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Traveling wave solutions of the heat flow of director fields

Fronts progressifs pour le flot des champs de vecteurs unitaires

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Abstract

We consider the simplest possible heat equation for director fields, $u_t = \Delta u + |\nabla u|^2 u$ ($|u| = 1$), and construct axially symmetric traveling wave solutions defined in an infinitely long cylinder. The traveling waves have a point singularity of topological degree 0 or 1.

Résumé

Dans ce papier nous considérons l'équation de la chaleur la plus simple pour champs de vecteurs unitaires, $u_t = \Delta u + |\nabla u|^2 u$ ($|u| = 1$), avec domaine donné par un cylindre infiniment long. Pour cette équation nous édifions des solutions axialement symétriques en forme de front progressif. Ces fronts progressifs ont un point de singularité avec degré topologique 0 ou 1.

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1. Introduction

We consider the heat flow of harmonic maps from an infinitely long vertical cylinder, $\Omega = \{(x_1, x_2, x_3)\colon x_1^2 + x_2^2 < 1\} \subset \mathbb{R}^3$, to the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$:

$$u_t - \Delta u = |\nabla u|^2 u \quad \text{in } \Omega \times \mathbb{R}. \tag{1}$$

It can be viewed as the simplest possible equation of a class of evolution equations for director fields which naturally arise in applications (see [20] and [5]). In cylindrical coordinates (r, θ, x_3), axially symmetric solutions can be represented in the form

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$$u(r,\theta,x_3,t) = (\cos\theta\sin h, \sin\theta\sin h, \cos h), \tag{2}$$

where $h = h(r,x_3,t)$, the so-called *angle function*, satisfies the scalar equation (see [9,10])

$$h_t = h_{rr} + h_{x_3x_3} + \frac{h_r}{r} - \frac{\sin(2h)}{2r^2} \quad \text{for } 0 < r < 1,\ x_3 \in \mathbb{R},\ t \in \mathbb{R}. \tag{3}$$

Recently [1,17] it has been shown that nonuniqueness of axially symmetric solutions of harmonic map flow, which was observed for the first time by Coron [8], is directly related to the occurrence of point singularities in the solutions: in the special case of the unit ball in $\mathbb{R}^3$ as spatial domain and the function $x/|x|$ as initial and boundary condition, the evolution of the point singularity on the vertical axis of the ball can be prescribed, i.e. given any function $\zeta_0(t) \in (-1,1)$ there exists an axially symmetric solution of the heat flow (with the same initial and boundary condition!) which is regular in its domain except of the set $\{(x_1,x_2,x_3,t) = (0,0,\zeta_0(t),t),\ t \geqslant 0\}$. The proof of this nonuniqueness phenomenon is based on the construction of quite complicated comparison functions for Eq. (3).

In a forthcoming paper [15], we shall see that for more general axially symmetric initial functions nonuniqueness results can still be obtained, but it is much harder to find appropriate comparison functions. Traveling wave solutions in infinitely long cylinders with a point singularity on the x_3-axis turn out to be very useful in this context.

This motivated us to look for traveling wave solutions of Eq. (3) with positive wave speed $c > 0$: $h(r,x_3,t) = \psi(r, x_3 - ct)$. With abuse of notation we shall denote the function $\psi(r,z)$ by $h(r,z)$, $z = x_3 - ct$. Then h satisfies the singular elliptic equation

$$h_{rr} + h_{zz} + \frac{h_r}{r} + ch_z - \frac{\sin(2h)}{2r^2} = 0 \quad \text{for } 0 < r < 1,\ z \in \mathbb{R}, \tag{4}$$

to which we add a boundary condition at $r = 1$:

$$h(1,z) = g(z). \tag{5}$$

Here g is a given function which satisfies, for some $z_0 < z_1$ and $0 < B < A$,

$$g \in C^4(\mathbb{R}),\ g' \leqslant 0 \quad \text{in } \mathbb{R}, \qquad g = A \quad \text{in } (-\infty, z_0), \qquad g = B \quad \text{in } (z_1, \infty). \tag{6}$$

At first glance condition (5) may seem artificial. In a way it forces solutions to move in the x_3-direction with prescribed speed $c > 0$, and one could argue that this trivially imposes the existence of traveling wave solutions with the same velocity. On the other hand this construction supplies exactly the type of comparison functions needed in [18]. In addition the techniques developed in the present paper can be used to construct solutions of (4), (5) in the more "natural" case that $g(z)$ is a constant independent of z [3], a construction which leads to new and unexpected nonuniqueness phenomena for traveling waves which provide significant new insight in the nonuniqueness for the general heat flow mentioned before [4].

To ensure that the traveling waves have a point singularity, we shall always choose $A > \pi$ and $0 < B < \pi/2$. The axial symmetry implies that point singularities necessarily belong to the z-axis (since the equation for h is regular for $r > 0$), and it is easy to show that $h(0,z)$ is necessarily a multiple of π for a.e. $z \in \mathbb{R}$. Since we shall construct traveling waves which are nonincreasing with respect to z, this means that point singularities occur at points $(r,z) = (0,\bar{z})$ at which h is discontinuous. The following two theorems, the main results of the paper, show that it is possible to have both singular points at which h jumps from π to 0 (Theorem 1) and ones at which h jumps from 2π to 0 (Theorem 2). In the first case the topological degree of the point singularity is 1, in the latter case it is 0.

Theorem 1. *Let $c > 0$ and let $g(z)$ be a given function satisfying* (6) *with*

$$\pi < A < 3\pi/2 \quad \textit{and} \quad 0 < B < \pi/2. \tag{7}$$

Then there exists a function $h_1 : [0,1] \times \mathbb{R} \to \mathbb{R}$ which is smooth in $(0,1] \times \mathbb{R}$ and satisfies Eqs. (4) *and* (5). *In addition the following properties are satisfied*:

(i) *there exists $\bar{z}_1$ such that h_1 is continuous in $\{(0,z)\colon z \neq \bar{z}_1\}$, $h_1(0,z) = 0$ if $z > \bar{z}_1$ and $h_1(0,z) = \pi$ if $z < \bar{z}_1$*;
(ii) *$h_1(r,z)$ is nonincreasing with respect to z*;
(iii) *$h_1(r,z) \to 2\arctan(br)$ uniformly with respect to $r \in [0,1]$ as $z \to \infty$, where b is defined by $2\arctan b = B$*;

(iv) $h_1(r,z) \to \pi + 2\arctan(a_1 r)$ *uniformly with respect to* $r \in [0,1]$ *as* $z \to -\infty$, *where* a_1 *is defined by* $\pi + 2\arctan a_1 = A$;
(v) h_1 *is real analytic in* $[0,1) \times \mathbb{R} \setminus \{(0,\bar{z}_1)\}$.

Theorem 2. *Let* $c > 0$ *and let* $g(z)$ *be a given function satisfying* (6) *with*

$$\pi < A < 3\pi \quad \text{and} \quad 0 < B < \pi/2. \tag{8}$$

Then there exists a function $h_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ *which satisfies Theorem* 1 *with properties* (i), (iv) *and* (v) *replaced by*:

(i) *there exists* $\bar{z}_2$ *such that* h_2 *is continuous in* $\{(0,z)\colon z \neq \bar{z}_2\}$, $h_2(0,z) = 0$ *if* $z > \bar{z}_2$ *and* $h_2(0,z) = 2\pi$ *if* $z < \bar{z}_2$;
(iv) $h_2(r,z) \to 2\pi + 2\arctan(a_2 r)$ *uniformly with respect to* $r \in [0,1]$ *as* $z \to -\infty$, *where* a_2 *is defined by* $2\pi + 2\arctan a_2 = A$;
(v) h_2 *is real analytic in* $[0,1) \times \mathbb{R} \setminus \{(0,\bar{z}_2)\}$.

Our approach will be variational. In the case of Theorem 2 the minimization problem involves a variant of the relaxed energy introduced by Bethuel, Brezis and Coron in [6] and used by Hardt, Lin and Poon [12] to construct axially symmetric harmonic maps with zero-degree singularities. The proof of the monotonicity of the solutions with respect to z relies on a rearrangement technique.

The paper is organized as follows. In Section 2 we introduce the two minimization problems. In Section 3 we collect some preliminary results. In Section 4 we prove the existence of minimizers and in Section 5 we show their monotonicity with respect to z. In Section 6 we prove that the minimizers have a singularity. In Section 7 we discuss the behavior of the singularities as $c \to \infty$.

2. Variational formulation

Let $c > 0$. Eq. (4) is the Euler–Lagrange equation of the functional

$$\Phi_c(f) = \int_{\mathbb{R}} \mathrm{d}z \int_0^1 \mathrm{d}r \left\{ \frac{r}{2} \mathrm{e}^{cz} \left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r) \right) \right\}. \tag{9}$$

The function $G_b(r)$ is chosen in such a way that $\Phi_c(f)$ is convergent as $z \to \infty$ for all functions f belonging to a suitable class which contains the function $2\arctan(br)$, describing the desired behavior of the traveling waves as $z \to \infty$ (see point (iii) of Theorems 1 and 2):

$$G_b(r) = \frac{\sin^2(2\arctan(br))}{r^2} + \left| \frac{\mathrm{d}}{\mathrm{d}r} \big(2\arctan(br) \big) \right|^2. \tag{10}$$

A straightforward calculation shows that

$$\int_0^1 \frac{r}{2} G_b(r)\, \mathrm{d}r = \frac{2b^2}{1+b^2}. \tag{11}$$

On the other hand, it is well known (see also Theorem 23 in Appendix A) that, if $0 < b < 1$,

$$\int_0^1 \frac{r}{2} \left(f_r^2 + \frac{\sin^2 f}{r^2} \right) \mathrm{d}r \geqslant \frac{2b^2}{1+b^2} \quad \text{if } f \in H^1_{\mathrm{loc}}\big((0,1]\big) \text{ and } f(1) = 2\arctan b. \tag{12}$$

We define the class of functions

$$\mathcal{W} = \left\{ v \in W^{1,2}_{\mathrm{loc}}\big(\mathbb{R}; L^2_r(0,1)\big) \cap L^2_{\mathrm{loc}}\big(\mathbb{R}; H^1_r(0,1)\big);\ \frac{\sin v}{r} \in L^2_{\mathrm{loc}}\big(\mathbb{R}; L^2_r(0,1)\big) \right\},$$

where the subscript r (in L^2_r, H^1_r etc.) indicates that the usual L^p or Sobolev spaces are to be considered with the weight function r. If $f \in \mathcal{W}$, then for a.e. $z \in \mathbb{R}$ the function $f(\cdot, z)$ is defined almost everywhere in $(0,1)$, $f(\cdot, z) \in H^1_r(0,1)$, and $(\sin f(\cdot, z))/r \in L^2_r(0,1)$. This implies (see [20]) that, for almost every $z \in \mathbb{R}$, $f(\cdot, z) \in C^0([0,1])$ and

$$f(0,z) = k(z)\pi \quad \text{for some } k(z) \in \mathbb{Z}. \tag{13}$$

If $f \in \mathcal{W}$, the trace of f at $r = 1$ is well-defined. If $f(1,z) \equiv g(z)$ for a.e. $z \in \mathbb{R}$, it follows from (6), (11), (12) and the monotone convergence theorem that

$$\Phi_c(f) = \lim_{\substack{\alpha \to -\infty \\ \beta \to \infty}} \int_\alpha^\beta \mathrm{d}z \int_0^1 \mathrm{d}r \left\{ \frac{r}{2} \mathrm{e}^{cz} \left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r) \right) \right\}$$

is well-defined and attains values in $(-\infty, \infty]$. More precisely, for such functions f we have that

$$\Phi_c(f) \geqslant -\int_{-\infty}^{z_1} \mathrm{d}z \int_0^1 \frac{r}{2} \mathrm{e}^{cz} G_b(r)\,\mathrm{d}r = -\frac{2b^2 \mathrm{e}^{cz_1}}{c(1+b^2)}. \tag{14}$$

We define, for each $c > 0$,

$$\mathcal{W}^c = \bigl\{ f \in \mathcal{W};\ f(1,z) \equiv g(z),\ \Phi_c(f) < \infty \bigr\}$$

(observe that $\mathcal{W}^c \neq \emptyset$; it contains the function $2\arctan(br) + (g(z) - 2\arctan b)r$). Since (14) holds in $\mathcal{W}^c$ we can formulate our first minimization problem:

First variational problem. Find $h_1 \in \mathcal{W}^c$ which minimizes Φ_c in $\mathcal{W}^c$.

Its solution will be the traveling wave of Theorem 1.

In order to prove Theorem 2 we need a suitable variant of the concept of relaxed energy, introduced in [6]. Let

$$\mathfrak{C} = \bigl\{ \xi \in C^1\bigl([0,1] \times \mathbb{R}\bigr); \operatorname{supp}(\xi) \subseteq [0,1] \times [-M, M] \text{ for some } M > 0 \bigr\}$$

and

$$\mathfrak{C}^c = \bigl\{ \xi \in \mathfrak{C}; \bigl|\nabla \xi(r,z)\bigr| \leqslant \mathrm{e}^{cz} \text{ in } [0,1] \times \mathbb{R} \bigr\}.$$

We define for every $f \in \mathcal{W}$ and $\xi \in \mathfrak{C}$,

$$L(f,\xi) := \frac{1}{2} \int_{\mathbb{R}} \mathrm{d}z \int_0^1 \sin f (f_z \xi_r - f_r \xi_z)\,\mathrm{d}r - \frac{1}{2} \int_{\mathbb{R}} \cos\bigl(f(1,z)\bigr) \xi_z(1,z)\,\mathrm{d}z. \tag{15}$$

We observe that $L(f,\xi)$ is well-defined and $L(f,-\xi) = -L(f,\xi)$. Hence

$$L_c(f) := \sup_{\xi \in \mathfrak{C}^c} L(f,\xi) \in [0,\infty] \quad \text{for } f \in \mathcal{W}.$$

It turns out that $L_c < \infty$ in $\mathcal{W}^c$:

Theorem 3. *Let $f \in \mathcal{W}^c$ and let $P_f = \{z \in \mathbb{R}; \cos(f(0,z)) = -1\}$. Then*

$$L_c(f) = \int_{P_f} \mathrm{e}^{cz}\,\mathrm{d}z < \infty.$$

We observe that, by (13), P_f is well-defined and Lebesgue-measurable. We shall prove Theorem 3 in Section 3.

Theorem 2 corresponds to the following minimization problem:

Second variational problem. Find $h_2 \in \mathcal{W}^c$ which minimizes $\Phi_c + 2L_c$ in $\mathcal{W}^c$.

3. Preliminaries, proof of Theorem 2.1

We introduce the following coordinate transformation:

$$x = \mathrm{e}^{cz} > 0 \leftrightarrow z = c^{-1} \log x. \tag{16}$$

It transforms Eq. (4) into

$$(rh_r)_r + c^2 r\left(x^2 h_x\right)_x - \frac{\sin(2h)}{2r} = 0 \quad \text{in } (0,1) \times \mathbb{R}^+,$$

which is the Euler–Lagrange equation of the functional

$$\Psi_c(f) = \frac{1}{2c} \int_0^\infty \mathrm{d}x \int_0^1 r\left(c^2 x^2 f_x^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r)\right) \mathrm{d}r.$$

Transformation (16) induces naturally a bijective map $T : \mathcal{W} \to T(\mathcal{W})$, $f(r,z) \mapsto f(r, c^{-1}\log x)$, and

$$T(\mathcal{W}) = \left\{ f \in W^{1,2}_{\mathrm{loc}}\left(\mathbb{R}^+; L^2_r(0,1)\right) \cap L^2_{\mathrm{loc}}\left(\mathbb{R}^+; H^1_r(0,1)\right);\ \frac{\sin v}{r} \in L^2_{\mathrm{loc}}\left(\mathbb{R}^+; L^2_r(0,1)\right) \right\}.$$

In particular

$$T\left(\mathcal{W}^c\right) = \left\{ f \in T(\mathcal{W});\ f(1,x) \equiv g\left(c^{-1}\log(x)\right), \Psi_c(f) < \infty \right\},$$

$$\Psi_c(f) = \lim_{\substack{\alpha \to 0^+ \\ \beta \to \infty}} \frac{1}{2c} \int_\alpha^\beta \mathrm{d}x \int_0^1 r\left(c^2 x^2 f_x^2 + f_r^2 + \frac{\sin^2 f}{r^2} - G_b(r)\right) \mathrm{d}r. \tag{17}$$

We observe that

$$\Phi_c(f) = \Psi_c\left(T(f)\right) \quad \text{for } f \in \mathcal{W}^c. \tag{18}$$

We set

$$\mathcal{L}(f,\xi) = \frac{1}{2} \int_{\mathbb{R}^+} \mathrm{d}x \int_0^1 \sin(f)(f_x \xi_r - f_r \xi_x)\, \mathrm{d}r - \frac{1}{2} \int_{\mathbb{R}^+} \cos\left(f(1,x)\right) \xi_x(1,x)\, \mathrm{d}x$$

for every $f \in T(\mathcal{W})$ and $\xi \in T(\mathfrak{C})$. It follows easily that

$$T(\mathfrak{C}) = \left\{ \xi \in C^1\left([0,1] \times \mathbb{R}^+\right):\ \mathrm{supp}(\xi) \subseteq [0,1] \times \left[M^{-1}, M\right] \text{ for some } M > 1 \right\},$$

$$T\left(\mathfrak{C}^c\right) = \left\{ \xi \in T(\mathfrak{C}):\ \frac{1}{x^2}\xi_r^2 + c^2 \xi_x^2 \leqslant 1 \text{ in } [0,1] \times \mathbb{R}^+ \right\},$$

and $\mathcal{L}(T(f), T(\xi)) = L(f,\xi)$ for each $\xi \in \mathfrak{C}$ and $f \in \mathcal{W}$. Hence, defining

$$\mathcal{L}_c(f) = \sup_{\xi \in T(\mathfrak{C}^c)} \mathcal{L}(f,\xi) \geqslant 0 \quad \text{for } f \in T(\mathcal{W}),$$

we obtain that

$$L_c(f) = \mathcal{L}_c\left(T(f)\right) \quad \text{for all } f \in \mathcal{W}. \tag{19}$$

In order to prove Theorem 3 we need the following result.

Proposition 4. *For all* $f \in T(\mathcal{W})$

$$\mathcal{L}(f,\xi) = -\frac{1}{2} \int_{\mathbb{R}^+} \cos\left(f(0,x)\right) \xi_x(0,x)\, \mathrm{d}x \quad \textit{for } \xi \in T(\mathfrak{C}),$$

and

$$\mathcal{L}_c(f) = \sup_{\{\lambda\in C_0^1(\mathbb{R}^+);\, |\lambda'|\leqslant 1/c\}} \left(-\frac{1}{2}\int_{\mathbb{R}^+} \cos\bigl(f(0,x)\bigr)\lambda'(x)\,\mathrm{d}x \right).$$

Proof. The first statement implies at once the second one. If ξ is sufficiently smooth, the first statement follows from an integration by parts in (15) (observe that for all $f \in \mathcal{W}$ we have, in addition to (13), that $\cos f(\cdot, z)$ is absolutely continuous in $[0, 1]$ for a.e. $z \in \mathbb{R}$, and $\cos f(r, \cdot)$ is locally absolutely continuous in $\mathbb{R}$ for a.e. $r \in (0, 1)$). A standard approximation argument completes the proof of the first statement. □

Proposition 5. *Let $w \in T(\mathcal{W}^c)$, let $E_w = \{x \in \mathbb{R}^+; \cos(w(0, x)) = -1\}$ and let μ denote the 1-dimensional Lebesgue measure. Then*

$$\mathcal{L}_c(w) = \frac{1}{c}\mu(E_w) < \infty.$$

Proof. First we prove that $\mu(E_w) < \infty$. Arguing by contradiction we suppose that $\mu(E_w) = \infty$. Let z_1 be defined by (6) and set $x_1 = \mathrm{e}^{cz_1}$. Then $\mu(E_w \cap (x_1, \infty)) = \infty$. For all $x \in E_w \cap (x_1, \infty)$ we have that $w(0, x) = k(x)\pi$, with $k(x)$ odd, and $w(1, x) = 2\arctan b$. Hence it follows from (11) and Lemma 20 that for any $x \in E_w \cap (x_1, \infty)$

$$\int_0^1 \frac{r}{2}\left(c^2x^2w_x^2 + w_r^2 + \frac{\sin^2 w}{r^2} - G_b(r)\right)\mathrm{d}r \geqslant \frac{2}{1+b^2} - \frac{2b^2}{1+b^2} = 2\frac{1-b^2}{1+b^2}.$$

On the other hand, by (12), the same integral is nonnegative if $x \geqslant x_1$ and uniformly bounded from below if $0 < x \leqslant x_1$. Since $\mu(E_w \cap (x_1, \infty)) = \infty$ and $1 - b^2 > 0$, this implies that $\Psi_c(w) = \infty$. Hence $w \notin T(\mathcal{W}^c)$ and we have found a contradiction.

Let $\lambda \in C_0^1(\mathbb{R}^+)$ such that $|\lambda'| \leqslant c^{-1}$. Then

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}^+} \cos\bigl(w(0,x)\bigr)\lambda'(x)\,\mathrm{d}x &= -\frac{1}{2}\int_{\mathbb{R}^+\setminus E_w} \lambda'(x)\,\mathrm{d}x + \frac{1}{2}\int_{E_w}\lambda'(x)\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{R}^+}\lambda'(x)\,\mathrm{d}x + \int_{E_w}\lambda'(x)\,\mathrm{d}x \\ &= \int_{E_w}\lambda'(x)\,\mathrm{d}x \leqslant \frac{\mu(E_w)}{c} \end{aligned}$$

and hence, by Proposition 4, $\mathcal{L}_c(w) \leqslant \mu(E_w)/c$.

It remains to prove that $\mathcal{L}_c(w) \geqslant \mu(E_w)/c$. Let $\varepsilon > 0$. Then there exists $x_\varepsilon > 0$ such that $\ell_\varepsilon \equiv \mu(E_w \cap (0, x_\varepsilon)) > \mu(E_w) - \varepsilon$. Let λ_ε be the function

$$\lambda_\varepsilon(x) = \begin{cases} \dfrac{x}{c} & \text{if } x \in (0, x_\varepsilon], \\ \dfrac{2x_\varepsilon - x}{c} & \text{if } x \in (x_\varepsilon, 2x_\varepsilon], \\ 0 & \text{if } x > 2x_\varepsilon. \end{cases}$$

It follows from Proposition 4 and a straightforward approximation argument that

$$\mathcal{L}_c(w) \geqslant -\frac{1}{2}\int_{\mathbb{R}^+} \cos\bigl(w(0,x)\bigr)\lambda_\varepsilon'(x)\,\mathrm{d}x.$$

Hence

$$\mathcal{L}_c(w) \geqslant -\frac{1}{2}\int_{\mathbb{R}^+} \cos\bigl(w(0,x)\bigr)\lambda_\varepsilon'(x)\,\mathrm{d}x = -\frac{1}{2c}\int_0^{x_\varepsilon} \cos\bigl(w(0,x)\bigr)\,\mathrm{d}x$$
$$+\frac{1}{2c}\int_{x_\varepsilon}^{2x_\varepsilon} \cos\bigl(w(0,x)\bigr)\,\mathrm{d}x = -\frac{1}{2c}\bigl(\mu\bigl((0,x_\varepsilon)\setminus E_w\bigr) - \mu\bigl((0,x_\varepsilon)\cap E_w\bigr)\bigr)$$
$$+\frac{1}{2c}\bigl(\mu\bigl((x_\varepsilon,2x_\varepsilon)\setminus E_w\bigr) - \mu\bigl((x_\varepsilon,2x_\varepsilon)\cap E_w\bigr)\bigr) > -\frac{1}{2c}(x_\varepsilon - 2\ell_\varepsilon) + \frac{1}{2c}(x_\varepsilon - 2\varepsilon),$$

since $\mu((x_\varepsilon,2x_\varepsilon)\cap E_w) \leqslant \mu(E_w\setminus(0,x_\varepsilon)) = \mu(E_w\setminus(E_w\cap(0,x_\varepsilon))) = \mu(E_w) - \ell_\varepsilon < \varepsilon$ and $\mu((x_\varepsilon,2x_\varepsilon)\setminus E_w) = x_\varepsilon - \mu((x_\varepsilon,2x_\varepsilon)\cap E_w) > x_\varepsilon - \varepsilon$. Hence $\mathcal{L}_c(w) > (\mu(E_w) - 2\varepsilon)/c$ and since $\varepsilon > 0$ can be chosen arbitrarily small the proof is complete. □

Theorem 3 follows at once from (19), Proposition 5, and the relation

$$\int_{P_f} \mathrm{e}^{cz}\,\mathrm{d}z = \frac{1}{c}\int_{E_{T(f)}} \mathrm{d}x = \frac{1}{c}\mu(E_{T(f)}).$$

We conclude this section with a technical result which we shall use in Section 6.

Proposition 6. *Let* $0 < b < 1$*,* $w \in T(\mathcal{W}^c)$*,* $k \in \mathbb{Z}\setminus\{0\}$ *and* $0 < \sigma < \sigma_b$*, where*

$$\sigma_b = \arccos\left(\frac{3b^2-1}{1+b^2}\right).$$

Then

$$\mu\bigl(\{x>0;\, w(0,x) = k\pi\}\bigr) = \lim_{r\to 0^+}\mu\bigl(\{x>0;\, k\pi - \sigma \leqslant w(r,x) < k\pi + \sigma\}\bigr) < \infty.$$

Proof. Let $n \in \mathbb{N}$ and $0 < r < 1$, and set

$$S_n = \{0 < x < n;\, w(0,x) = k\pi\},$$
$$S_{r,n} = \{0 < x < n;\, k\pi - \sigma \leqslant w(r,x) < k\pi + \sigma\},$$
$$F_{r,n} = \{x > n;\, k\pi - \sigma \leqslant w(r,x) < k\pi + \sigma\}.$$

Since, for a.e. $x > 0$, $w(\cdot,x) \in C^0([0,1])$ and $w(0,x) = j(x)\pi$ for some $j(x) \in \mathbb{Z}$, the characteristic function of the set $\{x > 0;\, k\pi - \sigma \leqslant w(r,x) < k\pi + \sigma\}$ converges a.e. to the characteristic function of $\{x > 0;\, w(0,x) = k\pi\}$ (here we have used that $\sigma < \pi$). Hence, by Lebesgue's theorem $\mu(S_{r,n}) \to \mu(S_n)$ as $r \to 0$ for all $n \in \mathbb{N}$.

It is easy to complete the proof if we show that for all $\varepsilon > 0$ there exists $\nu \in \mathbb{N}$ such that $\mu(F_{r,n}) < \varepsilon$ for all $n \geqslant \nu$ and $0 \leqslant r \leqslant 1$.

Arguing by contradiction we suppose that there exists $\varepsilon > 0$ such that for every $\nu \in \mathbb{N}$ there exist $n = n(\nu) \geqslant \nu$ and $0 \leqslant r_n \leqslant 1$ such that $\mu(F_{r_n,n}) \geqslant \varepsilon$. Choosing $\nu \geqslant x_1 \equiv \mathrm{e}^{cz_1}$, $w(1,x) = 2\arctan b$ for every $x \in F_{r_n,n}$. On the other hand, since $k \neq 0$, $w(r_n,x) \geqslant \pi - \sigma$ or $w(r_n,x) < -\pi + \sigma$ if $x \in F_{r_n,n}$. Hence, by Lemma 20, for all $n = n(\nu)$ and $x \in F_{r_n,n}$

$$\int_{r_n}^{1} \frac{r}{2}\left(w_r^2(r,x) + \frac{\sin^2 w(r,x)}{r^2}\right)\mathrm{d}r \geqslant \bigl|\cos(2\arctan b) + \cos(\sigma)\bigr|.$$

In view of (11) it is natural to require that the right-hand side is larger than $2b^2/(1+b^2)$, which leads at once to the condition $\sigma < \sigma_b$. Hence there exists $C = C(b,\sigma) > 0$ such that for $n = n(\nu)$ and $\nu \geqslant x_1$

$$\int_n^\infty \mathrm{d}x \int_0^1 \frac{r}{2}\left(w_r^2(r,x) + \frac{\sin^2 w(r,x)}{r^2} - G_b(r)\right)\mathrm{d}r \geqslant C\mu(F_{r_n,n}) \geqslant C\varepsilon.$$

On the other hand, since $w \in T(\mathcal{W}^c)$, the latter integral vanishes as $n \to \infty$, and we have found a contradiction. □

4. Existence of minimizers

In this section we prove the following result:

Theorem 7. *Let g satisfy* (6), *with $0 < B < \frac{\pi}{2}$ and $A > B$, and let $b \in (0,1)$ be defined by $2\arctan b = B$. Then the first and the second variational problem have a solution, h_1 and h_2 respectively, which satisfy the following properties*:

- (i) *h_1 and h_2 are real analytic in $(0,1) \times \mathbb{R}$ and continuous up to $r = 1$, and satisfy Eqs.* (4) *and* (5).
- (ii) *If $\pi < A < \frac{3\pi}{2}$, then $2\arctan(br) < h_1(r,z) < \pi + 2\arctan(a_1 r)$ for $(r,z) \in (0,1) \times \mathbb{R}$, where $a_1 \in (0,1)$ is defined by $\pi + 2\arctan a_1 = A$.*
- (iii) *If $\pi < A < 3\pi$, then $2\arctan(br) < h_2(r,z) < 2\pi + 2\arctan(a_2 r)$ for $(r,z) \in (0,1) \times \mathbb{R}$, where $a_2 \in \mathbb{R}$ is defined by $2\pi + 2\arctan a_2 = A$.*
- (iv) *$h_i(r,z) \to 2\arctan(br)$ $(i = 1,2)$, uniformly with respect to $r \in [0,1]$ as $z \to \infty$.*

Proof. We only sketch the proof in case of the second variational problem. Since great parts of it are standard, we omit all details except of the less standard ones. We set

$$\mathcal{I} = \inf\{\Phi_c(h) + 2L_c(h);\, h \in \mathcal{W}^c\}.$$

By (14),

$$\mathcal{I} \geqslant -\frac{2b^2 \mathrm{e}^{cz_1}}{c(1+b^2)}.$$

Let $\{h_n\}$ be a minimizing sequence and let $\sigma > 0$. We set, for all $f \in W_r^{1,2}((0,1) \times (-\sigma,\sigma))$,

$$E_{c,\sigma}(f) = \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \left\{ \frac{r}{2}\mathrm{e}^{cz}\left(f_z^2 + f_r^2 + \frac{\sin^2 f}{r^2} \right) \right\},$$

$$\Phi_{c,\sigma}(f) = E_{c,\sigma}(f) - \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \left(\frac{r}{2}\mathrm{e}^{cz} G_b(r) \right).$$

Then $\Phi_{c,\sigma}(h_n)$ is uniformly bounded with respect to both σ and n. In addition, $\{h_n\}$ is bounded in $W_r^{1,2}((0,1) \times (-\sigma,\sigma))$ for all σ, and, by a standard diagonal procedure, there exist h, belonging to $W_r^{1,2}((0,1) \times (-\sigma,\sigma))$ for all $\sigma > 0$, and a subsequence of $\{h_n\}$, which we shall denote again by $\{h_n\}$, such that

$$h(1,z) = g(z) \quad \text{for a.e. } z \in \mathbb{R},$$
$$h_n \rightharpoonup h \quad \text{in } W_r^{1,2}\big((0,1) \times (-\sigma,\sigma)\big) \quad \text{and} \quad h_n \to h \quad \text{a.e. in } (0,1) \times \mathbb{R},$$

and

$$\frac{\sin h_n}{r} \rightharpoonup \frac{\sin h}{r} \quad \text{in } L^2\big((-\sigma,\sigma); L_r^2(0,1)\big)$$

(indeed, $\frac{\sin h_n}{r}$ is uniformly bounded in $L^2((-\sigma,\sigma); L_r^2(0,1))$ and the weak convergence follows from Lebesgue's Dominated Convergence Theorem applied to the sequence

$$\{f \sin h_n\} = \left\{ \frac{\sin h_n}{\sqrt{r}} \sqrt{r}\, f \right\},$$

with $f \in L^2((-\sigma,\sigma); L_r^2(0,1)))$.

Setting $f_n = h_n - h$, the identity $E_{c,\sigma}(h_n) = E_{c,\sigma}(f_n) + E_{c,\sigma}(h) + R$, with

$$R = \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \left\{ r\mathrm{e}^{cz}\left(f_{nr}h_r + f_{nz}h_z + \frac{\sin f_n \sin h \cos h_n}{r^2} \right) \right\} \mathrm{d}r,$$

implies that

$$E_{c,\sigma}(h_n) = E_{c,\sigma}(h) + E_{c,\sigma}(h_n - h) + \mathrm{o}(1) \quad \text{as } n \to \infty. \tag{20}$$

We fix $\sigma > 0$ and $\xi \in \mathfrak{C}^c$ such that $\operatorname{supp}(\xi) \subseteq [0,1] \times [-\sigma,\sigma]$. We claim that

$$2L(h_n,\xi) - 2L(h,\xi) \geqslant -E_{c,\sigma}(h_n - h) + \mathrm{o}(1) \quad \text{as } n \to \infty. \tag{21}$$

This inequality follows easily from the decomposition $2L(h_n,\xi) - 2L(h,\xi) = I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n}$, where

$$I_{1,n} = \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \bigl\{\sin f_n \cos h (f_{nz}\xi_r - f_{nr}\xi_z)\bigr\},$$

$$I_{2,n} = \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \bigl\{\sin h (\cos f_n - 1)(f_{nz}\xi_r - f_{nr}\xi_z)\bigr\},$$

$$I_{3,n} = \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \bigl\{\sin h (f_{nz}\xi_r - f_{nr}\xi_z)\bigr\},$$

$$I_{4,n} = -\int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \mathrm{d}r \bigl\{(\sin h - \sin h_n)(h_z\xi_r - h_r\xi_z)\bigr\};$$

$$|I_{1,n}| \leqslant \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 r \left\{ \frac{|\sin f_n|}{r} |\nabla f_n| \, |\nabla \xi| \right\} \mathrm{d}r \leqslant \int_{-\sigma}^{\sigma} \mathrm{d}z \int_0^1 \frac{r}{2} \mathrm{e}^{cz} \left\{ \frac{\sin^2 f_n}{r^2} + |\nabla f_n|^2 \right\} \mathrm{d}r = E_{c,\sigma}(f_n),$$

and $I_{i,n} \to 0$ as $n \to \infty$ for $i = 2,3,4$.

Combining (20) and (21) and taking σ and ξ as before, we have that

$$\Phi_{c,\sigma}(h) + 2L(h,\xi) \leqslant \Phi_{c,\sigma}(h_n) + 2L(h_n,\xi) + \mathrm{o}(1) \leqslant \Phi_{c,\sigma}(h_n) + 2L_c(h_n) + \mathrm{o}(1). \tag{22}$$

Arguing as in the proof of (14), we obtain that $\Phi_c(h_n) \geqslant \Phi_{c,\sigma}(h_n) - 2b^2\mathrm{e}^{-c\sigma}/(c(1+b^2))$ for all $\sigma > z_1$. Since $\Phi_c(h_n) + 2L_c(h_n) \to \mathcal{I}$ as $n \to \infty$, it follows from (22) that $\Phi_{c,\sigma}(h) + 2L(h,\xi) \leqslant \mathcal{I} + 2b^2\mathrm{e}^{-c\sigma}/(c(1+b^2))$ for all $\xi \in \mathfrak{C}^c$ and $\sigma > z_1$ such that $\operatorname{supp}(\xi) \subseteq [0,1] \times [-\sigma,\sigma]$. Letting $\sigma \to \infty$ we find that $\Phi_c(h) + 2L(h,\xi) \leqslant \mathcal{I}$ for all $\xi \in \mathfrak{C}^c$, and hence h solves the second variational problem.

It remains to prove points (i)–(iv) of Theorem 7. The proof of (i) is standard. The proofs of (ii) and (iii) are similar and we omit the one of (ii).

Proof of (iii). Let $f_1(r,z) = \max\{2\arctan(br), h_2(r,z)\}$. Then $f_1 \in \mathcal{W}$, $f_1(1,z) = g(z)$ for a.e. $z \in \mathbb{R}$, $|f_{1r}| \leqslant \max(|h_{2r}|, 2b/(1+b^2r^2))$, and $|f_{1z}| \leqslant |h_{2z}|$. We fix $z \in \mathbb{R}$ arbitrarily. Since $h_2(r,z) - 2\arctan(br)$ is real analytic in $(0,1)$, we may write

$$E_-(z) \equiv \bigl\{r \in (0,1);\, h_2(r,z) < 2\arctan(br)\bigr\} = \bigcup_{n \in \mathcal{T} \subseteq \mathbb{Z}} (\alpha_n, \beta_n), \tag{23}$$

where $0 \leqslant \alpha_n < \beta_n \leqslant \alpha_{n+1} < \beta_{n+1} \leqslant 1$ for $n, n+1 \in \mathcal{T}$. We observe that, for all $n \in \mathcal{T}$, $h_2(\beta_n,z) = 2\arctan(b\beta_n)$ and, if $\alpha_n > 0$, $h_2(\alpha_n,z) = 2\arctan(b\alpha_n)$. We set

$$H(r,z;u) = \frac{r}{2}\left(u_r^2(r,z) + u_z^2(r,z) + \frac{\sin^2 u(r,z)}{r^2}\right).$$

Then

$$\int_0^1 \big(H(r,z;f_1)-H(r,z;h_2)\big)\,\mathrm{d}r = \int_{E_-(z)} \big(H\big(r,z;2\arctan(br)\big)-H(r,z;h_2)\big)\,\mathrm{d}r$$
$$= \sum_{n\in\mathcal{T}} \int_{\alpha_n}^{\beta_n} \big(H\big(r,z;2\arctan(br)\big)-H(r,z;h_2)\big)\,\mathrm{d}r.$$

By Corollary 21

$$\int_{\alpha_n}^{\beta_n} \big(H\big(r,z;2\arctan(br)\big)-H(r,z;h_2)\big)\,\mathrm{d}r \leqslant 0 \quad \text{if } \alpha_n > 0. \tag{24}$$

We observe that $\alpha_n = 0$ may happen for at most one value of n, and if so we may assume without loss of generality that $\alpha_0 = 0$. Since $0 < b < 1$, it follows in this case from Theorem 23 that also

$$\int_0^{\beta_0} \big(H\big(r,z;2\arctan(br)\big)-H(r,z;h_2)\big)\,\mathrm{d}r \leqslant 0 \quad \text{if } \alpha_0 = 0. \tag{25}$$

Hence, by (24) and (25),

$$\int_0^1 H(r,z;f_1)\,\mathrm{d}r - \int_0^1 H(r,z;h_2)\,\mathrm{d}r \leqslant 0. \tag{26}$$

Since (26) holds for a.e. $z\in\mathbb{R}$ we conclude that $\Phi_c(f_1)\leqslant\Phi_c(h_2)$. In particular $f_1\in\mathcal{W}^c$. In addition it follows from Theorem 3 that $L_c(f_1)\leqslant L_c(h_2)$. This implies that f_1 is a solution of the second variational problem. By standard regularity theory f_1 is smooth in $(0,1)\times\mathbb{R}$ and, by the strong maximum principle, $f_1(r,z) > 2\arctan(br)$ for all $(r,z)\in(0,1)\times\mathbb{R}$. Hence $f_1 = h_2$ in $(0,1)\times\mathbb{R}$ and we have proved the first inequality in (iii).

Similarly we define $f_2 = \min\{2\pi + 2\arctan(a_2 r), h_2(r,z)\}$. Arguing as before, with $E_-(z)$ replaced by $E_+(z) = \{r\in(0,1);\, h_2(r,z) > 2\pi + 2\arctan(a_2 r)\}$, only the inequality (25) needs to be slightly modified. So we suppose that there exist $z\in\mathbb{R}$ and $\beta_0\in(0,1]$ such that

$$h_2(r,z) > 2\pi + 2\arctan(a_2 r) \quad \text{for } 0<r<\beta_0 \quad \text{and} \quad h_2(\beta_0,z) = 2\pi + 2\arctan(a_2\beta_0). \tag{27}$$

In view of (13) we may assume without loss of generality that $h_2(0,z) = k_0(z)\pi$ for some $k_0(z)\in\mathbb{Z}$. By (27) we have that $k_0(z)\geqslant 2$. If $k_0(z)=2$ or if $k_0(z)\geqslant 4$, we obtain from Lemma 20 that (25) still holds, with $2\arctan(br)$ replaced by $2\pi + 2\arctan(a_2 r)$. In the remaining case, $k_0(z)=3$, (25) is replaced by the inequality

$$\int_0^{\beta_0} \big(H\big(r,z;2\pi + 2\arctan(a_2 r)\big)-H(r,z;h_2)\big)\,\mathrm{d}r \leqslant 2 \quad \text{if } h_2(0,z) = 3\pi,$$

which follows easily from Lemma 20. This means that the inequality $\Phi_c(f_2)\leqslant\Phi_c(h_2)$ is not necessarily valid, but since $\cos(h_2(0,z)) = -1$ if $k_0(z)=3$, it follows easily from Theorem 3 that the inequality $\Phi_c(f_2)+2L_c(f_2)\leqslant\Phi_c(h_2)+2L_c(h_2)$ holds.

Proof of (iv). We only prove the result for h_1, which we shall denote by h. It follows from (11) and (12) that

$$U(z) \equiv \int_0^1 \frac{r}{2}\left(h_r^2 + \frac{\sin^2 h}{r^2} - G_b(r)\right)\bigg|_z \mathrm{d}r \geqslant 0 \quad \text{if } z\geqslant z_1,$$

where z_1 is defined by (6). Since $\int_{z_1}^{\infty} \mathrm{e}^{cz}U(z)\,\mathrm{d}z \leqslant \Phi_c(h) + 2b^2\mathrm{e}^{cz_1}/(c(1+b^2)) < \infty$, there exists a sequence $z_n\to\infty$ such that $U(z_n)\to 0$ as $n\to\infty$. Hence, by Theorem 24, $h(r,z_n)\to 2\arctan(br)$ uniformly with respect to $r\in[0,1]$ as $n\to\infty$.

By standard Schauder estimates, for any $\rho > 0$ the function $V_\rho(z) \equiv \int_\rho^1 h_z^2(r,z)\,\mathrm{d}r$ is Lipschitz continuous in $\mathbb{R}$. On the other hand, the inequality

$$\int_{z_1}^{\infty} \mathrm{d}z \int_0^1 \frac{r\mathrm{e}^{cz}}{2} h_z^2 \,\mathrm{d}r \leqslant \Phi_c(h) + \frac{2b^2\mathrm{e}^{cz_1}}{c(1+b^2)}$$

implies $\int_{z_1}^\infty V_\rho(z)\mathrm{e}^{cz}\,\mathrm{d}z < \infty$ and then $V_\rho(z)\mathrm{e}^{\frac{c}{2}z} \to 0$ as $z \to \infty$.

By Schauder estimates, from here follows the existence of $K, \delta > 0$ such that $\|h_z(\cdot,z)\|_{L^\infty(\rho,1)} \leqslant K\mathrm{e}^{-\delta z}$. Hence $\lim_{z\to\infty} h(r,z)$ exists for all $r \in (0,1]$ and it is equal to $\lim_{n\to\infty} h(r,z_n) = 2\arctan(br)$. Obviously, for any $\rho > 0$,

$$\lim_{z\to\infty} h(r,z) = 2\arctan(br) \quad \text{uniformly with respect to } r \in [\rho,1]. \tag{28}$$

It remains to show that the limit is uniform with respect to $r \in (0,1]$. In the next section we shall show that we may assume that h is decreasing with respect to z (in the proof we shall use (28)). Hence the uniform convergence follows at once from the uniform convergence along the subsequence $\{z_n\}$. □

5. Monotonicity properties of minimizers

In this section we shall show that our two variational problems have solutions which are decreasing with respect to z. We use a one-dimensional monotone rearrangement technique [14] applied to the variable $x = \mathrm{e}^{cz}$.

Throughout this section $f(r,x)$ will denote a C^1-function defined in $(0,1)\times\mathbb{R}^+$ satisfying the following four properties:

(1) for all $r \in (0,1)$, $C \in \mathbb{R}$ and $0 < \alpha < \beta$ the sets $\{x \in [\alpha,\beta];\ f(r,x) = C\}$ and $\{x \in [\alpha,\beta];\ f_x(r,x) = 0\}$ are finite;
(2) $f_r \in L^\infty((\rho,1)\times\mathbb{R}^+)$ and $f_x \in L^\infty((\rho,1)\times(\rho,\infty))$ for all $\rho > 0$;
(3) $f \in L^\infty((0,1)\times\mathbb{R}^+)$ and

$$\ell(r) \equiv \inf_{x>0} f(r,x) < L(r) \equiv \sup_{x>0} f(r,x) \quad \text{for } 0 < r < 1;$$

(4) for any $\rho > 0$, $\lim_{x\to\infty} f(r,x) = \ell(r)$ uniformly with respect to $r \in [\rho,1)$.

In view of Theorem 7, (28) and standard Schauder estimates applied to Eq. (4), the functions

$$f_1 \equiv T(h_1) \quad \text{and} \quad f_2 \equiv T(h_2) \tag{29}$$

satisfy these four properties, with $\ell(r) = 2\arctan(br)$. Here T is the operator induced by the transformation $x = \mathrm{e}^{cz}$ and introduced in Section 3.

Given f, we set

$$\Omega_{d,r} = \{x > 0;\ f(r,x) \geqslant d\} \quad \text{for } d \in \mathbb{R},\ 0 < r < 1,$$
$$f^*(r,x) = \sup\{d \in \mathbb{R} \mid x \leqslant \mu(\Omega_{d,r})\} \quad \text{for } 0 < r < 1,\ x > 0,$$

where μ is the one-dimensional Lebesgue measure. By construction, the rearrangement f^* of f is nonincreasing with respect to x, $\lim_{x\to\infty} f^*(r,x) = \ell(r)$ uniformly with respect to $r \in [\rho,1)$ for $\rho > 0$, and for all $0 < r < 1$, $d_1 < d_2$

$$\mu(\{x > 0;\ d_1 \leqslant f^*(r,x) < d_2\}) = \mu(\{x > 0;\ d_1 \leqslant f(r,x) < d_2\}). \tag{30}$$

Now we are ready to formulate the main result of this section and its first consequence.

Theorem 8. *Let T be defined as in Section 3 and let f_1 and f_2 be defined by (29). Then $T^{-1}(f_1^*)$ and $T^{-1}(f_2^*)$ are solutions of, respectively, the first and second variational problem.*

Corollary 9. *We may assume that the functions h_1 and h_2, defined in Theorem 7, are strictly decreasing with respect to z in $(0,1)\times\mathbb{R}$, and that for all $\rho > 0$*

$$h_1(r,z) \to \pi + 2\arctan(a_1 r) \quad \textit{uniformly with respect to } r \in [\rho,1] \textit{ as } z \to -\infty. \tag{31}$$

The first part of Corollary 9 follows at once from Theorem 8 and the monotonicity of the rearranged functions. The monotonicity of h_1 implies the existence of the limit in (31), which we denote by $v(r)$. It easily follows that v is a solution of the equation $v_{rr} + \frac{1}{r}v_r - \sin(2v)/(2r^2) = 0$ in the interval $(0,1)$, with boundary condition $v(1) = g(-\infty) = \pi + 2\arctan a_1$. In addition it follows from Theorem 7(ii) that $2\arctan(br) \leqslant v(r) \leqslant \pi + 2\arctan(a_1 r)$ in $(0,1)$. The only function v satisfying all these conditions is the function $\pi + 2\arctan(a_1 r)$. It follows at once from Schauder estimates that the convergence is uniform in the sets $[\rho, 1]$ for $\rho > 0$, which completes the proof of Corollary 9.

We observe that, arguing as before, we need the condition that $a_2 \geqslant 0$ to obtain a result similar to (31) for the function h_2:

$$h_2(r,z) \to 2\pi + 2\arctan(a_2 r) \quad \text{uniformly with respect to } r \in [\rho, 1] \text{ as } z \to -\infty. \tag{32}$$

Indeed, if $a_2 < 0$ the same procedure leads to two possible limit functions in (32): $2\pi + 2\arctan(a_2 r)$ and $\pi - 2\arctan(r/a_2)$. Only in Section 6 we shall be able to exclude the latter possibility.

It remains to prove Theorem 8.

We define, for $0 < r < 1$, $d \in \mathbb{R}$, and $0 < \sigma < \tau$, the sets

$$\Omega_{\sigma,d,r} = \{x \geqslant \sigma;\ f(r,x) \geqslant d\}, \quad \Omega^\tau_{d,r} = \{x \in (0,\tau];\ f(r,x) \geqslant d\}, \quad \Omega^\tau_{\sigma,d,r} = \{x \in [\sigma,\tau];\ f(r,x) \geqslant d\},$$

and, in $(0,1) \times \mathbb{R}^+$, the rearranged functions

$$f^{*\sigma}(r,x) = \begin{cases} \sup\{d \in \mathbb{R};\ \mu(\Omega_{\sigma,d,r}) > 0\} & \text{if } x \leqslant \sigma, \\ \sup\{d \in \mathbb{R};\ x - \sigma \leqslant \mu(\Omega_{\sigma,d,r})\} & \text{if } x > \sigma, \end{cases}$$

$$f^*_\tau(r,x) = \begin{cases} \sup\{d \in \mathbb{R};\ x \leqslant \mu(\Omega^\tau_{d,r})\} & \text{if } x \leqslant \tau, \\ \sup\{d \in \mathbb{R};\ \tau \leqslant \mu(\Omega^\tau_{d,r})\} & \text{if } x > \tau, \end{cases}$$

$$f^{*\sigma}_\tau(r,x) = \begin{cases} \sup\{d \in \mathbb{R};\ \mu(\Omega^\tau_{\sigma,d,r}) > 0\} & \text{if } x \leqslant \sigma, \\ \sup\{d \in \mathbb{R};\ x - \sigma \leqslant \mu(\Omega^\tau_{\sigma,d,r})\} & \text{if } x \in (\sigma,\tau], \\ \sup\{d \in \mathbb{R};\ \tau - \sigma \leqslant \mu(\Omega^\tau_{\sigma,d,r})\} & \text{if } x > \tau. \end{cases}$$

It follows at once from the definition of $f^{*\sigma}$ that for all $x \leqslant \sigma$

$$f^{*\sigma}(r,x) = L_{\sigma,r} := \sup_{x \geqslant \sigma} f(r,x).$$

The proofs of the following propositions are based on standard techniques for one-dimensional rearrangements (see [14]). In particular we remind the reader that it is well known that f^* and $f^{*\sigma}$ are continuous and a.e. differentiable in $(0,1) \times \mathbb{R}^+$, and that, for all $0 < \rho < 1$ and $\sigma > 0$,

$$\|(f^*)_r\|_{L^\infty(R_\rho)}, \|(f^{*\sigma})_r\|_{L^\infty(R_\rho)} \leqslant \|f_r\|_{L^\infty(R_\rho)},$$
$$\|(f^*)_x\|_{L^\infty(R_{\rho,\sigma})}, \|(f^{*\sigma})_x\|_{L^\infty(R_\rho)} \leqslant \|f_x\|_{L^\infty(R_{\rho,\sigma})},$$

where $R_\rho = (\rho,1) \times \mathbb{R}^+$ and $R_{\rho,\sigma} = (\rho,1) \times (\sigma,\infty)$.

Proposition 10. *For all* $0 < \sigma < \tau$

$$f^*(r,x) \leqslant f^{*\sigma}(r,x) \leqslant f^*(r,x-\sigma) \quad \textit{if } 0 < r < 1 \textit{ and } x > \sigma,$$
$$f^*_\tau(r,x) \leqslant f^{*\sigma}_\tau(r,x) \leqslant f^*_\tau(r,x-\sigma) \quad \textit{if } 0 < r < 1 \textit{ and } \sigma \leqslant x \leqslant \tau.$$

In particular $f^{*\sigma} \to f^*$ *uniformly on* $[\alpha,1) \times [\alpha,\infty)$ $(\alpha > 0)$ *as* $\sigma \to 0^+$, *and* $f^{*\sigma}_\tau \to f^*_\tau$ *in* $C_{\mathrm{loc}}((0,1) \times (0,\tau))$ *as* $\sigma \to 0^+$.

Proposition 11. *Let* $F : (0,1) \times \mathbb{R} \to \mathbb{R}$ *be continuous and nonnegative, and let* $G : [0,\infty) \to [0,\infty)$ *be convex and nondecreasing* (*and hence continuous*). *Then, for all* $0 < \sigma < \tau$ *and* $0 < \rho < 1$,

$$\int_\sigma^\tau F(\rho, f^{*\sigma}_\tau(\rho,x)) G(|(f^{*\sigma}_\tau)_r(\rho,x)|)\,dx \leqslant \int_\sigma^\tau F(\rho, f(\rho,x)) G(|f_r(\rho,x)|)\,dx. \tag{33}$$

For the proof it is sufficient to apply Lemma 2.6 and Remark 2.22 of [14] to the function $f(\rho, x)$, with $x \in [\sigma, \tau]$.

In Proposition 11 it is important that the function F does not depend on x. This explains why we cannot apply the rearrangement technique directly to the functional Φ_c in the original z variable. On the other hand, the form of the functional Ψ_c, defined in (17), and the following key inequality applied to the function $P(x) = x^2$ make the method work in the x variable:

Proposition 12. *Let $P(x)$ be a nonnegative and nondecreasing function defined for $x > 0$. Then, for all $\sigma > 0$ and $0 < \rho < 1$,*

$$\int_0^\infty P(x)\big(f^{*\sigma}\big)_x^2(\rho, x)\,\mathrm{d}x \leqslant \int_0^\infty P(x) f_x^2(\rho, x)\,\mathrm{d}x. \tag{34}$$

Proof. We fix $\sigma > 0$ and $0 < \rho < 1$ and set

$$\mathcal{A} = \big\{f(\rho, x);\, x \geqslant \sigma \text{ and } f_x(\rho, x) = 0\big\}.$$

In view of the properties of f the set $\mathcal{A}$ is either finite or countable. We give the proof only in the latter case. So we assume that $A = \{a_n\}$, where $a_n > a_{n+1}$ for all $n \geqslant 0$ and $\lim_{n\to\infty} a_n = \ell(\rho)$. Of course, $\sup_{n\in\mathbb{N}} a_n = a_0 \leqslant L_{\sigma,\rho} := \sup_{x\geqslant\sigma} f(r, x)$.

We define the open sets $D_n = \{x > \sigma;\, a_{n+1} < f(\rho, x) < a_n\}$ and $D_n^* = \{x > \sigma;\, a_{n+1} < f^{*\sigma}(\rho, x) < a_n\}$. For each n we can decompose D_n in a finite number, k_n, of disjoint open intervals $\gamma_{n,j}$ $(j = 1, \ldots, k_n)$ on each of which $f_x(\rho, \cdot)$ is either strictly positive or strictly negative. We label these intervals according to their distance to the origin by taking $\gamma_{n,1}$ as the farthest one. Then $\operatorname{sgn}(f_x(\rho, \cdot)) = (-1)^j$ on $\gamma_{n,j}$ for all $j = 1, \ldots, k_n$, and there exists for all $j = 1, \ldots, k_n$ and $\lambda \in (a_{n+1}, a_n)$ a unique $x_j = x_j(\rho, \lambda) \in \gamma_{n,j}$ such that $f(\rho, x_j(\rho, \lambda)) = \lambda$. By the implicit function theorem, x_j can be thought as a smooth function defined in an open set containing $\{\rho\} \times (a_{n+1}, a_n)$. Similarly there exists for all $\lambda \in (\ell(\rho), L_{\sigma,\rho}]$ a unique $x^*(\rho, \lambda) \geqslant \sigma$ such that $f^{*\sigma}(\rho, x^*(\rho, \lambda)) = \lambda$. By construction, $x^*(\rho, \lambda) = \mu(\Omega_{\sigma,\lambda,\rho}) + \sigma$, x^* is strictly decreasing with respect to λ, and $x^*(\rho, \lambda) \in D_n^*$ if $a_{n+1} < \lambda < a_n$. It is easy to check that

$$x^*(\rho, \lambda) = p(k_n)\sigma + \sum_{j=1}^{k_n} (-1)^{j+1} x_j(\rho, \lambda), \tag{35}$$

where $p(k_n) = 0$ if k_n is odd and $p(k_n) = 1$ if k_n even. A simple computation yields that for every n and for almost all $\lambda \in (a_{n+1}, a_n)$

$$\big|\big(f^{*\sigma}\big)_x\big(\rho, x^*(\rho, \lambda)\big)\big| = \left(\sum_{j=1}^{k_n} \big|(x_j)_\lambda(\rho, \lambda)\big|\right)^{-1},$$

$$\big|f_x\big(\rho, x_j(\rho, \lambda)\big)\big| = \big|(x_j)_\lambda(\rho, \lambda)\big|^{-1}, \quad j = 1, \ldots, k_n.$$

These equalities imply that for every n

$$\int_{D_n} P(x) f_x^2(\rho, x)\,\mathrm{d}x = \sum_{j=1}^{k_n} \int_{\gamma_{n,j}} P(x) f_x^2(\rho, x)\,\mathrm{d}x = \int_{a_{n+1}}^{a_n} \left(\sum_{j=1}^{k_n} \frac{P(x_j(\rho, \lambda))}{|(x_j)_\lambda(\rho, \lambda)|}\right) \mathrm{d}\lambda \tag{36}$$

and

$$\int_{D_n^*} P(x)\big(f^{*\sigma}\big)_x^2(\rho, x)\,\mathrm{d}x = \int_{a_{n+1}}^{a_n} \frac{P(x^*(\rho, \lambda))}{\sum_{j=1}^{k_n} |(x_j)_\lambda(\rho, \lambda)|}\,\mathrm{d}\lambda. \tag{37}$$

On the other hand we know that $x^*(\rho, \lambda) = \mu(\Omega_{\sigma,\lambda,\rho}) + \sigma \leqslant x_1(\rho, \lambda)$, since $\Omega_{\sigma,\lambda,\rho} \subseteq [\sigma, x_1(\rho, \lambda)]$ by the definition of $x_1(\rho, \lambda)$. Hence it follows from (36) and (37) that

$$\int_{D_n^*} P(x)\big(f^{*\sigma}\big)_x^2(\rho, x)\,\mathrm{d}x \leqslant \int_{D_n} P(x) f_x^2(\rho, x)\,\mathrm{d}x. \tag{38}$$

We remind that $a_0 = \max_{n\in\mathbb{N}} a_n$ and $L_{\sigma,\rho} = \sup_{x\geqslant\sigma} f(r,x)$. If $a_0 = L_{\sigma,\rho}$, then we have that

$$(\sigma,\infty)\Big\backslash\bigcup_{n\in\mathbb{N}} D_n \quad\text{and}\quad (\sigma,\infty)\Big\backslash\bigcup_{n\in\mathbb{N}} D_n^* \tag{39}$$

are sets of zero Lebesgue measure, and the inequality (34) follows at once from (38) (we have used that $f^{*\sigma}(\rho,\cdot)$ is constant for $x \in (0,\sigma]$).

It remains to consider the case in which $a_0 < L_{\sigma,\rho}$. Then there exists $\bar{x} > \sigma$ such that $\{x \geqslant \sigma;\, a_0 < f(\rho,x) < L_{\sigma,\rho})\} = (\sigma,\bar{x})$ and $f_x(\rho,\cdot) < 0$ in $(\sigma,\bar{x})$. Hence $f^{*\sigma}(\rho,x) = f(\rho,x)$ for all $\sigma \leqslant x \leqslant \bar{x}$. Arguing now as in (39) with (σ,∞) replaced by $(\bar{x},\infty)$, we obtain (34). □

Proposition 13. *Let f_1 and f_2 be given by* (29). *Then, for $i = 1, 2$,*

$$\int_{\mathbb{R}^+} \mathrm{d}x \int_0^1 r x^2 \big(f_i^*\big)_x^2 \,\mathrm{d}r \leqslant \int_{\mathbb{R}^+} \mathrm{d}x \int_0^1 r x^2 (f_i)_x^2 \,\mathrm{d}r < \infty.$$

Proof. Since h_i is a minimizer, it follows from (18) that the latter integral is finite. We omit the subscript i. It is sufficient to prove that for any $\rho \in (0,1)$

$$\int_{\mathbb{R}^+} x^2 (f^*)_x^2(\rho,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^+} x^2 f_x^2(\rho,x)\,\mathrm{d}x. \tag{40}$$

Without loss of generality we may assume that the right-hand side is finite, i.e. $xf_x \in L^2(\mathbb{R}^+)$. Let $\sigma_n \to 0^+$. By (34) the sequence $\{x(f^{*\sigma_n})_x\}$ is bounded in $L^2(\mathbb{R}^+)$ and, up to a subsequence, there exists $v \in L^2(\mathbb{R}^+)$ such that $x(f^{*\sigma_n})_x \to v$ weakly in $L^2(\mathbb{R}^+)$. It follows easily from Proposition 10 and the regularity properties of $f^{*\sigma}$ and f^* that $v(\rho,x) = x(f^*)_x(\rho,x)$ for a.e. $x \in \mathbb{R}^+$. Hence (40) follows from (34). □

Proposition 14. *Let f_1 and f_2 be given by* (29). *Then, for $i = 1, 2$ and for every $M \geqslant x_1 \equiv \mathrm{e}^{cz_1}$,*

$$\int_0^M \mathrm{d}x \int_0^1 r\left(\big(f_i^*\big)_r^2 + \frac{\sin^2 f_i^*}{r^2} - G_b(r)\right)\mathrm{d}r \leqslant \int_0^\infty \mathrm{d}x \int_0^1 r\left((f_i)_r^2 + \frac{\sin^2 f_i}{r^2} - G_b(r)\right)\mathrm{d}r < \infty.$$

Proof. Since h_i is a minimizer, it follows from (18) that the latter integral is finite. We omit the subscript i. For any $\tau > 0$ we set

$$q_\tau(x) = \int_0^1 r\left(\big(f_\tau^*\big)_r^2(r,x) + \frac{\sin^2 f_\tau^*(r,x)}{r^2} - G_b(r)\right)\mathrm{d}r \quad \text{for } x > 0.$$

Observe that q_τ is a measurable function with values in $\mathbb{R} \cup \{\infty\}$ and that, by Theorem 23, $q_\tau(x) \geqslant 0$ for a.e. $x > x_1$ if $\tau \geqslant x_1$. Similarly, the function

$$q(x) = \int_0^1 r\left((f^*)_r^2(r,x) + \frac{\sin^2 f^*(r,x)}{r^2} - G_b(r)\right)\mathrm{d}r \quad \text{for } x > 0$$

is nonnegative a.e. in (x_1,∞). Observing that for any $\rho > 0$ and $M > 0$ there exists $\tau(\rho,M)$ such that $f_\tau^* = f^*$ in $(\rho,1)\times(0,M)$ if $\tau > \tau(\rho,M)$, it follows that $f_\tau^* \to f^*$ and $(f_\tau^*)_r \to (f^*)_r$ locally uniformly on $(0,1)\times\mathbb{R}^+$, and hence, by Fatou's lemma, $q(x) \leqslant \liminf_{\tau\to\infty} q_\tau(x)$ for all $x > 0$. In particular

$$\int_0^{x_1} q(x)\,\mathrm{d}x \leqslant \liminf_{\tau\to\infty} \int_0^{x_1} q_\tau(x)\,\mathrm{d}x.$$

Since $q, q_\tau \geqslant 0$ a.e. in $[x_1, \infty)$ if $\tau \geqslant x_1$, it follows again from Fatou's lemma that for all $M \geqslant x_1$

$$\int_{x_1}^{M} q(x)\,\mathrm{d}x \leqslant \int_{x_1}^{\infty} q(x)\,\mathrm{d}x \leqslant \liminf_{\tau\to\infty} \int_{x_1}^{\tau} q_\tau(x)\,\mathrm{d}x.$$

The proof is complete if we show that, for every $\tau > 0$,

$$\int_0^\tau \mathrm{d}x \int_0^1 \frac{\sin^2 f_\tau^*}{r}\,\mathrm{d}r \leqslant \int_0^\tau \mathrm{d}x \int_0^1 \frac{\sin^2 f}{r}\,\mathrm{d}r, \tag{41}$$

$$\int_0^\tau \mathrm{d}x \int_0^1 r\big(f_\tau^*\big)_r^2\,\mathrm{d}r \leqslant \int_0^\tau \mathrm{d}x \int_0^1 r f_r^2\,\mathrm{d}r. \tag{42}$$

Inequality (41) follows at once from (33), with $G \equiv 1$ and $F = \sin^2(f) r^{-1}$, Proposition 10 and Fatou's lemma. Applying (33) with $F = r$ and $G(v) = v^2$ we find that for all $0 < \sigma < \tau$

$$\int_0^1\!\!\int_\sigma^\tau r\big(f_{\sigma,\tau}^*\big)_r^2\,\mathrm{d}x\,\mathrm{d}r \leqslant \int_0^1\!\!\int_\sigma^\tau r f_r^2\,\mathrm{d}x\,\mathrm{d}r. \tag{43}$$

Letting $\sigma \to 0^+$ and arguing as in the previous proof we easily obtain (42). □

Proof of Theorem 8. It follows at once from Propositions 13 and 14 that $\Psi_c(f_i^*) \leqslant \Psi_c(f_i)$, and hence, by (18), $\Phi_c(T^{-1}(f_i^*)) \leqslant \Phi_c(h_i)$ for $i = 1, 2$.

In view of (19) it remains to prove that $\mathcal{L}_c(f_2^*) = \mathcal{L}_c(f_2)$. By Theorem 7(iii) and Propositions 5 and 6, this is equivalent to proving that, for $\sigma > 0$ small enough,

$$\lim_{r\to 0^+} \mu\big(\big\{x > 0; \pi - \sigma \leqslant f_2^*(r,x) < \pi + \sigma\big\}\big) = \lim_{r\to 0^+} \mu\big(\big\{x > 0; \pi - \sigma \leqslant f_2(r,x) < \pi + \sigma\big\}\big).$$

The latter equality follows at once from (30). □

6. Existence of a point singularity

By Theorems 7 and 8, both variational problems have a minimizer which is strictly decreasing with respect to z in $(0,1) \times \mathbb{R}$. In this section we complete the proofs of Theorems 1 and 2. In particular we shall prove that both minimizers have exactly one singular point at the axis $r = 0$ and we shall determine the behavior of the minimizers as $z \to -\infty$.

Theorem 15. *Let h_1 and h_2 be a minimizer of, respectively, the first and second variational problem which is strictly decreasing with respect z for all $0 < r < 1$.*

(i) *There exists $\bar z_1 \in \mathbb{R}$ such that $h_1(0,z) = \pi$ if $z < \bar z_1$ and $h_1(0,z) = 0$ if $z > \bar z_1$.*
(ii) *$h_1(r,z) \to \pi + 2\arctan(a_1 r)$ uniformly with respect to $r \in [0,1]$ as $z \to -\infty$, where a_1 is defined by $\pi + 2\arctan a_1 = A$.*
(iii) *There exists $\bar z_2 \in \mathbb{R}$ such that $h_2(0,z) = 2\pi$ if $z < \bar z_2$ and $h_2(0,z) = 0$ if $z > \bar z_2$.*
(iv) *$h_2(r,z) \to 2\pi + 2\arctan(a_2 r)$ uniformly with respect to $r \in [0,1]$ as $z \to -\infty$, where a_2 is defined by $2\pi + 2\arctan a_2 = A$.*
(v) *h_i is continuous in $[0,1] \times \mathbb{R} \setminus \{(0, \bar z_i)\}$ and real analytic in $[0,1) \times \mathbb{R} \setminus \{(0, \bar z_i)\}$ $(i = 1, 2)$.*

The proof of (i) is based on the following lemma. We omit its proof, which is based on straightforward computations and estimates.

Lemma 16. *Let $p < q$ and $\alpha \in C^1((p,q])$ be such that*

$$\alpha > 0 \quad \text{in } (p,q], \qquad \frac{(\alpha')^2}{\alpha^3} \in L^1(p,q), \qquad \alpha(z) \to \infty \quad \text{and} \quad \frac{\alpha'(z)}{\alpha^2(z)} \to 0 \quad \text{as } z \to p^+.$$

Then the function $v \in C^1((0,1] \times (p,q))$, defined by

$$v(r,z) = 2\arctan\left(\frac{\alpha(z)r^2}{r+1}\right) \quad \text{for } (r,z) \in (0,1] \times (p,q],$$

satisfies

(i) $\int_0^1 r v_r^2(r,z)\,\mathrm{d}r \leqslant 12$, $\int_0^1 \frac{\sin^2 v(r,z)}{r}\,\mathrm{d}r \leqslant 6$, $\int_0^1 r v_z^2(r,z)\,\mathrm{d}r \leqslant 8(\alpha'(z))^2\alpha^{-3}(z)$ *for* $p < z \leqslant q$;
(ii) $v_z \in L^2((p,q); L_r^2(0,1))$;
(iii) *for all* $0 < \rho < 1$, $v(r,z) \to \pi$ *and* $v_r(r,z), v_z(r,z) \to 0$ *uniformly in* $[\rho,1]$ *as* $z \to p^+$.

Proof of Theorem 15(i). By (13), $h_1(0,z) = k(z)\pi$ for some integer $k(z)$ for a.e. z. By Theorem 7(ii) and (iv), $k(z) = 0$ or $k(z) = 1$ for a.e. z, and $k(z) = 0$ for z large enough. Since h_1, and hence also k, is nonincreasing with respect to z, it remains to show that $k \not\equiv 0$ in $\mathbb{R}$. We argue by contradiction and suppose that $h_1(0,z) = 0$ for all $z \in \mathbb{R}$.

Given $n \in \mathbb{N}$ and $0 < r_n < 1$, by (31) there exists $q_n \leqslant z_0$ such that $h_1(r,z) \geqslant \pi$ for $z \leqslant q_n$ and $r \in [r_n,1]$. We define $p_n = q_n - \frac{1}{n}$, $\alpha_n(z) = (z-p_n)^{-2}$ and

$$h_{1,n}(r,z) = \begin{cases} h_1(r,z), & z > q_n, \\ \max\{h_1(r,z), v_n(r,z)\}, & z \in (p_n,q_n], \\ \max\{\pi, h_1(r,z)\}, & z \leqslant p_n, \end{cases}$$

where

$$v_n(r,z) = 2\arctan\left(\frac{\alpha_n(z)r^2}{r+1}\right).$$

Choosing $r_n = b/(n^2-b)$, which is a root of the equation $v_n(r,q_n) = 2\arctan(br)$, it follows easily from Lemma 16 and the definition of p_n and q_n that $h_{1,n} \in \mathcal{W}^c$.

We claim that $\Phi_c(h_{1,n}) < \Phi_c(h_1)$ for n large enough, which is a contradiction since h_1 is a minimizer of Φ_c in $\mathcal{W}^c$.

Given a measurable set $S \subset (0,1) \times \mathbb{R}$ and $f \in \mathcal{W}$, we set

$$E_S(f) := \iint_S \frac{r}{2}\mathrm{e}^{cz}\left(f_r^2 + f_z^2 + \frac{\sin^2 f}{r^2}\right)\mathrm{d}r\,\mathrm{d}z.$$

Then $\Phi_c(h_{1,n}) - \Phi_c(h_1) = I_{1,n} - I_{2,n}$, where

$$I_{1,n} = E_{[0,1]\times(p_n,q_n)}(h_{1,n}) - E_{[0,1]\times(p_n,q_n)}(h_1),$$
$$I_{2,n} = E_{[0,1]\times(-\infty,p_n)}(h_{1,n}) - E_{[0,1]\times(-\infty,p_n)}(h_1).$$

By Lemma 16, $I_{1,n} \leqslant E_{[0,1]\times(p_n,q_n)}(v_n) \leqslant \frac{25}{c}\mathrm{e}^{cp_n}(\mathrm{e}^{c/n}-1)$.

We define $\rho(z) = \inf\{r \in (0,1];\ h_1(r,z) \geqslant \pi\}$ for $z \leqslant z_0$. Then $0 < \rho(z) < 1$, since $h_1(0,z) = 0$ and $h_1(1,z) = \pi + 2\arctan a_1$ if $z \leqslant z_0$. We set

$$A_n = \{(r,z);\ 0 < r < \rho(z), z < p_n\} \quad \text{and} \quad B_n = \{(r,z);\ \rho(z) < r < 1, z < p_n\}.$$

Since $h_{1,n} = \pi$ in A_n, it follows from Lemma 20 that

$$E_{A_n}(h_{1,n}) - E_{A_n}(h_1) = -E_{A_n}(h_1) \leqslant -2\int_{-\infty}^{p_n} \mathrm{e}^{cz}\,\mathrm{d}z = -\frac{2}{c}\mathrm{e}^{cp_n}.$$

Since $|(h_{1,n})_r| \leqslant |h_{1r}|$, $|(h_{1,n})_z| \leqslant |h_{1z}|$ and $|\sin h_{1,n}| \leqslant |\sin h_1|$ in B_n, this implies that $I_{2,n} \leqslant -\frac{2}{c}\mathrm{e}^{cp_n}$.

We conclude that

$$\Phi_c(h_{1,n}) - \Phi_c(h_1) \leqslant \frac{e^{cp_n}}{c}\big(25e^{c/n} - 25 - 2\big) < 0$$

for n large enough, and we have proved our claim. □

Proof of Theorem 15(ii). The uniform convergence follows at once from (31), Theorem 15(i), the monotonicity in z and the upper bound in Theorem 7(ii). □

In the proof of part (iii) we shall use an auxiliary lemma which is based on the following proposition.

Proposition 17. *Let h_i be as in Theorem 15, I be an open nonempty interval and $k \in \mathbb{Z}$ a constant such that $h_i(0, z) = k\pi$ for $z \in I$. Then h_i is real analytic in $[0, 1) \times I$.*

Proof. It is enough to prove that h_i is real analytic in a neighborhood of $(0, z)$ for all $z \in I$. The monotonicity with respect to z implies that h_i is continuous in $[0, 1) \times I$. Then the function

$$u_i(x_1, x_2, z) := \left(\frac{x_1}{r}\sin h_i(r, z), \frac{x_2}{r}\sin h_i(r, z), \cos h_i(r, z)\right), \quad r = \sqrt{x_1^2 + x_2^2},$$

(see also (2)) is a continuous weak solution of $\Delta u + |\nabla u|^2 u + cu_z = 0$ in $D \times I$, where D indicates the unit disk. It is well known (see [13] and [11]) that weak solutions are real analytic in open sets in which they are continuous, and hence u_i is analytic in $D \times I$. Since the first component of u_i, $(u_i)_1$, vanishes in $\{(0, 0)\} \times I$ and $(u_i)_1(r, 0, z) = \sin(h_i(r, z))$, the analyticity of the function arcsin in a neighborhood of the origin implies that, given $z \in I$, h_i is real analytic in a neighborhood of $(0, z)$. □

Lemma 18. *Let h_i, I and k be as in Proposition 17. Then there exists $\tilde{z} \in I$ such that $(h_i)_r(0, \tilde{z}) \neq 0$.*

Proof. Omitting the subscript i and arguing by contradiction we suppose that $h_r(0, z) = 0$ for all $z \in I$. We claim that for all positive integers α

$$\frac{\partial^\alpha h}{\partial r^\alpha}(0, z) \equiv 0 \quad \text{for } z \in I. \tag{44}$$

This leads immediately to a contradiction: by Proposition 17 and (44) h is constant in $(0, 1) \times I$, which is impossible since h is strictly decreasing with respect to z in $(0, 1) \times \mathbb{R}$.

In order to prove (44) we argue by induction. We know that (44) is true for $\alpha = 1$. Suppose that it is true for $\alpha = 1, \ldots, \beta$ for some $\beta \geqslant 1$. Using a Taylor expansion we obtain that for all $z \in I$ and $\alpha = 1, \ldots, \beta$

$$\frac{\partial^\alpha}{\partial r^\alpha}\big(\sin(2h)\big)\bigg|_{r=0} = 0 \quad \text{and} \quad \frac{\partial^{\beta+1}}{\partial r^{\beta+1}}\big(\sin(2h)\big)\bigg|_{r=0} = 2\frac{\partial^{\beta+1} h}{\partial r^{\beta+1}}(0, z),$$
$$h_{rr}(r, z) = \frac{1}{(\beta - 1)!}\frac{\partial^{\beta+1} h}{\partial r^{\beta+1}}(0, z) r^{\beta-1} + \mathrm{O}\big(r^\beta\big),$$
$$h_r(r, z) r^{-1} = \frac{1}{\beta!}\frac{\partial^{\beta+1} h}{\partial r^{\beta+1}}(0, z) r^{\beta-1} + \mathrm{O}\big(r^\beta\big),$$
$$\frac{1}{2}\sin\big(2h(r, z)\big) r^{-2} = \frac{1}{(\beta + 1)!}\frac{\partial^{\beta+1} h}{\partial r^{\beta+1}}(0, z) r^{\beta-1} + \mathrm{O}\big(r^\beta\big),$$
$$h_z(r, z) = \mathrm{O}\big(r^{\beta+1}\big), \qquad h_{zz}(r, z) = \mathrm{O}\big(r^{\beta+1}\big).$$

Substituting these equalities in Eq. (4), we find that (44) holds for $\alpha = \beta + 1$. □

Proof of Theorem 15(iii). The proof consists of two steps. In the first one we exclude the possibility that $h_2(0, z) = 0$ for all $z \in \mathbb{R}$. In the second one we show that $h_2(0, z) \neq \pi$ for a.e. $z \in \mathbb{R}$. Since h_2 is nonincreasing with respect to z, the proof is then completed by Theorem 7(iii)–(iv).

Step 1. We only give the proof in the case that $a_2 < 0$ (if $a_2 \geqslant 0$ the proof can be considerably simplified). As in the proof of part (i) we argue by contradiction and suppose that $h_2(0, z) = 0$ for all $z \in \mathbb{R}$.

Given $n \in \mathbb{N}$ and $\rho_n = b/(n^2 - b)$, the statement which follows formula (32) (which treats the case $a_2 < 0$) implies that there exists $q_n \leqslant z_0$ such that

$$h_2(r, z) \geqslant \pi + 2\arctan\left(\frac{r}{2|a_2|}\right) \quad \text{if } z \leqslant q_n \text{ and } \rho_n \leqslant r \leqslant 1. \tag{45}$$

We set

$$p_n = q_n - \frac{1}{n}, \quad z_n = p_n - 1, \quad r_n \in [\rho_n, 1],$$

and we define for all $0 \leqslant r \leqslant 1$

$$\begin{aligned}
&v_n(r, z) = 2\arctan\left(\frac{\alpha_n(z) r^2}{r+1}\right), \quad p_n < z \leqslant q_n,\\
&w_n(r, z) = \pi + 2\arctan\bigl(\beta_n(z) r\bigr), \quad z_n \leqslant z \leqslant p_n,\\
&\omega_n(r, z) = \max\left\{2\pi - 2\arctan\bigl(\gamma_n(z) r\bigr), \pi + 2\arctan\left(\frac{r}{2|a_2|}\right)\right\}, \quad z_n - r_n \leqslant z < z_n,\\
&\chi_n(r) = \omega_n(r, z_n - r_n),
\end{aligned}$$

where

$$\alpha_n(z) = \frac{1}{(z - p_n)^2}, \quad \beta_n(z) = \frac{p_n - z}{2|a_2|}, \quad \gamma_n(z) = \frac{2|a_2|}{(z_n - z)^2}.$$

Finally, we set, for $0 \leqslant r \leqslant 1$,

$$h_{2,n}(r, z) = \begin{cases}
h_2(r, z) & \text{if } z > q_n,\\
\max\bigl\{h_2(r, z), v_n(r, z)\bigr\} & \text{if } z \in (p_n, q_n],\\
\max\bigl\{h_2(r, z), w_n(r, z)\bigr\} & \text{if } z \in [z_n, p_n],\\
\max\bigl\{h_2(r, z), \omega_n(r, z)\bigr\} & \text{if } z \in [z_n - r_n, z_n),\\
\max\bigl\{h_2(r, z), \chi_n(r)\bigr\} & \text{if } z < z_n - r_n.
\end{cases}$$

It is easy to show that $h_{2,n}$ is locally Lipschitz continuous in $(0, 1] \times \mathbb{R}$ and belongs to $\mathcal{W}^c$. To obtain a contradiction it is enough to show that

$$\Phi_c(h_{2,n}) + 2L_c(h_{2,n}) < \Phi_c(h_2) + 2L_c(h_2) \quad \text{for } n \text{ large enough.} \tag{46}$$

Defining $E_S(f)$ as in the proof of part (i), we write

$$\Phi_c(h_{2,n}) - \Phi_c(h_2) = I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n},$$

where

$$\begin{aligned}
&I_{1,n} := E_{(0,1)\times(p_n,q_n)}(h_{2,n}) - E_{(0,1)\times(p_n,q_n)}(h_2),\\
&I_{2,n} := E_{(0,1)\times(z_n,p_n)}(h_{2,n}) - E_{(0,1)\times(z_n,p_n)}(h_2),\\
&I_{3,n} := E_{(0,1)\times(z_n-r_n,z_n)}(h_{2,n}) - E_{(0,1)\times(z_n-r_n,z_n)}(h_2),\\
&I_{4,n} := E_{(0,1)\times(-\infty,z_n-r_n)}(h_{2,n}) - E_{(0,1)\times(-\infty,z_n-r_n)}(h_2).
\end{aligned}$$

By Lemma 16,

$$I_{1,n} \leqslant E_{(0,1)\times(p_n,q_n)}(v_n) \leqslant \frac{25}{c} \mathrm{e}^{c p_n} \bigl(\mathrm{e}^{\frac{c}{n}} - 1\bigr). \tag{47}$$

Since $w_n(r, z) \leqslant \pi + 2\arctan(r/(2|a_2|))$ if $0 < r < 1$ and $z_n < z < p_n$, it follows from (45) that

$$I_{2,n} \leqslant E_{(0,\rho_n)\times(z_n,p_n)}(h_{2,n}) - E_{(0,\rho_n)\times(z_n,p_n)}(h_2).$$

Hence, by Corollary 21 and a straightforward calculation,

$$I_{2,n} \leqslant \int_{z_n}^{p_n} e^{cz}\left(-2+\int_0^{\rho_n}\frac{1}{2}r(h_{2,n})_z^2\,dr\right)dz$$
$$=\int_{z_n}^{p_n} e^{cz}\left(-2+\left(\frac{\beta_n'(z)}{\beta_n^2(z)}\right)^2\left(\log\left(1+\beta_n^2\rho_n^2\right)-1+\frac{1}{1+\beta_n^2\rho_n^2}\right)\right)dz,$$

and there exists a constant $C_2>0$ which does not depend on n such that

$$I_{2,n}\leqslant\frac{-2+C_2\rho_n^4}{c}\left(e^{cp_n}-e^{cz_n}\right). \tag{48}$$

Since $\gamma_n(z)r\geqslant 2|a_2|/r_n$ for all $r\geqslant r_n(\geqslant\rho_n)$ and $z\in[z_n-r_n,z_n)$, it follows from (45) that

$$2\pi-2\arctan\left(\gamma_n(z)r\right)\leqslant\pi+2\arctan\left(\frac{r}{2|a_2|}\right)\leqslant h_2(r,z)\quad\text{if } r\geqslant r_n.$$

Hence

$$I_{3,n}=E_{(0,r_n)\times(z_n-r_n,z_n)}(h_{2,n})-E_{(0,r_n)\times(z_n-r_n,z_n)}(h_2),$$

and, by Corollary 21,

$$I_{3,n}\leqslant\int_{z_n-r_n}^{z_n} e^{cz}\,dz\int_0^{r_n}\frac{1}{2}r\left((h_{2,n})_z^2-(h_2)_z^2\right)dr$$
$$\leqslant\int_{z_n-r_n}^{z_n} e^{cz}\,dz\int_0^{r_n} r\left(\arctan\left(\gamma_n(z)r\right)\right)_z^2\,dr$$
$$=\int_{z_n-r_n}^{z_n} e^{cz}\left(\frac{\gamma_n'(z)}{\gamma_n^2(z)}\right)^2\left(\log\left(1+\gamma_n^2r_n^2\right)-1+\frac{1}{1+\gamma_n^2r_n^2}\right)dz.$$

Since $\log(1+s^2)\leqslant 4\sqrt{s}$ for $s>0$, it follows easily that there exists a constant $C_3>0$ which does not depend on n such that

$$I_{3,n}\leqslant C_3r_n^2\sqrt{r_n}\,e^{cz_n}. \tag{49}$$

Since $\chi_n(r)=\pi+2\arctan(r/(2|a_2|))$ for $r\geqslant r_n\geqslant\rho_n$,

$$I_{4,n}=E_{(0,r_n)\times(-\infty,z_n-r_n)}(h_{2,n})-E_{(0,r_n)\times(-\infty,z_n-r_n)}(h_2).$$

On the other hand, $\chi_n(r)=2\pi-2\arctan(2|a_2|r/r_n^2)$ for $r\leqslant r_n$, and hence, setting

$$S_-:=\left\{(r,z)\in(0,r_n)\times(-\infty,z_n-r_n);\ h_2(r,z)<\chi_n(r)\right\},$$

$$I_{4,n}\leqslant\iint_{S_-}\frac{1}{2}re^{cz}\left(\left((\chi_n)_r^2+\frac{\sin^2\chi_n}{r^2}\right)-\left((h_2)_r^2+\frac{\sin^2h_2}{r^2}\right)\right)dr\,dz$$
$$\leqslant\int_{-\infty}^{z_n-r_n} e^{cz}\left(J_{1,n}(z)-J_{2,n}(z)\right)dz,$$

where

$$J_{1,n}(z):=\int_0^{r_n}\frac{1}{2}r\left((\chi_n)_r^2+\frac{\sin^2\chi_n}{r^2}\right)dr=2-\frac{2r_n^2}{r_n^2+4|a_2|^2},$$

and

$$J_{2,n}(z) := \int_0^{\rho(z)} \frac{1}{2} r \left((h_2)_r^2 + \frac{\sin^2 h_2}{r^2} \right) \mathrm{d}r$$

with $\rho(z) := \inf\{r \in [0, r_n];\ h_2(r,z) \geqslant \pi\}$. By Lemma 20 $J_{2,n}(z) \geqslant 2$, and hence there exists a constant $C_4 > 0$ which does not depend on n such that

$$I_{4,n} \leqslant -C_4 r_n^2 \mathrm{e}^{c(z_n - r_n)}. \tag{50}$$

Since $L_c(h_{2,n}) = (\mathrm{e}^{cp_n} - \mathrm{e}^{cz_n})/c$ (by Theorem 3), it follows from (48), (49) and (50) that there exists $\delta > 0$ such that if $\rho_n \leqslant r_n \leqslant \delta$ then

$$I_{2,n} + I_{3,n} + I_{4,n} + 2L_c(h_{2,n}) \leqslant -\frac{1}{2} C_4 r_n^2 \mathrm{e}^{c(z_n - r_n)}.$$

Hence, by (47), we can choose $r_n = \delta$ and n so large that

$$\Phi_c(h_{2,n}) + 2L_c(h_{2,n}) < \Phi_c(h_2),$$

and (46) follows.

Step 2. We argue by contradiction and suppose that there exist $p < q$ such that $h_2(0,z) = \pi$ if $p < z < q$. In view of Lemma 18 and the monotonicity of h_2 with respect to z, we may assume, without loss of generality, that for some $k_0 > 0$ either

$$(h_2)_r(0,z) \geqslant (h_2)_r(0,q) > k_0 > 0 \quad \text{if } z < q \tag{51}$$

or

$$(h_2)_r(0,z) \leqslant (h_2)_r(0,p) < -k_0 < 0 \quad \text{if } p < z. \tag{52}$$

One way to obtain a contradiction is to modify the proof of a more general result in [12]. Alternatively, we can use the approach used in the proof of part (iii): if (51) holds, we can construct a function h_2^* such that $h_2^*(0,z) = 2\pi$ if $z < q$ and $\Phi_c(h_2^*) + 2L_c(h_2^*) < \Phi_c(h_2) + 2L_c(h_2)$; if (52) holds, a similar function h_2^* exists such that $h_2^*(0,z) = 0$ if $z > p$. For example, in the first case we can choose h_2^* of the type

$$h_2^*(r,z) = \begin{cases} h_2(r,z) & \text{if } 0 < r < 1, z \geqslant q \text{ or } r^* < r < 1, z < q, \\ \max\{h_2(r,z), \omega(r,z)\} & \text{if } 0 < r < r^*, z \in [q - z^*, q), \\ \max\{h_2(r,z), \omega(r, q - z^*)\} & \text{if } 0 < r < r^*, z < q - z^*, \end{cases}$$

where $\omega(r,z) = 2\pi - 2\arctan(\gamma(z)r)$, $\gamma(z) = C^*(q - z)^{-2}$, and r^*, z^* and C^* are constants to be chosen appropriately. We leave the details to the interested reader. □

Proof of Theorem 15(iv). The uniform convergence follows at once from formula (32) (which holds only if $a_2 \geqslant 0$) and the sentence immediately after (32) (which holds if $a_2 < 0$), Theorem 15(iii), the monotonicity in z and the upper bound in Theorem 7(iii). □

Proof of Theorem 15(v). The proof is an immediate consequence of Proposition 17. □

7. Position of the singularity when $c \to \infty$

Let $c > 0$ and let h_1 and h_2 be the solutions given by, respectively, Theorems 1 and 2 with a point singularity in $(0, \bar{z}_1)$ and $(0, \bar{z}_2)$. In this section we consider the behavior of $\bar{z}_i$ as $c \to \infty$. We shall often add the subscript c and use the notation $h_{i,c}$ and $\bar{z}_{i,c}$ $(i = 1, 2)$.

We first give a heuristic argument and set

$$\tau = -\frac{z}{c}, \quad \tau_{i,c} = -\frac{\bar{z}_{i,c}}{c} \quad \text{and} \quad q_{i,c}(r,\tau) = h_{i,c}(r, -c\tau). \tag{53}$$

Then $q_{i,c}$ is smooth in $[0,1] \times \mathbb{R} \setminus \{(0, \tau_{i,c})\}$ and is a solution of the equation

$$q_\tau = \frac{q_{\tau\tau}}{c^2} + q_{rr} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \quad \text{in } (0,1) \times \mathbb{R}. \tag{54}$$

In addition $q_{i,c}$ satisfies the properties:

$$\begin{cases} q_{i,c}(r,\infty) = i\pi + 2\arctan(a_i r), & r \in [0,1], \\ q_{i,c}(r,-\infty) = 2\arctan(br), & r \in [0,1], \\ q_{i,c}(1,\tau) = g(-c\tau), & \tau \in \mathbb{R}, \\ q_{i,c}(0,\tau) = 0, & \tau < \tau_{i,c}, \\ q_{i,c}(0,\tau) = i\pi, & \tau > \tau_{i,c}. \end{cases} \tag{55}$$

If $q_{i,c}$ converges to some limit function q_i as $c \to \infty$, it is plausible that q_i satisfies the parabolic equation

$$q_\tau = q_{rr} + \frac{q_r}{r} - \frac{\sin(2q)}{2r^2} \quad \text{in } (0,1) \times \mathbb{R} \tag{56}$$

with the following conditions at $\tau = -\infty$ and $r = 1$:

$$\begin{cases} q_i(r,-\infty) = 2\arctan(br), & r \in [0,1], \\ q_i(1,\tau) = g(\infty) = B, & \tau < 0, \\ q_i(1,\tau) = g(-\infty) = A, & \tau > 0. \end{cases} \tag{57}$$

So q_i is a solution of the harmonic map flow on the unit disk, with τ playing the role of time. The problem for q_i can be easily split up in two separate problems: one for $\tau < 0$, with the trivial solution

$$q_i(r,\tau) = 2\arctan(br) \quad \text{if } 0 \leqslant r \leqslant 1,\ \tau < 0, \tag{58}$$

and the other one for $\tau > 0$ with an initial condition at $\tau = 0$ inherited from (58):

$$\begin{cases} q_\tau = q_{rr} + \dfrac{q_r}{r} - \dfrac{\sin(2q)}{2r^2}, & 0 < r < 1,\ \tau > 0, \\ q(r,0) = 2\arctan(br), & 0 < r < 1, \\ q(1,\tau) = g(-\infty) = A, & \tau > 0. \end{cases} \tag{59}$$

Since $A > \pi$ it is known (see [7]) that (59) has a classical solution q which blows up after finite time $\bar{\tau} > 0$, satisfying

$$q(0,\tau) = 0 \quad \text{if } \tau < \bar{\tau} \text{ and } q(0,\bar{\tau}) = \pi. \tag{60}$$

In [2,19] it has been shown that this solution can be continued for $\tau > \bar{\tau}$ in at least 2 different ways: for $\tau > \bar{\tau}$, q satisfies either $q(0,\tau) = \pi$ or $q(0,\tau) = 2\pi$. The latter property explains the difference between the limit functions q_1 and q_2. In particular we claim that $\bar{z}_{1,c}$ and $\bar{z}_{2,c}$ have the same limiting behavior as $c \to \infty$:

Theorem 19. *Let $h_{1,c}$ and $h_{2,c}$ be the solutions constructed in Theorems* 1 *and* 2*, and let $(0, \bar{z}_{1.c})$ and $(0, \bar{z}_{2,c})$ be their singularities. Then*

$$\bar{z}_{i,c} = -\bar{\tau}c\big(1 + \mathrm{o}(1)\big) \to -\infty \quad \text{as } c \to \infty \ (i = 1,2), \tag{61}$$

where $\bar{\tau} > 0$ is defined by (60).

The rigorous proof of this result is quite lengthy, and below we only sketch its structure.

It is not difficult to show that for all compact subsets Ω of $(0,1) \times \mathbb{R}$ there exists a constant $K = K(\Omega)$ which does not depend on c such that for all $c \geqslant 1$

$$\iint_\Omega \left(\left| \frac{\partial q_{i,c}}{\partial r} \right|^2 + \left| \frac{\partial q_{i,c}}{\partial \tau} \right|^2 \right) \mathrm{d}\tau\, \mathrm{d}r \leqslant K.$$

Hence there exist $q_i \in H^1_{\mathrm{loc}}((0,1) \times \mathbb{R})$ such that, up to subsequences,

$$q_{i,c} \rightharpoonup q_i \quad \text{in } H^1_{\mathrm{loc}}\big((0,1) \times \mathbb{R}\big) \text{ as } c \to \infty.$$

By standard regularity theory, q_i is a smooth solution of Eq. (56) in $(0,1)\times\mathbb{R}$. In addition q_i is increasing with respect to τ and satisfies $2\arctan(br) < q_i(r,\tau) < i\pi + 2\arctan(a_i r)$.

Using that

$$\Phi_c(h_{i,c}) \leqslant \Phi_c\big(2\arctan(br) + \big(g(z) - 2\arctan b\big)r\big) \leqslant \frac{K}{c}\mathrm{e}^{cz_1}$$

for some K which does not depend on c one can prove that, for any $M>0$ and $\varepsilon>0$,

$$\int_{-M-\varepsilon}^{-M} f_{i,c}(\tau)\,\mathrm{d}\tau \leqslant \frac{K}{c^2}\mathrm{e}^{cz_1-c^2M} \to 0 \quad \text{as } c\to\infty, \tag{62}$$

where

$$f_{i,c}(\tau) := \int_0^1 \frac{r}{2}\left((q_{i,c})_r^2 + \frac{\sin^2(q_{i,c})}{r^2} - G_b(r)\right)\mathrm{d}r.$$

By Theorem 23 $f_{i,c}(\tau) \geqslant 0$ in $(-\infty, -z_1/c)$. Using the monotonicity with respect to τ, it follows easily from (62) and Theorem 24 that $q_{i,c} \to 2\arctan(br)$ uniformly in $[0,1]\times(-\infty,-M]$ for all $M>0$, and (58) follows.

The rest of the proof is based on some detailed information about the minimal solution, $q_{\min}(r,\tau)$ $(\tau\geqslant 0)$, of (59). In particular $q_{\min}$ satisfies (60), $q_{\min}(0,\tau)=\pi$ if $\tau\geqslant\bar\tau$, and $q_{\min}$ is increasing with respect to τ (since the initial function is a subsolution). Lap-number theory (see [16]) implies that for all $0<\tau<\bar\tau$ there exists a unique $r(\tau)$ such that $q_{\min}(r(\tau),\tau)=\pi$. In addition $r(\tau)$ is decreasing with respect to τ and $r(\tau)\to 0$ as $\tau\to\bar\tau$. Finally $q_{\min}>\pi$ in $(0,1)\times(\bar\tau,\infty)$ and $(q_{\min})_r(0,\tau)>0$ if $\tau>\bar\tau$.

Arguing by contradiction, we use these properties and the fact that h_i is a minimizer to prove that

(i) $q_i = q_{\min}$ in $(0,1)\times(0,\bar\tau)$ and for all $\varepsilon>0$ there exists $c_{\varepsilon,1}$ such that $-\bar z_{i,c} > (\bar\tau-\varepsilon)c$ for all $c>c_{\varepsilon,1}$;
(ii) for all $\varepsilon>0$ there exists $c_{\varepsilon,2}$ such that $-\bar z_{i,c} < (\bar\tau+\varepsilon)c$ for all $c>c_{\varepsilon,2}$.

The proofs of (i) and (ii) are based on the construction of functions which are similar to the ones used in the previous section (the functions $h_{1,n}$ and $h_{2,n}$). We omit their construction, which is rather delicate and lengthy.

Appendix A

Lemma 20. *For all $w\in H^1_{\mathrm{loc}}(0,\infty)\subset C^0((0,\infty))$ and $0<\rho_1<\rho_2$*

$$\int_{\rho_1}^{\rho_2} \frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{\mathrm{d}w}{\mathrm{d}r}\right|^2\right)\mathrm{d}r \geqslant \left|\cos\big(w(\rho_2)\big) - \cos\big(w(\rho_1)\big)\right|.$$

Proof.

$$\begin{aligned}\left|\cos\big(w(\rho_2)\big) - \cos\big(w(\rho_1)\big)\right| &= \left|\int_{\rho_1}^{\rho_2} -\sin w\frac{\mathrm{d}w}{\mathrm{d}r}\,\mathrm{d}r\right| \leqslant \int_{\rho_1}^{\rho_2}\frac{r}{2}\left(2\left|\frac{\sin w}{r}\right|\left|\frac{\mathrm{d}w}{\mathrm{d}r}\right|\right)\mathrm{d}r\\ &\leqslant \int_{\rho_1}^{\rho_2}\frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{\mathrm{d}w}{\mathrm{d}r}\right|^2\right)\mathrm{d}r. \qquad \square\end{aligned}$$

A straightforward calculation leads to the following consequence:

Corollary 21. *Let $0<\alpha<\beta$, $k\in\mathbb{Z}$ and $b\in\mathbb{R}$. Let*

$$E_\alpha^\beta(w) = \int_\alpha^\beta \frac{r}{2}\left(\left|\frac{\mathrm{d}w}{\mathrm{d}r}\right|^2 + \frac{\sin^2 w}{r^2}\right)\mathrm{d}r \quad \textit{for } w\in H^1(\alpha,\beta). \tag{A.1}$$

Then

$$E_\alpha^\beta(w) \geqslant E_\alpha^\beta\bigl(k\pi + 2\arctan(br)\bigr) = \frac{2}{1+b^2\alpha^2} - \frac{2}{1+b^2\beta^2}$$

for all $w \in H^1(\alpha,\beta)$ *satisfying* $w(\alpha) = k\pi + 2\arctan(b\alpha)$ *and* $w(\beta) = k\pi + 2\arctan(b\beta)$.

Lemma 22. *Let* $0 < \alpha < \beta$, $w \in H^1(\alpha,\beta)$ *and let* $E_\alpha^\beta(w)$ *be defined by* (A.1). *If* k_1, k_2 *are integers satisfying* $w(\alpha) \in [k_1\pi, (k_1+1)\pi)$, $w(\beta) \in [k_2\pi, (k_2+1)\pi)$, *then*

$$E_\alpha^\beta(w) \geqslant \begin{cases} 2(k_2-k_1-1) + \bigl|\cos\bigl(w(\beta)\bigr) - (-1)^{k_2}\bigr| + \bigl|(-1)^{k_1+1} - \cos\bigl(w(\alpha)\bigr)\bigr| & \text{if } k_2 > k_1,\\ \bigl|\cos\bigl(w(\beta)\bigr) - \cos\bigl(w(\alpha)\bigr)\bigr| & \text{if } k_2 = k_1,\\ 2(k_1-k_2-1) + \bigl|\cos\bigl(w(\alpha)\bigr) - (-1)^{k_1}\bigr| + \bigl|(-1)^{k_2+1} - \cos\bigl(w(\beta)\bigr)\bigr| & \text{if } k_2 < k_1. \end{cases}$$

Proof. If $k_2 = k_1$ the conclusion follows directly from Lemma 20. If $k_2 > k_1$ and so $w(\beta) > w(\alpha)$, it is sufficient to apply Lemma 20 to the partition $\alpha < R_0 < \cdots < R_{k_2-k_1-1} < \beta$ of $[\alpha,\beta]$, where $w(R_j) = (k_1+1+j)\pi$ for all $j = 0, 1, \ldots, k_2-k_1-1$. The case $k_2 < k_1$ is similar. □

Theorem 23. *Let* $R > 0$, $0 < b < 1$ *and* $w \in H^1_{\mathrm{loc}}((0,R])$. *If* $w(R) = 2\arctan b$ *then*

$$E_0^R(w) = \int_0^R \frac{r}{2}\left(\frac{\sin^2 w}{r^2} + \left|\frac{\mathrm{d}w}{\mathrm{d}r}\right|^2\right)\mathrm{d}r \geqslant E_0^R\left(2\arctan\left(\frac{br}{R}\right)\right) = \frac{2b^2}{1+b^2}.$$

Proof. The latter equality is trivial. To prove the inequality, we observe that, since $0 < b < 1$, if $\lim_{\rho\to0^+} w(\rho) = k\pi$ for some $k \in \mathbb{Z}$, Lemma 20 implies that $E_0^R(w) \geqslant 2/(1+b^2) > 2b^2/(1+b^2)$ if k is odd, and $E_0^R(w) \geqslant 2b^2/(1+b^2)$ if k is even. It is easy to prove that in all other cases $E_0^R(w) = \infty$. Indeed, if $\lim_{\rho\to0^+} w(\rho)$ exists and is finite but not equal to a multiple of π, then $(\sin^2 w)/r$ is not integrable at $r = 0$; if $\lim_{\rho\to0^+} w(\rho)$ is infinite or does not exist it is enough to apply (repeatedly in the latter case) Lemma 22. □

Setting

$$\mathcal{S}_b(R) = \left\{ w \in H_r^1(0,R);\ \frac{\sin w}{r} \in L_r^2(0,R),\ w(R) = 2\arctan b \right\},$$

Theorem 23 implies that if $0 < b < 1$ the function $2\arctan(br/R)$ is a minimum of the functional $E_0^R(w)$ on $\mathcal{S}_b(R)$. Since any minimum satisfies the Euler–Lagrange equation

$$w_{rr} + \frac{1}{r}w_r - \frac{\sin(2w)}{2r^2} = 0,$$

it is easy to show that $2\arctan(br/R)$ is the unique minimum. Using the estimates obtained in this appendix it is very easy to show a slightly sharper result, of which we omit the proof:

Theorem 24. *Let* $R > 0$, $0 < b < 1$ *and let* $\{w_n\}$ *be a minimizing sequence for* $E_0^R(w)$ *on* $\mathcal{S}_b(R)$. *Then* $w_n(0) = 0$ *for* n *large enough and* $w_n(r) \to 2\arctan(br/R)$ *uniformly in* $[0,R]$ *as* $n \to \infty$.

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