## $L^{1}$ existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions

# Résultats d'existence et d'unicité dans $L^{1}$ pour des équations elliptiques quasi-linéaires avec des conditions au bord non linéaires 

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#### Abstract

In this paper we study the questions of existence and uniqueness of weak and entropy solutions for equations of type $-\operatorname{div} \mathbf{a}(x, D u)+\gamma(u) \ni \phi$, posed in an open bounded subset $\Omega$ of $\mathbb{R}^{N}$, with nonlinear boundary conditions of the form $\mathbf{a}(x, D u)$. $\eta+\beta(u) \ni \psi$. The nonlinear elliptic operator $\operatorname{div} \mathbf{a}(x, D u)$ is modeled on the $p$-Laplacian operator $\Delta_{p}(u)=\operatorname{div}\left(|D u|^{p-2} D u\right)$, with $p>1, \gamma$ and $\beta$ are maximal monotone graphs in $\mathbb{R}^{2}$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, and the data $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$. © 2006 Elsevier Masson SAS. All rights reserved.


## Résumé

Dans ce papier nous étudions les questions d'existence et d'unicité de solution faibles et entropiques pour des équations elliptiques de la forme $-\operatorname{div} \mathbf{a}(x, D u)+\gamma(u) \ni \phi$, dans un domaine borné $\Omega \subset \mathbb{R}^{N}$, avec des conditions au bord générales de la forme $\mathbf{a}(x, D u) \cdot \eta+\beta(u) \ni \psi$. L'opérateur div $\mathbf{a}(x, D u)$ généralise l'opérateur $p$-Laplacien $\Delta_{p}(u)=\operatorname{div}\left(|D u|^{p-2} D u\right)$, avec $p>1, \gamma$ et $\beta$ sont des graphes maximaux monotones dans $\mathbb{R}^{2}$ tels que $0 \in \gamma(0) \cap \beta(0)$, et les données $\phi$ et $\psi$ sont des fonctions $L^{1}$. © 2006 Elsevier Masson SAS. All rights reserved.

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## 1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^{N}$ with smooth boundary $\partial \Omega$ and $p>1$, and let $\mathbf{a}: \Omega \times \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}$ be a Carathéodory function satisfying

[^0]$\left(\mathrm{H}_{1}\right)$ there exists $\lambda>0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geqslant \lambda|\xi|^{p}$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{N}$,
$\left(\mathrm{H}_{2}\right)$ there exists $\sigma>0$ and $\theta \in L^{p^{\prime}}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leqslant \sigma\left(\theta(x)+|\xi|^{p-1}\right)$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{N}$, where $p^{\prime}=\frac{p}{p-1}$,
$\left(\mathrm{H}_{3}\right)\left(\mathbf{a}\left(x, \xi_{1}\right)-\mathbf{a}\left(x, \xi_{2}\right)\right) \cdot\left(\xi_{1}-\xi_{2}\right)>0$ for a.e. $x \in \Omega$ and for all $\xi_{1}, \xi_{2} \in \mathbb{R}^{N}, \xi_{1} \neq \xi_{2}$.

The hypotheses $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{3}\right)$ are classical in the study of nonlinear operators in divergence form (cf. [23] or [5]). The model example of function a satisfying these hypotheses is $\mathbf{a}(x, \xi)=|\xi|^{p-2} \xi$. The corresponding operator is the $p$-Laplacian operator $\Delta_{p}(u)=\operatorname{div}\left(|D u|^{p-2} D u\right)$.

We are interested in the study of existence and uniqueness of weak and entropy solutions for the elliptic problem

$$
\left(S_{\phi, \psi}^{\gamma, \beta}\right) \quad \begin{cases}-\operatorname{div} \mathbf{a}(x, D u)+\gamma(u) \ni \phi & \text { in } \Omega \\ \mathbf{a}(x, D u) \cdot \eta+\beta(u) \ni \psi & \text { on } \partial \Omega\end{cases}
$$

where $\eta$ is the unit outward normal on $\partial \Omega, \psi \in L^{1}(\partial \Omega)$ and $\phi \in L^{1}(\Omega)$. The nonlinearities $\gamma$ and $\beta$ are maximal monotone graphs in $\mathbb{R}^{2}$ (see, e.g. [12]) such that $0 \in \gamma(0)$ and $0 \in \beta(0)$. In particular, they may be multivalued and this allows to include the Dirichlet condition (taking $\beta$ to be the monotone graph $D$ defined by $D(0)=\mathbb{R}$ ) and the Neumann condition (taking $\beta$ to be the monotone graph $N$ defined by $N(r)=0$ for all $r \in \mathbb{R}$ ) as well as many other nonlinear fluxes on the boundary that occur in some problems in Mechanic and Physics (see, e.g., [16] or [11]). Note also that, since $\gamma$ may be multivalued, problems of type $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ appears in various phenomena with changes of state like multiphase Stefan problem (cf. [14]) and in the weak formulation of the mathematical model of the so called Hele-Shaw problem (cf. [15] and [17]).

Particular instances of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ have been studied in [9,5,3] and [1]. Let us describe their results in some detail. The work of Bénilan, Crandall and Sacks [9] was pioneer in this kind of problems. They study problem $\left(S_{\phi, 0}^{\gamma, \beta}\right)$ for any $\gamma$ and $\beta$ maximal monotone graphs in $\mathbb{R}^{2}$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, for the Laplacian operator, i.e., for $\mathbf{a}(x, \xi)=\xi$, and prove, between other results, that, for any $\phi \in L^{1}(\Omega)$ satisfying the range condition

$$
\inf \{\operatorname{Ran}(\gamma)\} \operatorname{meas}(\Omega)+\inf \{\operatorname{Ran}(\beta)\} \operatorname{meas}(\partial \Omega)<\int_{\Omega} \phi<\sup \{\operatorname{Ran}(\gamma)\} \operatorname{meas}(\Omega)+\sup \{\operatorname{Ran}(\beta)\} \operatorname{meas}(\partial \Omega)
$$

there exists a unique, up to a constant for $u$, named weak solution, $[u, z, w] \in W^{1,1}(\Omega) \times L^{1}(\Omega) \times L^{1}(\partial \Omega), z(x) \in$ $\gamma(u(x))$ a.e. in $\Omega, w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that

$$
\int_{\Omega} D u \cdot D v+\int_{\Omega} z v+\int_{\partial \Omega} w v=\int_{\Omega} \phi v
$$

for all $v \in W^{1, \infty}(\Omega)$. For nonhomogeneous boundary condition, i.e. $\psi \not \equiv 0$, one can see [18] for $\psi \in \operatorname{Ran}(\beta)$, and [19,20] for some particular situations of $\beta$ and $\gamma$.

Another important work in the $L^{1}$-theory for $p$-Laplacian type equations is [5], where problem

$$
\left(D_{\phi}^{\gamma}\right) \quad \begin{cases}-\operatorname{div} \mathbf{a}(x, D u)+\gamma(u) \ni \phi & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

is studied for any $\gamma$ maximal monotone graph in $\mathbb{R}^{2}$ such that $0 \in \gamma(0)$. It is proved that, for any $\phi \in L^{1}(\Omega)$, there exists a unique, named entropy solution, $[u, z] \in \mathcal{T}_{0}^{1, p}(\Omega) \times L^{1}(\Omega), z(x) \in \gamma(u(x))$ a.e. in $\Omega$, such that

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}(\cdot, D u) \cdot D T_{k}(u-v)+\int_{\Omega} z T_{k}(u-v) \leqslant \int_{\Omega} \phi T_{k}(u-v) \quad \forall k>0 \tag{1}
\end{equation*}
$$

for all $v \in L^{\infty}(\Omega) \cap W_{0}^{1, p}(\Omega)$ (see Section 2 for the definition of $\mathcal{T}_{0}^{1, p}(\Omega)$ ). Following [5], problems $\left(S_{\phi, 0}^{\mathrm{id}, \beta}\right.$ ) and $\left(S_{\phi, \psi}^{\mathrm{id}, \beta}\right)$, where $\mathrm{id}(r)=r$ for all $r \in \mathbb{R}$, are studied in [3] and [1] respectively, for any $\beta$ maximal monotone graph in $\mathbb{R}^{2}$ with closed domain such that $0 \in \beta(0)$. It is proved that, for any $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$, there exists a unique $u \in \mathcal{T}_{\text {tr }}^{1, p}(\Omega)$, and there exists $w \in L^{1}(\partial \Omega), w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that

$$
\int_{\Omega} \mathbf{a}(\cdot, D u) \cdot D T_{k}(u-v)+\int_{\Omega} u T_{k}(u-v)+\int_{\partial \Omega} w T_{k}(u-v) \leqslant \int_{\partial \Omega} \psi T_{k}(u-v)+\int_{\Omega} \phi T_{k}(u-v) \quad \forall k>0
$$

for all $v \in L^{\infty}(\Omega) \cap W^{1, p}(\Omega), v(x) \in \beta(u(x))$ a.e. in $\partial \Omega$.
Our aim is to prove existence and uniqueness of weak and entropy solutions for the general elliptic problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. The main interest in our work is that we are dealing with general nonlinear operators - div $\mathbf{a}(x, D u)$ with nonhomogeneous boundary conditions and general nonlinearities $\beta$ and $\gamma$. As in [9], a range condition relating the average of $\phi$ and $\psi$ to the range of $\beta$ and $\gamma$ is necessary for existence of weak solution and entropy solution (see Remark 3.3). However, in contrast to the smooth homogeneous case, a smooth and $\psi=0$, for the nonhomogeneous case this range condition is not sufficient for the existence of weak solution. Indeed, in general, the intersection of the domains of $\beta$ and $\gamma$ seems to create some obstruction phenomena for the existence of these solutions. In general, even if $D(\beta)=\mathbb{R}$, it does not exist weak solution, as the following example shows. Let $\gamma$ be such that $D(\gamma)=[0,1]$, $\beta=\mathbb{R} \times\{0\}$, and let $\phi \in L^{1}(\Omega), \phi \leqslant 0$ a.e. in $\Omega$, and $\psi \in L^{1}(\partial \Omega), \psi \leqslant 0$ a.e. in $\partial \Omega$. If there exists $[u, z, w]$ weak solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ (see Definition 3.1), then $z \in \gamma(u)$, therefore $0 \leqslant u \leqslant 1$ a.e. in $\Omega, w=0$, and it holds that for any $v \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$,

$$
\int_{\Omega} \mathbf{a}(x, D u) D v+\int_{\Omega} z v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v
$$

Taking $v=u$, as $u \geqslant 0$, we get

$$
0 \leqslant \int_{\Omega} \mathbf{a}(x, D u) D u+\int_{\Omega} z u=\int_{\partial \Omega} \psi u+\int_{\Omega} \phi u \leqslant 0
$$

Therefore, we obtain that $\int_{\Omega}|D u|^{p}=0$, so $u$ is constant and

$$
\int_{\Omega} z v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v
$$

for any $v \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$, and in particular, for any $v \in W_{0}^{1, p}(\Omega) \cap L^{\infty}(\Omega)$. Consequently, $\phi=z$ a.e. in $\Omega$, and $\psi$ must be 0 a.e. in $\partial \Omega$.

The main applications we have in mind is the study of doubly nonlinear evolution problems of elliptic-parabolic type and degenerate parabolic problems of Stefan or Hele-Shaw type, with nonhomogeneous boundary conditions and/or dynamical boundary conditions (see [2]). Notice that in all these applications one has $D(\gamma)=\mathbb{R}$, which is sufficiently covered in this paper.

The results we obtain have an interpretation in terms of accretive operators. Indeed, we can define the (possibly multivalued) operator $\mathcal{B}^{\gamma, \beta}$ in $X:=L^{1}(\Omega) \times L^{1}(\partial \Omega)$ as

$$
\mathcal{B}^{\gamma, \beta}:=\left\{((v, w),(\hat{v}, \widehat{w})) \in X \times X: \exists u \in \mathcal{T}_{\text {tr }}^{1, p}(\Omega), \text { with }[u, v, w] \text { an entropy solution of }\left(S_{v+\hat{v}, w+\widehat{w}}^{\gamma, \beta}\right)\right\} .
$$

Then, under certain assumptions, $\mathcal{B}^{\gamma, \beta}$ is an $m-\mathrm{T}$-accretive operator in $X$. Therefore, by the theory of evolution equations governed by accretive operators (see, [4,8] or [13]), for any $\left(v_{0}, w_{0}\right) \in \overline{D\left(\mathcal{B}^{\gamma, \beta}\right)^{X}}$ and any $(f, g) \in$ $L^{1}\left(0, T ; L^{1}(\Omega)\right) \times L^{1}\left(0, T ; L^{1}(\partial \Omega)\right)$, there exists a unique mild-solution of the problem

$$
V^{\prime}+\mathcal{B}^{\gamma, \beta}(V) \ni(f, g), \quad V(0)=\left(v_{0}, w_{0}\right)
$$

which rewrites, as an abstract Cauchy problem in $X$, the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

$$
D P(\gamma, \beta) \quad\left\{\begin{array}{l}
v_{t}-\operatorname{div} \mathbf{a}(x, D u)=f, \quad v \in \gamma(u), \text { in } \Omega \times(0, T) \\
w_{t}+\mathbf{a}(x, D u) \cdot \eta=g, \quad w \in \beta(u), \text { on } \partial \Omega \times(0, T) \\
v(0)=v_{0} \quad \text { in } \Omega, \quad w(0)=w_{0} \quad \text { in } \partial \Omega
\end{array}\right.
$$

In principle, it is not clear how these mild solutions have to be interpreted respect to the problem $D P(\gamma, \beta)$. In a next paper [2] we characterize these mild solutions.

Let us briefly summarize the contents of the paper. In Section 2 we fix the notation and give some preliminaries. In Section 3 we give the definitions of the different concepts of solution we use and state the main results. The next section is devoted to prove the uniqueness results. In the last section we prove the existence results. First, we study the existence of solutions of approximated problems, next we prove the existence of weak solutions for data in $L^{p^{\prime}}$ and finally the existence of entropy solutions for data in $L^{1}$.

## 2. Preliminaries

For $1 \leqslant p<+\infty, L^{p}(\Omega)$ and $W^{1, p}(\Omega)$ denote respectively the standard Lebesgue space and Sobolev space, and $W_{0}^{1, p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1, p}(\Omega)$. For $u \in W^{1, p}(\Omega)$, we denote by $u$ or $\tau(u)$ the trace of $u$ on $\partial \Omega$ in the usual sense and by $W^{1 / p^{\prime}, p}(\partial \Omega)$ the set $\tau\left(W^{1, p}(\Omega)\right)$. Recall that $\operatorname{Ker}(\tau)=W_{0}^{1, p}(\Omega)$.

In [5], the authors introduce the set

$$
\mathcal{T}^{1, p}(\Omega)=\left\{u: \Omega \rightarrow \mathbb{R} \text { measurable such that } T_{k}(u) \in W^{1, p}(\Omega) \forall k>0\right\},
$$

where $T_{k}(s)=\sup (-k, \inf (s, k))$. They also prove that given $u \in \mathcal{T}^{1, p}(\Omega)$, there exists a unique measurable function $v: \Omega \rightarrow \mathbb{R}^{N}$ such that

$$
D T_{k}(u)=v \chi_{\{|v|<k\}} \quad \forall k>0 .
$$

This function $v$ will be denoted by $D u$. It is clear that if $u \in W^{1, p}(\Omega)$, then $v \in L^{p}(\Omega)$ and $v=D u$ in the usual sense.

As in [3], $\mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega)$ denotes the set of functions $u$ in $\mathcal{T}^{1, p}(\Omega)$ satisfying the following conditions, there exists a sequence $u_{n}$ in $W^{1, p}(\Omega)$ such that
(a) $u_{n}$ converges to $u$ a.e. in $\Omega$,
(b) $D T_{k}\left(u_{n}\right)$ converges to $D T_{k}(u)$ in $L^{1}(\Omega)$ for all $k>0$,
(c) there exists a measurable function $v$ on $\partial \Omega$, such that $u_{n}$ converges to $v$ a.e. in $\partial \Omega$.

The function $v$ is the trace of $u$ in the generalized sense introduced in [3]. In the sequel, the trace of $u \in \mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega)$ on $\partial \Omega$ will be denoted by $\operatorname{tr}(u)$ or $u$. Let us recall that in the case where $u \in W^{1, p}(\Omega), \operatorname{tr}(u)$ coincides with the trace of $u, \tau(u)$, in the usual sense, and the space $\mathcal{T}_{0}^{1, p}(\Omega)$, introduced in [5] to study $\left(D_{\phi}^{\gamma}\right)$, is equal to $\operatorname{Ker}(\operatorname{tr})$. Moreover, for every $u \in \mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega)$ and every $k>0, \tau\left(T_{k}(u)\right)=T_{k}(\operatorname{tr}(u))$, and, if $\phi \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$, then $u-\phi \in \mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega)$ and $\operatorname{tr}(u-\phi)=\operatorname{tr}(u)-\tau(\phi)$.

We denote

$$
V^{1, p}(\Omega):=\left\{\phi \in L^{1}(\Omega): \exists M>0 \text { such that } \int_{\Omega}|\phi v| \leqslant M\|v\|_{W^{1, p}(\Omega)} \forall v \in W^{1, p}(\Omega)\right\}
$$

and

$$
V^{1, p}(\partial \Omega):=\left\{\psi \in L^{1}(\partial \Omega): \exists M>0 \text { such that } \int_{\partial \Omega}|\psi v| \leqslant M\|v\|_{W^{1, p}(\Omega)} \forall v \in W^{1, p}(\Omega)\right\} .
$$

$V^{1, p}(\Omega)$ is a Banach space endowed with the norm

$$
\|\phi\|_{V^{1, p}(\Omega)}:=\inf \left\{M>0: \int_{\Omega}|\phi v| \leqslant M\|v\|_{W^{1, p}(\Omega)} \forall v \in W^{1, p}(\Omega)\right\},
$$

and $V^{1, p}(\partial \Omega)$ is a Banach space endowed with the norm

$$
\|\psi\|_{V^{1, p}(\partial \Omega)}:=\inf \left\{M>0: \int_{\partial \Omega}|\psi v| \leqslant M\|v\|_{W^{1, p}(\Omega)} \forall v \in W^{1, p}(\Omega)\right\} .
$$

Observe that, Sobolev embeddings and trace theorems imply, for $1 \leqslant p<N$,

$$
L^{p^{\prime}}(\Omega) \subset L^{(N p /(N-p))^{\prime}}(\Omega) \subset V^{1, p}(\Omega)
$$

and

$$
L^{p^{\prime}}(\partial \Omega) \subset L^{((N-1) p /(N-p))^{\prime}}(\partial \Omega) \subset V^{1, p}(\partial \Omega)
$$

Also,

$$
\begin{aligned}
& V^{1, p}(\Omega)=L^{1}(\Omega) \quad \text { and } \quad V^{1, p}(\partial \Omega)=L^{1}(\partial \Omega) \quad \text { when } p>N, \\
& L^{q}(\Omega) \subset V^{1, N}(\Omega) \quad \text { and } \quad L^{q}(\partial \Omega) \subset V^{1, N}(\partial \Omega) \quad \text { for any } q>1 .
\end{aligned}
$$

We say that a is smooth (see [3]) when, for any $\phi \in L^{\infty}(\Omega)$ such that there exists a bounded weak solution $u$ of the homogeneous Dirichlet problem
(D) $\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega,\end{cases}$ there exists $\psi \in L^{1}(\partial \Omega)$ such that $u$ is also a weak solution of the Neumann problem
(N) $\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega, \\ \mathbf{a}(x, D u) \cdot \eta=\psi & \text { on } \partial \Omega .\end{cases}$

Functions a corresponding to linear operators with smooth coefficients and $p$-Laplacian type operators are smooth (see [11] and [22]). The smoothness of the Laplacian operator is even stronger than this, in fact, there is a bounded linear mapping $T: L^{1}(\Omega) \rightarrow L^{1}(\partial \Omega)$, such that the weak solution of (D) for $\phi \in L^{1}(\Omega)$ is also a weak solution of (N) for $\psi=T(\phi)$ (see [9]).

For a maximal monotone graph $\gamma$ in $\mathbb{R} \times \mathbb{R}$ and $r \in \mathbb{N}$ we denote by $\gamma_{r}$ the Yosida approximation of $\gamma$, given by $\gamma_{r}=r\left(I-\left(I+\frac{1}{r} \gamma\right)^{-1}\right)$. The function $\gamma_{r}$ is maximal monotone and Lipschitz. We recall the definition of the main section $\gamma^{0}$ of $\gamma$

$$
\gamma^{0}(s):= \begin{cases}\text { the element of minimal absolute } \\ +\infty & \text { if }[s,+\infty) \cap D(\gamma)=\emptyset \\ -\infty & \text { if }(-\infty, s] \cap D(\gamma)=\emptyset\end{cases}
$$

If $s \in D(\gamma),\left|\gamma_{r}(s)\right| \leqslant\left|\gamma^{0}(s)\right|$ and $\gamma_{r}(s) \rightarrow \gamma^{0}(s)$ as $r \rightarrow+\infty$, and if $s \notin D(\gamma),\left|\gamma_{r}(s)\right| \rightarrow+\infty$ as $r \rightarrow+\infty$.
We will denote by $P_{0}$ the following set of functions,

$$
P_{0}=\left\{q \in C^{\infty}(\mathbb{R}): 0 \leqslant q^{\prime} \leqslant 1, \operatorname{supp}\left(q^{\prime}\right) \text { is compact, and } 0 \notin \operatorname{supp}(q)\right\} .
$$

In [7] the following relation for $u, v \in L^{1}(\Omega)$ is defined,

$$
u \ll v \quad \text { if } \int_{\Omega}(u-k)^{+} \leqslant \int_{\Omega}(v-k)^{+} \quad \text { and } \quad \int_{\Omega}(u+k)^{-} \leqslant \int_{\Omega}(v+k)^{-} \quad \text { for any } k>0,
$$

and the following facts are proved.
Proposition 2.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^{N}$.
(i) For any $u, v \in L^{1}(\Omega)$, if $\int_{\Omega} u q(u) \leqslant \int_{\Omega} v q(u)$ for all $q \in P_{0}$, then $u \ll v$.
(ii) If $u, v \in L^{1}(\Omega)$ and $u \ll v$, then $\|u\|_{p} \leqslant\|v\|_{p}$ for any $p \in[1,+\infty]$.
(iii) If $v \in L^{1}(\Omega)$, then $\left\{u \in L^{1}(\Omega): u \ll v\right\}$ is a weakly compact subset of $L^{1}(\Omega)$.

## 3. The main results

In this section we give the different concepts of solutions we use and state the main results.
Definition 3.1. Let $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$. A triple of functions $[u, z, w] \in W^{1, p}(\Omega) \times L^{1}(\Omega) \times L^{1}(\partial \Omega)$ is a weak solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ if $z(x) \in \gamma(u(x))$ a.e. in $\Omega, w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, and

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}(x, D u) \cdot D v+\int_{\Omega} z v+\int_{\partial \Omega} w v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v, \tag{2}
\end{equation*}
$$

for all $v \in L^{\infty}(\Omega) \cap W^{1, p}(\Omega)$.

In general, as it is remarked in [5], for $1<p \leqslant 2-\frac{1}{N}$, there exists $f \in L^{1}(\Omega)$ such that the problem

$$
u \in W_{\mathrm{loc}}^{1,1}(\Omega), \quad u-\Delta_{p}(u)=f \quad \text { in } \mathcal{D}^{\prime}(\Omega),
$$

has no solution. In [5], to overcome this difficulty and to get uniqueness, it was introduced a new concept of solution, named entropy solution. As in [3] or [1], following these ideas, we introduce the following concept of solution.

Definition 3.2. Let $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$. A triple of functions $[u, z, w] \in \mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega) \times L^{1}(\Omega) \times L^{1}(\partial \Omega)$ is an entropy solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ if $z(x) \in \gamma(u(x))$ a.e. in $\Omega, w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$ and

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}(x, D u) \cdot D T_{k}(u-v)+\int_{\Omega} z T_{k}(u-v)+\int_{\partial \Omega} w T_{k}(u-v) \leqslant \int_{\partial \Omega} \psi T_{k}(u-v)+\int_{\Omega} \phi T_{k}(u-v) \quad \forall k>0, \tag{3}
\end{equation*}
$$

for all $v \in L^{\infty}(\Omega) \cap W^{1, p}(\Omega)$.
Obviously, every weak solution is an entropy solution and an entropy solution with $u \in W^{1, p}(\Omega)$ is a weak solution.
Remark 3.3. If we take $v=T_{h}(u) \pm 1$ as test functions in (3) and let $h$ go to $+\infty$, we get that

$$
\int_{\Omega} z+\int_{\partial \Omega} w=\int_{\partial \Omega} \psi+\int_{\Omega} \phi
$$

Then necessarily $\phi$ and $\psi$ must satisfy

$$
\mathcal{R}_{\gamma, \beta}^{-} \leqslant \int_{\partial \Omega} \psi+\int_{\Omega} \phi \leqslant \mathcal{R}_{\gamma, \beta}^{+}
$$

where

$$
\mathcal{R}_{\gamma, \beta}^{+}:=\sup \{\operatorname{Ran}(\gamma)\} \operatorname{meas}(\Omega)+\sup \{\operatorname{Ran}(\beta)\} \operatorname{meas}(\partial \Omega)
$$

and

$$
\mathcal{R}_{\gamma, \beta}^{-}:=\inf \{\operatorname{Ran}(\gamma)\} \operatorname{meas}(\Omega)+\inf \{\operatorname{Ran}(\beta)\} \operatorname{meas}(\partial \Omega)
$$

We will write $\left.\mathcal{R}_{\gamma, \beta}:=\right] \mathcal{R}_{\gamma, \beta}^{-}, \mathcal{R}_{\gamma, \beta}^{+}\left[\right.$when $\mathcal{R}_{\gamma, \beta}^{-}<\mathcal{R}_{\gamma, \beta}^{+}$.
Remark 3.4. Let $\phi \in V^{1, p}(\Omega)$ and $\psi \in V^{1, p}(\partial \Omega)$. Then, if $[u, z, w]$ is a weak solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$, it is easy to see that

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D u+\int_{\Omega} z u+\int_{\partial \Omega} w u=\int_{\partial \Omega} \psi u+\int_{\Omega} \phi u .
$$

Moreover, if $D(\gamma) \neq\{0\}$ and $D(\beta) \neq\{0\}$, it follows that $z \in V^{1, p}(\Omega), w \in V^{1, p}(\partial \Omega)$ and

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D v+\int_{\Omega} z v+\int_{\partial \Omega} w v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v,
$$

for any $v \in W^{1, p}(\Omega)$.
In fact, let $v \in W^{1, p}(\Omega)$ and take $T_{k}(|v|) \frac{1}{r} T_{r}(u)$ as test function in (2). Then, letting $r$ go to 0 , there exists $M_{1}>0$ such that

$$
\int_{\{x \in \Omega: u(x) \neq 0\}}|z| T_{k}(|v|)+\int_{\{x \in \partial \Omega: u(x) \neq 0\}}|w| T_{k}(|v|) \leqslant M_{1}\|v\|_{W^{1, p}(\Omega)} .
$$

Letting now $k$ go to $+\infty$, applying Fatou's Lemma, we get

$$
\int_{\{x \in \Omega: u(x) \neq 0\}}|z||v|+\int_{\{x \in \partial \Omega: u(x) \neq 0\}}|w||v| \leqslant M_{1}\|v\|_{W^{1, p}(\Omega)} .
$$

If $\beta(0)$ is bounded, there exists $M_{2}>0$ such that

$$
\int_{\{x \in \partial \Omega: u(x)=0\}}|w||v| \leqslant M_{2}\|v\|_{W^{1, p}(\Omega)} .
$$

In the case $\beta(0)$ is unbounded from above (a similar argument can be done in the case of being unbounded from below) let us take $T_{k}(|v|) S_{r}(u)$ as test function in (2), where $S_{r}(s):=\frac{s+r}{r} \chi_{[-r, 0]}(s)+\chi_{[0,+\infty[ }(s)$, then, letting $r$ go to 0 , there exists $M_{2}>0$ such that

$$
\int_{\{x \in \partial \Omega: u(x)=0\}} w T_{k}(|v|) \leqslant M_{2}\|v\|_{W^{1, p}(\Omega)},
$$

and consequently, since $\beta(0)$ must be bounded from below (because $D(\beta) \neq\{0\}$ ), there exists $M_{3}>0$ such that

$$
\int_{\{x \in \partial \Omega: u(x)=0\}}|w| T_{k}(|v|) \leqslant M_{3}\|v\|_{W^{1, p}(\Omega)} .
$$

Letting now $k$ go to $+\infty$, applying Fatou's Lemma, we get

$$
\int_{\{x \in \partial \Omega: u(x)=0\}}|w||v| \leqslant M_{4}\|v\|_{W^{1, p}(\Omega)} .
$$

Similarly, there exists $M_{5}>0$ such that

$$
\int_{\{x \in \Omega: u(x)=0\}}\left|z\left\|v \mid \leqslant M_{5}\right\| v \|_{W^{1, p}(\Omega)} .\right.
$$

We shall state now the uniqueness result of entropy solutions. Since every weak solution is an entropy solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$, the same result is true for weak solutions.

Theorem 3.5. Let $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$, and let $\left[u_{1}, z_{1}, w_{1}\right]$ and $\left[u_{2}, z_{2}, w_{2}\right]$ be entropy solutions of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. Then, there exists a constant $c \in \mathbb{R}$ such that

$$
\begin{array}{ll}
u_{1}-u_{2}=c & \text { a.e. in } \Omega, \\
z_{1}-z_{2}=0 & \text { a.e. in } \Omega . \\
w_{1}-w_{2}=0 & \text { a.e. in } \partial \Omega .
\end{array}
$$

Moreover, if $c \neq 0$, there exists a constant $k \in \mathbb{R}$ such that $z_{1}=z_{2}=k$.
Respect to the existence of weak solutions we obtain the following results.
Theorem 3.6. Assume $D(\gamma)=\mathbb{R}$ and $\mathcal{R}_{\gamma, \beta}^{-}<\mathcal{R}_{\gamma, \beta}^{+}$. Let $D(\beta)=\mathbb{R}$ or a smooth.
(i) For any $\phi \in V^{1, p}(\Omega)$ and $\psi \in V^{1, p}(\partial \Omega)$ with

$$
\begin{equation*}
\int_{\Omega} \phi+\int_{\partial \Omega} \psi \in \mathcal{R}_{\gamma, \beta}, \tag{4}
\end{equation*}
$$

there exists a weak solution $[u, z, w]$ of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$.
(ii) For any $\left[u_{1}, z_{1}, w_{1}\right]$ weak solution of problem $\left(S_{\phi_{1}, \psi_{1}}^{\gamma, \beta}\right), \phi_{1} \in V^{1, p}(\Omega)$ and $\psi_{1} \in V^{1, p}(\partial \Omega)$ satisfying (4), and any $\left[u_{2}, z_{2}, w_{2}\right]$ weak solution of problem $\left(S_{\phi_{2}, \psi_{2}}^{\gamma, \beta}\right), \phi_{2} \in V^{1, p}(\Omega)$ and $\psi_{2} \in V^{1, p}(\partial \Omega)$ satisfying (4), we have that

$$
\int_{\Omega}\left(z_{1}-z_{2}\right)^{+}+\int_{\partial \Omega}\left(w_{1}-w_{2}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

In the case $\mathcal{R}_{\gamma, \beta}^{-}=\mathcal{R}_{\gamma, \beta}^{+}$, that is, when $\gamma(r)=\beta(r)=0$ for any $r \in \mathbb{R}$, existence and uniqueness of weak solutions are also obtained.

Theorem 3.7. For any $\phi \in V^{1, p}(\Omega)$ and $\psi \in V^{1, p}(\partial \Omega)$ with

$$
\begin{equation*}
\int_{\Omega} \phi+\int_{\partial \Omega} \psi=0 \tag{5}
\end{equation*}
$$

there exists a unique (up to a constant) weak solution $u \in W^{1, p}(\Omega)$ of the problem

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega, \\ \mathbf{a}(x, D u) \cdot \eta=\psi & \text { on } \partial \Omega\end{cases}
$$

in the sense that

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v,
$$

for all $v \in W^{1, p}(\Omega)$.
In the line of Proposition C(iv) of [9] given for the Laplacian operator, as a consequence of Theorem 3.6 we have the following result.

Corollary 3.8. a is smooth if and only if for any $\phi \in V^{1, p}(\Omega)$ there exists $T(\phi) \in V^{1, p}(\partial \Omega)$ such that the weak solution $u$ of

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

is a weak solution of

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega, \\ \mathbf{a}(x, D u) \cdot \eta=T(\phi) & \text { on } \partial \Omega .\end{cases}
$$

Moreover, the map $T: V^{1, p}(\Omega) \rightarrow V^{1 . p}(\partial \Omega)$ satisfies

$$
\int_{\Omega}\left(T\left(\phi_{1}\right)-T\left(\phi_{2}\right)\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+},
$$

for all $\phi_{1}, \phi_{2} \in V^{1, p}(\Omega)$.
In the case $\psi=0$ we have the following result without imposing any condition on $\gamma$, in the same line to the one obtained by Bénilan, Crandall and Sack in [9] for the Laplacian operator and $L^{1}$-data.

Theorem 3.9. Assume $D(\beta)=\mathbb{R}$ or a smooth. Let $\mathcal{R}_{\gamma, \beta}^{-}<\mathcal{R}_{\gamma, \beta}^{+}$.
(i) For any $\phi \in V^{1, p}(\Omega)$ such that $\int_{\Omega} \phi \in \mathcal{R}_{\gamma, \beta}$, there exists a weak solution $[u, z, w]$ of problem $\left(S_{\phi, 0}^{\gamma, \beta}\right)$, with $z \ll \phi$.
(ii) For any $\left[u_{1}, z_{1}, w_{1}\right]$ weak solution of problem $\left(S_{\phi_{1}, 0}^{\gamma, \beta}\right), \phi_{1} \in V^{1, p}(\Omega), \int_{\Omega} \phi_{1} \in \mathcal{R}_{\gamma, \beta}$, and any $\left[u_{2}, z_{2}, w_{2}\right]$ weak solution of problem $\left(S_{\phi_{2}, 0}^{\gamma, \beta}\right), \phi_{2} \in V^{1, p}(\Omega), \int_{\Omega} \phi_{2} \in \mathcal{R}_{\gamma, \beta}$, we have that

$$
\int_{\Omega}\left(z_{1}-z_{2}\right)^{+}+\int_{\partial \Omega}\left(w_{1}-w_{2}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

For Dirichlet boundary condition we have the following result.

Theorem 3.10. Assume $D(\beta)=\{0\}$. For any $\phi \in V^{1, p}(\Omega)$, there exists a unique $[u, z]=\left[u_{\phi, \psi}, z_{\phi, \psi}\right] \in W_{0}^{1, p}(\Omega) \times$ $V^{1, p}(\Omega), z \in \gamma(u)$ a.e. in $\Omega$, such that

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D v+\int_{\Omega} z v=\int_{\Omega} \phi v,
$$

for all $v \in W_{0}^{1, p}(\Omega)$.
Moreover, if $\phi_{1}, \phi_{2} \in V^{1, p}(\Omega)$, then

$$
\begin{equation*}
\int_{\Omega}\left(z_{\phi_{1}, \psi_{1}}-z_{\phi_{2}, \psi_{2}}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} \tag{6}
\end{equation*}
$$

Let us now state the existence results of entropy solutions for data in $L^{1}$.
Theorem 3.11. Assume $D(\gamma)=\mathbb{R}$, and $D(\beta)=\mathbb{R}$ or $\mathbf{a}$ smooth. Let also assume that, if $[0,+\infty[\subset D(\beta)$,

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} \gamma^{0}(k)=+\infty \quad \text { and } \quad \lim _{k \rightarrow+\infty} \beta^{0}(k)=+\infty \tag{7}
\end{equation*}
$$

and if $]-\infty, 0] \subset D(\beta)$,

$$
\begin{equation*}
\lim _{k \rightarrow-\infty} \gamma^{0}(k)=-\infty \quad \text { and } \quad \lim _{k \rightarrow-\infty} \beta^{0}(k)=-\infty \tag{8}
\end{equation*}
$$

Then,
(i) for any $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$, there exists an entropy solution $[u, z, w]$ of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$.
(ii) For any $\left[u_{1}, z_{1}, w_{1}\right]$ entropy solution of problem $\left(S_{\phi_{1}, \psi_{1}}^{\gamma, \beta}\right), \phi_{1} \in L^{1}(\Omega), \psi_{1} \in L^{1}(\partial \Omega)$, and any $\left[u_{2}, z_{2}, w_{2}\right]$ entropy solution of problem $\left(S_{\phi_{2}, \psi_{2}}^{\gamma, \beta}\right), \phi_{2} \in L^{1}(\Omega), \psi_{2} \in L^{1}(\partial \Omega)$, we have that

$$
\int_{\Omega}\left(z_{1}-z_{2}\right)^{+}+\int_{\partial \Omega}\left(w_{1}-w_{2}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Taking into account Theorem 3.11 and Corollary 3.8 we have the following result.
Corollary 3.12. a is smooth if and only if for any $\phi \in L^{1}(\Omega)$ there exists $T(\phi) \in L^{1}(\partial \Omega)$ such that the entropy solution $u$ of

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

is an entropy solution of

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)=\phi & \text { in } \Omega, \\ \mathbf{a}(x, D u) \cdot \eta=T(\phi) & \text { on } \partial \Omega\end{cases}
$$

Moreover, the map $T: L^{1}(\Omega) \rightarrow L^{1}(\partial \Omega)$ satisfies

$$
\int_{\Omega}\left(T\left(\phi_{1}\right)-T\left(\phi_{2}\right)\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+},
$$

for all $\phi_{1}, \phi_{2} \in L^{1}(\Omega)$, and $T\left(V^{1, p}(\Omega)\right) \subset V^{1, p}(\partial \Omega)$.
In the homogeneous case without any condition on $\gamma$ we also obtain the following result.
Theorem 3.13. Assume $D(\beta)=\mathbb{R}$ or $\mathbf{a}$ is smooth. Let also assume that, if $[0,+\infty[\subset D(\gamma) \cap D(\beta)$ the assumption (7) holds, and, if $]-\infty, 0] \subset D(\gamma) \cap D(\beta)$ the assumption (8) holds. Then,
(i) for any $\phi \in L^{1}(\Omega)$, there exists an entropy solution $[u, z, w]$ of problem $\left(S_{\phi, 0}^{\gamma, \beta}\right)$, with $z \ll \phi$.
(ii) For any $\left[u_{1}, z_{1}, w_{1}\right]$ entropy solution of problem $\left(S_{\phi_{1}, 0}^{\gamma, \beta}\right), \phi_{1} \in L^{1}(\Omega)$, and any $\left[u_{2}, z_{2}, w_{2}\right]$ entropy solution of problem $\left(S_{\phi_{2}, 0}^{\gamma, \beta}\right), \phi_{2} \in L^{1}(\Omega)$, we have that

$$
\int_{\Omega}\left(z_{1}-z_{2}\right)^{+}+\int_{\partial \Omega}\left(w_{1}-w_{2}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+}
$$

As we mention in Remark 5.9, different conditions to (7) and (8) can be used in order to get Theorems 3.11 and 3.13 .

We also obtain the following result given by Bénilan et al. in [5] for Dirichlet boundary condition.
Theorem 3.14. Assume $D(\beta)=\{0\}$. For any $\phi \in L^{1}(\Omega)$, there exists a unique entropy solution $[u, z]$ of

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D u)+\gamma(u) \ni \phi & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

in the sense given by Bénilan et al. in [5] (see (1) in the Introduction).

## 4. Proof of the uniqueness result

This section deals with the proof of the uniqueness result Theorem 3.5. We firstly need the following lemma.
Lemma 4.1. Let $[u, z, w]$ be an entropy solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. Then, for all $h>0$,

$$
\lambda \int_{\{h<|u|<h+k\}}|D u|^{p} \leqslant k \int_{\partial \Omega \cap\{|u| \geqslant h\}}|\psi|+k \int_{\Omega \cap\{|u| \geqslant h\}}|\phi| .
$$

Proof. Taking $T_{h}(u)$ as test function in (3), we have

$$
\begin{aligned}
& \int_{\Omega} \mathbf{a}(x, D u) \cdot D T_{k}\left(u-T_{h}(u)\right)+\int_{\Omega} z T_{k}\left(u-T_{h}(u)\right)+\int_{\partial \Omega} w T_{k}\left(u-T_{h}(u)\right) \\
& \quad \leqslant \int_{\partial \Omega} \psi T_{k}\left(u-T_{h}(u)\right)+\int_{\Omega} \phi T_{k}\left(u-T_{h}(u)\right) .
\end{aligned}
$$

Now, using $\left(\mathrm{H}_{1}\right)$ and the positivity of the second and third terms, it follows that

$$
\lambda \int_{\{h<|u|<h+k\}}|D u|^{p} \leqslant k \int_{\partial \Omega \cap\{|u| \geqslant h\}}|\psi|+k \int_{\Omega \cap\{|u| \geqslant h\}}|\phi| .
$$

Proof of Theorem 3.5. Let $\left[u_{1}, z_{1}, w_{1}\right]$ and $\left[u_{2}, z_{2}, w_{2}\right]$ be entropy solutions of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. For every $h>0$, we have that

$$
\begin{aligned}
& \int_{\Omega} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right)+\int_{\Omega} z_{1} T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right)+\int_{\partial \Omega} w_{1} T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right) \\
& \quad \leqslant \int_{\partial \Omega} \psi T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right)+\int_{\Omega} \phi T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\Omega} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right)+\int_{\Omega} z_{2} T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right)+\int_{\partial \Omega} w_{2} T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right) \\
& \quad \leqslant \int_{\partial \Omega} \psi T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right)+\int_{\Omega} \phi T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right)
\end{aligned}
$$

Adding both inequalities and taking limits when $h$ goes to $\infty$, on account of the monotonicity of $\gamma$ and $\beta$, if

$$
I_{h, k}:=\int_{\Omega} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-T_{h}\left(u_{2}\right)\right)+\int_{\Omega} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-T_{h}\left(u_{1}\right)\right)
$$

we get

$$
\begin{equation*}
\limsup _{h \rightarrow \infty} I_{h, k} \leqslant-\int_{\Omega}\left(z_{1}-z_{2}\right) T_{k}\left(u_{1}-u_{2}\right)-\int_{\partial \Omega}\left(w_{1}-w_{2}\right) T_{k}\left(u_{1}-u_{2}\right) \leqslant 0 \tag{9}
\end{equation*}
$$

Let us see that

$$
\begin{equation*}
\liminf _{h \rightarrow \infty} I_{h, k} \geqslant 0 \quad \text { for any } k \tag{10}
\end{equation*}
$$

To prove this, we split

$$
I_{h, k}=I_{h, k}^{1}+I_{h, k}^{2}+I_{h, k}^{3}+I_{h, k}^{4}
$$

where

$$
\begin{aligned}
I_{h, k}^{1} & :=\int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right|<h\right\}}\left(\mathbf{a}\left(x, D u_{1}\right)-\mathbf{a}\left(x, D u_{2}\right)\right) \cdot D T_{k}\left(u_{1}-u_{2}\right), \\
I_{h, k}^{2} & :=\int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-h \operatorname{sign}\left(u_{2}\right)\right)+\int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-u_{1}\right) \\
& \geqslant \int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-u_{1}\right), \\
I_{h, k}^{3} & :=\int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right|<h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-u_{2}\right)+\int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right|<h\right\}} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-h \operatorname{sign}\left(u_{1}\right)\right) \\
& \geqslant \int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right|<h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-u_{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{h, k}^{4}:= & \int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-h \operatorname{sign}\left(u_{2}\right)\right) \\
& +\int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-h \operatorname{sign}\left(u_{1}\right)\right) \geqslant 0 .
\end{aligned}
$$

Combining the above estimates we get

$$
\begin{equation*}
I_{h, k} \geqslant I_{h, k}^{1}+L_{h, k}^{1}+L_{h, k}^{2} \tag{11}
\end{equation*}
$$

where

$$
\begin{aligned}
L_{h, k}^{1} & :=\int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right| \geqslant h\right\}} \mathbf{a}\left(x, D u_{2}\right) \cdot D T_{k}\left(u_{2}-u_{1}\right), \\
L_{h, k}^{2} & :=\int_{\left\{\left|u_{1}\right| \geqslant h,\left|u_{2}\right|<h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot D T_{k}\left(u_{1}-u_{2}\right)
\end{aligned}
$$

and $I_{h, k}^{1}$ is nonnegative and nondecreasing in $h$. Now, if we set

$$
C(h, k):=\left\{h<\left|u_{1}\right|<k+h\right\} \cap\left\{h-k<\left|u_{2}\right|<h\right\},
$$

we have

$$
\begin{aligned}
\left|L_{h, k}^{2}\right| & \leqslant \int_{\left\{\left|u_{1}-u_{2}\right|<k,\left|u_{1}\right| \geqslant h,\left|u_{2}\right|<h\right\}}\left|\mathbf{a}\left(x, D u_{1}\right) \cdot\left(D u_{1}-D u_{2}\right)\right| \\
& \leqslant \int_{C(h, k)}\left|a\left(x, D u_{1}\right) \cdot D u_{1}\right|+\int_{C(h, k)}\left|\mathbf{a}\left(x, D u_{1}\right) \cdot D u_{2}\right| .
\end{aligned}
$$

Then, by Hölder's inequality, we get

$$
\left|L_{h, k}^{2}\right| \leqslant\left(\int_{C(h, k)}\left|\mathbf{a}\left(x, D u_{1}\right)\right|^{p^{\prime}}\right)^{1 / p^{\prime}}\left(\left(\int_{C(h, k)}\left|D u_{1}\right|^{p}\right)^{1 / p}+\left(\int_{C(h, k)}\left|D u_{2}\right|^{p}\right)^{1 / p}\right) .
$$

Now, by $\left(\mathrm{H}_{2}\right)$,

$$
\begin{aligned}
\left(\int_{C(h, k)}\left|\mathbf{a}\left(x, D u_{1}\right)\right|^{p^{\prime}}\right)^{1 / p^{\prime}} & \leqslant\left(\int_{C(h, k)} \sigma^{p^{\prime}}\left(\theta(x)+\left|D u_{1}\right|^{p-1}\right)^{p^{\prime}}\right)^{1 / p^{\prime}} \\
& \leqslant \sigma 2^{1 / p}\left(\|\theta\|_{p^{\prime}}^{p^{\prime}}+\int_{\left\{h<\left|u_{1}\right|<k+h\right\}}\left|D u_{1}\right|^{p}\right)^{1 / p^{\prime}}
\end{aligned}
$$

On the other hand, by Lemma 4.1, we obtain

$$
\int_{\left\{h<\left|u_{1}\right|<k+h\right\}}\left|D u_{1}\right|^{p} \leqslant \frac{k}{\lambda}\left(\int_{\left\{\left|u_{1}\right| \geqslant h\right\}}|\psi|+\int_{\left\{\left|u_{1}\right| \geqslant h\right\}}|\phi|\right)
$$

and

$$
\int_{\left\{h-k<\left|u_{2}\right|<h\right\}}\left|D u_{2}\right|^{p} \leqslant \frac{k}{\lambda}\left(\int_{\left\{\left|u_{2}\right| \geqslant h-k\right\}}|\psi|+\int_{\left\{\left|u_{2}\right| \geqslant h-k\right\}}|\phi|\right) .
$$

Then, since $\phi \in L^{1}(\Omega), \psi \in L^{1}(\partial \Omega)$ and having in mind that

$$
\lim _{r \rightarrow+\infty} \operatorname{meas}\left\{x \in \Omega:\left|u_{i}(x)\right| \geqslant r\right\}=0
$$

and

$$
\lim _{r \rightarrow+\infty} \operatorname{meas}\left\{x \in \partial \Omega:\left|u_{i}(x)\right| \geqslant r\right\}=0,
$$

since $u_{i} \in \mathcal{T}_{\text {tr }}^{1, p}(\Omega)$, we obtain that

$$
\lim _{h \rightarrow \infty} L_{h, k}^{2}=0
$$

Similarly, $\lim _{h \rightarrow \infty} L_{h, k}^{1}=0$. Therefore by (11), (10) holds. Now, from (10), (9) and (11), we have that

$$
\lim _{h \rightarrow+\infty} \int_{\left\{\left|u_{1}\right|<h,\left|u_{2}\right|<h\right\}}\left(\mathbf{a}\left(x, D u_{1}\right)-\mathbf{a}\left(x, D u_{2}\right)\right) \cdot D T_{k}\left(u_{1}-u_{2}\right)=0 .
$$

Therefore, for any $h>0, D T_{h}\left(u_{1}\right)=D T_{h}\left(u_{2}\right)$ a.e. in $\Omega$. Consequently, there exists a constant $c$ such that

$$
u_{1}-u_{2}=c \quad \text { a.e. in } \Omega .
$$

Moreover, by (9) and (10), we have

$$
\begin{equation*}
\int_{\Omega}\left(z_{1}-z_{2}\right) T_{k}\left(u_{1}-u_{2}\right)+\int_{\partial \Omega}\left(w_{1}-w_{2}\right) T_{k}\left(u_{1}-u_{2}\right)=0 \quad \forall k>0, \tag{12}
\end{equation*}
$$

from where it follows that

$$
\left(w_{1}-w_{2}\right) \chi_{\left\{u_{1}-u_{2} \neq 0\right\}}=0 \quad \text { a.e. in } \partial \Omega,
$$

and

$$
\left(z_{1}-z_{2}\right) \chi_{\left\{u_{1}-u_{2} \neq 0\right\}}=0 \quad \text { a.e. in } \Omega .
$$

Then, if $c \neq 0$ it follows that $w_{1}=w_{2}$, and $z_{1}=z_{2}$.
In order to see that $z_{1}=z_{2}$ in the case $c=0$, we take $T_{h}\left(u_{1}\right)-\varphi$ and $T_{h}\left(u_{1}\right)+\varphi, \varphi \in D(\Omega)$, as test functions in (3) for the solution $\left[u_{1}, z_{1}, w_{1}\right]$ and $\left[u_{1}, z_{2}, w_{2}\right]$, respectively, adding these inequalities and letting $h$ go to $+\infty$, if $k>\|\varphi\|_{\infty}$, we get

$$
\lim _{h \rightarrow \infty} J_{h, k}+\int_{\Omega}\left(z_{1}-z_{2}\right) \varphi \leqslant 0
$$

where

$$
\begin{aligned}
J_{h, k} & =\int_{\Omega} \mathbf{a}\left(x, D u_{1}\right) \cdot\left[D T_{k}\left(u_{1}-T_{h}\left(u_{1}\right)+\varphi\right)+D T_{k}\left(u_{1}-T_{h}\left(u_{1}\right)-\varphi\right)\right] \\
& =\int_{\left\{\left|u_{1}\right|>h\right\}} \mathbf{a}\left(x, D u_{1}\right) \cdot\left[D T_{k}\left(u_{1}-T_{h}\left(u_{1}\right)+\varphi\right)+D T_{k}\left(u_{1}-T_{h}\left(u_{1}\right)-\varphi\right)\right] .
\end{aligned}
$$

Then, using Hölder's inequality and Lemma 4.1, we obtain that

$$
\lim _{h \rightarrow \infty} J_{h, k}=0
$$

Hence

$$
\int_{\Omega} z_{1} \varphi \leqslant \int_{\Omega} z_{2} \varphi .
$$

Similarly,

$$
\int_{\Omega} z_{2} \varphi \leqslant \int_{\Omega} z_{1} \varphi .
$$

Therefore $z_{1}=z_{2}$.
If $c \neq 0$, following the arguments of Lemma 3.5 of [6], we have that $z_{1}=z_{2}$ is constant. In fact, let $j(r)=$ $\int_{0}^{r} \gamma^{0}(s) \mathrm{d} s$, therefore, $\gamma=\partial j$, the subdifferential of $j$. Now, $z_{1}(x) \in \gamma\left(u_{1}(x)\right) \cap \gamma\left(u_{1}(x)+c\right)$ a.e. $x \in \Omega$, consequently, $j\left(u_{1}(x)+c\right)-j\left(u_{1}(x)\right)=c z_{1}(x)$ a.e. in $\Omega$. Moreover, if $\gamma(\mathbb{R})$ is bounded, $j$ is Lipschitz continuous, $j\left(T_{k}\left(u_{1}\right)+c\right), j\left(T_{k}\left(u_{1}\right)\right) \in W^{1, p}(\Omega)$ and $\nabla\left(j\left(T_{k}\left(u_{1}\right)+c\right)-j\left(T_{k}\left(u_{1}\right)\right)\right)=0$ a.e. in $\Omega$. The above identity is obvious when $\left|u_{1}\right| \geqslant k$, and in the case $\left|u_{1}\right|<k$, we have $\nabla\left(j\left(u_{1}+c\right)-j\left(u_{1}\right)\right)=0$. Therefore $j\left(T_{k}\left(u_{1}\right)+c\right)-j\left(T_{k}\left(u_{1}\right)\right)$ is constant (this constant, in fact, does not depend on $k$ ) and consequently $c z_{1}$ is constant. As $c \neq 0, z_{1}$ is constant. In the case $\gamma$ is not bounded, we work, again as in Lemma 3.5 of [6], truncating $\gamma$.

Finally, in order to see that $w_{1}=w_{2}$, we use the fact that we can take as test function in (3), for the corresponding $\left(S_{\phi, \psi}^{\gamma, \beta}\right), v=T_{h}\left(u_{i}\right) \pm \varphi$, for any $\varphi \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$. Then, since $u_{1}=u_{2}+c$ and $z_{1}=z_{2}$, we get

$$
\int_{\partial \Omega} w_{1} \varphi=\int_{\partial \Omega} w_{2} \varphi .
$$

Therefore $w_{1}=w_{2}$.

## 5. Proofs of the existence results

In this section we give the proofs of the existence results. In order to get the existence of weak solutions, the main idea is to consider the approximated problem

$$
\left(S_{\phi_{m, n}, \psi_{m, n}}^{\gamma_{m, n}, \beta_{m, n}}\right) \quad \begin{cases}-\operatorname{div} \mathbf{a}(x, D u)+\gamma_{m, n}(u) \ni \phi_{m, n} & \text { in } \Omega, \\ \mathbf{a}(x, D u) \cdot \eta+\beta_{m, n}(u) \ni \psi_{m, n} & \text { on } \partial \Omega,\end{cases}
$$

where $\gamma_{m, n}$ and $\beta_{m, n}$ are approximations of $\gamma$ and $\beta$ given by

$$
\gamma_{m, n}(r)=\gamma(r)+\frac{1}{m} r^{+}-\frac{1}{n} r^{-}
$$

and

$$
\beta_{m, n}(r)=\beta(r)+\frac{1}{m} r^{+}-\frac{1}{n} r^{-}
$$

respectively, $m, n \in \mathbb{N}$, and we are approximating $\phi$ and $\psi$ by

$$
\phi_{m, n}=\sup \{\inf \{m, \phi\},-n\}
$$

and

$$
\psi_{m, n}=\sup \{\inf \{m, \psi\},-n\}
$$

respectively, $m, n \in \mathbb{N}$. For these approximated problems we obtain existence of weak solutions with appropriated estimates and monotone properties, which allow us to pass to the limit.

### 5.1. Approximated problems

Proposition 5.1. Assume $D(\gamma)=D(\beta)=\mathbb{R}$. Let $m, n \in \mathbb{N}, m \leqslant n$. Then, the following hold.
(i) For $\phi \in L^{\infty}(\Omega)$ and $\psi \in L^{\infty}(\partial \Omega)$, there exist $u=u_{\phi, \psi, m, n} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega), z=z_{\phi, \psi, m, n} \in L^{\infty}(\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in $\Omega$, and $w=w_{\phi, \psi, m, n} \in L^{\infty}(\partial \Omega), w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that $\left[u, z+\frac{1}{m} u^{+}{ }_{-}\right.$ $\left.\frac{1}{n} u^{-}, w+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}\right]$is a weak solution of $\left(S_{\phi, \psi}^{\gamma_{m, n}, \beta_{m, n}}\right)$.
Moreover, if $M:=\|\phi\|_{\infty}+\|\psi\|_{\infty}$,

$$
\begin{aligned}
& -n M \leqslant u \leqslant n M, \\
& -\gamma^{0}(-n M) \leqslant z \leqslant \gamma^{0}(n M),
\end{aligned}
$$

and there exists $c(\Omega, N, p)>0$ such that

$$
\|D u\|_{L^{p}(\Omega)}^{p-1} \leqslant \frac{c(\Omega, N, p)}{\lambda}\left(\|\phi\|_{V^{1, p}(\Omega)}+\|\psi\|_{V^{1, p}(\partial \Omega)}\right) .
$$

(ii) If $m_{1} \leqslant m_{2} \leqslant n_{2} \leqslant n_{1}, \phi_{1}, \phi_{2} \in L^{\infty}(\Omega), \psi_{1}, \psi_{2} \in L^{\infty}(\partial \Omega)$ then

$$
\int_{\Omega}\left(z_{\phi_{1}, \psi_{1}, m_{1}, n_{1}}-z_{\phi_{2}, \psi_{2}, m_{2}, n_{2}}\right)^{+}+\int_{\partial \Omega}\left(w_{\phi_{1}, \psi_{1}, m_{1}, n_{1}}-w_{\phi_{2}, \psi_{2}, m_{2}, n_{2}}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Proof. Observe that $\frac{1}{m} s^{+}-\frac{1}{n} s^{-}=\frac{1}{m} s+\left(\frac{1}{m}-\frac{1}{n}\right) s^{-}=\left(\frac{1}{m}-\frac{1}{n}\right) s^{+}+\frac{1}{n} s$.
Let us take

$$
c_{r}>\sup \left\{n M, \gamma_{r}(n M),-\gamma_{r}(-n M), \beta_{r}(n M),-\beta_{r}(-n M)\right\},
$$

where $\gamma_{r}$ and $\beta_{r}$ are the Yosida approximations of $\gamma$ and $\beta$, respectively. For $r \in \mathbb{N}$, it is easy to see that the operator $B_{r}: W^{1, p}(\Omega) \rightarrow\left(W^{1, p}(\Omega)\right)^{\prime}$ defined by

$$
\begin{aligned}
\left\langle B_{r} u, v\right\rangle= & \int_{\Omega} \mathbf{a}(x, D u) \cdot D v+\int_{\Omega} T_{c_{r}}\left(\gamma_{r}(u)\right) v+\frac{1}{r} \int_{\Omega}|u|^{p-2} u v+\frac{1}{m} \int_{\Omega} T_{c_{r}}\left(u^{+}\right) v-\frac{1}{n} \int_{\Omega} T_{c_{r}}\left(u^{-}\right) v \\
& +\int_{\partial \Omega} T_{c_{r}}\left(\beta_{r}(u)\right) v+\frac{1}{m} \int_{\partial \Omega} T_{c_{r}}\left(u^{+}\right) v-\frac{1}{n} \int_{\partial \Omega} T_{c_{r}}\left(u^{-}\right) v-\int_{\partial \Omega} \psi v-\int_{\Omega} \phi v,
\end{aligned}
$$

is bounded, coercive, monotone and hemicontinuous. Then, by a classical result of Browder [21], there exists $u_{r}=$ $u_{\phi, \psi, m, n, r} \in W^{1, p}(\Omega)$, such that

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{r}\right) \cdot D v+\int_{\Omega} T_{c_{r}}\left(\gamma_{r}\left(u_{r}\right)\right) v+\frac{1}{r} \int_{\Omega}\left|u_{r}\right|^{p-2} u_{r} v+\frac{1}{m} \int_{\Omega} T_{c_{r}}\left(\left(u_{r}\right)^{+}\right) v-\frac{1}{n} \int_{\Omega} T_{c_{r}}\left(\left(u_{r}\right)^{-}\right) v \\
& \quad+\int_{\partial \Omega} T_{c_{r}}\left(\beta_{r}\left(u_{r}\right)\right) v+\frac{1}{m} \int_{\partial \Omega} T_{c_{r}}\left(\left(u_{r}\right)^{+}\right) v-\frac{1}{n} \int_{\partial \Omega} T_{c_{r}}\left(\left(u_{r}\right)^{-}\right) v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v, \tag{13}
\end{align*}
$$

for all $v \in W^{1, p}(\Omega)$.
Taking $v=T_{k}\left(\left(u_{r}-m M\right)^{+}\right)$in (13), misleading nonnegative terms, dividing by $k$, and taking limits as $k$ goes to 0 , we get

$$
\begin{aligned}
& \frac{1}{m} \int_{\Omega} T_{c_{r}}\left(u_{r}\right) \operatorname{sign}^{+}\left(u_{r}-m M\right)+\frac{1}{m} \int_{\partial \Omega} T_{c_{r}}\left(u_{r}\right) \operatorname{sign}^{+}\left(u_{r}-m M\right) \\
& \quad \leqslant \int_{\partial \Omega} \psi \operatorname{sign}^{+}\left(u_{r}-m M\right)+\int_{\Omega} \phi \operatorname{sign}^{+}\left(u_{r}-m M\right)
\end{aligned}
$$

Consequently

$$
\begin{aligned}
& \int_{\Omega}\left(T_{c_{r}}\left(u_{r}\right)-m M\right) \operatorname{sign}^{+}\left(u_{r}-m M\right)+\int_{\partial \Omega}\left(T_{c_{r}}\left(u_{r}\right)-m M\right) \operatorname{sign}^{+}\left(u_{r}-m M\right) \\
& \leqslant \int_{\partial \Omega}(m \psi-m M) \operatorname{sign}^{+}\left(u_{r}-m M\right)+\int_{\Omega}(m \phi-m M) \operatorname{sign}^{+}\left(u_{r}-m M\right) \leqslant 0 .
\end{aligned}
$$

Therefore, since $m \leqslant n$,

$$
u_{r}(x) \leqslant n M \quad \text { a.e. in } \Omega .
$$

Similarly, taking $v=T_{k}\left(\left(u_{r}+n M\right)^{-}\right)$in (13), we get

$$
u_{r}(x) \geqslant-n M \quad \text { a.e. in } \Omega .
$$

Consequently,

$$
\begin{equation*}
\left\|u_{r}\right\|_{\infty} \leqslant n M \tag{14}
\end{equation*}
$$

and (13) yields

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{r}\right) \cdot D v+\int_{\Omega} \gamma_{r}\left(u_{r}\right) v+\frac{1}{r} \int_{\Omega}\left|u_{r}\right|^{p-2} u_{r} v+\frac{1}{m} \int_{\Omega} u_{r}^{+} v-\frac{1}{n} \int_{\Omega} u_{r}^{-} v \\
& \quad+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right) v+\frac{1}{m} \int_{\partial \Omega} u_{r}^{+} v-\frac{1}{n} \int_{\partial \Omega} u_{r}^{-} v=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v \tag{15}
\end{align*}
$$

for all $v \in W^{1, p}(\Omega)$.
Taking $v=T_{k}\left(\left(u_{r}\right)^{+}\right)$in (15), disregarding some positive terms, dividing by $k$ and letting $k$ go to $\infty$ we get that

$$
\begin{equation*}
\frac{1}{m} \int_{\Omega} u_{r}^{+}+\int_{\Omega} \gamma_{r}\left(u_{r}\right)^{+}+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right)^{+} \leqslant \int_{\Omega} \phi^{+}+\int_{\partial \Omega} \psi^{+} \tag{16}
\end{equation*}
$$

and, similarly, taking $T_{k}\left(\left(u_{r}\right)^{-}\right)$we get

$$
\begin{equation*}
\frac{1}{n} \int_{\Omega} u_{r}^{-}+\int_{\Omega} \gamma_{r}\left(u_{r}\right)^{-}+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right)^{-} \leqslant \int_{\Omega} \phi^{-}+\int_{\partial \Omega} \psi^{-} . \tag{17}
\end{equation*}
$$

Taking $v=u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}$ as test function in (15) and having in mind that

$$
\begin{aligned}
\int_{\partial \Omega} \beta_{r}\left(u_{r}\right)\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right)= & \int_{\partial \Omega}\left(\beta_{r}\left(u_{r}\right)-\beta_{r}\left(\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right)\right)\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \geqslant 0 ; \\
\int_{\Omega} \gamma_{r}\left(u_{r}\right)\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right)= & \int_{\Omega}\left(\gamma_{r}\left(u_{r}\right)-\gamma_{r}\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}\right)\right)\left(u_{r}-\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}\right) \\
& -\int_{\Omega} \gamma_{r}\left(u_{r}\right)\left(\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}-\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}\right) \\
\geqslant & -\int_{\Omega} \gamma_{r}\left(u_{r}\right)\left(\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}-\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}\right)
\end{aligned}
$$

and working similarly with the other terms, we get

$$
\begin{aligned}
\lambda \int_{\Omega}\left|D u_{r}\right|^{p} \leqslant & \int_{\Omega} \phi\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right)+\int_{\partial \Omega} \psi\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \\
& -\int_{\Omega} \gamma_{r}\left(u_{r}\right)\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \\
& -\frac{1}{m} \int_{\Omega} u_{r}^{+}\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \\
& +\frac{1}{n} \int_{\Omega} u_{r}^{-}\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) .
\end{aligned}
$$

Now, by Poincare's inequality and the trace theorem, there exists $c_{1}=c_{1}(\Omega, N, p)>0$ such that

$$
\int_{\Omega} \phi\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \leqslant c_{1}\|\phi\|_{V^{1, p}(\Omega)}\left\|D u_{r}\right\|_{L^{p}(\Omega)},
$$

and

$$
\int_{\partial \Omega} \psi\left(u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \leqslant c_{1}\|\psi\|_{V^{1, p}(\partial \Omega)}\left\|D u_{r}\right\|_{L^{p}(\Omega)} .
$$

On the other hand, by (16) and (17),

$$
\begin{aligned}
& -\int_{\Omega} \gamma_{r}\left(u_{r}\right)\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right)-\frac{1}{m} \int_{\Omega} u_{r}^{+}\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \\
& \quad+\frac{1}{n} \int_{\Omega} u_{r}^{-}\left(\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right) \\
& \leqslant \\
& \quad 2\left(\int_{\partial \Omega}|\psi|+\int_{\Omega}|\phi|\right)\left|\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right| .
\end{aligned}
$$

Moreover, applying again the generalized Poincaré inequality, there exists $c_{2}=c_{2}(\Omega, N, p)>0$ such that

$$
\begin{aligned}
& \left|\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right| \\
& \quad \leqslant \frac{1}{\operatorname{meas}(\Omega)^{1 / p}}\left(\left\|u_{r}-\frac{1}{\operatorname{meas}(\Omega)} \int_{\Omega} u_{r}\right\|_{L^{p}(\Omega)}+\left\|u_{r}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{r}\right\|_{L^{p}(\Omega)}\right) \\
& \leqslant c_{2}\left\|D u_{r}\right\|_{L^{p}(\Omega)} .
\end{aligned}
$$

Therefore, there exists $c_{3}=c_{3}(\Omega, N, p)>0$, such that

$$
\begin{equation*}
\left\|D u_{r}\right\|_{L^{p}(\Omega)}^{p-1} \leqslant \frac{c_{3}}{\lambda}\left(\|\phi\|_{V^{1, p}(\Omega)}+\|\psi\|_{V^{1, p}(\partial \Omega)}\right) . \tag{18}
\end{equation*}
$$

As a consequence of (14) and (18) we can suppose that there exists a subsequence, still denoted $u_{r}$, such that
$u_{r}$ converges weakly in $W^{1, p}(\Omega)$ to $u \in W^{1, p}(\Omega)$,
$u_{r}$ converges in $L^{q}(\Omega)$ and a.e. in $\Omega$ to $u$, for any $q \geqslant 1$,
$u_{r}$ converges in $L^{p}(\partial \Omega)$ and a.e. to $u$,
with

$$
\begin{equation*}
-n M \leqslant u \leqslant n M . \tag{19}
\end{equation*}
$$

Taking into account (19), we get that $\left|\gamma_{r}\left(u_{r}\right)\right|$ is uniformly bounded. Consequently, we can assume that $\gamma_{r}\left(u_{r}\right) \rightarrow$ $z \in L^{\infty}(\Omega)$ weakly*, moreover

$$
-\gamma^{0}(-n M) \leqslant z \leqslant \gamma^{0}(n M)
$$

Since $u_{r} \rightarrow u$ in $L^{1}(\Omega)$, applying [9, Lemma G], it follows that $z(x) \in \gamma(u(x))$ a.e. on $\Omega$.
On the other hand, since $\beta_{r}\left(u_{r}\right)$ is also uniformly bounded, we can assume that $\beta_{r}\left(u_{r}\right) \rightarrow w \in L^{\infty}(\partial \Omega)$ weakly*. Again, applying [9, Lemma G], it follows that $w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$.

Let us see now that $\left\{D u_{r}\right\}$ converges in measure to $D u$. We follow the technique used in [10] (see also [3]). Since $D u_{r}$ converges to $D u$ weakly in $L^{p}(\Omega)$, it is enough to show that $\left\{D u_{r}\right\}$ is a Cauchy sequence in measure. Let $t$ and $\epsilon>0$. For some $A>1$, we set

$$
C(x, A, t):=\inf \{(\mathbf{a}(x, \xi)-\mathbf{a}(x, \eta)) \cdot(\xi-\eta):|\xi| \leqslant A,|\eta| \leqslant A,|\xi-\eta| \geqslant t\} .
$$

Having in mind that the function $\psi \rightarrow a(x, \psi)$ is continuous for almost all $x \in \Omega$ and the set $\{(\xi, \eta):|\xi| \leqslant A,|\eta| \leqslant A$, $|\xi-\eta| \geqslant t\}$ is compact, the infimum in the definition of $C(x, A, t)$ is a minimum. Hence, by $\left(\mathrm{H}_{3}\right)$, it follows that

$$
\begin{equation*}
C(x, A, t)>0 \quad \text { for almost all } x \in \Omega . \tag{20}
\end{equation*}
$$

Now, for $r, s \in \mathbb{N}$ and any $k>0$, the following inclusion holds

$$
\begin{equation*}
\left\{\left|D u_{r}-D u_{s}\right|>t\right\} \subset\left\{\left|D u_{r}\right| \geqslant A\right\} \cup\left\{\left|D u_{s}\right| \geqslant A\right\} \cup\left\{\left|u_{r}-u_{s}\right| \geqslant k^{2}\right\} \cup\{C(x, A, t) \leqslant k\} \cup G, \tag{21}
\end{equation*}
$$

where

$$
G=\left\{\left|u_{r}-u_{s}\right| \leqslant k^{2}, C(x, A, t) \geqslant k,\left|D u_{r}\right| \leqslant A,\left|D u_{s}\right| \leqslant A,\left|D u_{r}-D u_{s}\right|>t\right\} .
$$

Since the sequence $D u_{r}$ is bounded in $L^{p}(\Omega)$ we can choose $A$ large enough in order to have

$$
\begin{equation*}
\operatorname{meas}\left(\left\{\left|D u_{r}\right| \geqslant A\right\} \cup\left\{\left|D u_{s}\right| \geqslant A\right\}\right) \leqslant \frac{\epsilon}{4} \quad \text { for all } r, s \in \mathbb{N} . \tag{22}
\end{equation*}
$$

By (20), we can choose $k$ small enough in order to have

$$
\begin{equation*}
\operatorname{meas}(\{C(x, A, t) \leqslant k\}) \leqslant \frac{\epsilon}{4} . \tag{23}
\end{equation*}
$$

On the other hand, if we use $T_{k}\left(u_{r}-u_{s}\right)$ and $T_{k}\left(u_{r}-u_{s}\right)$ as test functions in (15) for $u_{r}$ and $u_{s}$ respectively, we obtain

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{r}\right) \cdot D T_{k}\left(u_{r}-u_{s}\right)+\int_{\Omega} \gamma_{r}\left(u_{r}\right) T_{k}\left(u_{r}-u_{s}\right)+\frac{1}{r} \int_{\Omega}\left|u_{r}\right|^{p-2} u_{r} T_{k}\left(u_{r}-u_{s}\right)+\frac{1}{m} \int_{\Omega} u_{r}^{+} T_{k}\left(u_{r}-u_{s}\right) \\
& \quad-\frac{1}{n} \int_{\Omega} u_{r}^{-} T_{k}\left(u_{r}-u_{s}\right)+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right) T_{k}\left(u_{r}-u_{s}\right)+\frac{1}{m} \int_{\partial \Omega} u_{r}^{+} T_{k}\left(u_{r}-u_{s}\right)-\frac{1}{n} \int_{\partial \Omega} u_{r}^{-} T_{k}\left(u_{r}-u_{s}\right) \\
& =\int_{\partial \Omega} \psi T_{k}\left(u_{r}-u_{s}\right)+\int_{\Omega} \phi T_{k}\left(u_{r}-u_{s}\right), \tag{24}
\end{align*}
$$

and

$$
\begin{align*}
& -\int_{\Omega} \mathbf{a}\left(x, D u_{s}\right) \cdot D T_{k}\left(u_{r}-u_{s}\right)-\int_{\Omega} \gamma_{s}\left(u_{s}\right) T_{k}\left(u_{r}-u_{s}\right)-\frac{1}{s} \int_{\Omega}\left|u_{s}\right|^{p-2} u_{s} T_{k}\left(u_{r}-u_{s}\right)-\frac{1}{m} \int_{\Omega} u_{s}^{+} T_{k}\left(u_{r}-u_{s}\right) \\
& \quad+\frac{1}{n} \int_{\Omega} u_{s}^{-} T_{k}\left(u_{r}-u_{s}\right)-\int_{\partial \Omega} \beta_{s}\left(u_{s}\right) T_{k}\left(u_{r}-u_{s}\right)-\frac{1}{m} \int_{\partial \Omega} u_{s}^{+} T_{k}\left(u_{r}-u_{s}\right)+\frac{1}{n} \int_{\partial \Omega} u_{s}^{-} T_{k}\left(u_{r}-u_{s}\right) \\
& =-\int_{\partial \Omega} \psi T_{k}\left(u_{r}-u_{s}\right)-\int_{\Omega} \phi T_{k}\left(u_{r}-u_{s}\right) . \tag{25}
\end{align*}
$$

Adding (24) and (25) and disregarding some positive terms, we get

$$
\begin{aligned}
& \int_{\Omega}\left(\mathbf{a}\left(x, D u_{r}\right)-\mathbf{a}\left(x, D u_{s}\right)\right) \cdot D T_{k}\left(u_{r}-u_{s}\right) \leqslant-\int_{\Omega}\left(\gamma_{r}\left(u_{r}\right)-\gamma_{s}\left(u_{s}\right)\right) T_{k}\left(u_{r}-u_{s}\right) \\
& \quad-\int_{\Omega}\left(\frac{1}{r}\left|u_{r}\right|^{p-2} u_{r}-\frac{1}{s}\left|u_{s}\right|^{p-2} u_{s}\right) T_{k}\left(u_{r}-u_{s}\right)-\int_{\partial \Omega}\left(\beta_{r}\left(u_{r}\right)-\beta_{s}\left(u_{s}\right)\right) T_{k}\left(u_{r}-u_{s}\right) .
\end{aligned}
$$

Consequently, there exists a constant $\widehat{M}$ independent of $r$ and $s$ such that

$$
\int_{\Omega}\left(\mathbf{a}\left(x, D u_{r}\right)-\mathbf{a}\left(x, D u_{s}\right)\right) \cdot D T_{k}\left(u_{r}-u_{s}\right) \leqslant k \widehat{M} .
$$

Hence

$$
\begin{align*}
\operatorname{meas}(G) & \leqslant \operatorname{meas}\left(\left\{\left|u_{r}-u_{s}\right| \leqslant k^{2},\left(\mathbf{a}\left(x, D u_{r}\right)-\mathbf{a}\left(x, D u_{s}\right)\right) \cdot D\left(u_{r}-u_{s}\right) \geqslant k\right\}\right) \\
& \leqslant \frac{1}{k} \int_{\left\{\left|u_{r}-u_{s}\right|<k^{2}\right\}}\left(\mathbf{a}\left(x, D u_{r}\right)-\mathbf{a}\left(x, D u_{s}\right)\right) \cdot D\left(u_{r}-u_{s}\right) \\
& =\frac{1}{k} \int_{\Omega}\left(\mathbf{a}\left(x, D u_{r}\right)-\mathbf{a}\left(x, D u_{s}\right)\right) \cdot D T_{k^{2}}\left(u_{r}-u_{s}\right) \leqslant \frac{1}{k} k^{2} \widehat{M} \leqslant \frac{\epsilon}{4} \tag{26}
\end{align*}
$$

for $k$ small enough.
Since $A$ and $k$ have been already chosen, if $r_{0}$ is large enough we have for $r, s \geqslant r_{0}$ the estimate meas $\left(\left\{\left|u_{r}-u_{s}\right| \geqslant\right.\right.$ $\left.\left.k^{2}\right\}\right) \leqslant \frac{\epsilon}{4}$. From here, using (21)-(23) and (26), we can conclude that

$$
\operatorname{meas}\left(\left\{\left|D u_{r}-D u_{s}\right| \geqslant t\right\}\right) \leqslant \epsilon \quad \text { for } r, s \geqslant r_{0} .
$$

From here, up to extraction of a subsequence, we also have $\mathbf{a}\left(\cdot, D u_{r}\right)$ converges in measure and a.e. to $\mathbf{a}(\cdot, D u)$. Now, by ( $\mathrm{H}_{2}$ ) and (18), $\mathbf{a}\left(\cdot, D u_{r}\right) \quad$ converges weakly in $L^{p^{\prime}}(\Omega)^{N}$ to $\mathbf{a}(\cdot, D u)$.

Finally, letting $r \rightarrow+\infty$ in (15), we prove (i).
In order to prove (ii), we write $u_{1, r}=u_{\phi_{1}, \psi_{1}, m_{1}, n_{1}, r}$ and $u_{2, r}=u_{\phi_{2}, \psi_{2}, m_{2}, n_{2}, r}$. Taking $T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)$, with $r$ large enough, as test function in (15) for $u_{1, r}, m=m_{1}$ and $n=n_{1}$, we get

$$
\begin{aligned}
& \int_{\Omega} \mathbf{a}\left(x, D u_{1, r}\right) \cdot D T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\int_{\Omega} \gamma_{r}\left(u_{1, r}\right) T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\frac{1}{r} \int_{\Omega}\left|u_{1, r}\right|^{p-2} u_{1, r} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& \quad+\frac{1}{m_{1}} \int_{\Omega} u_{1, r}^{+} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\frac{1}{n_{1}} \int_{\Omega} u_{1, r}^{-} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\int_{\partial \Omega} \beta_{r}\left(u_{1, r}\right) T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& \quad+\frac{1}{m_{1}} \int_{\partial \Omega} u_{1, r}^{+} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\frac{1}{n_{1}} \int_{\partial \Omega} u_{1, r}^{-} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& =\int_{\partial \Omega} \psi_{1} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\int_{\Omega} \phi_{1} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right),
\end{aligned}
$$

and taking $T_{k}\left(u_{1, r}-u_{2, r}\right)^{+}$as test function in (15) for $u_{2, r}, m=m_{2}$ and $n=n_{2}$, we get

$$
\begin{aligned}
& -\int_{\Omega} \mathbf{a}\left(x, D u_{2, r}\right) \cdot D T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\int_{\Omega} \gamma_{r}\left(u_{2, r}\right) T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& \quad-\frac{1}{r} \int_{\Omega}\left|u_{2, r}\right|^{p-2} u_{2, r} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\frac{1}{m_{2}} \int_{\Omega} u_{2, r}^{+} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\frac{1}{n_{2}} \int_{\Omega} u_{2, r}^{-} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& \quad-\int_{\partial \Omega} \beta_{r}\left(u_{2, r}\right) T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\frac{1}{m_{2}} \int_{\partial \Omega} u_{2, r}^{+} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)+\frac{1}{n_{2}} \int_{\partial \Omega} u_{2, r}^{-} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) \\
& =-\int_{\partial \Omega} \psi_{2} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right)-\int_{\Omega} \phi_{2} T_{k}\left(\left(u_{1, r}-u_{2, r}\right)^{+}\right) .
\end{aligned}
$$

Adding these two inequalities, misleading some nonnegative terms, dividing by $k$, and letting $k \rightarrow 0$, we get

$$
\begin{equation*}
\int_{\Omega}\left(\gamma_{r}\left(u_{1, r}\right)-\gamma_{r}\left(u_{2, r}\right)\right)^{+}+\int_{\partial \Omega}\left(\beta_{r}\left(u_{1, r}\right)-\beta_{r}\left(u_{2, r}\right)\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} . \tag{27}
\end{equation*}
$$

Therefore, taking into account the above convergence, (ii) is obtained.
In the homogeneous case without any condition on $\gamma$ we obtain the following result.
Proposition 5.2. Assume $D(\beta)=\mathbb{R}$. Let $m, n \in \mathbb{N}, m \leqslant n$. Then, the following hold.
(i) For $\phi \in L^{\infty}(\Omega)$, there exist $u=u_{\phi, m, n} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega), z=z_{\phi, m, n} \in L^{\infty}(\Omega), z(x) \in \gamma(u(x))$ a.e. in $\Omega$, and $w=w_{\phi, m, n} \in L^{\infty}(\partial \Omega), w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that $\left[u, z+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}, w+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}\right]$is a weak solution of problem $\left(S_{\phi, 0}^{\gamma_{m, n}, \beta_{m, n}}\right)$, and $z \ll \phi$.
(ii) If $m_{1} \leqslant m_{2} \leqslant n_{2} \leqslant n_{1}, \phi_{1}, \phi_{2} \in L^{\infty}(\Omega)$, then

$$
\int_{\Omega}\left(z_{\phi_{1}, m_{1}, n_{1}}-z_{\phi_{2}, m_{2}, n_{2}}\right)^{+}+\int_{\partial \Omega}\left(w_{\phi_{1}, m_{1}, n_{1}}-w_{\phi_{2}, m_{2}, n_{2}}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Proof. Following the proof of Proposition 5.1 there exists $u_{r}=u_{\phi, m, n, r} \in W^{1, p}(\Omega)$, such that $\left\|u_{r}\right\|_{\infty} \leqslant n\|\phi\|_{\infty}$,
and

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{r}\right) \cdot D v+\frac{1}{r} \int_{\Omega}\left|u_{r}\right|^{p-2} u_{r} v+\int_{\Omega} \gamma_{r}\left(u_{r}\right) v+\frac{1}{m} \int_{\Omega} u_{r}^{+}\left(u_{r}-v\right)-\frac{1}{n} \int_{\Omega} u_{r}^{-} v \\
& \quad+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right) v+\frac{1}{m} \int_{\partial \Omega} u_{r}^{+} v-\frac{1}{n} \int_{\partial \Omega} u_{r}^{-} v=\int_{\Omega} \phi v, \tag{28}
\end{align*}
$$

for all $v \in W^{1, p}(\Omega)$.

We can finish the proof as in Propositions 5.1 if we prove that $\gamma_{r}\left(u_{r}\right)$ is weakly convergent in $L^{1}(\Omega)$. Taking $v=q\left(\gamma_{r}\left(u_{r}\right)\right), q \in P_{0}$, as test function in (28) we have that, after misleading nonnegative terms,

$$
\int_{\Omega} \gamma_{r}\left(u_{r}\right) q\left(\gamma_{r}\left(u_{r}\right)\right) \leqslant \int_{\Omega} \phi q\left(\gamma_{r}\left(u_{r}\right)\right),
$$

which implies, $\gamma_{r}\left(u_{r}\right) \ll \phi$. In particular, see Proposition 2.1, $\left\|\gamma_{r}\left(u_{r}\right)\right\|_{\infty} \leqslant\|\phi\|_{\infty}$ and $\gamma_{r}\left(u_{r}\right) \rightarrow z \in L^{\infty}(\Omega)$ weakly in $L^{1}(\Omega)$, with $z \ll \phi$.

Remark 5.3. Observe that if $D(\beta)=\{0\}$, for any $\gamma$, if we rewrite the proof of Proposition 5.2, using $W_{0}^{1, p}(\Omega)$ instead of $W^{1, p}(\Omega)$, we find $u=u_{\phi, m, n} \in W_{0}^{1, p}(\Omega) \cap L^{\infty}(\Omega), z=z_{\phi, m, n} \in L^{\infty}(\Omega), z(x) \in \gamma(u(x))$ a.e. in $\Omega$, such that

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D v+\int_{\Omega} z v+\frac{1}{m} \int_{\Omega} u^{+} v-\frac{1}{n} \int_{\Omega} u^{-} v=\int_{\Omega} \phi v,
$$

for all $v \in W_{0}^{1, p}(\Omega)$. Moreover, if $m_{1} \leqslant m_{2} \leqslant n_{2} \leqslant n_{1}, \phi_{1}, \phi_{2} \in L^{\infty}(\Omega)$, then

$$
\int_{\Omega}\left(z_{\phi_{1}, \psi_{1}, m_{1}, n_{1}}-z_{\phi_{2}, \psi_{2}, m_{2}, n_{2}}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Proposition 5.4. Assume $D(\gamma)=\mathbb{R}$ and a smooth. Let $m, n \in \mathbb{N}, m \leqslant n$. Then, the following hold.
(i) For $\phi \in L^{\infty}(\Omega)$ and $\psi \in L^{\infty}(\partial \Omega)$, there exist $u=u_{\phi, \psi, m, n} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega), z=z_{\phi, \psi, m, n} \in L^{\infty}(\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in $\Omega$, and $w=w_{\phi, \psi, m, n} \in L^{1}(\partial \Omega), w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that $\left[u, z+\frac{1}{m} u^{+}-\right.$ $\left.\frac{1}{n} u^{-}, w+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}\right]$is a weak solution of $\left(S_{\phi, \psi}^{\gamma_{m, n}, \beta_{m, n}}\right)$.
Moreover, there exists $c(\Omega, N, p)>0$ such that

$$
\|D u\|_{L^{p}(\Omega)}^{p-1} \leqslant \frac{c(\Omega, N, p)}{\lambda}\left(\|\phi\|_{V^{1, p}(\Omega)}+\|\psi\|_{V^{1, p}(\partial \Omega)}\right) .
$$

(ii) If $m_{1} \leqslant m_{2} \leqslant n_{2} \leqslant n_{1}, \phi_{1}, \phi_{2} \in L^{\infty}(\Omega), \psi_{1}, \psi_{2} \in L^{\infty}(\partial \Omega)$ then

$$
\int_{\Omega}\left(z_{\phi_{1}, \psi_{1}, m_{1}, n_{1}}-z_{\phi_{2}, \psi_{2}, m_{2}, n_{2}}\right)^{+}+\int_{\partial \Omega}\left(w_{\phi_{1}, \psi_{1}, m_{1}, n_{1}}-w_{\phi_{2}, \psi_{2}, m_{2}, n_{2}}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Proof. Applying Proposition 5.1 to $\beta_{r}$, the Yosida approximation of $\beta$, there exists $u_{r}=u_{\phi, \psi, m, n, r} \in W^{1, p}(\Omega) \cap$ $L^{\infty}(\Omega)$ and $z_{r}=z_{\phi, \psi, m, n, r} \in L^{\infty}(\Omega), z_{r} \in \gamma\left(u_{r}\right)$ a.e. in $\Omega$, such that

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{r}\right) \cdot D v+\int_{\Omega} z_{r} v+\int_{\partial \Omega} \beta_{r}\left(u_{r}\right) v+\frac{1}{m} \int_{\Omega} u_{r}^{+} v-\frac{1}{n} \int_{\Omega} u_{r}^{-} v+\frac{1}{m} \int_{\partial \Omega} u_{r}^{+} v-\frac{1}{n} \int_{\partial \Omega} u_{r}^{-} v \\
& \quad=\int_{\partial \Omega} \psi v+\int_{\Omega} \phi v, \tag{29}
\end{align*}
$$

for all $v \in W^{1, p}(\Omega)$. Moreover, $\left|u_{r}\right|$ is uniformly bounded by $n M, M:=\|\phi\|_{\infty}+\|\psi\|_{\infty}$,

$$
-\gamma^{0}(-n M) \leqslant z_{r} \leqslant \gamma^{0}(n M),
$$

and

$$
\int_{\Omega} z_{r}^{ \pm}+\int_{\partial \Omega} w_{r}^{ \pm} \leqslant \int_{\partial \Omega} \psi^{ \pm}+\int_{\Omega} \phi^{ \pm} .
$$

Let now $\hat{u} \in L^{\infty}(\Omega)$ and $\hat{z} \in \gamma(\hat{u}), \hat{z} \in L^{\infty}(\Omega)$, be such that $\hat{u}$ is solution of the Dirichlet problem (see Remark 5.3)

$$
\begin{cases}-\operatorname{div} \mathbf{a}(x, D \hat{u})+\hat{z}+\frac{1}{m} \hat{u}^{+}-\frac{1}{n} \hat{u}^{-}=\phi & \text { in } \Omega \\ \hat{u}=0 & \text { on } \partial \Omega\end{cases}
$$

Since a is smooth, there exists $\hat{\psi} \in L^{1}(\partial \Omega)$ such that

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}(x, D \hat{u}) \cdot D v+\int_{\Omega} \hat{z} v+\frac{1}{m} \int_{\Omega} \hat{u}^{+} v-\frac{1}{n} \int_{\Omega} \hat{u}^{-} v=\int_{\partial \Omega} \hat{\psi} v+\int_{\Omega} \phi v, \tag{30}
\end{equation*}
$$

for any $v \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$.
Taking $v=q\left(\beta_{r}\left(u_{r}-\hat{u}\right)\right), q \in P_{0}$, as test function in (29), and $q\left(\beta_{r}\left(u_{r}-\hat{u}\right)\right)$ as test function in (30), and adding both equalities we get, after misleading nonnegative terms, that

$$
\int_{\partial \Omega} \beta_{r}\left(u_{r}\right) q\left(\beta_{r}\left(u_{r}\right)\right) \leqslant \int_{\partial \Omega}(\psi-\hat{\psi}) q\left(\beta_{r}\left(u_{r}\right)\right)
$$

i.e., $\beta_{r}\left(u_{r}\right) \ll \psi-\hat{\psi}$, which implies (see Proposition 2.1) that

$$
\beta_{r}\left(u_{r}\right) \rightarrow w \in L^{1}(\partial \Omega) \quad \text { weakly in } L^{1}(\partial \Omega)
$$

Now, arguing as in the proof of Proposition 5.1, we obtain (i).
To prove (ii), Proposition 5.1 implies, denoting $u_{i, r}=u_{\phi_{i}, \psi_{i}, m_{i}, n_{i}, r}$ and $z_{i, r}=z_{\phi_{i}, \psi_{i}, m_{i}, n_{i}, r}, i=1,2$,

$$
\begin{equation*}
\int_{\Omega}\left(z_{1, r}-z_{2, r}\right)^{+}+\int_{\partial \Omega}\left(\beta_{r}\left(u_{1, r}\right)-\beta_{r}\left(u_{2, r}\right)\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} . \tag{31}
\end{equation*}
$$

Taking limits in (31) when $r$ goes to $+\infty$, (ii) holds.
In the case $\psi=0$, we have the following result.
Proposition 5.5. Assume a smooth. Let $m, n \in \mathbb{N}, m \leqslant n$. Then, the following hold.
(i) For $\phi \in L^{\infty}(\Omega)$, there exist $u=u_{\phi, m, n} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega), z=z_{\phi, m, n} \in L^{\infty}(\Omega), z(x) \in \gamma(u(x))$ a.e. in $\Omega$, and $w=w_{\phi, m, n} \in L^{1}(\partial \Omega), w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$, such that $\left[u, z+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}, w+\frac{1}{m} u^{+}-\frac{1}{n} u^{-}\right]$is a weak solution of problem $\left(S_{\phi, 0}^{\gamma_{m, n}, \beta_{m, n}}\right)$, with $z \ll \phi$.
(ii) If $m_{1} \leqslant m_{2} \leqslant n_{2} \leqslant n_{1}, \phi_{1}, \phi_{2} \in L^{\infty}(\Omega)$, then

$$
\int_{\Omega}\left(z_{\phi_{1}, m_{1}, n_{1}}-z_{\phi_{2}, m_{2}, n_{2}}\right)^{+}+\int_{\partial \Omega}\left(w_{\phi_{1}, m_{1}, n_{1}}-w_{\phi_{2}, m_{2}, n_{2}}\right)^{+} \leqslant \int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

5.2. Existence of weak solutions

Proof of Theorem 3.6. We approximate $\phi$ and $\psi$ by

$$
\phi_{m, n}=\sup \{\inf \{m, \phi\},-n\}
$$

and

$$
\psi_{m, n}=\sup \{\inf \{m, \psi\},-n\},
$$

respectively. We have, $\phi_{m, n} \in L^{\infty}(\Omega), \psi_{m, n} \in L^{\infty}(\partial \Omega)$, are nondecreasing in $m$, nonincreasing in $n,\left\|\phi_{m, n}\right\|_{L^{p^{\prime}}(\Omega)} \leqslant$ $\|\phi\|_{L^{p^{\prime}}(\Omega)}$ and $\left\|\psi_{m, n}\right\|_{L^{p^{\prime}}(\partial \Omega)} \leqslant\|\psi\|_{L^{p^{\prime}}(\partial \Omega)}$. Then, if $m \leqslant n$, by Propositions 5.1 or 5.4, there exist $u_{m, n} \in W^{1, p}(\Omega) \cap$ $L^{\infty}(\Omega), z_{m, n} \in L^{\infty}(\Omega), z_{m, n}(x) \in \gamma\left(u_{m, n}(x)\right)$ a.e. in $\Omega$ and $w_{m, n} \in L^{1}(\partial \Omega), w_{m, n}(x) \in \beta\left(u_{m, n}(x)\right)$ a.e. on $\partial \Omega$, such that

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{m, n}\right) \cdot D v+\int_{\Omega} z_{m, n} v+\int_{\partial \Omega} w_{m, n} v+\frac{1}{m} \int_{\Omega} u_{m, n}^{+} v-\frac{1}{n} \int_{\Omega} u_{m, n}^{-} v+\frac{1}{m} \int_{\partial \Omega} u_{m, n}^{+} v-\frac{1}{n} \int_{\partial \Omega} u_{m, n}^{-} v \\
& \quad=\int_{\partial \Omega} \psi_{m, n} v+\int_{\Omega} \phi_{m, n} v \tag{32}
\end{align*}
$$

for any $v \in W^{1, p}(\Omega)$. Moreover,

$$
\begin{equation*}
\int_{\Omega} z_{m, n}^{ \pm}+\int_{\partial \Omega} w_{m, n}^{ \pm} \leqslant \int_{\Omega} \phi^{ \pm}+\int_{\partial \Omega} \psi^{ \pm} \tag{33}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|D u_{m, n}\right\|_{L^{p}(\Omega)}^{p-1} \leqslant \frac{c(\Omega, N, p)}{\lambda}\left(\|\phi\|_{V^{1, p}(\Omega)}+\|\psi\|_{V^{1, p}(\partial \Omega)}\right) . \tag{34}
\end{equation*}
$$

Fixed $m \in \mathbb{N}$, by Propositions 5.1 or 5.4 (ii), $\left\{z_{m, n}\right\}_{n=m}^{\infty}$ and $\left\{w_{m, n}\right\}_{n=m}^{\infty}$ are monotone nonincreasing. Then, by (33) and the Monotone Convergence Theorem, there exists $\hat{z}_{m} \in L^{1}(\Omega), \widehat{w}_{m} \in L^{1}(\partial \Omega)$ and a subsequence $n(m)$, such that

$$
\left\|z_{m, n(m)}-\hat{z}_{m}\right\|_{1} \leqslant \frac{1}{m}
$$

and

$$
\left\|w_{m, n(m)}-\widehat{w}_{m}\right\|_{1} \leqslant \frac{1}{m} .
$$

Thanks again to Proposition 5.1 or 5.4(ii), $\hat{z}_{m}$ and $\widehat{w}_{m}$ are nondecreasing in $m$. Now, by (33), we have that $\int_{\Omega}\left|\hat{z}_{m}\right|$ and $\int_{\partial \Omega}\left|\widehat{w}_{m}\right|$ are bounded. Using again the Monotone Convergence Theorem, there exist $z \in L^{1}(\Omega)$ and $w \in L^{1}(\partial \Omega)$ such that

$$
\hat{z}_{m} \text { converges a.e. and in } L^{1}(\Omega) \text { to } z
$$

and

$$
\widehat{w}_{m} \text { converges a.e. and in } L^{1}(\partial \Omega) \text { to } w .
$$

Consequently,

$$
\begin{equation*}
z_{m}:=z_{m, n(m)} \text { converges to } z \text { a.e. and in } L^{1}(\Omega) \tag{35}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{m}:=w_{m, n(m)} \text { converges to } w \text { a.e. and in } L^{1}(\partial \Omega) . \tag{36}
\end{equation*}
$$

If we set $u_{m}:=u_{m, n(m)}, \phi_{m}:=\phi_{m, n(m)}$ and $\psi_{m}:=\psi_{m, n(m)}$, then we have

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D v+\int_{\Omega} z_{m} v+\int_{\partial \Omega} w_{m} v+\frac{1}{m} \int_{\Omega} u_{m}^{+} v-\frac{1}{n(m)} \int_{\Omega} u_{m}^{-} v+\frac{1}{m} \int_{\partial \Omega} u_{m}^{+} v-\frac{1}{n(m)} \int_{\partial \Omega} u_{m}^{-} v \\
& \quad=\int_{\partial \Omega} \psi_{m} v+\int_{\Omega} \phi_{m} v, \tag{37}
\end{align*}
$$

for any $v \in W^{1, p}(\Omega)$.
As a consequence of (34),

$$
\begin{equation*}
\left\{u_{m}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}\right\}_{m} \quad \text { is bounded in } W^{1, p}(\Omega) \tag{38}
\end{equation*}
$$

Let us see that

$$
\begin{equation*}
\left\{\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}: m \in \mathbb{N}\right\} \quad \text { is a bounded sequence. } \tag{39}
\end{equation*}
$$

If (39) does not hold, then, extracting a subsequence if necessary, we can suppose that $\int_{\partial \Omega} u_{m}$ converges to $+\infty$ (or $-\infty$, respectively). Suppose first that $\int_{\partial \Omega} u_{m}$ converges to $+\infty$. Hence, by (38) we have
$u_{m}$ converges to $+\infty$ a.e. in $\Omega$, and a.e. in $\partial \Omega$.

Moreover, since for $m$ large enough

$$
u_{m}^{-} \leqslant\left(u_{m}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}\right)^{-}+\left(\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}\right)^{-}=\left(u_{m}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}\right)^{-},
$$

by (38), we get

$$
\left\{\int_{\partial \Omega} u_{m}^{-}\right\}_{m \in \mathbb{N}} \text { is bounded }
$$

and, similarly,

$$
\left\{\int_{\Omega} u_{m}^{-}\right\}_{m \in \mathbb{N}} \text { is bounded. }
$$

In the case $\int_{\partial \Omega} u_{m}$ converges to $-\infty$, we similarly obtain that

$$
u_{m} \text { converges to }-\infty \text { a.e. in } \Omega, \text { and a.e. in } \partial \Omega,
$$

and

$$
\left\{\int_{\partial \Omega} u_{m}^{+}\right\}_{m \in \mathbb{N}} \text { and }\left\{\int_{\Omega} u_{m}^{+}\right\}_{m \in \mathbb{N}} \text { are bounded. }
$$

Therefore, we have $z=\sup \{\operatorname{Ran}(\gamma)\}(z=\inf \{\operatorname{Ran}(\gamma)\}$, respectively) and $w=\sup \{\operatorname{Ran}(\beta)\}(w=\inf \{\operatorname{Ran}(\beta)\}$, respectively). Now, taking $v=1$ as test function in (37), we get

$$
\frac{1}{m} \int_{\Omega} u_{m}^{+}-\frac{1}{n(m)} \int_{\Omega} u_{m}^{-}+\frac{1}{m} \int_{\partial \Omega} u_{m}^{+}-\frac{1}{n(m)} \int_{\partial \Omega} u_{m}^{-}=\int_{\Omega} \phi_{m}+\int_{\partial \Omega} \psi_{m}-\int_{\Omega} z_{m}-\int_{\partial \Omega} w_{m},
$$

and we get a contradiction with (4). Hence, (39) is true. By (38) and (39), we have $\left\{\left\|u_{m}\right\|_{W^{1, p}(\Omega)}\right\}_{m}$ is bounded. Therefore, there exists a subsequence, that we denote equal, such that

$$
\begin{array}{ll}
u_{m} \rightarrow u & \text { weakly in } W^{1, p}(\Omega), \\
u_{m} \rightarrow u & \text { in } L^{p}(\Omega) \text { and a.e. in } \Omega, \\
u_{m} \rightarrow u & \text { in } L^{p}(\partial \Omega) \text { and a.e. in } \partial \Omega .
\end{array}
$$

Moreover, arguing as in Proposition 5.1, it is not difficult to see that $\left\{D u_{m}\right\}$ is a Cauchy sequence in measure. Then, up to extraction of a subsequence, $D u_{m}$ converges to $D u$ a.e. in $\Omega$. Consequently, we obtain that
$\mathbf{a}\left(\cdot, D u_{m}\right)$ converges weakly in $L^{p^{\prime}}(\Omega)^{N}$ and a.e. in $\Omega$ to $\mathbf{a}(\cdot, D u)$.
From these convergences, we finish the proof of existence.
The proof of (ii) is a consequence of the existence result, Propositions 5.1(ii) or 5.4(ii), and the uniqueness result.

Remark 5.6. For positive data $\phi$ and $\psi$, it is not necessary the assumption $D(\gamma)=D(\beta)=\mathbb{R}$, that is, we can improve the above result in the following way. Assume $\left[0,+\infty\left[\subset D(\gamma)\right.\right.$ and $\mathcal{R}_{\gamma, \beta}^{+}>0$. Let $[0,+\infty[\subset D(\beta)$ or a smooth. For any $\left.0 \leqslant \phi \in V^{1, p} \Omega\right)$ and $0 \leqslant \psi \in V^{1, p}(\partial \Omega)$ with $\int_{\Omega} \phi+\int_{\partial \Omega} \psi<\mathcal{R}_{\gamma, \beta}^{+}$, there exists a weak solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. A similar result holds for nonpositive data.

Proof of Theorem 3.7. Let us approximate $\phi$ by $\phi_{m}=T_{m}(\phi)-\frac{1}{\operatorname{meas}(\Omega)} \alpha_{m}$ and $\psi$ by $\psi_{m}=T_{m}(\psi)$, where $\alpha_{m}=$ $\int_{\Omega} T_{m}(\phi)+\int_{\partial \Omega} T_{m}(\psi)$. Observe that

$$
\begin{equation*}
\lim _{m \rightarrow+\infty} \alpha_{m}=0 \tag{40}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega} \phi_{m}+\int_{\partial \Omega} \psi_{m}=0 . \tag{41}
\end{equation*}
$$

By Proposition 5.1, there exist $u_{m} \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega)$ such that

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D v+\frac{1}{m} \int_{\Omega} u_{m} v+\frac{1}{m} \int_{\partial \Omega} u_{m} v=\int_{\partial \Omega} \psi_{m} v+\int_{\Omega} \phi_{m} v, \tag{42}
\end{equation*}
$$

for any $v \in W^{1, p}(\Omega)$.
Taking $v=u_{m}$ as test function in (42), using (40) and the Poincaré inequality, it is easy to see that

$$
\begin{equation*}
\left\{u_{m}-\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}\right\}_{m} \quad \text { is bounded in } W^{1, p}(\Omega) \tag{43}
\end{equation*}
$$

Let us also see that

$$
\begin{equation*}
\left\{\frac{1}{\operatorname{meas}(\partial \Omega)} \int_{\partial \Omega} u_{m}: m \in \mathbb{N}\right\} \quad \text { is a bounded sequence. } \tag{44}
\end{equation*}
$$

If (44) does not hold, then, extracting a subsequence if necessary, we can suppose that $\int_{\partial \Omega} u_{m}$ converges to $+\infty$ (or $-\infty$, respectively). Suppose first that $\int_{\partial \Omega} u_{m}$ converges to $+\infty$. Hence, as in the proof of Theorem 3.6, we have

$$
\left\{\int_{\Omega} u_{m}^{-}\right\}_{m \in \mathbb{N}} \text { is bounded. }
$$

Now, taking $v=m$ in (42) and using (41), it follows that

$$
\lim _{m \rightarrow+\infty} \int_{\Omega} u_{m}^{-}=+\infty
$$

which is a contradiction. Similarly, we get a contradiction in the case $\int_{\partial \Omega} u_{m}$ converging to $-\infty$. Hence, (44) is true. By (43) and (44), we have $\left\{\left\|u_{m}\right\|_{W^{1, p}(\Omega)}\right\}_{m}$ is bounded, and we can finish as in the proof of Theorem 3.6.

Remark 5.7. Taking into account the arguments used in Remark 3.4, we get that $[u, z, w]$ in the above results (including also the case $\beta=D$ ) satisfies

$$
\int_{\Omega}|z v|+\int_{\partial \Omega}|w v| \leqslant \int_{\Omega}|\phi v|+\int_{\partial \Omega}|\psi v|+\sigma\left(\|g\|_{L^{p^{\prime}}(\Omega)}+\|D u\|_{L^{p}(\Omega)}^{p-1}\right)\|D v\|_{L^{p}(\Omega)}
$$

for all $v \in W^{1, p}(\Omega)$, and

$$
\|D u\|_{L^{p}(\Omega)}^{p-1} \leqslant \frac{c(\Omega, N, p)}{\lambda}\left(\|\phi\|_{V^{1, p}(\Omega)}+\|\psi\|_{V^{1, p}(\partial \Omega)}\right),
$$

for some $c(\Omega, N, p)>0$.
Taking $\beta=D, \gamma(r)=0$ for all $r \in \mathbb{R}$, and a smooth in Theorem 3.6, by Remark 5.7, it follows Corollary 3.8. The proof of Theorem 3.9 follows in a similar way to the proof of Theorem 3.6 taking into account Propositions 5.2 and 5.5. Finally, on account of Remark 5.3, it follows Theorem 3.10.

### 5.3. Existence of entropy solutions

Proof of Theorem 3.11. Observe that, under the assumptions of the theorem, we have $\mathcal{R}_{\gamma, \beta}=\mathbb{R}$. We divide the proof in several steps.

Step 1. Let us approximate $\phi$ by $\phi_{m}:=T_{m}(\phi)$ and $\psi$ by $\psi_{m}:=T_{m}(\psi)$. Then, by Theorem 3.6, there exist $u_{m} \in$ $W^{1, p}(\Omega), z_{m} \in V^{1, p}(\Omega), z_{m}(x) \in \gamma\left(u_{m}(x)\right)$ a.e. in $\Omega$, and $w_{m} \in V^{1, p}(\partial \Omega), w_{m}(x) \in \beta\left(u_{m}(x)\right)$ a.e. on $\partial \Omega$, such that

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D v+\int_{\Omega} z_{m} v+\int_{\partial \Omega} w_{m} v=\int_{\partial \Omega} \psi_{m} v+\int_{\Omega} \phi_{m} v \tag{45}
\end{equation*}
$$

for any $v \in W^{1, p}(\Omega)$.
Moreover,

$$
\begin{equation*}
\int_{\Omega} z_{m}^{ \pm}+\int_{\partial \Omega} w_{m}^{ \pm} \leqslant \int_{\partial \Omega} \psi_{m}^{ \pm}+\int_{\Omega} \phi_{m}^{ \pm} \tag{46}
\end{equation*}
$$

and

$$
\int_{\Omega}\left|z_{n}-z_{m}\right|+\int_{\partial \Omega}\left|w_{n}-w_{m}\right| \leqslant \int_{\partial \Omega}\left|\psi_{n}-\psi_{m}\right|+\int_{\Omega}\left|\phi_{n}-\phi_{m}\right| .
$$

Consequently

$$
\begin{align*}
& z_{m} \rightarrow z \quad \text { in } L^{1}(\Omega) \\
& w_{m} \rightarrow w \quad \text { in } L^{1}(\partial \Omega) \tag{47}
\end{align*}
$$

Taking $v=T_{k}\left(u_{m}\right)$ in (45), we obtain

$$
\begin{equation*}
\lambda \int_{\Omega}\left|D T_{k}\left(u_{m}\right)\right|^{p} \leqslant k\left(\|\phi\|_{1}+\|\psi\|_{1}\right), \quad \forall k \in \mathbb{N} . \tag{48}
\end{equation*}
$$

By (48), we have $\left\{T_{k}\left(u_{m}\right)\right\}$ is bounded in $W^{1, p}(\Omega)$. Then, we can suppose that there exists $\sigma_{k} \in W^{1, p}(\Omega)$ such that
$T_{k}\left(u_{m}\right)$ converges to $\sigma_{k}$ weakly in $W^{1, p}(\Omega)$,
$T_{k}\left(u_{m}\right)$ converges to $\sigma_{k}$ in $L^{p}(\Omega)$ and a.e. in $\Omega$
and
$T_{k}\left(u_{m}\right)$ converges to $\sigma_{k}$ in $L^{p}(\partial \Omega)$ and a.e. in $\partial \Omega$.
Step 2 . Let us see that $u_{m}$ converges almost every where in $\Omega$.
If $D(\beta)$ is bounded from above by $r_{1}$, using the Poincaré inequality and (48),

$$
\begin{aligned}
\operatorname{meas}\left\{x \in \Omega: \sigma_{k}^{+}(x)=k\right\} & \leqslant \int_{\Omega} \frac{\left(\sigma_{k}^{+}\right)^{p^{*}}}{k p^{*}} \leqslant \liminf _{m} \int_{\Omega} \frac{\left.\left(T_{k}\left(\left(u_{m}\right)^{+}\right)\right)\right)^{p^{*}}}{k^{p^{*}}} \\
& \leqslant \frac{C_{1}}{k p^{*}} \liminf _{m}\left(\int_{\partial \Omega} T_{k}\left(\left(u_{m}\right)^{+}\right)+\left(\int_{\Omega}\left|D T_{k}\left(\left(u_{m}\right)^{+}\right)\right|^{p}\right)^{1 / p}\right)^{p^{*}} \\
& \leqslant \frac{C_{1}}{k p^{*}}\left(r_{1} \operatorname{meas}(\partial \Omega)+\left(\frac{\|\phi\|_{1}+\|\psi\|_{1}}{\lambda} k\right)^{1 / p}\right)^{p^{*}} \quad \forall k>0
\end{aligned}
$$

where $p^{*}=\frac{N p}{N-p}$ and $C_{1}$ is independent of $k$ and $m$.
If $D(\beta)$ is unbounded from above, then, we are supposing $\lim _{k \rightarrow+\infty} \gamma^{0}(k)=+\infty$. Therefore, for $k>0$ large enough (in order to have $\gamma^{0}(k)>0$ ), by (46) we have

$$
\begin{aligned}
\operatorname{meas}\left\{x \in \Omega: \sigma_{k}^{+}(x)=k\right\} & =\int_{\left.\left\{x \in \Omega: \sigma_{k}^{+}(x)=k\right\}\right\}} \frac{\gamma^{0}\left(\sigma_{k}^{+}(x)\right)}{\gamma^{0}(k)} \leqslant \frac{1}{\gamma^{0}(k)} \liminf _{m} \int_{\Omega} \gamma^{0}\left(T_{k}\left(\left(u_{m}\right)^{+}\right)\right) \\
& \leqslant \frac{1}{\gamma^{0}(k)}\left(\|\phi\|_{1}+\|\psi\|_{1}\right) .
\end{aligned}
$$

Consequently, in any case, there exists $g(k)>0, \lim _{k \rightarrow+\infty} g(k)=0$, such that

$$
\begin{equation*}
\operatorname{meas}\left\{x \in \Omega: \sigma_{k}^{+}(x)=k\right\} \leqslant g(k) \quad \forall k>0 . \tag{49}
\end{equation*}
$$

Similarly, if $D(\beta)$ is bounded from below or assumption (8) holds, we can prove that there exists $g(k)$ as above such that

$$
\begin{equation*}
\operatorname{meas}\left\{x \in \Omega: \sigma_{k}^{-}(x)=k\right\} \leqslant g(k) \quad \forall k>0 . \tag{50}
\end{equation*}
$$

Note that we have proved (49) and (50) in any case. Consequently, there exists $g(k)>0$ with $\lim _{k \rightarrow+\infty} g(k)=0$, such that

$$
\operatorname{meas}\left\{x \in \Omega:\left|\sigma_{k}(x)\right|=k\right\} \leqslant g(k) \quad \forall k>0 .
$$

Therefore, if we define $u(x)=\sigma_{k}(x)$ on $\left\{x \in \Omega:\left|\sigma_{k}(x)\right|<k\right\}$, then
$u_{m}$ converges to $u$ a.e. in $\Omega$,
and we have that

$$
\begin{aligned}
& T_{k}\left(u_{m}\right) \text { converges weakly in } W^{1, p}(\Omega) \text { to } T_{k}(u), \\
& T_{k}\left(u_{m}\right) \text { converges in } L^{p}(\Omega) \text { and a.e. in } \Omega \text { to } T_{k}(u)
\end{aligned}
$$

and

$$
T_{k}\left(u_{m}\right) \text { converges in } L^{p}(\partial \Omega) \text { and a.e. in } \partial \Omega \text { to } T_{k}(u) .
$$

Consequently, $u \in \mathcal{T}^{1, p}(\Omega)$.
Arguing as in Proposition 5.1, it is not difficult to see that $\left\{D u_{m}\right\}$ is a Cauchy sequence in measure. Similarly, we can prove that $D T_{k}\left(u_{m}\right)$ converges in measure to $D T_{k}(u)$. Then, up to extraction of a subsequence, $D u_{m}$ converges to $D u$ a.e. in $\Omega$. Consequently, we obtain that

$$
\begin{equation*}
\mathbf{a}\left(\cdot, D T_{k}\left(u_{m}\right)\right) \text { converges weakly in } L^{p^{\prime}}(\Omega)^{N} \text { and a.e. in } \Omega \text { to } \mathbf{a}\left(\cdot, D T_{k}(u)\right) \text {. } \tag{52}
\end{equation*}
$$

Step 3. Let us see now that $u \in \mathcal{T}_{\text {tr }}^{1, p}(\Omega)$. On the one hand we have that $u_{m} \rightarrow u$ a.e. in $\Omega$. On the other hand, since $D T_{k}\left(u_{m}\right)$ is bounded in $L^{p}(\Omega)$ and $D T_{k}\left(u_{m}\right) \rightarrow D T_{k}(u)$ in measure, it follows from [5, Lemma 6.1] that $D T_{k}\left(u_{m}\right) \rightarrow D T_{k}(u)$ in $L^{1}(\Omega)$. Next, let us see that $u_{m}$ converges a.e. in $\partial \Omega$. Let suppose first that $D(\beta)$ is bounded from above by $r_{1}$, then, by (48), there exists a constant $C_{3}$ such that

$$
\operatorname{meas}\left\{x \in \partial \Omega: \sigma_{k}^{+}(x)=k\right\} \leqslant \int_{\partial \Omega} \frac{\sigma_{k}^{+}}{k} \leqslant \liminf _{m} \int_{\partial \Omega} \frac{T_{k}\left(\left(u_{m}\right)^{+}\right)}{k} \leqslant \frac{r_{1} \operatorname{meas}(\partial \Omega)}{k} \quad \forall k>0 .
$$

If $D(\beta)$ is unbounded from above, then, we are supposing $\lim _{k \rightarrow+\infty} \beta^{0}(k)=+\infty$. Therefore, for $k>0$ large enough (in order to have $\beta^{0}(k)>0$ ), by (46) we have

$$
\begin{aligned}
\operatorname{meas}\left\{x \in \partial \Omega: \sigma_{k}^{+}(x)=k\right\} & =\int_{\left.\left\{x \in \partial \Omega: \sigma_{k}^{+}(x)=k\right\}\right\}} \frac{\beta^{0}\left(\sigma_{k}^{+}(x)\right)}{\beta^{0}(k)} \leqslant \frac{1}{\beta^{0}(k)} \liminf _{m} \int_{\partial \Omega} \beta^{0}\left(T_{k}\left(\left(u_{m}\right)^{+}\right)\right) \\
& \leqslant \frac{1}{\beta^{0}(k)}\left(\|\phi\|_{1}+\|\psi\|_{1}\right) .
\end{aligned}
$$

We work similarly if $D(\beta)$ is bounded from below or assumption (8) holds, and, in any case, there exists $\hat{g}(k)>0$, $\lim _{k \rightarrow+\infty} \hat{g}(k)=0$, such that

$$
\operatorname{meas}\left\{x \in \partial \Omega:\left|\sigma_{k}(x)\right|=k\right\} \leqslant \hat{g}(k) \quad \forall k>0 .
$$

Hence, if we define $v(x)=T_{k}(u)(x)$ on $\left\{x \in \partial \Omega:\left|T_{k}(u)(x)\right|<k\right\}$, then
$u_{m}$ converges to $v$ a.e. in $\partial \Omega$.
Consequently, $u \in \mathcal{T}_{\text {tr }}^{1, p}(\Omega)$.

Since $z_{m}(x) \in \gamma\left(u_{m}(x)\right)$ a.e. in $\Omega$ and $w_{m}(x) \in \beta\left(u_{m}(x)\right)$ a.e. in $\partial \Omega$, from (47), (51), (53) and from the maximal monotonicity of $\gamma$ and $\beta$, we deduce that $z(x) \in \gamma(u(x))$ a.e. in $\Omega$ and $w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$.

Step 4. Finally, let us prove that $[u, z, w]$ is an entropy solution relative to $D(\beta)$ of $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. To do that, we introduce the class $\mathcal{F}$ of functions $S \in C^{2}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ satisfying

$$
\begin{aligned}
& S(0)=0, \quad 0 \leqslant S^{\prime} \leqslant 1, \quad S^{\prime}(s)=0 \quad \text { for } s \text { large enough, } \\
& S(-s)=-S(s), \quad \text { and } \quad S^{\prime \prime}(s) \leqslant 0 \quad \text { for } s \geqslant 0 .
\end{aligned}
$$

Let $v \in W^{1, p}(\Omega) \cap L^{\infty}(\Omega), v(x) \in D(\beta)$ a.e. in $\partial \Omega$, and $S \in \mathcal{F}$. Taking $S\left(u_{m}-v\right)$ as test function in (45), we get

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D S\left(u_{m}-v\right)+\int_{\Omega} z_{m} S\left(u_{m}-v\right)+\int_{\partial \Omega} w_{m} S\left(u_{m}-v\right) \\
& \quad=\int_{\partial \Omega} \psi_{m} S\left(u_{m}-v\right)+\int_{\Omega} \phi_{m} S\left(u_{m}-v\right) . \tag{54}
\end{align*}
$$

We can write the first term of (54) as

$$
\begin{equation*}
\int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D u_{m} S^{\prime}\left(u_{m}-v\right)-\int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D v S^{\prime}\left(u_{m}-v\right) . \tag{55}
\end{equation*}
$$

Since $u_{m} \rightarrow u$ and $D u_{m} \rightarrow D u$ a.e., Fatou's Lemma yields

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D u S^{\prime}(u-v) \leqslant \liminf _{m \rightarrow \infty} \int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D u_{m} S^{\prime}\left(u_{m}-v\right)
$$

The second term of (55) is estimated as follows. Let $r:=\|v\|_{\infty}+\|S\|_{\infty}$. By (52)

$$
\begin{equation*}
\mathbf{a}\left(x, D T_{r} u_{m}\right) \rightarrow \mathbf{a}\left(x, D T_{r} u\right) \quad \text { weakly in } L^{p^{\prime}}(\Omega) \tag{56}
\end{equation*}
$$

On the other hand,

$$
\left|D v S^{\prime}\left(u_{m}-v\right)\right| \leqslant|D v| \in L^{p}(\Omega) .
$$

Then, by the Dominated Convergence Theorem, we have

$$
\begin{equation*}
D v S^{\prime}\left(u_{m}-v\right) \rightarrow D v S^{\prime}(u-v) \quad \text { in } L^{p}(\Omega)^{N} . \tag{57}
\end{equation*}
$$

Hence, by (56) and (57), it follows that

$$
\lim _{m \rightarrow \infty} \int_{\Omega} \mathbf{a}\left(x, D u_{m}\right) \cdot D v S^{\prime}\left(u_{m}-v\right)=\int_{\Omega} \mathbf{a}(x, D u) \cdot D v S^{\prime}(u-v)
$$

Therefore, applying again the Dominated Convergence Theorem in the other terms of (54), we obtain

$$
\int_{\Omega} \mathbf{a}(x, D u) \cdot D S(u-v)+\int_{\Omega} z S(u-v)+\int_{\partial \Omega} w S(u-v) \leqslant \int_{\partial \Omega} \psi S(u-v)+\int_{\Omega} \phi S(u-v) .
$$

From here, to conclude, we only need to apply the technique used in the proof of [5, Lemma 3.2].
The proof of (ii) is a consequence of the existence result, Theorem 3.6(ii), and the uniqueness result.
Theorems 3.13 and 3.14 follows in a similar way taking into account Theorems 3.9 and 3.10 respectively.
Remark 5.8. In Theorem 3.11, if the data $\phi$ and $\psi$ are nonnegative (nonpositive, respectively), then assumption (8) ((7), respectively) is not necessary. That is, only assuming $[0,+\infty[\subset D(\gamma),[0,+\infty[\subset D(\beta)$ or a smooth, and assumption (7) if $\left[0,+\infty\left[\subset D(\beta)\right.\right.$, for any $0 \leqslant \phi \in L^{1}(\Omega)$ and $0 \leqslant \psi \in L^{1}(\partial \Omega)$, there exists an entropy solution of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. A similar result holds for nonpositive data.

Remark 5.9. In Theorems 3.11 and 3.13, it is not difficult to see that (7) can be substituted by one of the following assumptions,
(7') $\exists 0<\alpha \leqslant 1, r_{0}>0: \gamma^{0}(r) \geqslant r^{\alpha} \forall r \geqslant r_{0}$,
(7") $\exists 0<\alpha \leqslant 1, r_{0}>0: \beta^{0}(r) \geqslant r^{\alpha} \forall r \geqslant r_{0}$;
and (8) can be substituted by one of the following assumptions,
( $8^{\prime}$ ) $\exists 0<\alpha \leqslant 1, r_{0}>0: \gamma^{0}(r) \leqslant-(-r)^{\alpha} \forall r \leqslant-r_{0}$,
( $\left.8^{\prime \prime}\right) \exists 0<\alpha \leqslant 1, r_{0}>0: \beta^{0}(r) \leqslant-(-r)^{\alpha} \forall r \leqslant-r_{0}$.

### 5.4. Some extensions

Following the ideas developed in this work, it is possible to find a larger class of entropy solutions when $\beta$ is only assumed to have closed domain.

Definition 5.10. Let $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$. A triple of functions $[u, z, w] \in \mathcal{T}_{\mathrm{tr}}^{1, p}(\Omega) \times L^{1}(\Omega) \times L^{1}(\partial \Omega)$ is an entropy solution relative to $D(\beta)$ of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$ if $z(x) \in \gamma(u(x))$ a.e. in $\Omega, w(x) \in \beta(u(x))$ a.e. in $\partial \Omega$ and

$$
\begin{align*}
& \int_{\Omega} \mathbf{a}(x, D u) \cdot D T_{k}(u-v)+\int_{\Omega} z T_{k}(u-v)+\int_{\partial \Omega} w T_{k}(u-v) \\
& \quad \leqslant \int_{\partial \Omega} \psi T_{k}(u-v)+\int_{\Omega} \phi T_{k}(u-v) \quad \forall k>0, \tag{58}
\end{align*}
$$

for all $v \in L^{\infty}(\Omega) \cap W^{1, p}(\Omega), v(x) \in D(\beta)$ a.e. in $\partial \Omega$.
For this concept of solution we can prove the following result.
Theorem 5.11. Assume $D(\beta)$ is closed and $D(\beta) \subset D(\gamma)$. Let also assume that if $[0,+\infty[\subset D(\beta)$ the assumption (7) holds, and if $]-\infty, 0] \subset D(\beta)$ the assumption (8) holds. Then,
(i) for any $\phi \in L^{1}(\Omega)$ and $\psi \in L^{1}(\partial \Omega)$ there exists an entropy solution $[u, z, w]=\left[u_{\phi, \psi}, z_{\phi, \psi}, w_{\phi, \psi}\right]$ relative to $D(\beta)$ of problem $\left(S_{\phi, \psi}^{\gamma, \beta}\right)$. Moreover,

$$
\beta^{0}(\inf D(\beta)) \leqslant w \leqslant \beta^{0}(\sup D(\beta))
$$

and

$$
\int_{\Omega} z^{ \pm}+\int_{\partial \Omega} w^{ \pm} \leqslant \int_{\partial \Omega} \psi^{ \pm}+\int_{\Omega} \phi^{ \pm} .
$$

(ii) Given $\phi_{1}, \phi_{2} \in L^{1}(\Omega)$ and $\psi_{1}, \psi_{2} \in L^{1}(\partial \Omega)$,

$$
\int_{\Omega}\left(z_{\phi_{1}, \psi_{1}}-z_{\phi_{2}, \psi_{2}}\right)^{+}+\int_{\partial \Omega}\left(w_{\phi_{1}, \psi_{2}}-w_{\phi_{2}, \psi_{2}}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

(iii) For any $\left[u_{1}, z_{1}, w_{1}\right]$ entropy solution relative to $D(\beta)$ of problem $\left(S_{\phi_{1}, \psi_{1}}^{\gamma, \beta}\right), \phi_{1} \in L^{1}(\Omega), \psi_{1} \in L^{1}(\partial \Omega)$, and any $\left[u_{2}, z_{2}, w_{2}\right]$ entropy solution relative to $D(\beta)$ of problem $\left(S_{\phi_{2}, \psi_{2}}^{\gamma, \beta}\right), \phi_{2} \in L^{1}(\Omega), \psi_{2} \in L^{1}(\partial \Omega)$, we have that

$$
\int_{\Omega}\left(z_{1}-z_{2}\right)^{+} \leqslant \int_{\partial \Omega}\left(\psi_{1}-\psi_{2}\right)^{+}+\int_{\Omega}\left(\phi_{1}-\phi_{2}\right)^{+} .
$$

Remark 5.12. In general, for this concept of solution we have not uniqueness of $w$, as the following example shows. Let $\gamma$ and $\beta$ be such that $\gamma(0)=[0,1]$ and $\beta(0)=]-\infty, 0]$ and let $0<\phi<1$ and $\psi \leqslant 0$. Then, for any $w$ such that $\psi \leqslant w \leqslant 0,[0, \phi, w]$ is an entropy solution relative to $D(\beta)$.

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