Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 851–874
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Type II collapsing of maximal solutions to the Ricci flow in $\mathbb{R}^2$

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Received 21 January 2005; accepted 7 June 2006

Available online 19 December 2006

Abstract

We consider the initial value problem $u_t = \Delta \log u$, $u(x,0) = u_0(x) \geqslant 0$ in $\mathbb{R}^2$, corresponding to the Ricci flow, namely conformal evolution of the metric $u(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ by Ricci curvature. It is well known that the maximal solution u vanishes identically after time $T = \frac{1}{4\pi}\int_{\mathbb{R}^2} u_0$. Assuming that u_0 is radially symmetric and satisfies some additional constraints, we describe precisely the Type II collapsing of u at time T: we show the existence of an inner region with exponentially fast collapsing and profile, up to proper scaling, a *soliton cigar solution*, and the existence of an outer region of persistence of a logarithmic cusp. This is the only Type II singularity which has been shown to exist, so far, in the Ricci Flow in any dimension. It recovers rigorously formal asymptotics derived by J.R. King [J.R. King, Self-similar behavior for the equation of fast nonlinear diffusion, Philos. Trans. R. Soc. London Ser. A 343 (1993) 337–375].

1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta \log u, & \text{in } \mathbb{R}^2 \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

for the *logarithmic fast diffusion* equation in $\mathbb{R}^2$, with initial data u_0 non-negative, integrable and $T > 0$.

It has been observed by S. Angenent and L. Wu [20,21] that Eq. (1.1) represents the evolution of the conformally equivalent metric $g_{ij} = u\,\mathrm{d}x_i\,\mathrm{d}x_j$ under the *Ricci Flow*

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1.2}$$

which evolves g_{ij} by its Ricci curvature. The equivalence follows easily from the observation that the conformal metric $g_{ij} = uI_{ij}$ has scalar curvature $R = -(\Delta \log u)/u$ and in two dimensions $R_{ij} = \frac{1}{2}Rg_{ij}$.

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[1] Partially supported by the NSF grants DMS-01-02252, DMS-03-54639 and the EPSRC in the UK.
[2] Partially supported by grants Fondecyt 1030840 and FONDAP.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.06.006

Eq. (1.1) arises also in physical applications, as a model for long Van-der-Wals interactions in thin films of a fluid spreading on a solid surface, if certain nonlinear fourth order effects are neglected, see [6,3,4].

We consider solutions with finite total

$$A = \int_{\mathbb{R}^2} u \, \mathrm{d}x < \infty.$$

Since u goes to zero when $|x|$ tends to infinity, the equation is not uniformly parabolic. It becomes singular when u is close to zero. This results to many interesting phenomena, in particular solutions are not unique [8]. It is shown in [8] that given an initial data $u_0 \geqslant 0$ with finite area and a constant $\gamma \geqslant 2$, there exists a solution u_γ of (1.1) with

$$\int_{\mathbb{R}^2} u_\lambda(x,t) \, \mathrm{d}x = \int_{\mathbb{R}^2} u_0 \, \mathrm{d}x - 2\pi\gamma t. \tag{1.3}$$

The solution u_γ exists up to the exact time $T = T_\gamma$, which is determined in terms of the initial area and γ by $T_\gamma = \frac{1}{2\pi\gamma}\int_{\mathbb{R}} u_0 \, \mathrm{d}x$. In addition, if $u_0 \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, for some $p > 1$, $u_0 \not\equiv 0$ and it is radially symmetric, then u_γ is unique and characterized by the flux-condition

$$\lim_{r\to+\infty} \frac{r u_r(r,t)}{u(r,t)} = -\gamma, \quad \text{as } r \to \infty \tag{1.4}$$

for all $0 < t < T_\gamma$.

We restrict our attention to *maximal solutions* u of (1.1), corresponding to the value $\gamma = 2$ in (1.3), which vanish at time

$$T = \frac{1}{4\pi} \int_{\mathbb{R}^2} u_0(x) \, \mathrm{d}x. \tag{1.5}$$

Before we proceed with statements of our main results, let us comment on the extinction behavior of the intermediate solutions u_γ of (1.1), corresponding to values $\gamma > 2$. This has been recently studied by S.Y. Hsu [17] (see also [16]). Let u_γ be the unique radially symmetric solution of (1.1) which satisfies (1.3) and (1.4). It has been shown in [17] that there exist unique constants $\alpha > 0$, $\beta > -1/2$, $\alpha = 2\beta + 1$, depending on γ, such that the rescaled function

$$v(y,\tau) = \frac{u(y/(T-t)^\beta, t)}{(T-t)^\alpha}, \quad \tau = -\log(T-t)$$

will converge uniformly on compact subsets of $\mathbb{R}^2$ to $\phi_{\lambda,\beta}(y)$, for some constant $\lambda > 0$, where $\phi_{\lambda,\beta}(y) = \phi_{\lambda,\beta}(r)$, $r = |y|$ is radially symmetric and satisfies the ODE

$$\frac{1}{r}\left(\frac{r\phi'}{\phi}\right)' + \alpha\phi + \beta r \phi' = 0, \quad \text{in } (0,\infty)$$

with

$$\phi(0) = \frac{1}{\lambda}, \qquad \phi'(0) = 0.$$

In the case where $\gamma = 4$ the above result simply gives the asymptotics

$$u(x,t) \approx \frac{8\lambda(T-t)}{(\lambda + |x|^2)^2}, \quad \text{as } t \to T$$

corresponding to the geometric result of R. Hamilton [12] and B. Chow [5] that under the Ricci Flow, a two-dimensional compact surface shrinks to a sphere. The extinction behavior of *non-radial* solutions of (1.1) satisfying (1.3) with $\gamma > 2$ is still an open question. Let us also point out that the asymptotic behavior, as $t \to \infty$, of maximal solutions of (1.1) when the initial data $u_0 \geqslant 0$, $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ has infinite area $\int_{\mathbb{R}^2} u_0(x) \, \mathrm{d}x = \infty$ and satisfies the specific bounds

$$\frac{\alpha}{|x|^2 + \beta_1} \leqslant u_0(x) \leqslant \frac{\alpha}{|x|^2 + \beta_2}, \quad x \in \mathbb{R},$$

for some $\alpha > 0$, $\beta_1, \beta_2 > 0$, has been studied by S.Y. Hsu in [14,15], extending previous geometric results by L.F. Wu [20,21].

The methods in [17] no longer apply for the maximal solution, which turns out to exhibit more delicate asymptotic behavior. This is due to the fact that the blow up of the curvature $R = -\Delta \log u/u$ at the vanishing time T of the maximal solution is of Type II ($R_{\max}(t)(T-t) \to \infty$, as $t \to T$) and not of the standard Type I ($R_{\max}(t)(T-t) \leqslant C < \infty$, as $t \to T$) which is shown to happen in all the other cases. This Type II blow up behavior of the maximal solution is proven, via geometric a priori estimates, by the first author and R. Hamilton in [9]. Note that this is the only case of Type II singularity which has been shown to exist in the Ricci Flow, in any dimension.

J.R. King [18] has formally analyzed the extinction behavior of maximal solutions u of (1.1), as $t \to T$, with $T = (1/4\pi)\int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x$. His analysis, for compactly supported initial data, suggests the existence of two regions of different behavior. In the *outer region* $(T-t)\ln r > T$ the "logarithmic cusp" exact solution $2t/|x|^2 \log^2|x|$ of equation $u_t = \Delta \log u$ persists. However, in the *inner region* $(T-t)\ln r \leqslant T$ the solution vanishes exponentially fast and approaches, after an appropriate change of variables, one of the soliton solutions U of equation $U_\tau = \Delta \log U$ on $-\infty < \tau < \infty$ given by $U(x,\tau) = 1/(\lambda|x|^2 + \mathrm{e}^{4\lambda\tau})$, with $\tau = 1/(T-t)$ and λ a constant which depends on the initial data u_0.

Our goal in this paper is to establish rigorously that behavior, under a set of geometrically natural constraints on the initial condition u_0.

We assume in what follows that $u_0 = u_0(|x|)$ is nonnegative, not identically zero, radially symmetric and bounded with

$$T \equiv \frac{1}{4\pi}\int_{\mathbb{R}^2} u_0\,\mathrm{d}x < +\infty \tag{1.6}$$

such that

$$u_0(r) \text{ is strictly decreasing on } r \geqslant r_0, \text{ for some } r_0 \gg 1 \tag{1.7}$$

and it satisfies the growth condition

$$u_0(x) = \frac{2\mu}{|x|^2 \log^2|x|}\big(1 + \mathrm{o}(1)\big), \quad \text{as } |x| \to \infty \tag{1.8}$$

for some positive constant μ. Since locally bounded weak solutions of (1.1) are strictly positive and smooth, we may assume without loss of generality that u_0 is strictly positive and smooth. The initial asymptotic behavior (1.8) is in fact natural, since it holds true for the maximal solution at any positive time prior to vanishing if the initial datum has compact support or fast decay. Moreover, according to the results in [8] and [19] (1.8) implies that the maximal solution u which extincts at time T also satisfies the asymptotic behavior

$$u(x,t) = \frac{2(t+\mu)}{|x|^2 \log^2|x|}\big(1 + \mathrm{o}(1)\big), \quad \text{as } |x| \to \infty,\ 0 \leqslant t < T, \tag{1.9}$$

this bound of course deteriorates as $t \to T$. Geometrically this corresponds to the condition that the conformal metric is complete. The manifold can be visualized as a surface of revolution with an unbounded cusp with finite area closing around its axis. Note that also condition (1.7) is not restrictive, because of the flux condition (1.4) which holds for any maximal solution with $\gamma = 2$.

The scalar curvature R_{cusp} of the logarithmic cusp $2\mu/|x|^2 \log^2|x|$, satisfies the lower bound $R_{\text{cusp}} \geqslant -1/\mu$. We assume the geometric condition that the initial curvature $R_0 = -\Delta \log u_0/u_0$ satisfies the lower bound

$$R_0(x) \geqslant -\frac{1}{\mu} \quad \text{on } \mathbb{R}. \tag{1.10}$$

Our main results describe the asymptotic behavior of the maximal solution u of (1.1) near $t = T$ as follows: Theorem 1.1 describes the *inner behavior* essentially as

$$u(x,t) \approx \frac{(T-t)^2}{\lambda|x|^2 + \mathrm{e}^{\frac{2(T+\mu)}{(T-t)}}}$$

for some $\lambda > 0$, whenever $|x| = O(e^{\frac{T+\mu}{T-t}})$, while the *outer behavior* is given, according to Theorem 1.2, by

$$u(x,t) \approx \frac{2(t+\mu)}{|x|^2 \log^2 |x|}$$

for $|x| \gg e^{\frac{T+\mu}{T-t}}$.

To make these statements precise, we perform the following change of variables:

$$\bar{u}(x,\tau) = \tau^2 u(x,t), \quad \tau = \frac{1}{T-t} \tag{1.11}$$

and

$$\tilde{u}(y,\tau) = \alpha(\tau)\bar{u}\big(\alpha(\tau)^{1/2} y, \tau\big), \tag{1.12}$$

with

$$\alpha(\tau) = \big[\bar{u}(0,\tau)\big]^{-1} = \big[(T-t)^{-2} u(0,t)\big]^{-1} \tag{1.13}$$

so that $\tilde{u}(0,\tau) = 1$.

A direct computation shows that the rescaled solution $\tilde{u}$ satisfies the equation

$$\tilde{u}_\tau = \Delta \log \tilde{u} + \frac{\alpha'(\tau)}{2\alpha(\tau)} \nabla(y \cdot \tilde{u}) + \frac{2\tilde{u}}{\tau}. \tag{1.14}$$

Then, following result holds:

Theorem 1.1 *(Inner behavior). Assume that the u_0 is radially symmetric, positive, smooth and satisfies* (1.6)–(1.8) *and* (1.10). *Then, for each sequence $\tau_k \to \infty$, there is a subsequence, $\tau_{k_l} \to \infty$ such that $\alpha'(\tau_{k_l})/2\alpha(\tau_{k_l}) \to 2\lambda$, for some constant $\lambda \geqslant (T+\mu)/2$ and along which the rescaled solution $\tilde{u}$ defined by* (1.11)–(1.13) *converges, uniformly on compact subsets of $\mathbb{R}^2$, to the solution $U_\lambda(x) = (\lambda|y|^2 + 1)^{-1}$ of the steady state equation*

$$\Delta \log U + 2\lambda \nabla(y \cdot U) = 0.$$

In addition

$$\lim_{\tau\to\infty} \frac{\log \alpha(\tau)}{\tau} = T + \mu. \tag{1.15}$$

To describe the vanishing behavior of $u(r,t)$ in the outer region we first perform the cylindrical change of variables

$$v(s,t) = r^2 u(r,t), \quad s = \log r \tag{1.16}$$

which transforms equation $u_t = \Delta \log u$ to the one-dimensional equation

$$v_t = (\log v)_{ss}, \quad -\infty < s < \infty. \tag{1.17}$$

We then perform a further scaling setting

$$\tilde{v}(\xi,\tau) = \tau^2 v(\tau\xi, t), \quad \tau = \frac{1}{T-t}. \tag{1.18}$$

A direct computation shows that $\tilde{v}$ satisfies the equation

$$\tau \tilde{v}_\tau = \frac{1}{\tau} (\log \tilde{v})_{\xi\xi} + \xi \tilde{v}_\xi + 2\tilde{v}. \tag{1.19}$$

The extinction behavior of u (or equivalently of v) in the outer region $\xi \geqslant T + \mu$, is described in the following result.

Theorem 1.2 *(Outer behavior). Assume that the initial data u_0 is positive, radially symmetric, smooth and satisfies* (1.6)–(1.8) *and* (1.10). *Then, the rescaled solution $\tilde{v}$ defined by* (1.18) *converges, as $\tau \to \infty$, to the steady state solution V of Eq.* (1.19) *given by*

$$V(\xi) = \begin{cases} \frac{2(T+\mu)}{\xi^2}, & \xi > \xi_\mu, \\ 0, & \xi < \xi_\mu \end{cases} \tag{1.20}$$

with

$$\xi_\mu = T + \mu.$$

Moreover, the convergence is uniform on the interval $(-\infty, \xi_\mu^-]$ *and on compact subsets of* $[\xi_\mu^+, +\infty)$*, for* $-\infty < \xi_\mu^- < \xi_\mu < \xi_\mu^+ < +\infty$.

This work is devoted to the proof of the above theorems. We conjecture that the limit λ in Theorem 1.1 is unique, along all subsequences and it is equal to $(T + \mu)/2$. We also conjecture that the results in this work are true without the assumption of radial symmetry. Condition (1.8) is necessary as it is evident from the above theorems that the extinction behavior of u depends on the constant μ.

The proof of the above results relies on sharp estimates on the geometric *width* W and on the *maximum curvature* $R_{\max}$ of maximal solutions near their extinction time T derived in [9] by the first author and R. Hamilton. In particular, it is found in [9] that the maximum curvature is proportional to $1/(T-t)^2$, which does not go along with the natural scaling of the problem which would entail blow-up of order $1/(T-t)$. One says that the collapsing is *of type II*. It is interesting to mention that construction of symmetric solutions to mean curvature flow exhibiting type II blow-up was achieved by Angenent and Velazquez in [2], where distinct geometric inner and outer behaviors are found as well. Rather than a general classification result like ours, their construction relies on carefully chosen, very special initial data.

2. Preliminaries

In this section we will collect a few preliminary results which will be used throughout the rest of the paper. For the convenience of the reader, we start with a brief description of the geometric estimates in [9] on which the proofs of Theorems 1.1 and 1.2 rely upon.

2.1. Geometric estimates

In [9] the first author and R. Hamilton established upper and lower bounds on the geometric *width* $W(t)$ of the maximal solution u of (1.1), given in the rotational symmetric case by $W(t) = \max_{r\geqslant 0} 2\pi|x|\sqrt{u}(x,t)$, and on the *maximum curvature* $R_{\max}(t) = \max_{x\in\mathbb{R}^2} R(x,t)$, with $R = -(\Delta \log u)/u$.

As we noted in the introduction the maximal solution u of (1.1) will exist only up to $T = (1/4\pi)\int_{\mathbb{R}^2} u_0(x)\,\mathrm{d}x$. The estimates in [9] depend on the time to collapse $T - t$. However, they do not scale in the usual way.

Theorem 2.1. [9] *There exist positive constants c and C for which*

$$c(T-t) \leqslant W(t) \leqslant C(T-t) \tag{2.1}$$

and

$$\frac{c}{(T-t)^2} \leqslant R_{\max}(t) \leqslant \frac{C}{(T-t)^2} \tag{2.2}$$

for all $0 < t < T$.

In the radially symmetric case (2.1) implies the pointwise bound

$$c(T-t) \leqslant \max_{r\geqslant 0} r\sqrt{u}(r,t) \leqslant C(T-t) \tag{2.3}$$

on the maximal solution u of (1.1), or the bound

$$c(T-t) \leqslant \max_{s\in\mathbb{R}} \sqrt{v}(s,t) \leqslant C(T-t) \tag{2.4}$$

for the solution $v = r^2 u(r,t)$, $s = \log r$, of the one-dimensional equation (1.17).

2.2. Eternal solutions

We will present now a classification result for radially symmetric solutions U of equation

$$\frac{\partial U}{\partial t} = \Delta \log U, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}. \tag{2.5}$$

Since the solutions U are defined for $-\infty < t < \infty$ they are called *eternal* solutions to the Ricci flow. This classification result will be crucial in Section 3, where we will show that rescaled solutions of Eq. (1.1) converge to eternal solutions of Eq. (2.5).

We assume that the solution U of (2.5) is smooth, strictly positive, radially symmetric, with uniformly bounded width, i.e.,

$$\max_{(x,t)\in\mathbb{R}^2\times\mathbb{R}} |x|^2 u(x,t) < \infty. \tag{2.6}$$

In addition, we assume that the scalar curvature $R = -\Delta \log U/U$ is nonnegative and satisfies

$$\max_{(x,t)\in\mathbb{R}\times[-\infty,\tau]} R(x,t) < \infty, \quad \forall \tau \in \mathbb{R}. \tag{2.7}$$

Since U is strictly positive at all $t < \infty$, it follows that $U(\cdot,t)$ must have infinite area, i.e.,

$$\int_{\mathbb{R}^2} U(x,t)\,\mathrm{d}x = \infty, \quad \forall t \in \mathbb{R}. \tag{2.8}$$

Otherwise, if $\int_{\mathbb{R}^2} U(x,t)\,\mathrm{d}x < \infty$, for some $t < \infty$, then by the results in [8] the solution U must vanish at time $t+T$, with $T = 1/4\pi \int_{\mathbb{R}^2} U(x,t)\,\mathrm{d}x$, or before.

Theorem 2.2. *Assume that U is a smooth, strictly positive, radially symmetric solution of Eq.* (2.5) *on* $\mathbb{R}^2\times\mathbb{R}$ *which satisfies conditions* (2.6) *and* (2.7)*. Then, U is a gradient soliton of the Ricci flow of the form*

$$U(x,\tau) = \frac{2}{\beta(|x|^2 + \delta e^{2\beta t})} \tag{2.9}$$

for some $\delta > 0$ and $\beta > 0$.

The above classification result has been recently shown by the first author and N. Sesum [7], without the assumption of radial symmetry and under certain necessary geometric assumptions. For the completeness of this work and the convenience of the reader we present here its simpler proof in the radially symmetric case.

Under the additional assumptions that the scalar curvature R is uniformly bounded on $\mathbb{R}^2\times\mathbb{R}$ and assumes its maximum at an interior point (x_0,t_0), with $-\infty < t_0 < \infty$, i.e., $R(x_0,t_0) = \max_{(x,t)\in\mathbb{R}\times\mathbb{R}^2} R(x,t) < \infty$, Theorem 2.2 follows from the result of R. Hamilton in [11], which also holds in the non-radial case. However, since in general $\partial R/\partial t \geqslant 0$, without this rather restrictive assumption, Hamilton's result does not apply.

Before we begin with the proof of Theorem 2.2, let us give a few remarks.

Remarks.

(i) The assumption (2.6) is necessary to rule out constant solutions, which appear as Type I blow up limits. We will show in Section 3 that Type II blow up limits satisfy condition (2.6).

(ii) Any eternal solution of Eq. (2.5) which satisfies condition (2.6) has $R > 0$. This is an immediate consequence of the Aronson–Bénilan inequality, which in the case of a solution on $\mathbb{R}^2\times[\tau,t)$ states as $u_t \leqslant u/(t-\tau)$. Letting, $\tau\to-\infty$, we obtain for a solution U of (2.5), the time derivative bound $U_t \leqslant 0$, which is equivalent to $R \geqslant 0$. Since, R evolves by $R_t = \Delta_g R + R^2$ the strong maximum principle guarantees that $R > 0$ or $R \equiv 0$ at all times. Solutions with $R \equiv 0$ (flat) violate condition (2.6). Hence, $R > 0$ at all $(x,t)\in\mathbb{R}^2\times\mathbb{R}$.

(iii) The proof of Theorem 2.2 relies heavily on the Harnack inequality satisfied by the curvature R, shown by R. Hamilton [12,13]. In the case of eternal solutions U of (2.5) with bounded curvature it states as

$$\frac{\partial \log R}{\partial t} \geqslant |D_g \log R|^2 \tag{2.10}$$

where $D_g R$ denotes the gradient with respect to the metric $g = U(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$. Equivalently, this gives the inequality

$$\frac{\partial R}{\partial t} \geqslant \frac{|DR|^2}{RU}. \tag{2.11}$$

Assuming that U is radially symmetric solution of Eq. (2.5), we perform the cylindrical change of coordinates

$$V(s,t) = r^2 U(r,t), \quad s = \log r \tag{2.12}$$

and notice once more that V satisfies the one-dimensional equation

$$\frac{\partial V}{\partial t} = (\log V)_{ss}, \quad (s,t) \in \mathbb{R} \times \mathbb{R}. \tag{2.13}$$

In addition the curvature R is given, in terms of V, by

$$R = -\frac{(\log V)_{ss}}{V}$$

so that the condition $R > 0$ implies that the function $\log V$ is concave. We will state in the next lemma several properties of the function V which will be used in the proof of Theorem 2.2.

Lemma 2.3. *The solution V of* (2.13) *enjoys the following properties*:

(i) $\max_{(s,t)\in\mathbb{R}^2} V(s,t) < \infty$.
(ii) *The limit* $\lim_{s\to+\infty}(\log V(s,t))_s = 0$, *for all* t.
(iii) *The limit* $C_\infty(t) := \lim_{s\to+\infty} V(s,t) > 0$, *for all* t.
(iv) *The limit* $\lim_{s\to+\infty}(\log V(s,t))_{ss} = 0$, *for all* t.

Proof. The fact that $V(\cdot,t)$ is bounded is a direct consequence of the width bound (2.6). The rest of the properties are easy consequences of the inequality $(\log V)_{ss} \leqslant 0$, the L^∞ bound on V and the infinite area condition (2.8). Let us prove (ii). Since $(\log V)_{ss} \leqslant 0$ either $\lim_{s\to+\infty}(\log V(s,t))_s = a$ or it is $-\infty$. The number a cannot be positive, otherwise V wouldn't be bounded. If $a < 0$ or $a = -\infty$ then for $s \gg 1$, $\log V(s,t) \leqslant -\mu s$, for some $\mu > 0$ which would imply that $U(r,t) \leqslant 1/r^{2+\mu}$, therefore integrable contradicting (2.8). Hence $\lim_{s\to+\infty}(\log V(s,t))_s = 0$, as desired.

Since $(\log V)_s$ is decreasing in s, (ii) implies that $(\log V)_s > 0$ for all s. Hence, the bound on V implies that the limit $C_\infty(t) = \lim_{s\to+\infty} V(s,t)$ exist and it is strictly positive. Note that $C_\infty(t)$ is the circumference at infinity of $\mathbb{R}^2$ with respect to the metric $\mathrm{d}s^2 = U(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$.

Since, $C_\infty(t) > 0$, the last property $\lim_{s\to+\infty}(\log V(s,t))_{ss} = 0$ is equivalent to $\lim_{r\to\infty} R(r,t) = 0$ and will be shown separately in the following lemma. □

Lemma 2.4. *Under the assumptions of Theorem* 2.2 *we have*

$$\lim_{r\to\infty} R(r,t) = 0, \quad \forall t.$$

Proof. We first observe that

$$\lim_{k\to\infty} \inf\{R(r,t)\colon 2^k \leqslant r \leqslant 2^{k+1}\} = 0, \quad \forall t \in \mathbb{R}. \tag{2.14}$$

Indeed, since $R = -(\log V)_{ss}/V$ and V is bounded from below away from zero near $+\infty$, the latter is equivalent to

$$\lim_{k\to\infty} \inf\{|(\log V)_{ss}|\colon k \leqslant s \leqslant k+1\} = 0$$

which readily follows from the fact that $(\log V)_{ss}$ is negative and integrable.

To show that actually $\lim_{r\to\infty} R(r,t) = 0$, we use the Harnack inequality (2.10). Let (x_1,t_1), (x_2,t_2) be any two points in $\mathbb{R}^2 \times \mathbb{R}$, with $t_2 > t_1$. Integrating (2.11) along the path $x(t) = x_1 + \frac{t-t_1}{t_2-t_1} x_2$, also using the bound (2.6), we find the more standard in PDE Harnack inequality

$$R(x_2,t_2) \geqslant R(x_1,t_1) \mathrm{e}^{-C(\frac{|x_2-x_1|^2}{|x_1|^2(t_2-t_1)})} \tag{2.15}$$

which in particular implies that

$$\lim_{k\to\infty} \sup\{R(r,t)\colon 2^k \leqslant r \leqslant 2^{k+1}\} \leqslant C \lim_{k\to\infty} \inf\{R(r,t+1)\colon 2^k \leqslant r \leqslant 2^{k+1}\}$$

therefore, combined with (2.14), showing that $\lim_{r\to\infty} R(r,t)=0$, as desired. □

Combining the above with classical derivative estimates for linear strictly parabolic equations, gives the following.

Lemma 2.5. *Under the assumptions of Theorem* 2.2 *we have*

$$\lim_{r\to\infty} r R_r(r,t) = 0, \quad \forall t\in\mathbb{R}.$$

Proof. For any $\rho>1$ we set $\widetilde{R}(r,t)=R(\rho r,t)$ and we compute from the evolution equation $R_t = U^{-1}\Delta R + R^2$ of R, that

$$\widetilde{R}_t = \left(\rho^2 U\right)^{-1}\Delta\widetilde{R} + \widetilde{R}^2.$$

Fix $\tau\in\mathbb{R}$ and consider the cylinder $Q=\{(r,t)\colon 1/2\leqslant r\leqslant 4,\ \tau-1\leqslant t\leqslant\tau\}$. Lemma 2.3 implies that $0<c(\tau)\leqslant \rho^2 u(r,t)\leqslant C(\tau)$, on Q, hence $\widetilde{R}$ satisfies a uniformly parabolic equation in Q. Classical derivative estimates then imply that

$$\left|(\widetilde{R})_r(r,t)\right| \leqslant C\|\widetilde{R}\|_{L^\infty(Q)}$$

for all $1\leqslant r\leqslant 2$, $\tau-1/2\leqslant t\leqslant\tau$, showing in particular that

$$\rho\left|R_r(r,\tau)\right| \leqslant C\|R\|_{L^\infty(Q_\rho)}$$

for all $\rho\leqslant r\leqslant 2\rho$, where $Q_\rho=\{(r,t)\colon \rho/2\leqslant r\leqslant 4\rho,\ \tau-1\leqslant t\leqslant\tau\}$. The proof now follows from Lemma 2.4. □

Proof of Theorem 2.2. Most of the computations below are known in the case that $U(\mathrm{d}x_1^2+\mathrm{d}x_2^2)$ defines a metric on a compact surface, see for instance [5]. However, in the non-compact case we deal with, exact account of the boundary terms at infinity should be made.

We begin by integrating the Harnack Inequality $R_t \geqslant |DR|^2/RU$ with respect to the measure $\mathrm{d}\mu = U\,\mathrm{d}x$. Since the measure $\mathrm{d}\mu$ has infinite area, we will integrate over a fixed ball B_ρ. At the end of the proof we will let $\rho\to\infty$. Using also that $R_t = U^{-1}\Delta R + R^2$ we find

$$\int_{B_\rho} \Delta R\,\mathrm{d}x + \int_{B_\rho} R^2 U\,\mathrm{d}x \geqslant \int_{B_\rho} \frac{|DR|^2}{R}\,\mathrm{d}x$$

and by Green's Theorem we conclude

$$\int_{B_\rho} \frac{|DR|^2}{R}\,\mathrm{d}x - \int_{B_\rho} R^2 U\,\mathrm{d}x \leqslant \int_{\partial B_\rho} \frac{\partial R}{\partial \nu}\,\mathrm{d}\sigma. \tag{2.16}$$

Following Chow [5], we consider the vector $X=\nabla R + R\nabla f$, where $f=-\log U$ is the potential function (defined up to a constant) of the scalar curvature, since it satisfies $\Delta_g f = R$, with $\Delta_g f = U^{-1}\Delta f$ denoting the Laplacian with respect to the conformal metric $g=U(\mathrm{d}x_1^2+\mathrm{d}x_2^2)$. As it was observed in [5], $X\equiv 0$ on Ricci solitons, i.e., Ricci solitons are gradient solitons in the direction of $\nabla_g f$. A direct computation shows

$$\int_{B_\rho} \frac{|X|^2}{R}\,\mathrm{d}x = \int_{B_\rho} \frac{|DR|^2}{R}\,\mathrm{d}x + 2\int_{B_\rho} \nabla R\cdot\nabla f\,\mathrm{d}x + \int_{B_\rho} R|Df|^2\,\mathrm{d}x.$$

Integration by parts implies

$$\int_{B_\rho} \nabla R\cdot\nabla f\,\mathrm{d}x = -\int_{B_\rho} R\Delta f\,\mathrm{d}x + \int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,\mathrm{d}\sigma = -\int_{B_\rho} R^2 U\,\mathrm{d}x + \int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,\mathrm{d}\sigma$$

since $\Delta f = RU$. Hence

$$\int_{B_\rho} \frac{|X|^2}{R}\,dx = \int_{B_\rho} \frac{|DR|^2}{R}\,dx - 2\int_{B_\rho} R^2U\,dx + \int_{B_\rho} R|Df|^2\,dx + 2\int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,d\sigma. \tag{2.17}$$

Combining (2.16) and (2.17) we find that

$$\int_{B_\rho} \frac{|X|^2}{R}\,dx \leqslant -\Big(\int_{B_\rho} R^2U\,dx - \int_{B_\rho} R|Df|^2\,dx\Big) + I_\rho = -M + I_\rho \tag{2.18}$$

where

$$I_\rho = \int_{\partial B_\rho} \frac{\partial R}{\partial n}\,d\sigma + 2\int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,d\sigma.$$

Lemmas 2.3–2.5 readily imply that

$$\lim_{\rho\to\infty} I_\rho = 0. \tag{2.19}$$

As in [5], we will show next that $M \geqslant 0$ and indeed a complete square which vanishes exactly on Ricci solitons. To this end, we define the matrix

$$M_{ij} = D_{ij}f + D_i f D_j f - \frac{1}{2}\big(|Df|^2 + Ru\big)I_{ij}$$

with I_{ij} denoting the identity matrix. A direct computation shows that $M_{ij} = \nabla_i\nabla_j f - \frac{1}{2}\Delta_g f g_{ij}$, with ∇_i denoting covariant derivatives. It is well known that the Ricci solitons are characterized by the condition $M_{ij} = 0$ (see in [12]).

Claim.

$$M := \int_{B_\rho} R^2U\,dx - \int_{B_\rho} R|Df|^2\,dx = 2\int_{B_\rho} |M_{ij}|^2\frac{1}{U}\,dx + J_\rho \tag{2.20}$$

where

$$\lim_{\rho\to\infty} J_\rho = 0.$$

To prove the claim we first observe that since $\Delta f = RU$

$$\int_{B_\rho} R^2U = \int_{B_\rho} \frac{(\Delta f)^2}{U}\,dx = \int_{B_\rho} D_{ii}f D_{jj}f\frac{1}{U}\,dx.$$

Integrating by parts and using again that $\Delta f = RU$, we find

$$\int_{B_\rho} D_{ii}f D_{jj}f\frac{1}{U}\,dx = -\int_{B_\rho} D_{jii}f D_j f\frac{1}{U}\,dx + \int_{B_\rho} \Delta f D_j f\frac{D_jU}{U^2}\,dx + \int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,d\sigma.$$

Integrating by parts once more we find

$$\int_{B_\rho} D_{jii}f D_j f\frac{1}{U}\,dx = -\int_{B_\rho} |D_{ij}f|^2\frac{1}{U}\,dx + \int_{B_\rho} D_{ij}f D_j f\frac{D_iU}{U^2}\,dx + \frac{1}{2}\int_{\partial B_\rho} \frac{\partial(|Df|^2)}{\partial n}\frac{1}{U}\,d\sigma$$

since

$$\int_{\partial B_\rho} D_{ij}f D_j f n_i\frac{1}{U}\,d\sigma = \frac{1}{2}\int_{\partial B_\rho} \frac{\partial(|Df|^2)}{\partial n}\frac{1}{U}\,d\sigma.$$

Combining the above and using that $Df = -U^{-1}DU$ and $\Delta f = RU$ we conclude

$$\int_{B_\rho} R^2 U \,\mathrm{d}x = \int_{B_\rho} |D_{ij} f|^2 \frac{1}{U}\,\mathrm{d}x + \int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x - \int_{B_\rho} R|Df|^2\,\mathrm{d}x + J_\rho^1 \tag{2.21}$$

where

$$J_\rho^1 = \int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,\mathrm{d}\sigma - \frac{1}{2}\int_{\partial B_\rho} \frac{\partial(|Df|^2)}{\partial n}\frac{1}{U}\,\mathrm{d}\sigma.$$

Hence

$$M = \int_{B_\rho} |D_{ij} f|^2 \frac{1}{U}\,\mathrm{d}x + \int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x - 2\int_{B_\rho} R|Df|^2\,\mathrm{d}x + J_\rho^1. \tag{2.22}$$

We will now integrate $|M_{ij}|^2$. A direct computation and $\Delta f = RU$ imply

$$\int_{B_\rho} |M_{ij}|^2 \frac{1}{U}\,\mathrm{d}x = \int_{B_\rho} |D_{ij} f|^2 \frac{1}{U}\,\mathrm{d}x + 2\int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x - \int_{B_\rho} R|Df|^2\,\mathrm{d}x$$
$$+ \frac{1}{2}\int_{B_\rho} |Df|^4 \frac{1}{U}\,\mathrm{d}x - \frac{1}{2}\int_{B_\rho} R^2 U\,\mathrm{d}x. \tag{2.23}$$

Combining (2.22) and (2.23) we then find

$$M - 2\int_{B_\rho} |M_{ij}|^2 \frac{1}{U}\,\mathrm{d}x = -\int_{B_\rho} |D_{ij} f|^2 \frac{1}{U}\,\mathrm{d}x - 3\int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x - \int_{B_\rho} |Df|^4 \frac{1}{U}\,\mathrm{d}x + \int_{B_\rho} R^2 U\,\mathrm{d}x + J_\rho^1.$$

Using (2.21) we then conclude that

$$M - 2\int_{B_\rho} |M_{ij}|^2 \frac{1}{U}\,\mathrm{d}x = -2\int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x - \int_{B_\rho} |Df|^4 \frac{1}{U}\,\mathrm{d}x - \int_{B_\rho} R|Df|^2\,\mathrm{d}x + J_\rho^2 \tag{2.24}$$

where

$$J_\rho^2 = \int_{\partial B_\rho} R\frac{\partial f}{\partial n}\,\mathrm{d}\sigma - \int_{\partial B_\rho} \frac{\partial(|Df|^2)}{\partial n}\frac{1}{U}\,\mathrm{d}\sigma.$$

We next observe that

$$2\int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x = \int_{B_\rho} D_i\big(|Df|^2\big) D_i f \frac{1}{U}$$

and integrate by parts using once more that $\Delta f = RU$ and that $D_i f = -U^{-1} D_i f$, to find

$$2\int_{B_\rho} D_{ij} f D_i f D_j f \frac{1}{U}\,\mathrm{d}x = -\int_{B_\rho} R|Df|^2\,\mathrm{d}x - \int_{B_\rho} |Df|^4 \frac{1}{U}\,\mathrm{d}x + J_\rho^3$$

where

$$J_\rho^3 = \lim_{\rho\to\infty} \int_{\partial B_\rho} |Df|^2 \frac{\partial f}{\partial n}\,\mathrm{d}\sigma.$$

Combining the above we conclude that

$$M - 2\int_{B_\rho} |M_{ij}|^2 \frac{1}{U}\,\mathrm{d}x = J_\rho$$

with

$$J_\rho = \int\limits_{\partial B_\rho} R\frac{\partial f}{\partial n}\,\mathrm{d}\sigma - \int\limits_{\partial B_\rho}\left(\frac{\partial(|Df|^2)}{\partial n} + |Df|^2\frac{\partial f}{\partial n}\right)\frac{1}{U}\,\mathrm{d}\sigma.$$

We will now show that $\lim_{\rho\to\infty} J_\rho = 0$. Clearly the first term tends to zero, because $r|Df(r,t)| = |2-(\log V(s,t))_s|$ is bounded by Lemma 2.3 and $R(r,t)\to 0$, as $r\to\infty$, by Lemma 2.4.

It remains to show that

$$\lim_{r\to\infty}\left(\frac{\partial(|Df|^2)}{\partial r} + |Df|^2\frac{\partial f}{\partial r}\right)\frac{r}{U} = 0.$$

To this end, we observe that since $f = -\log U$ and $V(s,t) = r^2U(r,t)$, with $s = \log r$

$$\left(\frac{\partial(|Df|^2)}{\partial r} + |Df|^2\frac{\partial f}{\partial r}\right)\frac{r}{U} = \frac{\partial}{\partial r}\left(\frac{|Df|^2}{r^2U}\right) = \frac{\partial}{\partial s}\left(\frac{[2-(\log V)_s]^2}{V}\right)$$

and

$$\frac{\partial}{\partial s}\left(\frac{[2-(\log V)_s]^2}{V}\right) = -\big[2-(\log V)_s\big]\frac{(\log V)_{ss}}{V} + \big[2-(\log V)_s\big]^2\frac{(\log V)_s}{V}.$$

Since both $R = -(\log V)_{ss}/V$ and $(\log V)_s$ tend to zero, as $s\to\infty$, our claim follows.

We will now conclude the proof of the theorem. From (2.18) and (2.20) it follows that

$$\int\limits_{B_\rho}\frac{|X|^2}{R}\,\mathrm{d}x + 2\int\limits_{B_\rho}|M_{ij}|^2\frac{1}{U}\,\mathrm{d}x \leqslant I_\rho + J_\rho$$

where both

$$\lim_{\rho\to\infty} I_\rho + J_\rho = 0.$$

This immediately gives that $X\equiv 0$ and $M_{ij}\equiv 0$ for all t showing that U is a gradient soliton. It has been shown by L.F. Wu [21] that there are only two types of gradient solitons on $\mathbb{R}^2$ the standard flat metric ($R\equiv 0$) which is stationary and the cigar solitons (2.5). This, in the radial symmetric case can be directly shown by integrating the equality $M_{ij} = 0$. The flat solitons violate condition (2.6). Hence, U must be of the form (2.9), finishing the proof of the theorem. □

2.3. Monotonicity of solutions

We will show next that radially symmetric solutions of Eq. (1.1) with initial data satisfying conditions (1.7) and (1.8) become radially decreasing near their vanishing time, as stated in the next lemma, which will be used in the next section.

Lemma 2.6. *Assume that u is a radially symmetric maximal solution of Eq.* (1.1) *with initial data u_0 positive satisfying conditions* (1.7) *and* (1.8). *Then, there exists a number $\tau_0 < T$ such that $u(\cdot,t)$ is radially decreasing for $\tau_0 \leqslant t < T$.*

Proof. Because $u_0 > 0$ is strictly decreasing for $r\geqslant r_0$, there exists a number δ_0 with the property: for all $\delta\leqslant\delta_0$ there is exactly one r such that $u_0(r) = \delta$. It then follows that for any number $\delta\leqslant\delta_0$ the number $J(\delta,t)$ of intersections between $u(\cdot,t)$ and the constant solution $S(r,t) = \delta$ satisfies $J(\delta,t)\leqslant 1$ (see in [1]). Since $u(\cdot,t)\to 0$ uniformly as $t\to T$, there exists a time t such that $u(\cdot,t) < \delta_0$. Define

$$\tau_0 = \inf\big\{t\in(0,T)\colon\ u(r,t)\leqslant\delta_0,\ \forall r>0\big\}.$$

Clearly, we can choose δ_0 sufficiently small so that $\tau_0 > 0$. Then, for $t<\tau_0$, $J(\delta_0,t) = 1$. Assume that $r_{\delta_0}(t)$ satisfies $u(r_{\delta_0}(t),t) = \delta_0$. Since, $u(r,t)\to 0$ as $r\to\infty$, it then follows that $u(r,t) > \delta_0$ for $r < r_{\delta_0}(t)$ and $u(r,t) < \delta_0$ for $r > r_{\delta_0}(t)$, for all $t<\tau_0$. It follows that $r_{\delta_0}(\tau_0) = 0$ and by the strong maximum principle $u_r(0,\tau_0) < 0$. We claim that $u(\cdot,\tau_0)$ is strictly decreasing. If not then, there exists $\delta<\delta_0$ such that the constant solution $S(r,t)=\delta$ intersects the graph of $u(\cdot,\tau_0)$ at least twice, contradicting our choice of δ_0. Hence, $u_r(r,\tau_0)\leqslant 0$, for all $r>0$, and actually by the strong maximum principle $u_r(\cdot,\tau_0)<0$, for all r. This inequality is preserved, by the maximum principle for all $\tau_0\leqslant t<T$, finishing the proof of the lemma. □

3. Inner region convergence

This section is devoted to the proof of the inner region convergence, Theorem 1.1 stated in the introduction. We assume, throughout this section, that u is a smooth, radially symmetric maximal solution of (1.1) with initial data satisfying (1.6)–(1.8) and (1.10). Because of Lemma 2.6 we may also assume, without loss of generality, that u is radially decreasing.

We begin by introducing the appropriate scaling.

3.1. Scaling and convergence

We introduce a new scaling on the solution u. We first set

$$\bar u(x,\tau)=\tau^2u(x,t),\quad \tau=\frac{1}{T-t},\ \tau\in\left(\frac{1}{T},\infty\right). \tag{3.1}$$

Then $\bar u$ satisfies the equation

$$\bar u_\tau=\Delta\log\bar u+\frac{2\bar u}{\tau},\quad \text{on } 1/T\leqslant\tau<\infty. \tag{3.2}$$

Notice that under this transformation, $\overline{R}:=-\Delta\log\bar u/\bar u$ satisfies the estimate

$$\overline{R}_{\max}(\tau)\leqslant C \tag{3.3}$$

for some constant $C<\infty$. This is a direct consequence of Theorem 2.1, since $\overline{R}_{\max}(\tau)=(T-t)^2R_{\max}(t)$.

For an increasing sequence $\tau_k\to\infty$ we set

$$\bar u_k(y,\tau)=\alpha_k\bar u\big(\alpha_k^{1/2}y,\tau+\tau_k\big),\quad (y,\tau)\in\mathbb{R}^2\times\left(-\tau_k+\frac{1}{T},\infty\right) \tag{3.4}$$

where

$$\alpha_k=\big[\bar u(0,\tau_k)\big]^{-1}$$

so that $\bar u_k(0,0)=1$, for all k. Then, $\bar u_k$ satisfies the equation

$$\bar u_\tau=\Delta\log\bar u+\frac{2\bar u}{\tau+\tau_k}. \tag{3.5}$$

Let

$$\overline{R}_k:=-\frac{\Delta\log\bar u_k}{\bar u_k}.$$

Then, by (3.3), we have

$$\max_{y\in\mathbb{R}^2}\overline{R}_k(y,\tau)\leqslant C,\quad -\tau_k+\frac{1}{T}<\tau<+\infty. \tag{3.6}$$

We will also derive a global bound from bellow on $\overline{R}_k$. The Aronson–Benilán inequality $u_t\leqslant u/t$, on $0\leqslant t<T$ gives the bound $R(x,t)\geqslant -1/t$ on $0\leqslant t<T$. In particular, $R(x,t)\geqslant -C$ on $T/2\leqslant t<T$, which in the new time variable $\tau=1/(T-t)$ implies the bound

$$\overline{R}(x,\tau)\geqslant -\frac{C}{\tau^2},\quad \frac{2}{T}<\tau<\infty.$$

Hence

$$\overline{R}_k(y,\tau) \geqslant -\frac{C}{(\tau+\tau_k)^2}, \quad -\tau_k + \frac{2}{T} < \tau < +\infty.$$

Combining the above inequalities we get

$$-\frac{C}{(\tau+\tau_k)^2} \leqslant \overline{R}_k(y,\tau) \leqslant C, \quad \forall (y,\tau) \in \mathbb{R}^2 \times \left(-\tau_k + \frac{2}{T}, +\infty\right). \tag{3.7}$$

Also, the width bound (2.3), implies the bound

$$\max_{y\in\mathbb{R}^2} |y|^2 \bar{u}_k(y,\tau) \leqslant C, \quad \forall (y,\tau) \in \mathbb{R}^2 \times \left(-\tau_k + \frac{2}{T}, +\infty\right). \tag{3.8}$$

Based on the above estimates we will now show the following convergence result.

Lemma 3.1. *For each sequence $\tau_k \to \infty$, there exists a subsequence τ_{k_l} of τ_k, for which the rescaled solution $\bar{u}_{\tau_{k_l}}$ defined by* (3.4) *converges, uniformly on compact subsets of* $\mathbb{R}^2 \times \mathbb{R}$, *to an eternal solution U of equation $U_\tau = \Delta \log U$ on $\mathbb{R}^2 \times \mathbb{R}$ with uniformly bounded curvature and uniformly bounded width. Moreover, the convergence is in $C^\infty(K)$, for any $K \subset \mathbb{R}^2 \times \mathbb{R}$ compact.*

Proof. Since $\bar{u}_k(0,0) = 1$ with $\bar{u}_k(\cdot,0) \leqslant 1$ (because each $u_k(\cdot,t)$ is radially decreasing) one may use standard arguments to show that $\bar{u}_k$ is uniformly bounded from above and below away from zero on any compact subset of $\mathbb{R}^2 \times \mathbb{R}$. Hence, by the classical regularity theory the sequence $\{\bar{u}_k\}$ is equicontinuous on compact subsets of $\mathbb{R}^2 \times \mathbb{R}$. It follows, that there exists a subsequence τ_{k_l} of τ_k such that $\bar{u}_{k_l} \to U$ on compact subsets of $\mathbb{R}^2 \times \mathbb{R}$, where U is an eternal solution of equation

$$U_\tau = \Delta \log U, \quad \text{on } \mathbb{R}^2 \times \mathbb{R} \tag{3.9}$$

with infinite area $\int_{\mathbb{R}^2} U(y,\tau) = \infty$ (since $\int_{\mathbb{R}^2} \bar{u}_k(y,\tau)\,\mathrm{d}y = 2(\tau+\tau_k)$). In addition the classical regularity theory of quasilinear parabolic equations implies that $\{u_{k_l}\}$ can be chosen so that $u_{k_l} \to U$ in $C^\infty(K)$, for any compact set $K \subset \mathbb{R}^2 \times (-\infty,\infty)$.

It then follows that $\overline{R}_{k_l} \to \widetilde{R} := -(\Delta \log U)/U$. Taking the limit $k_l \to \infty$ on both sides of (3.7) we obtain the bounds

$$0 \leqslant \widetilde{R} \leqslant C, \quad \text{on } \mathbb{R}^2 \times (-\infty,\infty). \tag{3.10}$$

Finally, to show that U has uniformly bounded width, we take the limit $k_l \to \infty$ in (3.8). □

A direct consequence of Lemma 3.1 and Theorem 2.2 is the following convergence result.

Theorem 3.2. *For each sequence $\tau_k \to \infty$, there exists a subsequence τ_{k_l} of τ_k and a number $\lambda > 0$ for which the rescaled solution $\bar{u}_{\tau_{k_l}}$ defined by* (3.4) *converges, uniformly on compact subsets of* $\mathbb{R}^2 \times \mathbb{R}$ *to the soliton solution U_λ of the Ricci Flow given by*

$$U(y,\tau) = \frac{1}{\lambda|y|^2 + \mathrm{e}^{4\lambda\tau}}. \tag{3.11}$$

Moreover, the convergence is in $C^\infty(K)$, for any $K \subset \mathbb{R}^2 \times \mathbb{R}$, compact.

Proof. From Lemma 3.1, $\bar{u}_{\tau_{k_l}} \to U$, where U is an eternal solution of Eq. (2.5), which satisfies the bounds (2.6) and (2.7). Applying Theorem 2.2 shows that the limiting solution U is a soliton of the form $U(y,\tau) = 2/\beta(|x|^2 + \delta \mathrm{e}^{2\beta t})$, with $\beta > 0$, $\delta > 0$, which under the condition $U(0,0) = 1$ takes the form (3.11), with $\lambda > 0$. □

Remark. The proof of Theorem 3.2 did not utilize the lower bound $R_{\max}(t) \geqslant c(T-t)^{-2} > 0$ proven in Theorem 2.1, which in particular shows that the blow up is of Type II. The Type II blow up is implicitly implied by the upper bound on the width (2.3).

3.2. Further behavior

We will now use the geometric properties of the rescaled solutions and their limit, to further analyze their vanishing behavior.

We begin by observing that rescaling back in the original (x,t) variables, Theorem 3.2 gives the following asymptotic behavior of the maximal solution u of (1.1).

Lemma 3.3. *Assuming that along a sequence $t_k \to T$, the sequence $\bar{u}_k$ defined by* (3.4) *with $\tau_k = (T - t_k)^{-1}$ converges to the soliton solution U_λ, on compact subsets of $\mathbb{R}^2 \times \mathbb{R}$, then along the sequence t_k the solution $u(x,t)$ of* (1.1) *satisfies the asymptotics*

$$u(x,t_k) \approx \frac{(T-t_k)^2}{\lambda|x|^2 + \alpha_k}, \quad \text{on } |x| \leqslant \alpha_k^{1/2} M \tag{3.12}$$

for all $M > 0$. In addition, the curvature $R(0,t_k) = -\Delta \log u(0,t_k)/u(0,t_k)$ satisfies

$$\lim_{t_k \to T} (T-t_k)^2 R(0,t_k) = 4\lambda. \tag{3.13}$$

Proof. From Lemma 3.1 we have

$$\alpha_k \bar{u}_k\big(\alpha_k^{1/2} y, \tau + \tau_k\big) \approx \frac{1}{\lambda|y|^2 + \mathrm{e}^{4\lambda\tau}}$$

for $|y| \leqslant M$, $|\tau| \leqslant M^2$, i.e., in terms of the variable x

$$\bar{u}(x,\tau+\tau_k) \approx \frac{\alpha_k^{-1}}{\lambda\alpha_k^{-1}|x|^2 + \mathrm{e}^{4\lambda(\tau)}} \approx \frac{1}{\lambda|x|^2 + \alpha_k \mathrm{e}^{4\lambda\tau}}$$

for $|x| \leqslant \alpha_k^{1/2} M$, $|\tau - \tau_k| \leqslant M^2$. In particular, when $\tau = 0$ this gives

$$\bar{u}(x,\tau_k) \approx \frac{1}{\lambda|x|^2 + \alpha_k}$$

for $|x| \leqslant \alpha_k^{1/2} M$, which in terms of the original variables gives

$$u(x,t_k) \approx \frac{(T-t_k)^2}{\lambda|x|^2 + \alpha_k}$$

for $|x| \leqslant \alpha_k^{1/2} M$, as desired. Since $\overline{R}_k(0,\tau)$ converges to the curvature of the cigar U_λ at the origin (its maximum curvature) and this is equal to 4λ, the limit (3.13) follows by simply observing that $(T-t_k)^2 R(0,t_k) = \overline{R}_k(0,0)$. □

The following lemma provides a sharp bound from below on the maximum curvature 4λ of the limiting solitons.

Lemma 3.4. *Under the assumptions of Theorem* 1.1 *the constant λ in each limiting solution* (3.11) *satisfies*

$$\lambda \geqslant \frac{T+\mu}{2}.$$

Proof. We are going to use the estimate proven in Section 2 of [9]. There it is shown that if at time t the solution u of (1.1) satisfies the scalar curvature bound $R(t) \geqslant -2k(t)$, then the width $W(t)$ of the metric $u(t)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$ (cf. in Section 2.1 for the definition) satisfies the bound

$$W(t) \leqslant \sqrt{k(t)}\, A(t) = 4\pi\sqrt{k(t)}\,(T-t).$$

Here $A(t) = 4\pi(T-t)$ denotes the area of the plane with respect to the conformal metric $u(t)(\mathrm{d}x_1^2 + \mathrm{d}x_2^2)$. Observing that for radially symmetric u the width $W(t) = \max_{r \geqslant 0} 2\pi r\sqrt{u}(r,t)$ we conclude the pointwise estimate

$$r\sqrt{u}(r,t) \leqslant 2\sqrt{k(t)}\,(T-t), \quad r \geqslant 0,\ 0 < t < T \tag{3.14}$$

when $R(x,t) = -\Delta \log u/u \geqslant -2k(t)$, for all x.

Observe next that the initial curvature lower bound (1.10) implies the bound

$$R(x,t) \geqslant -\frac{1}{t+\mu}.$$

This easily follows by the maximum principle, since $-1/(t+\mu)$ is an exact solution of the scalar equation $R_t = \Delta_g R + R^2$. Hence, we can take $k(t) = 1/2(t+\mu)$ in (3.14) and conclude the bound

$$r\sqrt{u}(r,t) \leqslant \frac{2(T-t)}{\sqrt{2(t+\mu)}}, \quad r \geqslant 0,\ 0 < t < T. \tag{3.15}$$

Assume now that $\{t_k\}$ is a sequence $t_k \to T$. Using (3.12) in (3.15) we find

$$\frac{r(T-t_k)}{\sqrt{\lambda r^2 + \alpha_k}} \leqslant \frac{2(T-t_k)}{\sqrt{2(t_k+\mu)}}, \quad r \leqslant M\alpha_k^{1/2}$$

where M is any positive constant. Hence, when $r = M\alpha_k^{1/2}$ we obtain the estimate

$$\frac{M\alpha_k^{1/2}}{\sqrt{\lambda M^2\alpha_k + \alpha_k}} \leqslant \frac{2}{\sqrt{2(t_k+\mu)}}$$

or

$$\frac{1}{\sqrt{\lambda + 1/M^2}} \leqslant \frac{2}{\sqrt{2(t_k+\mu)}}.$$

Letting $t_k \to T$ and taking squares on both sides, we obtain

$$\frac{1}{\lambda + 1/M^2} \leqslant \frac{2}{T+\mu}.$$

Since $M > 0$ is an arbitrary number, we finally conclude $\lambda \geqslant (T+\mu)/2$, as desired. □

We will next provide a bound on the behavior of $\alpha(\tau) = \tau^2 \bar{u}(0,\tau)$, as $\tau \to \infty$. In particular, we will prove (1.15). Notice that since

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} = (T-t)^2 \alpha(\tau)^{-1}, \quad \tau = 1/(T-t)$$

this bound shows the vanishing behavior of $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$, as $t \to T$. We begin by a simple consequence of Lemma 3.4.

Lemma 3.5. *Under the assumptions of Theorem* 3.2 *we have*

$$\liminf_{\tau\to\infty} \frac{\alpha'(\tau)}{\alpha(\tau)} \geqslant 4\lambda_\mu \tag{3.16}$$

with $\lambda_\mu = (T+\mu)/2$.

Proof. We argue by contradiction. If (3.16) does not hold, then there exists a sequence $\tau_k \to \infty$ for which

$$\lim_{k\to\infty} \frac{\alpha'(\tau_k)}{\alpha(\tau_k)} < 4\lambda_\mu. \tag{3.17}$$

Next notice that by the definition of $\alpha(\tau)$ we have

$$\big[\log \alpha(\tau)\big]_\tau = -\big[\log \bar{u}(0,\tau)\big]_\tau = -\frac{\Delta \log \bar{u}}{\bar{u}} - \frac{2}{\tau} = \overline{R}(0,\tau) - \frac{2}{\tau}. \tag{3.18}$$

Now, because of Theorem 3.2, may assume without loss of generality that, as $\tau_k \to \infty$, we have $\bar{u}_k(y,\tau) \to 1/(\lambda|y|^2 + e^{4\lambda\tau})$, with $\bar{u}_k$ given by (3.4) and for some constant λ which according to Lemma 3.4 satisfies the inequality $\lambda \geqslant \lambda_\mu$. But then

$$\lim_{k\to\infty} \overline{R}(0,\tau_k) = 4\lambda \geqslant 4\lambda_\mu$$

which in combination with (3.18) contradicts (3.17), finishing the proof of the lemma. □

Corollary 3.6. *Under the hypotheses of Theorem* 3.2, *we have*

$$\alpha(\tau) \geqslant \mathrm{e}^{4\lambda_\mu \tau + \mathrm{o}(\tau)}, \quad \text{as } \tau \to \infty \tag{3.19}$$

with $\lambda_\mu = (T+\mu)/2$.

Proof. By the previous lemma we have

$$\big[\log \alpha(\tau)\big]_\tau \geqslant 4\lambda_\mu + \mathrm{o}(1), \quad \text{as } \tau \to \infty$$

which implies that

$$\log \alpha(\tau) \geqslant 4\lambda_\mu \tau + \mathrm{o}(\tau), \quad \text{as } \tau \to \infty$$

showing the corollary. □

We will now show (1.15) as stated in the next proposition. This bound will be crucial in establishing the outer region behavior of u.

Proposition 3.7. *Under the hypotheses of Theorem* 1.1, *we have*

$$\lim_{\tau\to\infty} \frac{\log \alpha(\tau)}{\tau} = 4\lambda_\mu \tag{3.20}$$

with $\lambda_\mu = (T+\mu)/2$.

Proof. We argue by contradiction. If (3.20) does not hold, then by Corollary 3.6 there exists a sequence $\tau_k \to \infty$ for which

$$\lim_{k\to\infty} \frac{\log \alpha(\tau_k)}{\tau_k} = 4\bar\lambda > 4\lambda_\mu.$$

Because of Theorem 3.2, we may assume that for the same sequence τ_k, we have

$$\alpha_k \bar u\big(\alpha_k^{1/2} y, \tau_k\big) \to \frac{1}{\lambda |y|^2 + 1}, \quad |y| \leqslant 1,\ |\tau| \leqslant 1$$

for some number $\lambda > \lambda_\mu$, with

$$\alpha_k = \alpha(\tau_k) = \mathrm{e}^{4\bar\lambda \tau_k + \mathrm{o}(\tau_k)}.$$

This implies the asymptotics

$$\bar u(x, \tau_k) \approx \frac{1}{\lambda |x|^2 + \mathrm{e}^{4\bar\lambda\tau_k}}, \quad |x| \leqslant \mathrm{e}^{2\bar\lambda \tau_k - \mathrm{o}(\tau_k)}. \tag{3.21}$$

We next perform the change of variables (1.16)–(1.18), namely $v(s,t) = r^2 u(r,t)$, $s = \log r$, and $\tilde v(\xi,\tau) = \tau^2 v(\tau\xi, t)$, $\tau = 1/(T-t)$. As we noted in the Introduction, the rescaled solution $\tilde v$ satisfies Eq. (1.19). An important for our purposes observation, is that the new scaling makes the area of $\tilde v$ to be constant in time, since

$$\int_{-\infty}^{\infty} \tilde v(\xi,\tau)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} \tau^2 v(\tau\xi, t)\,\mathrm{d}\xi = \tau \int_{-\infty}^{\infty} v(x,t)\,\mathrm{d}x = 2. \tag{3.22}$$

Here we have used

$$\int_{-\infty}^{\infty} v(s,t)\,\mathrm{d}s = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x,t)\,\mathrm{d}x = 2(T-t).$$

Claim. $\tilde v(\xi, \tau_k) \to 0$, *uniformly on* $(-\infty, \bar\xi]$, *for any* $\bar\xi < 2\bar\lambda$.

Indeed, expressing (3.21) in terms of $\tilde{v}$ gives

$$\tilde{v}(\xi,\tau_k) \approx \frac{e^{2\tau_k \xi}}{\lambda e^{2\tau_k \xi} + e^{4\bar{\lambda}\tau_k}}, \quad \xi < 2\bar{\lambda}$$

which immediately implies the claim.

To finish the proof of the proposition, assuming that $\bar{\lambda} > \lambda_\mu$, we choose $\bar{\xi}$ such that $2\lambda_\mu < \bar{\xi} < 2\bar{\lambda}$ so that $\tilde{v}(\xi,\tau_k) \to 0$, uniformly on $(-\infty,\bar{\xi}]$. We will show that this violates the area condition

$$\int_{-\infty}^{\infty} \tilde{v}(\xi,\tau)\,d\tau = 2.$$

We first observe that there exists constants s_0 and $\bar{s}$, independent of t, such that

$$v(s,t) \leqslant \frac{2(t+\mu)}{(s+\bar{s})^2}, \quad s \geqslant s_0.$$

This readily follows from initial condition (1.8) and the maximum principle, since $2(t+\mu)/(s+\bar{s})^2$ is an exact solution of Eq. (1.17). This, in terms of $\tilde{v}$, gives the bound

$$\tilde{v}(\xi,\tau) \leqslant \frac{2(T+\mu-1/\tau)}{(\xi+\tau^{-1}\bar{s})^2}.$$

Also

$$2 = \int_{-\infty}^{\infty} \tilde{v}(\xi,\tau)\,d\xi = \int_{-\infty}^{\bar{\xi}} \tilde{v}(\xi,\tau)\,d\xi + \int_{\bar{\xi}}^{\infty} \tilde{v}(\xi,\tau)\,d\xi.$$

Since

$$\int_{-\infty}^{\bar{\xi}} \tilde{v}(\xi,\tau_k)\,d\xi \to 0$$

we conclude that

$$2 \leqslant \lim_{k\to\infty} \int_{\bar{\xi}}^{\infty} \frac{2(T+\mu-1/\tau_k)}{\xi^2+\tau_k^{-1}\bar{s}} = \frac{2(T+\mu)}{\bar{\xi}}.$$

This implies the bound

$$\bar{\xi} \leqslant (T+\mu) = 2\lambda_\mu$$

violating our choice of $\bar{\xi}$ to be larger than $2\lambda_\mu$, therefore finishing the proof of the proposition. □

We have actually shown the following result, which will be used in the next section.

Corollary 3.8. *Under the assumptions of Lemma* 3.1 *the rescaled solution* $\tilde{v}$ *defined by* (1.18) *satisfies*

$$\lim_{\tau\to\infty} \tilde{v}(\xi,\tau) = 0, \quad \text{uniformly on } (-\infty,\xi_\mu^-]$$

for all $\xi_\mu^- < \xi_\mu$, *with*

$$\xi_\mu = 2\lambda_\mu = T+\mu.$$

3.3. Proof of Theorem 1.1

We finish this section with the proof of Theorem 1.1 which will easily follow from the results in Sections 3.1 and 3.2. Fixing a sequence $\tau_k \to \infty$, we first observe that because of the curvature bound (2.2), $\alpha'(\tau_k)/2\alpha(\tau_k)$ is bounded from above and hence, passing to a subsequence, still denoted by τ_k, we have

$$\frac{\alpha'(\tau_k)}{\alpha(\tau_k)} \to 4\lambda \tag{3.23}$$

for some constant $\lambda < \infty$, which according to Lemma 3.5, it satisfies $\lambda \geqslant \lambda_\mu = (T+\mu)/2$.

For the same sequence τ_k, we have $\tilde{u}(y,\tau_k) = \bar{u}_k(y,0)$, with $\bar{u}_k$ defined as in (3.4). Since, by Theorem 3.2, $\bar{u}_k \to U_{\bar{\lambda}}$, for some $\bar{\lambda} \geqslant \lambda_\mu$, we conclude that $\tilde{u}(y,\tau_k) \to U_{\bar{\lambda}}(y,0) = 1/(\bar{\lambda}|y|^2+1)$. To finish the proof of the theorem, we observe that $\bar{\lambda} = \lambda$, by (3.23), since

$$\lim_{k\to\infty} \frac{\alpha'(\tau_k)}{\alpha(\tau_k)} = -\lim_{k\to\infty} \frac{\Delta \log \bar{u}_k(0,0)}{\bar{u}_k(0,0)} = -\frac{\Delta \log U_{\bar{\lambda}}(0,0)}{U_{\bar{\lambda}}(0,0)} = 4\bar{\lambda}. \qquad \square$$

4. Outer region asymptotic behavior

We assume, throughout this section, that u is a positive, smooth, radially symmetric solution of (1.1) and we consider as in Introduction the solution $v(s,t) = r^2 u(r,t)$, $s = \log r$, of the one-dimensional equation (1.17). We next set

$$\bar{v}(s,\tau) = \tau^2 v(s,t), \quad \tau = \frac{1}{T-t} \tag{4.1}$$

and

$$\tilde{v}(\xi,\tau) = \bar{v}(\tau\xi,\tau). \tag{4.2}$$

The function $\tilde{v}$ satisfies the equation

$$\tau \tilde{v}_\tau = \frac{1}{\tau}(\log \tilde{v})_{\xi\xi} + \xi \tilde{v}_\xi + 2\tilde{v}. \tag{4.3}$$

As we computed in (3.22), under the above scaling the area of $\tilde{v}$ remains constant, in particular

$$\int_{-\infty}^{\infty} \tilde{v}(\xi,\tau)\,d\xi = 2, \quad \forall \tau. \tag{4.4}$$

We shall show that, $\tilde{v}(\cdot,\tau)$ converges, as $\tau \to \infty$, to a steady state of Eq. (4.3), namely to a solution of the linear first order equation

$$\xi V_\xi + 2V = 0. \tag{4.5}$$

The area condition (4.4) shall imply that

$$\int_{-\infty}^{\infty} V(\xi)\,d\xi = 2. \tag{4.6}$$

Positive solutions of Eq. (4.5) are of the form

$$V(\xi) = \frac{\eta}{\xi^2} \tag{4.7}$$

where $\eta > 0$ is any constant. These solutions become singular at $\xi = 0$ and in particular are non-integrable at $\xi = 0$, so that they do not satisfy the area condition (4.6). However, it follows from Corollary 3.8 that V must vanish in the interior region $\xi < \xi_\mu$, with $\xi_\mu = T + \mu$. We will show that although $\tilde{v}(\xi,\tau) \to 0$, as $\tau \to \infty$ on $(-\infty,\xi_\mu)$, $\tilde{v}(\xi,\tau) \geqslant c > 0$, for $\xi > \xi_\mu$ and that actually $\tilde{v}(\xi,\tau) \to 2(T+\mu)/\xi^2$, on (ξ_μ,∞), as stated in Theorem 1.2.

The rest of the section is devoted to the proof of Theorem 1.2. We begin by showing the following properties of the rescaled solution $\tilde{v}$.

Lemma 4.1. *The rescaled solution $\tilde{v}$ given by* (4.1)–(4.2) *has the following properties*:

(i) $\tilde{v}(\xi,\tau) \leqslant C$, $\forall \xi \in \mathbb{R}$, *for a constant C independent of τ.*
(ii) *Let $\xi_\mu = T + \mu$. Then, for any $\xi_\mu^- < \xi_\mu$, $\tilde{v}(\cdot,\tau) \to 0$, as $\tau \to \infty$, uniformly on $(-\infty, \xi_\mu^-]$.*
(iii) *Let $\xi(\tau) = (\log \alpha(\tau))/2\tau$, with $\alpha(\tau)$ as in (1.12), i.e., $\alpha(\tau) = [\tau^2 u(0,t)]^{-1}$. Then, there is a constant $\eta > 0$, independent of τ, such that*

$$\tilde{v}(\xi,\tau) \geqslant \frac{\eta}{\xi^2}, \quad \text{on } \xi \geqslant \xi(\tau),\ \tau \geqslant \frac{1}{T}.$$

In addition

$$\xi(\tau) = \xi_\mu + \mathrm{o}(1), \quad \text{as } \tau \to \infty. \tag{4.8}$$

(iv) *$\tilde{v}(\xi,\tau)$ also satisfies the upper bound*

$$\tilde{v}(\xi,\tau) \leqslant \frac{C}{\xi^2}, \quad \text{on } \xi > 0,\ \tau \geqslant \frac{1}{T}$$

for some constant $C > 0$.

Proof. (i) The estimate of $\tilde{v} \leqslant C$ is a direct consequence of the width estimate (2.4).
(ii) This is shown in Corollary 3.8.
(iii)

Claim. *There is a constant $\eta > 0$ for which $\tilde{v}(\xi(\tau),\tau) \geqslant \eta$, for all $1/T \leqslant \tau < \infty$.*

To show the claim, we argue by contradiction. If it is not correct, then there exists a sequence $\tau_k \to \infty$ for which $\tilde{v}(\xi(\tau_k),\tau_k) \to 0$. Because of the interior convergence Theorem 3.2, we may assume, without loss of generality, that for the same sequence τ_k the rescaled solution $\bar{u}(x,\tau) = \tau^2 u(x,t)$, $\tau = 1/(T-t)$, defined by (3.1) satisfies the asymptotics $\bar{u}(x,\tau_k) \approx 1/(\lambda|x|^2 + \alpha_k(\tau_k))$, when $|x| \leqslant \sqrt{\alpha_k(\tau_k)}$. In particular for $|x| = \sqrt{\alpha(\tau_k)}$ we then have

$$\bar{u}\big(\sqrt{\alpha(\tau_k)}, \tau_k\big) \approx \frac{\alpha(\tau_k)^{-1}}{\lambda + 1}$$

and hence using that $\xi(\tau_k) = (\log \alpha(\tau_k))/2\tau_k$ and the transformations (4.1)–(4.2), we conclude that

$$\tilde{v}\big(\xi(\tau_k),\tau_k\big) = \mathrm{e}^{2\xi(\tau_k)\tau_k}\bar{u}\big(\mathrm{e}^{\xi(\tau_k)\tau_k},\tau_k\big) \approx \frac{1}{1+\lambda}$$

contradicting our assumption that $\tilde{v}(\xi(\tau_k),\tau_k) \to 0$, therefore proving the claim.

Let us observe next that (4.8) readily follows from Proposition 3.7. Hence, it remains to show $\tilde{v} \geqslant \eta/\xi^2$, on $[\xi(\tau),\infty)$, $1/T \leqslant \tau < \infty$. To this end, we will compare $\tilde{v}$ with the subsolution $V_\eta(\xi) = \eta/\xi^2$ of Eq. (4.3). According to our claim above, there exists a constant $\eta > 0$, so that

$$V_\eta\big(\xi(\tau)\big) = \frac{\eta}{\xi(\tau)^2} \leqslant \tilde{v}\big(\xi(\tau),\tau\big).$$

Moreover, since the initial data u_0 is strictly positive, radially decreasing and satisfies the growth condition (1.8), we can make

$$\tilde{v}_0\left(\xi, \frac{1}{T}\right) > \frac{\eta}{\xi^2}, \quad \text{on } \xi \geqslant \xi\left(\frac{1}{T}\right)$$

by choosing η sufficiently small. Hence we can apply the comparison principle on the set $\{(\xi,\tau):\ \xi \geqslant \xi(\tau), 1/T < \tau < \infty\}$, to conclude that $\tilde{v}(\xi,\tau) \geqslant \eta/\xi^2$, for $\xi \geqslant \xi(\tau)$. Since the set $\{(\xi,\tau):\ \xi \geqslant \xi(\tau), 1/T < \tau < \infty\}$ is not a cylinder, to justify the application of the maximum principle, we set $w = \tilde{v} - V_\eta$, so that w satisfies the differential inequality

$$w_\tau \geqslant \frac{1}{\tau^2}\big(A(\xi,\tau)w\big)_{\xi\xi} + \frac{1}{\tau}(\xi w_\xi + 2w), \quad \text{on } \xi \geqslant \xi(\tau)$$

with

$$A(\xi,\tau)=\begin{cases}\dfrac{\log\tilde v-\log V_\eta}{\tilde v-V_\eta}, & \tilde v\neq V_\eta,\\[2mm] \dfrac{1}{V_\eta} & \tilde v=V_\eta.\end{cases}$$

We next set $\bar w(\xi,\tau)=w(\xi+\xi(\tau),\tau)$ and we compute that $\bar w$ satisfies

$$\overline{w}_\tau\geqslant\frac{1}{\tau^2}\big(A\big(\xi+\xi(\tau),\tau\big)\overline{w}\big)_{\xi\xi}+\frac{1}{\tau}\big(\big(\xi+\xi(\tau)\big)\overline{w}_\xi+2\overline{w}\big)+\xi'(\tau)\overline{w}_\xi$$

on $\xi\geqslant 0$. We may choose η sufficiently small so that $\overline{w}>0$ on $\xi=0$, $\tau\geqslant 1/T$, and on $\xi\geqslant 0$, $\tau=1/T$ and also as $\xi\to\infty$. It follows then by the maximum principle that $\overline{w}\geqslant 0$ on $\{(\xi,\tau)\colon\ \xi\geqslant 0,\ 1/T\leqslant\tau<\infty\}$, implying that $\tilde v\geqslant V_\eta$ on the same set, as desired.

(iv) Since u_0 satisfies (1.8), there exists a constant A such that $u_0(r)\leqslant 2A/r^2\log^2 r$, on $r>1$. Then, for all time $0<t<T$, we will have $u(r,t)\leqslant 2(t+A)/r^2\log^2 r$, on $r>1$, which readily implies the desired bound on $\tilde v$, with $C=2(A+T)$. □

We will next show a first order derivative bound for the rescaled solution $\tilde v$. We begin by observing that the Aronson–Bénilan inequality $u_t\leqslant u/t$ implies the bounds

$$\frac{(\log\tilde v(\xi,\tau))_{\xi\xi}}{\tilde v(\xi,\tau)}\leqslant C,\quad \text{on } \xi\in\mathbb{R},\ \tau\geqslant\frac{2}{T}.$$

Since, also $\tilde v\leqslant C$ by our width estimate, we conclude that $\omega=\log\tilde v$ satisfies the bound

$$\omega_{\xi\xi}(\xi,\tau)\leqslant C,\quad \text{on } \xi\in\mathbb{R},\ \tau\geqslant\frac{2}{T}.$$

This bound combined with the previous lemma gives the following.

Lemma 4.2. *For any $K\subset(\xi_\mu,\infty)$ compact, there exists a constant C for which*

$$\big|\omega_\xi(\xi,\tau)\big|\leqslant C,\quad \forall\xi\in K,\ \tau\geqslant\frac{2}{T}.$$

Proof. It is enough to prove the lemma for $K=[a,b]$ a compact interval, with $a>\xi_\mu$. Fix a $\tau\geqslant 2/T$ and observe first that from the previous lemma, the bound $|\omega|\leqslant M$, holds on $[a,b+1]$.

Let $\xi_0\in K$. The bound $|\omega|\leqslant M$ on $[\xi_0,\xi_0+1]$ implies that there exists a $\tilde\xi\in(\xi_0,\xi_0+1)$ for which $\omega_\xi(\tilde\xi,\tau)\geqslant -2M$. Hence the upper bound $\omega_{\xi\xi}\leqslant C$ readily implies the lower bound $\omega_\xi(\xi_0,\tau)\geqslant -2M-C$. For the upper bound, let $\xi_0'=\xi_0-\alpha$ with $\alpha=(a-\xi_\mu)/2$, so that still $|\omega(\xi_0',\tau)|\leqslant M$, for a possibly larger constant M. Then, there exists a $\tilde\xi\in(\xi_0',\xi_0)$ for which $\omega_\xi(\tilde\xi,\tau)\leqslant 2M/\alpha$. Hence the upper bound $\omega_{\xi\xi}\leqslant C$ readily implies the upper bound $\omega_\xi(\xi_0,\tau)\leqslant 2M/\alpha+C$. □

We will next use Bernstein type estimates for singularly perturbed first-order equations to show a second derivative bound for $\tilde v$. Before we do so, we introduce a new time variable

$$s=\log\tau=-\log(T-t),\quad s\geqslant-\log T.$$

To simplify the notation we still call $\tilde v(\xi,s)$ the solution $\tilde v$ in the new time scale. Then, it is easy to compute that $\tilde v(\xi,s)$ satisfies the equation

$$\tilde v_s=\mathrm{e}^{-s}(\log\tilde v)_{\xi\xi}+\xi\,\tilde v_\xi+2\tilde v. \tag{4.9}$$

Lemma 4.3. *For any compact sub-interval $K\subset(\xi_\mu,\infty)$, there exists a constant $C=C(K)<\infty$, for which*

$$\big|\tilde v_{\xi\xi}(\xi,s)\big|\leqslant C(K)\mathrm{e}^{s/2},\quad \forall\xi\in K,\ s\geqslant-\log\frac{T}{2}. \tag{4.10}$$

Proof. We will show that in spite of the singularity of Eq. (4.9) as $s \to \infty$, the classical Bernstein technique for establishing derivative estimates for solutions of quasilinear parabolic equations can be applied in this case. This has already been observed in other similar instances, cf. in [10] and the references therein. We will only give an outline of the estimate, referring to [10], Section 5.11, for further details.

Before we proceed with the proof, let us observe that since $K \subset (\xi_\mu, \infty)$, Lemmas 4.1 and 4.2 imply the bounds

$$0 < c \leqslant \tilde{v}^{-1}(\xi, s) \leqslant C < \infty \quad \text{and} \quad \left|\tilde{v}_\xi(\xi, s)\right| \leqslant C \tag{4.11}$$

for all $\xi \in K$, $s \geqslant -\log T$, for some constants c, C depending on K.

We will use the maximum principle to bound $\tilde{v}_{\xi\xi}$, as in the classical Bernstein estimates for the porous medium equation. We first differentiate Eq. (4.9) with respect to ξ to find that $w = \tilde{v}_\xi$ satisfies the equation

$$w_s = \mathrm{e}^{-s}\left\{\frac{1}{\tilde{v}} w_{\xi\xi} - \frac{3}{\tilde{v}^2} w w_\xi + \frac{2}{\tilde{v}^3} w^3\right\} + \xi w_\xi + 3w.$$

We next set $w = \phi(\theta)$, for a function $\phi > 0$ to be determined in the sequel, and use the equation for w to find that θ satisfies the equation

$$\theta_s = \mathrm{e}^{-s}\left\{\frac{1}{\tilde{v}}\theta_{\xi\xi} + \frac{\phi''}{\tilde{v}\phi'}\theta_\xi^2 - \frac{3\phi}{\tilde{v}^2}\theta_\xi + \frac{2\phi^3}{\tilde{v}^3\phi'}\right\} + \xi\theta_\xi + \frac{3\phi}{\phi'}.$$

Differentiating once more with respect to ξ we find the following evolution equation for $z = \theta_\xi$

$$\begin{aligned} z_s = \mathrm{e}^{-s}\bigg\{&\frac{1}{\tilde{v}} z_{\xi\xi} + \left[\frac{2\phi'' z}{\tilde{v}\phi'} - \frac{3\phi}{\tilde{v}^2} - \frac{\phi}{\tilde{v}^2}\right] z_\xi + \frac{1}{\tilde{v}}\left(\frac{\phi''}{\phi'}\right)' z^3 + \left[-\frac{\phi\phi''}{\tilde{v}^2\phi'} - \frac{3\phi'}{\tilde{v}^2}\right] z^2 \\ &+ \frac{2}{\tilde{v}^3}\left(\frac{\phi^3}{\phi'}\right)' z - \frac{6\phi^4}{\tilde{v}^4\phi'}\bigg\} + \xi z_\xi + \left[\left(\frac{3\phi}{\phi'}\right)' + 1\right] z. \end{aligned}$$

Finally, we set $Z = z^2$ and find that Z satisfies the equation

$$\begin{aligned} Z_s = \mathrm{e}^{-s}\bigg\{&\frac{1}{\tilde{v}} Z_{\xi\xi} - \frac{1}{2\tilde{v}}\frac{Z_\xi^2}{Z} + \left[\frac{2\phi'' z}{\tilde{v}\phi'} - \frac{3\phi}{\tilde{v}^2} - \frac{\phi}{\tilde{v}^2}\right] Z_\xi + \frac{2}{\tilde{v}}\left(\frac{\phi''}{\phi'}\right) Z'^2 + \left[-\frac{2\phi\phi''}{\tilde{v}^2\phi'} - \frac{6\phi'}{\tilde{v}^2}\right] Z^{\frac{3}{2}} \\ &+ \frac{4}{\tilde{v}^3}\left(\frac{\phi^3}{\phi'}\right)' Z - \frac{12\phi^4}{\tilde{v}^4\phi'} Z^{\frac{1}{2}}\bigg\} + \xi Z_\xi + \left[\left(\frac{6\phi}{\phi'}\right)' + 2\right] Z. \end{aligned} \tag{4.12}$$

Notice that we can bound the coefficients of the above equation from the constants c and C in (4.11) and the function ϕ. Let us now choose ϕ in the form

$$\phi(\theta) = \theta(\theta + 1)$$

so that

$$\left(\frac{\phi''}{\phi'}\right)' = -\frac{4}{(2\theta + 1)^2} \leqslant -c_1$$

and

$$\left(\frac{6\phi}{\phi'}\right)' \leqslant C_1$$

for fixed constants c_1 and C_1 depending only on the bounds of $|\tilde{v}_\xi|$ in (4.11). We can then assume that at the maximum point of Z, where also $Z_{\xi\xi} \leqslant 0$ and $Z_\xi = 0$, the highest order powers of Z dominate in (4.12) so that we have

$$Z_s \leqslant -c_2\mathrm{e}^{-s} Z^2 + C_2 Z.$$

We conclude that Z is bounded by constants, depending only on the bounds in (4.11), unless

$$-c_2\mathrm{e}^{-s} Z^2 + C_2 Z \geqslant 0$$

i.e., unless

$$Z \leqslant \sqrt{C_2 {c_2}^{-1}}\, \mathrm{e}^{s/2}$$

which readily implies the desired bound on $\tilde{v}_{\xi\xi}$. To make the above proof completely rigorous one needs to localize the above argument by setting $Z = \chi^2(\xi)z^2$, where χ is an appropriate cut off function. However, this introduces only harmless terms and does not change the argument. We refer the reader to the proof Section 5.11, Step 3 of [10] for the details. □

Combining the previous two lemmas gives the following.

Corollary 4.4. *For any compact $K \subset (\xi_\mu, \infty)$, there exists a constant $C = C(K)$ such that*

$$\left|\tilde{v}_\xi(\xi, s)\right| \leqslant C, \quad \left|\tilde{v}_s(\xi, s)\right| \leqslant C \quad \forall \xi \in K, s \geqslant -\log T/2. \tag{4.13}$$

For an increasing sequence of times $s_k \to \infty$, we let

$$\tilde{v}_k(\xi, s) = \tilde{v}(\xi, s + s_k), \quad -s_k - \log T < s < \infty.$$

Then each $\tilde{v}_k$ satisfies the equation

$$(\tilde{v}_k)_s = \mathrm{e}^{-(s+s_k)}(\log \tilde{v}_k)_{\xi\xi} + \xi(\tilde{v}_k)_\xi + 2\tilde{v}_k \tag{4.14}$$

and the area condition

$$\int_{-\infty}^{\infty} \tilde{v}_k(\xi, s)\,\mathrm{d}\xi = 2. \tag{4.15}$$

Lemma 4.5. *Passing to a subsequence, $\tilde{v}_k(\xi, s)$ converges uniformly on compact subsets of $(\xi_\mu, \infty) \times (-\infty, \infty)$ to a solution V of the equation*

$$V_s = \xi V_\xi + 2V, \quad (\xi, s) \in (\xi_\mu, \infty) \times (-\infty, +\infty) \tag{4.16}$$

with

$$\int_{\xi_\mu}^{\infty} V(\xi, s)\,\mathrm{d}\xi = 2. \tag{4.17}$$

Proof. Let $K \subset (\xi_\mu, \infty) \times (-\infty, \infty)$ compact. Then according to the previous lemma, the sequence $\tilde{v}_k$ is equicontinuous on K, hence passing to a subsequence it converges to a function V, which satisfies the bounds

$$\left|V_\xi(\xi, s)\right| \leqslant C, \quad \left|V_s(\xi, s)\right| \leqslant C, \quad \forall \xi \in K, s \geqslant -\log T/2. \tag{4.18}$$

In addition, the estimate (4.10) readily implies that V is a solution of the first order equation (4.16).

On the other hand, Lemma 4.1 implies that $\tilde{v}_k(\cdot, s) \to 0$, uniformly on $(-\infty, \xi_\mu^-]$, for any $\xi_\mu^- < \xi_\mu$, $s \in \mathbb{R}$. In addition $\tilde{v}_k \leqslant C$ uniformly in space and time, by our width estimate (2.4). Hence, we can pass to the limit in (4.15) to conclude that V satisfies the area condition (4.17). □

Lemma 4.6. *Assume that V is a positive, locally Lipschitz, solution of the equation*

$$V_s = \xi V_\xi + 2V, \quad \text{on } (\xi_\mu, \infty) \times (-\infty, \infty) \tag{4.19}$$

with

$$\int_{\xi_\mu}^{\infty} V(\xi, s)\,\mathrm{d}\xi = 2, \quad \forall s \in (-\infty, \infty). \tag{4.20}$$

Then,

$$V(\xi, s) = \frac{2\xi_\mu}{\xi^2}, \quad \xi \geqslant \xi_\mu$$

with $\xi_\mu = T + \mu$.

Proof. The basic idea is that the fixed area condition (4.20) completely determines V. This is better understood by setting

$$W(\zeta, s) = \int_{\zeta}^{\infty} V(\xi, s)\,\mathrm{d}\xi, \quad \zeta \geqslant \xi_\mu$$

and observing that W satisfies the equation

$$W_s = (\zeta W)_\zeta, \quad \text{on } [\xi_\mu, \infty) \times (-\infty, \infty)$$

with

$$W(\xi_\mu, s) = 2, \quad -\infty < s < \infty$$

and

$$W(\zeta, s) \to 0, \quad \text{as } \zeta \to \infty, -\infty < s < \infty.$$

We will show that W is completely determined by its boundary values at $\zeta = \xi_\mu$. Indeed, integrating along characteristics we easily find that W satisfies

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\mathrm{e}^{-s} W\left(\mathrm{e}^{-s}\zeta, s\right)\right] = 0.$$

Hence, for any $\zeta \geqslant \xi_\mu$ and $s, \bar{s}$ we have

$$\mathrm{e}^{-s} W\left(\mathrm{e}^{-s}\zeta, s\right) = \mathrm{e}^{-\bar{s}} W\left(\mathrm{e}^{-\bar{s}}\zeta, \bar{s}\right)$$

or equivalently

$$\mathrm{e}^{\bar{s}-s} W\left(\mathrm{e}^{\bar{s}-s}\zeta, s\right) = W(\zeta, \bar{s}).$$

Fixing a point $\overline{P} = (\bar{\zeta}, \bar{s})$, there exists a unique characteristic line passing through $\overline{P}$, which intersects the boundary $\zeta = \xi_\mu$ at the point (ξ_μ, s_0) with $s_0 = \bar{s} + \log(\bar{\zeta}/\xi_\mu)$. Hence

$$W(\zeta, s) = 2\mathrm{e}^{s-s_0} = \frac{2\xi_\mu}{\zeta}, \quad \forall \zeta \geqslant \xi_\mu,\ \forall s \in \mathbb{R}.$$

Differentiating with respect to ζ we then obtain that $V = 2\xi_\mu/\xi^2$, as desired. □

As an immediate consequence of the previous two lemmas we obtain the following.

Corollary 4.7. *The sequence* $\{\tilde{v}_k(\xi, s)\}$ *converges uniformly on compact subsets of* $(\xi_\mu, \infty) \times (-\infty, \infty)$ *to the function* $V_\mu(\xi) = 2(T + \mu)/\xi^2$.

The proof of Theorem 1.2, stated in the Introduction, is an immediate consequence of Lemma 4.1, part (ii), and Corollary 4.7.

Proof of Theorem 1.2. Let $\xi_\mu^- < \xi_\mu = T + \mu$. By Lemma 4.1, part (ii), $\tilde{v}(\cdot, \tau)$ converges to zero, as $\tau \to \infty$, uniformly on $(-\infty, \xi_\mu^-]$. On the other hand, for $\xi_\mu^+ > \xi_\mu$, Corollary 4.7 implies that $\tilde{v}(\xi, s)$ converges to $V_\mu(\xi) = 2\xi_\mu/\xi^2$, as $s \to \infty$ uniformly on compact sets of $[\xi_\mu^+, \infty)$. Changing back to the $\tau = \mathrm{e}^s$ variable we readily conclude that $\tilde{v}(\xi, \tau) \to 2\xi_\mu/\xi^2$, as $\tau \to \infty$, uniformly on compact sets of $[\xi_\mu^+, \infty)$, finishing the proof of the theorem. □

Acknowledgements

We are grateful to Richard Hamilton for many enlightening discussions in the course of this work. This paper was completed while the first author was a Visiting Fellow in the Department Mathematics, Imperial College, London, partially supported by ESPRC in the UK. She is grateful to this institution for its hospitality and support.

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