ a.e. in $\mathbf{R}^n$. Hence, by the definition of the additive eigenvalue c, we have $c \leqslant -\varepsilon/2$, which contradicts our assumption that $c = 0$.

Using again the fact that $\phi_1 \in \mathcal{S}_H^-$ satisfies $H[\phi_1] \leqslant -1$ in $\mathbf{R}^n \setminus B(0, R)$ in the viscosity sense, we see from Lemma 8.5 that $\mathcal{A}_H \subset B(0, R)$.

It remains to show that $\mathcal{A}_H$ is a closed set. Let $\{y_k\} \subset \mathcal{A}_H$ be a sequence converging to $y \in \mathbf{R}^n$. Because of the coercivity assumption (A2), the sequence $\{d_H(\cdot, y_k)\}$ is locally equi-Lipschitz on $\mathbf{R}^n$. In particular, there is a constant $C > 0$ such that $\max\{d_H(y_k, y), d_H(y, y_k)\} \leqslant C|y_k - y|$ for all $k \in \mathbf{N}$. By the triangle inequality for d_H, we have

$$\big|d_H(x, y) - d_H(x, y_k)\big| \leqslant \max\big\{d_H(y_k, y), d_H(y, y_k)\big\} \leqslant C|y_k - y| \quad \text{for all } x \in \mathbf{R}^n.$$

Consequently, as $k \to \infty$, we have $d_H(x, y_k) \to d_H(x, y)$ uniformly for $x \in \mathbf{R}^n$. By the stability of the viscosity property under uniform convergence, we find that $d_H(\cdot, y) \in \mathcal{S}_H$, proving that $y \in \mathcal{A}_H$ and therefore that $\mathcal{A}_H$ is a closed set. □

Theorem 8.6. *Let $v \in \mathcal{S}_H^-$ and $w \in \mathcal{S}_H^+ \cap \Phi_0$. Assume that $v \leqslant w$ on $\mathcal{A}_H$. Then $v \leqslant w$ on $\mathbf{R}^n$.*

Proof. Fix any $\varepsilon > 0$. Choose a compact neighborhood V of $\mathcal{A}_H$ so that $v(x) \leqslant w(x) + \varepsilon$ for all $x \in V$. Fix a constant $R > 0$ so that $H[\phi_1] \leqslant -1$ a.e. in $\mathbf{R}^n \setminus B(0, R)$. By Lemma 8.4, there are a function $\psi \in C^{0+1}(\mathbf{R}^n)$ such that $H[\psi] \leqslant 0$ a.e. in $\mathbf{R}^n$ and $H[\psi] \leqslant -\delta$ a.e. in $B(0, R) \setminus V$ for some constant $\delta \in (0, 1)$. We set $g(x) = \frac{1}{2}(\phi_1(x) + \psi(x))$ for all $x \in \mathbf{R}^n$ and observe that $H[g] \leqslant -\frac{\delta}{2}$ a.e. in $\mathbf{R}^n \setminus V$. Let $\lambda \in (0, 1)$ and set $v_\lambda(x) = (1 - \lambda)v(x) + \lambda g(x) - 2\varepsilon$ for $x \in \mathbf{R}^n$. Observe that $H[v_\lambda] \leqslant -\frac{\lambda\delta}{2}$ in $\mathbf{R}^n \setminus V$ and that for $\lambda \in (0, 1)$ sufficiently small, $v_\lambda(x) \leqslant w(x)$ for all $x \in V$. We apply Theorem 3.2, to get $v_\lambda(x) \leqslant w(x)$ for all $x \in \mathbf{R}^n \setminus V$ and all λ sufficiently small. That is, if $\lambda \in (0, 1)$ is sufficiently small, then we have $v_\lambda(x) \leqslant w(x)$ for all $x \in \mathbf{R}^n$. From this, we find that $v(x) \leqslant w(x)$ for all $x \in \mathbf{R}^n$. □

The above theorem has the following corollary.

Corollary 8.7. *Let $u \in \mathcal{S}_H$. Then*

$$u(x) = \inf\bigl\{u(y) + d_H(x, y) \mid y \in \mathcal{A}_H\bigr\} \quad \textit{for all } x \in \mathbf{R}^n. \tag{8.5}$$

We refer to [15,17] for previous results related to Corollary 8.7. Also we refer to [22] for a recent result which generalizes the above representation formula.

Proof. We write $v(x)$ for the right-hand side of (8.6). Since v is defined as the pointwise infimum of a family of viscosity solutions, the function v is a viscosity solution of $H[v] = 0$ in $\mathbf{R}^n$. Since $u(x) - u(y) \leqslant d_H(x, y)$ for all $x, y \in \mathbf{R}^n$, we see that $u(x) \leqslant v(x)$ for all $x \in \mathbf{R}^n$. On the other hand, for any $x \in \mathcal{A}_H$, we have $u(x) = u(x) + d_H(x, x) \geqslant v(x)$. Hence Theorem 8.6 guarantees that $u(x) \geqslant v(x)$ for all $x \in \mathbf{R}^n$. □

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. We write $w(x)$ for the right-hand side of (8.2) and set $w_0(x) = \inf\{d_H(x, y) + u_0(y) \mid y \in \mathbf{R}^n\}$ for $x \in \mathbf{R}^n$. Also we write $u(x, t) = S_t u_0(x)$ for $(x, t) \in \mathbf{R}^n \times [0, \infty)$.

By the definition of w_0, it is clear that $w_0(x) \leqslant u_0(x)$ for all $x \in \mathbf{R}^n$. Since $d_H(\cdot, y) \in \mathcal{S}_H^-$ for all $y \in \mathbf{R}^n$, we see that $w_0 \in \mathcal{S}_H^-$. Noting that the function $z(x, t) := w_0(x)$ is a viscosity subsolution of (1.1), we find by Theorem 4.1 that $z(x, t) \leqslant u(x, t)$ for all $(x, t) \in \mathbf{R}^n \times [0, \infty)$, which implies that $w_0 \leqslant v_0$ on $\mathbf{R}^n$. Since $w \leqslant w_0 \leqslant v_0$ on $\mathcal{A}_H$, by Theorem 8.6 we obtain $w \leqslant v_0$ on $\mathbf{R}^n$.

Next we fix any $x \in \mathbf{R}^n$, $y \in \mathcal{A}_H$, and $z \in \mathbf{R}^n$. Note that $d_H(\cdot, y) \in \mathcal{S}_H \cap \Phi_0$. By Corollary 6.2, we may choose a curve $\gamma \in \mathcal{E}((-\infty, 0], d_H(\cdot, y))$ so that $\gamma(0) = x$. By Lemma 6.5, there is a constant $M > 0$ such that $\gamma(t) \in B(0, M)$ for all $t \leqslant 0$. We choose any divergent sequence $\{t_j\}_{j\in\mathbf{N}} \subset (0, \infty)$ such that $\{\gamma(-t_j)\}_{j\in\mathbf{N}}$ is convergent. Let $x_0 \in \mathbf{R}^n$ be the limit of the sequence $\{\gamma(-t_j)\}$.

Arguing as in the last part of the proof of Theorem 1.3, with $d_H(\cdot, y)$ in place of u^-, we obtain

$$u(x, t_j) \leqslant d_H(x, y) - d_H\bigl(\gamma(-t_j), y\bigr) + u\bigl(\gamma(-t_j), t\bigr) + \frac{tt_j}{t_j - t}\omega\left(\frac{t}{t_j - t}\right)$$

for any $t > 0$ if j is large enough, where ω is the modulus from Proposition 7.1. Sending $j \to \infty$ yields

$$v_0(x) \leqslant d_H(x, y) - d_H(x_0, y) + u(x_0, t) \quad \text{for } t > 0.$$

By the variational formula (5.1), we have

$$u(x_0, t) \leqslant \int_0^t L\bigl(\xi(s), \dot{\xi}(s)\bigr)\,\mathrm{d}s + u_0\bigl(\xi(0)\bigr) \quad \text{for any } \xi \in \mathcal{C}(x_0, t).$$

Hence we have

$$v_0(x) \leqslant d_H(x, y) - d_H(x_0, y) + \int_0^t L\big(\xi(s), \dot{\xi}(s)\big)\,\mathrm{d}s + u_0\big(\xi(0)\big)$$

for all $t > 0$ and $\xi \in \mathcal{C}(x_0, t)$. Consequently, we have

$$v_0(x) \leqslant d_H(x, y) - d_H(x_0, y) + \int_0^t L\big(\xi(\sigma), \dot{\xi}(\sigma)\big)\,\mathrm{d}\sigma + \int_0^s L\big(\eta(\sigma), \dot{\eta}(\sigma)\big)\,\mathrm{d}\sigma + u_0(z)$$

for any $t > 0$, $s > 0$, $\xi \in \mathcal{C}(x_0, t; y, 0)$, and $\eta \in \mathcal{C}(y, s; z, 0)$. Therefore, by Proposition 8.2, we get

$$\begin{aligned} v_0(x) &\leqslant d_H(x, y) - d_H(x_0, y) + d_H(x_0, y) + d_H(y, z) + u_0(z) \\ &= d_H(x, y) + d_H(y, z) + u_0(z). \end{aligned}$$

Thus we have $v_0(x) \leqslant w(x)$ for all $x \in \mathbf{R}^n$. The proof is now complete. □

9. Examples

We give two sufficient conditions for H to satisfy (A.4).

Let $H_0 \in C(\mathbf{R}^n \times \mathbf{R}^n)$ and $f \in C(\mathbf{R}^n)$. Set $H(x, p) = H_0(x, p) - f(x)$ for $(x, p) \in \mathbf{R}^n \times \mathbf{R}^n$. We assume that

$$\lim_{|x|\to\infty} f(x) = \infty, \tag{9.1}$$

and that there exists a $\delta > 0$ such that

$$\sup_{\mathbf{R}^n \times B(0,\delta)} |H_0| < \infty. \tag{9.2}$$

Fix such a $\delta > 0$ and set

$$C_\delta = \sup_{\mathbf{R}^n \times B(0,\delta)} |H_0|.$$

Then we define $\phi_i \in C^{0+1}(\mathbf{R}^n)$, with $i = 0, 1$, by setting

$$\phi_0(x) = -\frac{\delta}{2}|x| \quad \text{and} \quad \phi_1(x) = -\delta|x|,$$

and observe that for $i = 0, 1$,

$$H_0\big(x, D\phi_i(x)\big) \leqslant C_\delta \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for $i = 0, 1$, we have

$$H_0\big(x, D\phi_i(x)\big) \leqslant \frac{1}{2} f(x) + C_\delta - \frac{1}{2}\min_{\mathbf{R}^n} f \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

If we set

$$\sigma_i(x) = \frac{1}{2} f(x) - C_\delta + \frac{1}{2}\min_{\mathbf{R}^n} f \quad \text{for } x \in \mathbf{R}^n \text{ and } i = 0, 1,$$

then H satisfies (A.4) with these ϕ_i and σ_i, $i = 0, 1$. It is clear that if H_0 satisfies (A.1)–(A.3), then so does H.

A smaller ϕ_0 yields a larger space Φ_0, and in applications of Theorems 1.1–1.3, it is important to have a larger Φ_0. We are thus interested in finding a smaller ϕ_0. A method better than the above in this respect is as follows. We assume that (9.1), (9.2), and (A.2) with H_0 in place of H hold and that for each $x \in \mathbf{R}^n$ the function: $p \mapsto H_0(x, p)$ is convex in $\mathbf{R}^n$. We fix a function $\theta \in C^1(\mathbf{R}^n)$ so that

$$\lim_{|x|\to\infty} \theta(x) = \infty \quad \text{and} \quad \lim_{|x|\to\infty} \big|D\theta(x)\big| = 0.$$

For instance, the function $\theta(x) = \log(|x|^2 + 1)$ has these properties. Fix an $\varepsilon > 0$ so that $\varepsilon|D\theta(x)| \leqslant \delta/2$ for all $x \in \mathbf{R}^n$. Fix any $\lambda \in (0, 1)$. Define the function $G \in C(\mathbf{R}^n \times \mathbf{R}^n)$ by

$$G(x, p) = \max\bigl\{H_0(x, p), H_0\bigl(x, p - \varepsilon D\theta(x)\bigr)\bigr\} - (1 - \lambda) f(x) - C_\delta + (1 - \lambda) \min_{\mathbf{R}^n} f.$$

We note that for each $x \in \mathbf{R}^n$ the function: $p \mapsto G(x, p)$ is convex in $\mathbf{R}^n$. Define the function $\psi \in C^{0+1}(\mathbf{R}^n)$ by

$$\psi(x) = \inf\bigl\{v(x) \mid v \in C^{0+1}\bigl(\mathbf{R}^n\bigr), G[Dv] \leqslant 0 \text{ a.e. in } \mathbf{R}^n,\ v(0) = 0\bigr\}.$$

Note that $v(x) := -\frac{\delta}{2}|x|$ has the properties: $G(x, Dv(x)) \leqslant 0$ a.e. $x \in \mathbf{R}^n$ and $v(0) = 0$. Hence we have $\psi(x) \leqslant -\frac{\delta}{2}|x|$ for all $x \in \mathbf{R}^n$. Because of the convexity of $G(x, p)$ in p, we see that ψ is a viscosity solution of $G[\psi] \leqslant 0$ in $\mathbf{R}^n$. This implies that ψ and $\psi - \varepsilon\theta$ are both viscosity solutions of

$$H(x, Dv) \leqslant -\lambda f(x) + C_\delta - (1 - \lambda) \min_{\mathbf{R}^n} f \quad \text{in } \mathbf{R}^n.$$

With functions $\phi_0 := \psi$, $\phi_1 := \psi - \varepsilon\theta$, and $\sigma_0 = \sigma_1 := \lambda f - C_\delta + (1 - \lambda) \min_{\mathbf{R}^n} f$, the function H satisfies all the conditions of (A.4). As is already noted, the function ψ satisfies the inequality $\psi(x) \leqslant -\frac{\delta}{2}|x|$ for all $x \in \mathbf{R}^n$. Moreover, for any $\gamma \in (1/2, 1)$, the function $v(x) := -\gamma\delta|x|$ satisfies

$$G\bigl(x, Dv(x)\bigr) \leqslant 0 \quad \text{a.e. } x \in \mathbf{R}^n \setminus B(0, R)$$

for some constant $R \equiv R(\gamma) > 0$. It is now easy to see that if $A > 0$ is large enough, then

$$\psi(x) \leqslant \min\left\{-\frac{\delta}{2}|x|, -\gamma\delta|x| + A\right\} \quad \text{for all } x \in \mathbf{R}^n.$$

Now we examine another class of Hamiltonians H. Let $\alpha > 0$ and let $H_0 \in C(\mathbf{R}^n)$ be a strictly convex function satisfying the superlinear growth condition

$$\lim_{|p| \to \infty} \frac{H_0(p)}{|p|} = \infty.$$

Let $f \in C(\mathbf{R}^n)$. We set

$$H(x, p) = \alpha x \cdot p + H_0(p) - f(x) \quad \text{for } (x, p) \in \mathbf{R}^n \times \mathbf{R}^n.$$

This class of Hamiltonians H is very close to that treated in [19].

Clearly, this function H satisfies (A.1), (A.2), and (A.3). Let L_0 denote the convex conjugate H_0^* of H_0. By the strict convexity of H_0, we see that $L_0 \in C^1(\mathbf{R}^n)$. Define the function $\psi \in C^1(\mathbf{R}^n)$ by

$$\psi(x) = -\frac{1}{\alpha} L_0(-\alpha x).$$

Then we have $D\psi(x) = DL_0(-\alpha x)$ and therefore, by the convex duality, $H_0(D\psi(x)) = D\psi(x) \cdot (-\alpha x) - L_0(-\alpha x)$ for all $x \in \mathbf{R}^n$. Consequently, for all $x \in \mathbf{R}^n$, we have

$$H\bigl(x, D\psi(x)\bigr) = \alpha x \cdot D\psi(x) + H_0\bigl(D\psi(x)\bigr) - f(x) = -L_0(-\alpha x) - f(x).$$

Now we assume that there is a convex function $l \in C(\mathbf{R}^n)$ such that

$$\lim_{|x| \to \infty} \bigl(l(-\alpha x) + f(x)\bigr) = \infty, \tag{9.3}$$

$$\lim_{|\xi| \to \infty} (L_0 - l)(\xi) = \infty. \tag{9.4}$$

Let h denote the convex conjugate of l. We define $\phi \in C^{0+1}(\mathbf{R}^n)$ by $\phi(x) = -\frac{1}{\alpha} l(-\alpha x)$ for $x \in \mathbf{R}^n$. This function ϕ is almost everywhere differentiable. Let $x \in \mathbf{R}^n$ be any point where ϕ is differentiable. By a computation similar to the above for ψ, we get

$$\alpha x \cdot D\phi(x) + h\bigl(D\phi(x)\bigr) - f(x) \leqslant -l(-\alpha x) - f(x). \tag{9.5}$$

By assumption (9.4), there is a constant $C > 0$ such that $L_0(\xi) \geqslant l(\xi) - C$ for all $\xi \in \mathbf{R}^n$. This inequality implies that $H_0 \leqslant h + C$ in $\mathbf{R}^n$. Hence, from (9.5), we get

$$H\bigl(x, D\phi(x)\bigr) \leqslant -l(-\alpha x) - f(x) + C.$$

We now conclude that the function H satisfies (A.4), with the functions $\phi_0 = \phi$, $\phi_1 = \psi$, $\sigma_0(x) = l(-\alpha x) + f(x) - C$, and $\sigma_1(x) = L(-\alpha x) + f(x)$.

It is assumed here that H_0 is strictly convex in $\mathbf{R}^n$, while it is only assumed in [19] that H_0 is just convex in $\mathbf{R}^n$, so that L_0 may not be a C^1 function. The reason why the strict convexity of H_0 is not needed in [19] is in the fact that Hamiltonians H in this class have a simple structure of the Aubry sets. Indeed, if c is the additive eigenvalue of H, then $\min_{p\in\mathbf{R}^n} H(x,p) = c$ for all $x \in \mathcal{A}_H$. Given such a simple property of the Aubry set, the proof of Theorem 1.3 can be simplified greatly and does not require the C^1 regularity of L_0, while such a regularity is needed in the proof of Lemma 7.2 in the general case. Any $x \in \mathcal{A}_H$ is called an *equilibrium point* if $\min_{p\in\mathbf{R}^n} H(x,p) = c$. A characterization of an equilibrium point $x \in \mathcal{A}_H$ is given by the condition that $L(x,0) = -c$. The property of Aubry sets $\mathcal{A}_H$ mentioned above can be stated that the set $\mathcal{A}_H$ comprises only of equilibrium points.

The following example tells us that such a nice property of Aubry sets is not always the case. Let $n = 2$ and here we write (x, y) for a generic point in $\mathbf{R}^2$. We choose a function $g \in C(\mathbf{R}^2)$ so that $g \geqslant 0$ in $\mathbf{R}^2$, $g(x,y) = 0$ for all $(x,y) \in \mathbf{R}^2 \setminus B((0,0),1)$, and $g(x,y) > 0$ for all $(x,y) \in \operatorname{int} B((0,0),1)$. Also, we choose a function $h \in C(\mathbf{R}^2)$ so that $h(x,y) \geqslant 0$ for all $(x,y) \in \mathbf{R}^2$, $h(x,y) = 0$ for all $(x,y) \in B((0,0),2)$, and $h(x,y) \geqslant x^2 + y^2 - 4$ for all $(x,y) \in \mathbf{R}^2$. We define the Hamiltonian $H \in C(\mathbf{R}^4)$ by

$$H(x,y,p,q) = \big(p - g(x,y)\big)^2 + q^2 - g(x,y)^2 - h(x,y).$$

It is clear that this Hamiltonian H satisfies (A.1)–(A.3). Note that (9.1) and (9.2) are satisfied with $H_0(x,y,p,q) = (p - g(x,y))^2 + q^2 - g(x,y)^2$ and $f = h$. Thus we see that H satisfies (A.4) as well. Note moreover that we may take the function: $(x,y) \mapsto \delta|(x,y)|$, with any $\delta > 0$, as ϕ_0 in (A.4).

Note that the zero function $z = 0$ is a viscosity solution of $H[z] \leqslant 0$ in $\mathbf{R}^2$ and that $\min_{(p,q)\in\mathbf{R}^2} H(x,y,p,q) = 0$ for all $(x,y) \in B((0,0),2)$. Therefore, in view of Proposition 3.4, we deduce that the additive eigenvalue c for H is zero.

Now we claim that $\mathcal{A}_H = B((0,0),2) \setminus \operatorname{int} B((0,0),1)$. Since the zero function $z = 0$ satisfies

$$H[z] = -h(x,y) < 0$$

in $\mathbf{R}^2 \setminus B((0,0),2)$, we see by Lemma 8.5 that $\mathcal{A}_H \subset B((0,0),2)$. Let $\phi \in C^{0+1}(\mathbf{R}^2)$ be any viscosity subsolution of $H[\phi] = 0$ in $\mathbf{R}^2$. Then, since $H(x,y,p,q) = (p - g(x,y))^2 + q^2 - g(x,y)^2$ for any $(x,y,p,q) \in \mathbf{R}^2 \times B((0,0),2)$, for almost all $(x,y) \in B((0,0),2)$ we have

$$0 \leqslant \frac{\partial \phi}{\partial x}(x,y) \leqslant 2g(x,y). \tag{9.6}$$

Since $g(x,y) = 0$ for all $(x,y) \in B((0,0),2) \setminus B((0,0),1)$, we find that $D\phi = 0$ a.e. in $B((0,0),2) \setminus B((0,0),1)$ and therefore that $\phi(x,y) = a$ for all $(x,y) \in B((0,0),2) \setminus B((0,0),1)$ and some constant $a \in \mathbf{R}$. The first inequality in (9.6) guarantees that for each $y \in (-1,1)$ the function: $x \mapsto \phi(x,y)$ is nondecreasing in $(-1,1)$. These observations obviously implies that $\phi(x,y) = a$ for all $(x,y) \in B((0,0),2)$. This shows that for any $(x_0,y_0) \in \operatorname{int} B((0,0),2)$, the function $d_H(x,y) \equiv 0$ in a neighborhood of (x_0,y_0) and hence it is a viscosity solution of $H[u] = 0$ in $\mathbf{R}^2$. Thus we see that $\operatorname{int} B((0,0),2) \subset \mathcal{A}_H$. By the fact that $\mathcal{A}_H$ is a closed set, we conclude that $\mathcal{A}_H = B((0,0),2)$.

Finally we remark that $H(x,y,g(x,y),0) = -g(x,y)^2 < 0$ for all $(x,y) \in \operatorname{int} B((0,0),1)$, which shows that any $(x,y) \in \operatorname{int} B((0,0),1)$ is an element of $\mathcal{A}_H$, but not an equilibrium point.

Next we examine another example whose Aubry set does not contain any equilibrium points. As before we consider the two-dimensional case. We fix $\alpha, \beta \in \mathbf{R}$ so that $0 < \alpha < \beta$ and choose a function $g \in C([0,\infty))$ so that $g(r) = 0$ for all $r \in [\alpha,\beta]$, $g(r) > 0$ for all $r \in [0,\alpha) \cup (\beta,\infty)$, and $\lim_{r\to\infty} g(r)/r^2 = \infty$. We define the functions $H_0, H \in C(\mathbf{R}^4)$ by

$$H_0(x,y,p,q) = (p - y)^2 - y^2 + (q + x)^2 - x^2,$$
$$H(x,y,p,q) = H_0(x,y,p,q) - g\big(\sqrt{x^2 + y^2}\big).$$

It is easily seen that this function H satisfies (A.1)–(A.3). Let $\delta > 0$ and set $\psi(x,y) = \delta(x^2 + y^2)$ for $(x,y) \in \mathbf{R}^2$. Writing $\psi_x = \partial\psi/\partial x$ and $\psi_y = \partial\psi/\partial x$, we observe that $\psi_x(x,y) = 2\delta x$, $\psi_y(x,y) = 2\delta y$, and $H_0(x,y,\psi_x,\psi_y) = 4\delta^2(x^2 + y^2)$ for all $(x,y) \in \mathbf{R}^2$. Therefore, for any $\delta > 0$, if we set $\phi_0(x,y) = -\delta(x^2 + y^2)$ and $\phi_1(x,y) = -2\delta(x^2 + y^2)$ for $(x,y) \in \mathbf{R}^2$, then (A.4) holds with these ϕ_0 and ϕ_1.

Noting that the zero function $z = 0$ is a viscosity subsolution of $H[z] = 0$ in $\mathbf{R}^2$, we find that the additive eigenvalue c for H is nonpositive. We fix any $r \in [\alpha, \beta]$ and consider the curve $\gamma \in \mathrm{AC}([0, 2\pi])$ given by $\gamma(t) \equiv (x(t), y(t)) := r(\cos t, \sin t)$. We denote by U the open annulus $\operatorname{int} B((0,0), \beta) \setminus B((0,0), \alpha)$ for notational simplicity. Let $\phi \in C^{0+1}(\mathbf{R}^2)$ be a viscosity solution of $H[\phi] = c$ in $\mathbf{R}^n$. Such a viscosity solution indeed exists according to Theorem 3.3. Due to Proposition 2.4, there are functions $p, q \in L^\infty(0, 2\pi, \mathbf{R}^2)$ such that for almost all $t \in (0, 2\pi)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi\big(\gamma(t)\big) = r\big(-p(t)\sin t + q(t)\cos t\big),$$
$$\big(p(t), q(t)\big) \in \partial_c \phi\big(\gamma(t)\big).$$

The last inclusion guarantees that $H(x(t), y(t), p(t), q(t)) \leqslant c$ a.e. $t \in (0, 2\pi)$. Hence, recalling that $\alpha \leqslant r \leqslant \beta$, we get

$$c \geqslant H_0\big(x(t), y(t), p(t), q(t)\big) = p(t)^2 - 2y(t)p(t) + q(t)^2 + 2x(t)q(t) \quad \text{a.e. } t \in (0, 2\pi).$$

We calculate that

$$\begin{aligned}\phi\big(\gamma(T)\big) - \phi\big(\gamma(0)\big) &= r\int_0^T \big(-p(t)\sin t + q(t)\cos t\big)\,\mathrm{d}t \\ &\leqslant \frac{1}{2}\int_0^T \big(c - p(t)^2 - q(t)^2\big)\,\mathrm{d}t \leqslant \frac{cT}{2} \quad \text{for all } T \in [0, 2\pi].\end{aligned}$$

This clearly implies that $c = 0$ and also that the function: $t \mapsto \phi(\gamma(t))$ is a constant. Thus we find that $\phi(x, y) = h(x^2 + y^2)$ for some function $h \in C^{0+1}([\alpha, \beta])$.

Next, we show that ϕ is a constant function in U. At any $r \in (\alpha, \beta)$ and any $(x, y) \in \partial B((0,0), r)$, we have

$$\phi_x(x, y) = 2xh'\big(x^2 + y^2\big) \quad \text{and} \quad \phi_y(x, y) = 2yh'\big(x^2 + y^2\big),$$

and, in particular, $y\phi_x(x, y) - x\phi_y(x, y) = 0$. Therefore, for almost all $(x, y) \in U$, we have

$$0 \geqslant H_0(x, y, \phi_x, \phi_y) = (\phi_x - y)^2 - y^2 + (\phi_y + x)^2 - x^2 = \phi_x^2 + \phi_y^2.$$

That is, we have

$$\phi_x(x, y) = \phi_y(x, y) = 0 \quad \text{a.e. } (x, y) \in U,$$

which assures that ϕ is a constant in U.

Now we know that for any $y \in U$, the function: $x \mapsto d_H(x, y)$ is a constant in a neighborhood of y, which guarantees that $U \subset \mathcal{A}_H$ and moreover that $\mathcal{A}_H = \overline{U}$.

Finally, we note that $H(x, y, y, -x) = H_0(x, y, y, -x) = -x^2 - y^2 < 0$ for all $(x, y) \in \overline{U}$, and conclude that any $(x, y) \in \mathcal{A}_H = \overline{U}$ is not an equilibrium points.

The following two propositions give sufficient conditions for points of the Aubry set $\mathcal{A}_H$ to be equilibrium points. Here we assume as usual that $c_H = 0$.

Proposition 9.1. *If y is an isolated point of $\mathcal{A}_H$, then it is an equilibrium point.*

Proof. Let y be an isolated point of $\mathcal{A}_H$. Since $d_H(\cdot, y) \in \mathcal{S}_H$, according to Corollary 6.2, there exists a curve $\gamma \in \mathcal{E}((-\infty, 0], d_H(\cdot, y))$ such that $\gamma(0) = y$.

We show that $\gamma(t) \in \mathcal{A}_H$ for all $t \leqslant 0$, which guarantees that

$$\gamma(t) = y \quad \text{for all } t \leqslant 0. \tag{9.7}$$

For this purpose we fix any $z \in \mathbf{R}^n \setminus \mathcal{A}_H$. By Lemma 8.4 there are two functions $\phi \in \mathcal{S}_H^- \cap \Phi_0$ and $\sigma \in C(\mathbf{R}^n)$ such that $H[\phi] \leqslant -\sigma$ in $\mathbf{R}^n$ in the viscosity sense, $\sigma \geqslant 0$ in $\mathbf{R}^n$, and $\sigma(z) > 0$. By Proposition 2.5, for any fixed $t > 0$, we have

$$\phi(y) - \phi\big(\gamma(-t)\big) \leqslant \int_{-t}^{0} L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s - \int_{-t}^{0} \sigma\big(\gamma(s)\big)\,\mathrm{d}s$$
$$= d_H(y, y) - d_H\big(\gamma(-t), y\big) - \int_{-t}^{0} \sigma\big(\gamma(s)\big)\,\mathrm{d}s.$$

Accordingly we have

$$\int_{-t}^{0} \sigma\big(\gamma(s)\big)\,\mathrm{d}s + d_H\big(\gamma(-t), y\big) \leqslant \phi\big(\gamma(-t)\big) - \phi\big(\gamma(0)\big) \leqslant d_H\big(\gamma(-t), y\big).$$

Hence we get

$$\int_{-t}^{0} \sigma\big(\gamma(s)\big)\,\mathrm{d}s \leqslant 0,$$

which implies that $\gamma(s) \neq z$ for all $s \leqslant 0$. Thus we conclude that (9.7) holds.

Now we have

$$0 = d_H(y, y) - d_H\big(\gamma(-1), y\big) = \int_{-1}^{0} L\big(\gamma(t), \dot\gamma(t)\big)\,\mathrm{d}t = L(y, 0),$$

which shows that y is an equilibrium point. □

Proposition 9.2. *Assume that there exists a viscosity solution $w \in C(\mathbf{R}^n)$ of $H(x, Dw) = \min_{p\in\mathbf{R}^n} H(x, p)$ in $\mathbf{R}^n$. Then $\mathcal{A}_H$ consists only of equilibrium points.*

For instance, if $H(x, 0) \leqslant H(x, p)$ for all $(x, p) \in \mathbf{R}^{2n}$, then $w = 0$ satisfies $H(x, Dw(x)) = \min_{p\in\mathbf{R}^n} H(x, p)$ for all $x \in \mathbf{R}^n$ in the viscosity sense. If H has the form $H(x, p) = \alpha x \cdot p + H_0(p) - f(x)$ as before, then H attains a minimum as a function of p at a unique point q satisfying $\alpha x + D^- H_0(q) \ni 0$, or equivalently $q = DL_0(-\alpha x)$, that is,

$$\min_{p\in\mathbf{R}^n} H(x, p) = \alpha x \cdot q + H_0(q) - f(x),$$

where L_0 denotes the convex conjugate H_0^* of H_0. Therefore, in this case, the function $w(x) := -(1/\alpha)L_0(-\alpha x)$ is a viscosity solution of $H[w] = \min_{p\in\mathbf{R}^n} H(x, p)$ in $\mathbf{R}^n$. In these two cases, the Aubry sets consist only of equilibrium points.

Proof. Since $c_H = 0$, we have $\min_{p\in\mathbf{R}^n} H(x, p) \leqslant 0$ for all $x \in \mathbf{R}^n$. Note that the function $\sigma(x) := -\min_{p\in\mathbf{R}^n} H(x, p)$ is continuous on $\mathbf{R}^n$ and that w is a viscosity solution of $H[w] = -\sigma$ in $\mathbf{R}^n$. Applying Lemma 8.5, we see that if $y \in \mathbf{R}^n$ and $\min_{p\in\mathbf{R}^n} H(y, p) < 0$, then $y \notin \mathcal{A}_H$. That is, if $y \in \mathcal{A}_H$, then $\min_{p\in\mathbf{R}^n} H(y, p) = 0$, which is equivalent that y is an equilibrium point. □

Acknowledgements

I would like to thank the referee for his suggestion on the treatment of the Cauchy problem (1.1) and (1.2) as well as other critical comments which helped me to improve the presentation. In the previous versions I took a generalized dynamical approach, which is very close to the one in [12], to proving main results. A strong indication for the new approach taken in the present version has come up in a recent joint work [20] with Dr. Naoyuki Ichihara. I would like to thank him for bringing me such a fortune as well as for his pointing out many errors and misprints in previous versions of this paper.

Appendix A

We show here that value functions, associated with given Hamiltonian H or its Lagrangian L, are viscosity solutions of $H = 0$.

Let $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$ be a function such that for each $x \in \mathbf{R}^n$ the function: $p \mapsto H(x, p)$ is convex in $\mathbf{R}^n$, and let L be its Lagrangian. Let S be a nonempty subset of $\mathbf{R}^n$ and v_0 a real-valued function on S. We define the function $v : \mathbf{R}^n \to [-\infty, \infty]$ by

$$v(x) = \inf\left\{ \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s + v_0\big(\gamma(0)\big) \,\Big|\, t > 0, \gamma \in \mathcal{C}(x, t), \gamma(0) \in S \right\}.$$

We define the upper and lower semicontinuous envelopes v^* and v_* of v, respectively, by

$$v^*(x) = \lim_{r \to +0} \sup\big\{v(y) \mid y \in B(x, r)\big\} \quad \text{and} \quad v_*(x) = \lim_{r \to +0} \inf\big\{v(y) \mid y \in B(x, r)\big\}.$$

As is well known, v^* and v_* are upper and lower semicontinuous in $\mathbf{R}^n$, respectively.

Theorem A.1. *Let Ω be an open subset of $\mathbf{R}^n$, and assume that v is locally bounded above in Ω. Then $u := v^*$ is a viscosity subsolution of $H[u] = 0$ in Ω.*

Proof. Let $(\varphi, z) \in C^1(\Omega) \times \Omega$ and assume that $v^* - \varphi$ attains a maximum at z. We show that $H(z, D\varphi(z)) \leqslant 0$. We may assume without loss of generality that $v^*(z) = \varphi(z)$, so that $v^* \leqslant \varphi$ in Ω. Define the multi-function $F : \Omega \to 2^{\mathbf{R}^n}$ by

$$F(x) = \big\{\xi \in \mathbf{R}^n \mid D\varphi(x) \cdot \xi \geqslant L(x, \xi) + H\big(x, D\varphi(x)\big)\big\}.$$

Since, for any $x \in \mathbf{R}^n$, the function: $p \mapsto H(x, p)$ is a real-valued convex function in $\mathbf{R}^n$, it is subdifferentiable everywhere, which shows that $F(x) \neq \emptyset$ for all $x \in \Omega$. Also, it is easily seen that $F(x)$ is a closed convex set for any $x \in \Omega$ and that the multi-function F is upper semicontinuous in Ω. Moreover, since $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$, the function $L(x, \xi)$ has a superlinear growth as $|\xi| \to \infty$. As a consequence, the multi-function is locally bounded in Ω. By a standard existence result for differential inclusions (see, e.g., [2, Theorem 2.1.3]), we see that there is a constant $\delta > 0$ such that for any $y \in B(z, \delta)$ there exists a curve $\eta_y \in \mathrm{AC}([0, \delta], \mathbf{R}^n)$ such that $\dot\eta_y(s) \in -F(\eta_y(s))$ a.e. $s \in (0, \delta)$ and $\eta_y(0) = y$. Fix such a $\delta > 0$ and for each $y \in B(z, \delta)$ a curve $\eta_y \in \mathrm{AC}([0, \delta], \mathbf{R}^n)$. We may assume, thanks to the local boundedness of the multi-function F, that $|\dot\eta_y(s)| \leqslant M$ a.e. $s \in (0, \delta)$ for all $y \in B(z, \delta)$ and for some $M > 0$ and that $\eta_y([0, \delta]) \subset \Omega$. Note that $|\eta_y(s) - y| \leqslant Ms$ for all $0 \leqslant s \leqslant \delta$.

Fix any $\varepsilon, \lambda \in (0, \delta)$ and $y \in B(z, \lambda)$. Noting that $v^* \leqslant \varphi$ in Ω, by the definition of v, we may choose $t > 0$ and $\gamma \in \mathcal{C}(\eta_y(\varepsilon), t)$ so that $\gamma(0) \in S$ and

$$\varphi\big(\eta_y(\varepsilon)\big) + \lambda > \int_0^t L\big(\gamma(s), \dot\gamma(s)\big)\,\mathrm{d}s + v_0\big(\gamma(0)\big).$$

We define the curve $\zeta \in \mathcal{C}(y, t + \varepsilon)$ by

$$\zeta(s) = \begin{cases} \gamma(s) & \text{for } s \in [0, t], \\ \eta_y(\varepsilon + t - s) & \text{for } s \in (t, t + \varepsilon]. \end{cases}$$

It is obvious that $\zeta(0) \in S$. Noting that

$$\dot\zeta(s) = -\dot\eta_y(\varepsilon + t - s) \in F\big(\eta_y(\varepsilon + t - s)\big) = F\big(\zeta(s)\big) \quad \text{a.e. } s \in (t, t + \varepsilon),$$

we have

$$D\varphi\big(\zeta(s)\big) \cdot \dot\zeta(s) = L\big(\zeta(s), \dot\zeta(s)\big) + H\big(\zeta(s), D\varphi\big(\zeta(s)\big)\big) \quad \text{a.e. } s \in (t, t + \varepsilon).$$

Hence we get

$$\begin{aligned}
\varphi(y) &= \varphi\big(\zeta(t+\varepsilon)\big) = \varphi\big(\zeta(t)\big) + \int_t^{t+\varepsilon} D\varphi\big(\zeta(s)\big)\cdot\dot\zeta(s)\,\mathrm{d}s \\
&= \varphi\big(\gamma(t)\big) + \int_t^{t+\varepsilon} \big[L\big(\zeta(s),\dot\zeta(s)\big) + H\big(\zeta(s), D\varphi\big(\zeta(s)\big)\big)\big]\,\mathrm{d}s \\
&> -\lambda + v_0\big(\gamma(0)\big) + \int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s + \int_t^{t+\varepsilon} \big[L\big(\zeta(s),\dot\zeta(s)\big) + H\big(\zeta(s), D\varphi\big(\zeta(s)\big)\big)\big]\,\mathrm{d}s \\
&= -\lambda + v_0\big(\zeta(0)\big) + \int_0^{t+\varepsilon} L\big(\zeta(s),\dot\zeta(s)\big)\,\mathrm{d}s + \int_t^{t+\varepsilon} H\big(\zeta(s), D\varphi\big(\zeta(s)\big)\big)\,\mathrm{d}s \\
&\geqslant -\lambda + v(y) + \varepsilon \min_{x\in B(y,M\varepsilon)} H\big(x, D\varphi(x)\big).
\end{aligned}$$

Hence, as $y \in B(z,\lambda)$ is arbitrary, we get

$$0 \geqslant -\lambda + \sup_{x\in B(z,\lambda)} (v-\varphi)(x) + \varepsilon \min_{x\in B(z,\lambda+M\varepsilon)} H\big(x, D\varphi(x)\big).$$

Sending $\lambda \to 0$ first, then dividing by ε, and letting $\varepsilon \to 0$ yield $H(z, D\varphi(z)) \leqslant 0$, completing the proof. □

Theorem A.2. *Let Ω be an open subset of $\mathbf{R}^n$ such that $S \cap \Omega = \emptyset$, and assume that v is locally bounded below in Ω. Then v_* is a viscosity supersolution of $H = 0$ in Ω.*

Proof. Let $(\varphi, z) \in C^1(\Omega) \times \Omega$ be such that $v_* - \varphi$ has a strict minimum at z. We will show that $H(z, D\varphi(z)) \geqslant 0$. To do this, we argue by contradiction and thus suppose that $H(z, D\varphi(z)) < 0$. We may assume as usual that $v_*(z) = \varphi(z)$. We choose a constant $r > 0$ so that $B(z,r) \subset \Omega$ and $H(x, D\varphi(x)) \leqslant 0$ for all $x \in B(z,r)$. We set $m = \min_{\partial B(z,r)}(v_* - \varphi)$. Note that $m > 0$ and $v_*(x) \geqslant \varphi(x) + m$ for all $x \in \partial B(z,r)$.

Fix any $y \in B(z,r)$. Pick any $t > 0$ and $\gamma \in \mathcal{C}(y,t)$ such that $\gamma(0) \in S$. Since $\gamma(0) \notin \Omega$, there is a constant $\tau \in (0,t]$ such that $\gamma(\tau) \in \partial B(z,r)$ and $\gamma(s) \in B(z,r)$ for all $s \in [\tau, t]$. We now compute that

$$\begin{aligned}
\varphi(y) &= \varphi\big(\gamma(t)\big) = \varphi\big(\gamma(\tau)\big) + \int_\tau^t D\varphi\big(\gamma(s)\big)\cdot\dot\gamma(s)\,\mathrm{d}s \\
&\leqslant v_*\big(\gamma(\tau)\big) - m + \int_\tau^t \big[L\big(\gamma(s),\dot\gamma(s)\big) + H\big(\gamma(s), D\varphi\big(\gamma(s)\big)\big)\big]\,\mathrm{d}s \\
&\leqslant v_0\big(\gamma(0)\big) + \int_0^\tau L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s + \int_\tau^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s - m \\
&\leqslant v_0\big(\gamma(0)\big) + \int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s - m.
\end{aligned}$$

Taking the infimum over $\gamma \in \mathcal{C}(y,t)$, with $\gamma(0) \in S$, and $t > 0$ in the above inequality, we get $\varphi(y) \leqslant v(y) - m$ for all $y \in B(z,r)$ and hence $\varphi(z) \leqslant v_*(z) - m$, which is a contradiction. This proves that $H(z, D\varphi(z)) \geqslant 0$. □

Remark. We may apply above theorems to (1.1) as follows. We introduce the Hamiltonian $\widetilde{H} \in C(\mathbf{R}^{n+1} \times \mathbf{R}^{n+1})$ defined by $\widetilde{H}(x,t,p,q) = q + H(x,p)$. The corresponding Lagrangian $\widetilde{L}$ is given by $\widetilde{L}(x,t,\xi,\eta) = L(x,\xi) + \delta_{\{1\}}(\eta)$, where L is the Lagrangian of H and $\delta_{\{1\}}$ denotes the indicator function of the set $\{1\} \subset \mathbf{R}$. We set $S = \mathbf{R}^n \times \{0\}$ and $\Omega = \mathbf{R}^n \times (0,\infty)$. Also, for given $u_0 \in C(\mathbf{R}^n)$, we define the function $v_0 \in C(S)$ by $v_0(x,0) = u_0(x)$. We then observe that

$$\inf\left\{\int_0^t L\big(\gamma(s),\dot\gamma(s)\big)\,\mathrm{d}s+u_0\big(\gamma(0)\big)\,\Big|\,\gamma\in\mathcal{C}(x,t)\right\}$$
$$=\inf\left\{\int_0^T \tilde L\big(\zeta(s),\dot\zeta(s)\big)\,\mathrm{d}s+v_0\big(\zeta(0)\big)\,\Big|\,T>0,\ \zeta\in\mathcal{C}\big((x,t),T\big),\ \zeta(0)\in S\right\}.$$

We give here a basic property of the Aubry set $\mathcal{A}_H$ (cf. [15,17]). We assume as usual that $c_H=0$.

Proposition A.3. *Let $y\in\mathbf{R}^n$. Then $y\in\mathcal{A}_H$ if and only if for any $\tau>0$,*

$$\inf\left\{\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s\,\Big|\,t\geqslant\tau,\ \gamma\in\mathcal{C}(y,t;y,0)\right\}=0. \tag{A.1}$$

Proof. We start by observing that for any $y\in\mathbf{R}^n$, $t>0$, and $\gamma\in\mathcal{C}(y,t;y,0)$,

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s\geqslant\phi_1\big(\gamma(t)\big)-\phi_1\big(\gamma(0)\big)=0.$$

We assume that $y\notin\mathcal{A}_H$, and will show that (A.1) does not hold for some $\tau>0$. In view of Proposition 8.3 and Lemma 8.4, there is a function $\psi\in\mathcal{S}_H^-\cap\Phi_0$ and a constant $\delta>0$ such that $H[\psi]\leqslant-\delta$ a.e. in $B(y,2\delta)$. Let $t>0$ and $\gamma\in\mathcal{C}(y,t;y,0)$ be such that

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s<1.$$

We select a function $f\in C(\mathbf{R}^n)$ so that $0\leqslant f\leqslant\delta$ in $\mathbf{R}^n$, $f(x)\geqslant\delta$ for all $x\in B(y,\delta)$, and $f(x)=0$ for all $x\in\mathbf{R}^n\setminus B(y,2\delta)$. Then, noting that $H[\psi]\leqslant-f$ in $\mathbf{R}^n$ in the viscosity sense, by virtue of Proposition 2.5, we have

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s\geqslant\psi\big(\gamma(t)\big)-\psi\big(\gamma(0)\big)+\int_0^t f\big(\gamma(s)\big)\,\mathrm{d}s\geqslant\delta|I|,$$

where $I=\{s\in[0,t]\mid\gamma(s)\in B(y,\delta)\}$ and $|I|$ denotes the one-dimensional Lebesgue measure of I. By Lemmas 6.4 and 6.5, there is a constant $C_\delta>0$, depending only on δ, H, y, and ϕ_1, such that

$$\int_0^t\big|\dot\gamma(s)\big|\,\mathrm{d}s\leqslant\frac{\delta}{2}+C_\delta t.$$

Therefore, setting $\tau=\delta/(2C_\delta)$, we see that if $t\geqslant\tau$, then $\gamma(s)\in B(y,\delta)$ for all $s\in[0,\tau]$. Accordingly, if $t\geqslant\tau$, we have

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s\geqslant\delta\tau.$$

This shows that (A.1) does not hold with our choice of τ.

Next we suppose that (A.1) does not hold for some $\tau>0$ and will show that $y\notin\mathcal{A}_H$. We see immediately from this assumption that $L(y,0)>0$, which implies that $\min_{p\in\mathbf{R}^n}H(y,p)=H(y,q)<0$ for some $q\in\mathbf{R}^n$. By Proposition 2.1, there are constants $\varepsilon>0$ and $C>0$ such that $L(x,p)\leqslant C$ for all $(x,p)\in B(y,\varepsilon)\times B(0,\varepsilon)$. We may assume as well that

$$d_H(x,y)<1\quad\text{and}\quad H(x,q)\leqslant0\quad\text{for all }x\in B(y,\varepsilon).$$

Let $r \in (0, \varepsilon)$ be a constant to be fixed later on. Fix $x \in B(y, r) \setminus \{y\}$, $t > 0$, and $\gamma \in \mathcal{C}(x, t; y, 0)$ so that

$$\int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s < 1.$$

According to Lemmas 6.4 and 6.5, there is a constant $C_\varepsilon > 0$, independent of the choice of γ, such that

$$\int_0^t \left|\dot\gamma(s)\right|\mathrm{d}s < \frac{\varepsilon}{2} + C_\varepsilon t.$$

In particular, there is a constant $\sigma > 0$ such that $\gamma(s) \in B(y, \varepsilon)$ for all $s \in [0, \min\{t, \sigma\}]$.

We may assume that $k\sigma = \tau$ for some $k \in \mathbf{N}$. Note that

$$k \inf\left\{\int_0^T L(\gamma, \dot\gamma)\,\mathrm{d}s \,\middle|\, T \geqslant \sigma, \gamma \in \mathcal{C}(y, t; y, 0)\right\} \geqslant \inf\left\{\int_0^T L(\gamma, \dot\gamma)\,\mathrm{d}s \,\middle|\, T \geqslant \tau, \gamma \in \mathcal{C}(y, t; y, 0)\right\} > 0.$$

We may choose a constant $a > 0$ so that

$$\inf\left\{\int_0^T L(\eta, \dot\eta)\,\mathrm{d}s \,\middle|\, T \geqslant \sigma, \eta \in \mathcal{C}(y, T; y, 0)\right\} > a.$$

We divide our considerations into two cases. The first case is when $t \leqslant \sigma$. Then we have $\gamma(s) \in B(y, \varepsilon)$ for all $s \in [0, t]$ and hence

$$\begin{aligned} q \cdot (x - y) = q \cdot \big(\gamma(t) - \gamma(0)\big) &= \int_0^t q \cdot \dot\gamma(s)\,\mathrm{d}s \\ &\leqslant \int_0^t \big[L\big(\gamma(s), \dot\gamma(s)\big) + H\big(\gamma(s), q\big)\big]\,\mathrm{d}s \leqslant \int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s. \end{aligned}$$

In the other case when $t > \sigma$, we define $\eta \in \mathcal{C}(y, t + \varepsilon^{-1}|y - x|; y, 0)$ by

$$\eta(s) = \begin{cases} \gamma(s) & \text{for } s \in [0, t], \\ x + (s - t)\varepsilon|y - x|^{-1}(y - x) & \text{for } s \in [t, t + \varepsilon^{-1}|x - y|]. \end{cases}$$

Noting that $(\eta(s), \dot\eta(s)) \in B(y, r) \times B(0, \varepsilon)$ for all $s \in (t, t + \varepsilon^{-1}|x - y|)$, we have

$$\begin{aligned} a &\leqslant \int_0^{t+\varepsilon^{-1}|x-y|} L(\gamma, \dot\gamma)\,\mathrm{d}s = \int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s + \int_t^{t+\varepsilon^{-1}|x-y|} L\big(\eta(s), \dot\eta(s)\big)\,\mathrm{d}s \\ &\leqslant \int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s + C\varepsilon^{-1}|x - y| \leqslant \int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s + C\varepsilon^{-1}r. \end{aligned}$$

Now we fix $r \in (0, \varepsilon)$ so that $C\varepsilon^{-1}r \leqslant \frac{a}{2}$. Consequently we get

$$\int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s \geqslant \frac{a}{2}.$$

Hence we have

$$\int_0^t L(\gamma, \dot\gamma)\,\mathrm{d}s \geqslant \min\left\{p \cdot (x - y), \frac{a}{2}\right\},$$

from which we get

$$\min\left\{q \cdot (x-y), \frac{a}{2}\right\} \leqslant d_H(x,y) \quad \text{for all } x \in B(y,r).$$

This shows that $q \in D_1^- d_H(y,y)$. Since $H(y,q) < 0$, we conclude that $y \notin \mathcal{A}_H$. $\square$

References

[1] O. Alvarez, Bounded-from-below viscosity solutions of Hamilton–Jacobi equations, Differential Integral Equations 10 (3) (1997) 419–436.
[2] J.-P. Aubin, A. Cellina, Differential Inclusions. Set-Valued Maps and Viability Theory, Grundlehren der Mathematischen Wissenschaften, vol. 264, Springer-Verlag, Berlin, 1984.
[3] G. Barles, Solutions de viscosité des équations de Hamilton–Jacobi, Mathématiques & Applications (Berlin), vol. 17, Springer-Verlag, Paris, 1994.
[4] M. Bardi, I. Capuzzo-Dolcetta, Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations, Systems & Control: Foundations & Applications, Birkhäuser Boston, Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.
[5] G. Barles, J.-M. Roquejoffre, Ergodic type problems and large time behaviour of unbounded solutions of Hamilton–Jacobi equations, Comm. Partial Differential Equations 31 (8) (2006) 1209–1225.
[6] G. Barles, P.E. Souganidis, On the large time behavior of solutions of Hamilton–Jacobi equations, SIAM J. Math. Anal. 31 (4) (2000) 925–939.
[7] G. Barles, P.E. Souganidis, Space–time periodic solutions and long-time behavior of solutions to quasi-linear parabolic equations, SIAM J. Math. Anal. 32 (6) (2001) 1311–1323.
[8] E.N. Barron, R. Jensen, Semicontinuous viscosity solutions for Hamilton–Jacobi equations with convex Hamiltonians, Comm. Partial Differential Equations 15 (12) (1990) 1713–1742.
[9] E.N. Barron, R. Jensen, Optimal control and semicontinuous viscosity solutions, Proc. Amer. Math. Soc. 113 (2) (1991) 397–402.
[10] F.H. Clarke, Optimization and Nonsmooth Analysis, SIAM, Philadelphia, 1983.
[11] M.G. Crandall, H. Ishii, P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–67.
[12] A. Davini, A. Siconolfi, A generalized dynamical approach to the large time behavior of solutions of Hamilton–Jacobi equations, SIAM J. Math. Anal. 38 (2) (2006) 478–502.
[13] A. Fathi, Théorème KAM faible et théorie de Mather pour les systèmes lagrangiens, C. R. Acad. Sci. Paris Sér. I Math. 324 (9) (1997) 1043–1046.
[14] A. Fathi, Sur la convergence du semi-groupe de Lax–Oleinik, C. R. Acad. Sci. Paris Sér. I Math. 327 (3) (1998) 267–270.
[15] A. Fathi, Weak KAM theorem in Lagrangian dynamics, in press.
[16] A. Fathi, A. Siconolfi, Existence of C^1 critical subsolutions of the Hamilton–Jacobi equation, Invent. Math. 155 (2) (2004) 363–388.
[17] A. Fathi, A. Siconolfi, PDE aspects of Aubry–Mather theory for quasiconvex Hamiltonians, Calc. Var. Partial Differential Equations 22 (2) (2005) 185–228.
[18] Y. Fujita, H. Ishii, P. Loreti, Asymptotic solutions of viscous Hamilton–Jacobi equations with Ornstein–Uhlenbeck operator, Comm. Partial Differential Equations 31 (6) (2006) 827–848.
[19] Y. Fujita, H. Ishii, P. Loreti, Asymptotic solutions of Hamilton–Jacobi equations in Euclidean n space, Indiana Univ. Math. J., in press.
[20] N. Ichihara, H. Ishii, in preparation.
[21] H. Ishii, A generalization of a theorem of Barron and Jensen and a comparison theorem for lower semicontinuous viscosity solutions, Proc. Roy. Soc. Edinburgh Sect. A 131 (1) (2001) 137–154.
[22] H. Ishii, H. Mitake, Representation formulas for solutions of Hamilton–Jacobi equations, preprint.
[23] P.-L. Lions, Generalized Solutions of Hamilton–Jacobi Equations, Research Notes in Mathematics, vol. 69, Pitman (Advanced Publishing Program), Boston, MA, 1982.
[24] P.-L. Lions, G. Papanicolaou, S. Varadhan, Homogenization of Hamilton–Jacobi equations, unpublished preprint.
[25] G. Namah, J.-M. Roquejoffre, Remarks on the long time behaviour of the solutions of Hamilton–Jacobi equations, Comm. Partial Differential Equations 24 (5–6) (1999) 883–893.
[26] T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[27] J.-M. Roquejoffre, Convergence to steady states or periodic solutions in a class of Hamilton–Jacobi equations, J. Math. Pures Appl. (9) 80 (1) (2001) 85–104.