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Stokes and Navier–Stokes equations with nonhomogeneous boundary conditions

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Abstract

In this paper, we study the existence and regularity of solutions to the Stokes and Oseen equations with nonhomogeneous Dirichlet boundary conditions with low regularity. We consider boundary conditions for which the normal component is not equal to zero. We rewrite the Stokes and the Oseen equations in the form of a system of two equations. The first one is an evolution equation satisfied by $P\mathbf{u}$, the projection of the solution on the Stokes space – the space of divergence free vector fields with a normal trace equal to zero – and the second one is a quasi-stationary elliptic equation satisfied by $(I-P)\mathbf{u}$, the projection of the solution on the orthogonal complement of the Stokes space. We establish optimal regularity results for $P\mathbf{u}$ and $(I-P)\mathbf{u}$. We also study the existence of weak solutions to the three-dimensional instationary Navier–Stokes equations for more regular data, but without any smallness assumption on the initial and boundary conditions.

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1. Introduction

Let Ω be a bounded and connected domain in $\mathbb{R}^N$, with $N=2$ or $N=3$, with a regular boundary Γ, and let T be positive. Set $Q=\Omega\times(0,T)$ and $\Sigma=\Gamma\times(0,T)$. We are interested in the following boundary value problems for the Navier–Stokes equations

$$\begin{aligned}&\frac{\partial\mathbf{u}}{\partial t}-\Delta\mathbf{u}+\kappa(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=0,\quad \operatorname{div}\mathbf{u}=0\quad\text{in } Q,\\ &\mathbf{u}=\mathbf{g}\quad\text{on }\Sigma,\qquad \mathbf{u}(0)=\mathbf{u}_0\quad\text{in }\Omega,\end{aligned}\tag{1.1}$$

where $\mathbf{g}$ is a nonhomogeneous boundary condition, $\mathbf{u}_0$ is the initial condition, and $\kappa=0$ or $\kappa=1$. For $\kappa=0$ Eq. (1.1) corresponds to the Stokes equations and for $\kappa=1$ to the Navier–Stokes equations. We are also interested in similar

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problems for the Oseen equations. Let us denote by $\mathbf{n}$ the outward unit normal to the boundary Γ. In the case when the normal component of $\mathbf{g}$ is equal to zero, that is to say if

$$\mathbf{g}(x,t)\cdot\mathbf{n}(x)=0 \quad \text{for a.e. } (x,t)\in\Gamma\times(0,T), \tag{1.2}$$

Eq. (1.1) can be studied by pseudo-differential techniques [13,14], and the regularity results for the Stokes equations are of the same type as for the heat equation [13,14,24]. Moreover, using the so-called Stokes operator A (see Section 2), when condition (1.2) is satisfied, Eq. (1.1) with $\kappa=0$ can be written in the form:

$$\mathbf{u}'=A\mathbf{u}+(-A)D\mathbf{g}, \qquad \mathbf{u}(0)=\mathbf{u}_0, \tag{1.3}$$

where, for almost all $t\in(0,T)$, $D\mathbf{g}(t)$ is the solution of the stationary Stokes problem with $\mathbf{g}(t)$ as nonhomogeneous boundary condition.

For engineering applications – see e.g. [15] – it is important to study Eq. (1.1) when condition (1.2) is not satisfied. However in that case the situation is more complicated because (1.1) cannot be written in the form of an evolution equation. Indeed, due to the incompressibility condition, if $\mathbf{u}$ is a solution to (1.1) we have

$$\int_\Omega \operatorname{div}\mathbf{u}(t)\,\mathrm{d}x=\big\langle\mathbf{u}(t)\cdot\mathbf{n},1\big\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}=0 \quad \text{for a.e. } t\in(0,T).$$

Thus we look for $\mathbf{u}(t)$ in the space

$$\mathbf{V}^0(\Omega)=\big\{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid \operatorname{div}\mathbf{u}=0,\ \langle\mathbf{u}\cdot\mathbf{n},1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}=0\big\}.$$

But the Stokes operator is defined as an unbounded operator in the space

$$\mathbf{V}^0_n(\Omega)=\big\{\mathbf{u}\in\mathbf{L}^2(\Omega)\mid \operatorname{div}\mathbf{u}=0,\ \mathbf{u}\cdot\mathbf{n}=0 \text{ in } H^{-1/2}(\Gamma)\big\}.$$

Consequently, Eq. (1.1) cannot be written as an evolution equation of the form (1.3), contrarily to the case when (1.2) is satisfied (see Section 2).

To overcome this difficulty Fursikov, Gunzburger and Hou [9,8] have first determined the trace spaces corresponding to some function spaces, before proving the existence of weak solutions. Thus, taking the trace $\mathbf{g}$ in the right space, using an extension procedure, they prove the existence of a solution in the space initially chosen.

Another approach is investigated in [7]. It consists in solving the stationary Stokes problem

$$-\Delta\mathbf{w}(t)+\nabla\pi(t)=0 \quad\text{and}\quad \operatorname{div}\mathbf{w}(t)=0 \quad \text{in } \Omega, \qquad \mathbf{w}(t)=\mathbf{g}(t) \quad \text{on } \Gamma,$$

for all $t\in[0,T]$, and looking for the equation satisfied by $\mathbf{u}-\mathbf{w}$. Farwig, Galdi and Sohr [7] prove new regularity results for the Stokes equations when $\mathbf{g}$ belongs to some classes of Banach spaces. The corresponding classes of Hilbert spaces are the following ones [7, Theorem 4 and Corollary 5]:

(i) $\mathbf{g}\in L^2(0,T;\mathbf{H}^{-1/2}(\Gamma))$ and $\langle\mathbf{g}(t),\mathbf{n}\rangle_{\mathbf{H}^{-1/2}(\Gamma),\mathbf{H}^{1/2}(\Gamma)}=0$,
(ii) $\mathbf{g}\in L^2(0,T;\mathbf{H}^{3/2}(\Gamma))\cap H^1(0,T;\mathbf{H}^{-1/2}(\Gamma))$ and $\int_\Gamma \mathbf{g}(t)\cdot\mathbf{n}=0$.

The existence of solutions to the Navier–Stokes equations is also proved in [7] for small data.

Here, motivated by stabilization problems [22,23], we would like to find optimal regularity results for the solution to the Stokes and the Oseen equations when $\mathbf{g}$ belongs to the space

$$\mathbf{V}^{s,s/2}(\Sigma)=L^2\big(0,T;\mathbf{V}^s(\Gamma)\big)\cap H^{s/2}\big(0,T;\mathbf{V}^0(\Gamma)\big),$$

with $s\geqslant 0$, and

$$\mathbf{V}^s(\Gamma)=\left\{\mathbf{u}\in\mathbf{H}^s(\Gamma)\ \middle|\ \int_\Gamma \mathbf{u}\cdot\mathbf{n}=0\right\}.$$

We are also interested in finding a sufficient condition on $\mathbf{g}$ so that a weak solution to Eq. (1.1) exists in the case where $\kappa=1$. This approach is an essential step to study the local feedback boundary stabilization of the Navier–Stokes equations [22,23].

The paper is organized as follows. We study the Stokes equation in Section 2. We give a new definition of weak solutions to Eq. (1.1) (in the case where $\kappa=0$) that we compare with the other ones existing in the literature. Thanks to this new definition we are able to prove optimal regularity results for $P\mathbf{u}$ and $(I-P)\mathbf{u}$, where $\mathbf{u}$ is the solution of the Stokes equations and P is the so-called Helmholtz projection operator (Theorems 2.3 and 2.7). In particular if $\mathbf{g}$ belongs to $\mathbf{V}^{s,s/2}(\Sigma)$, $P\mathbf{u}_0$ belongs to $\mathbf{V}_n^{s-1/2}(\Omega)$, and if they satisfy some compatibility conditions, we first prove that $P\mathbf{u}$ belongs to $\mathbf{V}^{s+1/2-\varepsilon,s/2+1/4-\varepsilon/2}(Q)$ for all $\varepsilon>0$ if $0\leqslant s\leqslant 2$, $s\neq 1$ (Theorem 2.3). The question of knowing if we can take $\varepsilon=0$ is not obvious in the case when $\mathbf{g}(t)\cdot\mathbf{n}\neq 0$. Using results already proved in Theorem 2.3, we answer positively to this question in Theorem 2.7. In Section 3, we prove that we can have $P\mathbf{u}\in\mathbf{V}^{s+1/2,s/2+1/4}(Q)$ and $(I-P)\mathbf{u}\in\mathbf{V}^{s+1/2,s/2+1/4}(Q)$ under conditions on $\mathbf{g}$ and $\mathbf{u}_0$ which are different from the ones in [9] (Theorem 3.1). In Section 4, we study the Oseen equations in two cases. The first one corresponds to the linearized Navier–Stokes equations around a stationary state, and the second one corresponds to a linearization around an instationary state. We extend the results of Section 2 to the first case. The second case, with homogeneous boundary conditions, is needed in Section 4 to study the Navier–Stokes equations with nonhomogeneous boundary conditions. We prove the existence of global weak solutions to the Navier–Stokes equations, in the 3D case, when $\mathbf{g}\in\mathbf{V}^{3/4,3/4}(\Sigma)=L^2(0,T;\mathbf{V}^{3/4}(\Gamma))\cap H^{3/4}(0,T;\mathbf{V}^0(\Gamma))$. To the best of our knowledge, this existence result for the three-dimensional Navier–Stokes equations seems to be new. Since we prove the existence of a weak solution, we are not able to establish uniqueness (the situation is the same as in the case of homogeneous boundary conditions), contrarily to the existence results obtained by a fixed point method with small data where uniqueness is directly proved [14,8,9,7].

In Appendix A we establish results needed for the stationary Stokes equations with nonhomogeneous boundary conditions. Their extension to the stationary Oseen equations are stated in Appendix B.

2. Stokes equation

Throughout the paper we assume that Ω is at least of class C^2. In this section we study the Stokes equations with a nonhomogeneous boundary condition:

$$\begin{aligned}&\frac{\partial\mathbf{u}}{\partial t}-\Delta\mathbf{u}+\nabla p=0,\quad \operatorname{div}\mathbf{u}=0\quad\text{in } Q,\\ &\mathbf{u}=\mathbf{g}\quad\text{on }\Sigma,\qquad \mathbf{u}(0)=\mathbf{u}_0\quad\text{in }\Omega.\end{aligned}\tag{2.1}$$

The main results of this section are stated in Theorems 2.3 and 2.7.

2.1. Notation

Let us introduce the following function spaces: $H^s(\Omega;\mathbb{R}^N)=\mathbf{H}^s(\Omega)$, $L^2(\Omega;\mathbb{R}^N)=\mathbf{L}^2(\Omega)$, the same notation conventions are used for the spaces $H_0^s(\Omega;\mathbb{R}^N)$, and the trace spaces $H^s(\Gamma;\mathbb{R}^N)$. Throughout what follows, for all $\mathbf{u}\in\mathbf{L}^2(\Omega)$ such that $\operatorname{div}\mathbf{u}\in L^2(\Omega)$, we denote by $\mathbf{u}\cdot\mathbf{n}$ the normal trace of $\mathbf{u}$ in $H^{-1/2}(\Gamma)$ [25]. Following [9], we use the letter $\mathbf{V}$ to define different spaces of divergence free vector functions and for some associated trace spaces:

$$\begin{aligned}&\mathbf{V}^s(\Omega)=\left\{\mathbf{u}\in\mathbf{H}^s(\Omega)\mid \operatorname{div}\mathbf{u}=0\text{ in }\Omega,\ \langle\mathbf{u}\cdot\mathbf{n},1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}=0\right\}\quad\text{for } s\geqslant 0,\\ &\mathbf{V}_n^s(\Omega)=\left\{\mathbf{u}\in\mathbf{H}^s(\Omega)\mid \operatorname{div}\mathbf{u}=0\text{ in }\Omega,\ \mathbf{u}\cdot\mathbf{n}=0\text{ on }\Gamma\right\}\quad\text{for } s\geqslant 0,\\ &\mathbf{V}_0^s(\Omega)=\left\{\mathbf{u}\in\mathbf{H}^s(\Omega)\mid \operatorname{div}\mathbf{u}=0\text{ in }\Omega,\ \mathbf{u}=0\text{ on }\Gamma\right\}\quad\text{for } s>\frac{1}{2},\\ &\mathbf{V}^s(\Gamma)=\left\{\mathbf{u}\in\mathbf{H}^s(\Gamma)\ \Big|\ \int_\Gamma\mathbf{u}\cdot\mathbf{n}=0\right\}\quad\text{for } s\geqslant 0.\end{aligned}$$

For $s<0$, $\mathbf{V}^s(\Gamma)$ is the dual space of $\mathbf{V}^{-s}(\Gamma)$, with $\mathbf{V}^0(\Gamma)$ as pivot space. For spaces of time dependent functions we set

$$\mathbf{V}^{s,\sigma}(Q)=H^\sigma\big(0,T;\mathbf{V}^0(\Omega)\big)\cap L^2\big(0,T;\mathbf{V}^s(\Omega)\big),$$

and

$$\mathbf{V}^{s,\sigma}(\Sigma)=H^\sigma\big(0,T;\mathbf{V}^0(\Gamma)\big)\cap L^2\big(0,T;\mathbf{V}^s(\Gamma)\big).$$

Observe that

$$\mathbf{V}^{s,\sigma}(Q)=\mathbf{H}^{s,\sigma}(Q)\cap L^2\big(0,T;\mathbf{V}^0(\Omega)\big)\quad\text{for all } s\geqslant 0\text{ and }\sigma\geqslant 0,$$

where $\mathbf{H}^{s,\sigma}(Q)=(H^{s,\sigma}(Q))^N$, and $H^{s,\sigma}(Q)$ corresponds to the notation in [20].

We denote by $\gamma_\tau\in\mathcal{L}(\mathbf{V}^0(\Gamma))$ and $\gamma_n\in\mathcal{L}(\mathbf{V}^0(\Gamma))$ the operators defined by

$$\gamma_n\mathbf{u}=(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\quad\text{and}\quad\gamma_\tau\mathbf{u}=\mathbf{u}-\gamma_n\mathbf{u}\quad\text{for all }\mathbf{u}\in\mathbf{V}^0(\Gamma).$$

As usual, for $s>1/2$, $\gamma_0\in\mathcal{L}(\mathbf{V}^s(\Omega),\mathbf{V}^{s-1/2}(\Gamma))$ denotes the trace operator. Throughout the paper, for all $\boldsymbol{\Phi}\in\mathbf{H}^{3/2+\varepsilon'}(\Omega)$ and all $\psi\in H^{1/2+\varepsilon'}(\Omega)$, with $\varepsilon'>0$, we denote by $c(\boldsymbol{\Phi},\psi)$, the constant defined by

$$c(\boldsymbol{\Phi},\psi)=-\frac{1}{|\Gamma|}\int_\Gamma\left(\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}}\cdot\mathbf{n}-\psi\right),\tag{2.2}$$

where $|\Gamma|$ is the $(N-1)$-dimensional Lebesgue measure of Γ. If moreover $\boldsymbol{\Phi}\in\mathbf{V}_0^{3/2+\varepsilon'}(\Omega)$, then $\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}}\cdot\mathbf{n}=0$ (see [3, Lemma 3.3.1]), and in that case we shall use the constant

$$c(\psi)=\frac{1}{|\Gamma|}\int_\Gamma\psi.\tag{2.3}$$

We also introduce the space

$$W\big(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega)\big)=\left\{\mathbf{u}\in L^2\big(0,T;\mathbf{V}^1(\Omega)\big)\;\middle|\;\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\in L^2\big(0,T;\mathbf{V}^{-1}(\Omega)\big)\right\},$$

where $\mathbf{V}^{-1}(\Omega)$ denotes the dual space of $\mathbf{V}_0^1(\Omega)$ with $\mathbf{V}_n^0(\Omega)$ as pivot space.

Let us denote by P the orthogonal projection operator in $\mathbf{L}^2(\Omega)$ on $\mathbf{V}_n^0(\Omega)$. Recall that the Stokes operator $A=P\Delta$, with domain $D(A)=\mathbf{H}^2(\Omega)\cap\mathbf{V}_0^1(\Omega)$ in $\mathbf{V}_n^0(\Omega)$, is the infinitesimal generator of a strongly continuous analytic semigroup $(\mathrm{e}^{tA})_{t\geqslant0}$ on $\mathbf{V}_n^0(\Omega)$. The operator P can be continuously extended to a bounded operator from $\mathbf{H}^{-1}(\Omega)$ to $\mathbf{V}^{-1}(\Omega)$, that we still denote by P.

We also introduce the Dirichlet operators $D\in\mathcal{L}(\mathbf{V}^0(\Gamma),\mathbf{V}^0(\Omega))$ and $D_p\in\mathcal{L}(\mathbf{V}^0(\Gamma),(H^1(\Omega)/\mathbb{R})')$ defined by

$$D\mathbf{g}=\mathbf{w}\quad\text{and}\quad D_p\mathbf{g}=\pi,$$

where $(\mathbf{w},\pi)$ is the solution to

$$-\Delta\mathbf{w}+\nabla\pi=0\quad\text{and}\quad\operatorname{div}\mathbf{w}=0\quad\text{in }\Omega,\qquad\mathbf{w}=\mathbf{g}\quad\text{on }\Gamma.$$

Notice that D can be extended to a bounded operator from $\mathbf{V}^{-1/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$ (see Corollary A.1).

2.2. Stokes equation

Fursikov, Gunzburger and Hou have studied the linearized Navier–Stokes and Navier–Stokes equations with non-homogeneous boundary conditions when the domain Ω is not necessarily bounded in $\mathbb{R}^3$ [8,9]. For that they first characterize the traces for functions belonging to spaces of the type

$$\mathcal{V}^{(s)}(Q)=\big\{\mathbf{u}\mid\mathbf{u}\text{ is the restriction to }Q\text{ of a function belonging to }\mathcal{H}^{(s)}\big(\mathbb{R}^{N+1}\big),\ \operatorname{div}\mathbf{u}=0\big\},$$

where

$$\mathcal{H}^{(s)}\big(\mathbb{R}^{N+1}\big)=L^2\big(\mathbb{R};\mathbf{H}^s\big(\mathbb{R}^N\big)\big)\cap H^1\big(\mathbb{R};\mathbf{H}^{s-2}\big(\mathbb{R}^N\big)\big).$$

Observe that for $s=2$, we have $\mathcal{V}^{(2)}(Q)=\mathbf{V}^{2,1}(Q)$. For $s=1$, the identity

$$\mathcal{H}^{(1)}\big(\mathbb{R}^{N+1}\big)=L^2\big(\mathbb{R};\mathbf{H}^1\big(\mathbb{R}^N\big)\big)\cap H^1\big(\mathbb{R};\mathbf{H}^{-1}\big(\mathbb{R}^N\big)\big)=W\big(\mathbb{R};\mathbf{H}^1\big(\mathbb{R}^N\big),\mathbf{H}^{-1}\big(\mathbb{R}^N\big)\big),$$

does not imply the corresponding identity for $\mathcal{H}^{(1)}(Q)$ or $\mathcal{V}^{(1)}(Q)$, that is

$$\mathcal{H}^{(1)}(Q)\subset W\big(0,T;\mathbf{H}^1(\Omega),\mathbf{H}^{-1}(\Omega)\big),\qquad\mathcal{H}^{(1)}(Q)\neq W\big(0,T;\mathbf{H}^1(\Omega),\mathbf{H}^{-1}(\Omega)\big),$$
$$\mathcal{V}^{(1)}(Q)\subset W\big(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega)\big)\quad\text{and}\quad\mathcal{V}^{(1)}(Q)\neq W\big(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega)\big).$$

It is proved in [8] that, if $N=3$, the trace space of functions in $\mathcal{V}^{(1)}(Q)$ is

$$G^1(\Sigma)=\big\{\mathbf{u}\in L^2\big(0,T;\mathbf{V}^{1/2}(\Gamma)\big)\mid\gamma_\tau\mathbf{u}\in H^{1/2}\big(0,T;\mathbf{V}^{-1/2}(\Gamma)\big),\ \gamma_n\mathbf{u}\in H^{3/4}\big(0,T;\mathbf{V}^{-1}(\Gamma)\big)\big\}.$$

In [8,9], solutions to Eq. (2.1) are defined as follows.

Definition 2.1. A function $\mathbf{u} \in \mathcal{V}^{(1)}(Q)$ is a solution to Eq. (2.1) if $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} = E\mathbf{g}$ (E is a continuous extension operator from $G^1(\Sigma)$ to $\mathcal{V}^{(1)}(Q)$), and $\mathbf{w}$ is the solution to the equation

$$\begin{aligned}&\frac{\partial \mathbf{w}}{\partial t} - \Delta \mathbf{w} + \nabla p = -\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v}, \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } Q,\\ &\mathbf{w} = 0 \quad \text{on } \Sigma, \qquad \mathbf{w}(0) = \mathbf{u}_0 - \mathbf{v}(0) \quad \text{in } \Omega.\end{aligned}$$

In [8,9], extension operators $E \in \mathcal{L}(G^1(\Sigma), \mathcal{V}^{(1)}(Q))$ are explicitly defined, but any continuous extension operator from $G^1(\Sigma)$ to $\mathcal{V}^{(1)}(Q)$ can be used to define solutions to Eq. (2.1). The theorem below is a direct consequence of results established in [8].

Theorem 2.1. *If* $\mathbf{g} \in G^1(\Sigma)$ *and if* $\gamma_n \mathbf{g}|_{t=0} = \gamma_n \mathbf{g}(0) = (\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}$, *then Eq.* (2.1) *admits a unique solution in* $\mathcal{V}^{(1)}(Q)$ *in the sense of Definition* 2.1, *and the following estimate holds*:

$$\|\mathbf{u}\|_{\mathcal{V}^{(1)}(Q)} \leqslant C\big(\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{1/2}(\Gamma))} + \|\gamma_\tau \mathbf{g}\|_{H^{1/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))} + \|\gamma_n \mathbf{g}\|_{H^{3/4}(0,T;\mathbf{V}^{-1}(\Gamma))} + \|\mathbf{u}_0\|_{\mathbf{V}^0(\Omega)}\big).$$

Let us state a simple proposition that will be useful in the following.

Proposition 2.1. *Assume that* $\mathbf{g} \in C([0,T]; \mathbf{V}^{-1/2}(\Gamma))$ *and* $\mathbf{u}_0 \in \mathbf{V}^0(\Omega)$. *Then the compatibility condition* $\gamma_n \mathbf{g}|_{t=0} = \gamma_n \mathbf{g}(0) = (\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}$ *is equivalent to* $(I - P)(\mathbf{u}_0 - D\mathbf{g}(0)) = 0$.

Proof. Assume that $\gamma_n \mathbf{g}(0) = (\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}$. Then $(I-P)(D((\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}) - D\mathbf{g}(0)) = 0$. Moreover $(I-P)D((\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}) = (I-P)\mathbf{u}_0$, because $D((\mathbf{u}_0 \cdot \mathbf{n})\mathbf{n}) - \mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$. Thus $(I-P)(\mathbf{u}_0 - D\mathbf{g}(0)) = 0$.

Conversely, if $(I-P)(\mathbf{u}_0 - D\mathbf{g}(0)) = 0$, then $\gamma_n((I-P)\mathbf{u}_0) = \gamma_n((I-P)D\mathbf{g}(0)) = \gamma_n \mathbf{g}(0)$. And $\gamma_n((I-P)\mathbf{u}_0) = \gamma_n \mathbf{u}_0$. The proof is complete. □

Observe that Definition 2.1 cannot be used to define weak solutions when $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$ or $\mathbf{g} \in L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$. In this case, following [18,1,2,7], solutions can be defined by a duality method (also called 'the transposition method' in [18–20]).

Definition 2.2. Assume that $\mathbf{g} \in L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$ and $\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$. A function $\mathbf{u} \in L^2(0,T;\mathbf{V}^0(\Omega))$ is a solution to the Stokes equations (2.1), defined by duality (or transposition), if and only if

$$\int_Q \mathbf{u}\mathbf{f} = \int_0^T \Big\langle -\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{n}}(t) + \psi(t)\mathbf{n}, \mathbf{g}(t)\Big\rangle_{\mathbf{V}^{1/2}(\Gamma),\mathbf{V}^{-1/2}(\Gamma)} \mathrm{d}t + \big\langle \mathbf{u}_0, \boldsymbol{\Phi}(0)\big\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} \tag{2.4}$$

for every $\mathbf{f} \in L^2(0,T;\mathbf{V}^0(\Omega))$, where $(\boldsymbol{\Phi}, \psi)$ is the solution to

$$\begin{aligned}&-\frac{\partial \boldsymbol{\Phi}}{\partial t} - \Delta \boldsymbol{\Phi} + \nabla \psi = \mathbf{f}, \quad \operatorname{div} \boldsymbol{\Phi} = 0 \quad \text{in } Q,\\ &\boldsymbol{\Phi} = 0 \quad \text{on } \Sigma, \qquad \boldsymbol{\Phi}(T) = 0 \quad \text{in } \Omega.\end{aligned} \tag{2.5}$$

Remark 2.1. Notice that $\langle \mathbf{u}_0, \boldsymbol{\Phi}(0)\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}^1_0(\Omega)} = \langle P\mathbf{u}_0, \boldsymbol{\Phi}(0)\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}^1_0(\Omega)}$. Thus only $P\mathbf{u}_0$ intervenes in the above definition. The above definition is slightly different from the one in [7, Definition 1]. Actually it can be shown that they are equivalent in the case when $\mathbf{g} \in L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$ and $\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$.

Theorem 2.2. *For all* $\mathbf{g} \in L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$ *and all* $\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$, *Eq.* (2.1) *admits a unique solution in* $L^2(0,T;\mathbf{V}^0(\Omega))$ *in the sense of Definition* 2.2, *and*

$$\|\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0(\Omega))} \leqslant C\big(\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))} + \|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)}\big). \tag{2.6}$$

Moreover if $\mathbf{g}$ *and* $\mathbf{u}_0$ *satisfy the assumptions of Theorem* 2.1, *then the solutions given by Theorems* 2.1 *and* 2.2 *coincide.*

Remark 2.2. The result stated in Theorem 2.2 will be completed by additional regularity results in Lemma 3.1. Since Definition 2.2 is equivalent to [7, Definition 1], we can observe that Theorem 2.2 and Lemma 3.1 are already stated in [7, Theorem 4]. Since our approach is slightly different from the one in [7], we prefer to give complete proofs for the convenience of the reader.

Proof. *Step 1*. Let $\mathbf{f}$ be in $L^2(0,T;\mathbf{V}^0(\Omega))$, the solution $(\boldsymbol{\Phi},\psi)$ to Eq. (2.5) belongs to

$$\mathbf{V}^{2,1}(Q)\times L^2\big(0,T;H^1(\Omega)/\mathbb{R}\big).$$

Let $\Lambda\in\mathcal{L}(L^2(0,T;\mathbf{V}^0(\Omega)),(L^2(0,T;\mathbf{V}^{1/2}(\Gamma))\times\mathbf{V}^1_0(\Omega)))$ be the operator defined by

$$\Lambda(\mathbf{f})=\left(-\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}}+\psi\mathbf{n}-c(\psi)\mathbf{n},\boldsymbol{\Phi}(0)\right),$$

where $(\boldsymbol{\Phi},\psi)$ is the solution to Eq. (2.5) and $c(\psi)$ is defined in (2.3). Eq. (2.4) can be rewritten in the form

$$(\mathbf{u},\mathbf{f})_{L^2(0,T;\mathbf{V}^0(\Omega))}=\big\langle\Lambda(\mathbf{f}),(\mathbf{g},P\mathbf{u}_0)\big\rangle_{L^2(0,T;\mathbf{V}^{1/2}(\Gamma))\times\mathbf{V}^1_0(\Omega),L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))\times\mathbf{V}^{-1}(\Omega)},$$

and we have

$$\big\langle\Lambda(\mathbf{f}),(\mathbf{g},P\mathbf{u}_0)\big\rangle_{L^2(0,T;\mathbf{V}^{1/2}(\Gamma))\times\mathbf{V}^1_0(\Omega),L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))\times\mathbf{V}^{-1}(\Omega)}=\big(\mathbf{f},\Lambda^*(\mathbf{g},P\mathbf{u}_0)\big)_{L^2(0,T;\mathbf{V}^0(\Omega))}.$$

Since Λ^* – the adjoint of Λ – belongs to $\mathcal{L}(L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))\times\mathbf{V}^{-1}(\Omega),L^2(0,T;\mathbf{V}^0(\Omega)))$, the function $\mathbf{u}=\Lambda^*(\mathbf{g},P\mathbf{u}_0)$ is clearly a solution to Eq. (2.1) in the sense of Definition 2.2, and the estimate of $\mathbf{u}$ follows from the continuity of Λ^*. To prove the uniqueness, we observe that if $\mathbf{u}$ is a solution corresponding to $(\mathbf{g},\mathbf{u}_0)=(0,0)$, setting $\mathbf{f}=\mathbf{u}$ in (2.4), we prove that $\mathbf{u}=0$.

Step 2. To compare the solutions corresponding to Definitions 2.1 and 2.2 we first consider the case of regular data. Assume that $\mathbf{g}\in C^1([0,T];\mathbf{V}^{3/2}(\Gamma))$. Let $(\mathbf{w}(t),\pi(t))\in\mathbf{V}^2(\Omega)\times H^1(\Omega)/\mathbb{R}$ be the solution to the equation:

$$-\Delta\mathbf{w}(t)+\nabla\pi(t)=0\quad\text{and}\quad\operatorname{div}\mathbf{w}(t)=0\quad\text{in }\Omega,\qquad\mathbf{w}(t)=\mathbf{g}(t)\quad\text{on }\Gamma.\tag{2.7}$$

It is clear that $(\mathbf{w},\pi)\in C^1([0,T];\mathbf{V}^2(\Omega)\times H^1(\Omega)/\mathbb{R})$. Let $(\mathbf{y},q)$ be the weak solution in $W(0,T;\mathbf{V}^1_0(\Omega),\mathbf{V}^{-1}(\Omega))\times L^2(0,T;L^2(\Omega)/\mathbb{R})$ to the equation

$$\begin{aligned}&\frac{\partial\mathbf{y}}{\partial t}-\Delta\mathbf{y}+\nabla q=-\frac{\partial\mathbf{w}}{\partial t},\quad\operatorname{div}\mathbf{y}=0\quad\text{in }Q,\\&\mathbf{y}=0\quad\text{on }\Sigma,\qquad\mathbf{y}(0)=P\big(\mathbf{u}_0-\mathbf{w}(0)\big)\quad\text{in }\Omega.\end{aligned}\tag{2.8}$$

We set $\mathbf{u}=\mathbf{w}+\mathbf{y}$. We can easily verify that $\mathbf{u}=\mathbf{w}+\mathbf{y}$ is a solution to Eq. (2.1) in the sense of Definitions 2.1 and 2.2.

Step 3. Let $\mathbf{g}$ be in $G^1(\Sigma)$, $\mathbf{u}_0\in\mathbf{V}^0(\Omega)$, and assume that $\gamma_n\mathbf{g}(0)=(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}$. Recall that $(I-P)\mathbf{u}_0=(I-P)D((\mathbf{u}_0\cdot\mathbf{n})\mathbf{n})$. Thus $(I-P)\mathbf{u}_0=(I-P)D\gamma_n\mathbf{g}(0)=(I-P)D\mathbf{g}(0)$. Let $(\mathbf{g}_k)_k$ be a sequence in $C^1([0,T];\mathbf{V}^{3/2}(\Gamma))$ converging to $\mathbf{g}$ in $G^1(\Sigma)$. Let $(\mathbf{w}_k(t),\pi_k(t))$ be the solution to Eq. (2.7) corresponding to $\mathbf{g}_k(t)$, and set $\mathbf{u}_{0,k}=P\mathbf{u}_0+(I-P)D\mathbf{g}_k(0)=P\mathbf{u}_0+(I-P)\mathbf{w}_k(0)$. Since $(\gamma_n\mathbf{g}_k(0))_k$ converges to $\gamma_n\mathbf{g}(0)$ in $\mathbf{V}^{-1/2}(\Gamma)$, $((I-P)\mathbf{w}_k(0))_k$ converges to $(I-P)\mathbf{u}_0$ in $\mathbf{V}^0(\Omega)$. Moreover from the definition of $\mathbf{u}_{0,k}$ it follows that $\mathbf{g}_k$ and $\mathbf{u}_{0,k}$ obey the compatibility condition $\gamma_n\mathbf{g}_k(0)=(\mathbf{u}_{0,k}\cdot\mathbf{n})\mathbf{n}$. Let $(\mathbf{y}_k,q_k)$ be the weak solution to Eq. (2.8) corresponding to $\mathbf{w}_k$, and set $\mathbf{u}_k=\mathbf{w}_k+\mathbf{y}_k$. We have

$$\mathbf{u}_k(0)=P\mathbf{u}_k(0)+(I-P)\mathbf{u}_k(0)=P\mathbf{w}_k(0)+\mathbf{y}_k(0)+(I-P)\mathbf{w}_k(0)=P\mathbf{u}_0+(I-P)\mathbf{w}_k(0)=\mathbf{u}_{0,k}.$$

Due to Step 2, $\mathbf{u}_k$ is the solution to Eq. (2.1) in the sense of Definitions 2.1 and 2.2. By a density argument and due to the estimates in Theorem 2.1 and to (2.6), it follows that the solutions $\mathbf{u}$ to Eq. (2.1) in the sense of Definitions 2.1 and 2.2 coincide if $\mathbf{g}\in G^1(\Sigma)$ and $\gamma_n\mathbf{g}(0)=(\mathbf{u}_0\cdot\mathbf{n})\mathbf{n}$. □

Definition 2.2 cannot be used to obtain optimal regularity results because $P\mathbf{u}$ and $(I-P)\mathbf{u}$ are not decoupled in the weak formulation (2.4).

We are going to define weak solutions to Eq. (2.1) in the case where $\mathbf{g}\in L^2(0,T;\mathbf{V}^0(\Gamma))$, by adapting to the case of the Stokes operator the extrapolation used in [4] and [5] for the heat equation. Before stating a new definition of weak solutions to Eq. (2.1), let us define solutions when the data are regular. Suppose that $\mathbf{g}\in C^1_c(]0,T];\mathbf{V}^{3/2}(\Gamma))$.

Denote by $(\mathbf{w}(t),\pi(t))\in\mathbf{V}^2(\Omega)\times H^1(\Omega)/\mathbb{R}$ the solution to Eq. (2.7), that is $(\mathbf{w}(t),\pi(t))=(D\mathbf{g}(t),D_p\mathbf{g}(t))$, and denote by $(\mathbf{y},q)$ the solution to Eq. (2.8). We already know that $(\mathbf{u},p)=(\mathbf{w}+\mathbf{y},\pi+q)$ is a solution to Eq. (2.1) in the sense of Definition 2.2. Eq. (2.8) can be rewritten in the form

$$\mathbf{y}'=A\mathbf{y}-P\mathbf{w}',\qquad \mathbf{y}(0)=P\mathbf{u}_0,$$

because $\mathbf{w}(0)=0$, and $\mathbf{y}$ is defined by

$$\mathbf{y}(t)=\mathrm{e}^{tA}P\mathbf{u}_0-\int_0^t \mathrm{e}^{(t-s)A}P\mathbf{w}'(s)\,\mathrm{d}s.$$

Integrating by parts we obtain

$$\mathbf{y}(t)=\mathrm{e}^{tA}P\mathbf{u}_0+\int_0^t (-A)\mathrm{e}^{(t-s)A}P\mathbf{w}(s)\,\mathrm{d}s-P\mathbf{w}(t).$$

Thus we have

$$P\mathbf{u}(t)=\mathbf{y}(t)+P\mathbf{w}(t)=\mathrm{e}^{tA}P\mathbf{u}_0+\int_0^t (-A)\mathrm{e}^{(t-s)A}PD\mathbf{g}(s)\,\mathrm{d}s.$$

With the extrapolation method, we can extend the operator A to an unbounded operator $\tilde{A}$ of domain $D(\tilde{A})=\mathbf{V}^0_n(\Omega)$ in $(D(A^*))'=(D(A))'$, in order that $(\tilde{A},D(\tilde{A}))$ be the infinitesimal generator of a strongly continuous semigroup $(\mathrm{e}^{t\tilde{A}})_{t\geqslant 0}$ on $(D(A^*))'$, satisfying $\mathrm{e}^{tA}\mathbf{u}_0=\mathrm{e}^{t\tilde{A}}\mathbf{u}_0$ for all $\mathbf{u}_0\in\mathbf{V}^0_n(\Omega)$. This means that $P\mathbf{u}$ is solution to the equation

$$P\mathbf{u}'=\tilde{A}P\mathbf{u}+(-\tilde{A})PD\mathbf{g},\qquad P\mathbf{u}(0)=P\mathbf{u}_0.$$

The equation satisfied by $(I-P)\mathbf{u}$ is nothing else than

$$(I-P)\mathbf{u}(t)=(I-P)\mathbf{w}(t)=(I-P)D\mathbf{g}(t).$$

One can easily verify that $D\gamma_\tau\mathbf{g}(t)\in\mathbf{V}^0_n(\Omega)$. Thus we have $(I-P)D\mathbf{g}(t)=(I-P)D\gamma_n\mathbf{g}(t)$. The operator $P\circ D$ is linear and continuous from $\mathbf{V}^0(\Gamma)$ to $\mathbf{V}^{1/2}_n(\Omega)$. Thus $(-\tilde{A})PD$ is linear and continuous from $\mathbf{V}^0(\Gamma)$ to $(D((-A^*)^{3/4+\varepsilon}))'$ for all $\varepsilon>0$. Consequently $(-\tilde{A})PD\mathbf{g}$ belongs to $L^2(0,T;(D((-A^*)^{3/4+\varepsilon}))')$ if $\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^0(\Gamma))$.

We can now state a new definition of weak solution.

Definition 2.3. A function $\mathbf{u}\in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to Eq. (2.1) if

$P\mathbf{u}$ is a weak solution of evolution equation

$$P\mathbf{u}'=\tilde{A}P\mathbf{u}+(-\tilde{A})PD\mathbf{g},\qquad P\mathbf{u}(0)=P\mathbf{u}_0,\tag{2.9}$$

and if $(I-P)\mathbf{u}$ is defined by

$$(I-P)\mathbf{u}(\cdot)=(I-P)D\gamma_n\mathbf{g}(\cdot)\quad\text{in } L^2\big(0,T;\mathbf{V}^0(\Omega)\big).\tag{2.10}$$

By definition of weak solutions to evolution equations [4], a function $P\mathbf{u}\in L^2(0,T;\mathbf{V}^0_n(\Omega))$ is weak solution to Eq. (2.9) if and only if, for all $\boldsymbol{\Phi}\in D(A^*)$, the mapping $t\mapsto\int_\Omega P\mathbf{u}(t)\boldsymbol{\Phi}$ belongs to $H^1(0,T)$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{u}(t)\boldsymbol{\Phi}=\int_\Omega P\mathbf{u}(t)A^*\boldsymbol{\Phi}+\big\langle(-\tilde{A})PD\mathbf{g}(t),\boldsymbol{\Phi}\big\rangle_{(D(A^*))',D(A^*)}.$$

Observe that $A^*=A$ and that

$$\big\langle(-\tilde{A})PD\mathbf{g}(t),\boldsymbol{\Phi}\big\rangle_{(D(A^*))',D(A^*)}=\int_\Gamma \mathbf{g}(t)\,D^*(-A^*)\boldsymbol{\Phi}.$$

Due to Lemma A.4, we have

$$\int_\Gamma \mathbf{g}(t)D^*(-A^*)\boldsymbol{\Phi} = \int_\Gamma \mathbf{g}(t)\bigg(-\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}\bigg),$$

where $\psi \in H^1(\Omega)/\mathbb{R}$ is determined by

$$\nabla\psi = (I-P)\Delta\boldsymbol{\Phi}.$$

Thus the variational equation satisfied by $P\mathbf{u}$ is nothing else than

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{u}(t)\boldsymbol{\Phi} = \int_\Omega P\mathbf{u}(t)A\boldsymbol{\Phi} + \int_\Gamma \mathbf{g}(t)\bigg(-\frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}\bigg) \quad \text{for all } \boldsymbol{\Phi} \in D(A^*). \tag{2.11}$$

Remark 2.3. If $\gamma_n\mathbf{g} = 0$ then $(I-P)\mathbf{u} = 0$ and $\mathbf{u} = P\mathbf{u}$ is only determined by the evolution equation $P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})D\mathbf{g}$, $P\mathbf{u}(0) = P\mathbf{u}_0$. If $\gamma_\tau\mathbf{g} = 0$, $P\mathbf{u}_0 = 0$, and $\gamma_n\mathbf{g} \neq 0$, we can ask if $P\mathbf{u} = 0$ or not. Due to Proposition A.1 we can claim that the answer is negative. Indeed if $\gamma_\tau\mathbf{g} = 0$ and $\gamma_n\mathbf{g} \neq 0$ then $PD\mathbf{g} \neq 0$. We also clarify the contribution of $\gamma_n\mathbf{g}$ to $P\mathbf{u}$ in Proposition 2.2.

Remark 2.4. Notice that in Definition 2.3, we do not require that $\mathbf{u}(0) = \mathbf{u}_0$, we only impose the initial condition $P\mathbf{u}(0) = P\mathbf{u}_0$. Indeed if $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$, then $(I-P)\mathbf{u} = (I-P)D\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^{1/2}(\Omega))$, $(I-P)\mathbf{u}(0)$ is not defined, and therefore the initial condition of $(I-P)\mathbf{u}$ cannot be defined. On the other hand if $\mathbf{g} \in H^s(0,T;\mathbf{V}^0(\Gamma))$ with $s > 1/2$, then $(I-P)\mathbf{u} = (I-P)D\mathbf{g}$ belongs to $H^s(0,T;\mathbf{V}^{1/2}(\Omega))$, and $(I-P)\mathbf{u}(0)$ is well defined in $\mathbf{V}^{1/2}(\Omega)$. If $(I-P)\mathbf{u}(0) = (I-P)\mathbf{u}_0$, then the solution defined in Definition 2.3 satisfies $\mathbf{u}(0) = \mathbf{u}_0$. Otherwise we only have $P\mathbf{u}(0) = P\mathbf{u}_0$. According to Proposition 2.1 the condition $(I-P)\mathbf{u}_0 = (I-P)\mathbf{u}(0)$ is equivalent to $(I-P)(\mathbf{u}_0 - D\mathbf{g}(0)) = 0$ because $(I-P)\mathbf{u}(0) = (I-P)D\mathbf{g}(0)$.

Therefore only the datum $P\mathbf{u}_0$ is needed to define the weak solutions of Eq. (2.1). When $(I-P)D\mathbf{g}(0)$ is well defined, it is natural to assume that $(I-P)\mathbf{u}_0 = (I-P)D\mathbf{g}(0)$. This is the reason why throughout what follows, we only state theorems with assumptions on $P\mathbf{u}_0$. The component $(I-P)\mathbf{u}(0)$, when it exists, is defined by $(I-P)D\mathbf{g}(0)$, and only in that case we assume that $(I-P)\mathbf{u}_0 = (I-P)D\mathbf{g}(0)$.

We are going to prove the main results of this section: Theorems 2.3, 2.5, 2.6, and 2.7.

Theorem 2.3.

(i) *For all* $P\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$ *and all* $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$ *Eq.* (2.1)*, admits a unique weak solution in* $L^2(0,T;\mathbf{V}^0(\Omega))$ *in the sense of Definition* 2.3*. This solution obeys*

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}_n^{1/2-\varepsilon}(\Omega))} + \|P\mathbf{u}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{1/2}(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big) \quad \text{for all } \varepsilon > 0.$$

(ii) *If* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$ *with* $0 \leqslant s \leqslant 2$*, and* Ω *is of class* C^3 *when* $3/2 < s \leqslant 2$*, then*

$$\big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{s+1/2}(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{H^{s/2}(0,T;\mathbf{V}^{1/2}(\Omega))} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}.$$

(iii) *If* Ω *is of class* C^3*,* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$ *and* $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$*, with* $0 \leqslant s < 1$ *and* $0\vee(s-1/2) = \max(0, s-1/2)$*, then*

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2-\varepsilon,s/2+1/4-\varepsilon/2}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big) \quad \text{for all } \varepsilon > 0. \tag{2.12}$$

(iv) *If* Ω *is of class* C^3*,* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$*,* $P\mathbf{u}_0 \in \mathbf{V}_n^{s-1/2}(\Omega)$*, with* $1 < s \leqslant 2$*, and if* $\mathbf{u}_0$ *and* $\mathbf{g}(0)$ *satisfy the compatibility condition*

$$\gamma_0\big(P\big(\mathbf{u}_0 - D\mathbf{g}(0)\big)\big) = 0, \tag{2.13}$$

then the estimate (2.12) *is satisfied.*

(v) *If Ω is of class C^3, $\mathbf{g}$ belongs to $\mathbf{V}^{1,1}(\Sigma)$ and $P\mathbf{u}_0 \in \mathbf{V}^1_n(\Omega)$, and if $P\mathbf{u}_0$ and $\mathbf{g}(0)$ satisfy the compatibility condition*

$$P\big(\mathbf{u}_0 - D\mathbf{g}(0)\big) \in \mathbf{V}^1_0(\Omega),$$

then $\mathbf{u}$ belongs to $\mathbf{V}^{2,1}(Q) + (L^2(0,T;\mathbf{V}^{3/2}(\Omega)) \cap H^1(0,T;\mathbf{V}^{1/2}(\Omega)))$, in particular $\mathbf{u}$ belongs to $C([0,T];\mathbf{V}^1(\Omega))$ and

$$\|\mathbf{u}\|_{C([0,T];\mathbf{V}^1(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^1_n(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{1,1}(\Sigma)}\big).$$

Remark 2.5. If $\gamma_n\mathbf{g} = 0$, it is proved in [14, Theorem 2.1] that we can take $\varepsilon = 0$ in estimate (2.12). We would like to know if we can still take $\varepsilon = 0$ in estimate (2.12) if $\gamma_n\mathbf{g} \neq 0$. This is not at all obvious because the condition $\gamma_n\mathbf{g} = 0$ plays a crucial role in the calculations in [13,14] (see e.g. identity [13, (A.27)]). We give a complete answer to this question in Theorems 2.5, 2.6, and 2.7.

The assumption 'Ω is of class C^3' is needed when we use regularity results for $D\mathbf{g}$ stated in Corollary A.1 for $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$ with $s > 3/2$. Since the results stated in (iii) and (iv) are obtained by interpolation this additional assumption for Ω is needed in all these cases.

Proof. *Step 1.* The system

$$P\mathbf{u}' = \tilde{A}P\mathbf{u}, \quad P\mathbf{u}(0) = 0, \quad \text{and} \quad (I-P)\mathbf{u} = 0,$$

admits $\mathbf{u} = 0$ as unique solution. Thus uniqueness of solution to Eq. (2.1) is obvious. Let us prove the existence. Let us first take $\mathbf{g} \in C^1([0,T];\mathbf{V}^{3/2}(\Gamma))$. We have already seen that the function $\mathbf{u} = \mathbf{w} + \mathbf{y}$, where $(\mathbf{w}(t), \pi(t))$ is the solution to (2.7) and $(\mathbf{y}, q)$ is the solution to (2.8), is a solution to Eq. (2.1) in the sense of Definition 2.2. Let us show that $\mathbf{u}$ is a solution to Eq. (2.1) in the sense of Definition 2.3. We notice that $(I-P)\mathbf{u} = (I-P)\mathbf{w} = (I-P)D\gamma_n\mathbf{g}$. For all $\boldsymbol{\Phi} \in D(A)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{u}(t)\boldsymbol{\Phi} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{u}(t)\boldsymbol{\Phi} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{w}(t)\boldsymbol{\Phi} + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{y}(t)\boldsymbol{\Phi} \\
&= \int_\Omega (\Delta\mathbf{y} - \nabla q + \Delta\mathbf{w} - \nabla\pi)\boldsymbol{\Phi} = \int_\Omega (\mathbf{y}+\mathbf{w})\Delta\boldsymbol{\Phi} - \int_\Gamma \mathbf{g}\cdot\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} \\
&= \int_\Omega P\mathbf{u}A\boldsymbol{\Phi} + \int_\Omega \mathbf{u}(I-P)\Delta\boldsymbol{\Phi} - \int_\Gamma \mathbf{g}\cdot\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} \\
&= \int_\Omega P\mathbf{u}A\boldsymbol{\Phi} + \int_\Omega \mathbf{u}\cdot\nabla\psi - \int_\Gamma \mathbf{g}\cdot\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} \\
&= \int_\Omega P\mathbf{u}A\boldsymbol{\Phi} + \int_\Gamma \mathbf{g}\cdot\left(-\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}\right),
\end{aligned}$$

where $\psi \in H^1(\Omega)/\mathbb{R}$ is defined by

$$\nabla\psi = (I-P)\Delta\boldsymbol{\Phi}.$$

According to the weak formulation (2.11), $\mathbf{u}$ is a solution to Eq. (2.1) in the sense of Definition 2.3. From the above calculation it also follows that

$$\int_\Omega P\Delta\mathbf{u}\boldsymbol{\Phi} = \int_\Omega \Delta(\mathbf{y}+\mathbf{w})\boldsymbol{\Phi} = \big\langle \tilde{A}P\mathbf{u} + (-\tilde{A})PD\mathbf{g}, \boldsymbol{\Phi}\big\rangle_{(D(A))',D(A)} \quad \text{for all } \boldsymbol{\Phi} \in D(A),$$

that is

$$P\Delta\mathbf{u} = \tilde{A}P\mathbf{u} + (-\tilde{A})PD\mathbf{g} \quad \text{in } L^2\big(0,T;\big(D(A)\big)'\big). \tag{2.14}$$

Now suppose that $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$. Let $(\mathbf{g}_k)_k$ be a sequence in $C^1([0,T];\mathbf{V}^{3/2}(\Gamma))$ converging to $\mathbf{g}$ in $L^2(0,T;\mathbf{V}^0(\Gamma))$. Let $(\mathbf{w}_k(t), \pi_k(t))$ be the solution to Eq. (2.7) corresponding to $\mathbf{g}_k(t)$, let $(\mathbf{y}_k, q_k)$ be the weak solution to Eq. (2.8) corresponding to $\mathbf{w}_k$, and set $\mathbf{u}_k = \mathbf{w}_k + \mathbf{y}_k$. We have already seen that $(\mathbf{u}_k)_k$ converges in $L^2(0,T;\mathbf{V}^0(\Omega))$ to the solution $\mathbf{u}$ to Eq. (2.1) in the sense of Definition 2.2. Moreover, passing to the limit when k tends to infinity in the equality $(I-P)\mathbf{u}_k = (I-P)D\gamma_n\mathbf{g}_k$, we obtain $(I-P)\mathbf{u} = (I-P)D\gamma_n\mathbf{g}$. Knowing that $(P\mathbf{u}_k)_k$ converges to $P\mathbf{u}$ in $L^2(0,T;\mathbf{V}^0_n(\Omega))$, and passing to the limit in the variational formulation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{u}_k(t)\boldsymbol{\Phi} = \int_\Omega P\mathbf{u}_k A\boldsymbol{\Phi} + \int_\Gamma \mathbf{g}_k\cdot\left(-\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n}\right),$$

we can show that $P\mathbf{u}$ is the solution of $P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PD\mathbf{g}$, $P\mathbf{u} = P\mathbf{u}_0$. Thus $\mathbf{u}$ is the solution of Eq. (2.1) in the sense of Definition 2.3.

Step 2. If $\mathbf{g}\in\mathbf{V}^{s,s/2}(\Sigma)$ with $0\leqslant s\leqslant 2$, and if Ω is of class C^3 when $3/2 < s \leqslant 2$, from Corollary A.1 it follows that

$$\big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{s+1/2}(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{H^{s/2}(0,T;\mathbf{V}^{1/2}(\Omega))} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}.$$

Step 3. To prove the statements (iii) and (iv) in the theorem, we follow the technique of proof used in [16] for the heat equation. We have

$$P\mathbf{u}(t) = \mathrm{e}^{tA}P\mathbf{u}_0 - A\int_0^t \mathrm{e}^{(t-s)A}PD\mathbf{g}(s)\,\mathrm{d}s. \tag{2.15}$$

If $\mathbf{g}\in L^2(0,T;\mathbf{V}^0(\Gamma))$, then $(-A)^{1/4-\varepsilon/4}PD\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^0_n(\Omega))$, and

$$\begin{aligned}
&\big\|(-A)^{1/4-\varepsilon/2}P\mathbf{u}(t)\big\|_{\mathbf{V}^0_n(\Omega)}\\
&\quad\leqslant \big\|\mathrm{e}^{tA}(-A)^{1/4-\varepsilon/2}P\mathbf{u}_0\big\|_{\mathbf{V}^0_n(\Omega)} + \int_0^t \big\|(-A)^{1-\varepsilon/4}\mathrm{e}^{(t-s)A}\big\|\,\big\|(-A)^{1/4-\varepsilon/4}PD\mathbf{g}(s)\big\|_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}s\\
&\quad\leqslant Ct^{-1/4+\varepsilon/2}\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)} + \int_0^t (t-s)^{-1+\varepsilon/4}\big\|(-A)^{1/4-\varepsilon/4}PD\mathbf{g}(s)\big\|_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}s.
\end{aligned}$$

From Young's inequality for convolutions, we deduce that

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{1/2-\varepsilon}(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^0(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big)\quad \text{for all } \varepsilon > 0. \tag{2.16}$$

Moreover

$$\begin{aligned}
\frac{\mathrm{d}P\mathbf{u}}{\mathrm{d}t} &= A\mathrm{e}^{tA}P\mathbf{u}_0 - APD\mathbf{g}(t) - A\int_0^t A\mathrm{e}^{(t-s)A}PD\mathbf{g}(s)\,\mathrm{d}s\\
&= -(-A)^{3/4+\varepsilon/2}\mathrm{e}^{tA}(-A)^{1/4-\varepsilon/2}P\mathbf{u}_0 + (-A)^{3/4+\varepsilon/2}(-A)^{1/4-\varepsilon/2}PD\mathbf{g}(t)\\
&\quad + (-A)^{3/4+\varepsilon/2}(-A)^{1/4-\varepsilon/2}\int_0^t A\mathrm{e}^{(t-s)A}PD\mathbf{g}(s)\,\mathrm{d}s\\
&= (-A)^{3/4+\varepsilon/2}\big[-(-A)^{1/4-\varepsilon/2}P\mathbf{u}(t) + (-A)^{1/4-\varepsilon/2}PD\mathbf{g}(t)\big],
\end{aligned}$$

that is

$$(-A)^{-3/4-\varepsilon/2}P\mathbf{u}' = -(-A)^{1/4-\varepsilon/2}P\mathbf{u}(t) + (-A)^{1/4-\varepsilon/2}PD\mathbf{g}(t).$$

Since $(-A)^{1/4-\varepsilon/2}P\mathbf{u}$ and $(-A)^{1/4-\varepsilon/2}PD\mathbf{g}$ belong to $L^2(0,T;\mathbf{V}^0_n(\Omega))$, we deduce that

$$\begin{aligned}
&\|P\mathbf{u}'\|_{L^2(0,T;[D((-A)^{3/4+\varepsilon/2})]')}\\
&\quad\leqslant C\big(\big\|(-A)^{1/4-\varepsilon/2}P\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))} + \big\|(-A)^{1/4-\varepsilon/2}PD\mathbf{g}\big\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))}\big)\\
&\quad\leqslant C\big(\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{1/2-\varepsilon}(\Omega))} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big) \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^0(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big),
\end{aligned} \tag{2.17}$$

for all $\varepsilon > 0$. By interpolation between (2.16) and (2.17), we obtain

$$\|P\mathbf{u}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0_n(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^0(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big), \quad \text{for all } \varepsilon > 0.$$

Step 4. Let us show that if $\mathbf{g} \in \mathbf{V}^{2,1}(\Sigma)$ and if $\mathbf{u}_0$ and $\mathbf{g}(0)$ obey the compatibility condition

$$P\big(\mathbf{u}_0 - D\mathbf{g}(0)\big) \in \mathbf{V}^{3/2}(\Omega) \cap \mathbf{V}^1_0(\Omega),$$

then $P\mathbf{u}$ belongs to $\mathbf{V}^{5/2-\varepsilon,5/4-\varepsilon/2}(Q)$ for all $\varepsilon > 0$. By integration by parts in Eq. (2.15), we have

$$P\mathbf{u}(t) = \mathrm{e}^{tA}\big(P\mathbf{u}_0 - PD\mathbf{g}(0)\big) + PD\mathbf{g}(t) - \int_0^t \mathrm{e}^{(t-s)A} PD\mathbf{g}'(s)\,\mathrm{d}s. \tag{2.18}$$

Since $\mathbf{g} \in L^2(0,T;\mathbf{V}^2(\Gamma))$, from Lemma A.1 it follows that $PD\mathbf{g} \in L^2(0,T;\mathbf{V}^{5/2}(\Omega))$. Moreover

$$\begin{aligned}\left\|(-A)^{5/4-\varepsilon/2}\int_0^t \mathrm{e}^{(t-s)A} PD\mathbf{g}'(s)\,\mathrm{d}s\right\|_{\mathbf{V}^0_n(\Omega)} &= \left\|\int_0^t (-A)^{1-\varepsilon/4}\mathrm{e}^{(t-s)A}(-A)^{1/4-\varepsilon/4} PD\mathbf{g}'(s)\,\mathrm{d}s\right\|_{\mathbf{V}^0_n(\Omega)} \\ &\leqslant C\int_0^t (t-s)^{-1+\varepsilon/4}\big\|(-A)^{1/4-\varepsilon/4} PD\mathbf{g}'(s)\big\|_{\mathbf{V}^0_n(\Omega)}\,\mathrm{d}s.\end{aligned}$$

From Young's inequality for convolutions it follows that

$$\left\|\int_0^{(\cdot)} \mathrm{e}^{(\cdot-s)A} PD\mathbf{g}'(s)\,\mathrm{d}s\right\|_{L^2(0,T;\mathbf{V}^{5/2-\varepsilon}(\Omega))} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{2,1}(\Sigma)}.$$

We also have:

$$\begin{aligned}\big\|(-A)^{5/4-\varepsilon/2}\mathrm{e}^{tA}\big(P\mathbf{u}_0 - PD\mathbf{g}(0)\big)\big\|_{\mathbf{V}^0_n(\Omega)} &\leqslant C\,t^{-1/2+\varepsilon/4}\big\|(-A)^{3/4-\varepsilon/4}\big(P\mathbf{u}_0 - PD\mathbf{g}(0)\big)\big\|_{\mathbf{V}^0_n(\Omega)} \\ &\leqslant Ct^{-1/2+\varepsilon/4}\big\|P\mathbf{u}_0 - PD\mathbf{g}(0)\big\|_{\mathbf{V}^{3/2-\varepsilon/2}(\Omega)\cap\mathbf{V}^1_0(\Omega)} \\ &\leqslant Ct^{-1/2+\varepsilon/4}\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{3/2}(\Omega)} + \big\|PD\mathbf{g}(0)\big\|_{\mathbf{V}^{3/2}(\Omega)}\big) \leqslant Ct^{-1/2+\varepsilon/4}\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{3/2}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{2,1}(\Sigma)}\big).\end{aligned}$$

Thus

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^{5/2-\varepsilon}(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{3/2}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{2,1}(\Sigma)}\big), \quad \text{for all } \varepsilon > 0.$$

By differentiating (2.18) we obtain

$$\frac{\mathrm{d}P\mathbf{u}}{\mathrm{d}t} = A\mathrm{e}^{tA}\big(P\mathbf{u}_0 - PD\mathbf{g}(0)\big) - A\int_0^t \mathrm{e}^{(t-s)A} PD\mathbf{g}'(s)\,\mathrm{d}s.$$

Since $\mathbf{g}' \in L^2(0,T;\mathbf{V}^0(\Gamma))$, from Step 3 we deduce that $A\int_0^t \mathrm{e}^{(t-s)A} PD\mathbf{g}'(s)\,\mathrm{d}s$ belongs to $H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0(\Omega))$. Moreover, since $P(\mathbf{u}_0 - D\mathbf{g}(0)) \in \mathbf{V}^{3/2}(\Omega) \cap \mathbf{V}^1_0(\Omega)$, $\mathrm{e}^{tA}(P\mathbf{u}_0 - PD\mathbf{g}(0))$ belongs to

$$L^2\big(0,T;\mathbf{V}^{5/2}(\Omega) \cap \mathbf{V}^1_0(\Omega)\big) \cap H^{5/4}\big(0,T;\mathbf{V}^0_n(\Omega)\big).$$

In particular $\mathrm{e}^{tA}(P\mathbf{u}_0 - PD\mathbf{g}(0))$ belongs to $H^{1/4}(0,T;\mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega))$. Thus $A\mathrm{e}^{tA}(P\mathbf{u}_0 - PD\mathbf{g}(0))$ belongs to $H^{1/4}(0,T;\mathbf{V}^0_n(\Omega))$. Therefore $\frac{\mathrm{d}P\mathbf{u}}{\mathrm{d}t} \in H^{1/4-\varepsilon/2}(0,T;\mathbf{V}^0_n(\Omega))$, and we have

$$\|P\mathbf{u}\|_{H^{5/4-\varepsilon/2}(0,T;\mathbf{V}^0_n(\Omega))} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{3/2}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{2,1}(\Sigma)}\big), \quad \text{for all } \varepsilon > 0.$$

The other estimates of the statements (iii) and (iv) in Theorem 2.3 can be obtained by interpolation.

Step 5. Suppose now that $\mathbf{g}$ belongs to $\mathbf{V}^{1,1}(\Sigma)$, $P\mathbf{u}_0 \in \mathbf{V}^1_n(\Omega)$, and that $\mathbf{u}_0$ and $\mathbf{g}(0)$ satisfy the compatibility condition $P(\mathbf{u}_0 - D\mathbf{g}(0)) \in \mathbf{V}^1_0(\Omega)$. From Corollary A.1 we deduce that

$$\big\|(I-P)\mathbf{u}\big\|_{H^1(0,T;\mathbf{V}^{1/2}(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{3/2}(\Omega))} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{1,1}(\Sigma)}.$$

Moreover as in Step 3 we have

$$P\mathbf{u}(t) = e^{tA}\big(P\mathbf{u}_0 - PD\mathbf{g}(0)\big) + PD\mathbf{g}(t) - \int_0^t e^{(t-s)A} PD\mathbf{g}'(s)\,ds.$$

Since $P\mathbf{u}_0 - PD\mathbf{g}(0) \in \mathbf{V}^1_0(\Omega)$, $e^{tA}(P\mathbf{u}_0 - PD\mathbf{g}(0))$ belongs to $\mathbf{V}^{2,1}(Q)$. Since

$$\mathbf{g} \in H^1\big(0,T;\mathbf{V}^0(\Gamma)\big) \cap L^2\big(0,T;\mathbf{V}^1(\Gamma)\big),$$

from Corollary A.1 it follows that $PD\mathbf{g}$ belong to

$$H^1\big(0,T;\mathbf{V}^{1/2}(\Omega)\big) \cap L^2\big(0,T;\mathbf{V}^{3/2}(\Omega)\big).$$

The term $\int_0^t e^{(t-s)A} PD\mathbf{g}'(s)\,ds$ belongs to $\mathbf{V}^{2,1}(Q)$ because $PD\mathbf{g}'$ belongs to $L^2(0,T;\mathbf{V}^0_n(\Omega))$ (actually it belongs to $L^2(0,T;\mathbf{V}^{1/2}_n(\Omega))$). It is clear that $H^1(0,T;\mathbf{V}^{1/2}(\Omega)) \cap L^2(0,T;\mathbf{V}^{3/2}(\Omega)) \hookrightarrow C([0,T];\mathbf{V}^1(\Omega))$ and $\mathbf{V}^{2,1}(Q) \hookrightarrow C([0,T];\mathbf{V}^1(\Omega))$, see [6, Chapter 18, Section 1.3, page 579]. Thus the proof is complete. □

Theorem 2.4. *For all $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$, the solutions given by Theorems* 2.2 *and* 2.3 *coincide.*

Proof. The proof is given in the first step in the proof of Theorem 2.3. □

Before ending this subsection we would like to give an equivalent formulation to Eq. (2.9) which allows us to use regularity results from [14].

Proposition 2.2. *Assume that Ω is of class C^3, $\mathbf{g} \in \mathbf{V}^{2,1}(\Sigma)$, $P\mathbf{u}_0 \in \mathbf{V}^1_n(\Omega)$, and $P(\mathbf{u}_0 - D\mathbf{g}(0)) \in \mathbf{V}^1_0(\Omega)$. A function $P\mathbf{u} \in \mathbf{V}^{2,1}(Q)$ is a weak solution to Eq.* (2.9) *if and only if the following conditions are satisfied*:

(i) $P\mathbf{u}(0) = P\mathbf{u}_0$. *There exists a function $\pi \in L^2(0,T;H^1(\Omega))$ such that*

$$\frac{\partial P\mathbf{u}}{\partial t} - \Delta P\mathbf{u} + \nabla\pi = 0, \tag{2.19}$$

in the sense of distributions in Q.

(ii) $P\mathbf{u}$ *satisfies the following boundary condition*:

$$P\mathbf{u}|_\Sigma = \gamma_\tau\mathbf{g} - \gamma_\tau(\nabla q), \tag{2.20}$$

where $q \in L^2(0,T;H^2(\Omega)/\mathbb{R})$ is the solution to the boundary problem

$$\Delta q(t) = 0 \quad \text{in } \Omega, \qquad \frac{\partial q(t)}{\partial \mathbf{n}} = \mathbf{g}(t)\cdot\mathbf{n} \quad \text{on } \Gamma, \qquad \text{for all } t \in [0,T]. \tag{2.21}$$

Proof. First prove (2.19). Let $\mathbf{u}$ be the solution of (2.1) and let $P\mathbf{u}$ be the solution of (2.9). Due to Theorem 2.3, we know that $P\mathbf{u} \in \mathbf{V}^{5/2-\varepsilon,5/4-\varepsilon/2}(Q)$ for all $\varepsilon > 0$, and that $(I-P)\mathbf{u} \in L^2(0,T;\mathbf{V}^{5/2}(\Omega)) \cap H^1(0,T;\mathbf{V}^{1/2}(\Omega))$. Thus the pressure in (2.1) belongs to $L^2(0,T;H^1(\Omega))$, and we have

$$\frac{\partial P\mathbf{u}}{\partial t} - \Delta P\mathbf{u} + \nabla p = \Delta(I-P)\mathbf{u} - \frac{\partial (I-P)\mathbf{u}}{\partial t}.$$

We know that $(I-P)\mathbf{u} = (I-P)D\mathbf{g}$, and from the characterization of $(I-P)$ (see [25]), it follows that $(I-P)D\mathbf{g} = \nabla q$, where q is the solution of (2.21). Since $\mathbf{g} \in \mathbf{V}^{2,1}(\Sigma)$, the function q belongs to $H^1(0,T;H^{3/2}(\Omega)) \cap L^2(0,T;H^3(\Omega))$ (we have only assumed that Ω is of class C^3, and we cannot hope to have a better regularity than $H^3(\Omega)$ even if the Neumann condition is in $H^2(\Gamma)$, see [26]). Thus $\Delta q \in L^2(0,T;H^1(\Omega))$ and $\frac{\partial q}{\partial t} \in L^2(0,T;H^{3/2}(\Omega))$. Since $\Delta\nabla q = \nabla\Delta q$ in the sense of distributions in Q, Eq. (2.19) is established with $\pi = p - \Delta q + \frac{\partial q}{\partial t}$.

To prove (2.20), we observe that

$$P\mathbf{u}|_\Sigma = \mathbf{u}|_\Sigma - (I-P)\mathbf{u}|_\Sigma = \mathbf{g} - (I-P)D\mathbf{g}|_\Sigma,$$

and that $(I-P)D\mathbf{g} = \nabla q$, where q is the solution of (2.21). Therefore (2.20) is proved because $\mathbf{g}(t) - \gamma_0(\nabla q(t)) = \gamma_\tau \mathbf{g}(t) - \gamma_\tau(\nabla q(t))$.

Now we assume that $P\mathbf{u} \in \mathbf{V}^{2,1}(Q)$ obeys the statements (i) and (ii) of the proposition. For all $\boldsymbol{\Phi} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega P\mathbf{u}(t)\boldsymbol{\Phi} = \int_\Omega \Delta P\mathbf{u}(t)\boldsymbol{\Phi} = \int_\Omega \mathbf{u}(t)A\boldsymbol{\Phi} - \int_\Gamma \big(\gamma_\tau \mathbf{g}(t) - \gamma_\tau\big(\nabla q(t)\big)\big)\cdot \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{n}}.$$

Introducing the function $\psi \in H^1(\Omega)/\mathbb{R}$ defined by $\nabla\psi = (I-P)\Delta\boldsymbol{\Phi}$, we obtain

$$\int_\Gamma \gamma_\tau\big(\nabla q(t)\big)\cdot\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} = \int_\Omega \Delta\boldsymbol{\Phi}\cdot\nabla q(t) = \int_\Omega \big((I-P)\Delta\boldsymbol{\Phi}\big)\cdot\nabla q(t) = \int_\Omega \nabla\psi\cdot\nabla q(t) = \int_\Gamma \psi\mathbf{g}(t)\cdot\mathbf{n}.$$

The first equality comes from the fact that $\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}}\cdot\mathbf{n} = 0$ and that $\int_\Omega \Delta\nabla q(t) = 0$. Thus, if $P\mathbf{u}$ obeys conditions (i) and (ii) in the proposition, then $P\mathbf{u}$ is the weak solution to Eq. (2.9) (see (2.11)). □

Proposition 2.3. *Assume that* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$ *and* $P\mathbf{u}_0 \in \mathbf{V}^{s-1/2}_n(\Omega)$ *for some* $s > 1$. *Let* q *be the solution of* (2.21). *The compatibility condition*

$$\gamma_\tau\mathbf{g}(0) - \gamma_\tau\nabla q(0) = \gamma_0 P\mathbf{u}_0, \tag{2.22}$$

is equivalent to (2.13).

Proof. We have

$$\gamma_0 PD\mathbf{g}(0) = \gamma_0 D\mathbf{g}(0) - \gamma_0\big((I-P)D\mathbf{g}(0)\big) = \gamma_0\mathbf{g}(0) - \gamma_0\nabla q(0) = \gamma_\tau\mathbf{g}(0) - \gamma_\tau\nabla q(0),$$

which proves that (2.13) and (2.22) are equivalent. □

Proposition 2.4. *There exists a constant* $C > 0$ *such that*

$$\big\|\gamma_\tau(\nabla q)\big\|_{\mathbf{V}^s(\Gamma)} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^s(\Gamma)} \quad \textit{for all } s \in\,]0,3], \textit{ and all } \mathbf{g}\in\mathbf{V}^s(\Gamma),$$

where q *is the solution of Eq.* (2.21), *and*

$$\big\|\nabla_\tau(\gamma_0 q)\big\|_{\mathbf{V}^0(\Gamma)} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^0(\Gamma)} \quad \textit{for all } \mathbf{g}\in\mathbf{V}^0(\Gamma),$$

where ∇_τ *denotes the tangential gradient operator. In the above statements we assume that*

$$\begin{aligned} &\Omega \textit{ is of class } C^3 \textit{ if } 0 \leqslant s \leqslant 3/2,\\ &\Omega \textit{ is of class } C^4 \textit{ if } 3/2 < s \leqslant 5/2,\\ &\textit{and } \Omega \textit{ is of class } C^5 \textit{ if } 5/2 < s \leqslant 3. \end{aligned} \tag{2.23}$$

Proof. If $\mathbf{g} \in \mathbf{V}^s(\Gamma)$ and $s \in\,]0,3]$, we know that $q \in H^{s+3/2}(\Omega)$, $\nabla q \in \mathbf{H}^{s+1/2}(\Omega)$, and $\gamma_0(\nabla q) \in \mathbf{H}^s(\Gamma)$, which provides the estimate of the proposition in the case when $s > 0$. For $s = 0$, we have $\gamma_0 q \in H^1(\Gamma)$, and $\nabla_\tau(\gamma_0 q) \in \mathbf{V}^0(\Gamma)$. The proof is complete. □

Remark 2.6. Since we use regularity results for the auxiliary problem (2.21), we need that Ω satisfies (2.23) (see [26, Exercise 3.11]). From Proposition 2.2 and a density argument it follows that the system (2.19)–(2.20) is equivalent to

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})D\big(\gamma_\tau\mathbf{g} - \gamma_\tau(\nabla q)\big), \qquad P\mathbf{u}(0) = P\mathbf{u}_0,$$

if $\mathbf{g} \in L^2(0,T;\mathbf{V}^s(\Gamma))$ and $s > 0$, and to

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})D\big(\gamma_\tau\mathbf{g} - \nabla_\tau(\gamma_0 q)\big), \qquad P\mathbf{u}(0) = P\mathbf{u}_0,$$

if $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$.

Theorem 2.5. *Assume that* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$*,* $P\mathbf{u}_0 \in \mathbf{V}_n^{s-1/2}(\Omega)$*, with* $3/2 \leqslant s < 3$*, and* Ω *satisfies* (2.23)*. If* $\mathbf{u}_0$ *and* $\mathbf{g}(0)$ *satisfy the compatibility condition* (2.22)*, then the solution* $P\mathbf{u}$ *to Eq.* (2.9) *satisfies the estimate*

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{s-1/2}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big). \tag{2.24}$$

Proof. By a density argument, it is sufficient to prove estimate (2.24) when $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma) \cap \mathbf{V}^{2,1}(\Sigma)$. In this way we can use Proposition 2.2. With Proposition 2.4, we can show that

$$\big\|\gamma_\tau(\nabla q)\big\|_{\mathbf{V}^{s,s/2}(\Sigma)} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)} \quad \text{for all } s \in]0,3], \text{ and all } \mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma),$$

where q is the solution of Eq. (2.21). Thus, the theorem is a direct consequence of the above estimate, of Proposition 2.2, and of known regularity results for the instationary Stokes equations with nonhomogeneous boundary conditions [24]. □

Theorem 2.6. *Assume that* Ω *is of class* C^3*,* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$*, and* $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$*, with* $s \in [0,1[$*. Then the solution* $P\mathbf{u}$ *to Eq.* (2.9) *satisfies the estimate*

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big). \tag{2.25}$$

Proof. By a density argument, it is sufficient to prove estimate (2.25) in the case when $\mathbf{g} \in \mathbf{V}^{2,1}(\Sigma)$ and $\mathbf{g}(0)$ and $P\mathbf{u}_0$ satisfy $\gamma_0(PD\mathbf{g}(0) - P\mathbf{u}_0) = 0$. If $\mathbf{g} \in \mathbf{V}^{2,1}(\Sigma)$, with Proposition 2.4 we can show that

$$\big\|\gamma_\tau(\nabla q)\big\|_{\mathbf{V}^{s,s/2}(\Sigma)} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)} \quad \text{for all } s \in [0,1[,$$

where q is the solution of Eq. (2.21). (For $s = 0$, we have to observe that $\nabla_\tau(\gamma_0 q) = \gamma_\tau(\nabla q)$.) Thus estimate (2.25) follows from Proposition 2.2, and from [14, Theorem 2.1] in the case where $0 \leqslant s < 1$. □

Theorem 2.7. *Assume that* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$*,* $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$*, with* $s \in [0,1[\,\cap\,]1,3[$*, and* Ω *satisfies* (2.23)*. If* $\mathbf{u}_0$ *and* $\mathbf{g}(0)$ *satisfy the compatibility condition* (2.13) *when* $1 < s < 3$*, then*

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big). \tag{2.26}$$

Proof. Estimate (2.26) is already proved for $s \in [0,1[$ and $s \in [3/2,3[$. For $s \in]1,3/2[$, it is obtained by interpolation between the regularity results stated in Theorems 2.5 and 2.6. □

Remark 2.7. In Theorems 2.5 and 2.7, for $s = 2$, we have to assume that Ω is of class C^4 (because we make use of Proposition 2.2), while in Theorem 2.3(iv) we only assume that Ω is of class C^3. Actually, combining the results stated in Theorem 2.6 for $s = 0$ with arguments in Step 4 of the proof of Theorem 2.3, we can show that (2.26) is still true when Ω is of class C^3.

Corollary 2.1. *Assume that* Ω *is of class* C^3*. If* $\mathbf{g}$ *belongs to* $\mathbf{V}^{3/4,3/4}(\Sigma)$*, if* $P\mathbf{u}_0 \in \mathbf{V}_n^{3/4}(\Omega)$*, and if* $P\mathbf{u}_0 = PD\mathbf{g}(0)$*, then*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{5/4,5/8}(Q)} + \big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{5/4}(\Omega))\cap H^{3/4}(0,T;\mathbf{V}^{1/2}(\Omega))} + \|\mathbf{u}\|_{C([0,T];\mathbf{V}^{3/4}(\Omega))}\\&\quad\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{3/4}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{3/4,3/4}(\Sigma)}\big).\end{aligned} \tag{2.27}$$

Proof. From Theorem 2.6 with $s = 3/4$, it follows that

$$\|P\mathbf{u}\|_{\mathbf{V}^{5/4,5/8}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{1/4}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{3/4,3/8}(\Sigma)}\big).$$

It is clear that $(I-P)\mathbf{u}$ belongs to

$$L^2\big(0,T;\mathbf{V}^{5/4}(\Omega)\big) \cap H^{3/4}\big(0,T;\mathbf{V}^{1/2}(\Omega)\big) \hookrightarrow C\big([0,T];\mathbf{V}^{3/4}(\Omega)\big).$$

Moreover $P\mathbf{u} \in C([0,T];\mathbf{V}_n^{3/4}(\Omega))$. Indeed, if $\mathbf{g} \in \mathbf{V}^{0,0}(\Sigma)$ and $P\mathbf{u}_0 \in \mathbf{V}_n^0(\Omega)$, then $P\mathbf{u} \in \mathbf{V}^{1/2,1/4}(Q)$. If $\mathbf{g} \in \mathbf{V}^{1,1}(\Sigma)$ and $P\mathbf{u}_0 = PD\mathbf{g}(0)$, then

$$P\mathbf{u}(t) = PD\mathbf{g}(t) - \int_0^t e^{(t-s)A} PD\mathbf{g}'(s)\,ds,$$

$$\int_0^t e^{(t-s)A} PD\mathbf{g}'(s)\,ds \in L^2\big(0,T;\mathbf{V}^2(\Omega)\big) \cap H^1\big(0,T;\mathbf{V}^{1/4}(\Omega)\big) \hookrightarrow H^{2/3}\big(0,T;\mathbf{V}^{5/6}(\Omega)\big),$$

$$PD\mathbf{g} \in L^2\big(0,T;\mathbf{V}^{3/2}(\Omega)\big) \cap H^1\big(0,T;\mathbf{V}^{1/2}(\Omega)\big) \hookrightarrow H^{2/3}\big(0,T;\mathbf{V}^{5/6}(\Omega)\big).$$

By interpolation we obtain that $P\mathbf{u} \in C([0,T];\mathbf{V}_n^{3/4}(\Omega))$ when the assumptions of the corollary are satisfied. The proof is complete. □

3. Other regularity results

In the previous section we have seen that, for all $s \in [0,3[$ with $s \neq 1$, if $\mathbf{g}$ belongs to $\mathbf{V}^{s,s/2}(\Sigma)$, $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$, and if $\gamma_0(P\mathbf{u}_0 - PD\mathbf{g}(0)) = 0$ when $s > 1$, then

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big).$$

This result generalizes to the Stokes equations the type of regularity results known for the nonhomogeneous heat equation. We would like to obtain regularity results different from the ones stated in Theorems 2.3 and 2.7, still in the case where $\gamma_n\mathbf{g} \neq 0$. From [8] or from Theorem 2.3, we know that the condition $\mathbf{g} \in \mathbf{V}^{3/2,3/4}(\Sigma)$ is not sufficient to guarantee that $\mathbf{u}$ belongs to $\mathbf{V}^{2,1}(Q)$ if $\gamma_n\mathbf{g} \neq 0$. The regularity of the normal trace of $\mathbf{g}$ must be better than what is needed for the tangential component. We show below that the regularity $\mathbf{u} \in \mathbf{V}^{s,s/2}(Q)$ can be obtained if $\mathbf{g} \in L^2(0,T;\mathbf{V}^{s-1/2}(\Gamma)) \cap H^{s/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))$. At the end of the section, we compare our result with the corresponding one in [8] in the case when $\gamma_n\mathbf{g} = \mathbf{g}$.

Theorem 3.1. *For all $s \in [0,2]$ with $s \neq \frac{3}{2}$, all $P\mathbf{u}_0 \in [\mathbf{V}^{-1}(\Omega),\mathbf{V}_n^1(\Omega)]_{s/2}$, and all $\mathbf{g} \in L^2(0,T;\mathbf{V}^{s-1/2}(\Gamma)) \cap H^{s/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))$ satisfying $\gamma_0(P\mathbf{u}_0 - PD\mathbf{g}(0)) = 0$ if $s > 3/2$, Eq.* (2.1)*, admits a unique weak solution in $L^2(0,T;\mathbf{V}^0(\Omega))$ in the sense of Definition* 2.3*. This solution obeys*

$$\begin{aligned}&\|P\mathbf{u}\|_{\mathbf{V}^{s,s/2}(Q)} + \|P\mathbf{u}\|_{H^1(0,T;[(D(A^*))',\mathbf{V}_n^0(\Omega)]_{s/2})}\\ &\quad\leqslant C\big(\|P\mathbf{u}_0\|_{[\mathbf{V}^{-1}(\Omega),\mathbf{V}_n^1(\Omega)]_{s/2}} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{s-1/2}(\Gamma))\cap H^{s/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big),\\ &\big\|(I-P)\mathbf{u}\big\|_{\mathbf{V}^{s,s/2}(Q)} \leqslant C\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{s-1/2}(\Gamma))\cap H^{s/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))}.\end{aligned}$$

Lemma 3.1. *For all $P\mathbf{u}_0 \in \mathbf{V}^{-1}(\Omega)$, and all $\mathbf{g} \in L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$, Eq.* (2.1) *admits a unique weak solution in $L^2(0,T;\mathbf{V}^0(\Omega))$ in the sense of Definition* 2.3*. This solution obeys*

$$\begin{aligned}&\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0(\Omega))} + \|P\mathbf{u}\|_{H^1(0,T;(D(A^*))')} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big),\\ &\big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^0(\Omega))} \leqslant C\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))}.\end{aligned}$$

Proof. If $\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))$, then $PD\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}_n^0(\Omega))$ (see Corollary A.1), $(-\tilde{A})PD\mathbf{g}$ belongs to $L^2(0,T;(D(A^*))')$, and we have

$$\big\|(-\tilde{A})PD\mathbf{g}\big\|_{L^2(0,T;(D(A^*))')} \leqslant C\|PD\mathbf{g}\|_{L^2(0,T;\mathbf{V}_n^0(\Omega))} \leqslant C\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))}.$$

Due to [4, Chapter 3, Theorem 2.2], the equation

$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PD\mathbf{g}, \qquad P\mathbf{u}(0) = P\mathbf{u}_0,$$

admits a unique solution in $L^2(0,T;\mathbf{V}_n^0(\Omega)) \cap H^1(0,T;(D(A^*))')$ and

$$\begin{aligned}\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}_n^0(\Omega))} + \|P\mathbf{u}\|_{H^1(0,T;(D(A^*))')} &\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)} + \big\|(-\tilde{A})PD\mathbf{g}\big\|_{L^2(0,T;(D(A^*))')}\big)\\ &\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^{-1}(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big). \qquad \square\end{aligned}$$

Lemma 3.2. *For all $P\mathbf{u}_0 \in \mathbf{V}_n^1(\Omega)$, and all $\mathbf{g} \in L^2(0,T;\mathbf{V}^{3/2}(\Gamma)) \cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))$ satisfying $P(\mathbf{u}_0 - D\mathbf{g}(0)) \in \mathbf{V}_0^1(\Omega)$, Eq.* (2.1) *admits a unique weak solution in $L^2(0,T;\mathbf{V}_n^0(\Omega))$ in the sense of Definition* 2.3. *This solution obeys*

$$\|\mathbf{u}\|_{\mathbf{V}^{2,1}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^1(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Gamma))\cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big).$$

Proof. Let $\mathbf{g}$ be in $L^2(0,T;\mathbf{V}^{3/2}(\Gamma)) \cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))$, and set $(\mathbf{w}(t),\pi(t)) = (D\mathbf{g}(t), D_p\mathbf{g}(t))$. It is clear that $(\mathbf{w},\pi) \in \mathbf{V}^{2,1}(Q) \times L^2(0,T;H^1(\Omega)/\mathbb{R})$ (see Corollary A.1), and that

$$\|\mathbf{w}\|_{\mathbf{V}^{2,1}(Q)} \leqslant C\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Gamma))\cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))}.$$

Let $(\mathbf{y},q)$ be the weak solution in $W(0,T;\mathbf{V}_0^1(\Omega),\mathbf{V}^{-1}(\Omega)) \times L^2(0,T;L^2(\Omega)/\mathbb{R})$ to the equation

$$\begin{aligned}&\frac{\partial \mathbf{y}}{\partial t} - \Delta\mathbf{y} + \nabla q = -\frac{\partial \mathbf{w}}{\partial t}, \quad \operatorname{div}\mathbf{y} = 0 \quad \text{in } Q,\\ &\mathbf{y} = 0 \quad \text{on } \Sigma, \qquad \mathbf{y}(0) = P\big(\mathbf{u}_0 - \mathbf{w}(0)\big) \quad \text{in } \Omega.\end{aligned} \tag{3.1}$$

We know that

$$\begin{aligned}\|\mathbf{y}\|_{\mathbf{V}^{2,1}(Q)} &\leqslant C\big(\big\|P\big(\mathbf{u}_0 - \mathbf{w}(0)\big)\big\|_{\mathbf{V}_0^1(\Omega)} + \|-P\mathbf{w}'\|_{L^2(0,T;\mathbf{V}_n^0(\Omega))}\big)\\ &\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^1(\Omega)} + \|\mathbf{w}\|_{C([0,T];\mathbf{V}^1(\Omega))} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Gamma))\cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big)\\ &\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^1(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^{3/2}(\Gamma))\cap H^1(0,T;\mathbf{V}^{-1/2}(\Gamma))}\big).\end{aligned}$$

Since $\mathbf{u} = \mathbf{w} + \mathbf{y}$ is a solution to Eq. (2.1) in the sense of Definition 2.3. The proof is complete. □

Remark 3.1. As mentioned in Remark 2.2, Lemma 3.1 is already stated in [7, Theorem 4]. We have given a short proof for the convenience of the reader. Observe that Lemma 3.2 is not a consequence of [7, Corollary 5] since we do not assume that $\mathbf{u}_0 \in \mathbf{H}^2(\Omega)$.

Proof of Theorem 3.1. The result stated in Theorem 3.1 can be derived by interpolation from Lemmas 3.1 and 3.2. Indeed we have (see [12]):

$$\begin{aligned}&\big[L^2\big(0,T;\mathbf{V}^{-1/2}(\Gamma)\big), L^2\big(0,T;\mathbf{V}^{3/2}(\Gamma)\big) \cap H^1\big(0,T;\mathbf{V}^{-1/2}(\Gamma)\big)\big]_{s/2}\\ &\quad = L^2\big(0,T;\mathbf{V}^{s-1/2}(\Gamma)\big) \cap H^{s/2}\big(0,T;\mathbf{V}^{-1/2}(\Gamma)\big),\\ &\big[H^1\big(0,T;\big(D(A^*)\big)'\big), H^1\big(0,T;\mathbf{V}_n^0(\Omega)\big)\big]_{s/2} = H^1\big(0,T;\big[\big(D(A^*)\big)',\mathbf{V}_n^0(\Omega)\big]_{s/2}\big),\end{aligned}$$

and

$$\big[L^2\big(0,T;\mathbf{V}^0(\Omega)\big),\mathbf{V}^{2,1}(Q)\big]_{s/2} = \mathbf{V}^{s,s/2}(Q),$$

for all $s \in [0,2]$. □

Before ending this section, we would like to compare the result stated in Theorem 3.1 with the one in [8, Theorem 6.1], in the case when $\mathbf{g} = \gamma_n\mathbf{g}$. Observe that the trace theorems proved in [8] are obtained when Ω is bounded or unbounded, but the regularity result in [8, Theorem 6.1] is stated for a bounded domain. Moreover when $\gamma_n\mathbf{g} = 0$, $P\mathbf{u} = \mathbf{u}$, and due to Theorem 2.7, it is sufficient to take $\gamma_\tau\mathbf{g} \in \mathbf{V}^{s-1/2,s/2-1/4}(\Sigma)$ to have $\mathbf{u} \in \mathbf{V}^{s,s/2}(Q)$. This means that the result stated in Theorem 3.1 is not optimal with respect to the tangential regularity needed for $\mathbf{g}$. Now let us consider the case when $\mathbf{g} = \gamma_n\mathbf{g}$. First of all, observe that only the case where $s \geqslant 1$ is studied in [8]. For $s = 2$ the result stated in Lemma 3.2 – when $\gamma_\tau\mathbf{g} = 0$ – is exactly the one corresponding to $s = 2$ in [8, Theorem 6.1]. For $s = 1$, we obtain $\mathbf{u} \in \mathbf{V}^{1,1/2}(Q)$ and $P\mathbf{u} \in W(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega))$ if $\mathbf{g} = \gamma_n\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^{1/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))$. The corresponding result stated in [8, Theorem 6.1] for $s = 1$ is different. It is assumed there that $\mathbf{g} = \gamma_n\mathbf{g}$ belongs to $L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^{3/4}(0,T;\mathbf{V}^{-1}(\Gamma)) \subset L^2(0,T;\mathbf{V}^{1/2}(\Gamma)) \cap H^{1/2}(0,T;\mathbf{V}^{-1/2}(\Gamma))$. Thus the assumption in [8, Theorem 6.1] for $s = 1$ is stronger than ours. But the solution is obtained in $\mathcal{V}^{(1)}(Q)$ which is strictly smaller than $W(0,T;\mathbf{V}^1(\Omega),\mathbf{V}^{-1}(\Omega))$. Therefore the two results cannot be completely compared. Observe that in Theorem 3.1 we also state results in the case when $s \in [0,1[$ which is not considered in [8, Theorem 6.1].

4. Oseen equation

4.1. Linearized Navier–Stokes equations around a stationary state

In this section, we want to extend the results of Section 2 to the equation

$$\begin{aligned} &\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + (\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{z} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } Q,\\ &\mathbf{u} = \mathbf{g} \quad \text{on } \Sigma, \qquad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{4.1}$$

where $\mathbf{z}$ belongs to $\mathbf{V}^1(\Omega)$.

To study Eq. (4.1) we introduce the unbounded operators $A_{\mathbf{z}}$ and $A^*_{\mathbf{z}}$ in $\mathbf{V}^0_n(\Omega)$ defined by

$$\begin{aligned} &D(A_{\mathbf{z}}) = \big\{\mathbf{u} \in \mathbf{V}^1_0(\Omega) \mid P\Delta\mathbf{u} - P\big((\mathbf{z}\cdot\nabla)\mathbf{u}\big) - P\big((\mathbf{u}\cdot\nabla)\mathbf{z}\big) \in \mathbf{V}^0_n(\Omega)\big\},\\ &D(A^*_{\mathbf{z}}) = \big\{\mathbf{u} \in \mathbf{V}^1_0(\Omega) \mid P\Delta\mathbf{u} + P\big((\mathbf{z}\cdot\nabla)\mathbf{u}\big) - P\big((\nabla\mathbf{z})^{\mathrm{T}}\mathbf{u}\big) \in \mathbf{V}^0_n(\Omega)\big\},\\ &A_{\mathbf{z}}\mathbf{u} = P\Delta\mathbf{u} - P\big((\mathbf{z}\cdot\nabla)\mathbf{u}\big) - P\big((\mathbf{u}\cdot\nabla)\mathbf{z}\big) \quad \text{and} \quad A^*_{\mathbf{z}}\mathbf{u} = P\Delta\mathbf{u} + P\big((\mathbf{z}\cdot\nabla)\mathbf{u}\big) - P\big((\nabla\mathbf{z})^{\mathrm{T}}\mathbf{u}\big). \end{aligned}$$

Throughout this section we assume that $\lambda_0 > 0$ is such that

$$\begin{aligned} &\int_\Omega \big(\lambda_0|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2 + \big((\mathbf{z}\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{u} + \big((\mathbf{u}\cdot\nabla)\mathbf{z}\big)\cdot\mathbf{u}\big)\,\mathrm{d}x \geqslant \frac{1}{2}\int_\Omega \big(|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2\big)\,\mathrm{d}x \quad \text{and}\\ &\int_\Omega \big(\lambda_0|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2 - \big((\mathbf{z}\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{u} + \big((\nabla\mathbf{z})^{\mathrm{T}}\mathbf{u}\big)\cdot\mathbf{u}\big)\,\mathrm{d}x \geqslant \frac{1}{2}\int_\Omega \big(|\mathbf{u}|^2 + |\nabla\mathbf{u}|^2\big)\,\mathrm{d}x \end{aligned} \tag{4.2}$$

for all $\mathbf{u} \in \mathbf{V}^1_0(\Omega)$.

Lemma 4.1. *The operator $(A_{\mathbf{z}} - \lambda_0 I)$ (respectively $(A^*_{\mathbf{z}} - \lambda_0 I)$) with domain $D(A_{\mathbf{z}} - \lambda_0 I) = D(A_{\mathbf{z}})$ (respectively $D(A^*_{\mathbf{z}} - \lambda_0 I) = D(A^*_{\mathbf{z}})$) is the infinitesimal generator of a bounded analytic semigroup on $\mathbf{V}^0_n(\Omega)$. Moreover, for all $0 \leqslant \alpha \leqslant 1$, we have*

$$D\big((\lambda_0 I - A_{\mathbf{z}})^\alpha\big) = D\big((\lambda_0 I - A^*_{\mathbf{z}})^\alpha\big) = D\big((\lambda_0 I - A)^\alpha\big) = D\big((-A)^\alpha\big).$$

Proof. The first part of the theorem is a direct consequence of (4.2) (see e.g. [4, Chapter 1, Theorem 1.12]). The characterization of the domains of $(\lambda_0 I - A_{\mathbf{z}})^\alpha$ and $(\lambda_0 I - A^*_{\mathbf{z}})^\alpha$ follows from [17]. □

Let us denote by $\tilde{A}_{\mathbf{z}}$ the extension of $A_{\mathbf{z}}$ to $(D(A^*_{\mathbf{z}}))' = (D(A^*))'$. Following what is done for the Stokes equations, we introduce the Dirichlet operators associated with $\lambda_0 I - A_{\mathbf{z}}$. For all $\mathbf{g} \in \mathbf{V}^0(\Gamma)$, we denote by $D_{\mathbf{z}}\mathbf{g} = \mathbf{w}$, and $D_{p,\mathbf{z}}\mathbf{g} = \pi$ the solution to the equation

$$\lambda_0\mathbf{w} - \Delta\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} + \nabla\pi = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.$$

Following what has been done for the Stokes equations, when $\mathbf{g} \in C^1_c(]0, T[; \mathbf{V}^{3/2}(\Gamma))$, we look for the solution $(\mathbf{u}, p)$ of Eq. (4.1) in the form $(\mathbf{u}, p) = (\mathbf{w}, \pi) + (\mathbf{y}, q)$, where $(\mathbf{w}(t), \pi(t)) = (D_{\mathbf{z}}\mathbf{g}(t), D_{p,\mathbf{z}}\mathbf{g}(t))$, and $(\mathbf{y}, q)$ is the solution of

$$\begin{aligned} &\frac{\partial \mathbf{y}}{\partial t} - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{z} + \nabla q = -\frac{\partial \mathbf{w}}{\partial t} + \lambda_0\mathbf{w}, \quad \operatorname{div}\mathbf{y} = 0 \quad \text{in } Q,\\ &\mathbf{y} = 0 \quad \text{on } \Sigma, \qquad \mathbf{y}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \end{aligned}$$

We have

$$\begin{aligned} \mathbf{y}(t) &= \mathrm{e}^{tA_{\mathbf{z}}}P\mathbf{u}_0 - \int_0^t \mathrm{e}^{(t-s)A_{\mathbf{z}}}P\mathbf{w}'(s)\,\mathrm{d}s + \lambda_0\int_0^t \mathrm{e}^{(t-s)A_{\mathbf{z}}}P\mathbf{w}(s)\,\mathrm{d}s\\ &= \mathrm{e}^{tA_{\mathbf{z}}}P\mathbf{u}_0 + (\lambda_0 I - A_{\mathbf{z}})\int_0^t \mathrm{e}^{(t-s)A_{\mathbf{z}}}P\mathbf{w}(s)\,\mathrm{d}s - P\mathbf{w}(t). \end{aligned}$$

Thus $P\mathbf{u}$ is defined by

$$P\mathbf{u}(t) = \mathrm{e}^{tA_\mathbf{z}} P\mathbf{u}_0 + \int_0^t (\lambda_0 I - A_\mathbf{z})\mathrm{e}^{(t-s)A_\mathbf{z}} P\mathbf{w}(s)\,\mathrm{d}s.$$

This leads to the following definition.

Definition 4.1. A function $\mathbf{u} \in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to Eq. (4.1) if

$$P\mathbf{u} \text{ is a weak solution of evolution equation}$$
$$P\mathbf{u}' = \tilde{A}_\mathbf{z} P\mathbf{u} + (\lambda_0 I - \tilde{A}_\mathbf{z}) P D_\mathbf{z}\mathbf{g}, \qquad P\mathbf{u}(0) = P\mathbf{u}_0, \tag{4.3}$$

and

$$(I-P)\mathbf{u}(\cdot) = (I-P)D_\mathbf{z}\gamma_n\mathbf{g}(\cdot) \quad \text{in } L^2\big(0,T;\mathbf{V}^0(\Omega)\big).$$

As in Section 2, we can establish the following theorem.

Theorem 4.1.

(i) *We assume that* $\mathbf{z} \in \mathbf{V}^1(\Omega)$. *For all* $P\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$ *and all* $\mathbf{g} \in L^2(0,T;\mathbf{V}^0(\Gamma))$ *Eq.* (4.1)*, admits a unique weak solution in* $L^2(0,T;\mathbf{V}^0(\Omega))$ *in the sense of Definition* 4.1*. This solution obeys*

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}_n^{1/2-\varepsilon}(\Omega))} + \|P\mathbf{u}\|_{H^{1/4-\varepsilon/2}(0,T;\mathbf{V}_n^0(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{1/2}(\Omega))}$$
$$\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^0(\Omega)} + \|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big) \quad \text{for all } \varepsilon > 0.$$

(ii) *If* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$ *with* $0 \leqslant s \leqslant 2$, *and if* Ω *is of class* C^3 *when* $3/2 < s \leqslant 2$, *then*

$$\big\|(I-P)\mathbf{u}\big\|_{L^2(0,T;\mathbf{V}^{s+1/2}(\Omega))} + \big\|(I-P)\mathbf{u}\big\|_{H^{s/2}(0,T;\mathbf{V}^{1/2}(\Omega))} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}. \tag{4.4}$$

(iii) *If* Ω *satisfies* (2.23)*,* $\mathbf{z} \in \mathbf{V}^{3/2\vee(s-1/2)}(\Omega)$*,* $\mathbf{g} \in \mathbf{V}^{s,s/2}(\Sigma)$*,* $P\mathbf{u}_0 \in \mathbf{V}_n^{0\vee(s-1/2)}(\Omega)$*, with* $s \in [0,1[\,\cup\,]1,3[$*, and if* $P\mathbf{u}_0$ *and* $\mathbf{g}(0)$ *satisfy the compatibility condition* (2.13) *when* $1 < s < 3$*, then*

$$\|P\mathbf{u}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(Q)} \leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Sigma)}\big). \tag{4.5}$$

Proof. To prove the estimate stated in (i), thanks to Lemma 4.1, it is sufficient to replace A, e^{tA}, and D by $A_\mathbf{z}$, $\mathrm{e}^{tA_\mathbf{z}}$, and $D_\mathbf{z}$ in the proof of Theorem 2.3. To prove (ii), we notice that $(I-P)(D_\mathbf{z}\gamma_n\mathbf{g} - D\gamma_n\mathbf{g}) = 0$. Indeed if $\mathbf{w} = D_\mathbf{z}\gamma_n\mathbf{g}$ and $\mathbf{v} = D\gamma_n\mathbf{g}$, then $\mathbf{w} - \mathbf{v} \in \mathbf{V}^0_n(\Omega)$ and $(I-P)(\mathbf{w}-\mathbf{v}) = 0$. We postpone the end of proof at the end of the section. □

Now we would like to show that $\mathbf{u} \in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to Eq. (4.1) if and only if

$$P\mathbf{u} \text{ is a weak solution of the evolution equation}$$
$$P\mathbf{u}' = \tilde{A}P\mathbf{u} + (-\tilde{A})PD\mathbf{g} + P\big(\operatorname{div}(\mathbf{z}\otimes\mathbf{u})\big) + P\big(\operatorname{div}(\mathbf{u}\otimes\mathbf{z})\big), \qquad P\mathbf{u}(0) = P\mathbf{u}_0, \tag{4.6}$$

and

$$(I-P)\mathbf{u}(\cdot) = (I-P)D\gamma_n\mathbf{g}(\cdot) \quad \text{in } L^2\big(0,T;\mathbf{V}^0(\Omega)\big). \tag{4.7}$$

We have already noticed that $(I-P)(D_\mathbf{z}\gamma_n\mathbf{g} - D\gamma_n\mathbf{g}) = 0$. Moreover if $\mathbf{u} \in L^2(0,T;\mathbf{V}^0(\Omega))$, then $\mathbf{z}\otimes\mathbf{u}$ and $\mathbf{u}\otimes\mathbf{z}$ belong to $(L^2(0,T;(\mathbf{L}^{3/2}(\Omega))^N)$. Thus $P(\operatorname{div}(\mathbf{z}\otimes\mathbf{u})) + P(\operatorname{div}(\mathbf{u}\otimes\mathbf{z}))$ is well defined in $L^2(0,T;(D(A^*))')$ by

$$\big\langle P\big(\operatorname{div}(\mathbf{z}\otimes\mathbf{u})\big) + P\big(\operatorname{div}(\mathbf{u}\otimes\mathbf{z})\big), \boldsymbol{\Phi}\big\rangle_{L^2(0,T;(D(A^*))'),L^2(0,T;D(A^*))}$$
$$= -\int_Q \big((\mathbf{u}\otimes\mathbf{z}) + (\mathbf{z}\otimes\mathbf{u})\big)\cdot\nabla\boldsymbol{\Phi}\,\mathrm{d}x\,\mathrm{d}t \quad \text{for all } \boldsymbol{\Phi} \in L^2\big(0,T;D(A^*)\big).$$

Therefore weak solutions to problem (4.6) may be defined as weak solutions in $L^2(0,T;(D(A^*))')$. To prove that weak solutions to Eqs. (4.3) and (4.6) are identical, we first study Eq. (4.6).

Proposition 4.1. *For all $P\mathbf{u}_0\in\mathbf{V}^0_n(\Omega)$ and all $\mathbf{g}\in L^2(0,T;\mathbf{V}^0(\Gamma))$ the problem* (4.6)*, admits a unique weak solution $P\mathbf{u}$ in $L^2(0,T;\mathbf{V}^0_n(\Omega))$ and it satisfies*

$$\|P\mathbf{u}\|_{L^2(0,T;\mathbf{V}^0_n(\Omega))}\leqslant C\big(\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)}+\|\mathbf{g}\|_{L^2(0,T;\mathbf{V}^0(\Gamma))}\big).$$

Proof. For all $\mathbf{v}\in L^2(0,T;\mathbf{V}^0_n(\Omega))$, $\mathbf{z}\otimes\mathbf{v}$ and $\mathbf{v}\otimes\mathbf{z}$ belong to $L^2(0,T;(\mathbf{L}^{3/2}(\Omega))^N)$ because $\mathbf{z}\in\mathbf{V}^1(\Omega)$. Thus, if $\mathbf{v}\in L^2(0,T;\mathbf{V}^0_n(\Omega))$, the evolution equation

$$\begin{aligned}&\mathbf{y}'=\tilde{A}\mathbf{y}+(-\tilde{A})PD\mathbf{g}+P\big(\operatorname{div}\big(\mathbf{z}\otimes\big((I-P)D\gamma_n\mathbf{g}\big)\big)\big)+P\big(\operatorname{div}\big(\big((I-P)D\gamma_n\mathbf{g}\big)\otimes\mathbf{z}\big)\big)\\&\qquad+P\big(\operatorname{div}(\mathbf{z}\otimes\mathbf{v})\big)+P\big(\operatorname{div}(\mathbf{v}\otimes\mathbf{z})\big),\\&P\mathbf{u}(0)=P\mathbf{u}_0,\end{aligned}$$

admits a unique solution $\mathbf{y}_\mathbf{v}$ in $L^2(0,T;\mathbf{V}^0_n(\Omega))$. More precisely we can show that

$$\|\mathbf{y}_\mathbf{v}\|_{L^2(0,T^*;\mathbf{V}^0_n(\Omega))}\leqslant C\big(\|\mathbf{z}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{z}\|_{L^\sigma(0,T^*;(\mathbf{L}^{3/2}(\Omega))^N)}+\|\mathbf{g}\|_{L^2(0,T^*;\mathbf{V}^0(\Gamma))}+\|P\mathbf{u}_0\|_{\mathbf{V}^0_n(\Omega)}\big),$$

for some $1<\sigma<2$, and for all $0<T^*\leqslant T$, where C is independent of T^*. Therefore, as in [21, Proposition 2.7] we can show that for $T^*>0$ small enough, the mapping

$$\mathbf{v}\longmapsto\mathbf{y}_\mathbf{v}$$

is a contraction in $L^2(0,T^*;\mathbf{V}^0_n(\Omega))$. Thus we have proved the existence of a unique local solution to Eq. (4.6). As in [21] we can iterate this process to prove the existence of a unique global in time solution in $L^2(0,T;\mathbf{V}^0(\Omega))$ to Eq. (4.6). The estimate of $P\mathbf{u}$ in $L^2(0,T;\mathbf{V}^0_n(\Omega))$ can be derived as in [21]. The estimate for $(I-P)\mathbf{u}=(I-P)D\gamma_n\mathbf{g}$ follows from the continuity of the operator $(I-P)D\gamma_n$. The proof is complete. □

Theorem 4.2. *A function $\mathbf{u}\in L^2(0,T;\mathbf{V}^0(\Omega))$ is a weak solution to Eq.* (4.1)*, in the sense of Definition* 4.1*, if and only if $\mathbf{u}$ is the weak solution to problem* (4.6)–(4.7)*.*

Proof. This equivalence can be easily shown in the case when $\mathbf{u}_0\in\mathbf{V}^1_0(\Omega)$ and $\mathbf{g}\in C^1_c(0,T;\mathbf{V}^{3/2}(\Gamma))$. Due to the estimates in Proposition 4.1 and in Theorem 4.1(i), the equivalence follows from a density argument.

End of proof of Theorem 4.1. To prove the estimate stated in (iii), we write Eq. (4.1) in the form

$$\begin{aligned}&P\mathbf{u}'=\tilde{A}P\mathbf{u}+(-\tilde{A})PD\mathbf{g}+P\big(\operatorname{div}\big(\mathbf{z}\otimes\big((I-P)D\gamma_n\mathbf{g}\big)\big)\big)+P\big(\operatorname{div}\big(\big((I-P)D\gamma_n\mathbf{g}\big)\otimes\mathbf{z}\big)\big)\\&\qquad+P\big(\operatorname{div}\big(\mathbf{z}\otimes P\mathbf{u}\big)\big)+P\big(\operatorname{div}(P\mathbf{u}\otimes\mathbf{z})\big),\\&P\mathbf{u}(0)=P\mathbf{u}_0,\qquad(I-P)\mathbf{u}=(I-P)D\gamma_n\mathbf{g},\end{aligned}$$

and we are going to use a fixed point method as in the proof of Proposition 4.1.

Step 1. Let us prove (4.5) for $0\leqslant s<1$. For $\mathbf{v}\in L^2(0,T;\mathbf{V}^{s+1/2}_n(\Omega))\cap H^{s/2+1/4}(0,T;\mathbf{V}^0_n(\Omega))$, we denote by $P\mathbf{y}_\mathbf{v}$ the solution to the equation

$$\begin{aligned}&P\mathbf{y}'=\tilde{A}P\mathbf{y}+(-\tilde{A})PD\mathbf{g}+P\big(\operatorname{div}\big(\mathbf{z}\otimes\big((I-P)D\gamma_n\mathbf{g}\big)\big)\big)+P\big(\operatorname{div}\big(\big((I-P)D\gamma_n\mathbf{g}\big)\otimes\mathbf{z}\big)\big)\\&\qquad+P\big(\operatorname{div}(\mathbf{z}\otimes\mathbf{v})\big)+P\big(\operatorname{div}(\mathbf{v}\otimes\mathbf{z})\big),\qquad P\mathbf{u}(0)=P\mathbf{u}_0.\end{aligned}$$

We have to prove that the mapping $\mathbf{v}\mapsto P\mathbf{y}_\mathbf{v}$ is a contraction in $L^2(0,T^*;\mathbf{V}^{s+1/2}_n(\Omega))\cap H^{s/2+1/4}(0,T^*;\mathbf{V}^0_n(\Omega))$ for $T^*>0$ small enough. For that we have to verify that if $\mathbf{v}$ belongs to $L^2(0,T^*;\mathbf{V}^{s+1/2}_n(\Omega))\cap H^{s/2+1/4}(0,T^*;\mathbf{V}^0_n(\Omega))$, then the two terms $\mathbf{v}\otimes\mathbf{z}$ and $\mathbf{z}\otimes\mathbf{v}$ belong to $H^\varepsilon(0,T^*;\mathbf{H}^{(s-1/2)\vee 0}(\Omega))\cap H^{(s/2-1/4+\varepsilon)\vee\varepsilon}(0,T^*;\mathbf{L}^2(\Omega))$ for some $\varepsilon>0$. This can be easily verified since $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$. Thus from classical results for the Stokes equations with homogeneous boundary conditions, and from Theorem 2.7 (to deal with the nonhomogeneous boundary condition), we obtain the estimate

$$\|P\mathbf{y_v}\|_{\mathbf{V}^{s+1/2,s/2+1/4}(\Omega\times(0,T^*))} \leqslant C\big(\|\mathbf{z}\otimes\mathbf{v}+\mathbf{v}\otimes\mathbf{z}\|_{H^\varepsilon(0,T^*;\mathbf{H}^{(s-1/2)\vee 0}(\Omega))\cap H^{(s/2-1/4+\varepsilon)\vee\varepsilon}(0,T^*;\mathbf{L}^2(\Omega))} + \|\mathbf{g}\|_{\mathbf{V}^{s,s/2}(\Gamma\times(0,T^*))} + \|P\mathbf{u}_0\|_{\mathbf{V}_n^{0\vee(s-1/2)}(\Omega)}\big),$$

where $C>0$ is independent of T^*. Thus we can show that the mapping $\mathbf{v}\mapsto P\mathbf{y_v}$ is a contraction in

$$L^2\big(0,T^*;\mathbf{V}_n^{s+1/2}(\Omega)\big)\cap H^{s/2+1/4}\big(0,T^*;\mathbf{V}_n^0(\Omega)\big)$$

for $T^*>0$ small enough. Next the estimate (4.5) can be obtained as in the proof of Proposition 4.1.

Step 2. Let us prove (4.5) for $1<s<3$. Since $\mathbf{z}$ belongs to $\mathbf{V}^{3/2\vee(s-1/2)}(\Omega)$, $\operatorname{div}(((I-P)D\gamma_n\mathbf{g})\otimes\mathbf{z})$ and $\operatorname{div}(\mathbf{z}\otimes((I-P)D\gamma_n\mathbf{g}))$ belong to $L^2(0,T;\mathbf{H}^{s-1/2}(\Omega))\cap H^{s/2-1/4}(0,T;\mathbf{L}^2(\Omega))$. The solution of the Stokes equation with a source term in $L^2(0,T;\mathbf{H}^{s-1/2}(\Omega))\cap H^{s/2-1/4}(0,T;\mathbf{L}^2(\Omega))$, belongs to $L^2(0,T;\mathbf{V}_n^{s+3/2}(\Omega))\cap H^{s/2+3/4}(0,T;\mathbf{V}_n^0(\Omega))\subset L^2(0,T;\mathbf{V}_n^{s+1/2}(\Omega))\cap H^{s/2+1/4}(0,T;\mathbf{V}_n^0(\Omega))$. Therefore these nonhomogeneous terms do not cause any difficulty. Similarly if $\mathbf{v}$ belongs to $L^2(0,T^*;\mathbf{V}_n^{s+1/2}(\Omega))\cap H^{s/2+1/4}(0,T^*;\mathbf{V}_n^0(\Omega))$, we can easily check that $P(\operatorname{div}(\mathbf{z}\otimes\mathbf{v}))+P(\operatorname{div}(\mathbf{v}\otimes\mathbf{z}))$ belongs to $H^\varepsilon(0,T^*;\mathbf{V}^{s-1/2}(\Omega))\cap H^{s/2-1/4+\varepsilon}(0,T^*;\mathbf{V}^0(\Omega))$ for some $\varepsilon>0$. Thus as in Step 1, we can conclude with a fixed point method. □

4.2. Linearized Navier–Stokes equations around an instationary state

In this section, we want to study the linearized Navier–Stokes equations around an instationary state $\mathbf{z}$, with homogeneous boundary conditions:

$$\begin{aligned}&\frac{\partial\mathbf{u}}{\partial t}-\Delta\mathbf{u}+(\mathbf{z}\cdot\nabla)\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{z}+\nabla p=\mathbf{f},\quad \operatorname{div}\mathbf{u}=0\quad\text{in } Q,\\&\mathbf{u}=0\quad\text{on }\Sigma,\qquad \mathbf{u}(0)=\mathbf{u}_0\quad\text{in }\Omega,\end{aligned}\tag{4.8}$$

in the case where $\mathbf{z}$ belongs to $L^2(0,T;\mathbf{V}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^4(\Omega))$, and $\mathbf{f}$ belongs to $L^2(0,T;\mathbf{H}^{-1}(\Omega))$.

For almost all $t\in(0,T)$, we define the operators $A_\mathbf{z}(t)\in\mathcal{L}(\mathbf{V}_0^1(\Omega),\mathbf{V}^{-1}(\Omega))$ and $A_\mathbf{z}^*(t)\in\mathcal{L}(\mathbf{V}_0^1(\Omega),\mathbf{V}^{-1}(\Omega))$ by

$$\big\langle A_\mathbf{z}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}_0^1(\Omega)}=\int_\Omega\big(-\nabla\mathbf{u}\cdot\nabla\mathbf{v}-\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{v}+\big((\mathbf{u}\cdot\nabla)\mathbf{v}\big)\cdot\mathbf{z}(t)\big)\,\mathrm{d}x,$$

$$\big\langle A_\mathbf{z}^*(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}_0^1(\Omega)}=\int_\Omega\big(-\nabla\mathbf{u}\cdot\nabla\mathbf{v}+\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{v}+\big((\mathbf{v}\cdot\nabla)\mathbf{u}\big)\cdot\mathbf{z}(t)\big)\,\mathrm{d}x,$$

for all $\mathbf{u}\in\mathbf{V}_0^1(\Omega)$ and all $\mathbf{v}\in\mathbf{V}_0^1(\Omega)$.

Let us still denote by P the continuous extension to $\mathbf{H}^{-1}(\Omega)$ of the Helmholtz projector, that is the bounded operator from $\mathbf{H}^{-1}(\Omega)$ onto $\mathbf{V}^{-1}(\Omega)$ defined by $\langle P\mathbf{f},\boldsymbol{\Phi}\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}_0^1(\Omega)}=\langle\mathbf{f},\boldsymbol{\Phi}\rangle_{\mathbf{H}^{-1}(\Omega),\mathbf{H}_0^1(\Omega)}$ for all $\mathbf{f}\in\mathbf{H}^{-1}(\Omega)$, and all $\boldsymbol{\Phi}\in\mathbf{V}_0^1(\Omega)$ (see e.g. [27, page xxiii] or [3, Appendix A.1]). Eq. (4.8) can be rewritten in the form

$$\mathbf{u}'=A_\mathbf{z}(t)\mathbf{u}+P\mathbf{f},\qquad \mathbf{u}(0)=\mathbf{u}_0.$$

Lemma 4.2. *Assume that* $\mathbf{z}$ *belongs to* $L^2(0,T;\mathbf{V}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^4(\Omega))$. *There exist* $\lambda_0>0$ *and* $M>0$ *such that*

$$\big|\big\langle A_\mathbf{z}(t)\mathbf{u},\mathbf{v}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}_0^1(\Omega)}\big|\leqslant M\|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)}\|\mathbf{v}\|_{\mathbf{V}_0^1(\Omega)}$$

and

$$\big\langle\lambda_0\mathbf{u}-A_\mathbf{z}(t)\mathbf{u},\mathbf{u}\big\rangle_{\mathbf{V}^{-1}(\Omega),\mathbf{V}_0^1(\Omega)}\geqslant\frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{V}_0^1(\Omega)},$$

for all $\mathbf{u}\in\mathbf{V}_0^1(\Omega)$, *all* $\mathbf{v}\in\mathbf{V}_0^1(\Omega)$ *and almost all* $t\in(0,T)$.

Proof. For all $\mathbf{u}\in\mathbf{V}_0^1(\Omega)$, almost all $t\in(0,T)$, and $\lambda_0>0$, we have:

$$\begin{aligned}\langle \lambda_0 \mathbf{u} - A_{\mathbf{z}}(t)\mathbf{u}, \mathbf{u}\rangle_{\mathbf{V}^{-1}(\Omega), \mathbf{V}_0^1(\Omega)} &= \int_\Omega \big(\lambda_0 |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + \big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big)\cdot \mathbf{u} - \big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)\cdot \mathbf{z}(t)\big)\,\mathrm{d}x \\ &= \int_\Omega \big(\lambda_0 |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2 - \big((\mathbf{u}\cdot\nabla)\mathbf{u}\big)\cdot \mathbf{z}(t)\big)\,\mathrm{d}x \\ &\geqslant \int_\Omega \big(\lambda_0 |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2\big)\,\mathrm{d}x - \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)} \|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{z}\|_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \\ &\geqslant \int_\Omega \big(\lambda_0 |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2\big)\,\mathrm{d}x - C\|\mathbf{u}\|^{1/4}_{\mathbf{V}_n^0(\Omega)} \|\mathbf{u}\|^{7/4}_{\mathbf{V}_0^1(\Omega)} \|\mathbf{z}\|_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \\ &\geqslant \int_\Omega \big(\lambda_0 |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2\big)\,\mathrm{d}x - \frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{V}_0^1(\Omega)} - \widetilde{C}\|\mathbf{z}\|^8_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \|\mathbf{u}\|^2_{\mathbf{V}_n^0(\Omega)},\end{aligned}$$

where $\widetilde{C} = \frac{7^7 C^8}{8\times 4^7}$. It is sufficient to choose $\lambda_0 = 1 + \widetilde{C}\|\mathbf{z}\|^8_{L^\infty(0,T;\mathbf{L}^4(\Omega))}$.

For all $\mathbf{u} \in \mathbf{V}_0^1(\Omega)$, all $\mathbf{v} \in \mathbf{V}_0^1(\Omega)$, and almost all $t \in (0,T)$, we have:

$$\begin{aligned}&\big|\langle A_{\mathbf{z}}(t)\mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}^{-1}(\Omega), \mathbf{V}_0^1(\Omega)}\big| \\ &\quad\leqslant \|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{v}\|_{\mathbf{V}_0^1(\Omega)} + \|\mathbf{z}\|_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} + \|\mathbf{u}\|_{\mathbf{L}^4(\Omega)} \|\mathbf{v}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{z}\|_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \\ &\quad\leqslant \|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{v}\|_{\mathbf{V}_0^1(\Omega)} + C\|\mathbf{z}\|_{L^\infty(0,T;\mathbf{L}^4(\Omega))} \|\mathbf{u}\|_{\mathbf{V}_0^1(\Omega)} \|\mathbf{v}\|_{\mathbf{V}_0^1(\Omega)}.\end{aligned}$$

The proof is complete. □

Theorem 4.3. *Assume that* $\mathbf{z}$ *belongs to* $L^2(0,T;\mathbf{V}^1(\Omega)) \cap L^\infty(0,T;\mathbf{L}^4(\Omega))$. *For all* $\mathbf{u}_0 \in \mathbf{V}_n^0(\Omega)$ *and all* $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$, *Eq.* (4.8) *admits a unique weak solution* $\mathbf{u}$ *in* $W(0,T;\mathbf{V}_0^1(\Omega),\mathbf{V}^{-1}(\Omega))$.

Proof. The theorem is a direct consequence of Lemma 4.2 and of a theorem by J.-L. Lions (see e.g. [6, Chapter 18, Section 3.2, Theorems 1 and 2]). □

With $A_{\mathbf{z}}(t)$ and $A^*_{\mathbf{z}}(t)$, we can associate two unbounded operators in $\mathbf{V}_n^0(\Omega)$, still denoted by $A_{\mathbf{z}}(t)$ and $A^*_{\mathbf{z}}(t)$ for simplicity, and defined by

$$\begin{aligned}&D\big(A_{\mathbf{z}}(t)\big) = \big\{\mathbf{u} \in \mathbf{V}_0^1(\Omega) \mid P\Delta\mathbf{u} - P\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big) - P\big((\mathbf{u}\cdot\nabla)\mathbf{z}(t)\big) \in \mathbf{V}_n^0(\Omega)\big\}, \\ &D\big(A^*_{\mathbf{z}}(t)\big) = \big\{\mathbf{u} \in \mathbf{V}_0^1(\Omega) \mid P\Delta\mathbf{u} + P\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big) - P\big((\nabla\mathbf{z}(t))^{\mathrm{T}}\mathbf{u}\big) \in \mathbf{V}_n^0(\Omega)\big\}, \\ &A_{\mathbf{z}}(t)\mathbf{u} = P\Delta\mathbf{u} - P\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big) - P\big((\mathbf{u}\cdot\nabla)\mathbf{z}(t)\big),\end{aligned}$$

and

$$A^*_{\mathbf{z}}(t)\mathbf{u} = P\Delta\mathbf{u} + P\big((\mathbf{z}(t)\cdot\nabla)\mathbf{u}\big) - P\big((\nabla\mathbf{z}(t))^{\mathrm{T}}\mathbf{u}\big).$$

5. The Navier–Stokes equation

In this section, we want to study the equation

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } Q, \\ &\mathbf{u} = \mathbf{g} \quad \text{on } \Sigma, \qquad P\mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega,\end{aligned} \tag{5.1}$$

where $\mathbf{g}$ belongs to $\mathbf{V}^{3/4,3/4}(\Sigma)$, and $\mathbf{u}_0 \in \mathbf{V}_n^0(\Omega)$.

One way to solve Eq. (5.1) is to look for a solution $\mathbf{u}$ of the form $\mathbf{u} = \mathbf{w} + \mathbf{v}$, where $\mathbf{w}$ is the solution of

$$-\Delta\mathbf{w}(t) + \nabla\pi(t) = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w}(t) = 0 \quad \text{in } \Omega, \qquad \mathbf{w}(t) = \mathbf{g}(t) \quad \text{on } \Gamma, \tag{5.2}$$

for all $t \in [0, T]$, and $\mathbf{v}$ is a solution to

$$\begin{aligned}&\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{w} + (\mathbf{v}\cdot\nabla)\mathbf{v} + (\mathbf{w}\cdot\nabla)\mathbf{w} + \nabla\rho = -\frac{\partial \mathbf{w}}{\partial t},\\ &\operatorname{div}\mathbf{v} = 0 \quad \text{in } Q, \quad \mathbf{v} = 0 \quad \text{on } \Sigma, \qquad \mathbf{v}(0) = \mathbf{u}_0 - PD\mathbf{g}(0) \quad \text{in } \Omega.\end{aligned} \tag{5.3}$$

If $\mathbf{w}$ is regular enough, we are going to see below that Eq. (5.3) can be solved by a Galerkin method as in the case of the Navier–Stokes equations (see e.g. [25, Chapter 3, Theorem 3.1]). For example if $\mathbf{g}$ belongs to $\mathbf{V}^{1,1}(\Sigma)$, then $\mathbf{w}$ belongs to $L^2(0, T; \mathbf{V}^{3/2}(\Omega)) \cap H^1(0, T; \mathbf{V}^{1/2}(\Omega))$. In particular $\mathbf{w}$ belongs to $L^2(0, T; \mathbf{V}^{3/2}(\Omega)) \cap C([0, T]; \mathbf{V}^1(\Omega))$, which is enough to prove the existence of solution to Eq. (5.3). The assumption $\mathbf{g} \in \mathbf{V}^{1,1}(\Sigma)$ is the one stated in [8,9] to prove the existence of a unique solution to Eq. (5.1) in the case of small data. The extension procedure in [9, Theorem 3.8]

$$\mathbf{g} \longmapsto \mathbf{w}$$

is different from the one corresponding to Eq. (5.2), but it leads to a similar regularity for $\mathbf{w}$.

Here we assume that $\mathbf{g}$ belongs to $\mathbf{V}^{3/4,3/4}(\Sigma)$. In that case the solution $\mathbf{w}$ to Eq. (5.2) belongs to

$$H^{3/4}\big(0, T; \mathbf{V}^{1/2}(\Omega)\big) \cap L^2\big(0, T; \mathbf{V}^{5/4}(\Omega)\big).$$

We think that in that case, because of the term $\mathbf{w}'$ in (5.3), neither the extension procedure considered in [9] nor the one corresponding to $\mathbf{w}$ determined by (5.2), may lead to a global existence result for Eq. (5.3). To overcome this difficulty we consider the extension determined by

$$\mathbf{g} \longmapsto \mathbf{z},$$

where $\mathbf{z}$ is the solution to equation

$$\begin{aligned}&P\mathbf{z}' = \tilde{A}P\mathbf{z} + (-\tilde{A})PD\mathbf{g}, \qquad P\mathbf{z}(0) = PD\mathbf{g}(0),\\ &(I-P)\mathbf{z}(\cdot) = (I-P)D\gamma_n\mathbf{g}(\cdot) \quad \text{in } L^2\big(0, T; \mathbf{V}^0(\Omega)\big).\end{aligned} \tag{5.4}$$

We look for a solution $\mathbf{u}$ to Eq. (5.1) in the form $\mathbf{u} = \mathbf{z} + \mathbf{y}$, where $\mathbf{y}$ is the solution of

$$\begin{aligned}&\frac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{z} + (\mathbf{y}\cdot\nabla)\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{z} + \nabla q = 0, \quad \operatorname{div}\mathbf{y} = 0 \quad \text{in } Q,\\ &\mathbf{y} = 0 \quad \text{on } \Sigma, \qquad \mathbf{y}(0) = \mathbf{u}_0 - PD\mathbf{g}(0) \quad \text{in } \Omega.\end{aligned} \tag{5.5}$$

Since $\mathbf{g} \in \mathbf{V}^{3/4,3/4}(\Sigma)$ and $\mathbf{V}^{3/4,3/4}(\Sigma) \hookrightarrow C([0, T]; \mathbf{V}^{1/4}(\Gamma))$, $PD\mathbf{g}(0)$ belongs to $\mathbf{H}^{3/4}(\Omega)$, and $P(\mathbf{z}(0) - D\mathbf{g}(0)) = 0$. According to Corollary 2.1, $\mathbf{z}$ belongs to $C([0, T]; \mathbf{V}^{3/4}(\Omega)) \cap L^2(0, T; \mathbf{V}^{5/4}(\Omega))$.

With the notation introduced in Section 4, we can rewrite Eq. (5.5) in the form

$$\mathbf{y}' = A_{\mathbf{z}}(t)\mathbf{y} - P\big((\mathbf{y}\cdot\nabla)\mathbf{y}\big) - P\big((\mathbf{z}\cdot\nabla)\mathbf{z}\big), \qquad \mathbf{y}(0) = \mathbf{u}_0 - PD\mathbf{g}(0). \tag{5.6}$$

Since $\mathbf{z}$ belongs to $L^2(0, T; \mathbf{V}^{5/4}(\Omega)) \cap C([0, T]; \mathbf{V}^{3/4}(\Omega))$, it is clear that $P((\mathbf{z}\cdot\nabla)\mathbf{z})$ belongs to $L^2(0, T; \mathbf{V}^{-1}(\Omega))$. Thus Eq. (5.6) is very similar to the three-dimensional Navier–Stokes equation with a source term belonging to $L^2(0, T; \mathbf{V}^{-1}(\Omega))$. The only difference is that the Stokes operator A is now replaced by $A_{\mathbf{z}}(t)$. Let us denote by $C_w([0, T]; \mathbf{V}^0_n(\Omega))$ the subspace in $L^\infty(0, T; \mathbf{V}^0_n(\Omega))$ of functions which are continuous from $[0, T]$ into $\mathbf{V}^0_n(\Omega)$ equipped with its weak topology.

Theorem 5.1. *For all* $\mathbf{u}_0 \in \mathbf{V}^0_n(\Omega)$ *and all* $\mathbf{g} \in \mathbf{V}^{3/4,3/4}(\Sigma)$*, Eq.* (5.5) *admits at least one weak solution in* $C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$.

Proof. Let λ_0 be the exponent appearing in Lemma 4.2. A function $\mathbf{y} \in C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ is a weak solution to (5.6) if and only if $\hat{\mathbf{y}}(t) = \mathrm{e}^{-\lambda_0 t}\mathbf{y}(t)$ is a solution in $C_w([0, T]; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ to

$$\hat{\mathbf{y}}' = A_{\mathbf{z}}(t)\hat{\mathbf{y}} - \lambda_0\hat{\mathbf{y}} - P\big(\mathrm{e}^{\lambda_0 t}(\hat{\mathbf{y}}\cdot\nabla)\hat{\mathbf{y}}\big) - P\big(\mathrm{e}^{-\lambda_0 t}(\mathbf{z}\cdot\nabla)\mathbf{z}\big), \qquad \hat{\mathbf{y}}(0) = \mathbf{u}_0 - PD\mathbf{g}(0). \tag{5.7}$$

Due to Lemma 4.2, the existence in $L^\infty(0, T; \mathbf{V}^0_n(\Omega)) \cap L^2(0, T; \mathbf{V}^1_0(\Omega))$ of a function $\hat{\mathbf{y}}$ satisfying the weak formulation of Eq. (5.7) may be proved as in the case of the Navier–Stokes equations (see e.g. [25, Chapter 3, Theorem 3.1]). Moreover we have

$$\frac{1}{2}\|\hat{\mathbf{y}}(t)\|^2_{\mathbf{V}^0_n(\Omega)} + \frac{1}{2}\int_0^t \|\hat{\mathbf{y}}(\tau)\|^2_{\mathbf{V}^1_0(\Omega)}\,\mathrm{d}\tau$$
$$\leqslant \frac{1}{2}\|\mathbf{u}_0 - PD\mathbf{g}(0)\|^2_{\mathbf{V}^0_n(\Omega)} - \big\langle P\big(\mathrm{e}^{-\lambda_0(\cdot)}(\mathbf{z}\cdot\nabla)\mathbf{z}\big), \hat{\mathbf{y}}\big\rangle_{L^2(0,t;\mathbf{V}^{-1}(\Omega)),L^2(0,t;\mathbf{V}^1_0(\Omega))}.$$

Thus $\hat{\mathbf{y}}$ obeys the estimate

$$\|\hat{\mathbf{y}}(t)\|^2_{\mathbf{V}^0_n(\Omega)} + \|\hat{\mathbf{y}}\|^2_{L^2(0,t;\mathbf{V}^1_0(\Omega))} \leqslant C\big(\big\|P\big((\mathbf{z}\cdot\nabla)\mathbf{z}\big)\big\|^2_{L^2(0,t;\mathbf{V}^{-1}(\Omega))} + \|\mathbf{u}_0 - PD\mathbf{g}(0)\|^2_{\mathbf{V}^0_n(\Omega)}\big), \tag{5.8}$$

for all $0 < t < T$, where C is independent of t and T. Moreover $\operatorname{div}(\mathbf{z}\otimes\mathbf{y})$ and $\operatorname{div}(\mathbf{y}\otimes\mathbf{z})$ belong to $L^2(0,T;\mathbf{V}^{-1}(\Omega))$, and $(\mathbf{y}\cdot\nabla)\mathbf{y}$ belongs to $L^2(0,T;\mathbf{L}^1(\Omega))$. We have $\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega)\hookrightarrow C_0(\Omega)$ with a dense embedding. Hence $\mathcal{M}_b(\Omega)\hookrightarrow(\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega))'$, and $(\mathbf{y}\cdot\nabla)\mathbf{y}$ which belongs to $L^2(0,T;\mathbf{L}^1(\Omega))$ can be identified with an element in $L^2(0,T;(\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega))')$. Thus

$$\mathbf{f} = \operatorname{div}(\mathbf{z}\otimes\mathbf{y}) + \operatorname{div}(\mathbf{y}\otimes\mathbf{z}) + (\mathbf{y}\cdot\nabla)\mathbf{y} \in L^2\big(0,T;\big(\mathbf{H}^2\cap\mathbf{H}^1_0(\Omega)\big)'\big).$$

Defining $P\mathbf{f}$ in $L^2(0,T;(\mathbf{V}^2\cap\mathbf{V}^1_0(\Omega))')$ by

$$\big\langle P\mathbf{f}(t),\boldsymbol{\Phi}\big\rangle_{(\mathbf{V}^2\cap\mathbf{V}^1_0(\Omega))',\mathbf{V}^2\cap\mathbf{V}^1_0(\Omega)} = \big\langle \mathbf{f}(t),\boldsymbol{\Phi}\big\rangle_{(\mathbf{H}^2\cap\mathbf{H}^1_0(\Omega))',\mathbf{H}^2\cap\mathbf{H}^1_0(\Omega)} \quad \text{for all } \boldsymbol{\Phi}\in\mathbf{V}^2\cap\mathbf{V}^1_0(\Omega),$$

with Eq. (5.5) we can prove that $\mathbf{y}' \in L^2(0,T;(\mathbf{V}^2\cap\mathbf{V}^1_0(\Omega))')$. Since $\mathbf{y}\in L^\infty(0,T;\mathbf{V}^0_n(\Omega))$, we can claim that $\mathbf{y}\in C_w([0,T];\mathbf{V}^0_n(\Omega))$, and the proof is complete. □

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Appendix A

Lemma A.1. *For all* $(\boldsymbol{\Phi},\mathbf{h})\in\mathbf{L}^2(\Omega)\times\mathbf{V}^{3/2}(\Gamma)$ *the equation*:

$$-\Delta\mathbf{v} + \nabla\pi = \boldsymbol{\Phi} \quad \textit{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \textit{in } \Omega, \qquad \mathbf{v} = \mathbf{h} \quad \textit{on } \Gamma, \tag{A.1}$$

admits a unique solution $(\mathbf{v},\pi)$ *in* $\mathbf{V}^1(\Omega)\times(L^2(\Omega)/\mathbb{R})$. *Moreover the following estimate holds*:

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} + \|\pi\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}\big).$$

If in addition Ω *is of class* C^3 *and* $(\boldsymbol{\Phi},\mathbf{h})\in\mathbf{H}^{1/2}(\Omega)\times\mathbf{V}^2(\Gamma)$*, then*

$$\|\mathbf{v}\|_{\mathbf{V}^{5/2}(\Omega)} + \|\pi\|_{H^{3/2}(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{H}^{1/2}(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^2(\Gamma)}\big).$$

This result can be deduced from [10, Theorem 6.1, Chapter 4].

Lemma A.2. *For all* $h\in H^1(\Omega)$ *obeying* $\int_\Omega h = 0$*, the equation*:

$$-\Delta\mathbf{u} + \nabla p = 0 \quad \textit{and} \quad \operatorname{div}\mathbf{u} = h \quad \textit{in } \Omega, \qquad \mathbf{u} = 0 \quad \textit{on } \Gamma, \tag{A.2}$$

admits a unique solution $(\mathbf{u}, p)$ *in* $\mathbf{H}^1_0(\Omega)\times(L^2(\Omega)/\mathbb{R})$. *It satisfies the estimate*:

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|h\|_{H^1(\Omega)/\mathbb{R}}.$$

This result is stated in [10, Exercise 6.2, Chapter 4].

Lemma A.3. *For all* $\mathbf{f}\in\mathbf{L}^2(\Omega)$ *and all* $\mathbf{g}\in\mathbf{V}^{3/2}(\Gamma)$*, the solution* $(\mathbf{w},q)$ *to equation*:

$$-\Delta\mathbf{w}+\nabla q=\mathbf{f}\quad\text{and}\quad \operatorname{div}\mathbf{w}=0\quad\text{in }\Omega,\qquad \mathbf{w}=\mathbf{g}\quad\text{on }\Gamma,$$

obeys the estimate:

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}+\left\|\frac{\partial\mathbf{w}}{\partial\mathbf{n}}-q\mathbf{n}+c(\mathbf{w},q)\mathbf{n}\right\|_{\mathbf{V}^{-3/2}(\Gamma)}+\|q\|_{(H^1(\Omega)/\mathbb{R})'}\leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}+\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big),$$

where $c(\mathbf{w},q)$ *is the constant corresponding to* $(\mathbf{w},q)$*, and defined in* (2.2).

Proof. (i) Let $(\boldsymbol{\Phi},\mathbf{h})$ be in $\mathbf{L}^2(\Omega)\times\mathbf{V}^{3/2}(\Gamma)$ and let $(\mathbf{v},\pi)$ be the solution to Eq. (A.1). The solutions $(\mathbf{w},q)$ and $(\mathbf{v},\pi)$ obeys the Green formula:

$$\begin{aligned}\int_\Omega \mathbf{w}\boldsymbol{\Phi}&=\int_\Omega \mathbf{f}\mathbf{v}+\int_\Gamma\Big(-\frac{\partial\mathbf{v}}{\partial\mathbf{n}}+\pi\mathbf{n}\Big)\mathbf{g}+\int_\Gamma\Big(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}-q\mathbf{n}\Big)\mathbf{h}\\&=\int_\Omega \mathbf{f}\mathbf{v}+\int_\Gamma\Big(-\frac{\partial\mathbf{v}}{\partial\mathbf{n}}+\pi\mathbf{n}-c(\mathbf{v},\pi)\mathbf{n}\Big)\mathbf{g}+\int_\Gamma\Big(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}-q\mathbf{n}+c(\mathbf{w},q)\mathbf{n}\Big)\mathbf{h}.\end{aligned}$$

Setting $\mathbf{h}=0$, with Lemma A.1 we obtain

$$\begin{aligned}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}&=\sup_{\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)}=1}\int_\Omega \mathbf{w}\boldsymbol{\Phi}\\&\leqslant\sup_{\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)}=1}\left(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}+\left\|-\frac{\partial\mathbf{v}}{\partial\mathbf{n}}+\pi\mathbf{n}-c(\mathbf{v},\pi)\mathbf{n}\right\|_{\mathbf{V}^{1/2}(\Gamma)}\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\right)\\&\leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}+\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).\end{aligned}$$

Setting $\boldsymbol{\Phi}=0$, we obtain

$$\begin{aligned}&\left\|\frac{\partial\mathbf{w}}{\partial\mathbf{n}}-q\mathbf{n}+c(\mathbf{w},q)\mathbf{n}\right\|_{\mathbf{V}^{-3/2}(\Gamma)}=\sup_{\|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}=1}\int_\Gamma\Big(\frac{\partial\mathbf{w}}{\partial\mathbf{n}}-q\mathbf{n}+c(\mathbf{w},q)\mathbf{n}\Big)\mathbf{h}\\&\quad\leqslant\sup_{\|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}=1}\left(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}+\left\|-\frac{\partial\mathbf{v}}{\partial\mathbf{n}}+\pi\mathbf{n}-c(\mathbf{v},\pi)\mathbf{n}\right\|_{\mathbf{V}^{1/2}(\Gamma)}\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\right)\\&\quad\leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}+\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).\end{aligned}$$

(ii) Let h be in $H^1(\Omega)$ obeying $\int_\Omega h=0$ and let $(\mathbf{u},p)$ be the solution to Eq. (A.2). The solutions $(\mathbf{w},q)$ and $(\mathbf{u},p)$ obey the Green formula:

$$\int_\Omega \mathbf{f}\mathbf{u}+\int_\Omega qh+\int_\Gamma\Big(-\frac{\partial\mathbf{u}}{\partial\mathbf{n}}+p\mathbf{n}\Big)\mathbf{g}=0.$$

With Lemma A.2 we have

$$\begin{aligned}\|q\|_{(H^1(\Omega)/\mathbb{R})'}&=\sup_{\|h\|_{H^1(\Omega)/\mathbb{R}}=1}\int_\Omega qh\\&=\sup_{\|h\|_{H^1(\Omega)/\mathbb{R}}=1}\left(\int_\Gamma\Big(\frac{\partial\mathbf{u}}{\partial\mathbf{n}}-p\mathbf{n}+c(\mathbf{u},p)\mathbf{n}\Big)\mathbf{g}-\int_\Omega\mathbf{f}\mathbf{u}\right)\\&\leqslant\sup_{\|h\|_{H^1(\Omega)/\mathbb{R}}=1}\left(\left\|\frac{\partial\mathbf{u}}{\partial\mathbf{n}}-p\mathbf{n}+c(\mathbf{u},p)\mathbf{n}\right\|_{\mathbf{V}^{1/2}(\Gamma)}\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}+\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)}\right)\\&\leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'}+\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).\end{aligned}$$

The proof is complete. □

We want to define $(\mathbf{w}, q)$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} + c(\mathbf{w}, q)\mathbf{n}$ in the case where $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and $\mathbf{g} \in \mathbf{V}^{-1/2}(\Gamma)$. For all $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and all $\mathbf{g} \in \mathbf{V}^{-1/2}(\Gamma)$, we consider the variational problem

determine $(\mathbf{w}, q, \boldsymbol{\Psi}) \in \mathbf{V}^0(\Omega) \times (H^1(\Omega)/\mathbb{R})' \times \mathbf{V}^{-3/2}(\Gamma)$ such that

$$\int_\Omega \mathbf{w}\boldsymbol{\Phi} = \langle \mathbf{f}, \mathbf{v}\rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} - \left\langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \pi\mathbf{n} + c(\mathbf{v}, \pi)\mathbf{n} \right\rangle_{\mathbf{V}^{-1/2}(\Gamma), \mathbf{V}^{1/2}(\Gamma)} + \langle \boldsymbol{\Psi}, \mathbf{h}\rangle_{\mathbf{V}^{-3/2}(\Gamma), \mathbf{V}^{3/2}(\Gamma)}$$
$$\text{for all } (\boldsymbol{\Phi}, \mathbf{h}) \in \mathbf{L}^2(\Omega) \times \mathbf{V}^{3/2}(\Gamma), \tag{A.3}$$

and

$$-\langle q, h\rangle_{(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}} = \langle \mathbf{f}, \mathbf{u}\rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} - \left\langle \mathbf{g}, \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} + c(\mathbf{u}, p)\mathbf{n} \right\rangle_{\mathbf{V}^{-1/2}(\Gamma), \mathbf{V}^{1/2}(\Gamma)}$$
$$\text{for all } h \in H^1(\Omega)/\mathbb{R} \text{ obeying } \int_\Omega h = 0,$$

where $(\mathbf{v}, \pi)$ is solution of Eq. (A.1) and $(\mathbf{u}, p)$ is solution of Eq. (A.2).

Remark A.1. When $\mathbf{f} = 0$, the estimate

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leqslant C\|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)},$$

is already stated in [11], but the estimate of $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} + c(\mathbf{w}, q)\mathbf{n}$ seems to be new.

Theorem A.1. *For all* $(\mathbf{f}, \mathbf{g}) \in (\mathbf{H}^2(\Omega))' \times \mathbf{V}^{-1/2}(\Gamma)$*, the variational problem* (A.3) *admits a unique solution* $(\mathbf{w}, q, \boldsymbol{\Psi}) \in \mathbf{V}^0(\Omega) \times (H^1(\Omega)/\mathbb{R})' \times \mathbf{V}^{-3/2}(\Gamma)$ *satisfying*

$$\|\mathbf{w}\|_{\mathbf{V}^0(\Omega)} + \|q\|_{(H^1(\Omega)/\mathbb{R})'} + \|\boldsymbol{\Psi}\|_{\mathbf{V}^{-3/2}(\Gamma)} \leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).$$

Proof. (i) Let us first prove the uniqueness. If $\mathbf{f} = 0$, $\mathbf{g} = 0$ and if $(\mathbf{w}, q, \boldsymbol{\Psi})$ is a corresponding solution to problem (A.3), choosing $(\boldsymbol{\Phi}, h, \mathbf{h}) = (\mathbf{w}, 0, 0)$ in (A.3), we obtain $\mathbf{w} = 0$. Choosing $(\boldsymbol{\Phi}, h, \mathbf{h}) = (0, h, 0)$ in (A.3), with any h in $H^1(\Omega)$ obeying $\int_\Omega h = 0$, we obtain $q = 0$ in $(H^1(\Omega)/\mathbb{R})'$. Choosing $(\boldsymbol{\Phi}, h, \mathbf{h}) = (0, 0, \mathbf{h})$ in (A.3), with any $\mathbf{h}$ in $\mathbf{V}^{-3/2}(\Gamma)$, we obtain $\boldsymbol{\Psi} = 0$.

(ii) The existence result relies on a density argument. Let $(\mathbf{f}, \mathbf{g})$ be in $(\mathbf{H}^2(\Omega))' \times \mathbf{V}^{-1/2}(\Gamma)$. The space $\mathbf{L}^2(\Omega) \times \mathbf{V}^{3/2}(\Gamma)$ being dense in $(\mathbf{H}^2(\Omega))' \times \mathbf{V}^{-1/2}(\Gamma)$, there exists a sequence $(\mathbf{f}_n, \mathbf{g}_n)_n \subset \mathbf{L}^2(\Omega) \times \mathbf{V}^{3/2}(\Gamma)$ converging to $(\mathbf{f}, \mathbf{g})$ in $(\mathbf{H}^2(\Omega))' \times \mathbf{V}^{-1/2}(\Gamma)$. Let $(\mathbf{w}_n, q_n)$ be the solution to the equation

$$-\Delta \mathbf{w}_n + \nabla q_n = \mathbf{f}_n \quad \text{and} \quad \operatorname{div} \mathbf{w}_n = 0 \quad \text{in } \Omega, \qquad \mathbf{w}_n = \mathbf{g}_n \quad \text{on } \Gamma.$$

We can easily verify that $(\mathbf{w}_n, q_n, \boldsymbol{\Psi}_n)$, with $\boldsymbol{\Psi}_n = \frac{\partial \mathbf{w}_n}{\partial \mathbf{n}} - q_n\mathbf{n} + c(\mathbf{w}_n, q_n)\mathbf{n}$, is solution to problem (A.3) corresponding to $(\mathbf{f}_n, \mathbf{g}_n)$. From Lemma A.3, we deduce that $(\mathbf{w}_n, q_n, \boldsymbol{\Psi}_n)_n$ converges to some $(\mathbf{w}, q, \boldsymbol{\Psi})$ in $\mathbf{V}^0(\Omega) \times (H^1(\Omega)/\mathbb{R})' \times \mathbf{V}^{-3/2}(\Gamma)$. To show that $(\mathbf{w}, q, \boldsymbol{\Psi})$ is solution to problem (A.3) corresponding to $(\mathbf{f}, \mathbf{g})$, it is sufficient to pass to the limit in the identities

$$\int_\Omega \mathbf{w}_n\boldsymbol{\Phi} = \int_\Omega \mathbf{f}_n\mathbf{v} + \int_\Gamma \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n} - c(\mathbf{v}, \pi)\mathbf{n}\right)\mathbf{g}_n + \int_\Gamma \left(\frac{\partial \mathbf{w}_n}{\partial \mathbf{n}} - q_n\mathbf{n} + c(\mathbf{w}_n, q_n)\mathbf{n}\right)\mathbf{h},$$

and

$$0 = \int_\Omega hq_n + \int_\Omega \mathbf{f}_n\mathbf{u} + \int_\Gamma \left(-\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n} - c(\mathbf{z}, p)\mathbf{n}\right)\mathbf{g}_n.$$

The proof is complete. □

Let us recall that for $\mathbf{g} \in \mathbf{V}^{1/2}(\Gamma)$, $(D\mathbf{g}, D_p\mathbf{g}) = (\mathbf{w}, q)$ is the unique solution in $\mathbf{V}^1(\Omega) \times (L^2(\Omega)/\mathbb{R})$ to the equation

$$-\Delta\mathbf{w} + \nabla q = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.$$

From Theorem A.1 we deduce the following corollary.

Corollary A.1. *The operator D is linear and continuous from $\mathbf{V}^s(\Gamma)$ into $\mathbf{V}^{s+1/2}(\Omega)$ for all $-1/2 \leqslant s \leqslant 3/2$, and the operator D_p is linear and continuous from $\mathbf{V}^s(\Gamma)$ into $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R}$ for all $-1/2 \leqslant s \leqslant 3/2$ (where by notational convention $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R} = (H^{-s+1/2}(\Omega)/\mathbb{R})'$ if $s - 1/2 < 0$, and $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R} = H^{s-1/2}(\Omega)/\mathbb{R}$ if $s - 1/2 \geqslant 0$).*

If in addition Ω is of class C^3 the above results are still true for $-1/2 \leqslant s \leqslant 2$.

Proof. Let us prove the result when Ω is of class C^2 and $-1/2 \leqslant s \leqslant 3/2$. The other case can be treated similarly. Due to Lemma A.1, D is continuous from $\mathbf{V}^{3/2}(\Gamma)$ into $\mathbf{V}^2(\Omega)$, and D_p is continuous from $\mathbf{V}^{3/2}(\Gamma)$ into $H^1(\Omega)/\mathbb{R}$. From Theorem A.1 it follows that D can be extended to a bounded operator from $\mathbf{V}^{-1/2}(\Gamma)$ into $\mathbf{V}^0(\Omega)$, and D_p can be extended to a bounded operator from $\mathbf{V}^{-1/2}(\Gamma)$ into $(H^1(\Omega)/\mathbb{R})'$. The result follows by interpolation. □

We define $D^* \in \mathcal{L}(\mathbf{V}^0(\Omega); \mathbf{V}^0(\Gamma))$ as the adjoint of $D \in \mathcal{L}(\mathbf{V}^0(\Gamma); \mathbf{V}^0(\Omega))$.

Lemma A.4. *For all $\mathbf{f} \in \mathbf{V}^0(\Omega)$, $D^*\mathbf{f}$ is defined by*

$$D^*\mathbf{f} = -\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n},$$

where $c(\pi)$ is the constant defined in (2.3), *and $(\mathbf{v}, \pi)$ is the solution to*

$$-\Delta\mathbf{v} + \nabla\pi = \mathbf{f} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v} = 0 \quad \text{on } \Gamma. \tag{A.4}$$

The operator D^* is bounded from $\mathbf{V}^s(\Omega)$ into $\mathbf{V}^{s+1/2}(\Gamma)$ for all $0 \leqslant s \leqslant 2$. Moreover, for all $\boldsymbol{\Phi} \in \mathbf{V}^2(\Omega) \cap \mathbf{V}^1_0(\Omega)$, we have:

$$D^*(-A)\boldsymbol{\Phi} = -\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n},$$

where $c(\psi)$ is the constant defined in (2.3), *and $\psi \in H^1(\Omega)/\mathbb{R}$ is determined by*

$$\nabla\psi = (I - P)\Delta\boldsymbol{\Phi}.$$

Proof. (i) For all $\mathbf{f} \in \mathbf{V}^0(\Omega)$, and all $\mathbf{g} \in \mathbf{V}^0(\Gamma)$, the solution $(\mathbf{v}, \pi)$ to Eq. (A.4) and $\mathbf{w} = D\mathbf{g}$ obey:

$$\int_\Omega D\mathbf{g} \cdot \mathbf{f} = \int_\Gamma \mathbf{g} \cdot \left(-\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n}\right).$$

Thus $D^*\mathbf{f}$ is well defined as indicated in the statement of the lemma. Due to regularity results for the Stokes equations we have

$$\|D^*\mathbf{f}\|_{\mathbf{V}^{s+1/2}(\Gamma)} = \left\|-\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n}\right\|_{\mathbf{V}^{s+1/2}(\Gamma)} \leqslant C\|\mathbf{f}\|_{\mathbf{V}^s(\Omega)}.$$

The first part of the lemma is proved.

(ii) From the first part of the proof it follows that

$$D^*(-A)\boldsymbol{\Phi} = -\frac{\partial\hat{\boldsymbol{\Phi}}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n},$$

where $(\hat{\boldsymbol{\Phi}}, \psi)$ is the solution of the equation

$$-\Delta\hat{\boldsymbol{\Phi}} + \nabla\psi = (-A)\boldsymbol{\Phi} \quad \text{and} \quad \operatorname{div}\hat{\boldsymbol{\Phi}} = 0 \quad \text{in } \Omega, \qquad \hat{\boldsymbol{\Phi}} = 0 \quad \text{on } \Gamma.$$

This equation is equivalent to

$$(-A)\hat{\boldsymbol{\Phi}} = (-A)\boldsymbol{\Phi} \quad \text{and} \quad \nabla\psi = (I - P)\Delta\hat{\boldsymbol{\Phi}}.$$

Thus $\hat{\boldsymbol{\Phi}} = \boldsymbol{\Phi}$ and $\nabla\psi = (I - P)\Delta\boldsymbol{\Phi}$. The proof is complete. □

Proposition A.1. *Let $\mathbf{g} \neq 0$ be in $\mathbf{V}^{1/2}(\Gamma)$. Assume that $\gamma_\tau \mathbf{g} = 0$. Then $PD\mathbf{g}$ is not equal to zero.*

Proof. Set $(\mathbf{w}, q) = (D\mathbf{g}, D_p\mathbf{g})$. We want to prove that $P\mathbf{w} \neq 0$, and we assume the contrary. If $P\mathbf{w} = 0$, then $\mathbf{w} = (I - P)\mathbf{w} = \nabla\varphi$, for some $\varphi \in H^2(\Omega)$. Since $\operatorname{div}\mathbf{w} = 0$, φ is solution to the elliptic problem

$$-\Delta\varphi = 0 \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}}\varphi = \mathbf{g}\cdot\mathbf{n} \neq 0 \quad \text{on } \Gamma.$$

Moreover $\nabla\varphi|_\Gamma = (I - P)\mathbf{w}|_\Gamma = \mathbf{w}|_\Gamma$, thus

$$\nabla\varphi = \mathbf{g} \quad \text{on } \Gamma \quad \text{and} \quad \gamma_\tau \mathbf{g} = 0.$$

Thus φ is equal to a constant C on Γ, and φ is also solution to the elliptic problem

$$-\Delta\varphi = 0 \quad \text{in } \Omega, \qquad \varphi = C \quad \text{on } \Gamma.$$

It yields that $\varphi = C$ in Ω and $\partial_{\mathbf{n}}\varphi = 0$ on Γ, which is in contradiction with $\mathbf{g}\cdot\mathbf{n} \neq 0$. The proof is complete. □

Appendix B

Throughout this appendix we assume that $\lambda_0 > 0$ satisfies (4.2), and that $\mathbf{z}$ belongs at least to $\mathbf{V}^1(\Omega)$, or is more regular than that.

Lemma B.1. *For all $(\boldsymbol{\Phi}, \mathbf{h}) \in \mathbf{L}^2(\Omega) \times \mathbf{V}^{3/2}(\Gamma)$ the equation*:

$$\lambda_0\mathbf{v} - \Delta\mathbf{v} + (\mathbf{z}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{z} + \nabla\pi = \boldsymbol{\Phi} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v} = \mathbf{h} \quad \text{on } \Gamma, \tag{B.1}$$

admits a unique solution $(\mathbf{v}, \pi)$ in $\mathbf{V}^1(\Omega) \times (L^2(\Omega)/\mathbb{R})$. Moreover the following estimate holds:

$$\|\mathbf{v}\|_{\mathbf{V}^2(\Omega)} + \|\pi\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}\big). \tag{B.2}$$

If in addition Ω is of class C^3, $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$, and $(\boldsymbol{\Phi}, \mathbf{h}) \in \mathbf{H}^{1/2}(\Omega) \times \mathbf{V}^2(\Gamma)$, then

$$\|\mathbf{v}\|_{\mathbf{V}^{5/2}(\Omega)} + \|\pi\|_{H^{3/2}(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{H}^{1/2}(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^2(\Gamma)}\big).$$

The above results are also true if we replace Eq. (B.1) *by the following one*

$$\lambda_0\mathbf{v} - \Delta\mathbf{v} - (\mathbf{z}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{z})^{\mathrm{T}}\mathbf{v} + \nabla\pi = \boldsymbol{\Phi} \quad \text{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v} = \mathbf{h} \quad \text{on } \Gamma. \tag{B.3}$$

Proof. The uniqueness result is obvious. We only prove the existence of a solution $(\mathbf{v}, \pi)$ satisfying (B.1), the other results can be proved similarly. Let $(\mathbf{w}, q)$ be the solution to the equation

$$\lambda_0\mathbf{w} - \Delta\mathbf{w} + \nabla q = \boldsymbol{\Phi} \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{h} \quad \text{on } \Gamma,$$

and let $(\mathbf{u}, p)$ be the solution to the equation

$$\lambda_0\mathbf{u} - \Delta\mathbf{u} + (\mathbf{z}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{z} + \nabla p = -(\mathbf{z}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{z} \quad \text{and} \quad \operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \Gamma.$$

Since

$$\|\mathbf{w}\|_{\mathbf{V}^2(\Omega)} + \|q\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}\big),$$

the term $-(\mathbf{z}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{z}$ belongs to $\mathbf{L}^2(\Omega)$. According to Lemma 4.1, we have

$$\|\mathbf{u}\|_{\mathbf{V}^2(\Omega)} \leqslant C\big\|(\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z}\big\|_{\mathbf{L}^2(\Omega)} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}\big).$$

We next deduce that

$$\|p\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\big(\|\boldsymbol{\Phi}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{h}\|_{\mathbf{V}^{3/2}(\Gamma)}\big).$$

It is clear that $(\mathbf{v}, \pi) = (\mathbf{w}, q) + (\mathbf{u}, p)$ is the solution to Eq. (B.1), and the estimate (B.2) is established.

If $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ and $\mathbf{v} \in \mathbf{V}^2(\Omega)$, then $(\mathbf{z}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{z})^{\mathrm{T}}\mathbf{v}$ belongs to $\mathbf{H}^{1/2}(\Omega)$ (see [13, Proposition B1]). Then the solution to Eq. (B.1) belongs to $\mathbf{V}^{5/2}(\Omega)$. □

Lemma B.2. *For all $h \in H^1(\Omega)$ obeying $\int_\Omega h = 0$, the equation*:

$$\lambda_0 \mathbf{u} - \Delta \mathbf{u} + (\mathbf{z} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{z} + \nabla p = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u} = h \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \Gamma, \tag{B.4}$$

admits a unique solution $(\mathbf{u}, p)$ belonging to $\mathbf{H}^1(\Omega) \times (L^2(\Omega)/\mathbb{R})$. It satisfies the estimate:

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_{H^1(\Omega)/\mathbb{R}} \leqslant C\|h\|_{H^1(\Omega)/\mathbb{R}}.$$

The above results are also true if we replace Eq. (B.4) *by the following one*

$$\lambda_0 \mathbf{u} - \Delta \mathbf{u} - (\mathbf{z} \cdot \nabla)\mathbf{u} + (\nabla \mathbf{z})^{\mathrm{T}}\mathbf{u} + \nabla p = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u} = h \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \Gamma. \tag{B.5}$$

Proof. The lemma can be proved by combining the results of Lemma A.2 and the same kind of arguments as in the proof of Lemma B.1. □

Lemma B.3. *For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and all $\mathbf{g} \in \mathbf{V}^{3/2}(\Gamma)$, the solution $(\mathbf{w}, q)$ to equation*:

$$\lambda_0 \mathbf{w} - \Delta \mathbf{w} + (\mathbf{z} \cdot \nabla)\mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{z} + \nabla q = \mathbf{f} \quad \text{and} \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma, \tag{B.6}$$

obeys the estimate:

$$\|\mathbf{w}\|_{\mathbf{V}^{1/2}(\Omega)} + \left\| \frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} + c(\mathbf{w}, q)\mathbf{n} \right\|_{\mathbf{V}^{-1}(\Gamma)} + \|q\|_{(H^{1/2}(\Omega)/\mathbb{R})'} \leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^{3/2}(\Omega))'} + \|\mathbf{g}\|_{\mathbf{V}^0(\Gamma)}\big),$$

where $c(\mathbf{w}, q)$ is the constant corresponding to $\mathbf{w}$, q, and defined in (2.2). *If in addition $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ then we also have*:

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \left\| \frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} + c(\mathbf{w}, q)\mathbf{n} \right\|_{\mathbf{V}^{-3/2}(\Gamma)} + \|q\|_{(H^1(\Omega)/\mathbb{R})'} \leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).$$

Proof. If $(\mathbf{v}, \pi)$ is the solution to Eq. (B.3), $(\mathbf{u}, p)$ is the solution to Eq. (B.5), and $(\mathbf{w}, q)$ to Eq. (B.6), then we have

$$\int_\Omega \mathbf{w}\boldsymbol{\Phi} = \int_\Omega \mathbf{f}\mathbf{v} + \int_\Gamma \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi\mathbf{n}\right)\mathbf{g} + \int_\Gamma \left(\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n}\right)\mathbf{h} - \int_\Gamma \mathbf{z} \cdot \mathbf{n}\mathbf{h} \cdot \mathbf{g},$$

and

$$\int_\Omega \mathbf{f}\mathbf{u} + \int_\Omega qh + \int_\Gamma \left(-\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + p\mathbf{n}\right)\mathbf{g} = 0.$$

Thus the proof can be performed as in the one of Lemma A.3. The assumption $\mathbf{z} \in \mathbf{V}^{3/2}(\Omega)$ is needed to estimate $\mathbf{z} \cdot \mathbf{n}\mathbf{h}$ in $\mathbf{H}^{1/2}(\Gamma)$ when $\mathbf{h}$ belongs to $\mathbf{V}^{3/2}(\Gamma)$. □

We want to define $(\mathbf{w}, q)$ and $\frac{\partial \mathbf{w}}{\partial \mathbf{n}} - q\mathbf{n} + c(\mathbf{w}, q)\mathbf{n}$ in the case where $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and $\mathbf{g} \in \mathbf{V}^{-1/2}(\Gamma)$. As in Appendix A, for all $\mathbf{f} \in (\mathbf{H}^2(\Omega))'$ and all $\mathbf{g} \in \mathbf{V}^{-1/2}(\Gamma)$, we consider the variational problem

$$\begin{aligned}
&\text{determine } (\mathbf{w}, q, \boldsymbol{\Psi}) \in \mathbf{V}^0(\Omega) \times \big(H^1(\Omega)/\mathbb{R}\big)' \times \mathbf{V}^{-3/2}(\Gamma) \text{ such that} \\
&\int_\Omega \mathbf{w}\boldsymbol{\Phi} = -\left\langle \mathbf{g}, \frac{\partial \mathbf{v}}{\partial \mathbf{n}} - \pi\mathbf{n} + \mathbf{z} \cdot \mathbf{n}\mathbf{h} + c(\mathbf{v}, \pi, \mathbf{z} \cdot \mathbf{n}\mathbf{h})\mathbf{n} \right\rangle_{\mathbf{V}^{-1/2}(\Gamma), \mathbf{V}^{1/2}(\Gamma)} \\
&\quad + \langle \mathbf{f}, \mathbf{v} \rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} + \langle \boldsymbol{\Psi}, \mathbf{h} \rangle_{\mathbf{V}^{-3/2}(\Gamma), \mathbf{V}^{3/2}(\Gamma)} \ \forall (\boldsymbol{\Phi}, \mathbf{h}) \in \mathbf{L}^2(\Omega) \times \mathbf{V}^{3/2}(\Gamma), \quad \text{and} \\
&-\langle q, h \rangle_{(H^1(\Omega)/\mathbb{R})', H^1(\Omega)/\mathbb{R}} = \langle \mathbf{f}, \mathbf{u} \rangle_{(\mathbf{H}^2(\Omega))', \mathbf{H}^2(\Omega)} - \left\langle \mathbf{g}, \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - p\mathbf{n} + c(\mathbf{u}, p)\mathbf{n} \right\rangle_{\mathbf{V}^{-1/2}(\Gamma), \mathbf{V}^{1/2}(\Gamma)} \\
&\quad \text{for all } h \in H^1(\Omega)/\mathbb{R} \text{ obeying } \int_\Omega h = 0,
\end{aligned} \tag{B.7}$$

where $(\mathbf{v}, \pi)$ is solution of Eq. (B.3), $(\mathbf{u}, p)$ is solution of Eq. (B.5), and

$$c(\mathbf{v}, \pi, \mathbf{z}\cdot\mathbf{nh}) = -\frac{1}{|\Gamma|}\int_\Gamma \left(\frac{\partial \mathbf{v}}{\partial \mathbf{n}}\cdot\mathbf{n} - \pi + \mathbf{z}\cdot\mathbf{nh}\cdot\mathbf{n}\right).$$

Observe that the term $\mathbf{z}\cdot\mathbf{nh}$ appears in the first equation of (B.7). This term was not present in (A.3). If $\mathbf{g}\in\mathbf{V}^0(\Gamma)$, we have to define $\mathbf{z}\cdot\mathbf{nh}$ in $\mathbf{L}^2(\Gamma)$. If $\mathbf{z}\in\mathbf{V}^1(\Omega)$, then $\mathbf{z}\cdot\mathbf{n}$ belongs to $H^{1/2}(\Gamma)\hookrightarrow L^4(\Gamma)$. If $\mathbf{h}\in\mathbf{V}^1(\Gamma)\hookrightarrow L^p(\Gamma)$ for all $1\leqslant p<\infty$, then $\mathbf{z}\cdot\mathbf{nh}$ is well defined in $\mathbf{L}^2(\Gamma)$, which leads to the first estimate in Theorem B.1.

To define $\mathbf{z}\cdot\mathbf{nh}$ in $\mathbf{H}^{1/2}(\Gamma)$ when $\mathbf{h}\in\mathbf{V}^{3/2}(\Gamma)$, we have to suppose that $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$. Indeed if $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$, $\mathbf{z}\cdot\mathbf{n}$ belongs to $H^1(\Gamma)$, and $\mathbf{z}\cdot\mathbf{nh}$ belongs $\mathbf{H}^1(\Gamma)$ [13, Proposition B1]. If we only suppose that $\mathbf{z}\in\mathbf{V}^1(\Omega)$, we can only prove that $\mathbf{z}\cdot\mathbf{nh}$ belongs $\mathbf{H}^s(\Gamma)$ for all $0\leqslant s<1/2$.

Theorem B.1. *For all* $(\mathbf{f},\mathbf{g})\in(\mathbf{H}^{3/2}(\Omega))'\times\mathbf{V}^0(\Gamma)$, *the variational problem* (B.7) *admits a unique solution* $(\mathbf{w}, q, \boldsymbol{\Psi})\in\mathbf{V}^{1/2}(\Omega)\times(H^{1/2}(\Omega)/\mathbb{R})'\times\mathbf{V}^{-1}(\Gamma)$ *satisfying*

$$\|\mathbf{w}\|_{\mathbf{V}^{1/2}(\Omega)} + \|q\|_{(H^{1/2}(\Omega)/\mathbb{R})'} + \|\boldsymbol{\Psi}\|_{\mathbf{V}^{-1}(\Gamma)} \leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^{3/2}(\Omega))'} + \|\mathbf{g}\|_{\mathbf{V}^0(\Gamma)}\big).$$

If in addition $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$ *then, for all* $(\mathbf{f},\mathbf{g})\in(\mathbf{H}^2(\Omega))'\times\mathbf{V}^{-1/2}(\Gamma)$, *the variational problem* (B.7) *admits a unique solution* $(\mathbf{w}, q, \boldsymbol{\Psi})\in\mathbf{V}^0(\Omega)\times(H^1(\Omega)/\mathbb{R})'\times\mathbf{V}^{-3/2}(\Gamma)$ *satisfying*

$$\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} + \|q\|_{(H^1(\Omega)/\mathbb{R})'} + \|\boldsymbol{\Psi}\|_{\mathbf{V}^{-3/2}(\Gamma)} \leqslant C\big(\|\mathbf{f}\|_{(\mathbf{H}^2(\Omega))'} + \|\mathbf{g}\|_{\mathbf{V}^{-1/2}(\Gamma)}\big).$$

Proof. The proof is similar to that of Theorem A.1. □

From Theorem B.1 we deduce the following corollary.

Corollary B.1. *The operator* $D_\mathbf{z}$ *is linear and continuous from* $\mathbf{V}^s(\Gamma)$ *into* $\mathbf{V}^{s+1/2}(\Omega)$ *for all* $0\leqslant s\leqslant 3/2$, *and the operator* $D_{\mathbf{z},p}$ *is linear and continuous from* $\mathbf{V}^s(\Gamma)$ *into* $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R}$ *for all* $0\leqslant s\leqslant 3/2$ (*where by notational convention* $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R} = (H^{-s+1/2}(\Omega)/\mathbb{R})'$ *if* $s-1/2<0$, *and* $\mathcal{H}^{s-1/2}(\Omega)/\mathbb{R} = H^{s-1/2}(\Omega)/\mathbb{R}$ *if* $s-1/2\geqslant 0$).

If $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$, *then the above results are still true for* $-1/2\leqslant s\leqslant 3/2$.

If Ω *is of class* C^3 *and if* $\mathbf{z}\in\mathbf{V}^{3/2}(\Omega)$, *then the above results are still true for* $-1/2\leqslant s\leqslant 2$.

Proof. See the proof of Corollary A.1. □

Let us recall that for $\mathbf{g}\in\mathbf{V}^{1/2}(\Gamma)$, $(D_\mathbf{z}\mathbf{g}, D_{p,\mathbf{z}}\mathbf{g}) = (\mathbf{w}, q)$ is the unique solution in $\mathbf{V}^1(\Omega)\times(L^2(\Omega)/\mathbb{R})$ to the equation

$$\lambda_0\mathbf{w} - \Delta\mathbf{w} + (\mathbf{z}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{z} + \nabla q = 0 \quad \text{and} \quad \operatorname{div}\mathbf{w} = 0 \quad \text{in } \Omega, \qquad \mathbf{w} = \mathbf{g} \quad \text{on } \Gamma.$$

We define $D_\mathbf{z}^*\in\mathcal{L}(\mathbf{V}^0(\Gamma);\mathbf{V}^0(\Omega))$ as the adjoint of $D_\mathbf{z}\in\mathcal{L}(\mathbf{V}^0(\Gamma);\mathbf{V}^0(\Omega))$.

Lemma B.4. *For all* $\mathbf{f}\in\mathbf{V}^0(\Omega)$, $D_\mathbf{z}^*\mathbf{f}$ *is defined by*

$$D_\mathbf{z}^*\mathbf{f} = -\frac{\partial\mathbf{v}}{\partial\mathbf{n}} + \pi\mathbf{n} - c(\pi)\mathbf{n},$$

where $c(\pi)$ *is the constant defined in* (2.3), *and* $(\mathbf{v},\pi)$ *is the solution to*

$$\lambda_0\mathbf{v} - \Delta\mathbf{v} - (\mathbf{z}\cdot\nabla)\mathbf{v} + (\nabla\mathbf{z})^{\mathrm{T}}\mathbf{v} + \nabla\pi = \mathbf{f} \quad \textit{and} \quad \operatorname{div}\mathbf{v} = 0 \quad \textit{in } \Omega, \qquad \mathbf{v} = 0 \quad \textit{on } \Gamma.$$

The operator $D_\mathbf{z}^*$ *is bounded from* $\mathbf{V}^s(\Omega)$ *into* $\mathbf{V}^{s+1/2}(\Gamma)$ *for all* $0\leqslant s\leqslant 2$. *Moreover, for all* $\boldsymbol{\Phi}\in\mathbf{V}^2(\Omega)\cap\mathbf{V}^1_0(\Omega)$, *we have*:

$$D_\mathbf{z}^*\big(\lambda_0 I - A_\mathbf{z}^*\big)\boldsymbol{\Phi} = -\frac{\partial\boldsymbol{\Phi}}{\partial\mathbf{n}} + \psi\mathbf{n} - c(\psi)\mathbf{n},$$

where $c(\psi)$ *is the constant defined in* (2.3), *and* $\psi\in H^1(\Omega)/\mathbb{R}$ *is determined by*

$$\nabla\psi = (I-P)\big(\Delta\boldsymbol{\Phi} + (\mathbf{z}\cdot\nabla)\boldsymbol{\Phi} - (\nabla\mathbf{z})^{\mathrm{T}}\boldsymbol{\Phi}\big).$$

Proof. (i) For all $\mathbf{f} \in \mathbf{V}^0(\Omega)$, and all $\mathbf{g} \in \mathbf{V}^0(\Gamma)$, the pairs $(D_{\mathbf{z}}\mathbf{g}, D_{p,\mathbf{z}}\mathbf{g}) = (\mathbf{w}, q)$ and $(\mathbf{v}, \pi)$ obey:

$$\int_\Omega D_{\mathbf{z}}\mathbf{g} \cdot \mathbf{f} = \int_\Gamma \mathbf{g} \cdot \left(-\frac{\partial \mathbf{v}}{\partial \mathbf{n}} + \pi \mathbf{n} - c(\pi)\mathbf{n}\right).$$

This identity gives the expression of $D_{\mathbf{z}}^*$. As in the proof of Lemma A.4, we can easily show that $D_{\mathbf{z}}^*$ is bounded from $\mathbf{V}^s(\Omega)$ into $\mathbf{V}^{s+1/2}(\Gamma)$ for all $0 \leqslant s \leqslant 2$.

(ii) From the first part of the proof it follows that

$$D_{\mathbf{z}}^*(\lambda_0 I - A_{\mathbf{z}}^*)\boldsymbol{\Phi} = -\frac{\partial \hat{\boldsymbol{\Phi}}}{\partial \mathbf{n}} + \psi \mathbf{n} - c(\psi)\mathbf{n},$$

where $(\hat{\boldsymbol{\Phi}}, \psi)$ is the solution of the equation

$$\lambda_0 \hat{\boldsymbol{\Phi}} - \Delta \hat{\boldsymbol{\Phi}} - (\mathbf{z} \cdot \nabla)\hat{\boldsymbol{\Phi}} - (\nabla \mathbf{z})^{\mathrm{T}} \hat{\boldsymbol{\Phi}} + \nabla \psi = (\lambda_0 I - A_{\mathbf{z}}^*)\boldsymbol{\Phi} \quad \text{and} \quad \operatorname{div} \hat{\boldsymbol{\Phi}} = 0 \quad \text{in } \Omega, \qquad \hat{\boldsymbol{\Phi}} = 0 \quad \text{on } \Gamma.$$

This equation is equivalent to

$$(\lambda_0 I - A_{\mathbf{z}})\hat{\boldsymbol{\Phi}} = (\lambda_0 I - A_{\mathbf{z}})\boldsymbol{\Phi} \quad \text{and} \quad \nabla \psi = (I - P)\big(\Delta \hat{\boldsymbol{\Phi}} + (\mathbf{z} \cdot \nabla)\hat{\boldsymbol{\Phi}} - (\nabla \mathbf{z})^{\mathrm{T}} \hat{\boldsymbol{\Phi}}\big).$$

Thus $\hat{\boldsymbol{\Phi}} = \boldsymbol{\Phi}$ and $\nabla \psi = (I - P)(\Delta \boldsymbol{\Phi} + (\mathbf{z} \cdot \nabla)\boldsymbol{\Phi} - (\nabla \mathbf{z})^{\mathrm{T}} \boldsymbol{\Phi})$. The proof is complete. □

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