

# Maslov index for homoclinic orbits of Hamiltonian systems

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## Abstract

A useful tool for studying nonlinear differential equations is index theory. For symplectic paths on bounded intervals, the index theory has been completely established, which revealed tremendous applications in the study of periodic orbits of Hamiltonian systems. Nevertheless, analogous questions concerning homoclinic orbits are still left open. In this paper we use a geometric approach to set up Maslov index for homoclinic orbits of Hamiltonian systems. On the other hand, a relative Morse index for homoclinic orbits will be derived through Fredholm index theory. It will be shown that these two indices coincide.

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## 0. Introduction

The Morse theory has widely been used in the calculus of variations to study the existence of multiple solutions of nonlinear differential equations. For the first order time periodic Hamiltonian system

$$\dot{z} = JH_z(t, z), \quad z \in \mathbb{R}^{2n}, \quad (0.1)$$

a periodic orbit of (0.1) is an extremal of the functional

$$\hat{I} = \int_0^\tau \frac{1}{2} (-J\dot{z}(t), z(t)) - H(t, z(t)) dt \quad (0.2)$$

over closed curves in the phase space; nevertheless, the strong indefiniteness of  $\hat{I}$  causes substantial difficulties in finding its critical points. Started with the pioneering work [33] of P. Rabinowitz, new techniques in critical point theory, such as intersection of sets involving linking [8,34], saddle point reduction and Galerkin approximation, have

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emerged as existence tools for studying various types of solutions of Hamiltonian systems. The interested reader may consult a survey article [34] for more complete references and related results.

The fundamental solution of a Hamiltonian system forms a path in the symplectic matrix group  $\text{Sp}(2n) = \{M \in \text{GL}(\mathbb{R}^{2n}) \mid M^T J M = J\}$ , where and throughout the paper

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

$I_n$  denotes the  $n \times n$  identity matrix and  $M^T$  the transpose of  $M$ . It is known that the extremals of  $\hat{I}$  always have infinite Morse index. A finite index representation for solutions of (0.1) seems to be more useful in applications. In 1984, C. Conley and E. Zehnder [14] established an index theory for non-degenerate paths in  $\text{Sp}(2n)$  for  $n \geq 2$ , where a path is said to be non-degenerate if its end points lie in  $\text{Sp}(2n)^* = \{M \mid M \in \text{Sp}(2n) \text{ and } \det(M - I_{2n}) \neq 0\}$ . The case of  $n = 1$  was settled by Y. Long and E. Zehnder [30]. The index theory for degenerate linear Hamiltonian systems was accomplished in 1990 by Y. Long [26] and C. Viterbo [43]. Later on, Long and his collaborators established iteration theory (cf. the book [27] of Y. Long and references therein) for the indices of symplectic paths on finite intervals. This index theory revealed tremendous interesting information for studying periodic orbits of Hamiltonian systems.

In this paper, we are concerned with homoclinic orbits of Hamiltonian systems. It is assumed that the function  $H$  satisfies the following conditions:

(H1)  $H \in C^2(\mathbb{R}^{2n+1}, \mathbb{R})$ ,  $\lim_{z \rightarrow 0} H_z(t, z) = 0$  and  $\lim_{z \rightarrow 0} H_{zz}(t, z) = B_*$  uniformly for  $t \in \mathbb{R}$ .

(H2) The spectrum of  $J B_*$  has no intersection with the imaginary axis; that is,  $\sigma(J B_*) \cap i\mathbb{R} = \emptyset$ .

In what follows, we use prime instead of subscript to denote differentiation with respect to  $z$ ; that is,  $H' = H_z$  and  $H'' = H_{zz}$ . A homoclinic orbit of (0.1) is a critical point of  $I$  defined by

$$I(z) = \int_{-\infty}^{\infty} \frac{1}{2} (-J\dot{z}(t), z(t)) - H(t, z(t)) dt.$$

Applying variational methods, a number of authors proved [5,15–17,23,38–40,42] the existence of homoclinic orbits of (0.1), under various conditions on  $H$ . Among these works, V. Coti Zelati, I. Ekeland and E. Séré [15] employed a dual variational method, while H. Hofer and K. Wysocki [23] used Fredholm operator theory for the first order elliptic systems to establish such existence results. In [38,39], E. Sere obtained the existence of infinitely many homoclinic orbits which are geometrically distinct. The convergence of subharmonic orbits to a homoclinic orbit of (0.1) was proved by K. Tanaka [42]. For the existence of homoclinic orbits of second order Lagrangian systems, may interesting results can be found in the references of [12,13,34].

Although the index theory for periodic orbits of Hamiltonian systems has been extensively studied for many years, analogous questions concerning index theory for homoclinic orbits are still left open. For a closed path rotating in  $\text{Sp}(2n)$ , it is a natural bridge to define index of a periodic orbit in connection with the winding number of a related symplectic path. Since a homoclinic orbit is a symplectic path on an infinite interval, setting up a way to count its winding number seems to be difficult.

A new approach is proposed in the paper to find a suitable way to define an index for homoclinic orbits of (0.1). This seems to be a quite natural approach in view of some interesting geometric features revealed in connection with the Maslov index. In 1965, V.P. Maslov [32] introduced an index for Lagrangian paths and it was interpreted by V.I. Arnold [6] as the net number of times for path passing through the singular cycle. The assumptions (H1) and (H2) indicate that 0 is a hyperbolic equilibrium. For the autonomous system, it will be seen that through the Hamiltonian flow generated by (1.7) the stable and unstable manifolds with respect to the equilibrium 0 are Lagrangian manifolds. Along the stable manifold of a homoclinic orbit, its tangent spaces forms a Lagrangian path. Likewise, the second Lagrangian path can be induced from the tangent spaces of unstable manifold. Such two Lagrangian paths will be used to set up a Maslov index for homoclinic orbits of (0.1). For a non-autonomous system, the index can be treated in a similar way. Detailed derivation will be carried out in Section 1.

There are many ways to define index for the paths of symplectic matrices. For the periodic orbits of (0.1), a relative Morse index has been studied in a number of articles [1–3,11,18–21,24,27,29,37,44]. Our aim in Section 2 is to derive a relative Morse index for a homoclinic orbit of (0.1) by making use of index theory for the Fredholm operators [2,31]

associated with the second Frechet derivative of  $I$ . This kind of methods [1,2,20,41] have been successfully employed to treat index theory for periodic orbits of Hamiltonian systems. Nevertheless, in stead of point spectrum in the case of periodic boundary conditions, the spectrum of  $-J \frac{d}{dt}$  in dealing with homoclinic orbits is the whole real line. A different way to interpret the relative Morse index is to consider the spectral flow of a family of self-adjoint Fredholm operators.

In Section 3, it will be shown that, for a orbit homoclinic to a hyperbolic equilibrium, its Maslov index indeed coincides with the relative Morse index derived in Section 2. Such kinds of results have been well established in case of periodic orbits (see e.g. [2,27]). Our results convince that both the analytic and geometric approaches can be unified for possibly building up Morse theory for homoclinic orbits of Hamiltonian systems. In case of Lagrangian system, the relative Morse index is not different from the Morse index. A verification will be given in Section 4.

To the best of our knowledge, the use of Lagrangian paths to study index theory of homoclinic orbits of (0.1) seems to be new. It looks like to have great potential in solving related problems in the future.

### 1. A Maslov index for homoclinic orbits

In this section, we are looking for a geometric approach to set up an index for homoclinic orbits of the first order Hamiltonian system

$$\dot{z} = JH'(t, z). \tag{1.1}$$

Let  $x(t)$  be a homoclinic orbit of (1.1) with the asymptotic behavior

$$\lim_{|t| \rightarrow \infty} x(t) = 0.$$

Set  $w_0(\xi, \eta) = \langle J\xi, \eta \rangle$ , the standard symplectic form on  $\mathbb{R}^{2n}$ . Denoted by  $V_s$  and  $V_u$  the stable and unstable manifolds with respect to 0 under the linear Hamiltonian flow

$$\dot{z}(t) = JB_*z(t). \tag{1.2}$$

It follows from (H2) that  $\mathbb{R}^{2n} = V_s \oplus V_u$ . Moreover, as an immediate consequence of Lemma 1, it will be seen that both the stable manifold  $V_s$  and the unstable manifold  $V_u$  are Lagrangian subspaces of  $(\mathbb{R}^{2n}, w_0)$ .

Let  $\text{Lag}(2n)$  be the set of Lagrangian subspaces of  $(\mathbb{R}^{2n}, w_0)$ . It is known that  $\text{Lag}(2n)$  is a manifold. For  $W \in \text{Lag}(2n)$ , set

$$O_j(W) = \{W_1 \mid W_1 \in \text{Lag}(2n) \text{ and } \dim(W_1 \cap W) = j\},$$

a submanifold with codimension  $j(j+1)/2$ . The union of all strata  $\bigcup_{j=1}^n O_j(W)$  is the closure of  $O_1(W)$ . As mentioned in [6], the closure of  $O_1(W)$  is a singular cycle with codimension 1. The top stratum  $O_1(W)$  has a canonical transverse orientation. To be more precise, for each  $\eta \in O_1(W)$ , the Lagrangian path  $\{e^{tJ}\eta, t \in (-\delta, \delta)\}$  crosses  $O_1(W)$  transversally, and as  $t$  increases the direction of this path points out the desired transverse orientation. Thus this singular cycle is two-sidedly imbedded in  $\text{Lag}(2n)$ , as stated in the fundamental lemma of [6]. Let  $\mathcal{L}(a, b) = C([a, b], \text{Lag}(2n))$ , the set of continuous Lagrangian paths on  $[a, b]$ . In [9] the Maslov index  $\mu(U_1, U_2)$  was defined as an integer invariant to a continuous one-parameter family  $\{(U_1(t), U_2(t)) \mid U_1, U_2 \in \mathcal{L}(a, b)\}$  of pairs of Lagrangian subspaces; indeed, four equivalent definitions of  $\mu(U_1, U_2)$  were discussed in [9] and a systematic and unified treatment has been worked out by the authors. An important property of Maslov index is homotopy invariance stated as follows.

**Proposition 1.** *Let  $U_1(\theta, \cdot), U_2(\theta, \cdot) \in \mathcal{L}(a, b)$  and  $\{(U_1(\theta, t), U_2(\theta, t)) \mid 0 \leq \theta \leq 1\}$  be a continuous family of pairs of Lagrangian paths. Suppose that both  $\dim(U_1(\theta, a) \cap U_2(\theta, a))$  and  $\dim(U_1(\theta, b) \cap U_2(\theta, b))$  are independent of  $\theta$ , then  $\mu(U_1(0, t), U_2(0, t)) = \mu(U_1(1, t), U_2(1, t))$ .*

The proof of Proposition 1 is omitted, since it easily follows from some basic properties of Maslov index [9].

To define the Maslov index of a homoclinic orbit of (1.1), we consider the Hamiltonian flow induced by

$$\dot{\phi} = JH''(t, x(t))\phi, \tag{1.3}$$

where  $x(t)$  is a homoclinic orbit under consideration. Let  $\phi(t, \nu)$  satisfy (1.3) and  $\phi(\nu, \nu) = I_{2n}$ . Clearly  $\phi$  satisfies a semigroup property; that is,  $\phi(t, \nu)\phi(\nu, \tau) = \phi(t, \tau)$ . For  $\nu \in \mathbb{R}$ , define

$$V_s(\nu) = \left\{ \xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow \infty} \phi(t, \nu)\xi = 0 \right\}, \quad (1.4)$$

and

$$V_u(\nu) = \left\{ \xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow -\infty} \phi(t, \nu)\xi = 0 \right\}. \quad (1.5)$$

We remark that

$$\lim_{\nu \rightarrow \infty} V_s(\nu) = V_s \text{ and } \lim_{\nu \rightarrow -\infty} V_u(\nu) = V_u. \quad (1.6)$$

**Lemma 1.** For each  $\nu \in \mathbb{R}$ , both  $V_s(\nu)$  and  $V_u(\nu)$  are Lagrangian subspaces of  $(\mathbb{R}^{2n}, w_0)$ .

**Proof.** For  $\xi, \eta \in V_s(\nu)$ , since  $\phi(t, \nu) \in \text{Sp}(2n)$ , it follows that

$$w_0(\xi, \eta) = \langle J\xi, \eta \rangle = \langle J\phi(t, \nu)\xi, \phi(t, \nu)\eta \rangle \text{ for all } t > \nu.$$

This together with (1.4) yields  $w_0(\xi, \eta) = 0$ . With only  $t$  being replaced by  $-t$ , the above argument shows that  $w_0(\xi, \eta) = 0$  if  $\xi, \eta \in V_u(\nu)$ . Since  $\dim V_s = \dim V_u = n$  and  $\phi(t, \nu)$  is a homeomorphism, we conclude that both  $V_s(\nu)$  and  $V_u(\nu)$  are Lagrangian subspaces of  $(\mathbb{R}^{2n}, w_0)$ .  $\square$

We are going to employ the pair of Lagrangian paths  $V_s(\nu)$  and  $V_u(\nu)$  to define the Maslov index for the homoclinic orbit  $x(t)$ ; here an extra care is needed in dealing with Lagrangian paths on infinite intervals. To give a better insight of its geometric interpretation, we treat the autonomous case first. Denoted by  $W_s$  and  $W_u$  the stable and unstable manifolds with respect to 0 under the flow

$$\dot{z} = JH'(z). \quad (1.7)$$

By Hadamard–Perron theorem,  $W_s$  and  $W_u$  exist in a neighborhood of 0. Moreover, they are  $C^1$  Lagrangian manifolds. For a homoclinic orbit  $x(t)$ , both  $T_{x(t)}W_s$  and  $T_{x(t)}W_u$  exist for all  $t \in \mathbb{R}$ ,  $\lim_{t \rightarrow \infty} T_{x(t)}W_s = V_s$ , and  $\lim_{t \rightarrow -\infty} T_{x(t)}W_u = V_u$ , where  $T_{x(t)}W_s$  and  $T_{x(t)}W_u$  denote respectively the tangent spaces of  $W_s$  and  $W_u$  at  $x(t)$ . Thus  $\tau_0$  can be chosen large enough to ensure that  $\dim(T_{x(\tau)}W_s \cap V_u) = 0$  for all  $\tau \geq \tau_0$ . Pick a fixed  $\tau \geq \tau_0$ . Then there is a  $\hat{\tau} < 0$  such that  $\dim(T_{x(\tau)}W_s \cap T_{x(t)}W_u) = 0$  if  $t < \hat{\tau}$ . Define

$$\mu_\tau(x) = \mu(T_{x(\tau)}W_s, T_{x(t)}W_u), \quad (1.8)$$

where  $T_{x(t)}W_u$  is a Lagrangian path on the interval  $[\hat{\tau}, \tau]$  and  $T_{x(\tau)}W_s$  stands for a constant path on  $[\hat{\tau}, \tau]$ . Since  $\dim(T_{x(\tau)}W_s \cap T_{x(t)}W_u) = 0$  if  $t < \hat{\tau}$ , it makes (1.8) no change if the Lagrangian paths under consideration are taken on the unbounded interval  $(-\infty, \tau]$ . This fact will be used in what follows without further comment. Furthermore, through the action of Hamiltonian flow,  $\dim(T_{x(t)}W_s \cap T_{x(t)}W_u)$  is independent of  $t$ . It follows from Proposition 1 that

$$\mu_{\tau_1}(x(t)) = \mu_{\tau_2}(x(t)) \text{ if } \tau_2 > \tau_1 \geq \tau_0;$$

in other words,  $\mu_\tau(x)$  is independent of the choice of  $\tau$  as long as  $\tau \in [\tau_0, \infty)$ . Thus the Maslov index of  $x(t)$  is well defined, as to be stated in Definition 1.

To extend the Maslov index of a homoclinic orbit in the general situation, we replace the Lagrangian path  $T_{x(\tau)}W_s$  by  $V_s(\tau)$  and  $T_{x(t)}W_u$  by  $V_u(t)$ , where  $\tau$  is a fixed large number so that  $V_s(\nu) \cap V_u = 0$  for all  $\nu \geq \tau$ . Then the Maslov index for a homoclinic orbit  $x(t)$  can be defined in the same manner with only (1.8) being changed to

$$\mu_\tau(x) = \mu(V_s(\tau), V_u(t)). \quad (1.9)$$

**Definition 1.** Let  $x(t)$  be a homoclinic orbit of (1.1). The Maslov index of  $x(t)$ , denoted by  $i_*(x)$ , is defined by

$$i_*(x) = \mu_\tau(x),$$

provided that  $\tau$  is taken to be sufficiently large in (1.9).

**Remark 1.** Let  $I''(x)$  denote the second Frechet derivative of  $I$  at  $x$ . If the null space of  $I''(x)$  is trivial then  $\dim(V_s(t) \cap V_u(t)) = 0$  for all  $t$ . Moreover, it is known [4] that  $\lim_{t \rightarrow \infty} V_u(t) = V_u$ . Letting  $\tau \rightarrow \infty$  in (1.9) and invoking (1.6), we see that  $i_*(x) = \mu(V_s, V_u(t))$ , where both  $V_u(t)$  and  $V_s$  are Lagrangian paths on  $(-\infty, \infty)$ .

## 2. A relative Morse index for homoclinic orbits

In this section, a different approach will be used to set up an index for homoclinic orbits of (1.1). This is so called relative Morse index of which many interesting properties and applications have been obtained [2,11,20,27] for periodic orbits of Hamiltonian systems.

To simplify notation, we set  $B(t) = H''(t, x(t)) - B_*$ . Let  $\|\cdot\|_2$  denote  $L^2$ -norm. Define  $\hat{A} = -J \frac{d}{dt} - B_*$ ,  $|\hat{A}| = (\hat{A}^2)^{1/2}$ , and

$$\|\xi\| = \|(I + |\hat{A}|)^{1/2} \xi\|_2 \tag{2.1}$$

if  $\xi \in H^{1/2}(\mathbb{R}, \mathbb{R}^{2n})$ . It is easy to check that (2.1) is equivalent to the graph norm of  $|\hat{A}|^{1/2}$  and the domain of  $|\hat{A}|^{1/2}$  equipped with (2.1) forms a Hilbert space  $E$ . Since the graph norm of  $|\hat{A}|^{1/2}$  can be taken as an equivalent norm of  $H^{1/2}(\mathbb{R}, \mathbb{R}^{2n})$ , we do not distinguish  $E$  from  $H^{1/2}(\mathbb{R}, \mathbb{R}^{2n})$ .

To find a relative Morse index for a homoclinic orbit  $x(t)$ , we are going to employ the index theory of Fredholm operators. Let  $\beta : E \times E \rightarrow \mathbb{R}$  be a continuous symmetric bilinear form and  $T$  be the self-adjoint operator induced by  $\langle T\xi, \eta \rangle = \beta(\xi, \eta)$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product inherited from  $E$ . For a Fredholm operator  $T$ , there is a unique  $T$ -invariant orthogonal splitting

$$E = E_+(T) \oplus E_-(T) \oplus E_0(T),$$

where  $E_0(T)$  is the null space of  $T$ ,  $\beta$  is positive definite on  $E_+(T)$  and negative definite on  $E_-(T)$ .

Let  $j$  be the imbedding from  $E$  to  $L^2(\mathbb{R}, \mathbb{R}^{2n})$  and  $(\cdot, \cdot)$  be the inner product in  $\mathbb{R}^{2n}$ . Set  $\hat{A}^+ = \hat{A}|_{E_+(\hat{A})}$  and  $\hat{A}^- = -\hat{A}|_{E_-(\hat{A})}$ . For  $\xi, \eta \in E$ , let  $A$  and  $F$  be linear operators defined by

$$\langle A\xi, \eta \rangle = \int_{-\infty}^{\infty} ((\hat{A}^+)^{1/2}\xi, (\hat{A}^+)^{1/2}\eta) dt - \int_{-\infty}^{\infty} ((\hat{A}^-)^{1/2}\xi, (\hat{A}^-)^{1/2}\eta) dt \tag{2.2}$$

and

$$\langle F\xi, \eta \rangle = \int_{-\infty}^{\infty} (B(t)j\xi, j\eta) dt. \tag{2.3}$$

Then

$$\langle I''(x)\xi, \eta \rangle = \langle A\xi, \eta \rangle - \langle F\xi, \eta \rangle,$$

where  $I''$  denotes the second Frechet derivative of  $I$  and  $x$  is the homoclinic orbit under consideration. It is known [10] that  $A$  and  $F$  are bounded operators on  $E$ . Moreover,  $A$  and  $F$  satisfy the following properties.

**Lemma 2.** *The operator  $A$  is reversible on  $E$ .*

**Lemma 3.** *The operator  $F : E \rightarrow E$  is compact.*

We refer to [23] for a detailed proof of Lemma 2. In the proof of Lemma 3, we will use the following proposition.

**Proposition 2.** *Let  $\chi$  be a bounded function in  $C^\infty(\mathbb{R})$  with bounded derivatives. If  $f \in E$  then the pointwise product of  $\chi$  and  $f$  is also a function in  $E$ .*

The proof of Proposition 2 can be found in [25].

**Proof of Lemma 3.** Let  $j^*$  be the adjoint operator of  $j$  and  $(\cdot, \cdot)_2$  denote the inner product in  $L^2(\mathbb{R}, \mathbb{R}^{2n})$ . Clearly  $F = j^* \sharp F j$  if  $\sharp F$  is defined by

$$(\sharp F \hat{\xi}, \hat{\eta})_2 = \int_{-\infty}^{\infty} (B(t)\hat{\xi}, \hat{\eta}) dt \quad \text{for } \hat{\xi}, \hat{\eta} \in L^2(\mathbb{R}, \mathbb{R}^{2n}).$$

Pick a sequence  $\{\chi_k\}$  of  $C^\infty$  functions on  $\mathbb{R}$  which satisfy  $\|\chi_k\|_\infty \leq 1$ ,  $\|\chi'_k\|_\infty \leq 2$  and

$$\chi_k(t) = \begin{cases} 1 & \text{if } |t| \leq k - 1, \\ 0 & \text{if } |t| \geq k, \end{cases}$$

where  $\|\cdot\|_\infty$  denotes  $L^\infty$ -norm. Set  $F_k = j^* \sharp F_k j$ , where

$$(\sharp F_k \hat{\xi}, \hat{\eta})_2 = \int_{-\infty}^{\infty} (B_k(t) \hat{\xi}, \hat{\eta}) dt \quad \text{for } \hat{\xi}, \hat{\eta} \in L^2(\mathbb{R}, \mathbb{R}^{2n}) \tag{2.4}$$

and

$$B_k(t) = \chi_k(t) B(t). \tag{2.5}$$

We claim that  $F_k$  is a compact operator on  $E$ . Define an operator  $\alpha_k$  by  $\alpha_k(\eta(t)) = \chi_k(t)\eta(t)$  for  $\eta \in E$ . It follows from Proposition 2 that  $\alpha_k$  is a bounded operator on  $E$ . Let  $\hat{C}_k = \{f \mid f \in C^1([-k, k], \mathbb{R}^{2n}) \text{ and } f(-k) = f(k)\}$ . Let  $L^2_k$  and  $E_k$  be the completions of  $\hat{C}_k$  under  $\|\cdot\|_2$  and  $\|\cdot\|$  respectively. Observe that  $\chi_k \eta \in E_k$  if  $\eta \in E$ . Since  $\text{supp}(B_k(t)) \subset [-k, k]$ , we know that  $j^* \sharp F_k j \eta(t) = j^* \sharp F_k j \chi_{k+1}(t) \eta(t)$  if  $\eta \in E$ . Thus it suffices to show that  $j^* \sharp F_k j \alpha_{k+1}$  is a compact operator. This is true due to the fact that the imbedding from  $E_k$  to  $L^2_k$  is compact.

Since  $\lim_{|t| \rightarrow \infty} x(t) = 0$ , it follows from (H1) that as  $|t| \rightarrow \infty$ ,  $H''(t, x(t)) \rightarrow B_*$  in the matrix norm and consequently, as a sequence of operators from  $L^2(\mathbb{R}, \mathbb{R}^{2n})$  to itself,  $\sharp F_k \rightarrow \sharp F$  in the operator norm. Furthermore, since  $j$  is a bounded operator,  $F_k \rightarrow F$  in the operator norm. Therefore,  $F$  is a compact operator on  $E$ .  $\square$

Let  $P_A$  denote the orthogonal projection from  $E$  to  $E_-(A)$ . It follows from Lemmas 2 and 3 that  $P_A - P_{A-F}$  is compact. Thus  $P_A|_{E_-(A-F)} : E_-(A-F) \rightarrow E_-(A)$  is a Fredholm operator. The Fredholm index of  $P_A|_{E_-(A-F)}$  is the relative dimension of  $E_-(A-F)$  with respect to  $E_-(A)$ . A detailed derivation for the relative Morse index of periodic orbits of Hamiltonian systems can be found in [2].

**Definition 2.** For a homoclinic orbit  $x$  of (1.1), a relative Morse index  $i(x)$  is defined to be the Fredholm index of  $P_A|_{E_-(A-F)}$ .

**Remark 2.** (a) For a Fredholm operator  $T$ , its Fredholm index will be denoted by  $\text{ind}(T)$ .

(b) The relative Morse index can be derived in different ways [2, 11, 44]. Such kinds of indices have been extensively studied in dealing with periodic orbits of first order Hamiltonian systems.

Next, we are going to investigate the relation between the relative Morse index  $i(x)$  and the Maslov index  $i_*(x)$  defined in Section 1. Our goal is to show that such two indices actually coincide.

**Theorem 1.** *If  $x(t)$  is an orbit homoclinic to a hyperbolic equilibrium, then  $i_*(x) = i(x)$ .*

**Remark 3.** Without loss of generality, we may assume that  $x(t)$  is homoclinic to 0. The hypotheses (H1) and (H2) indicate that 0 is a hyperbolic equilibrium of (1.1).

The proof of Theorem 1 will be carried out in the next section, in which the notion of spectral flow will be used. As being well known, the concept of spectral flow was introduced by Atiyah, Patodi and Singer [7]. Since then, many interesting properties and applications of spectral flow [9, 22, 35, 36, 44] have been subsequently established. Here, for convenience to the reader, a number of basic properties of spectral flow will be collected in the remainder of this section. Let  $\{A_\theta \mid \theta \in [0, 1]\}$  be a continuous path of self-adjoint Fredholm operators on a Hilbert space  $E$ . The spectral flow of  $A_\theta$  represents the net change in the number of negative eigenvalues of  $A_\theta$  as  $\theta$  runs from 0 to 1, where the counting follows from the rule that each negative eigenvalue crossing to the positive axis contributes +1 and each positive eigenvalue crossing to the negative axis contributes -1, and for each crossing the multiplicity of eigenvalue is taken into account. In the calculation of spectral flow, a crossing operator introduced in [36] will be used. Take a  $C^1$  path  $\{A_\theta \mid \theta \in [0, 1]\}$  and let  $\wp$  be the projection from  $E$  to  $E_0(A_\theta)$ . When eigenvalue crossing occurs at  $A_\theta$ , the operator

$$\wp \frac{\partial}{\partial \theta} A_\theta \wp : E_0(A_\theta) \rightarrow E_0(A_\theta) \tag{2.6}$$

is called a crossing operator, denoted by  $C_r[A_\theta]$ . As mentioned in [36], an eigenvalue crossing at  $A_\theta$  is said to be regular if the null space of  $C_r[A_\theta]$  is trivial. In this case, we define

$$\text{sign } C_r[A_\theta] = \dim E_+(C_r[A_\theta]) - \dim E_-(C_r[A_\theta]). \tag{2.7}$$

A crossing occurs at  $A_\theta$  is called simple crossing if  $\dim E_0(A_\theta) = 1$ .

Consider the case where all the crossings are regular. Let  $\mathcal{D}$  be the set containing all the points in  $[0, 1]$  at which the crossing occurs. The set  $\mathcal{D}$  contains only finitely many points. The spectral flow of  $A_\theta$  is

$$Sf(A_\theta, 0 \leq \theta \leq 1) = \sum_{\theta \in \mathcal{D}_*} \text{sign } C_r[A_\theta] - \dim E_-(C_r[A_0]) + \dim E_+(C_r[A_1]), \tag{2.8}$$

where  $\mathcal{D}_* = \mathcal{D} \cap (0, 1)$ . In what follows, the spectral flow of  $A_\theta$  will be simply denoted by  $Sf(A_\theta)$  when the starting and end points of the flow are clear from the contents.

**Remark 4.** (a) As indicated in [7,44],  $Sf(A_\theta) = Sf(A_\theta + \epsilon \text{id})$  if  $\text{id}$  is the identity operator on  $E$  and  $\epsilon$  is a sufficiently small positive number. Furthermore, by Theorem 4.22 of [36], there exist some  $\epsilon \in (0, 1)$  such that all the eigenvalue crossings occurred in  $\{A_\theta + \epsilon \text{id} \mid \theta \in [0, 1]\}$  are regular, and this property indeed holds for almost every  $\epsilon \in (0, 1)$ . Using the property of homotopy invariance of spectral flow, we may assume, without loss of generality, that  $\{A_\theta \mid \theta \in [0, 1]\}$  is continuously differentiable in  $\theta$  and all the eigenvalue crossings at  $A_\theta$  are regular crossings. Detailed analysis can be found in [36,44].

(b) Through the paper for an operator  $\mathcal{K}$  we let  $E_0(\mathcal{K})$  denote the null space of  $\mathcal{K}$ .

(c) Although we let  $E$  denote the Hilbert space  $H^{1/2}(\mathbb{R}, \mathbb{R}^{2n})$ , the spectral flow can be defined for self-adjoint Fredholm operators on other Hilbert spaces as well. This fact will be used later without further comment.

In the next proposition  $P_{A_\theta}$  will be simply denoted by  $P_\theta$ , so does in Lemma 4.

**Proposition 3.** *Suppose that, for each  $\theta_1 \in [0, 1]$ ,  $A_{\theta_1} - A_0$  is a compact operator on  $E$ , then*

$$\text{ind}(P_0|_{E_-(A_1)}) = -Sf(A_\theta, 0 \leq \theta \leq 1). \tag{2.9}$$

The following lemma will be used in the proof of Proposition 3.

**Lemma 4.** *If there is no eigenvalue crossing for all  $\theta \in [\theta_1, \theta_2]$ , then  $\text{ind}(P_{\theta_1}|_{E_-(A_{\theta_2})}) = 0$ .*

**Proof.** Since  $P_\theta$  is continuous in  $\theta$ , it directly follows from the continuity of Fredholm index.  $\square$

**Proof of Proposition 3.** As noted in Remark 4, it is sufficient to consider the case where  $\{A_\theta \mid \theta \in [0, 1]\}$  is continuously differentiable in  $\theta$  and all the eigenvalue crossings are regular. Let  $0 \leq \theta_1 < \theta_2 < \dots < \theta_k \leq 1$  be the points at which eigenvalue crossing occurs. By (2.7), (2.8) and Lemma 4, it suffices to show that

$$\text{ind}(P_{\theta_i - \epsilon}|_{E_-(A_{\theta_i})}) = -\dim E_+(C_r[A_{\theta_i}]) \tag{2.10}$$

and

$$\text{ind}(P_{\theta_i}|_{E_-(A_{\theta_i + \epsilon})}) = \dim E_-(C_r[A_{\theta_i}]) \tag{2.11}$$

if  $\epsilon$  is a sufficiently small positive number. We carry out the proof of (2.10) only, the other is analogue. Pick  $\delta$  and  $\epsilon$  sufficiently small so that  $\{-\delta, \delta\} \cap \sigma(A_\theta)$  is empty for all  $\theta \in [\theta_i - \epsilon, \theta_i + \epsilon]$ . Let  $\sigma_\delta(\theta) = (-\delta, \delta) \cap \sigma(A_\theta)$  and  $\Omega_\delta(\theta)$  be the space spanned by the corresponding eigenfunctions associated with the eigenvalues lying in  $\sigma_\delta(\theta)$ . Indeed,  $\delta$  and  $\epsilon$  can be chosen small enough so that

$$\dim \Omega_\delta(\theta) = \dim E_0(A_{\theta_i}) \quad \text{for all } \theta \in [\theta_i - \epsilon, \theta_i + \epsilon].$$

Let  $Z_\theta$  be the orthogonal projection from  $E$  to  $\Omega_\delta(\theta)$ . Using the facts that  $A_{\theta_i} Z_{\theta_i} = 0$  and  $Z_{\theta_i} A_{\theta_i} = 0$ , we get

$$\frac{d}{d\theta} (Z_\theta A_\theta Z_\theta)|_{\theta=\theta_i} = Z_{\theta_i} \dot{A}_{\theta_i} Z_{\theta_i}, \tag{2.12}$$

where dot denotes differentiation with respect to  $\theta$ . Thus, for  $|\theta - \theta_i|$  being small enough,

$$\dim E_+(Z_\theta A_\theta Z_\theta) = \dim E_+(C_r[A_{\theta_i}])$$

and

$$\dim E_-(Z_\theta A_\theta Z_\theta) = \dim E_-(C_r[A_{\theta_i}])$$

if  $\theta - \theta_i > 0$ . Likewise, in case  $\theta - \theta_i < 0$  it turns out to be

$$\dim E_-(Z_\theta A_\theta Z_\theta) = \dim E_+(C_r[A_{\theta_i}])$$

and

$$\dim E_+(Z_\theta A_\theta Z_\theta) = \dim E_-(C_r[A_{\theta_i}]).$$

Let  $\tilde{Z}_\theta$  be the orthogonal projection from  $E$  to  $E_-(Z_\theta A_\theta Z_\theta)$ . For fixed  $i$ , if  $\hat{Z} = \lim_{\theta \rightarrow \theta_i^-} \tilde{Z}_\theta$  and  $\hat{P} = \lim_{\theta \rightarrow \theta_i^-} P_\theta$ , then

$$\hat{P} = P_{\theta_i} + \hat{Z}$$

and

$$\text{ind}(P_{\theta_i - \varepsilon} |_{E_-(A_{\theta_i})}) = \text{ind}(P_{\theta_i - \varepsilon} |_{R(\hat{P})}) + \text{ind}(\hat{P} |_{E_-(A_{\theta_i})}),$$

where  $R(\hat{P})$  is the range of  $\hat{P}$ . Now (2.11) follows from the facts that  $\text{ind}(P_{\theta_i - \varepsilon} |_{R(\hat{P})}) = 0$  and  $\text{ind}(\hat{P} |_{E_-(A_{\theta_i})}) = -\dim E_+(C_r[A_{\theta_i}])$ . The proof is complete.  $\square$

### 3. Proof of Theorem 1

In the proof of Theorem 1, the case of  $E_0(A - F) = \{0\}$  will be treated first. We start with some preliminary lemmas. As indicated in the proof of Lemma 2,  $F_k \rightarrow F$  in the operator norm. This implies that  $E_0(A - F_k) = \{0\}$  for  $k \geq k_0$ , if  $k_0$  is chosen large enough.

Let  $\mathcal{E}_k^* = \{\xi \mid \xi \in H^1([-k, k], \mathbb{R}^{2n}), \xi(-k) \in V_u \text{ and } \xi(k) \in V_s\}$  and  $\mathcal{E}_k$  be the completion of  $\mathcal{E}_k^*$  under the graph norm of  $|\hat{A}|^{1/2}$ .

**Lemma 5.** Assume that  $E_0(A - F) = \{0\}$ . Then there is a  $k_0 \in \mathbb{N}$  such that

$$i(x) = -Sf((A - \sigma F_k)|_{\mathcal{E}_k}, 0 \leq \sigma \leq 1)$$

for any fixed  $k > k_0$ .

**Proof.** Note that by Definition 2 and Proposition 3,  $i(x) = -Sf(A - \sigma F, 0 \leq \sigma \leq 1)$ . For large  $k$ ,  $F_k$  is a small perturbation of  $F$ , so there is a  $k_0 > 0$  such that  $Sf(A - \sigma F_k, 0 \leq \sigma \leq 1) = Sf(A - \sigma F, 0 \leq \sigma \leq 1)$  if  $k \geq k_0$ . Let  $E_0((A - \sigma F_k)|_{\mathcal{E}_k})$  be the null space of  $(A - \sigma F_k)|_{\mathcal{E}_k}$ . It is not difficult to show that if  $\xi \in E_0((A - \sigma F_k)|_{\mathcal{E}_k})$  then  $\xi \in \mathcal{E}_k^*$ . For a given  $\xi \in \mathcal{E}_k^*$ , we denote  $\tilde{\xi}$  to be an extension of  $\xi$  defined by

$$\tilde{\xi}(t) = \begin{cases} \xi & \text{if } |t| \leq k, \\ e^{(t-k)JB_*} \xi(k) & \text{if } t > k, \\ e^{(t+k)JB_*} \xi(-k) & \text{if } t < -k. \end{cases} \tag{3.1}$$

It is easy to see that  $\xi \in E_0((A - \sigma F_k)|_{\mathcal{E}_k})$  if and only if  $\tilde{\xi} \in E_0(A - \sigma F_k)$ . Furthermore, direct calculation on the crossing operators shows that

$$Sf(A - \sigma F_k, 0 \leq \sigma \leq 1) = Sf((A - \sigma F_k)|_{\mathcal{E}_k}, 0 \leq \sigma \leq 1).$$

This together with  $i(x) = -Sf(A - \sigma F, 0 \leq \sigma \leq 1)$  completes the proof.  $\square$



**Remark 5.** (a) The proof of Lemma 5 indicates that  $\xi \in E_0((A - F_k)|_{\mathcal{E}_k})$  if and only if  $\tilde{\xi} \in E_0(A - F_k)$ .

(b) In [36,44] the authors dealt with spectral flow of unbounded operators. Let  $\sharp A_\sigma = -J \frac{d}{dt} - B_* - \sigma B_k(t)$ . Clearly  $\sharp A_\sigma : \mathcal{E}_k^* \rightarrow L^2([-k, k], \mathbb{R}^{2n})$  and it is an unbounded self-adjoint operator on  $L^2([-k, k], \mathbb{R}^{2n})$ . Straightforward calculation on the crossing operators shows that

$$Sf(\sharp A_\sigma) = Sf((A - \sigma F_k)|_{\mathcal{E}_k}). \tag{3.2}$$

Next, we are going to show

$$i_*(x) = -Sf((A - \sigma F_k)|_{\mathcal{E}_k}, 0 \leq \sigma \leq 1) \tag{3.3}$$

if  $k$  is sufficiently large. An interesting property of Maslov index proved in [36] will be used in the proof of (3.3). As indicated in [28], for a pair of Lagrangian paths  $\{(U_1(t), U_2(t)) \mid U_1, U_2 \in \mathcal{L}(a, b)\}$ , a variant of Maslov index  $\mu_{RS}$  defined in [35,36] can be formulated as

$$\mu_{RS}(U_1, U_2) = \mu(U_2, U_1) + \frac{1}{2} [\dim(U_1(b) \cap U_2(b)) - \dim(U_1(a) \cap U_2(a))]. \tag{3.4}$$

**Remark 6.** (a) In (3.4),  $\mu(U_2, U_1)$  denotes a Maslov index defined in [9], which has been used in the definition of  $i_*(x)$ .

(b) As mentioned in Section 1, a systematic treatment of Maslov index has been worked out in [9]. On the other hand, the works of Robbin and Salamon [35,36] illustrate a more convenient way in dealing with the calculation of Maslov index. This advantage will be used in the proof of Theorem 1.

Let  $U(\sigma)$  be a Lagrangian path in  $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, -w_0 \oplus w_0)$  and  $\Gamma \in C([a, b] \times \mathbb{R}, S(2n))$ , where  $S(2n)$  is the set of symmetric linear transformations from  $\mathbb{R}^{2n}$  to itself. Consider the following boundary value problem for a perturbed Cauchy–Riemann operator:

$$\begin{aligned} \bar{\partial}z &= \frac{\partial z}{\partial \sigma} - J \frac{\partial z}{\partial t} + \Gamma z, \quad a \leq t \leq b, \sigma \in \mathbb{R}, \\ (z(a, \sigma), z(b, \sigma)) &\in U(\sigma). \end{aligned}$$

Set  $H_\sigma^1 = \{\zeta \mid \zeta \in H^1([a, b], \mathbb{R}^{2n}), (\zeta(a), \zeta(b)) \in U(\sigma)\}$ . For fixed  $\sigma$ ,  $-J \frac{d}{dt} + \Gamma(t, \sigma) : H_\sigma^1 \rightarrow L^2$  is a self-adjoint operator on  $L^2$ . For  $\sigma \in \mathbb{R}$ , let  $\psi(t, \sigma)$  be a family of symplectic paths determined by  $J \frac{d\psi}{dt} = \Gamma(t, \sigma)\psi$  and  $\psi(a, \sigma) = I_{2n}$ .

In the next proposition,  $\Gamma(t, \sigma) = \Gamma(t, 1)$  if  $\sigma \geq 1$  and  $\Gamma(t, \sigma) = \Gamma(t, 0)$  if  $\sigma \leq 0$ . Moreover, it is assumed that the null space of  $-J \frac{d}{dt} + \Gamma(t, \sigma)$  is trivial if  $\sigma = 0, 1$ . We note that  $\{\xi, M\xi \mid \xi \in \mathbb{R}^{2n}\}$  is a Lagrangian subspace of  $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, -w_0 \oplus w_0)$  if  $M \in \text{Sp}(2n)$ . Thus the graph of a symplectic path  $\psi$  is a Lagrangian path, which will be denoted by  $G_r\psi$ .

**Proposition 4.** *The perturbed Cauchy–Riemann operator  $\bar{\partial}$  is a Fredholm operator. Moreover, if  $\hat{\psi}(\sigma) = \psi(b, \sigma)$ , the Fredholm index of  $\bar{\partial}$  satisfies the following property:*

$$\text{ind } \bar{\partial} = -\mu_{RS}(G_r\hat{\psi}, U) = -Sf\left(J \frac{d}{dt} - \Gamma(t, \sigma), 0 \leq \sigma \leq 1\right).$$

We refer to [36] for a detailed proof of Proposition 4.

Let  $\gamma_k$  be a symplectic path on  $[-k, k]$  which satisfies

$$\dot{\gamma} = J(B_* + B_k(t))\gamma, \quad \gamma(-k) = I_{2n}.$$

**Lemma 6.** *Assume that  $E_0(A - F) = \{0\}$ . Then there is a  $k_1 \in \mathbb{N}$  such that*

$$\mu_{RS}(\gamma_k V_u, V_s) = i_*(x) \tag{3.5}$$

if  $k > k_1$ .

**Proof.** Let  $k \in \mathbb{N}$  and  $\Phi_k(t, v)$  be the solution of the initial value problem

$$\frac{d}{dt}\Phi = J(B_* + B_k(t))\Phi, \quad \Phi(v, v) = I_{2n}. \tag{3.6}$$

Define  $V_s^k(v) = \{\xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow \infty} \Phi_k(t, v)\xi = 0\}$  and  $V_u^k(v) = \{\xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow -\infty} \Phi_k(t, v)\xi = 0\}$ . In what follows  $V_u^k(t; a, b)$  denotes the part of Lagrangian path  $V_u^k(t)$  truncated on a subinterval  $[a, b]$ , so does  $V_s(t; a, b)$ .

For a fixed large  $k$ , set  $\hat{F}_\theta(t) = \theta F + (1 - \theta)F_k$ ,  $\theta \in [0, 1]$ . If  $k$  is sufficiently large,  $\hat{F}_\theta$  is a small perturbation of  $F$  and thus  $E_0(A - \hat{F}_\theta) = 0$  for all  $\theta \in [0, 1]$ . Applying Proposition 1 yields

$$\mu(V_s(\tau), V_u(t; -\tau, \tau)) = \mu(V_s^k(\tau), V_u^k(t; -\tau, \tau)), \tag{3.7}$$

where as noted in Remark 1, (3.7) is valid for pairs of Lagrangian paths taken on any truncated subinterval  $[-\tau, \tau]$  as long as  $\tau$  is large enough. Recall that  $V_u^k(t) = V_u$  if  $t \leq -k$  and  $V_s^k(t) = V_s$  if  $t \geq k$ . It follows from Proposition 1 and (3.4) that

$$\begin{aligned} \mu(V_s^k(\tau), V_u^k(t; -\tau, \tau)) &= \mu(V_s^k(k), V_u^k(t; -k, k)) \\ &= \mu_{RS}(V_u^k(t; -k, k), V_s^k(k)) \\ &= \mu_{RS}(\gamma_k V_u, V_s). \end{aligned}$$

This together with (3.7) shows that

$$\mu_{RS}(\gamma_k V_u, V_s) = \mu(V_s(\tau), V_u(t; -\tau, \tau)) = i_*(x).$$

The proof is complete.  $\square$

**Proof of Theorem 1.** In the first step we treat the case where the null space of  $A - F$  is trivial. Set  $\Gamma(t, \sigma) = -(B_* + \sigma B_k(t))$  for  $\sigma \in [0, 1]$ . Taking  $a = -k$ ,  $b = k$  and invoking Proposition 4, we get

$$Sf\left(J \frac{d}{dt} - \Gamma(t, \sigma), 0 \leq \sigma \leq 1\right) = \mu_{RS}(G_r \hat{\psi}, V_u \oplus V_s), \tag{3.8}$$

where  $\hat{\psi}$  is a symplectic path defined by  $\hat{\psi}(\sigma) = \psi(k, \sigma)$  and, as a function of  $t$ ,  $\psi(t, \sigma)$  satisfies the initial value problem

$$\frac{d\psi}{dt} = J(B_* + \sigma B_k(t))\psi, \quad \psi(-k, \sigma) = I_{2n}.$$

Applying Theorem 3.2 of [35] yields

$$\mu_{RS}(G_r \hat{\psi}, V_u \oplus V_s) = \mu_{RS}(\hat{\psi} V_u, V_s). \tag{3.9}$$

Next, take a homotopy with the following form:

$$\tilde{B}_\sigma = \begin{cases} 2\sigma B_* & \text{if } 0 \leq \sigma \leq \frac{1}{2}, \\ B_* + (2\sigma - 1)B_k & \text{if } \frac{1}{2} \leq \sigma \leq 1. \end{cases}$$

Let  $\tilde{\phi}(t, \sigma)$  satisfy

$$\frac{d\tilde{\phi}}{dt} = J\tilde{B}_\sigma \tilde{\phi}, \quad \tilde{\phi}(-k, \sigma) = I_{2n}.$$

By direct calculation

$$\tilde{\phi}(k, \sigma) = \begin{cases} \exp(4k\sigma J B_*) & \text{if } 0 \leq \sigma \leq \frac{1}{2}, \\ \hat{\psi}(2\sigma - 1) & \text{if } \frac{1}{2} \leq \sigma \leq 1. \end{cases}$$

Observe that  $\gamma_k$  is a symplectic path on  $[-k, k]$ ,  $\tilde{\phi}(k, \cdot)$  is a symplectic path on  $[0, 1]$ , and by homotopy invariance

$$\mu_{RS}(\gamma_k V_u, V_s) = \mu_{RS}(\tilde{\phi}(k, \cdot) V_u, V_s). \tag{3.10}$$

Clearly,  $\exp(JB_*)V_u = V_u$ . Hence  $\mu_{RS}(\tilde{\phi}(k, \cdot)V_u, V_s) = \mu_{RS}(\hat{\psi}V_u, V_s)$ . This together with (3.8)–(3.10) leads to

$$\mu_{RS}(\gamma_k V_u, V_s) = -Sf\left(-J\frac{d}{dt} + \Gamma(t, \sigma), 0 \leq \sigma \leq 1\right). \tag{3.11}$$

Combining (3.11) with (3.2) gives

$$\mu_{RS}(\gamma_k V_u, V_s) = -Sf((A - \sigma F_k)|_{\varepsilon_k}, 0 \leq \sigma \leq 1).$$

Invoking Lemma 6 yields (3.3). This together with Lemma 5 completes the proof of Theorem 1 in case  $E_0(A - F) = \{0\}$ .

If  $E_0(A - F)$  is non-trivial,  $\dim(V_u(t) \cap V_s(t)) = \dim(E_0(A - F))$  for  $t \in \mathbb{R}$ . In this case, the assertion of theorem follows from a perturbation argument as follows. Set  $\langle G_\epsilon \xi, \eta \rangle = \int_{-\infty}^{\infty} ((B(t) - \epsilon h(t)I_{2n})\xi, \eta) dt$ , where  $h \in C^2$ ,  $h(t) > 0$  if  $t \in (0, 1)$  and  $h(t) = 0$  if  $t \notin (0, 1)$ . It is easy to check that  $E_0(A - G_\epsilon) = 0$  if  $\epsilon \in (0, \epsilon_0]$  and  $\epsilon_0$  is sufficiently small. We claim

$$\text{ind}(P_{A-F}|_{E_-(A-G_\epsilon)}) = 0 \quad \text{if } \epsilon \in (0, \epsilon_0]. \tag{3.12}$$

Indeed, in view of the definition of  $G_\epsilon$ , it follows from direct calculation that  $E_-(C_r[A - G_0]) = 0$ . This together with Proposition 3, Lemma 4 and (2.8) gives (3.12). Since  $\text{ind}(P_A|_{E_-(A-G_\epsilon)}) = \text{ind}(P_A|_{E_-(A-F)}) + \text{ind}(P_{A-F}|_{E_-(A-G_\epsilon)})$ , it follows that

$$\text{ind}(P_A|_{E_-(A-F)}) = \text{ind}(P_A|_{E_-(A-G_\epsilon)}) \quad \text{if } \epsilon \in (0, \epsilon_0]. \tag{3.13}$$

On the other hand, for  $\epsilon \in [0, \epsilon_0]$ , let  $\Psi_\epsilon(t, v)$  satisfy

$$\frac{d\Psi_\epsilon}{dt} = J(B_* + B(t) - \epsilon h(t)I_{2n})\Psi_\epsilon, \quad \Psi_\epsilon(v, v) = I_{2n}. \tag{3.14}$$

Since  $E_0(A - G_\epsilon) = 0$ , applying the results obtained in step 1 gives

$$\text{ind}(P_A|_{E_-(A-G_\epsilon)}) = \mu(\hat{V}_s^\epsilon(\tau), \hat{V}_u^\epsilon(t; -\infty, \tau)),$$

where  $\tau$  is a sufficiently large number,  $\hat{V}_s^\epsilon$  and  $\hat{V}_u^\epsilon$  are Lagrangian subspaces of  $\mathbb{R}^{2n}$  defined by

$$\hat{V}_s^\epsilon(v) = \{\xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow \infty} \Psi_\epsilon(t, v)\xi = 0\}$$

and

$$\hat{V}_u^\epsilon(v) = \{\xi \mid \xi \in \mathbb{R}^{2n} \text{ and } \lim_{t \rightarrow -\infty} \Psi_\epsilon(t, v)\xi = 0\}.$$

As noted above,

$$\mu(\hat{V}_s^\epsilon(\tau), \hat{V}_u^\epsilon(t; -\infty, \tau)) = \mu(\hat{V}_s^\epsilon(\tau), \hat{V}_u^\epsilon(t; -\tau_1, \tau))$$

provided that  $\tau_1$  is large enough. Since  $h(t) = 0$  if  $t \notin (0, 1)$ , it follows that  $\hat{V}_s^\epsilon(t) = V_s(t)$  if  $t > 1$  and  $\hat{V}_u^\epsilon(t) = V_u(t)$  if  $t < 0$ . Consequently

$$\mu(\hat{V}_s^\epsilon(\tau), \hat{V}_u^\epsilon(t; -\tau_1, \tau)) = \mu(V_s(\tau), \hat{V}_u^\epsilon(t; -\tau_1, \tau))$$

and  $\tau, \tau_1$  can be chosen independent of  $\epsilon$ . Observe that  $V_u(t) = \phi(t, -\tau_1)V_u(-\tau_1)$  and  $\hat{V}_u^\epsilon(t) = \Psi_\epsilon(t, -\tau_1)V_u(-\tau_1)$ . Let  $\hat{U}$  be a Lagrangian path defined by  $\hat{U} = \{\hat{V}_u^\epsilon(\tau) \mid 0 \leq \epsilon \leq \epsilon_0\}$ . By homotopy invariance and path additivity of Maslov index,

$$\mu(V_s(\tau), \hat{V}_u^\epsilon(t; -\tau_1, \tau)) = \mu(V_s(\tau), V_u(t; -\tau_1, \tau)) + \mu(V_s(\tau), U_*),$$

where  $\epsilon \in (0, \epsilon_0)$  and  $U_*$  is the truncation of  $\hat{U}$  on  $[0, \epsilon]$ .

It remains to show  $\mu(V_s(\tau), U_*) = 0$  to complete the proof. For  $\epsilon > 0$  and sufficiently small, we know  $E_0(A - G_\epsilon) = 0$  and  $\dim(\hat{V}_s^\epsilon(\tau) \cap \hat{V}_u^\epsilon(\tau)) = 0$ . This together with Theorem 3.1(ii) of [28] implies that

$$\mu(V_s(\tau), U_*) = m^+(\Gamma(U_*, V_s(\tau), 0)),$$

where  $\Gamma(U_*, V_s(\tau), 0) = -(\Psi_\epsilon(\tau))^T J \frac{\partial}{\partial \epsilon} \Psi_\epsilon(\tau)|_{\epsilon=0}$  is a crossing form defined in [28,35] and  $m^+(\Gamma)$  is the number of positive eigenvalues of  $\Gamma$ . Differentiating (3.14) with respect to  $\epsilon$  and multiplying by  $-\Psi_\epsilon^T J$ , we obtain

$$-\Psi_\epsilon^T J \frac{\partial^2 \Psi_\epsilon}{\partial \epsilon \partial t} = -\Psi_\epsilon^T h(t) I_{2n} \Psi_\epsilon + \Psi_\epsilon^T (B_* + B(t) - \epsilon h(t) I_{2n}) \frac{\partial \Psi_\epsilon}{\partial \epsilon}.$$

Hence by direct calculation

$$\begin{aligned} - \int_{-\tau_1}^{\tau} \Psi_\epsilon^T h(t) I_{2n} \Psi_\epsilon dt &= -\Psi_\epsilon^T J \frac{\partial \Psi_\epsilon}{\partial \epsilon} \Big|_{-\tau_1}^{\tau} + \int_{-\tau_1}^{\tau} \frac{\partial \Psi_\epsilon^T}{\partial t} J \frac{\partial \Psi_\epsilon}{\partial \epsilon} dt - \int_{-\tau_1}^{\tau} \Psi_\epsilon^T (B_* + B(t) - \epsilon h(t) I_{2n}) \frac{\partial \Psi_\epsilon}{\partial \epsilon} dt \\ &= -(\Psi_\epsilon(\tau))^T J \frac{\partial \Psi_\epsilon}{\partial \epsilon}(\tau), \end{aligned}$$

from which we know  $\mu(V_s(\tau), U_*) = 0$ . The proof is complete.  $\square$

#### 4. Lagrangian systems

The aim of this section turns to the Morse index of homoclinic orbits of Lagrangian system. Consider

$$\mathcal{F}(q) = \int_{-\infty}^{\infty} L(t, q, \dot{q}) dt, \tag{4.1}$$

where  $L$  satisfies the Legendre convexity condition:

$$\left( \frac{\partial^2 L}{\partial v^2}(t, u, v)w, w \right) > 0 \quad \text{for } w \in \mathbb{R}^n \setminus \{0\}, (u, v) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{4.2}$$

Assumed that  $\frac{\partial L}{\partial u}(t, 0, 0) = \frac{\partial L}{\partial v}(t, 0, 0) = 0$  for all  $t \in \mathbb{R}$ . As above, 0 is an equilibrium of

$$\frac{d}{dt} \frac{\partial L}{\partial v}(t, q, \dot{q}) - \frac{\partial L}{\partial u}(t, q, \dot{q}) = 0, \tag{4.3}$$

where and throughout this section dot denotes differentiation with respect to  $t$ . Suppose  $q_0$  is an orbit homoclinic to 0. The linearization of (4.3) at  $q_0$  is given by

$$-\frac{d}{dt} (N_0(t)\dot{y} + Q_0(t)y) + Q_0^T(t)\dot{y} + R_0(t)y = 0, \tag{4.4}$$

where  $N_0(t) = \frac{\partial^2 L}{\partial v^2}(t, q_0(t), \dot{q}_0(t))$ ,  $Q_0(t) = \frac{\partial^2 L}{\partial u \partial v}(t, q_0(t), \dot{q}_0(t))$  and  $R_0(t) = \frac{\partial^2 L}{\partial u^2}(t, q_0(t), \dot{q}_0(t))$ . Using Legendre transform  $p = \frac{\partial L}{\partial v}(t, q, \dot{q})$  and  $H(t, p, q) = p \cdot \dot{q} - L(q, \dot{q}, t)$ , (4.3) can be converted to

$$\dot{x} = JH'(t, x) \tag{4.5}$$

if  $x(t) = (\frac{\partial L}{\partial v}(t, q_0(t), \dot{q}_0(t)), q_0(t))$ . Straightforward calculation shows that the linearization of (4.5) at  $x(t)$  is  $\dot{z} = JY(t)z$ , where

$$Y(t) = \begin{pmatrix} N_0^{-1}(t) & -N_0^{-1}(t)Q_0(t) \\ -Q_0^T N_0^{-1}(t) & Q_0^T N_0^{-1}(t)Q_0(t) - R_0(t) \end{pmatrix}. \tag{4.6}$$

Remark that  $q_0(t) \rightarrow 0$  and  $\dot{q}_0(t) \rightarrow 0$  as  $|t| \rightarrow \infty$ . Set  $N = \lim_{t \rightarrow \infty} N_0(t)$ ,  $Q = \lim_{t \rightarrow \infty} Q_0(t)$ ,  $R = \lim_{t \rightarrow \infty} R_0(t)$  and

$$D = \begin{pmatrix} N & Q \\ Q^T & R \end{pmatrix}.$$

**Lemma 7.** *If  $D$  is positive definite, then (H2) holds with  $B_* = \lim_{t \rightarrow \infty} Y(t)$ .*

Under the hypothesis of Lemma 7, we have the following result:

**Theorem 2.** Let  $q_0$  be a homoclinic orbit of (4.3) and  $i_{\#}(q_0)$  be its Morse index. If  $x(t) = (\frac{\partial L}{\partial v}(t, q_0(t), \dot{q}_0(t)), q_0(t))$ , then  $x(t)$  is a homoclinic orbit of (4.5) and  $i(x) = i_{\#}(q_0)$ .

**Proof.** It is easy to verify that  $x(t)$  is a homoclinic orbit of (4.5). We are going to use some properties of spectral flow to prove  $i(x) = i_{\#}(q_0)$ . Consider, for  $0 \leq \theta \leq 1$ , a family of Lagrangian system  $\mathcal{F}_{\theta}(q) = \int_{-\infty}^{\infty} L_{\theta}(t, q, \dot{q})$ , where

$$L_{\theta}(t, u, v) = \frac{1}{2} \left( \begin{pmatrix} N_{\theta}(t) & Q_{\theta}(t) \\ Q_{\theta}^T(t) & R_{\theta}(t) \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}, \begin{pmatrix} v \\ u \end{pmatrix} \right), \tag{4.7}$$

and  $(N_{\theta}(t), Q_{\theta}(t), R_{\theta}(t))$  is a homotopy from  $(N_0(t), Q_0(t), R_0(t))$  to  $(N, Q, R)$ . To deal with the spectral flow of  $\mathcal{F}_{\theta}$ , we may assume that each eigenvalue crossing is a regular crossing. This can be achieved by choosing a suitable homotopy as noted in Remark 4(a). Let  $Y_{\theta}(t)$  be a matrix function defined as in the form of (4.6) with only  $N_0(t), Q_0(t), R_0(t)$  being replaced by  $N_{\theta}(t), Q_{\theta}(t), R_{\theta}(t)$  respectively. By Legendre transform,  $y_{\theta} \in E_0(\mathcal{F}_{\theta}'')$  if and only if  $z_{\theta} = (\frac{\partial L_{\theta}}{\partial v}(t, y_{\theta}, \dot{y}_{\theta}), y_{\theta}) \in E_0(X_{\theta})$ , where as in (2.2)  $\hat{X}_{\theta} = -J \frac{d}{dt} - Y_{\theta}$  and

$$\langle X_{\theta} \xi, \eta \rangle = \int_{-\infty}^{\infty} ((\hat{X}_{\theta}^+)^{1/2} \xi, (\hat{X}_{\theta}^+)^{1/2} \eta) dt - \int_{-\infty}^{\infty} ((\hat{X}_{\theta}^-)^{1/2} \xi, (\hat{X}_{\theta}^-)^{1/2} \eta) dt$$

for  $\xi, \eta \in H^{1/2}(\mathbb{R}, \mathbb{R}^{2n})$ .

To show  $i(x) = i_{\#}(q_0)$ , it suffices to prove

$$Sf(X_{\theta}, 0 \leq \theta \leq 1) = Sf(\mathcal{F}_{\theta}'', 0 \leq \theta \leq 1). \tag{4.8}$$

By (2.8), we see (4.8) holds if  $\text{sign } C_r[X_{\theta}] = \text{sign } C_r[\mathcal{F}_{\theta}'']$  whenever  $y_{\theta} \in E_0(\mathcal{F}_{\theta}'')$ . In view of

$$H_{\theta}(p, q) = p \cdot \dot{q} - L_{\theta}(t, q, \dot{q}) = \frac{1}{2} \left( Y_{\theta} \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right),$$

it follows from direct calculation that

$$\text{sign } C_r[X_{\theta}] = -\text{sign} \left( P_{\theta} \frac{\partial}{\partial \theta} Y_{\theta} P_{\theta} \right) = \text{sign} \left( P_{\theta} \frac{\partial}{\partial \theta} L_{\theta} P_{\theta} \right) = \text{sign } C_r[\mathcal{F}_{\theta}'']. \quad \square$$

**Proof of Lemma 7.** Let

$$D_1 = \begin{pmatrix} (Q^T - Q)N^{-1} & R \\ N^{-1} & 0 \end{pmatrix} \quad \text{and} \quad D_2 = \begin{pmatrix} I_n & -Q \\ 0 & I_n \end{pmatrix}.$$

It is easy to check that

$$D_2^{-1} = \begin{pmatrix} I_n & Q \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad D_1 = D_2 J B_* D_2^{-1}.$$

Suppose there exist  $\lambda \in \mathbb{R}$  and  $\eta = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^n$  such that  $D_1 \eta = i \lambda \eta$ . Then it follows from straightforward calculation that

$$N^{-1} \xi_1 = i \lambda \xi_2, \tag{4.9}$$

$$[\lambda^2 N + R + i \lambda (Q^T - Q)] \xi_2 = 0 \tag{4.10}$$

and

$$([\lambda^2 N + R + i \lambda (Q^T - Q)] \xi_2, \xi_2) = (D \zeta, \zeta) \quad \text{if } \zeta = (\lambda \xi_2, -i \xi_2). \tag{4.11}$$

Since  $D$  is positive definite, (4.10) and (4.11) imply  $\xi_2 = 0$ . This together with (4.9) and (4.2) yields  $\eta = 0$ . Now the proof is complete, due to the fact that  $\sigma(JB_*) = \sigma(D_1)$ .  $\square$

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