Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 677–710
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Shear layer solutions of incompressible MHD and dynamo effect

Solutions discontinues de la MHD incompressible et effet dynamo

David Gérard-Varet [a], Frédéric Rousset [b,*]

[a] *D.M.A, E.N.S., 45, rue d'Ulm, 75230 Paris Cedex 05, France*
[b] *Laboratoire J.A. Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, France*

Received 27 June 2005; received in revised form 26 January 2006; accepted 26 April 2006

Available online 19 December 2006

Abstract

This paper is devoted to the equations of incompressible magnetohydrodynamics (MHD). Its general concern is the "dynamo effect", i.e. the growth of magnetic field through the movement of a conducting fluid. Motivated by the so-called "stretch-diffuse mechanism", we study the nonlinear stability of solutions $(u, b = 0)$ where the velocity u is a regularized vortex sheet, and the magnetic component b is zero. We prove that dynamo effect is possible when both curvature of the sheet and magnetic diffusion are non-zero, and impossible otherwise.

Résumé

Cet article est consacré aux équations de la magnétohydrodynamique (MHD) incompressible. Le sujet principal en est l'effet dynamo, c'est-à-dire l'instabilité du champ magnétique, due aux mouvements du fluide conducteur. Motivés par la compréhension du mécanisme dit d'« étirement-diffusion », nous étudions la stabilité non-linéaire de solutions de la forme $(u, b = 0)$ où la vitesse u est une feuille de tourbillon régularisée, et la composante magnétique b est nulle. Nous prouvons que l'effet dynamo est possible si la courbure de la feuille de tourbillon et la diffusion magnétique sont non-nulles, et impossible sinon.

Keywords: WKB expansions; Nonlinear instability; Incompressible MHD

1. Introduction

This paper deals with shear layer solutions of viscous incompressible MHD equations. It is motivated by dynamo theory. Before we state precisely our main result, let us first specify the general framework.

* Corresponding author.
E-mail address: frousset@math.unice.fr (F. Rousset).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.04.005

The incompressible MHD equations read in a dimensionless form:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \dfrac{1}{Re}\Delta u = \operatorname{curl} b \times b + f, \\ \partial_t b - \operatorname{curl}(u \times b) - \dfrac{1}{Rm}\Delta b = 0, \\ \operatorname{div} u = \operatorname{div} b = 0. \end{cases} \tag{1}$$

They describe the evolution of an incompressible and electrically conducting fluid. They are derived from the incompressible Navier–Stokes equations, the Maxwell equations, and the Ohm's law in a conducting medium (see [12]). Functions $u = u(t, \mathrm{x}) \in \mathbb{R}^3$, $b = b(t, \mathrm{x}) \in \mathbb{R}^3$ are the fluid velocity and the magnetic field respectively. The source term $f = f(t, \mathrm{x}) \in \mathbb{R}^3$ models an external forcing, due for instance to natural convection or mechanical constraints. The space and time variables are $t \in \mathbb{R}^+$, $\mathrm{x} = (\mathrm{x}_h, z) = (x, y, z) \in \Omega \subset \mathbb{R}^3$. We denote

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \qquad \nabla = (\partial_x, \partial_y, \partial_z),$$

and for any $v = (v_x(\mathrm{x}), v_y(\mathrm{x}), v_z(\mathrm{x})) \in \mathbb{R}^3$,

$$\operatorname{div} v = \partial_x v_x + \partial_y v_y + \partial_z v_z, \qquad \operatorname{curl} v = (\partial_y v_z - \partial_z v_y, \partial_z v_x - \partial_x v_z, \partial_x v_y - \partial_y v_x).$$

Re and *Rm* are positive constants called the hydrodynamic and magnetic Reynolds numbers. Note that the divergence free condition on b is preserved by Eq. $(1)_\mathrm{b}$. As soon as it is satisfied initially, it is satisfied for all positive times.

In brief, dynamo theory deals with the stability of solutions

$$(u, b) = \big(u(t, \mathrm{x}), 0\big)$$

of system (1). More precisely, it studies the generation of magnetic field from the fluid flow u. The basic idea is that the "self-excited" term $\operatorname{curl}(u \times b)$ may amplify the magnetic field through an exponential instability. As long as the fluid motion is strong enough, this transfer from kinetic to magnetic energy may thus prevent the decay of the magnetic field, despite the dissipation term $-(Rm^{-1})\Delta b$.

It is widely accepted that dynamo action takes place in the Earth, in the Sun, and in many other planets and stars. Therefore, the understanding of dynamo mechanisms is a major physical issue. It has been the matter of a huge literature: we refer to the recent review papers by Gilbert [7] and Fearn [3] for a good introduction and appropriate lists of references. Note that most of these references are limited to "kinematic dynamos": the Laplace force is neglected, and only the induction equation $(1)_\mathrm{b}$ is considered, at imposed velocity u.

Among the kinematic dynamos that have been suggested, one of the simplest is the *Ponomarenko dynamo* [11]: in cylindrical coordinates (r, τ, z), the velocity field is

$$U_P = \begin{cases} (0, r\Omega, U_z), & r < 1, \\ 0, & r > 1, \end{cases}$$

where Ω and U_z are positive constants. The basic idea is the following: due to the shear of u at $r = 1$, a weak radial magnetic field b_r is stretched (through the $\operatorname{curl}(u \times b)$ term) and gives strong azimuthal field b_τ. The radial component is then renewed by the azimuthal one through diffusion in curved geometry. This is the so-called "stretch-diffuse" mechanism.

Although it has been the basis of a successful experiment [4], the Ponomarenko dynamo lacks realism. The velocity field is discontinuous, which is unphysical. Eq. $(1)_\mathrm{a}$ is not taken into account, which hides the possible influence of the Laplace force. More generally, it does not account for nonlinearities of system (1). The aim of the present paper is to remove these restrictions, and to study mathematically the stretch-diffuse mechanism.

We will investigate the stability of solutions $(u^\varepsilon, 0)$ of (1), for which the fluid velocity u is some regularized vortex sheet. More precisely, we will consider flows of the following two types

1. *Regularized planar vortex sheets*

$$u^\varepsilon = \big(0, u_y(x/\varepsilon), u_z(x/\varepsilon)\big), \quad 0 < \varepsilon \ll 1, \tag{2}$$

where $u_y = u_y(\zeta)$ and $u_z = u_z(\zeta)$ are smooth, constant for $|\zeta| \geqslant 1$, and satisfy

$$\lim_{\zeta \to -\infty} (u_y, u_z) = (0, 0), \qquad \lim_{\zeta \to +\infty} (u_y, u_z) = (U_y, U_z).$$

2. *Regularized helical vortex sheets*

$$u^\varepsilon = \big(\omega(s/\varepsilon)\,\mathrm{x}_h^\perp,\ u_z(s/\varepsilon)\big), \quad 0 < \varepsilon \ll 1, \tag{3}$$

where $s = \sqrt{x^2+y^2} - 1$, $\omega = \omega(\zeta)$ and $u_z = u_z(\zeta)$ are smooth, constant for $|\zeta| \geqslant 1$, and satisfy

$$\lim_{\zeta\to+\infty}(\omega, u_z) = (0,0), \qquad \lim_{\zeta\to-\infty}(\omega, u_z) = (\Omega, U_z).$$

Flows of type (2) and (3), are regularized vortex sheets. Their curl varies in a strip of small width ε, around the surface Γ: $\{y=0\}$, resp. Γ: $\{s=0\}$. Note that type (3) corresponds to smooth versions of Ponomarenko flows. Note also that u^ε is divergence free in both cases.

Remark 1. The results which follow could probably be applied to a larger class of velocity fields, typically

$$u^\varepsilon = \big(\omega\big(\varphi(\mathrm{x}_h)/\varepsilon\big)\,\nabla^\perp\varphi(\mathrm{x}_h),\ u_z\big(\varphi(\mathrm{x}_h)/\varepsilon\big)\big).$$

Indeed, most arguments used in the sequel depend on the local properties of Γ: $\{\varphi(\mathrm{x}_h)=0\}$, and could probably extend to general functions φ. However, it would involve much more technicalities, so that we do not address this question here.

Remark 2. This article is reminiscent of the former paper [5] by one of the authors. In [5], the emphasis was put on another dynamo mechanism, generated by solutions $(u^\varepsilon, 0)$ of the type:

$$u^\varepsilon = U\big(\mathrm{x}/\varepsilon\big),$$

with periodic functions $U = U(\theta)$. It was shown that these high frequency oscillations were nonlinearly unstable, leading to instabilities of the magnetic field ("the G.O. Roberts dynamo"). Thus, our paper can be seen as a complement on small-scale dynamo mechanisms, focusing on concentration effects rather than oscillations.

Substituting $u = u^\varepsilon + v$ into (1), we will rather work with the system:

$$\begin{cases} \partial_t v + u^\varepsilon\cdot\nabla v + v\cdot\nabla u^\varepsilon + v\cdot\nabla v + \nabla p - \dfrac{1}{Re}\Delta v = \operatorname{curl} b\times b, \\ \partial_t b - \operatorname{curl}(u^\varepsilon\times b) - \operatorname{curl}(v\times b) - \dfrac{1}{Rm}\Delta b = 0, \\ \operatorname{div} v = \operatorname{div} b = 0, \end{cases} \tag{4}$$

We will consider domains without boundaries, either $\Omega = \mathbb{R}\times\mathbb{T}^2_{y,z}$, or $\Omega = \mathbb{R}^2\times\mathbb{T}_z$, or $\Omega = \mathbb{T}^3$. Note that we consider regularized vortex sheets which are constant outside a compact set of $(-\pi,\pi)^3$, so that they can be in particular considered as periodic. This kind of periodic boundary conditions is very classical in stability problems in fluid mechanics: it allows to study local instabilities and to avoid complications due to boundary layers. For any given $\varepsilon > 0$, classical existence theory for Navier–Stokes equations extends straightforwardly to system (4). In particular, for any (v_0, b_0) in $H^1(\Omega)^3$, divergence-free, there exists a unique maximal *strong solution*

$$v, b \in \mathcal{C}\big([0,T); H^1(\Omega)^3\big), \quad T = T(\varepsilon)$$

of system (4), with initial data v_0, b_0.

As briefly described above, the stretch-diffuse mechanism depends on three physical ingredients: the strength of the shear (for stretching of the magnetic field), the diffusion and the curvature of the sheet (for the renewal process). We will show rigorously the necessity of these ingredients, through several stability and instability results.

1.1. Antidynamo results

We state here antidynamo results, i.e. stability estimates. We start with the case of planar sheets. In this case, we work in the domain $\mathbb{R}_x\times\mathbb{T}^2_{y,z}$. With a suitable choice of the pressure, we can always assume that the solution verifies

$$\int_{y,z} v_x = 0.$$

We refer to [1] for details. We remind that this condition is necessary to have divergence free vector fields of finite energy. Therefore, we shall also consider solutions for which the magnetic field verifies

$$\int_{y,z} b_x = 0.$$

Note that such condition on the magnetic field is preserved by the evolution. Indeed, the condition gives that $\int_{y,z} b_x$ is independent of x and the equation gives

$$\partial_t \int_{y,z} b_x = \int_{y,z} \partial_y\big((u^\varepsilon + v) \times b\big)_z - \partial_z\big((u^\varepsilon + v) \times b\big)_y = 0.$$

We can prove the following stability result.

Theorem 1 *(Regularized planar vortex sheets). Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *of type* (2). *There exists* $\delta, C_0, C > 0$*, such that*: *for all* $\varepsilon \leqslant 1$*, for all* $Rm \in [1, +\infty)$*, and for all*

$$Re^{-1} > C_0 \int_\zeta \big|(u_y, u_z)'(\zeta)\big| \,\mathrm{d}\zeta,$$

system (4) *is stable, in the following sense*: *the strong solutions of* (4) *are global in time and satisfy*

$$E(v, b)(t) \leqslant C E(v, b)(0), \quad \forall t \geqslant 0,$$

if

$$E(v, b)(0) \leqslant \frac{\delta \varepsilon^{1/2}}{Rm^{7/2}},$$

where

$$E(v, b)(t) := \big\|v(t)\big\|^2 + \big\|b(t)\big\|^2 + Rm^2\big\|b_x(t)\big\|^2 + \frac{\varepsilon}{Rm}\|\nabla b\|^2 + \varepsilon\|\nabla v\|^2.$$

Remark 3. The smallness condition on *Re* ensures the hydrodynamic stability of the flow.

Remark 4. In the planar geometry, diffusion does not couple the transverse and tangential components of the magnetic field. This explains why the magnetic energy does not grow.

We now turn to the case of cylindrical sheets, with $\Omega = \mathbb{R}^2 \times \mathbb{T}_z$. The stability estimate of Theorem 1 degenerates as expressed in

Theorem 2 *(Regularized helical vortex sheets). Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *of type* (3). *There exists,* $C > 0$*,* $\gamma > 0$*,* $\delta > 0$ *such that*: *for all* $\varepsilon \leqslant 1$*, for* $Rm = 1/\varepsilon^q$*,* $q > 0$*, and for all* $Re = \mathcal{O}(1)$*, system* (4) *is stable in the following sense*: *all strong solutions* v, b *satisfy*

$$E(v, b)(t) \leqslant C \exp(\gamma t) E(v, b)(0), \quad \forall t \in [0, T),$$

if

$$E(v, b)(0) \leqslant \frac{\delta \varepsilon^{1/2}}{Rm^{5/2}},$$

where

$$T = T\big(\varepsilon, Rm, E(v, b)(0)\big) \geqslant \big|\log\big(C\varepsilon^{-1/2} Rm^{5/2} E(v, b)(0)\big)\big|$$

and

$$E(v, b)(t) := \big\|v(t)\big\|^2 + \big\|b(t)\big\|^2 + Rm\big\|\chi b_s(t)\big\|^2 + \frac{\varepsilon}{Rm}\big(\big\|\nabla b(t)\big\|^2 + \big\|\nabla v(t)\big\|^2\big),$$

for some smooth function $\chi = \chi(s)$ *with compact support near* $s = 0$.

Remark 5. In a curved geometry, diffusion can transfer energy from azimuthal to radial component. This allows amplification of the magnetic field, so that we only get an exponential bound. Namely, for *Rm* and ε almost independent (since q is arbitrary), the growth of $E(b)$ is controlled by $e^{\gamma t}$ for γ independent of ε. Hence, we cannot get a dynamo effect in times less than $|\log E(b)(0)|$.

The estimates can be improved when the magnetic diffusion is weaker. We shall consider the more favorable geometry $\Omega = \mathbb{T}^3$ so that we can assume that $\int v = \int b = 0$, and we will make the same assumption as in Theorem 1 for the Reynolds number *Re*. We can prove:

Theorem 3 *(Weak magnetic diffusion). Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *of type* (3). *There exists,* $C_0, C > 0$, $\gamma > 0$ *and* $\delta > 0$ *such that*: *for all* $\varepsilon \leqslant 1$, *for* $Rm \leqslant 1/(\varepsilon^2 \nu(\varepsilon))$ *with* $\lim_{\varepsilon\to 0} \nu(\varepsilon) = 0$ *and for all Re*

$$Re^{-1} > C_0 \int_\zeta |\omega'| + |u_z'|\,\mathrm{d}\zeta$$

system (4) *is stable, in the following sense*: *all strong solutions* v, b *satisfy*

$$E(v,b)(t) \leqslant C \exp\bigl(\gamma \nu^{1/3} t\bigr) E(v,b)(0), \quad \forall t \in [0,T),$$

if

$$E(v,b)(0) \leqslant \delta \varepsilon^{11/2} \nu^{13/6},$$

where

$$T = T\bigl(\varepsilon, Rm, E(v,b)(0)\bigr) \geqslant \frac{1}{\gamma \nu^{1/3}} \bigl|\log\bigl(C \varepsilon^{-11/2} \nu^{-13/6} E(v,b)(0)\bigr)\bigr|$$

and

$$E(v,b)(t) := \bigl\|v(t)\bigr\|^2 + \bigl\|b(t)\bigr\|^2 + \frac{\|\chi b_s(t)\|^2}{\varepsilon^2 \nu^{2/3}} + \nu \varepsilon^3 \bigl\|\nabla b(t)\bigr\|^2 + \varepsilon \|\nabla v\|^2,$$

for some smooth function $\chi = \chi(s)$ *with compact support near* $s = 0$.

Remark 6. Theorem 3 shows stability up to times $T = O(\nu^{-1/3})$, with respect to small enough initial perturbations. Note that T_ν increases when ν goes to zero. It may appear strange at first sight: weakening diffusion has usually a destabilizing effect. Again, this has to do with the stretch-diffuse mechanism, in which magnetic diffusivity enhances the magnetic field.

Remark 7. We obtain in Theorem 3 an exponential bound with growth rate $O(\nu^{1/3})$ for small ν. This is consistent with formal computations of Gilbert [6]. Gilbert studies in [6] linear equation $(1)_{\mathrm{b}}$, for some smooth axisymmetric flow u (that is $\varepsilon = 1$), and builds formally an exponentially growing solution b, with growth rate $O(Rm^{-1/3})$. This suggests that the energy bound in Theorem 3 is optimal.

Remark 8. Again, in Theorem 3, the assumption on the geometry ($\Omega = \mathbb{T}^3$) and the assumption on the Reynolds number *Re* are sufficient conditions for the hydrodynamical stability of the fluid. It will be clear from the proof that we can get the same result by keeping the domain $\Omega = \mathbb{R}^2 \times \mathbb{T}_z$ and by assuming the restrictive condition $Re = \nu^{2/3}$.

1.2. Dynamo results

We now state a dynamo result, when geometry is curved (u^ε of type (3)), $\Omega = \mathbb{R}^2 \times \mathbb{T}_z$, and magnetic diffusion is large enough (we choose $Rm = \varepsilon^{-1}$). Note that in this case, Theorem 2 roughly states that when the initial weighted energy $E(v,b)(0)$ is under the form ε^{3+2m} for $m > 0$, we have an estimate of the energy as long as it stays under the amplitude ε^3. This critical amplitude ε^3 is the one for which the nonlinear terms in the system begin to play an important part in the qualitative behavior. We shall prove that the stretch-diffuse mechanism can indeed lead to dynamo effect: an initial data with energy of order ε^{3+2m} can reach the energy size ε^3 on times $O(m|\ln\varepsilon|)$. After these considerations on the scaling, it is natural to study solutions of (1) under the form $b = \varepsilon^{3/2}\tilde{b}$, $u = u^\varepsilon + \varepsilon^{3/2}\tilde{v}$ so that forgetting the $\tilde{}$, we shall rather work on the system

$$\begin{cases} \partial_t v + u^\varepsilon \cdot \nabla v + v \cdot \nabla u^\varepsilon + \varepsilon^{3/2} v \cdot \nabla v + \nabla p - \dfrac{1}{Re}\Delta v = \varepsilon^{3/2}\,\mathrm{curl}\, b \times b, \\ \partial_t b - \mathrm{curl}(u^\varepsilon \times b) - \varepsilon^{3/2}\,\mathrm{curl}(v \times b) - \dfrac{1}{Rm}\Delta b = 0, \\ \mathrm{div}\, v = \mathrm{div}\, b = 0, \end{cases} \tag{5}$$

The dynamo effect is proved by the following nonlinear instability result on (5)

Theorem 4 *(Exponential instability). Let* $\{u^\varepsilon\}_{\varepsilon>0}$ *of type* (3), $p, s > 0$, $Re > 0$. *Assume* $Rm = \varepsilon^{-1}$, *and that* $\Omega, U_z \neq 0$, *then for every* $\kappa_0 > 0$ *such that*

$$\frac{|\Omega|^{2/3}}{2^{5/3}} - \left(1 + \frac{\Omega^2}{U_z^2}\right) > \kappa_0,$$

there exists $\kappa_1 > 0$ *such that for*

$$\left| \int_{-1}^{1} \omega - \frac{\Omega}{U_z} \int_{-1}^{1} u_z \right| \leqslant \kappa_1,$$

one can find $\eta = \eta(Re, \omega, u_z) > 0$, *times* $t(\varepsilon) = \mathrm{O}(|\ln(\varepsilon)|)$, *and families of smooth solutions* $\{(v^\varepsilon, b^\varepsilon)^t\}_{\varepsilon>0}$ *of* (5) *with*

$$\left\| (v^\varepsilon, b^\varepsilon)|_{t=0} \right\|_{H^s} = \mathrm{O}(\varepsilon^p), \quad \varepsilon \to 0$$

and

$$\|b^\varepsilon|_{t=t(\varepsilon)}\|_{L^2}^2 \geqslant \eta\sqrt{\varepsilon}, \qquad \|b^\varepsilon|_{t=t(\varepsilon)}\|_{L^\infty}^2 \geqslant \eta.$$

Remark 9. The solutions of Theorem 4 grow exponentially in time, with growth rate O(1). This shows that instability develops for times bigger than $\mathrm{O}(|\ln(E(b)(0))|)$. Thus, the stability result of Theorem 2 cannot be extended to longer times. Note that the lower bound applies to b^ε, and not only to $(v^\varepsilon, b^\varepsilon)$. Thus, it is exactly the mathematical expression of a dynamo: small-scale velocity u^ε generates destabilization of $b = 0$.

Remark 10. As will be clear later on, the instability is localized in a boundary layer of size $\mathrm{O}(\sqrt{\varepsilon})$ around Γ. This explains why the lower bound of the theorem is $\mathrm{O}(\sqrt{\varepsilon})$ in the L^2 norm and O(1) in the L^∞ norm. The derivation of an O(1) instability in L^2 seems much more difficult. Indeed this would require to follow the system on larger times to see if the layer instability fills the domain.

Remark 11. Again, our proof of dynamo instability is consistent with the seminal paper of Ponomarenko [11], in the context of linear equation $(1)_\mathrm{b}$, with $u = U_P$ (see also the first part of [6]). In this simplified setting, the solution b can be explicitly calculated through Bessel functions. In our framework ($\varepsilon > 0$, nonlinear) such analysis is no longer possible.

The paper is divided into two main sections. Section 2 is devoted to the antidynamo results, with the proof of Theorems 1–3. Next Section 3 is devoted to the study of the dynamo effect and the proof of Theorem 4.

2. Stability results

2.1. Proof of Theorem 1

The first step is to use the triangular structure of the singular term for the equation on the magnetic field in (4). We remind that for divergence-free vector fields v, b,

$$\mathrm{curl}(v \times b) = b \cdot \nabla v - v \cdot \nabla b, \qquad \mathrm{curl}\, b \times b = b \cdot \nabla b + \nabla\left(\frac{1}{2}|b^2|\right).$$

The equation for the component b_x is

$$\partial_t b_x + u^\varepsilon \cdot \nabla b_x + v \cdot \nabla b_x = \frac{1}{Rm}\Delta b_x + b \cdot \nabla v_x$$

and since $\nabla \cdot (u^\varepsilon + v) = 0$, we get by a standard energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|b_x\|^2 + \frac{1}{Rm}\|\nabla b_x\|^2 \leqslant \|b_x b\|\|\nabla v_x\|.$$

Hence after an integration in time, we get

$$Rm^2\|b_x\|^2 + Rm\int_0^t \|\nabla b_x\|^2 \leqslant Rm^2\big\|b_x(0)\big\|^2 + CRm^2\int_0^t \|b_x b\|\|\nabla v_x\|. \tag{6}$$

Next, we use the usual energy estimate for the full system (4), we get since

$$(\operatorname{curl} b \times b, v) + \big(\operatorname{curl}(v \times b), b\big) = 0$$

the estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|b\|^2 + \|v\|^2\big) + \frac{1}{Rm}\|\nabla b\|^2 + \frac{1}{Re}\|\nabla v\|^2 \leqslant \frac{1}{\varepsilon}\left|\int b_x(u'_y b_y + u'_z b_z)\right| + \frac{1}{\varepsilon}\left|\int v_x(u'_y v_y + u'_z v_z)\right|. \tag{7}$$

To estimate the singular terms, we shall make a crucial use of our geometrical setting and hence of our assumption

$$\int_{y,z} v_x = 0, \qquad \int_{y,z} b_x = 0. \tag{8}$$

To estimate the singular term, we decompose v into

$$v = \bar{v} + \tilde{v}, \quad \bar{v} = \int_{y,z} v,$$

we can write with $k = y, z$

$$\int v_x u'_k v_k = \int_x u'_k \int_{y,z} v_x v_k = \int_x u'_k \int_{y,z} v_x \bar{v}_k + \int_x u'_k \int_{y,z} v_x \tilde{v}_k$$

and we notice that the first integral vanishes since thanks to (8), we have

$$\int_{y,z} v_x \bar{v}_k \,\mathrm{d}y\,\mathrm{d}z = \bar{v}_k \int_{y,z} v_x = 0.$$

To estimate the singular term involving the velocity, we can use the following Sobolev and Poincaré inequalities

$$\big|f(t,x,\cdot)\big|_{L^2(y,z)} \leqslant C\|f\|^{1/2}\|\partial_x f\|^{1/2}, \tag{9}$$

$$\big|f(t,x,\cdot)\big|_{L^2(y,z)} \leqslant \big|\nabla f(t,x,\cdot)\big|_{L^2(y,z)}, \tag{10}$$

the second inequality being true if $\int_{y,z} f = 0$. Consequently, we can write

$$\begin{aligned}
\frac{1}{\varepsilon}\left|\int v_x(u'_y v_y + u'_z v_z)\right| &\leqslant \frac{1}{\varepsilon}\int_x \left|u'\left(\frac{x}{\varepsilon}\right)\right| \big|\tilde{v}(t,x)\big|_{L^2(y,z)} \big|v_x(t,x)\big|_{L^2(y,z)}\,\mathrm{d}x \\
&\leqslant \|\partial_x v_x\|^{1/2}\|v_x\|^{1/2}\|\partial_x \tilde{v}\|^{1/2}\|\tilde{v}\|^{1/2}\int_\zeta \big|u'(\zeta)\big|\,\mathrm{d}\zeta \\
&\leqslant \|\nabla v\|^2 \int_\zeta \big|u'(\zeta)\big|\,\mathrm{d}\zeta.
\end{aligned} \tag{11}$$

A similar computation for the singular term involving the magnetic field leads to

$$\frac{1}{\varepsilon}\left|\int b_x(u'_y b_y + u'_z b_z)\right| \leqslant C\|\nabla b_x\|\|\nabla b\| \leqslant \frac{1}{2Rm}\|\nabla b\|^2 + CRm\|\nabla b_x\|^2. \tag{12}$$

When

$$\int_\zeta \left|u'(\zeta)\right| \mathrm{d}\zeta \leqslant \frac{1}{2\,Re},$$

we deduce, using (7) together with (11), (12) that

$$\left\|b(t)\right\|^2 + \left\|v(t)\right\|^2 + \frac{1}{2Rm}\int_0^t \|\nabla b\|^2 + \frac{1}{2\,Re}\int_0^t \|\nabla v\|^2 \leqslant CRm\int_0^t \|\nabla b_x\|^2. \tag{13}$$

By appropriate linear combination of (6) and (13), we get

$$E_0(v,b)(t) + \int_0^t D_0(v,b) \leqslant CE_0(v,b)(0) + CRm^2\int_0^t \|b_x b\|\|\nabla v_x\|,$$

where

$$E_0(v,b)(t) = \left\|v(t)\right\|^2 + \left\|b(t)\right\|^2 + Rm^2\left\|b_x(t)\right\|^2,$$
$$D_0(v,b)(t) = \frac{1}{Re}\|\nabla v\|^2 + \frac{1}{Rm}\|\nabla b\|^2 + Rm\|\nabla b_x\|^2$$

which gives thanks to the Young inequality

$$E_0(v,b)(t) + \int_0^t D_0(v,b) \leqslant CE_0(v,b)(0) + CRm^4\int_0^t \|b_x b\|^2. \tag{14}$$

The next step is to perform estimates of higher order derivatives. We multiply the equation for b of (4) by $-\Delta b$, this yields thanks to the Young inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla b\|^2 + \frac{1}{2Rm}\|\Delta b\|^2 \leqslant \left|(u^\varepsilon\cdot\nabla b, \Delta b)\right| + C\,Rm\|b\cdot\nabla u^\varepsilon\|^2 + C\,Rm\big(\|v\cdot\nabla b\|^2 + \|b\cdot\nabla v\|^2\big). \tag{15}$$

Next, we use that

$$Rm\|b\cdot\nabla u^\varepsilon\|^2 \leqslant \frac{Rm}{\varepsilon^2}\|b_x u'\|^2 \leqslant \frac{C\,Rm}{\varepsilon}\|\nabla b_x\|^2 \tag{16}$$

thanks to (9), (10). Additionally, for $k = x, y, z$,

$$\big(u^\varepsilon\cdot\nabla b, \partial_k^2 b\big) = -(\partial_k u^\varepsilon\cdot\nabla b, \partial_k b) - (u^\varepsilon\cdot\nabla\partial_k b, \partial_k b) = -(\partial_k u^\varepsilon\cdot\nabla b, \partial_k b)$$

since $\nabla\cdot u^\varepsilon = 0$. This yields the estimate

$$\left|(u^\varepsilon\cdot\nabla b, \Delta b)\right| \leqslant \frac{C}{\varepsilon}\|\nabla b\|^2. \tag{17}$$

By multiplying the equation for v in (4) by Δv, and by using the same kind of estimates as above, we easily get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + \frac{1}{Re}\|\Delta v\|^2 \leqslant \frac{C}{\varepsilon}\|\nabla v\|^2 + C\big(\|b\cdot\nabla b\|^2 + \|v\cdot\nabla v\|^2\big). \tag{18}$$

Finally, by appropriate linear combination of (14), (15), and (18), using (16) and (17), we get

$$E(v,b)(t)+\int_0^t D(v,b)$$
$$\leqslant CE(v,b)(0)+C\int_0^t Rm^4\|b_x b\|^2+\varepsilon\big(\|v\cdot\nabla b\|^2+\|b\cdot\nabla v\|^2+\|b\cdot\nabla b\|^2+\|v\cdot\nabla v\|^2\big), \tag{19}$$

where

$$E(v,b)(t)=\big\|v(t)\big\|^2+\big\|b(t)\big\|^2+Rm^2\big\|b_x(t)\big\|^2+\frac{\varepsilon}{Rm}\|\nabla b\|^2+\varepsilon\|\nabla v\|^2,$$
$$D(v,b)(t)=\frac{1}{Re}\|\nabla v\|^2+\frac{1}{Rm}\|\nabla b\|^2+Rm\|\nabla b_x\|^2+\frac{\varepsilon}{Rm^2}\|\Delta b\|^2+\frac{\varepsilon}{Re}\|\Delta v\|^2.$$

To end the proof of Theorem 1, it remains to estimate the nonlinear terms in (19). We shall use extensively the Gagliardo–Nirenberg inequality

$$\|f\|_{L^4}^2\leqslant C\big(\|f\|^{1/2}\|\nabla f\|^{3/2}+\|f\|_2^2\big)\leqslant C\|f\|^{1/2}\|f\|_{H^1}^{3/2}. \tag{20}$$

Note that lower order terms in (20) disappear as soon as f has zero mean with respect to y,z. Thus, we have

$$\varepsilon\int_0^t\|b\cdot\nabla b\|^2\leqslant C\varepsilon\int_0^t\|b\|^{1/2}\|b\|_{H^1}^{3/2}\|\nabla b\|^{1/2}\|\nabla b\|_{H^1}^{3/2}$$
$$\leqslant C\varepsilon^{1/4}Rm^{7/4}\sup_{[0,t]}\big(\|b\|^{1/2}\|b\|_{H^1}^{3/2}\big)\int_0^t\frac{\varepsilon}{Rm^2}\|\Delta b\|^2+\frac{1}{Rm}\|\nabla b\|^2$$
$$\leqslant C\varepsilon^{-1/2}Rm^{5/2}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t E(v,b). \tag{21}$$

In a similar way, we find

$$\varepsilon\int_0^t\|v\cdot\nabla b\|^2\leqslant C\varepsilon^{-1/2}Rm^{7/4}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t D(v,b), \tag{22}$$
$$\varepsilon\int_0^t\|b\cdot\nabla v\|^2\leqslant C\varepsilon^{-1/2}Rm^{3/4}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t D(v,b), \tag{23}$$
$$\varepsilon\int_0^t\|v\cdot\nabla v\|^2\leqslant C\varepsilon^{-1/2}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t D(v,b). \tag{24}$$

The only nonlinear term in (19) which requires some care is the term $\|b_x b\|^2$. Since $\int_{y,z}b_x=0$, we can write

$$Rm^4\int_0^t\|b_x b\|^2\leqslant Rm^4\int_0^t\|b_x\|^{1/2}\|\nabla b_x\|^{3/2}\|b\|^{1/2}\big(\|b\|^{3/2}+\|\nabla b\|^{3/2}\big)$$
$$\leqslant Rm^4\left(\sup_{[0,t]}\|b\|^2\int_0^t\|\nabla b_x\|^2+\sup_{[0,t]}\|b_x\|^{1/2}\|\nabla b\|\|b\|^{1/2}\int_0^t\|\nabla b\|^{1/2}\|\nabla b_x\|^{3/2}\right)$$
$$\leqslant\varepsilon^{-1/2}Rm^{7/2}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t D(v,b).$$

By collecting all the previous estimates, we find that

$$E(v,b)(t)+\int_0^t D(v,b)\leqslant CE(v,b)(0)+C\varepsilon^{-1/2}Rm^{7/2}\sup_{[0,t]}\big(E(v,b)(t')\big)\int_0^t D(v,b)$$

and hence, we easily conclude that if

$$\varepsilon^{-1/2}Rm^{7/2}E(v,b)(0)\leqslant\delta$$

for some δ sufficiently small, then we keep the estimate

$$E(v,b)(t)\leqslant CE(v,b)(0)$$

for every positive time.

2.2. *Proof of Theorem 2*

We turn to the stability estimates for curved interfaces. We shall use the local coordinates $(s:=r-1,\tau,z)$ and the cylindrical orthonormal frame (e_s,e_τ,e_z). For any vector v, we denote (v_s,v_τ,v_z) the components of v in this moving frame. Note that we are working with an orthonormal basis, so that $e_\tau=\frac{1}{1+s}\partial_\tau$. Consequently, the expressions that we give below are different from the usual expressions of differential geometry in local coordinates, where the local basis is chosen as $(\partial_r,\partial_\tau,\partial_z)$.

We recall the Frénet formula

$$e_x\partial_x+e_y\partial_y=\frac{1}{1+s}e_\tau\partial_\tau+e_s\partial_s.$$

It allows to compute the operators of (4) in cylindrical coordinates.

- For smooth vector fields w,c, we compute

$$w\cdot\nabla c=\left(w\cdot\nabla c_s-\frac{1}{1+s}w_\tau c_\tau,\ w\cdot\nabla c_\tau+\frac{1}{1+s}w_\tau c_s,\ w\cdot\nabla c_z\right),\tag{25}$$

  where, for any *scalar* function f,

$$w\cdot\nabla f=w_s\partial_s f+\frac{w_\tau}{1+s}\partial_\tau f+w_z\partial_z f.$$

- The Laplacian reads in local coordinates

$$\Delta w=\left(\Delta w_s-\frac{2}{(1+s)^2}\partial_\tau w_\tau-\frac{1}{(1+s)^2}w_s,\ \Delta w_\tau+\frac{2}{(1+s)^2}\partial_\tau w_s-\frac{1}{(1+s)^2}w_\tau,\ \Delta w_z\right),$$

  where, for any *scalar* function f,

$$\Delta f=\frac{1}{1+s}\partial_\tau\left(\frac{1}{1+s}\partial_\tau\right)f+\frac{1}{1+s}\partial_s f+\partial_s^2 f+\partial_z^2 f.$$

- The divergence operator is

$$\operatorname{div} v=\partial_s w_s+\frac{1}{1+s}\partial_\tau w_\tau+\partial_z w_z+\frac{1}{1+s}w_s.\tag{26}$$

Let $\chi=\chi(s)$ a smooth function, such that $\chi=1$ on $[-1/4,1/4]$, and $\chi=0$ outside $(-1/2,1/2)$. The first step is to perform an energy estimate on χb_s which is better thanks to the structure of the singular term. By using cylindrical coordinates, we find that the equation for b_s is

$$\partial_t b_s+u^\varepsilon\cdot\nabla b_s+v\cdot\nabla b_s=\frac{1}{Rm}\left(\Delta b_s-\frac{2}{(1+s)^2}\partial_\tau b_\tau-\frac{1}{(1+s)^2}b_s\right)+b\cdot\nabla v_s.$$

We perform an energy estimate on χb_s multiplying this equation by $\chi^2 b_s(1+s)$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi b_s\|^2 + \frac{1}{Rm}\big\|\nabla(\chi b_s)\big\|^2 = \frac{1}{Rm}\int (1+s)\chi'^2 b_s^2 - \frac{2}{Rm}\int \frac{\partial_\tau b_\tau - b_s}{1+s}\chi^2 b_s$$
$$- \frac{3}{Rm}\int \frac{b_s}{1+s}\chi^2 b_s + \int (1+s)\chi'\chi b_s^2 v_s + \int b\cdot\nabla v_s(1+s)\chi^2 b_s.$$

We deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\chi b_s\|^2 + \frac{1}{Rm}\big\|\nabla(\chi b_s)\big\|^2 \leqslant C\left(\frac{1}{Rm}\|b\|^2 + \frac{1}{Rm}\|\nabla b\|\|\chi b_s\| + \big\|(\chi b_s)b\big\|\big(\|\nabla v\| + \|v\|\big)\right)$$

since χ vanishes in the vicinity of the singularity at $s=-1$. The last estimate can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{\varepsilon^q}\|\chi b\|_s^2 + \frac{1}{\varepsilon^q\, Rm}\big\|\nabla(\chi b_s)\big\|^2 \leqslant \frac{1}{8Rm}\|\nabla b\|^2 + \frac{C}{\varepsilon^q\, Rm}\frac{\|\chi b_s\|^2}{\varepsilon^q} + C\varepsilon^{-q}\big\|(\chi b_s)b\big\|\big(\|\nabla v\| + \|v\|\big) \tag{27}$$

thanks to the Young inequality.

Next, we perform a classical energy estimate on the full system (4). This yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|b\|^2 + \|v\|^2\big) + \frac{1}{Rm}\|\nabla b\|^2 + \frac{1}{Re}\|\nabla v\|^2 \leqslant \left|\int v\cdot\nabla u^\varepsilon\cdot v\right| + \left|\int b\cdot\nabla u^\varepsilon\cdot b\right|.$$

Next we notice that thanks to (3), we get

$$\int b\cdot\nabla u^\varepsilon\cdot b = \int\left(\frac{1+s}{\varepsilon}\omega' b_s b_\tau + \frac{1}{\varepsilon}u_z' b_s b_z\right)\mathrm{d}x.$$

Remind that ω', u_z' have compact support, so that the truncation function χ can be introduced in the last singular term. To control this singular term on the right-hand side, we use the equivalent of (9) in cylindrical coordinates

$$\big|f(s,\cdot)\big|_{L^2((1+s)\,\mathrm{d}\tau\,\mathrm{d}z)} \leqslant C\|f\|^{1/2}\big\|\partial_s(f)\big\|^{1/2}. \tag{28}$$

Consequently, by using (28), we get

$$\left|\int b\cdot\nabla u^\varepsilon\cdot b\right| \leqslant C\|\chi b_s\|^{1/2}\big\|\nabla(\chi b_s)\big\|^{1/2}\|b\|^{1/2}\|\nabla b\|^{1/2}\int_\zeta |U'|\,\mathrm{d}\zeta$$
$$\leqslant C\left((\varepsilon^q\, Rm)^{1/2}\frac{\|\chi b_s\|}{\varepsilon^{q/2}}\right)^{1/2}\left(\frac{\|\nabla(\chi b_s\|}{\varepsilon^{q/2}Rm^{1/2}}\right)^{1/2}\left(\frac{\|\nabla b\|}{Rm^{1/2}}\right)^{1/2}(\varepsilon^q\, Rm)^{1/4}\|b\|^{1/2}$$
$$\leqslant C\varepsilon^q\, Rm\left(\|b\|^2 + \frac{\|\chi b_s\|^2}{\varepsilon^q}\right) + \frac{1}{4Rm}\|\nabla b\|^2 + \frac{1}{4\varepsilon^q\, Rm}\big\|\nabla(\chi b_s)\big\|^2,$$

where in the last line, we have used the inequality

$$abcd \leqslant \frac{1}{4}\big(a^4 + b^4 + c^4 + d^4\big).$$

Thanks to the inequality (28), we prove in a similar way that

$$\left|\int v\cdot\nabla u^\varepsilon\cdot v\right| \leqslant \frac{1}{2Re}\|\nabla v\|^2 + C\|v\|^2.$$

Consequently, we finally get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|b\|^2 + \|v\|^2\big) + \frac{3}{2Rm}\|\nabla b\|^2 + \frac{1}{Re}\|\nabla v\|^2$$
$$\leqslant C\big(\|b\|^2 + \|v\|^2\big) + C\varepsilon^q\, Rm\left(\|b\|^2 + \frac{\|\chi b_s\|^2}{\varepsilon^q}\right) + \frac{1}{4\varepsilon^q\, Rm}\big\|\nabla(\chi b_s)\big\|^2. \tag{29}$$

We make a linear combination of (27) and (29) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_0(v,b)+\frac{3}{4\varepsilon^q\,Rm}\|\nabla b_s\|^2+\frac{11}{8}\|\nabla b\|^2+\frac{1}{2\,Re}\|\nabla v\|^2\\ \leqslant C\left(1+\varepsilon^q Rm+\frac{1}{\varepsilon^q\,Rm}\right)E_0(v,b)+C\varepsilon^{-2q}\left\|(\chi b_s)b\right\|^2\\ \leqslant C E_0(v,b)+Rm^2\left\|(\chi b_s)b\right\|^2\end{aligned}$$

with the choice $\varepsilon^q\,Rm=1$, where

$$E_0(v,b)=\|v\|^2+\|b\|^2+\varepsilon^{-q}\|\chi b_s\|^2=\|v\|^2+\|b\|^2+Rm\|\chi b_s\|^2,$$
$$D_0(v,b)=\frac{1}{Rm}\|\nabla b\|^2+\frac{1}{Re}\|\nabla v\|^2+\left\|\nabla(\chi b_s)\right\|^2.$$

The next step is to get an estimate on ∇v and ∇b. We use the same technique as in Section 2 and hence we shall not give all the details.

We just point out the estimate

$$Rm\|b\cdot\nabla u^\varepsilon\|^2\leqslant C\left(Rm\|b\|^2+\frac{Rm}{\varepsilon}\left(\|\chi b_s\|^2+\left\|\nabla(\chi b_s)\right\|^2\right)\right).$$

We get

$$E(v,b)(t)+\int_0^t D(v,b)\leqslant C E(v,b)(0)+C\int_0^t E(v,b)+\int_0^t Rm^2\left\|(\chi b_s)b\right\|^2$$
$$+\varepsilon\int_0^t\|v\cdot\nabla b\|^2+\|b\cdot\nabla v\|^2+\|b\cdot\nabla b\|^2+\|v\cdot\nabla v\|^2,$$

where

$$E(v,b)(t)=E_0(v,b)+\frac{\varepsilon}{Rm}\|\nabla b\|^2+\frac{\varepsilon}{Rm}\|\nabla v\|^2,$$
$$D(v,b)(t)=D_0(v,b)+\frac{\varepsilon}{Rm^2}\|\Delta b\|^2+\frac{\varepsilon}{Rm}\|\Delta v\|^2.$$

To conclude, we need to estimate the last term in the previous inequality. We use (21)–(25) and we also note that

$$\int_0^t Rm^2\left\|(\chi b_s)b\right\|^2\leqslant C\,Rm^2\int_0^t\|\chi b_s\|^{1/2}\|\chi b_s\|_{H^1}^{3/2}\|\kappa b\|^{1/2}\|\kappa b\|_{H^1}^{3/2},\tag{30}$$

where κ is smooth compactly supported and such that $\kappa\chi=\chi$. Next, we write

$$\|\chi b_s\|_{L^2}^2\leqslant\int_z\left(\int_{x,y}|\chi b_s|^2\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z\leqslant\int_z|\operatorname{Supp}\chi|\sup_r\left(|\chi b_s|_{L^2(r\,\mathrm{d}\theta)}^2\right)\mathrm{d}z$$

and hence, thanks to (28), we get

$$\|\chi b_s\|_{L^2}\leqslant C\left\|\nabla(\chi b_s)\right\|.\tag{31}$$

With a slight variation, we also get the estimate

$$\|\kappa b\|_{L^2}^2\leqslant\int_z\left(\int_{x,y}|\kappa b|^2\,\mathrm{d}x\,\mathrm{d}y\right)\mathrm{d}z\leqslant\int_z|\operatorname{Supp}\kappa|\sup_s\left(|b|_{L^2(r\,\mathrm{d}\theta)}^2\right)\mathrm{d}z$$

and hence thanks to a new use of (28), we get

$$\|\kappa b\|_{L^2}^2\leqslant C\|b\|\|\nabla b\|.\tag{32}$$

Note that the same estimate holds with κ replaced by $\nabla\kappa$. Consequently, thanks to (30)–(32), we get

$$\int_0^t Rm^2 \big\|(\chi b_s)b\big\|^2 \leqslant C\,Rm^2 \sup_{[0,t]}\big(\|\chi b_s\|^{1/2}\|b\|_{H^1}\|b\|^{1/2}\big) \int_0^t \big\|\nabla(\chi b_s)\big\|^{3/2}\|\nabla b\|^{1/2}$$

$$\leqslant C\varepsilon^{-1/2}\,Rm^{5/2} \sup_{[0,t]} E(v,b) \int_0^t D(v,b). \tag{33}$$

Since we also have

$$\varepsilon \iint_0^t \big(\|v\cdot\nabla b\|^2 + \|b\cdot\nabla v\|^2 + \|b\cdot\nabla b\|^2 + \|v\cdot\nabla v\|^2\big) \leqslant \varepsilon^{-1/2}\,Rm^{7/4} \sup_{[0,t]} E(v,b) \int_0^t D(v,b),$$

we finally get

$$E(v,b)(t) \leqslant C E(v,b)(0) + C\int_0^t E(v,b)$$

and hence thanks to the Gronwall inequality

$$E(v,b)(t) \leqslant C\,\mathrm{e}^{Ct} E(v,b)(0)$$

as long as

$$\varepsilon^{-1/2}\,Rm^{5/2} \sup_{[0,t]} E(v,b) \leqslant \delta$$

for some $\delta > 0$ sufficiently small and the result of Theorem 2 easily follows.

2.3. Proof of Theorem 3

Up to now, since we work with v, b which have zero mean on $\mathbb{T}^3$, we shall make a constant use of the Poincare inequality

$$\|v\| \leqslant C\|\nabla v\|, \qquad \|b\| \leqslant C\|\nabla b\|.$$

Again the first step is to use an estimate on χb_s. Thanks to an integration by parts and the use of the Young inequality, we write it in the slightly different form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\chi b_s\|^2 + \frac{1}{2Rm}\big\|\nabla(\chi b_s)\big\|^2 \leqslant \frac{C}{Rm}\|b_s\|^2 + \frac{C}{Rm}\|b_\tau\|^2 + \big\|(\chi b_s)b\big\|\big(\|\nabla v\| + \|v\|\big).$$

Indeed, we have used that

$$\int \frac{\partial_\tau b_\tau b_s \chi^2}{1+s} = -\int \frac{b_\tau \partial_\tau b_s \chi^2}{1+s} \leqslant \frac{1}{2}\big\|\nabla(\chi b_s)\big\|^2 + C\big(\|b_s\|^2 + \|b_\tau\|^2\big).$$

Next, we find

$$\frac{\|\chi b_s\|^2}{\varepsilon^2\nu^{2/3}} + \nu^{1/3}\int_0^t \big\|\nabla(\chi b_s)\big\|^2$$

$$\leqslant \frac{\|\chi b_s(0)\|^2}{\varepsilon^2\nu^{2/3}} + C\nu^{1/3}\int_0^t \big(\|b_\tau\|^2 + \|b_s\|^2\big) + \frac{2}{\varepsilon^2\nu^{2/3}}\int_0^t \|b_s b\|\big(\|\nabla v\| + \|v\|\big). \tag{34}$$

Next the classical energy estimate on the full system (4) gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|^2 + \|b\|^2\big) + \frac{1}{Rm}\|\nabla v\|^2 + \frac{1}{Re}\|\nabla b\|^2 \leqslant \frac{1}{\varepsilon}\|\chi b_s\|\|b\| - \int (v\cdot\nabla)u^\varepsilon\cdot v\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where we have used again that u_z', ω' have compact support to localize through χ. Next we use that

$$\frac{1}{\varepsilon}\|\chi b_s\|\|b\| \leqslant \frac{\|\chi b_s\|^2}{2\varepsilon^2\nu^{1/3}} + \frac{1}{2}\nu^{1/3}\|b\|^2 \leqslant C\nu^{1/3}\left(\|b\|^2 + \frac{\|\chi b_s\|^2}{\varepsilon^2\nu^{2/3}}\right).$$

Moreover, by using cylindrical coordinates, the singular term involving the velocity is under the form

$$\int v\cdot\nabla u^\varepsilon\cdot v = \int\left(\frac{1+s}{\varepsilon}\omega' v_s v_\tau + \frac{1}{\varepsilon}u_z' v_s v_z\right).$$

By using (28) and the fact that $\int v = 0$ to replace L^2 norms by L^2 norms of the gradient, we get

$$\left|\int v\cdot\nabla u^\varepsilon\cdot v\right| \leqslant C\|\nabla v\|^2\int_\zeta |\omega'| + |u_z'|\,\mathrm{d}\zeta.$$

Consequently, thanks to (34) and our assumption on the Reynolds number *Re*, we easily get by collecting all the estimates that

$$E_0(v,b)(t) + \int_0^t D_0(v,b) \leqslant CE_0(v,b)(0) + C\nu^{1/3}\int_0^t E_0(v,b) + \frac{C}{\varepsilon^4\nu^{4/3}}\int_0^t \big\|(\chi b_s)b\big\|^2, \tag{35}$$

where

$$E_0(v,b) = \|v\|^2 + \|b\|^2 + \frac{\|b_s\|^2}{\varepsilon^2\nu^{2/3}},$$
$$D_0(v,b) = \frac{1}{Re}\|\nabla v\|^2 + \frac{1}{Rm}\|\nabla b\|^2 + \nu^{1/3}\big\|\nabla(\chi b_s)\big\|^2.$$

As usual the next step is to get an estimate on the derivatives. We shall not detail this part since most of the useful estimates are actually given in the previous sections. We get

$$E(v,b)(t) + \int_0^t D(v,b) \leqslant CE(v,b)(0) + \nu^{1/3}\int_0^t E(v,b) + C\int_0^t \frac{1}{\varepsilon^4\nu^{4/3}}\|\chi b_s b\|^2$$
$$+ C\varepsilon\int_0^t \big(\|\operatorname{curl} b\times b\|^2 + \|v\cdot\nabla v\|^2 + \|v\cdot\nabla b\|^2 + \|b\cdot\nabla v\|^2\big), \tag{36}$$

where

$$E(v,b) = E_0(v,b) + \frac{\varepsilon}{Rm}\|\nabla b\|^2 + \varepsilon\|\nabla v\|^2, \qquad D(v,b) = D_0(v,b) + \frac{\varepsilon}{Rm^2}\|\Delta b\|^2 + \varepsilon\|\Delta v\|^2.$$

The only term which requires some details in the derivation of the last estimate is the term

$$\frac{\varepsilon}{Rm}(b\cdot\nabla u^\varepsilon, \Delta b).$$

Indeed, the estimate (16) does not hold in the curved geometric setting since there is also a term without derivatives of u^ε in $b\cdot\nabla u^\varepsilon$. Since $\Delta b = -\operatorname{curl}\operatorname{curl} b$, we can write after an integration by parts

$$\int b\cdot\nabla u^\varepsilon\cdot\Delta b = -\int b\cdot\nabla\operatorname{curl} u^\varepsilon\cdot\operatorname{curl} b + \mathcal{O}(1)\int |\nabla u^\varepsilon||\nabla b|^2$$

and we notice that in cylindrical coordinates

$$\operatorname{curl} u^\varepsilon = -\frac{1}{\varepsilon}u_z' e_\tau + \left(2\omega + \frac{(1+s)}{\varepsilon}\omega'\right)e_z$$

so that

$$|b\cdot\nabla\operatorname{curl} u^\varepsilon| \leqslant C\big(\varepsilon^{-1}|b| + \varepsilon^{-2}\big(|\omega''| + |u_z''|\big)|\chi b_s|\big).$$

Consequently, by using again that $\int b = 0$, and (28), we get

$$\left|\int b \cdot \nabla u^\varepsilon \cdot \Delta b\right| \leqslant C\left(\varepsilon^{-1}\|\nabla b\|^2 + \varepsilon^{-3/2}\|\nabla b\|\left\|\nabla(\chi b_s)\right\|\right) \leqslant C\left(\varepsilon^{-1}\|\nabla b\|^2 + \varepsilon^{-2}\left\|\nabla(\chi b_s)\right\|^2\right)$$

so that

$$\frac{\varepsilon}{Rm}\left|\int b \cdot \nabla u^\varepsilon \cdot \Delta b\right| \leqslant C D_0(v, b).$$

which is a good estimate towards the derivation of (36).

To conclude, it remains to use the nonlinear terms, we use the estimates (21)–(24): since $1/Rm = \nu\varepsilon^2$, note that

$$\varepsilon^{-1/2} Rm^{7/4} = \frac{1}{\varepsilon^4 \nu^{7/4}}$$

so that the worst estimate in (21)–(24) is given by

$$\frac{1}{\varepsilon^2 \nu^{7/4}} \sup_{[0,t]} E(v, b) \int_0^t D(v, b).$$

Since by using the analogous of (33) we have

$$\frac{1}{\varepsilon^4 \nu^{4/3}} \int_0^t \|\chi b_s b\|^2 \leqslant \frac{C}{\varepsilon^{11/2} \nu^{13/6}} \sup_{[0,t]} E(v, b) \int_0^t D(v, b),$$

we get thanks to the Gronwall inequality

$$E(v, b)(t) \leqslant C \exp(\gamma \nu^{1/3} t) E(v, b)(0)$$

as long as

$$\sup_{[0,t]} E(v, b) \leqslant \delta \varepsilon^{11/2} \nu^{13/6}$$

for some $\delta > 0$ sufficiently small and hence the result of Theorem 3 follows easily.

3. The dynamo

This section is devoted to the description of the dynamo effect. The dynamo mechanism is connected to some special solutions (v, b) of (5). These solutions are localized near $s = 0$, and oscillate with respect to τ, z. To understand the structure of such solutions, we will begin with a WKB analysis. Throughout the text, we will use the following notation: for all smooth functions a, b defined on an open set U, depending on the parameter ε,

$$a \approx b$$

will mean that, for all m, for all compact subset K of U,

$$a - b = \mathrm{O}(\varepsilon^m), \quad \text{in } C^\infty(K).$$

3.1. WKB expansions

We construct here approximate solutions of (5). They involve three types of expansions, corresponding to different regions of $\mathbb{R}^3$.

- The inner expansion

It will hold in a vicinity of size ε around the surface $\{s=0\}$. It depends on s/ε, to match the variations of u^ε. It also involves tangential oscillations, at high frequency $\sqrt{\varepsilon}^{-1}$. Precisely,

$$\begin{aligned}
&\left(v_{\mathrm{app}}^{\mathrm{in}}, b_{\mathrm{app}}^{\mathrm{in}}\right) \approx \sqrt{\varepsilon}^{2m} \sum \sqrt{\varepsilon}^{i} \left(V^i, B^i\right)\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right), \\
&p_{\mathrm{app}}^{\mathrm{in}} \approx \sqrt{\varepsilon}^{2m-1} \sum \sqrt{\varepsilon}^{i} P^i\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right),
\end{aligned} \tag{37}$$

where the inner profiles are smooth:

$$\left(V^i, B^i, P^i\right) = \left(V^i, B^i, P^i\right)(t, \zeta, \theta, \lambda) \in \mathcal{C}^\infty\left(\mathbb{R}_+ \times \mathbb{R}^3\right)$$

and periodic in θ, λ:

$$\left(V^i, B^i, P^i\right)(t, \zeta, \theta, \lambda) = \sum_{\substack{m\in\mathbb{Z},\\ k\in K\mathbb{Z}}} \mathrm{e}^{\mathrm{i}(m\theta+k\lambda)} \mathcal{F}\left(V^i, B^i, P^i\right)(t, \zeta, m, k), \quad M, K > 0.$$

- The outer expansion

It will hold in the region $s = \mathrm{O}(\sqrt{\varepsilon})$, $s \gg \varepsilon$. This expansion will correct errors due to truncation of the inner expansion and describe the boundary layer where the instability takes place. Namely,

$$\begin{aligned}
&\left(v_{\mathrm{app}}^{\mathrm{out}}, b_{\mathrm{app}}^{\mathrm{out}}\right) \approx \sqrt{\varepsilon}^{2m} \sum_{i\geqslant 0} \sqrt{\varepsilon}^{i} \left(v^i, b^i\right)\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right), \\
&p_{\mathrm{app}}^{\mathrm{out}} \approx \sqrt{\varepsilon}^{2m-1} \sum_{i\geqslant 0} \sqrt{\varepsilon}^{i} p^i\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right),
\end{aligned} \tag{38}$$

where the outer "profiles"

$$\left(v^i, b^i, p^i\right) = \left(v^i, b^i, p^i\right)(t, \xi, \theta, \lambda) \in \mathcal{C}^\infty\left(\mathbb{R}_+ \times \mathbb{R}_* \times \mathbb{R}^2\right)$$

are smooth on each side of $\xi = 0$, and again periodic in θ, λ:

$$\left(v^i, b^i, p^i\right)(t, \xi, \theta, \lambda) = \sum_{\substack{m\in\mathbb{Z},\\ k\in K\mathbb{Z}}} \mathrm{e}^{\mathrm{i}(m\theta+k\lambda)} \mathcal{F}\left(v^i, b^i, p^i\right)(t, \xi, m, k).$$

These profiles will be shown to decrease fastly at infinity, except for the mean velocity and pressure

$$\bar{v}^i := \int_{\theta,\lambda} v^i, \qquad \overline{p}^i := \int_{\theta,\lambda} p^i.$$

- The external expansion

It will hold for $s \gg \sqrt{\varepsilon}$, and will correct $\mathrm{O}(1)$ terms due to the truncation of $\bar{v}^i$, $\overline{p}^i$. Namely,

$$\begin{aligned}
&v_{\mathrm{app}}^{\mathrm{ext}} \approx \sqrt{\varepsilon}^{2m} \sum_{i\geqslant 0} \sqrt{\varepsilon}^{i} w^i(t, s), \\
&p_{\mathrm{app}}^{\mathrm{ext}} \approx \sqrt{\varepsilon}^{2m-1} \sum_{i\geqslant 0} \sqrt{\varepsilon}^{i} q^i(t, s),
\end{aligned} \tag{39}$$

where the external profiles w^i, q^i are smooth outside $s = 0$.

Following [8], these expansions will be linked with the method of matched asymptotics. Precisely, we expect both inner and outer expansions to be valid in a matching zone, close to $s = 0$, of typical length $\varepsilon \ll l \ll \sqrt{\varepsilon}$. We can express the outer solution in terms of ζ, and through Taylor series obtain the following matching conditions: For all (t,θ,λ), for all i,

$$\big(V^i, B^i, P^i\big)(t,\zeta,\theta,\lambda) \sim \sum_{j=0}^{i} \frac{\zeta^j}{j!}\partial_\xi^j\big(v^{i-j}, b^{i-j}, p^{i-j}\big)(t,\xi=\pm 0,\theta,\lambda), \quad \zeta\to\pm\infty. \tag{40}$$

Similarly, the outer and external expansions should correspond for $\sqrt{\varepsilon} \ll s \ll 1$. This means

$$\big(\overline{v}^i, \overline{p}^i\big)(t,\xi) \sim \sum_{j=0}^{i} \frac{\xi^j}{j!}\partial_s^j\big(w^{i-j}, q^{i-j}\big)(t,s=\pm 0), \quad \xi\to\pm\infty. \tag{41}$$

3.1.1. Equations

We plug the various expansions in system (5). The resulting equations are ordered according to powers of $\sqrt{\varepsilon}$, and coefficients of the different powers of $\sqrt{\varepsilon}$ are set equal to 0. It leads to a collection of equations on the profiles. To lighten notations, we set

$$\big(V^i, P^i, B^i, v^i, p^i, b^i, w^i, q^i\big) \equiv 0 \quad \text{for } i<0.$$

- Inner equations

At order $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-4})$ in Eq. $(5)_{\mathrm{a}}$, we get

$$-\frac{1}{Re}\partial_\zeta^2 V^i + \big(0, w'(\zeta), U_s'(\zeta)\big)V_s^{i-2} + \big(\partial_\zeta P^{i-1}, 0, 0\big) - \frac{1}{Re}\big(\partial_\theta^2 + \partial_\lambda^2\big)V^{i-2} - \frac{1}{Re}\partial_\zeta V^{i-2} = F_v^i, \tag{42}$$

where F_v^i depends on (V^k, B^k), $k \leqslant i-3$, and P^k, $k\leqslant i-2$.

Similarly, order $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-2})$ in $(5)_{\mathrm{b}}$ provides

$$-\big(0, w'(\zeta), u_z'(\zeta)\big)B_s^i - \partial_\zeta^2 B^i + \big(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\big)B^{i-1} = F_b^i, \tag{43}$$

where F_b^i depends on (V^k, B^k), $k\leqslant i-2$.

Finally, $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-2})$ terms in divergence free conditions $(5)_{\mathrm{c}}$ lead to

$$\partial_\zeta V_s^i + \partial_\theta V_\theta^{i-1} + \partial_\lambda V_z^{i-1} = G_v^i, \tag{44}$$

$$\partial_\zeta B_s^i + \partial_\theta B_\theta^{i-1} + \partial_\lambda B_z^{i-1} = G_b^i, \tag{45}$$

where $G_{v,b}^i$ depend on (V^k, B^k), $k\leqslant i-2$.

- Outer equations

At order $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-2})$ in Eq. $(5)_{\mathrm{a}}$, we get

$$\begin{aligned}&-\frac{1}{Re}\big(\partial_\xi^2+\partial_\theta^2+\partial_\lambda^2\big)v^i + \big(\partial_\xi,\partial_\theta,\partial_\lambda\big)p^i + \mathbf{1}_{\mathbb{R}_-}(\xi)\big(\Omega\partial_\theta + U_z\partial_\lambda\big)v^{i-1}\\ &\quad + \frac{1}{Re}\big(2\partial_\theta v_\tau^{i-1}, -2\partial_\theta v_s^{i-1}, 0\big) + \frac{2}{Re}\xi\partial_\theta^2 v^{i-1} - \frac{1}{Re}\partial_\xi v^{i-1} = f_v^i,\end{aligned} \tag{46}$$

where f_v^i depends on v^k, b^k, $k\leqslant i-2$, and p^k, $k\leqslant i-1$.

At order $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-1})$ in Eq. $(5)_{\mathrm{b}}$, we get

$$\begin{aligned}&\mathbf{1}_{\mathbb{R}_-}(\xi)(\Omega\partial_\theta + U_z\partial_\lambda)b^i + \partial_t b^{i-1} - \big(\partial_\xi^2+\partial_\theta^2+\partial_\lambda^2\big)b^{i-1}\\ &\quad + \big(2\partial_\theta b_\tau^{i-2}, -2\partial_\theta b_s^{i-2}, 0\big) + 2\xi\partial_\theta^2 b^{i-2} - \partial_\xi b^{i-2} = f_b^i,\end{aligned} \tag{47}$$

where f_b^i depends on (v^k, b^k), $k\leqslant i-3$.

At order $\mathrm{O}(\sqrt{\varepsilon}^{2m+i-1})$ in $(5)_{\mathrm{c}}$, we recover

$$\partial_\xi v_s^i + \partial_\theta v_\tau^i + \partial_\lambda v_z^i + v_s^{i-1} - \xi \partial_\theta v_\tau^{i-1} = g_v^i, \tag{48}$$

$$\partial_\xi b_s^i + \partial_\theta b_\tau^i + \partial_\lambda b_z^i + b_s^{i-1} - \xi \partial_\theta b_\tau^{i-1} = g_b^i, \tag{49}$$

where $g_{v,b}^i$ depend on (v^k, b^k), $k \leqslant i-2$.

- External equations

They resume to $w_s^i = 0$, together with

$$\begin{aligned}
&\partial_s q^i = 2\mathbf{1}_{\{s<0\}}(s)\Omega w^{i-1} + \sum_{k+k'=i-2m-4} \frac{1}{1+s} w_\tau^k w_\tau^{k'}, \\
&\partial_t w_\tau^i - \frac{1}{Re}\Delta w_\tau^i + \frac{1}{Re\,(1+s)^2} w_\tau^i = 0, \\
&\partial_t w_z^i - \frac{1}{Re}\Delta w_z^i = 0.
\end{aligned} \tag{50}$$

Note that the nonlinear term $v \cdot \nabla v$ does not play any part in the equations for w_τ^i and w_z^i: it vanishes because of the radial symmetry of the functions.

3.1.2. First profiles

- Velocity

Outer part, $i = 0$. Eqs. (46) read:

$$\begin{cases} -\dfrac{1}{Re}\left(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\right) v^0 + (\partial_\xi, \partial_\theta, \partial_\lambda) p^0 = 0, \\ \operatorname{div} v^0 = 0. \end{cases} \tag{51}$$

Inner part, $i = 0$. Eq. (44) (which implies $(42)_\mathrm{a}$) yields

$$V_s^0 = V_s^0(t, \theta, \lambda),$$

and thanks to the matching condition (40),

$$\left[v_s^0\right]\big|_{\xi=0} = 0, \qquad V_s^0 = v_s^0(\xi = 0). \tag{52}$$

Eqs. $(42)_\mathrm{b}$, $(42)_\mathrm{c}$ imply that

$$\left(V_\tau^0, V_z^0\right) = C_1 \zeta + C_2,$$

where $C_j = C_j(t, \theta, \lambda)$. Using matching condition (40),

$$\left[\left(v_\tau^0, v_z^0\right)\right]\big|_{\xi=0} = 0, \qquad \left(V_\tau^0, V_z^0\right) = \left(v_\tau^0, v_z^0\right)(\xi = 0). \tag{53}$$

Inner part, $i = 1$. Proceeding with (44), we obtain

$$V_s^1 = C_1 \zeta + C_2, \quad C_j = C_j(t, \theta, \lambda),$$

so that

$$\left[\partial_\xi v_s^0\right]\big|_{\xi=0} = 0.$$

In the same way, $(42)_\mathrm{b}$, $(42)_\mathrm{c}$, $i = 1$ lead to

$$\left[\partial_\xi \left(v_\tau^0, v_z^0\right)\right]\big|_{\xi=0} = 0.$$

In turn, Eq. $(42)_\mathrm{a}$ implies that $\partial_\zeta P^0 = 0$. Using matching condition (40), we deduce

$$\left[p^0\right]\big|_{\xi=0} = 0, \quad P^0 = p^0(t, 0, \theta, \lambda).$$

Combining Stokes equations (51) with the previous jump conditions, we deduce

$$v^0 = 0, \qquad p^0 = 0.$$

Back to the inner profiles, we deduce from (52), (53) that $V^0 = 0$.

External part, $i = 0$. We get from (50) and the matching condition (41) that

$$w^0 = 0, \qquad q^0 = 0.$$

- Magnetic field

Outer part, $i = 0$. We get from (47)

$$\mathbf{1}_{\mathbb{R}_-}(\xi)(\Omega\partial_\theta + U_z\partial_\lambda)b^0 = 0. \tag{54}$$

We define the orthogonal projector Π as follows:

$$\Pi b^0 = b^0, \quad \xi > 0,$$
$$\Pi b^0 = \sum_{\substack{m,k \\ \Omega m + U_z k = 0}} \mathrm{e}^{\mathrm{i}(m\theta + k\lambda)} \mathcal{F} b^0, \quad \xi < 0,$$

so that (54) is equivalent to

$$\Pi b^0 = b^0 \tag{55}$$

Outer part, $i = 1$. Eq. (47) provides

$$\mathbf{1}_{\mathbb{R}_-}(\xi)(\Omega\partial_\theta + U_z\partial_\lambda)b^1 + \partial_t b^0 - \left(\partial_\zeta^2 + \partial_\theta^2 + \partial_\lambda^2\right)b^0 = 0. \tag{56}$$

Applying $(I - \Pi)$, we obtain that $\Pi b^1 = b^1$. Applying Π, it becomes

$$\partial_t b^0 - \left(\partial_\zeta^2 + \partial_\theta^2 + \partial_\lambda^2\right)b^0 = 0. \tag{57}$$

Outer part, $i = 2$. We appy Π to (47), and take its first component, to obtain

$$\partial_t b_s^1 - \left(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\right)b_s^1 = -2\partial_\theta b_\tau^0 - 2\xi\partial_\theta b_s^0 + \partial_\xi b_s^0. \tag{58}$$

Eqs. (57) and (58) will be linked through the jump conditions at $\xi = 0$. As for the velocity profiles, these jump conditions will be deduced from the inner equations.

Inner part, $i = 0$. Eq. $(43)_\mathrm{a}$ yields

$$\partial_\zeta^2 B_s^0 = 0.$$

We deduce that

$$\left[b_s^0\right]\big|_{\xi=0} = 0, \qquad B_s^0 = b_s^0(\xi = 0). \tag{59}$$

The last two components of (43) give

$$\left(\omega'(\zeta), u_z'(\zeta)\right)B_s^0 - \partial_\zeta^2\left(B_\tau^0, B_z^0\right) = 0.$$

We integrate to find:

$$\left(B_\tau^0, B_z^0\right) = b_s^0(\xi = 0)\int_0^\zeta \left(\omega(\zeta'), u_z(\zeta')\right)\mathrm{d}\zeta' + C_1\zeta + C_2,$$

where $C_j = C_j(t, \theta, \lambda)$. Using (40), we end up with $b_s^0(\xi = 0) = 0$, and

$$\left[b_{\tau,z}^0\right]\big|_{\xi=0} = 0, \qquad B_{\tau,z}^0 = b_{\tau,z}^0(\xi = 0). \tag{60}$$

Inner part, $i = 1$. Eq. $(43)_\mathrm{a}$ yields

$$-\partial_\zeta^2 B_s^1 + \left(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\partial_\theta\right)B_s^0 = 0,$$

so that

$$B_s^1 = C_1 + C_2\zeta, \quad C_j = C_j(t,\theta,\lambda).$$

From (40), (60), we get that

$$\left[\partial_\xi b_s^0\right]\big|_{\xi=0} = \left[b_s^1\right]\big|_{\xi=0} = 0, \qquad B_s^1 = \partial_\xi b_s^0(\xi=0)\zeta + b_s^1(\xi=0). \tag{61}$$

The last two components of (43) are:

$$\left(\omega'(\zeta), u_z'(\zeta)\right)B_s^1 - \partial_\zeta^2\left(B_\tau^1, B_z^1\right) + \left(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\right)\left(B_\tau^0, B_z^0\right) = 0.$$

Remind that ω and u_z are constant for $|\zeta| \geqslant 1$. In particular, for $|\zeta| \geqslant 1$,

$$\left(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\right)B_{\tau,z}^0 = \mathbf{1}_{\mathbb{R}^-}(\zeta)\left(\Omega\partial_\theta + U_z(\zeta)\partial_\lambda\right)b_{\tau,z}^0(\xi=0) = 0,$$

using (55). Integration leads to: for all (t,θ,λ), for all $\pm\zeta \geqslant 1$,

$$\left(B_\tau^1, B_z^1\right) = \int_0^\zeta\int_0^\eta \left(\omega'(\eta), u_z'(\eta)\right)B_s^1(\eta)\,\mathrm{d}\eta + \zeta\int_0^{\pm 1}\mathrm{d}\zeta\left(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\right)\left(b_\tau^0, b_z^0\right)(\xi=0) + C_1\zeta + C_2,$$

with $C_j = C_j(t,\theta,\lambda)$. Together with (40), (61), it gives

$$\begin{aligned}\left[\partial_\xi\left(b_\tau^0, b_z^0\right)\right]\big|_{\xi=0} = {}& \left(\int_{-1}^{1}\zeta\left(\omega'(\zeta), u_z'(\zeta)\right)\mathrm{d}\zeta\right)\partial_\xi b_s^0(\xi=\pm 0) + \left(\Omega, U_z\right)\Pi b_s^1(\xi=\pm 0)\\ &+ \left(\int_{-1}^{1}\omega(\zeta)\,\mathrm{d}\zeta\,\partial_\theta + \int_{-1}^{1}u_z(\zeta)\,\mathrm{d}\zeta\,\partial_\lambda\right)\left(b_\tau^0, b_z^0\right)(\xi=0).\end{aligned}$$

Inner part, $i=2$. In order to close the system, we need to have one more jump condition on $\partial_\xi b_s^1$. As above, we use $(43)_\mathrm{a}$, which reads

$$-\partial_\zeta^2 B_s^2 + \left(\omega(\zeta)\partial_\theta + u_z(\zeta)\partial_\lambda\partial_\theta\right)B_s^1 = 0,$$

from which we deduce

$$\left[\partial_\xi b_s^1\right]\big|_{\xi=0} = 0.$$

Gathering previous results, we see that (b_0, b_s^1) satisfies the following system:

$$\begin{cases}\Pi\left(b^0, b_s^1\right) = \left(b^0, b_s^1\right),\\ \partial_t b^0 - \left(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\right)b^0 = 0,\\ \partial_t b_s^1 - \left(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\right)b_s^1 = -2\partial_\theta b_\tau^0 - 2\xi\,\partial_\theta b_s^0 + \partial_\xi b_s^0,\end{cases}$$

with the following conditions at the interface $s=0$:

$$\begin{cases}b_s^0(\xi=0) = 0, \quad \left[b_{\tau,z}^0\right]\big|_{\xi=0} = 0,\\ \left[\partial_\xi\left(b_\tau^0, b_z^0\right)\right]\big|_{\xi=0} = \left(\int_{-1}^{1}\zeta\left(\omega'(\zeta), u_z'(\zeta)\right)\mathrm{d}\zeta\right)\partial_\xi b_s^0(\xi=\pm 0) + (\Omega, U_z)b_s^1(\xi=\pm 0)\\ \qquad + \left(\int_{-1}^{1}\omega(\zeta)\,\mathrm{d}\zeta\,\partial_\theta + \int_{-1}^{1}u_z(\zeta)\,\mathrm{d}\zeta\,\partial_\lambda\right)\left(b_\tau^0, b_z^0\right)(\xi=0),\\ \left[b_s^1\right]\big|_{\xi=0} = \left[\partial_\xi b_s^1\right]\big|_{\xi=0} = 0.\end{cases}$$

We shall limit ourselves to solutions with $b_s^0|_{t=0} = 0$. In this case,

$$b_s^0 \equiv 0$$

and (b_τ^0, b_z^0, b_s^1) is uniquely determined by the initial data $\Pi(b_\tau^0, b_z^0, b_s^1)|_{t=0}$ and by the equations

$$\begin{cases} \Pi\big(b_\tau^0, b_z^0, b_s^1\big) = \big(b_\tau^0, b_z^0, b_s^1\big), \\ \partial_t\big(b_\tau^0, b_z^0\big) - \big(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\big)\big(b_\tau^0, b_z^0\big) = 0, \\ \partial_t b_s^1 - \big(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\big) b_s^1 = -2\partial_\theta b_\tau^0, \end{cases} \tag{62}$$

$$\begin{cases} \big[b_{\tau,z}^0\big] = 0, \\ \big[\partial_\xi\big(b_\tau^0, b_z^0\big)\big]\big|_{\xi=0} = (\Omega, U_z) b_s^1(\xi = \pm 0) + \left(\displaystyle\int_{-1}^{1} \omega(\zeta)\,\mathrm{d}\zeta\,\partial_\theta + \int_{-1}^{1} u_z(\zeta)\,\mathrm{d}\zeta\,\partial_\lambda\right)\big(b_\tau^0, b_z^0\big)(\xi = 0) \\ \big[b_s^1\big]\big|_{\xi=0} = \big[\partial_\xi b_s^1\big]\big|_{\xi=0} = 0. \end{cases} \tag{63}$$

Back to (59), (60), it determines the inner term B^0.

Remark 12. In the previous lines, we did not take into account Eqs. (49) and (45) to derive the magnetic terms b^0, B^0, Πb_s^1. Indeed, the collections of Eqs. (47), (49), and (43), (45) are partially redundant. The reason is that the original system (5) is itself redundant, as the divergence free condition on b is preserved by Eq. $(5)_\mathrm{b}$. This will be clarified in the next subsection.

3.1.3. Higher order profiles

In the previous subsection, we have derived the first profile, namely

$$X^0 = \big(V^0, P^0, B^0, v^0, p^0, b^0, b_s^1, w^0, q^0\big).$$

The derivation of higher order profiles

$$X^i = \big(V^i, P^i, B^i, v^i, p^i, b^i, b_s^{i+1}, w^i, q^i\big)$$

follows the same lines. Indeed, they satisfy the same type of equations up to source terms coming from lower order profiles. For instance, the equations on the outer magnetic terms read

$$\begin{cases} \partial_t b_{\tau,z}^i - \big(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\big) b_{\tau,z}^i = f_{\tau,z}^i, \\ \partial_t b_s^{i+1} - \big(\partial_\xi^2 + \partial_\theta^2 + \partial_\lambda^2\big) b_s^{i+1} = -2\partial_\theta b_\tau^i + f_s^{i+1}, \end{cases} \tag{64}$$

$$\begin{cases} \big[b_{\tau,z}^i\big]\big|_{\xi=0} = g_{\tau,z}^i, \quad \big[\partial_\xi\big(b_\tau^i, b_z^i\big)\big]\big|_{\xi=0} = (\Omega, U_z) b_s^{i+1}(\xi = +0) \\ \quad + \left(\displaystyle\int_{-1}^{1} \omega(\zeta)\,\mathrm{d}\zeta\,\partial_\theta + \int_{-1}^{1} u_z(\zeta)\,\mathrm{d}\zeta\,\partial_\lambda\right)\big(b_\tau^i, b_z^i\big)(\xi = +0) + \big(h_\tau^i, h_z^i\big), \\ \big[b_s^{i+1}\big]\big|_{\xi=0} = k^{i+1}, \quad \big[\partial_\xi b_s^{i+1}\big]\big|_{\xi=0} = l^{i+1}. \end{cases} \tag{65}$$

In this construction, we point out that all the magnetic terms b^i will satisfy $\Pi b^i = b^i$. Indeed, the range of Π is stable by product, derivation, or multiplication by a function of ξ. As all the equations are built from such operations, we obtain inductively, using (47) that

$$\mathbf{1}_{\mathbb{R}_-}(\xi)(\Omega\partial_\theta + U_z\partial_\lambda) b^i = 0, \quad \text{i.e.} \quad \Pi b^i = b^i.$$

The outer velocity and pressure terms v^i, p^i will satisfy Stokes systems, with jump conditions on $[v^i]|_{\xi=0}$, $[\partial_\xi v^i]|_{\xi=0}$, and $[p^i]|_{\xi=0}$. The oscillatory part

$$\tilde{v}^i := v^i - \overline{v}^i, \qquad \tilde{p}^i := p^i - \overline{p}^i$$

will decay fast as $\xi \to \pm\infty$. The average terms $\bar{v}^i$, $\bar{p}^i$ will satisfy

$$\bar{v}^i_s = 0, \quad -\frac{1}{Re}\partial^2_\xi \bar{v}^i_{\tau,z} = f^i_{\tau,z}, \quad \partial_\xi \bar{p}^i = f^i_s,$$

with jump conditions on $[\bar{v}^i_{\tau,z}]|_{\xi=0}$, $[\partial_\xi \bar{v}^i_{\tau,z}]|_{\xi=0}$, and $[\bar{p}^i]|_{\xi=0}$. Even for localized source terms, the solution will not in general decay as $\xi \to \pm\infty$. However, we will have a decomposition

$$\bar{v}^i = \mathcal{V}^i + \overline{w}^i, \qquad \bar{p}^i = \mathcal{P}^i + \bar{q}^i, \tag{66}$$

where $\mathcal{V}^i(t,\xi)$ and $\mathcal{P}^i(t,\xi)$ will decay fast as $\xi \to \pm\infty$, whereas $\overline{w}^i(t,\xi)$ and $\bar{q}^i(t,\xi)$ will be polynomial in ξ. This will be detailed in the next section, when proving Proposition 1.

Such polynomials require to add the external expansion (39). Profiles q^i, w^i will satisfy (50), with given jump conditions on $[w^i]|_{s=0}$, $[\partial_s w^i]|_{s=0}$, and $[q^i]|_{s=0}$. One will have solutions such that when considered as functions defined over $\mathbb{R}^2$, we have

$$w^i(t,x_1,x_2) \in L^\infty\big(0,T;H^1\big(\mathbb{R}^2\backslash\mathcal{C}\big)\big), \qquad \partial_s q^i \in L^\infty\big(0,T;L^2\big(\mathbb{R}^2\backslash\mathcal{C}\big)\big), \tag{67}$$

where $\mathcal{C}$ is the unit circle, moreover, we also have

$$\frac{1}{1+s} w^i_\tau(t,s) \in L^\infty\big(0,T;L^2\big((-1,+\infty)\backslash\{0\}\big)\big).$$

Indeed, let $\chi = \chi(s)$ smooth, compactly supported such that $\chi(s) = 1$ near $s = 0$. We make the change of variables

$$w^i_{\tau,z} := w^i_{\tau,z} - \mathbf{1}_{\{s<0\}}(s)\chi(s)\big(\big[w^i_{\tau,z}\big]\big|_{s=0} + s\big[\partial_s w^i_{\tau,z}\big]\big|_{s=0}\big).$$

The equations become

$$\partial_t w^i_\tau - \frac{1}{Re}\Delta w^i_\tau + \frac{1}{Re(1+s)^2} w^i_\tau = g^i_\tau, \qquad \partial_t w^i_z - \frac{1}{Re}\Delta w^i_z = g^i_z, \tag{68}$$

where $g^i_{\tau,z}$ are compactly supported. Moreover, the jump conditions become homogeneous

$$\big[w^i_{\tau,z}\big]\big|_{s=0} = 0, \qquad \big[\partial_s w^i_{\tau,z}\big]\big|_{s=0} = 0. \tag{69}$$

System (68), (69) will have a unique solution with zero initial data, such that

$$w^i(t,x_1,x_2) \in L^\infty\big(0,T;H^1\big(\mathbb{R}^2\big)\big), \qquad w^i_\tau \in L^\infty\left(0,T;L^2\left(\frac{1}{1+s}\,\mathrm{d}s\right)\right). \tag{70}$$

It follows from the classical a priori estimate

$$\begin{aligned}
&\frac{1}{2}\left(\big\|w^i_\tau(t)\big\|^2_{H^1} + \int_0^t \big(\big\|\partial_t w^i_\tau\big\|^2_{L^2} + \big\|\Delta w^i_\tau\big\|^2_{L^2}\big) + \big\|w^i_\tau(t)\big\|^2_{L^2(\frac{1}{1+s}\,\mathrm{d}s)}\right) \leqslant C\int_0^t \big\|g^i_\tau\big\|^2_{L^2},\\
&\frac{1}{2}\left(\big\|w^i_z(t)\big\|^2_{H^1} + \int_0^t \big(\big\|\partial_t w^i_z\big\|^2_{L^2} + \big\|\Delta w^i_z\big\|^2_{L^2}\big)\right) \leqslant C\int_0^t \big\|g^i_z\big\|^2_{L^2}.
\end{aligned} \tag{71}$$

Moreover, we easily get that $w^i_{\tau,z}$ are smooth up to $s = 0$ in a vicinity of each side of the circle (actually there are smooth on every interval I such that $\bar{I} \subset [0,+\infty)$, or $\bar{I} \subset (-1,0]$).

Back to the original variables, we obtain (67), as the properties of the pressure term are straightforward consequences of those of w^k_τ, $k \leqslant i$.

We emphasize that, throughout the construction of higher order profiles, one has inductively

$$\big(V^i,B^i,P^i\big)(t,\zeta,\theta,\lambda) = \sum_{j=0}^{i} \frac{\zeta^j}{!j}\partial^j_\xi\big(v^{i-j},b^{i-j},p^{i-j}\big)(t,\pm 0,\theta,\lambda), \quad \pm\zeta \geqslant 1, \tag{72}$$

which of course implies (40). Indeed, suppose that relation (72) is satisfied for all indices less than $i-1$. Let us take the limit of $(43)_{\mathrm{a}}$ as $\zeta \to \pm\infty$: using the induction assumption together with $(47)_{\mathrm{a}}$, we deduce easily that

$$\partial_\zeta^2 B_s^i = \partial_\zeta^2 \left(\sum_{j=0}^{i} \frac{\zeta^j}{!j} \partial_\xi^j b_s^{i-j}(t, \pm 0, \theta, \lambda) \right), \quad \pm\zeta \geqslant 1,$$

which means that

$$B_s^i = \sum_{j=0}^{i} \frac{\zeta^j}{!j} \partial_\xi^j b_s^{i-j}(t, \pm 0, \theta, \lambda) + C_1 + C_2\zeta, \quad \pm\zeta \geqslant 1.$$

Moreover, in the construction process of B^i, one ensures that $C_1 = C_2 = 0$. The result follows for B_s^i. Reasoning with $(43)_{\mathrm{b,c}}$ yields the same for B_τ^i, B_z^i. Similar arguments apply to the velocity and pressure fields.

We finally remark that profiles X^i, as we build them, satisfy only (42), (44), (43), (46), (48), (47), and (50). It remains a priori to show that Eqs. (49) and (45) are also fulfilled. In terms of the inner and outer approximations, these conditions read

$$\operatorname{div} b_{\mathrm{app}}^{\mathrm{out}} \approx 0, \qquad \operatorname{div} b_{\mathrm{app}}^{\mathrm{in}} \approx 0.$$

We will ensure that

(1) Equation $\operatorname{div} b_{\mathrm{app}}^{\mathrm{out}}|_{t=0} \approx 0$ is satisfied.
(2) Equation $\operatorname{div} b_{\mathrm{app}}^{\mathrm{in}}|_{t=0} \approx 0$ is satisfied,

which will be sufficient for our purpose.

By an appropriate choice of $b^i|_{t=0} = \Pi b^i|_{t=0}$, we ensure that condition (1) is satisfied. Moreover, thanks to (47), we deduce that

$$\partial_t b_{\mathrm{app}}^{\mathrm{out}} + \left(u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}^{\mathrm{out}}\right) \cdot \nabla b_{\mathrm{app}}^{\mathrm{out}} - b_{\mathrm{app}}^{\mathrm{out}} \cdot \nabla\left(u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}^{\mathrm{out}}\right) - \varepsilon \Delta b_{\mathrm{app}}^{\mathrm{out}} \approx 0.$$

By (48), we know that $\operatorname{div}(u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}^{\mathrm{out}}) \approx 0$. This yields, taking the divergence of the last equation

$$\partial_t \operatorname{div} b_{\mathrm{app}}^{\mathrm{out}} - \left(u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}^{\mathrm{out}}\right) \cdot \nabla \operatorname{div} b_{\mathrm{app}}^{\mathrm{out}} - \varepsilon \Delta \operatorname{div} b_{\mathrm{app}}^{\mathrm{out}} \approx 0.$$

From condition (1) and this last relation, we get that $(\partial_t^\alpha \operatorname{div} b_{\mathrm{app}}^{\mathrm{out}})|_{t=0} \approx 0$ for all $\alpha \in \mathbb{N}$. This reads

$$\partial_t^\alpha L_i^{\mathrm{out}}(t, \xi, \theta, \lambda)|_{t=0} = 0, \quad \forall \alpha, \forall i, \tag{73}$$

where equation $L_i^{\mathrm{out}} = 0$ is exactly (49).

From (43), (44), we also have

$$\partial_t \operatorname{div} b_{\mathrm{app}}^{\mathrm{in}} - \left(u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}^{\mathrm{in}}\right) \cdot \nabla \operatorname{div} b_{\mathrm{app}}^{\mathrm{in}} - \varepsilon \Delta \operatorname{div} b_{\mathrm{app}}^{\mathrm{in}} \approx 0.$$

In the same time, we can write

$$\operatorname{div} b_{\mathrm{app}}^{\mathrm{in}} \approx \sum_{i \geqslant 0} \sqrt{\varepsilon}^{2m+i-2} L_i^{\mathrm{in}}\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right),$$

where for all i, $L_i^{\mathrm{in}}(t, \zeta, \tau, \theta) = 0$ is exactly (45). Combining these relations, we infer that

$$0 \approx \sqrt{\varepsilon}^{2m-4} \partial_\zeta^2 L_0^{\mathrm{in}} + \sqrt{\varepsilon}^{2m-3}\left(\partial_\zeta^2 L_1^{\mathrm{in}} + F_1\right) + \sqrt{\varepsilon}^{2m-2}\left(\partial_\zeta^2 L_2^{\mathrm{in}} + F_2\right) + \cdots, \tag{74}$$

where F_i involves L_k^{in}, $\partial_t L_k^{\mathrm{in}}$ and their spatial derivatives, $k \leqslant i-1$. Using this last relation, we can prove inductively that $(\partial_t^\alpha L_i^{\mathrm{in}})|_{t=0} = 0$ for all α, i:

- First, we notice that $L_0^{\mathrm{in}} = \partial_\zeta^2 B_s^0 = 0$ satisfies $(\partial_t^\alpha L_0^{\mathrm{in}})|_{t=0} = 0$.

- Let us then assume that $(\partial_t^\alpha L_k^{\text{in}})|_{t=0} = 0$ for all α, $k < i$. With relation (74), we deduce that

  $$\left(\partial_t^\alpha \partial_\zeta^2 L_i^{\text{in}}\right)\big|_{t=0} = 0,$$

  which yields

  $$\left(\partial_t^\alpha \partial_\zeta L_i^{\text{in}}\right)\big|_{t=0} = C(\theta, \lambda).$$

  For $|\zeta| > 1$, thanks to identity (72), this last equation becomes

  $$\left(\partial_t^\alpha L_{i-1}^{\text{out}}\right)\big|_{t=0} = C(\theta, \lambda).$$

  By relation (73), we deduce $C \equiv 0$. One more integration provides

  $$\left(\partial_t^\alpha L_i^{\text{in}}\right)\big|_{t=0} = C'(\theta, \lambda).$$

  which for $|\zeta| > 1$, reads

  $$\left(\partial_t^\alpha L_i^{\text{out}}\right)\big|_{t=0} = C'(\theta, \lambda),$$

  and condition (2) follows. This ends the part on WKB profiles.

3.2. Spectral analysis

We can now describe the exponential instability, leading to Theorem 1. It is connected to the solutions of Eqs. (62), (63), on which we will perform a spectral analysis. We state

Proposition 1. *There exists $\sigma > 0$, and a family of profiles*

$$X^i = \left(V^i, P^{i-1}, B^i, v^i, p^i, b^i, b_s^{i+1}, w^i, q^i\right), \quad i \geqslant 0,$$

such that:

(i) (First profile) *For all $p \in \mathbb{N}$,*

$$\left\| \left(b_\tau^0, b_z^0\right)(t) \right\|_{H^p_{\xi,\theta,\lambda}} \sim C_p \mathrm{e}^{\sigma t}, \quad t \to +\infty.$$

(ii) (Higher order profiles) *For all $p \in \mathbb{N}$, for all $i = 2km + l$, for all $l \in [|0, \ldots, 2m|]$, for all $\bar{\delta} > 0$,*

$$\left\|\left| X^i(t) \right\|\right|_p \leqslant C_{i,p,\bar{\delta}} \mathrm{e}^{(k+1)\sigma t} \mathrm{e}^{l\bar{\delta} t}, \quad t \geqslant 0, \tag{75}$$

where

$$\begin{aligned}\left\|\left| X^i(t) \right\|\right|_p = {} & \left\| \left(V^i, P^{i-1}, B^i\right)(t) \right\|_{H^p(\{|\zeta| \leqslant 2\} \times \mathbb{R}^2)} + \left\| w^i(t) \right\|_{H^1(\mathbb{R}^*)} + \left\| \partial_s q^i(t) \right\|_{L^2(\mathbb{R}^*)} \\ & + \left\| \left(1 + |\xi|^2\right)^p \left(\tilde{v}^i, \mathcal{V}^i, \tilde{p}^i, \mathcal{P}^i, b^i, b_s^{i+1}\right)(t) \right\|_{H^p(\mathbb{R}^* \times \mathbb{R}^2)}\end{aligned}$$

and $\mathcal{V}^i$, $\mathcal{P}^i$ are given by (66). *The polynomials $\overline{w}^i$ and $\bar{q}^i$ satisfy for all $i = 2km + 4 + l$, $l \in [|0, \ldots, 2m|]$, for all $\bar{\delta} > 0$,*

$$\left| \overline{w}^i(t, \xi) \right| + \left| \bar{q}^i(t, \xi) \right| \leqslant C_{i,p,\bar{\delta}} \sum_{j=0}^{k} \left(\mathrm{e}^{(j+1)\sigma t} \left(1 + |\xi|^2\right)^{2(k-j)m} \right) \mathrm{e}^{l\bar{\delta} t} \left(1 + |\xi|^2\right)^{1+l}. \tag{76}$$

Proof. We focus first on part (i) of the proposition, i.e. the existence of a first profile

$$b^0 = \left(0, b_\tau^0, b_z^0\right),$$

which grows exponentially with time.

We thus concentrate on Eqs. (62) and (63), and look for modal solutions of the type:

$$\begin{aligned} b_\tau^0 &= \mathrm{e}^{\sigma t}\mathrm{e}^{\mathrm{i}(m\theta+k\lambda)}\hat{b}_\tau^0(\xi) + \text{c.c}, \\ b_z^0 &= \mathrm{e}^{\sigma t}\mathrm{e}^{\mathrm{i}(m\theta+k\lambda)}\hat{b}_z^0(\xi) + \text{c.c}, \\ \Pi b_s^1 &= \mathrm{e}^{\sigma t}\mathrm{e}^{\mathrm{i}(m\theta+k\lambda)}\hat{b}_s^1(\xi) + \text{c.c}, \end{aligned} \tag{77}$$

where c.c is the complex conjugate. Eq. $(62)_\mathrm{a}$ becomes

$$(m\Omega + U_z k)\hat{b}^0 = 0, \quad \xi > 0.$$

In order to get increasing solutions, we need

$$m\Omega + U_z k = 0. \tag{78}$$

Eq. $(62)_\mathrm{a}$ together with $(62)_\mathrm{b}$ leads to

$$\big(\hat{b}_\tau^0, \hat{b}_z^0\big) = \exp(\pm q\xi)\big(\hat{b}_\tau^0(0), \hat{b}_z^0(0)\big), \quad \pm\xi < 0, \tag{79}$$

where

$$q = \big(\sigma + m^2 + k^2\big)^{1/2} = \left(\sigma + m^2\left(1 + \frac{\Omega^2}{U_z^2}\right)\right)^{1/2}$$

is the square root with positive real part. We then solve Eq. $(62)_\mathrm{c}$, using (79) and the fact that $[\Pi b_s^1]|_{\xi=0} = 0$:

$$\hat{b}_s^1 = \exp(\pm q\xi)\left(\hat{b}_s^1(0) \pm \frac{\mathrm{i}m}{q}\xi\hat{b}_\tau^0(0)\right), \quad \pm\xi < 0.$$

We inject this expression in the jump condition $[\partial_\xi \Pi b_s^1]|_{\xi=0} = 0$, and obtain

$$\hat{b}_s^1(0) = -\frac{\mathrm{i}m}{q^2}\hat{b}_\tau^0(0).$$

Finally, using this relation and Eq. (79) in the jump conditions $(63)_\mathrm{a}$, $(63)_\mathrm{b}$, we derive the following system on $(\hat{b}_\tau^0(0), \hat{b}_z^0(0))$:

$$\begin{aligned} -2q\hat{b}_\tau^0(0) &= +\frac{-\mathrm{i}m\Omega}{q^2}\hat{b}_\tau^0(0) + \mathrm{i}m\alpha\hat{b}_\tau^0(0), \\ -2q\hat{b}_z^0(0) &= +\frac{-\mathrm{i}mU_z}{q^2}\hat{b}_\tau^0(0) + \mathrm{i}m\alpha\hat{b}_z^0(0), \end{aligned} \tag{80}$$

where

$$\alpha = \int_{-1}^{1} \omega(\zeta)\,\mathrm{d}\zeta - \frac{\Omega}{U_z}\int_{-1}^{1} u_z(\zeta)\,\mathrm{d}\zeta.$$

This system has nontrivial solutions if and only if

$$\big(2q + \mathrm{i}\alpha m\big)\left(2q + \mathrm{i}\alpha m - \frac{\mathrm{i}m\Omega}{q^2}\right) = 0.$$

We are interested in solutions σ of positive real part, necessarily solving

$$2q^3 + \mathrm{i}\alpha m q^2 - \mathrm{i}m\Omega = 0. \tag{81}$$

Since

$$\sigma = q^2 - m^2\left(1 + \frac{\Omega^2}{U_z^2}\right),$$

we want to show that this equation in q has at least one root such that σ has positive real part for a suitable choice of the parameters α, Ω, U_z. At first, we choose Ω and U_z so that

$$\frac{|\Omega|^{2/3}}{2^{5/3}} - \left(1 + \frac{\Omega^2}{U_z^2}\right)0. \tag{82}$$

Let us focus on the case where $m = 1$. We shall investigate the behavior of the solutions of (81) with Ω and U_z fixed by the previous condition in the limit where α tends to zero. By the implicit function theorem, we have a root q such that

$$q = \left(\frac{|\Omega|}{2}\right)^{1/3} \exp\left(\frac{\mathrm{i}\pi \operatorname{sign}(\Omega)}{6}\right) + \mathcal{O}(\alpha).$$

This yields

$$Re(\sigma) = \frac{|\Omega|^{2/3}}{2^{5/3}} - \left(1 + \frac{\Omega^2}{U_z^2}\right) + \mathcal{O}(\alpha).$$

Consequently, thanks to (82), we have found an unstable solution of positive real part if α is sufficiently small.

Now let us consider Ω, U_z and α fixed by the previous argument, the next step is to study the real part of $\sigma(m)$ when the parameter m is in $\mathbb{Z}$ or equivalently the three roots $q(m)$ of (81). This will allow us to choose the solution with the maximal growth rate. Thanks to our suitable choice of the parameters, we have already shown that there is a root q such that $Re\,\sigma(1) > 0$. Moreover, we notice that the solutions of (81) are such that $-q(m) = q(-m)$ and hence we have that $\sigma(-m) = \sigma(m)$, so that it suffices to study the roots for m positive. We shall prove that for every solution of (81), $Re\,\sigma(m) \to -\infty$ when $m \to +\infty$. Since the roots depend continuously on m, we can choose m such that one of the $\sigma(m)$ has maximal positive real part. It remains to prove that for all the solutions of (81), we have $Re\,\sigma(m) \to -\infty$ when $m \to +\infty$. Let us set $q = mQ$, then Q is a root of

$$2Q^3 + \mathrm{i}\alpha Q^2 - \mathrm{i}\frac{\Omega}{m^2} = 0. \tag{83}$$

In the limit $m \to +\infty$, we find the equation

$$2Q^3 + \mathrm{i}\alpha Q^2 = 0.$$

Consequently, one root of (83) tend to $-\mathrm{i}\alpha/2$ and the two other ones cross at zero. We can handle very easily the two roots which vanish. Indeed they verify $|Q| \leqslant C/m$, consequently, we have $|q|^2 \leqslant C$ and hence we get

$$Re\,\sigma \leqslant C - m^2\left(1 + \Omega^2/U_z^2\right) \to -\infty.$$

The root which does not vanish verifies

$$Q = -\frac{\mathrm{i}\alpha}{2} + \mathcal{O}(m^{-2}),$$

and this yields

$$q^2 = -\frac{\alpha^2}{4}m^2 + \mathcal{O}(1)$$

and hence

$$Re\,\sigma = -\frac{\alpha^2}{4}m^2 - m^2\left(1 + \Omega^2/U_z^2\right) + \mathcal{O}(1) \to -\infty.$$

This finally shows the existence of an unstable mode with maximal growth rate σ. Note that the divergence-free condition (49), $i = 0$ is well-satisfied, as

$$\partial_\xi b_s^0 + \partial_\theta b_\tau^0 + \partial_\lambda b_z^0 = \mathrm{i}m\hat{b}_\tau^0 + \mathrm{i}k\hat{b}_z^0 = 0,$$

where we have used (78), (80).

With part (i) established, to obtain recursively higher order profiles with appropriate growth is classical, and has been performed in various stability studies, for instance [9,2,5]. Our situation is even simpler: we deal with discrete tangential Fourier modes, which avoids the construction of localized wavepackets. For the sake of brevity, we only remind the key elements of the process, and refer to [2] for all necessary details.

- The outer magnetic terms are deduced from a Laplace transform in time of Eqs. (64), (65). For each tangential Fourier mode

$$b^i = \mathrm{e}^{\mathrm{i}m\theta}\mathrm{e}^{\mathrm{i}k\lambda}\mathcal{F}\left(b^i\right)(t, \xi),$$

we get an integral formula

$$\mathcal{F}\big(b^i\big)(t,\xi) = \int_\Gamma e^{\lambda t}\hat{b}^i(\lambda,\xi)\,d\lambda, \tag{84}$$

where Γ is a contour in the complex domain. We can take for example a parabola $Re\,\lambda = -A(\mathcal{I}m\,\lambda)^2 + (\sigma+\delta)$ such that the zeros of the dispersion relation are in the left hand-side of Γ. The classical theory for analytic semigroups (see textbook [10] for example) gives that the semigroup $S(t)$ associated to the evolution equation (62), (63) enjoys the estimate

$$\big\|S(t)\big\|_{\mathcal{L}(H^p)} \leqslant C_p e^{(\sigma+\delta)t} \tag{85}$$

where $\delta > 0$ can be chosen arbitrarily small.
The term $\hat{b}^i(\lambda,\xi)$ is given by the same type of computations as for $\hat{b}^0$, accounting for the additional source terms. Hence, the exponential bound of (75) follows from the estimate on the semigroup (85).

- The control of $\tilde{v}^i$, $\tilde{p}^i$ is easy: they have zero average with respect to θ, λ (so that Poincaré's inequality holds) and satisfy Stokes type equations.
- The control of w^i, q^i is deduced recursively from (71).
- We remind decomposition (66) on the average velocity and pressure terms. The terms $\mathcal{V}^i$, $\mathcal{P}^i$ satisfy

$$\partial_\xi \mathcal{P}^i = f_s^i, \qquad -\frac{1}{Re}\partial_\xi^2 \mathcal{V}^i_{\tau,z} = f^i_{\tau,z},$$

where f_s^i and $f^i_{\tau,z}$ involve quadratically $\tilde{v}^k$, b^k and linearly $\mathcal{V}^k$, $\mathcal{P}^k$ for $k \leqslant i-1$. They decrease fast as $\xi \to \infty$, which means

$$\mathcal{P}^i = \int_{\pm\infty}^{\xi} f_s^i, \quad \mathcal{V}^i_{\tau,z} = -Re\int_{\pm\infty}^{\xi}\int_{\pm\infty}^{\xi'} f^i_{\tau,z}, \quad \pm\xi > 0.$$

The exponential bound (75) follows from the ones on the $\tilde{v}^k$'s and $\tilde{b}^k$'s. The other terms $\overline{w}^i$ and $\bar{q}^i$ are polynomial in ξ. They satisfy equations of the type

$$\partial_\xi \bar{q}^i = 2\Omega\,\overline{w}^{i-2} + \sum_{i=2m+k+k'+k''+5} (-1)^k \xi^k\,\overline{w}_\tau^{k'}\,\overline{w}_\tau^{k''},$$

$$-\frac{1}{Re}\partial_\xi^2 \overline{w}^i_{\tau,z} = \sum_{k=0}^{i-2} b_{k,\tau,z}\xi^k\,\overline{w}^{i-k-2}_{\tau,z} + \sum_{k=0}^{i-1} c_{k,\tau,z}\xi^k\,\overline{w}^{i-k-1}_{\tau,z}.$$

Jump conditions read

$$\big[\overline{w}^i_{\tau,z}\big]\big|_{\xi=0} = \alpha^i_{\tau,z}, \quad \big[\partial_\xi \overline{w}^i_{\tau,z}\big]\big|_{\xi=0} = \beta^i_{\tau,z}, \quad \big[\bar{q}^i\big]\big|_{\xi=0} = \gamma^i,$$

where $\alpha^i_{\tau,z}$, $\beta^i_{\tau,z}$ and γ^i involve V^i, V^{i+1}, P^i (coming from the inner expansion) and $\mathcal{V}^i$, $\mathcal{P}^i$. Note that $\bar{v}^i$ and $\bar{q}^i$ get non-zero as soon as $\operatorname{curl} b \times b$ is responsible for a quadratic term in the right-hand side. This happens for

$$2m + i - 2 = 2 + 4m + k + k', \quad k, k' \in \mathbb{N},$$

i.e. $i \geqslant 2m+4$. Hence, we have

$$\big|\overline{w}^{2m+4}(t,\xi)\big| + \big|\overline{w}^{2m+4}(t,\xi)\big| \leqslant C e^{2\sigma t}\big(1+|\xi|^2\big).$$

The general bound is then shown recursively. □

3.3. Conclusion

3.3.1. Unstable approximate solution

Let $m > 5$. Thanks to point (i) of Proposition 1, there exists $C_0 > 0$, such that

$$\big\|\big(b^0_\tau, b^0_z\big)(t)\big\|_{L^2_{\xi,\theta,\lambda}} \geqslant C_0 \exp(\sigma t).$$

We introduce

$$\mathcal{E}(t) = C_0\varepsilon^m \exp(\sigma t), \quad T_\varepsilon \text{ such that } \mathcal{E}(T_\varepsilon) = 1.$$

Using point (ii) of Proposition 1, we deduce that for all $i = 2km + l$, for all $l \in [|0, \dots, 2m|]$, for all $\bar\delta > 0$,

$$\begin{aligned}\left\|\left|\sqrt{\varepsilon}^{2m+i} X^i(t)\right\|\right|_p &\leqslant C_{i,p,\bar\delta}\sqrt{\varepsilon}^{2m+i}\exp\bigl((k+1)\sigma t\bigr)\exp(l\bar\delta t)\\ &\leqslant C_{i,p,\bar\delta}\bigl(\varepsilon^m\exp(\sigma t)\bigr)^{k+1}\bigl(\sqrt{\varepsilon}\exp(\bar\delta t)\bigr)^l.\end{aligned}$$

As $T_\varepsilon = -(m\ln(C_0\varepsilon))/\sigma$, we get

$$\left\|\left|\sqrt{\varepsilon}^{2m+i} X^i(t)\right\|\right|_p \leqslant C'_{i,p,\bar\delta}\mathcal{E}(t)^{k+1}\bigl(\varepsilon^{1/2-m\bar\delta/\sigma}\bigr)^l. \tag{86}$$

Let $\bar\delta < \sigma/(6m)$, N a positive integer to be chosen later, and $n = 2mN + 2m$. Thanks to the choice of n and inequality (86), we get, for all $0 \leqslant k \leqslant 2m$, and for all $0 \leqslant t \leqslant T_\varepsilon$,

$$\left\|\left|\sqrt{\varepsilon}^{2m+n-k} X^{n-k}\right\|\right|_p \leqslant C_{n,p}\mathcal{E}(t)^{N+1}\varepsilon^{(2m-k)/3}. \tag{87}$$

Besides, the approximate solution will involve terms like

$$\sqrt{\varepsilon}^{2m+i}\Psi\left(\frac{s}{\varepsilon^\gamma}\right)\left(\overline{w}^i\left(\frac{s}{\sqrt{\varepsilon}}\right), \bar q^i\left(\frac{s}{\sqrt{\varepsilon}}\right)\right)$$

for $0 < \gamma < 1/2$, and some smooth compactly supported function Ψ. By estimates (76) for $i = 2km + 4 + l$, we obtain

$$\begin{aligned}&\left|\sqrt{\varepsilon}^{2m+i}\Psi\left(\frac{s}{\varepsilon^\gamma}\right)\left(\overline{w}^i\left(\frac{s}{\sqrt{\varepsilon}}\right), \bar q^i\left(\frac{s}{\sqrt{\varepsilon}}\right)\right)\right|\\ &\quad\leqslant C\sqrt{\varepsilon}^{2m+i}\sum_{j=0}^{k} \mathrm{e}^{(j+1)\sigma t}\varepsilon^{(4\gamma-2)(k-j)m}\mathrm{e}^{l\bar\delta t}\varepsilon^{(2\gamma-1)(1+l)}\\ &\quad\leqslant C\varepsilon^m\mathrm{e}^{\sigma t}\sum_{j=0}^{k}\bigl(\varepsilon^{jm}\mathrm{e}^{j\sigma t}\bigr)\bigl(\varepsilon^{(k-j)m}\varepsilon^{(4\gamma-2)(k-j)m}\bigr)\bigl(\sqrt{\varepsilon}^l\mathrm{e}^{l\bar\delta t}\bigr)\bigl(\varepsilon^2\varepsilon^{(2\gamma-1)(1+l)}\bigr)\\ &\quad\leqslant C\mathcal{E}(t)\sum_{j=0}^{k}\mathcal{E}(t)^j\varepsilon^{(4\gamma-1)(k-j)m}\varepsilon^{l/3}\bigl(\varepsilon^2\varepsilon^{(2\gamma-1)(1+l)}\bigr).\end{aligned} \tag{88}$$

We choose γ such that

$$4\gamma - 1 > 1/2, \qquad 2 + (2\gamma - 1)(1 + 2m) > 0 \tag{89}$$

to obtain

$$\begin{aligned}\left|\sqrt{\varepsilon}^{2m+i}\Psi\left(\frac{s}{\varepsilon^\gamma}\right)\left(\overline{w}^i\left(\frac{s}{\sqrt{\varepsilon}}\right), \bar q^i\left(\frac{s}{\sqrt{\varepsilon}}\right)\right)\right| &\leqslant C\mathcal{E}(t)\sum_{j=0}^{k}\mathcal{E}(t)^j\mathcal{E}(t)^{(k-j)/2}\varepsilon^{l/3}\\ &\leqslant C\mathcal{E}(t)^{k/2+1}.\end{aligned} \tag{90}$$

As above, we deduce that for all $-4 \leqslant k \leqslant 2m - 4$,

$$\left|\sqrt{\varepsilon}^{2m+n-k}\Psi\left(\frac{s}{\varepsilon^\gamma}\right)\left(\overline{w}^{n-k}\left(\frac{s}{\sqrt{\varepsilon}}\right), \bar q^{n-k}\left(\frac{s}{\sqrt{\varepsilon}}\right)\right)\right| \leqslant C_{n,p}\mathcal{E}(t)^{N/2+1}\varepsilon^{(2m-k-4)/3}. \tag{91}$$

We now follow the approach of Goodman and Xin [8]. We introduce some smooth function $\chi = \chi(\alpha)$ with

$$\chi = 1 \quad \text{for } |\alpha| \leqslant 1, \qquad \chi = 0 \quad \text{for } |\alpha| \geqslant 2.$$

We set, in polar coordinates,

$$b_{\mathrm{app}} = \chi\left(\frac{s}{\varepsilon}\right)B\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon}\right)\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)b\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right),$$
$$v_{\mathrm{app}} = \chi\left(\frac{s}{\varepsilon}\right)V\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon}\right)\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)v\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon^\gamma}\right)\right)w(t,s),$$
$$p_{\mathrm{app}} = \chi\left(\frac{s}{\varepsilon}\right)P\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon}\right)\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)p\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon^\gamma}\right)\right)q(t,s),$$

where

$$(V,B,P)(t,\zeta,\theta,\lambda) = \sum_{i=0}^{n} \sqrt{\varepsilon}^{2m+i}\left(V^i, B^i, \varepsilon^{-1/2}P^i\right)(t,\zeta,\theta,\lambda),$$
$$(v,b,p)(t,\xi,\theta,\lambda) = \sum_{i=0}^{n} \sqrt{\varepsilon}^{2m+i}\left(v^i, b^i, \varepsilon^{-1/2}p^i\right)(t,\xi,\theta,\lambda),$$
$$(w,q)(t,s) = \sum_{i=0}^{n} \sqrt{\varepsilon}^{2m+i}\left(w^i, \varepsilon^{-1/2}q^i\right)(t,s).$$

We inject these smooth approximations in (5). We obtain

$$\partial_t v_{\mathrm{app}} + u^\varepsilon\cdot\nabla v_{\mathrm{app}} + v_{\mathrm{app}}\cdot\nabla u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}\cdot\nabla v_{\mathrm{app}} + \nabla p - \frac{1}{Re}\Delta v_{\mathrm{app}} = \varepsilon^{3/2} b_{\mathrm{app}}\cdot\nabla b_{\mathrm{app}} + \mathcal{R}_v,$$
$$\partial_t b_{\mathrm{app}} + u^\varepsilon\cdot\nabla b_{\mathrm{app}} - b_{\mathrm{app}}\cdot\nabla u^\varepsilon + \varepsilon^{3/2} v_{\mathrm{app}}\cdot\nabla b_{\mathrm{app}} - \varepsilon^{3/2} b_{\mathrm{app}}\cdot\nabla v_{\mathrm{app}} - \frac{1}{Rm}\Delta b_{\mathrm{app}} = \mathcal{R}_b.$$

It remains to estimate $\mathcal{R}_v$ and $\mathcal{R}_b$. We claim the following bounds

Proposition 2. *We have*

$$\|\mathcal{R}_b\|_{L^2} \leqslant C\varepsilon\mathcal{E}(t)^{N/2}, \quad \|\nabla\mathcal{R}_b\|_{L^2} \leqslant C\mathcal{E}(t)^{N/2}, \quad \|\mathcal{R}_v\|_{L^2} \leqslant C\varepsilon\mathcal{E}(t)^{N/2}.$$

Proof. The remainder $\mathcal{R}_b$ reads

$$\mathcal{R}_b = \chi\left(\frac{s}{\varepsilon}\right)\mathcal{R}_b^1\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \left(1-\chi\left(\frac{s}{\varepsilon}\right)\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)\mathcal{R}_b^2\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) + \mathcal{R}_b^3.$$

In this expression, $\mathcal{R}_b^1$, resp. $\mathcal{R}_b^2$ involves the inner profiles (V^i, B^i, P^i), resp. the outer profiles (v^i, b^i, p^i). It is small because these profiles satisfy (42), (43), resp. (46), (47). Note that v^i is involved only through the quadratic term $\operatorname{curl} v \times b$. Hence, the polynomial growth of $\overline{w}^i$ with respect to ξ is killed by the decrease of the b^k's. Consequently, with estimates (86), (87), one has easily

$$\begin{aligned} &\left\|\mathcal{R}_b^1(t,\cdot)\right\|_{H^1(\zeta,\theta,\lambda)} \leqslant \sqrt{\varepsilon}\mathcal{E}(t)^{N+1}, \\ &\left\|\mathcal{R}_b^2(t,\cdot)\right\|_{H^1(\xi,\theta,\lambda)} \leqslant \varepsilon\mathcal{E}(t)^{N+1}. \end{aligned} \tag{92}$$

The third term $\mathcal{R}_b^3$ comes from the truncation errors. It is made of two parts: one has support in $\{\varepsilon \leqslant |s| \leqslant 2\varepsilon\}$ (truncation of the inner expansion), the other has support in $\{\varepsilon^\gamma \leqslant |s| \leqslant 2\varepsilon^\gamma\}$ (truncation of the outer expansion). The latter part is $O(\varepsilon^\infty)$, as the outer magnetic terms have fast decrease in ξ (see estimate (75)). The former part is small because of the matching condition. Indeed,

$$\begin{aligned} \mathcal{R}_b^3 = {}& -\varepsilon^{-1}\chi''\left(\frac{s}{\varepsilon}\right)\left(B\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) - b\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right)\right) \\ & - 2\chi'\left(\frac{s}{\varepsilon}\right)\partial_s\left(B\left(t, \frac{s}{\varepsilon}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right) - b\left(t, \frac{s}{\sqrt{\varepsilon}}, \frac{\tau}{\sqrt{\varepsilon}}, \frac{z}{\sqrt{\varepsilon}}\right)\right) + \cdots. \end{aligned}$$

From the matching condition (72), we have for all $\varepsilon \leqslant |s| \leqslant 2\varepsilon$,

$$B\left(t, \frac{s}{\varepsilon}, \theta, \lambda\right) = \sum_{j+k\leqslant n} \frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k-j}\,\partial_\xi^j b^k\left(t, \operatorname{sign}(s)0, \theta, \lambda\right).$$

On the other hand, a Taylor expansion yields

$$b\left(t,\frac{s}{\sqrt{\varepsilon}},\theta,\lambda\right)=\sum_{j=0}^{n}\sum_{k=0}^{n}\frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k-j}\partial_\xi^j b^k\big(t,\operatorname{sign}(s)0,\theta,\lambda\big)+\frac{s^{n+1}}{(n+1)!}\sum_{k=0}^{n}\sqrt{\varepsilon}^{2m+k-n-1}\partial_\xi^{n+1}b^k\big(t,\xi_s^k,\theta,\lambda\big),$$

where $\xi_s^k\in[0,s/\sqrt{\varepsilon}]$. We thus obtain

$$\begin{aligned}B\left(t,\frac{s}{\varepsilon},\theta,\lambda\right)-b\left(t,\frac{s}{\sqrt{\varepsilon}},\theta,\lambda\right)&=\sum_{j+k>n}\frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k-j}\partial_\xi^j b^k\big(t,\operatorname{sign}(s)0,\theta,\lambda\big)\\&\quad+\frac{s^{n+1}}{(n+1)!}\sum_{k=0}^{n}\sqrt{\varepsilon}^{2m+k-n-1}\partial_\xi^{n+1}b^k\big(t,\xi_k^s,\theta,\lambda\big).\end{aligned}$$

with all $0\leqslant j,k\leqslant n$. The second term in the r.h.s. is bounded through

$$\left|\frac{s^{n+1}}{(n+1)!}\sum_{k=0}^{n}\sqrt{\varepsilon}^{2m+k-n-1}\partial_\xi^{n+1}b^k(t,\xi_k^s,\theta,\lambda)\right|\leqslant C\sqrt{\varepsilon}^{n+1}\mathcal{E}(t)\leqslant C\sqrt{\varepsilon}^{2m+1}\mathcal{E}(t)^{N+1}.$$

The first sum at the r.h.s. can be divided into

$$\left|\sum_{\substack{j\geqslant n/2,\\ j+k>n}}\frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k-j}\partial_\xi^j b^k\big(t,\operatorname{sign}(s)0,\theta,\lambda\big)\leqslant\sqrt{\varepsilon}^{\frac{n}{2}(\gamma-1)}\right|\leqslant C\sqrt{\varepsilon}^{n/2}\mathcal{E}(t)\leqslant C\sqrt{\varepsilon}^{m}\mathcal{E}(t)^{N/2+1},$$

$$\left|\sum_{\substack{j\leqslant m/2,\\ j+k>n}}\frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k}\partial_\xi^j b^k\big(t,\operatorname{sign}(s)0,\theta,\lambda\big)\right|\leqslant C\varepsilon^{m/3}\mathcal{E}(t)^{N+1}\varepsilon^{\gamma}\mathcal{E}(t)^{N+1},$$

using (87), and

$$\left|\sum_{\substack{n/2\geqslant j>m/2\\ j+k>n}}\frac{s^j}{j!}\sqrt{\varepsilon}^{2m+k}\partial_\xi^j b^k\big(t,\operatorname{sign}(s)0,\theta,\lambda\big)\right|\leqslant C\varepsilon^{m/4}\mathcal{E}(t)^{N/2}$$

using that all k in the last sum are more than $n/2$. We finally get

$$\left|B\left(t,\frac{s}{\varepsilon},\theta,\lambda\right)-b\left(t,\frac{s}{\sqrt{\varepsilon}},\theta,\lambda\right)\right|\leqslant C\varepsilon^{m/3}\mathcal{E}(t)^{N/2}.$$

Proceeding in the same way with the other terms, we end up with

$$\|\mathcal{R}_b^3\|_{L^2}+\|\nabla\mathcal{R}_b^3\|_{L^2}\leqslant C\varepsilon^{m/5}\mathcal{E}(t)^{N/2},$$

and together with (92), it yields the estimate on $\mathcal{R}_b$.

The remainder $\mathcal{R}_v$ is similar. Namely,

$$\begin{aligned}\mathcal{R}_v&=\chi\left(\frac{s}{\varepsilon}\right)\mathcal{R}_v^1\left(t,\frac{s}{\varepsilon},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right)+\left(1-\chi\left(\frac{s}{\varepsilon}\right)\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)\mathcal{R}_v^2\left(t,\frac{s}{\sqrt{\varepsilon}},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right)\\&\quad+\left(1-\chi\left(\frac{s}{\varepsilon^\gamma}\right)\right)\mathcal{R}_v^3(t,s)+\mathcal{R}_v^4.\end{aligned}$$

In this expression, $\mathcal{R}_v^1$ involves the inner profiles and satisfies

$$\|\mathcal{R}_v^1(t,\cdot)\|_{H^1(\zeta,\theta,\lambda)}\leqslant C\sqrt{\varepsilon}\mathcal{E}(t)^{N+1}.$$

The outer term $\mathcal{R}_v^2$ involves the polynomials $\overline{w}^i,\bar{q}^i$, truncated at scale ε^γ. Using estimates (88)–(91), we get

$$\|\mathcal{R}_v^1(t,\cdot)\|_{H^1(\xi,\theta,\lambda)}\leqslant C\varepsilon\mathcal{E}(t)^{N/2+1}.$$

The third term $\mathcal{R}_v^3$ involves the external profiles. It satisfies

$$\left\|\mathcal{R}_v^3(t,\cdot)\right\|_{L^2((1+s)\,\mathrm{d}s)} \leqslant C\varepsilon^{3/2}\mathcal{E}(t)^{N+1}.$$

The last term is due to truncation errors, and is treated again thanks to the matching conditions. We obtain

$$\left\|\mathcal{R}_v^4\right\|_{L^2} + \left\|\nabla\mathcal{R}_v^4\right\|_{L^2} \leqslant C\varepsilon^{m/5}\mathcal{E}(t)^{N/2}$$

and the estimate on $\mathcal{R}_v$ follows. □

At this point, we still need to add a divergence-free corrector to v_{app}, b_{app}, so that the divergence-free conditions hold.

Proposition 3. *There exists $\tilde{v} = \tilde{v}(t, \mathrm{x})$, $\tilde{b} = \tilde{b}(\mathrm{x})$, such that*

$$\operatorname{div}\tilde{v} = -\operatorname{div} v_{\mathrm{app}}, \qquad \operatorname{div}\tilde{b} = -\operatorname{div} b_{\mathrm{app}}|_{t=0},$$

and

$$\left\|\partial_t^\alpha\tilde{v}\right\|_{H^\beta} + \left\|\partial_t^\alpha\tilde{b}\right\|_{H^\beta} \leqslant C\varepsilon^{-\beta}\varepsilon^{m/5}\mathcal{E}(t)^{N/2},$$
$$\left\|\partial_t^\alpha\tilde{v}\right\|_{W^{\beta,\infty}} + \left\|\partial_t^\alpha\tilde{b}\right\|_{W^{\beta,\infty}} \leqslant C\varepsilon^{-\beta}\varepsilon\mathcal{E}(t)^{N/2}.$$

Proof. We have

$$\begin{aligned}\operatorname{div} v_{\mathrm{app}} &= \chi\left(\frac{s}{\varepsilon}\right)\mathcal{D}^1\left(t,\frac{s}{\varepsilon},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right) + (1-\chi)\left(\frac{s}{\varepsilon}\right)\chi\left(\frac{s}{\varepsilon^\gamma}\right)\mathcal{D}^2\left(t,\frac{s}{\sqrt{\varepsilon}},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right)\\ &\quad + \frac{1}{\varepsilon}\chi'\left(\frac{s}{\varepsilon}\right)\left(U_s\left(t,\frac{s}{\varepsilon},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right) - u_s\left(t,\frac{s}{\sqrt{\varepsilon}},\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right)\right) + \frac{1}{\varepsilon^\gamma}\chi'\left(\frac{s}{\varepsilon^\gamma}\right)u_s\\ &= \mathcal{D}\left(t,s,\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right).\end{aligned}$$

As $\mathcal{D}$ does not involve the average terms $\overline{w}^i(t,\xi)$ and $w^i(t,s)$ (which satisfy notably $\overline{w}^i_s = 0$, $w^i_s = 0$), we have

$$\int_{\theta,\lambda}\mathcal{D} = 0. \tag{93}$$

Moreover, reasoning as for $\mathcal{R}_{v,b}$, we get

$$\left(\int\left\|\partial_t^\alpha\partial_s^\beta\mathcal{D}(t,s,\cdot)\right\|^2_{H^m_{\theta,\lambda}}(1+s)\,\mathrm{d}s\right)^{1/2} \leqslant \frac{C_{\alpha,\beta}}{\varepsilon^\beta}\varepsilon^{m/5}\mathcal{E}(t)^{N/2}.$$

We then look for a lift $\tilde{v}$ of the following type:

$$\tilde{v} = \mathcal{V}\left(t,s,\frac{\tau}{\sqrt{\varepsilon}},\frac{z}{\sqrt{\varepsilon}}\right), \quad \mathcal{V} = \left(\partial_s, \sqrt{\varepsilon}^{-1}\frac{\partial_\theta}{1+s}, \sqrt{\varepsilon}^{-1}\partial_\lambda\right)\phi.$$

This leads to

$$\partial_s^2\phi + \frac{1}{1+s}\partial_s\phi + \frac{1}{\varepsilon(1+s)^2}\partial_\theta^2\phi + \frac{1}{\varepsilon}\partial_\lambda^2\phi = \mathcal{D}(t,s,\theta,\lambda). \tag{94}$$

$\mathcal{D}$ has a finite number of non-zero tangential Fourier modes (m,k), all satisfying $Um + \Omega k = 0$. Thanks to (93), we have $(m,k) \neq (0,0)$, so that $k \neq 0$. The Fourier transform of (94) then gives

$$\partial_s^2\hat{\phi} + \frac{1}{1+s}\partial_s\hat{\phi} - \frac{k^2}{\varepsilon}\left(\frac{\Omega^2}{U^2(1+s)^2} + 1\right)\hat{\phi} = \hat{\mathcal{D}}(t,s,\theta,\lambda).$$

Standard energy estimates provide the bounds on $\mathcal{V}$. The same reasoning holds for the magnetic part. □

As the final step of our process, we define

$$v^\varepsilon_{\mathrm{app}} = v_{\mathrm{app}} + \tilde{v}, \qquad b^\varepsilon_{\mathrm{app}} = b_{\mathrm{app}} + \tilde{b}.$$

It is straightforward from previous work that

$$\partial_t v^\varepsilon_{\text{app}} + u^\varepsilon \cdot \nabla v^\varepsilon_{\text{app}} + v^\varepsilon_{\text{app}} \cdot \nabla u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}} \cdot \nabla v^\varepsilon_{\text{app}} + \nabla p - \frac{1}{Re}\Delta v^\varepsilon_{\text{app}} = \varepsilon^{3/2} b^\varepsilon_{\text{app}} \cdot \nabla b^\varepsilon_{\text{app}} + \mathcal{R}^\varepsilon_v(t,s,\tau,z),$$
$$\partial_t b^\varepsilon_{\text{app}} + u^\varepsilon \cdot \nabla b^\varepsilon_{\text{app}} - b^\varepsilon_{\text{app}} \cdot \nabla u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}} \cdot \nabla b^\varepsilon_{\text{app}} - \varepsilon^{3/2} b^\varepsilon_{\text{app}} \cdot \nabla v^\varepsilon_{\text{app}} - \frac{1}{Rm}\Delta b^\varepsilon_{\text{app}} = \mathcal{R}^\varepsilon_b(t,s,\tau,z)$$

with

$$\left\|\mathcal{R}^\varepsilon_b\right\|_{L^2} \leqslant C\varepsilon \mathcal{E}(t)^{N/2}, \quad \left\|\nabla \mathcal{R}^\varepsilon_b\right\|_{L^2} \leqslant C \mathcal{E}(t)^{N/2}, \quad \left\|\mathcal{R}^\varepsilon_v\right\|_{L^2} \leqslant C\varepsilon \mathcal{E}(t)^{N/2}. \tag{95}$$

3.3.2. Proof of Theorem 4

We can now prove the exponential instability result. Let p, s large, and $m = p + s + 1$. We introduce $(v^\varepsilon, b^\varepsilon)$ the solution of (5) with initial data $(v^\varepsilon_{\text{app}}, b^\varepsilon_{\text{app}})|_{t=0}$. It is possible because both $v^\varepsilon_{\text{app}}$ and $b^\varepsilon_{\text{app}}$ are divergence-free at $t = 0$. Note that

$$\left\|(v^\varepsilon, b^\varepsilon)|_{t=0}\right\|_{H^s} \leqslant \varepsilon^p.$$

We set $v = v^\varepsilon - v^\varepsilon_{\text{app}}$, $b = b^\varepsilon - b^\varepsilon_{\text{app}}$. They satisfy

$$\begin{aligned}
&\partial_t v + \left(u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}}\right) \cdot \nabla v + v \cdot \nabla\left(u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}}\right) + \varepsilon^{3/2} v \cdot \nabla v \\
&\quad + \nabla p - \frac{1}{Re}\Delta v = \varepsilon^{3/2} b^\varepsilon_{\text{app}} \cdot \nabla b + \varepsilon^{3/2} b \cdot \nabla b^\varepsilon_{\text{app}} + \varepsilon^{3/2} b \cdot \nabla b + \mathcal{R}^\varepsilon_v\left(t,s,\tau,z\right), \quad \operatorname{div} v^\varepsilon_{\text{app}} = 0, \\
&\partial_t b + \left(u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}}\right) \cdot \nabla b - b \cdot \nabla\left(u^\varepsilon + \varepsilon^{3/2} v^\varepsilon_{\text{app}}\right) + \varepsilon^{3/2} v \cdot \nabla b^\varepsilon_{\text{app}} - \varepsilon^{3/2} b^\varepsilon_{\text{app}} \cdot \nabla v \\
&\quad + \varepsilon^{3/2} v \cdot \nabla b - \varepsilon^{3/2} b \cdot \nabla v - \frac{1}{Rm}\Delta b = \mathcal{R}^\varepsilon_b\left(t,s,\tau,z\right).
\end{aligned}$$

We take back notations of Theorem 2. Let us split the velocity field $v^\varepsilon_{\text{app}}$ into a smooth part and a singular part

$$v^\varepsilon_{\text{app}} = v^r + v^s, \quad v^s = \left(1 - \chi(s/4)\right) w(t,s)$$

so that v^r is smooth, as v^s contains the external expansion which is locally smooth but only enjoys the global property coming from (70) that is because of the truncation

$$v^s \in L^\infty\left(0, T, H^1\left(\mathbb{R}^2\right)\right). \tag{96}$$

With minor modifications of its proof, accounting for additional source terms, we obtain

$$\begin{aligned}
E(v,b,t) + \int_0^t D(v,b) &\leqslant \mathcal{C}\left(|U| + |U'| + \left\|\varepsilon\left(b^\varepsilon_{\text{app}}, v^r_{\text{app}}\right)\right\|_{W^{1,\infty}}\right) \int_0^t E(v,b)(s)\,\mathrm{d}s \\
&\quad + \int_0^t S + C \int_0^t \left(\|\mathcal{R}_v\|^2 + \varepsilon^{-1}\|\mathcal{R}_b\|^2\right)\mathrm{d}s
\end{aligned}$$

for some increasing function $\mathcal{C}$. Note that the term $\|\varepsilon(b^\varepsilon_{\text{app}}, v^\varepsilon_{\text{app}})\|_{W^{1,\infty}}$ and not $\|\varepsilon^{3/2}(b^\varepsilon_{\text{app}}, v^\varepsilon_{\text{app}})\|_{W^{1,\infty}}$ appears in this estimate because of the anisotropic weight in the energy $E(v,b)$. It remains to estimate the term S which contains all the terms involving the singular term v^s:

$$S = \varepsilon^{3/2}\left|\left(v \cdot \nabla v^s, v\right)\right| + \varepsilon^{3/2}\left|\left(b \cdot \nabla v^s, b\right)\right| + \varepsilon^3\left(\left\|v \cdot \nabla v^s\right\|^2 + \left\|v^s \cdot \nabla v\right\|^2\right) + \varepsilon^4\left(\left\|b \cdot \nabla v^s\right\|^2 + \left\|v^s \cdot \nabla b\right\|^2\right).$$

We easily estimate these new terms by using the Gagliardo–Nirenberg–Sobolev inequalities, for example, we write

$$\begin{aligned}
\varepsilon^4 \int_0^t \left\|b \cdot \nabla v^s\right\|^2 &\leqslant C\varepsilon^4 \sup_{[0,t]} \left\|\nabla v^s\right\|^2 \int_0^t \|b\|^2_{L^\infty} \leqslant C\varepsilon^4 \sup_{[0,t]} \left\|\nabla v^s\right\|^2 \int_0^t \|b\|^2_{H^2} \\
&\leqslant C\varepsilon \sup_{[0,t]} \left\|\nabla v^s\right\|^2 \int_0^t \left(E(v,b) + D(v,b)\right),
\end{aligned}$$

$$\varepsilon^4 \int_0^t \|v^s \cdot \nabla b\|^2 \leqslant C\varepsilon^4 \int_0^t \|v^s\|_{L^4}^4 \|\nabla b\|_{L^4}^4 \leqslant C\varepsilon^4 \int_0^t \|v^s\|_{H^1}^2 \|b\|_{H^2}^2$$

$$\leqslant C\varepsilon \sup_{[0,t]} \|v^s\|_{H^1}^2 \int_0^t \big(E(v,b) + D(v,b)\big),$$

$$\varepsilon^{3/2} \big|\big(b \cdot \nabla v^s, b\big)\big| \leqslant C\varepsilon^{3/2} \sup_{[0,t]} \|\nabla v^s\|^2 \int_0^t \|b\|_{L^4}^2 \leqslant C\varepsilon^{3/2} \sup_{[0,t]} \|\nabla v^s\|^2 \int_0^t \|b\|^{1/2} \|b\|_{H^1}^{3/2}$$

$$\leqslant C\varepsilon^{3/4} \sup_{[0,t]} \|\nabla v^s\|^2 \int_0^t E(v,b) + D(v,b).$$

All the other terms can be handled in a similar way, and hence we get

$$E(v,b,t) + \int_0^t D(v,b) \leqslant \mathcal{C}\Big(|U| + |U'| + \big\|\varepsilon\big(b_{\mathrm{app}}^\varepsilon, v_{\mathrm{app}}^r\big)\big\|_{W^{1,\infty}} + \varepsilon^{3/4} \sup_{[0,t]}\big(\|v^s\|_{H^1}^2\big)\Big) \int_0^t E(v,b)(s)\,\mathrm{d}s$$

$$+ \varepsilon^{3/4} \sup_{[0,t]}\big(\|v^s\|_{H^1}^2\big) \int_0^t D(v,b)(s)\,\mathrm{d}s + C \int_0^t \big(\|\mathcal{R}_v\|^2 + \varepsilon^{-1}\|\mathcal{R}_b\|^2\big)\,\mathrm{d}s.$$

Let $\mathcal{C}_0 > \mathcal{C}\big(|U| + |U'| + 1\big)$. We fix N such that $N\sigma > \mathcal{C}_0$.

We have

$$\big\|\varepsilon\big(b_{\mathrm{app}}^\varepsilon, v_{\mathrm{app}}^\varepsilon\big)\big\|_{W^{1,\infty}} \leqslant \sum_{i=0}^n \lambda_i \mathcal{E}(t)^{i+1} \leqslant \frac{1}{2}, \qquad \varepsilon^{3/4} \sup_{[0,t]}\big(\|v^s\|_{H^1}^2\big) \leqslant \frac{1}{2}$$

for $t \leqslant T^\varepsilon - \tau_0$, τ_0 large enough independent of ε. Thanks to the last energy estimate, we deduce by using Proposition 2 and (95) that for all $t \leqslant T^\varepsilon - \tau_0$,

$$E(v,b)(t) \leqslant \mathcal{C}_0 \int_0^t E(v,b)(s)\,\mathrm{d}s + C\varepsilon\varepsilon^{mN} \int_0^t \exp(N\sigma s)\,\mathrm{d}s.$$

The Gronwall's lemma implies

$$E(v,b)(t) \leqslant C\varepsilon\varepsilon^{mN} \exp(N\sigma t).$$

We get, for all $t \leqslant T^\varepsilon - \tau_0$,

$$\begin{aligned}
\big\|b^\varepsilon(t)\big\|_{L^2(|s|\leqslant\sqrt{\varepsilon})}^2 &\geqslant \big\|b_{\mathrm{app}}^\varepsilon(t)\big\|_{L^2(|s|\leqslant\sqrt{\varepsilon})}^2 - \big\|b(t)\big\|_{L^2}^2 \\
&\geqslant \left\|b^0\left(t, \frac{\cdot}{\sqrt{\varepsilon}}\right)\right\|_{L^2(|s|\leqslant\sqrt{\varepsilon})}^2 - \left\|b_{\mathrm{app}}^\varepsilon(t) - b^0\left(t, \frac{\cdot}{\sqrt{\varepsilon}}\right)\right\|_{L^2}^2 - E(v,b) \\
&\geqslant C_0\sqrt{\varepsilon}\mathcal{E}(t)^2 - C_1\sqrt{\varepsilon}\mathcal{E}(t)^4 \\
&\geqslant \eta\sqrt{\varepsilon}
\end{aligned}$$

for $\eta > 0$, and $t \leqslant T^\varepsilon - \tau_1$, τ_1 large enough independent of ε. This ends the proof of the L^2 instability. Since

$$\|b^\varepsilon\|_{L^2(|s|\leqslant\sqrt{\varepsilon})}^2 \leqslant \sqrt{\varepsilon}\|b^\varepsilon\|_{L^\infty}^2,$$

we also get the instability in the L^∞ norm.

Acknowledgement

We thank warmly the anonymous referee for his comments that led to substantial improvements of the original manuscript.

References

[1] P. Chossat, G. Iooss, The Couette–Taylor Problem, Applied Mathematical Sciences, vol. 102, Springer-Verlag, New York, 1994.
[2] B. Desjardins, E. Grenier, Linear instability implies nonlinear instability for various types of viscous boundary layers, Ann. Inst. H. Poincaré Anal. Non Linéaire 20 (1) (2003) 87–106.
[3] D. Fearn, Hydromagnetic flows in planetary cores, Rep. Prog. Phys. 61 (1998) 175–235.
[4] A. Gailitis, O. Lielausis, E. Platacis, G. Gerbeth, F. Stefani, Riga dynamo experiment and its theoretical background, Phys. Plasmas 11 (5) (2004) 2838–2843.
[5] D. Gérard-Varet, Oscillating solutions of incompressible MHD and dynamo effect, SIAM J. Math. Anal. 37 (3) (2005) 815–840.
[6] A.D. Gilbert, Fast dynamo action in the Ponomarenko dynamo, Geophys. Astrophys. Fluid Dynamics 44 (1988) 241–258.
[7] A.D. Gilbert, Dynamo theory, in: Handbook of Mathematical Fluid Dynamics, vol. II, North-Holland, Amsterdam, 2003, pp. 355–441.
[8] J. Goodman, Z.P. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, Arch. Rational Mech. Anal. 121 (3) (1992) 235–265.
[9] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, Comm. Pure Appl. Math. 53 (9) (2000) 1067–1091.
[10] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.
[11] Y. Ponomarenko, On the theory of hydromagnetic dynamos, Zh. Prikl. Mekh. Tekh. Fiz. (USSR) 44 (1973) 47–51.
[12] P. Roberts, An Introduction to Magnetohydrodynamics, Longmans, Green, 1967.