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Large time behavior for nonlinear higher order convection–diffusion equations

Comportement asymptotique pour des equations de convection–diffusion nonlinéaires d'ordre supérieur

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Abstract

We study the large time asymptotic behavior, in L^p $(1 \leqslant p \leqslant \infty)$, of higher derivatives $D^\gamma u(t)$ of solutions of the nonlinear equation

$$\begin{cases} u_t + \mathcal{T}u = a \cdot \nabla^\theta(\psi(u)) & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 \in L^1(\mathbb{R}^n), \end{cases} \tag{1}$$

where the integers n and θ are bigger than or equal to 1, a is a constant vector in $\mathbb{R}^p$ with $p = \binom{\theta+n-1}{n-1} = \frac{(\theta+n-1)!}{\theta!(n-1)!}$. The function ψ is a nonlinearity such that $\psi \in \mathcal{C}^\theta(\mathbb{R})$ and $\psi(0) = 0$, and $\mathcal{T}$ is a higher order elliptic operator with nonsmooth bounded measurable coefficients on $\mathbb{R}^n$. We also establish faster decay when $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
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Résumé

Nous étudions le comportement asymptotique, dans L^p $(1 \leqslant p \leqslant \infty)$, des dérivées d'ordre supérieur $D^\gamma u(t)$ des solutions de l'équation nonlinéaire (1), où $n \in \mathbb{N}^*$, $\theta \in \mathbb{N}^*$ et a est un vecteur constant de $\mathbb{R}^p$ avec $p = \binom{\theta+n-1}{n-1}$. La fonction ψ est nonlinéaire vérifiant $\psi \in \mathcal{C}^\theta(\mathbb{R})$ et $\psi(0) = 0$, et $\mathcal{T}$ est un opérateur elliptique d'ordre supérieur à coefficients peu réguliers dans $\mathbb{R}^n$. Nous étudions également le cas particulier où $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.
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1. Introduction

Our aim is to study the asymptotic behavior of higher derivatives of solutions of the Cauchy problem for the generalized convection–diffusion equation (1) using sufficiently smooth nonlinearities ψ. A typical example of (1) is given by

$$\begin{cases} u_t + \big(\Delta^m A \Delta^m\big)u = a \cdot \nabla^\theta\big(\psi(u)\big) & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 \in L^1\big(\mathbb{R}^n\big), \end{cases}$$

where A is a bounded measurable positive function independent of time t.

Let us start by mentioning some works which inspired ours. Escobedo and Zuazua studied in [3] the large time behavior of solutions of the Cauchy problem for the convection–diffusion equation (1) with $\mathcal{T} = -\Delta$, $\psi(u) = |u|^{q-1}u$ and $\theta = 1$. They proved, for $q > 1$, the existence and the uniqueness of a classical solution $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^n))$ such that $u \in \mathcal{C}((0,\infty); W^{2,p}(\mathbb{R}^n)) \cap \mathcal{C}^1((0,\infty); L^p(\mathbb{R}^n))$ for every $p \in (1,\infty)$. The argument used relies essentially on the classical Banach fixed point theorem. They also showed that, for t large, this solution behaves like the heat kernel K_t which can be regarded as the fundamental solution of the heat equation with the Dirac mass as initial data. More precisely, under the assumption $q > 1 + 1/n$, if M denotes the mass of u_0 ($M = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x$) then the solution u satisfies for all $p \in [1,\infty]$,

$$\lim_{t\to+\infty} t^{\frac{n}{2}(1-\frac{1}{p})} \big\| u(t) - M K_t \big\|_p = 0.$$

They also obtained a faster decay in the particular case when the initial data u_0 belongs to $L^1(\mathbb{R}^n; 1+|x|) \cap L^q(\mathbb{R}^n)$. The techniques used rely on standard heat kernel estimates on the integral representation of the solution. Subsequently, they finished their work by an extension to the more general equation $u_t - \Delta u = a \cdot \nabla(\psi(u))$ where ψ is an arbitrary sufficiently smooth nonlinearity.

In [2], Biller, Karch and Woyczyński studied the large time behavior of solutions of the Lévy conservation laws $u_t + \mathcal{L}u + \nabla \cdot \psi(u) = 0$ with initial data u_0, where ψ is a nonlinearity and $(-\mathcal{L})$ is the generator of a positivity-preserving symmetric Lévy semigroup on $L^1(\mathbb{R}^n)$. In particular, they showed that in the case where the symbol $\mathcal{A}$ of the operator $\mathcal{L}$ satisfies $\mathcal{A}(\zeta) \sim |\zeta|^\iota$ for $|\zeta| < 1$ $(0 < \iota < 2)$ and $\mathcal{A}(\zeta) \sim |\zeta|^2$ for $|\zeta| > 1$, and under the assumptions $\psi \in \mathcal{C}^2$ with $\psi'(0) = 0$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the following holds

$$\lim_{t\to+\infty} t^{\frac{n}{\iota}(1-\frac{1}{p})} \big\| u(t) - \mathrm{e}^{-t\mathcal{L}} u_0 \big\|_p = 0 \quad \text{for every } p \in [1,\infty]$$

as for the corresponding linear equation, and if $F := \int_0^\infty \int_{\mathbb{R}^n} \psi(u(y,s))\,\mathrm{d}y\,\mathrm{d}s$,

$$\lim_{t\to+\infty} t^{\frac{n}{\iota}(1-\frac{1}{p})+\frac{1}{\iota}} \big\| u(t) - \mathrm{e}^{-t\mathcal{L}} u_0 + F \cdot \big(\nabla \mathrm{e}^{-t\mathcal{L}}\big) \big\|_p = 0 \quad \text{for every } p \in (1,\infty]$$

resulting from the presence of the nonlinear term.

In [4], we considered the equation

$$\begin{cases} u_t + \mathcal{L}_t u = a\nabla u & \text{on } \mathbb{R}^n \times (0,\infty), \\ u(0) = u_0 \in L^1\big(\mathbb{R}^n\big), \end{cases} \tag{2}$$

where $\mathcal{L}_t = L_0^* b L_0$ (see below for the definition of L_0) and b is a positive bounded function such that $b(x,t) = b(x+at)$. We showed that the derivatives $D^\gamma u$ of order less than or equal to $2m-1$ of the solution to (2) have an asymptotic behavior similar to the one of the corresponding derivatives of the heat kernel with speed a,

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \big\| D_x^\gamma u(x,t) - M D_x^\gamma K_t(x+at) \big\|_p = 0, \tag{3}$$

for all $p \in [1,\infty]$. The method used to derive this result was the simple change of variables $v(x,t) := u(x-at,t)$ which reduces the study to the heat equation

$$\begin{cases} v_t + \mathcal{L}_0 v = 0 & \text{on } \mathbb{R}^n \times (0,\infty), \\ v(0) = v_0 = u_0. \end{cases}$$

We have shown (Proposition 5, [4]) that the solution v satisfies

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \left\| D_x^\gamma v - M D_x^\gamma K_t \right\|_p = 0, \tag{4}$$

for all $p \in [1,\infty]$ and all $\gamma \in \mathbb{N}^n$ such that $|\gamma| \leqslant 2m-1$.

Taking into account this linear transformation, the principal key to obtain (4) (and then (3)) was the following estimates ([4], Theorem 7),

$$\left\| D_x^\gamma K(t) * u_0 - M D_x^\gamma K(t) \right\|_p \leqslant \begin{cases} C t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}-\frac{1}{4m}} \|u_0\|_{L^1(\mathbb{R}^n;|x|)} & \text{if } |\gamma| < 2m-1, \\ C t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}-\frac{\nu}{8m}} \|u_0\|_{L^1(\mathbb{R}^n;|x|^{\nu/2})} & \text{if } |\gamma| = 2m-1 \end{cases} \tag{5}$$

valid for $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$, $\nu \in (0,1)$, for all $t > 0$ and all $p \in [1,\infty]$. The exact value of ν is given in [1,4]. The result is then extended to the case $u_0 \in L^1(\mathbb{R}^n)$ by a simple density argument.

It is worth mentioning that in the same paper, we pointed out that (3) also holds for equations of type (2) associated to higher order operators of the form $\sum_{|\alpha|=|\beta|=m} D^\alpha a_{\alpha\beta} D^\beta$ with bounded uniformly continuous (BUC), vanish mean oscillation (VMO) or bounded mean oscillation (BMO) coefficients $a_{\alpha\beta}$. For the last case, a small BMO-norm for the coefficients is required.

In this paper, we deal with the general equation (1). In order to make our presentation clear, we divide our study into two parts. The first one concerns the asymptotic behavior when u_0 belongs to $L^1(\mathbb{R}^n)$. More precisely, under some assumptions on ψ (respectively ψ'), the solution of (1) verifies for all $p \in [1,\infty]$ and all $t > 0$,

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \left\| D_x^\gamma u(x,t) - M D_x^\gamma K_t(x) \right\|_p = 0,$$

for all $\gamma \in \mathbb{N}^n$ such that $|\gamma| + \theta \leqslant 2m-1$ (respectively $|\gamma| + \theta = 2m$). The techniques used are different of those used in [4] and are in the same spirit as those appearing in, e.g., [3,2].

In the second part, we consider the case $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and we establish a faster decay of order $t^{-1/4m}$ under the condition $|\psi(t)| \leqslant C t^{1+(4m-\theta)/n} \xi(t)$, where ξ is a continuous and nondecreasing function which vanishes when t goes to 0 (see Proposition 2.2). We also deal with the asymptotics due to the nonlinearity ψ,

$$t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|+\theta}{4m}} \left\| D_x^\gamma u(t) - D_x^\gamma \mathrm{e}^{-t\mathcal{T}} u_0 + \left(\iint_{\mathcal{D}} \psi\big(u(y,s)\big)\,\mathrm{d}y\,\mathrm{d}s \right) a \nabla^\theta \big(D_x^\gamma \mathrm{e}^{-t\mathcal{T}} \big) \right\|_p \to 0$$

when $t \to \infty$ and where $\mathcal{D} = [0,\infty] \times \mathbb{R}^n$. This is valid under assumptions $p > 1$, $|\gamma| + \theta \leqslant 2m-1$ and $|\psi(t)| \leqslant C t^q$ for $q > 1 + (4m)/n$.

Now, after describing the problem and before stating our results, let us supply a few notations which will be used throughout this paper. A part of them was used above.

For a multi-index $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n$, we set $|\lambda| = \lambda_1 + \cdots + \lambda_n$. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any multi-index $\lambda \in \mathbb{N}^n$, we have $D_x^\lambda = \frac{\partial^{\lambda_1}}{\partial x_1^{\lambda_1}} \cdots \frac{\partial^{\lambda_n}}{\partial x_n^{\lambda_n}}$ and we will often write D^λ instead of D_x^λ when there is no risk of confusion. By $\nabla^m u$ we denote the vector $(D^\lambda u)_{|\lambda|=m}$.

We shall use the classical definition for the Sobolev space $W^{m,p}$, $m \in \mathbb{Z}$ and $p \in [1,\infty]$. In particular, the notation H^m stands for $W^{m,2}$. Norms in L^p-spaces will be denoted by $\|\cdot\|_p$. We shall also use the weighted space $L^1(\mathbb{R}^n; 1+|x|) = \{f \in L^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |f(x)|(1+|x|)\,\mathrm{d}x < \infty\}$ equipped with the norm $\|f\|_{L^1(\mathbb{R}^n;|x|)} = \int_{\mathbb{R}^n} |f(x)||x|\,\mathrm{d}x$.

To complete our notation, C will denote a generic constant whose value may change from line to line.

In accordance with the notation above, we give some properties related to the class of higher order elliptic operators studied here. More details can be found in [1] and [4]. For the reader's convenience, we recall the construction of the class of operators and the related properties useful for our study.

Let $m \in \mathbb{N}^*$ and $a_{\alpha\beta}(x)$ be bounded measurable functions on $\mathbb{R}^n$ where $\alpha, \beta \in \mathbb{N}^n$ are of length m. Set

$$\mathcal{Q}(u,v) = \int_{\mathbb{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\beta u(x) D^\alpha v(x)\,\mathrm{d}x$$

for all $u, v \in H^m(\mathbb{R}^n)$. The form $\mathcal{Q}$ is continuous on $H^m(\mathbb{R}^n)$ and then by a variation on the Lax–Milgram lemma, there exists a unique operator $L : H^m(\mathbb{R}^n) \to H^{-m}(\mathbb{R}^n)$ linear and continuous such that for all $u, v \in H^m(\mathbb{R}^n)$, we have $\langle Lu, v\rangle = \mathcal{Q}(u, v)$. We write $L = (-1)^m \sum_{|\alpha|=|\beta|=m} D^\alpha(a_{\alpha\beta} D^\beta)$. Here, $\langle\ ,\ \rangle$ stands for the usual scalar product on L^2.

We suppose that the class of operators L is elliptic in the sense of the Gårding inequality (i.e., there exists a constant $\delta > 0$ such that for all $u \in H^m(\mathbb{R}^n)$, $\mathcal{Q}(u, u) \geqslant \delta\|\nabla^m u\|_2^2$). Then, as a consequence of this inequality, the operator L restricted to the domain $\{u \in H^m(\mathbb{R}^n) \mid Lu \in L^2(\mathbb{R}^n)\}$ is maximal accretive and $(-L)$ is the generator of a contraction semigroup e^{-tL} on $L^2(\mathbb{R}^n)$.

Now, let us come to our class of operators $\mathcal{T}$ appearing in (1). They are defined by $\mathcal{T} = L_0^* A L_0$ where $A \in L^\infty(\mathbb{R}^n, \mathbb{R})$ is such that $A \geqslant \sigma > 0$, L_0^* is the adjoint of L_0 and

$$L_0 = (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} D^{\alpha+\beta}$$

is a particular case of operators L defined above and is associated to the *positive constants* coefficients $a_{\alpha\beta}$.

By $K_t(x, y) \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$, we denote the distributional kernel of the semigroup $\mathrm{e}^{-t\mathcal{T}}$ and we refer to it as to the heat kernel of $\mathcal{T}$. We have shown in ([1], Proposition 51) the following useful estimates on the heat kernel.

Proposition 1.1. *There exist constants C and $c > 0$ such that for all $t \in (0, \infty)$ and all $x \in \mathbb{R}^n$, we have*

$$\left|D_x^\lambda K_t(x, y)\right| \leqslant C t^{-\frac{n+|\lambda|}{4m}} \mathcal{G}_{m,c}\left(\frac{|x-y|}{t^{1/4m}}\right), \tag{G}$$

for any multi-index $\lambda \in \mathbb{N}^n$ such that $|\lambda| \leqslant 2m - 1$, and where $\mathcal{G}_{m,\delta}(y) = \exp(-\delta y^{\frac{4m}{4m-1}})$ for $\delta > 0$.

Note that details on the restriction $|\lambda| \leqslant 2m - 1$ can be found in ([1], Proposition 51 and/or [4], Proposition 2). Also, the reader interested in full informations on the properties of the semigroup $\mathrm{e}^{-t\mathcal{T}}$ (and then of the heat kernel $K_t(x, y)$) can see [1] and [5]. In particular, let us mention the following useful result which will be used to obtain estimates on the time derivatives of the heat kernel (see, for example, [1] Lemma 33).

Lemma 1.1. *If the kernel of $\mathrm{e}^{-t\mathcal{T}}$ satisfies the estimates* (G)*, then the same estimates hold for the kernel of $t\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-t\mathcal{T}}$.*

Also, we have established in [1] that $D^\lambda \mathrm{e}^{-t\mathcal{T}}$ maps L^2 into L^∞ with estimates

$$\left\|D^\lambda \mathrm{e}^{-t\mathcal{T}}\right\|_{2,\infty} \leqslant C t^{-\frac{n}{8m}-\frac{|\lambda|}{4m}},$$

where $\|\cdot\|_{p,q}$ denotes the norm from L^p into L^q. Since the same holds for the adjoint operator $\mathcal{T}^*$, the duality then entails the ultracontractivity property $D^\lambda \mathrm{e}^{-t\mathcal{T}} : L^1 \to L^\infty$ with estimates

$$\left\|D^\lambda \mathrm{e}^{-t\mathcal{T}}\right\|_{1,\infty} \leqslant C t^{-\frac{n+|\lambda|}{4m}}.$$

We have also shown the Hölder regularity $D^\lambda \mathrm{e}^{-t\mathcal{T}} : L^1 \to \dot{\mathcal{C}}^{0,\nu}$ for $|\lambda| = 2m - 1$ and $\nu \in (0, 1)$, where $\dot{\mathcal{C}}^{0,\nu}$ is the homogeneous Hölder space.

In [5], a part of our study has concerned the extension to the derivatives taken simultaneously with respect to x and y and we have obtained

$$\left\|D^\lambda \mathrm{e}^{-t\mathcal{T}} D^\mu\right\|_{1,\infty} \leqslant C t^{-\frac{n+|\lambda|+|\mu|}{4m}}$$

with the corresponding estimates on the kernel

$$\left|D_x^\lambda D_y^\mu K_t(x, y)\right| \leqslant C t^{-\frac{n+|\lambda|+|\mu|}{4m}} \mathcal{G}_{m,c}\left(\frac{|x-y|}{t^{1/4m}}\right)$$

for all $|\lambda|, |\mu| \leqslant 2m - 1$, and

$$\left\|D^\lambda \mathrm{e}^{-t\mathcal{T}} D^\mu\right\|_{L^1 \to \dot{\mathcal{C}}^{0,\nu}} \leqslant C t^{-\frac{n+|\lambda|+|\mu|+\nu}{4m}}$$

when $|\lambda| = 2m - 1$, $|\mu| \leqslant 2m - 1$ or $|\lambda| \leqslant 2m - 1$, $|\mu| = 2m - 1$.

From now on, $K_t(x) = K(t, x)$ stands for $K_t(x, 0)$.

2. Main results

2.1. Cauchy problem

Regarding the existence, uniqueness and regularity results, the classical Banach fixed point argument (see for example [3,2,6]) yields

Proposition 2.1. *There exists a unique solution* $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}^n))$ *to* (1) *such that* $u \in \mathcal{C}((0,\infty); W^{4m,p}(\mathbb{R}^n)) \cap \mathcal{C}^1((0,\infty); L^p(\mathbb{R}^n))$ *for all* $p \in (1,\infty)$. *This solution satisfies the conservation of mass property*:

$$\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x \quad \text{for all } t \geqslant 0,$$

and the L^1*-contraction property*:

$$\|u(t)\|_1 \leqslant \|u_0\|_1 \quad \text{for all } t \geqslant 0.$$

To show the conservation integral property, we integrate (1) with respect to x and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x + \int_{\mathbb{R}^n} \mathcal{T}u(x,t)\,\mathrm{d}x = 0$$

since $\int_{\mathbb{R}^n} a\cdot\nabla^\theta(\psi(u(x,t)))\,\mathrm{d}x = 0$. On the other hand,

$$\int_{\mathbb{R}^n} \mathcal{T}u(x,t)\,\mathrm{d}x = \widehat{L_0^* w}(0,t) = \mathcal{P}(0)\widehat{w}(0,t) = 0,$$

where $\hat{f}$ denotes the Fourier transform of f in $\mathbb{R}^n$, $\mathcal{P}(\zeta) = \prod_j \zeta_j^{\alpha_i+\beta_i}$ and $w = AL_0u$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = 0$.

The proof of the L^1-contraction property is a simple adaptation of the classical argument used in ([3], Proposition 1).

2.2. Asymptotics with initial data $u_0 \in L^1(\mathbb{R}^n)$

Lemma 2.1. *Let* $u_0 \in L^1(\mathbb{R}^n)$. *Then, for all* $p \in [1,\infty]$ *there exists* $C = C(p,n,m) > 0$ *such that the solution* u *of* (1) *satisfies for all* $t > 0$,

$$\|u(t)\|_p \leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})}\|u_0\|_1.$$

The proof of this result is a simple adaptation of the argument used in ([2], Theorems 3.2–3.3 and Corollary 3.2) since $\|\mathrm{e}^{-t\mathcal{T}}\|_{2,\infty} \leqslant Ct^{-\frac{n}{8m}}$ and $\|\mathrm{e}^{-t\mathcal{T}}\|_{1,\infty} \leqslant Ct^{-\frac{n}{4m}}$. The later implies $\|u(t)\|_2 \leqslant ct^{-\frac{n}{8m}}\|u_0\|_1$ and then $\|u(t)\|_\infty \leqslant c(t/2)^{-\frac{n}{8m}}\|u(t/2)\|_2 \leqslant ct^{-\frac{n}{4m}}\|u_0\|_1$. Eventually, by interpolation and the L^1-contraction property, we obtain $\|u(t)\|_p \leqslant \|u(t)\|_1^{\frac{1}{p}}\|u(t)\|_\infty^{1-\frac{1}{p}} \leqslant ct^{-\frac{n}{4m}(1-\frac{1}{p})}\|u_0\|_1^{1-\frac{1}{p}}\|u_0\|_1^{\frac{1}{p}} = ct^{-\frac{n}{4m}(1-\frac{1}{p})}\|u_0\|_1$ (c is a generic constant).

Theorem 2.1 *(Gradient* L^p*-estimates). Suppose* $u_0 \in L^1(\mathbb{R}^n)$, $m \geqslant 2$ *and*

$$|\psi(t)| \leqslant C|t|^{1+\frac{4m-\theta}{n}} \quad \text{for all } t \in \{s \in \mathbb{R} \mid |s| \leqslant 1\}. \tag{6}$$

Then, for all $p \in [1,\infty]$ *there exists* $C = C(p,n,m) > 0$ *such that the solution* u *of* (1) *satisfies for all* $t > 0$ *and all* $\gamma \in \mathbb{N}^n$ *such that* $|\gamma| \geqslant 1$,

$$\|D_x^\gamma u(t)\|_p \leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\mathcal{K}_0 \tag{7}$$

provided $|\gamma| + \theta \leqslant 2m - 1$ *and where* $\mathcal{K}_0 = \max(\|u_0\|_1,\ \|u_0\|_1^{(n+4m-\theta)/n})$. *If in addition,* $\theta > 1$ *and*

$$\left|\psi'(t)\right| \leqslant C|t|^{\frac{4m-\theta}{n}} \quad \text{for all } t \in \left\{s \in \mathbb{R} \mid |s| \leqslant 1\right\}, \tag{8}$$

then (7) *also holds when* $|\gamma| + \theta = 2m$ *with* $\mathcal{K}_0 = \max(\|u_0\|_1, \|u_0\|_1^{(4m-\theta)/n})$.

Proof. The solution u of (1) satisfies the integral equation

$$u(t) = K(t) * u_0 + \int_0^t K(t-s) * a \cdot \nabla^\theta\big(\psi\big(u(s)\big)\big)\,\mathrm{d}s$$

and then

$$D^\gamma u(t) = D^\gamma K(t) * u_0 + \int_0^t D^\gamma K(t-s) * a \cdot \nabla^\theta\big(\psi\big(u(s)\big)\big)\,\mathrm{d}s, \tag{9}$$

where $*$ is the convolution symbol with respect to the space variable x.

Assume that $|\gamma| + \theta \leqslant 2m - 1$. It follows from (9) that

$$D^\gamma u(t) = D^\gamma K(t) * u_0 + \int_0^t a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big) * \psi\big(u(s)\big)\,\mathrm{d}s. \tag{10}$$

Hence, using Young inequality yields

$$\begin{aligned}
\left\|D^\gamma u(t)\right\|_p &\leqslant \left\|D^\gamma K(t) * u_0\right\|_p + \int_0^t \left\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big) * \psi\big(u(s)\big)\right\|_p \mathrm{d}s \\
&\leqslant \left\|D^\gamma K(t)\right\|_p \|u_0\|_1 + \int_0^{t/2} \left\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big)\right\|_p \left\|\psi\big(u(s)\big)\right\|_1 \mathrm{d}s \\
&\quad + \int_{t/2}^t \left\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big)\right\|_1 \left\|\psi\big(u(s)\big)\right\|_p \mathrm{d}s.
\end{aligned}$$

On the one hand, (G) implies that

$$\left\|D_x^\lambda K(t)\right\|_p \leqslant C_{p,m} t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}} \tag{11}$$

for all $p \in [1,\infty]$ and all $\lambda \in \mathbb{N}^n$ such that $|\lambda| \leqslant 2m-1$. On the other hand, by using successively (11), (6) and Lemma 2.1 we get

$$\begin{aligned}
\int_0^{t/2} \left\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big)\right\|_p \left\|\psi\big(u(s)\big)\right\|_1 \mathrm{d}s &\leqslant C|a| \int_0^{t/2} (t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \left\|u(s)\right\|_{(n+4m-\theta)/n}^{(n+4m-\theta)/n} \mathrm{d}s \\
&\leqslant C|a| \|u_0\|_1^{(n+4m-\theta)/n} \int_0^{t/2} (t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} s^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s \\
&\leqslant C|a| \|u_0\|_1^{(n+4m-\theta)/n} \left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \int_0^{t/2} s^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s \\
&\leqslant C|a| \left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}} \|u_0\|_1^{(n+4m-\theta)/n}
\end{aligned}$$

and

$$\int_{t/2}^{t} \left\|a\cdot\nabla^{\theta}\big(D^{\gamma}K(t-s)\big)\right\|_1 \left\|\psi\big(u(s)\big)\right\|_p \,\mathrm{d}s \leqslant C|a|\int_{t/2}^{t}(t-s)^{-\frac{|\gamma|+\theta}{4m}}\left\|u(s)\right\|_{p(n+4m-\theta)/n}^{(n+4m-\theta)/n}\,\mathrm{d}s$$
$$\leqslant C|a|\|u_0\|_1^{(n+4m-\theta)/n}\int_{t/2}^{t}(t-s)^{-\frac{|\gamma|+\theta}{4m}}s^{-\frac{1}{4mp}(p(n+4m-\theta)-n)}\,\mathrm{d}s$$
$$\leqslant C|a|\|u_0\|_1^{(n+4m-\theta)/n}\left(\frac{t}{2}\right)^{-\frac{1}{4mp}(p(n+4m-\theta)-n)}\left(\frac{t}{2}\right)^{1-\frac{|\gamma|+\theta}{4m}}$$
$$= C|a|\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\|u_0\|_1^{(n+4m-\theta)/n}.$$

Eventually, combining the last estimates to (11) yields (7) in the case $|\gamma|+\theta\leqslant 2m-1$.

To deal with the case $|\gamma|+\theta=2m$, we rewrite (10) as

$$D^{\gamma}u(t)=D^{\gamma}K(t)*u_0+\int_0^t a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)*\nabla\psi\big(u(s)\big)\,\mathrm{d}s$$
$$=D^{\gamma}K(t)*u_0+\int_0^t a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)*\psi'\big(u(s)\big)\nabla u(s)\,\mathrm{d}s. \tag{12}$$

The same decomposition as in the first case yields

$$\left\|D^{\gamma}u(t)\right\|_p\leqslant\left\|D^{\gamma}K(t)\right\|_p\|u_0\|_1+\int_0^{t/2}\left\|a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)\right\|_p\left\|\psi'\big(u(s)\big)\nabla u(s)\right\|_1\mathrm{d}s$$
$$+\int_{t/2}^{t}\left\|a\cdot\nabla^{\theta}\big(D^{\gamma}K(t-s)\big)\right\|_1\left\|\psi'\big(u(s)\big)\nabla u(s)\right\|_p\mathrm{d}s.$$

Using (8), (7) (for the case $|\gamma|=1$), (11) (since $|\gamma|+\theta-1=2m-1$) and Lemma 2.1 implies

$$\int_0^{t/2}\left\|a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)\right\|_p\left\|\psi'\big(u(s)\big)\nabla u(s)\right\|_1\mathrm{d}s$$
$$\leqslant\int_0^{t/2}\left\|a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)\right\|_p\left\|\psi'\big(u(s)\big)\right\|_1\left\|\nabla u(s)\right\|_\infty\mathrm{d}s$$
$$\leqslant\int_0^{t/2}\left\|a\cdot\nabla^{\theta-1}\big(D^{\gamma}K(t-s)\big)\right\|_p\left\|u(s)\right\|_{(4m-\theta)/n}^{(4m-\theta)/n}\left\|\nabla u(s)\right\|_\infty\mathrm{d}s$$
$$\leqslant C|a|\|u_0\|_1^{(4m-\theta)/n}\int_0^{t/2}(t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta-1}{4m}}s^{-(1-\frac{n+\theta}{4m})}s^{-\frac{n+1}{4m}}\,\mathrm{d}s$$
$$\leqslant C|a|\|u_0\|_1^{(4m-\theta)/n}\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta-1}{4m}}\int_0^{t/2}s^{-(1-\frac{\theta-1}{4m})}\,\mathrm{d}s$$
$$\leqslant C|a|\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\|u_0\|_1^{(4m-\theta)/n}$$

and

$$\int_{t/2}^{t} \left\| a \cdot \nabla^{\theta-1}\left(D^{\gamma} K(t-s)\right)\right\|_1 \left\|\psi'\left(u(s)\right)\nabla u(s)\right\|_p \,\mathrm{d}s$$

$$\leqslant \int_{t/2}^{t} \left\| a \cdot \nabla^{\theta-1}\left(D^{\gamma} K(t-s)\right)\right\|_1 \left\|\psi'\left(u(s)\right)\right\|_p \left\|\nabla u(s)\right\|_\infty \,\mathrm{d}s$$

$$\leqslant \int_{t/2}^{t} \left\| a \cdot \nabla^{\theta-1}\left(D^{\gamma} K(t-s)\right)\right\|_1 \left\|u(s)\right\|_{p(4m-\theta)/n}^{(4m-\theta)/n} \left\|\nabla u(s)\right\|_\infty \,\mathrm{d}s$$

$$\leqslant C|a|\|u_0\|_1^{(4m-\theta)/n} \int_{t/2}^{t} (t-s)^{-\frac{|\gamma|+\theta-1}{4m}} s^{-\frac{1}{4mp}(p(4m-\theta)-n)} s^{-\frac{n+1}{4m}} \,\mathrm{d}s$$

$$\leqslant C|a|\|u_0\|_1^{(4m-\theta)/n} \left(\frac{t}{2}\right)^{-\frac{1}{4mp}(p(4m-\theta)-n)-\frac{n+1}{4m}} \left(\frac{t}{2}\right)^{1-\frac{|\gamma|+\theta-1}{4m}}$$

$$= C|a|\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}} \|u_0\|_1^{(4m-\theta)/n}.$$

Therefore, (7) is verified in the case $|\gamma|+\theta = 2m$, $\theta > 1$ and this ends the proof of Theorem 2.1. □

Remarks 2.1. 1. It is worth mentioning that (7) is verified for $m \geqslant 1$ in the case $|\gamma|+\theta \leqslant 2m-1$ ($m=1$ is a particular case of Lemma 2.1 since $\theta \geqslant 1$, $|\gamma|+\theta \leqslant 1$ and then $|\gamma|=0$). The choice $m \geqslant 2$ is adopted in order to guarantee the estimates on the gradient $\|\nabla u(t)\|_p$ used in the proof of the second case $|\gamma|+\theta = 2m$.

2. The condition (6) (resp. (8)) implies that for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$\left|\psi(t)\right| \leqslant C_\delta |t|^{(n+4m-\theta)/n}$$

(resp. $|\psi'(t)| \leqslant C_\delta |t|^{(4m-\theta)/n}$) for all $t \in \{s \in \mathbb{R} \mid |s| \leqslant \delta\}$.

3. It seems possible that under additional assumptions (like (6) and (8)) on the successive derivatives of ψ, we can establish (7) for $|\gamma| \leqslant 2m-1$. This remark also applies to (14) in the forthcoming theorem.

The property (7) suggests to study the problem of the large time behavior of $t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} D_x^\gamma u(t,x)$ in L^p-norm. Indeed, in the following result we show that, for t large, the higher derivatives $D_x^\gamma u(t)$ of the solution behave like the corresponding derivatives $D_x^\gamma K(t)$ of the heat kernel. More precisely,

Theorem 2.2. *Assume that $m \geqslant 2$ and*

$$\lim_{|t|\to 0} \frac{\psi(t)}{|t|^{1+\frac{4m-\theta}{n}}} = 0. \tag{13}$$

Then, for all $u_0 \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = M$, the solution u of (1) *satisfies for all $p \in [1,\infty]$,*

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \left\| D_x^\gamma u(x,t) - M D_x^\gamma K_t(x)\right\|_p = 0 \tag{14}$$

for all multi-index $\gamma \in \mathbb{N}^n$ such that $|\gamma|+\theta \leqslant 2m-1$. If in addition

$$\lim_{|t|\to 0} \frac{\psi'(t)}{|t|^{\frac{4m-\theta}{n}}} = 0, \tag{15}$$

then the property (14) *remains valid when $|\gamma|+\theta = 2m$ and $\theta > 1$.*

Proof. Suppose that $|\gamma| + \theta \leqslant 2m - 1$ and write

$$D^\gamma u(t) - M D^\gamma K(t+1) = \big(D^\gamma u(t+1) - M D^\gamma K(t)\big) - M\big(D^\gamma K(t+1) - D^\gamma K(t)\big).$$

It follows from (10) that

$$D^\gamma u(t+1) - M D^\gamma K(t+1) = \mathcal{A}_1(t) - M\mathcal{A}_2(t) + \mathcal{A}_3(t),$$

where

$$\mathcal{A}_1(t) := D^\gamma K(t) * u(1) - M D^\gamma K(t),$$
$$\mathcal{A}_2(t) := D^\gamma K(t+1) - D^\gamma K(t),$$
$$\mathcal{A}_3(t) := \int_0^t a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big) * \psi\big(u(s+1)\big)\,\mathrm{d}s.$$

Since $\int_{\mathbb{R}^n} u(x,1)\,\mathrm{d}x = \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = M$, it then follows by (4) that

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\big\|\mathcal{A}_1(t)\big\|_p = 0. \tag{16}$$

On the other hand, Lemma 1.1 yields $\|\mathcal{A}_2(t)\|_p \leqslant t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}-1}$ and then

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\big\|\mathcal{A}_2(t)\big\|_p = 0. \tag{17}$$

Note that (16) and (17) are verified for all $\gamma \in \mathbb{N}^n$ such that $|\gamma| \leqslant 2m-1$ and all $p \in [1,\infty]$.

Taking into account these estimates, it remains to show that

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\big\|\mathcal{A}_3(t)\big\|_p = 0 \tag{18}$$

for all $p \in [1,\infty]$ and all $\gamma \in \mathbb{N}^n$ such that $|\gamma| + \theta \leqslant 2m-1$.

We have $\|\mathcal{A}_3(t)\|_p \leqslant \mathcal{B}_1 + \mathcal{B}_2$ where

$$\mathcal{B}_1 := \int_0^{t/2} \big\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big)\big\|_p \big\|\psi\big(u(s+1)\big)\big\|_1\,\mathrm{d}s$$

$$\mathcal{B}_2 := \int_{t/2}^{t} \big\|a \cdot \nabla^\theta\big(D^\gamma K(t-s)\big)\big\|_1 \big\|\psi\big(u(s+1)\big)\big\|_p\,\mathrm{d}s.$$

If we set $\xi_0(s) := \psi(s)/|s|^{(n+4m-\theta)/n}$, then

$$\big\|\psi\big(u(s)\big)\big\|_p \leqslant \big\|\xi_0\big(u(s)\big)\big\|_\infty \big\|u(s)\big\|^{(n+4m-\theta)/n}_{p(n+4m-\theta)/n}. \tag{19}$$

Therefore, using successively (11), (19) and Lemma 2.1 involves estimates on $\mathcal{B}_1$ and $\mathcal{B}_2$ as follows

$$\begin{aligned}
\mathcal{B}_1 &\leqslant C|a| \int_0^{t/2} (t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty \big\|u(s+1)\big\|^{(n+4m-\theta)/n}_{(n+4m-\theta)/n}\,\mathrm{d}s \\
&\leqslant C|a| \int_0^{t/2} (t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty (s+1)^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s \\
&\leqslant C|a| \left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \int_0^{t/2} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty (s+1)^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s \\
&:= C|a| \left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \mathcal{X}(t)
\end{aligned} \tag{i}$$

and

$$\begin{aligned}
\mathcal{B}_2 &\leqslant C|a| \int_{t/2}^{t} (t-s)^{-\frac{|\gamma|+\theta}{4m}} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty \big\|u(s+1)\big\|_{p(n+4m-\theta)/n}^{(n+4m-\theta)/n} \,\mathrm{d}s \\
&\leqslant C|a| \int_{t/2}^{t} (t-s)^{-\frac{|\gamma|+\theta}{4m}} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty (s+1)^{-\frac{1}{4mp}(p(n+4m-\theta)-n)} \,\mathrm{d}s \\
&\leqslant C|a| \sup_{s\geqslant t/2+1} \big\|\xi_0\big(u(s)\big)\big\|_\infty \left(\frac{t}{2}\right)^{-\frac{1}{4mp}(p(n+4m-\theta)-n)} \left(\frac{t}{2}\right)^{1-\frac{|\gamma|+\theta}{4m}} \\
&\leqslant C|a| \sup_{s\geqslant t/2+1} \big\|\xi_0\big(u(s)\big)\big\|_\infty t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}. \qquad \text{(ii)}
\end{aligned}$$

It follows from (ii) that $\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\mathcal{B}_2 = 0$ since $\lim_{t\to\infty}\sup_{s\geqslant t/2+1}\|\xi_0(u(s))\|_\infty = 0$ and therefore, to obtain (18), it is enough to prove that $\lim_{t\to\infty} t^{-\frac{\theta}{4m}}\mathcal{X}(t) = 0$ since $t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\mathcal{B}_1 \leqslant C|a|t^{-\frac{\theta}{4m}}\mathcal{X}(t)$ thanks to (i).

On the one hand, since

$$\lim_{t\to\infty}\big\|\xi_0\big(u(t)\big)\big\|_\infty = 0$$

then for all $\varepsilon > 0$ there exists $B > 0$ such that $\|\xi_0(u(s+1))\|_\infty \leqslant \varepsilon$ for all $s \geqslant B$ and it then follows that

$$t^{-\frac{\theta}{4m}} \int_{B}^{t/2} \big\|\xi_0\big(u(s+1)\big)\big\|_\infty s^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s \leqslant \frac{4m\varepsilon}{\theta} t^{-\frac{\theta}{4m}} \left(\left(\frac{t}{2}\right)^{\frac{\theta}{4m}} - B^{\frac{\theta}{4m}}\right) \leqslant 4m\theta^{-1}2^{-\frac{\theta}{4m}}\varepsilon.$$

On the other hand

$$\lim_{t\to\infty} t^{-\frac{\theta}{4m}} \int_0^B \big\|\xi_0\big(u(s+1)\big)\big\|_\infty s^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s = 0.$$

Therefore $\lim_{t\to\infty} t^{-\frac{\theta}{4m}}\mathcal{X}(t) = 0$ and (14) follows.

For the case $|\gamma| + \theta = 2m$, we derive from (12)

$$D^\gamma u(t+1) - MD^\gamma K(t+1) = \mathcal{F}_1(t) - M\mathcal{F}_2(t) + \mathcal{F}_3(t),$$

where

$$\begin{aligned}
\mathcal{F}_1(t) &:= \mathcal{A}_1(t), \\
\mathcal{F}_2(t) &:= \mathcal{A}_2(t), \\
\mathcal{F}_3(t) &:= \int_0^t a\cdot\nabla^{\theta-1}\big(D^\gamma K(t-s)\big) * \psi'\big(u(s+1)\big)\nabla u(s+1)\,\mathrm{d}s.
\end{aligned}$$

Then, as for the first case, we show that

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}}\big\|\mathcal{F}_3(t)\big\|_p = 0 \qquad (20)$$

for all $p \in [1,\infty]$ and all $\gamma \in \mathbb{N}^n$ such that $|\gamma| + \theta = 2m$.

The same decomposition used in the proof of Theorem 2.1 implies that $\|\mathcal{F}_3(t)\|_p \leqslant \mathcal{M}_1 + \mathcal{M}_2$ where

$$\mathcal{M}_1 := \int_0^{t/2} \big\|a\cdot\nabla^{\theta-1}\big(D^\gamma K(t-s)\big)\big\|_p \big\|\psi'\big(u(s+1)\big)\big\|_1 \big\|\nabla u(s+1)\big\|_\infty\,\mathrm{d}s,$$

$$\mathcal{M}_2 := \int_{t/2}^{t} \left\| a \cdot \nabla^{\theta-1}\left(D^\gamma K(t-s)\right)\right\|_1 \left\|\psi'\left(u(s+1)\right)\right\|_p \left\|\nabla u(s+1)\right\|_\infty \mathrm{d}s,$$

and the same computations yield

$$\mathcal{M}_1 \leqslant C|a|\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta-1}{4m}} \mathcal{Z}(t)$$

and

$$\begin{aligned}\mathcal{M}_2 &\leqslant C|a|\int_{t/2}^{t} (t-s)^{-\frac{|\gamma|+\theta-1}{4m}} \left\|\xi_1\left(u(s+1)\right)\right\|_\infty (s+1)^{-\frac{1}{4mp}(p(4m-\theta)-n)}(s+1)^{-\frac{n+1}{4m}} \mathrm{d}s \\ &\leqslant C|a| \sup_{s\geqslant t/2+1} \left\|\xi_1\left(u(s)\right)\right\|_\infty t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}},\end{aligned}$$

where

$$\mathcal{Z}(t) := \int_0^{t/2} \left\|\xi_1\left(u(s+1)\right)\right\|_\infty (s+1)^{-(1-\frac{\theta-1}{4m})} \mathrm{d}s,$$

$$\xi_1(s) := \frac{\psi'(s)}{|s|^{(4m-\theta)/n}}.$$

Eventually, as for $\mathcal{B}_1$ and $\mathcal{B}_2$, $\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|}{4m}} \mathcal{M}_i = 0$ $(i = 1,2)$ since $\lim_{t\to\infty} t^{-\frac{\theta-1}{4m}} \mathcal{Z}(t) = 0$ and $\lim_{t\to\infty} \sup_{s\geqslant t/2+1} \|\xi_1(u(s))\|_\infty = 0$. This implies (20) and ends the proof of Theorem 2.2. □

Remark 2.1. See Remarks 2.1.1 and 2.1.3 respectively for the case $m = 1$ and for the possible extension of (14) to derivatives $D^\gamma u$ of order $|\gamma| \leqslant 2m-1$.

2.3. Faster decay when initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

In this section, we intend to obtain faster decay rate for higher derivatives when $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. For this purpose, let us first give a lemma which will be used to state our results.

Lemma 2.2. *Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then, the solution u of* (1) *satisfies*

$$\left\|u(t)\right\|_p \leqslant C(1+t)^{-\frac{n}{4m}(1-\frac{1}{p})}$$

for all $t > 0$ and all $p \in [1,\infty]$, and where $C > 0$ is a constant depending on $\|u_0\|_1$ and $\|u_0\|_p$.

As for Lemma 2.1, the proof is a straightforward adaptation of the argument used in ([2], Corollary 3.2).

2.3.1. Asymptotics like for the linear equation

Theorem 2.3. *Let ψ satisfying*

$$\left|\psi(t)\right| \leqslant C|t|^q \quad \text{for all } t \in \left\{s \in \mathbb{R} \mid |s| \leqslant 1\right\}, \tag{21}$$

with $q > 1 + \frac{4m-\theta}{n}$ and $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = M$.

Under these assumptions, there exists $\nu \in (0,1)$ such that the solution u of (1) *satisfies for all $t > 0$,*

$$\left\|D_x^\gamma u(x,t) - M D_x^\gamma K_t(x)\right\|_p \leqslant C t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}} \mathcal{S}(t), \tag{22}$$

where

$$\mathcal{S}(t)=\begin{cases}\begin{cases}t^{-\frac{1}{4m}} & \text{if } q>1+\frac{4m-\theta+1}{n},\\ t^{-\frac{1}{4m}(n(q-1)-(4m-\theta))} & \text{if } q\in(1+\frac{4m-\theta}{n},1+\frac{4m-\theta+1}{n}),\\ t^{-\frac{1}{4m}}\ln(t+2) & \text{if } q=1+\frac{4m-\theta+1}{n}\end{cases}\\ \text{provided } |\gamma|+\theta<2m-1,\\ \begin{cases}t^{-\frac{\nu}{8m}} & \text{if } q>1+\frac{4m-\theta+(\nu/2)}{n},\\ t^{-\frac{1}{4m}(n(q-1)-(4m-\theta))} & \text{if } q\in(1+\frac{4m-\theta}{n},1+\frac{4m-\theta+(\nu/2)}{n}),\\ t^{-\frac{\nu}{8m}}\ln(t+2) & \text{if } q=1+\frac{4m-\theta+(\nu/2)}{n}\end{cases}\\ \text{provided } |\gamma|+\theta=2m-1.\end{cases}$$

Proof. Since $L^1(\mathbb{R}^n;1+|x|)$ is dense into $L^1(\mathbb{R}^n)$, we prove (22) for $u_0\in L^1(\mathbb{R}^n;1+|x|)\cap L^\infty(\mathbb{R}^n)$ and the density argument used in ([4], Section 4, step 3) extends the result to the case $u_0\in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$.

Assume that $u_0\in L^1(\mathbb{R}^n;1+|x|)\cap L^\infty(\mathbb{R}^n)$. The solution of (1) verifies

$$D^\gamma u(t)=D^\gamma K(t)*u_0+\int_0^t a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\,\mathrm{d}s.$$

Since $u_0\in L^1(\mathbb{R}^n;1+|x|)$ then there exists $\nu\in(0,1)$ such that (5) holds for all $t>0$ and all $p\in[1,\infty]$. Therefore, it suffices to show that

$$\left\|\int_0^t a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\,\mathrm{d}s\right\|_p\leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\mathcal{S}(t).$$

Notice first that according to Remarks 2.1.2, the condition (21) implies that for all $\delta>0$ there exists $C_\delta>0$ such that $|\psi(t)|\leqslant C_\delta|t|^q$ for all $t\in\{s\in\mathbb{R}\mid |s|\leqslant\delta\}$. Hence, as in the previous section, using successively Young's inequality, (21), (11) and Lemma 2.2 we obtain

$$\begin{aligned}\int_{t/2}^t\big\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\big\|_p\,\mathrm{d}s&\leqslant\int_{t/2}^t\big\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)\big\|_1\big\|\psi\big(u(s)\big)\big\|_p\,\mathrm{d}s\\&\leqslant C|a|\int_{t/2}^t(t-s)^{-\frac{|\gamma|+\theta}{4m}}\big\|u(s)\big\|_{pq}^q\,\mathrm{d}s\\&\leqslant C|a|\int_{t/2}^t(t-s)^{-\frac{|\gamma|+\theta}{4m}}(s+1)^{-\frac{n}{4mp}(pq-1)}\,\mathrm{d}s\\&\leqslant C|a|t^{-\frac{n}{4mp}(pq-1)}t^{1-\frac{|\gamma|+\theta}{4m}}\\&=C|a|t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\,t^{-\frac{1}{4m}(n(q-1)-(4m-\theta))}\\&\leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\,\mathcal{S}(t)\end{aligned}$$

since $t^{-\frac{1}{4m}(n(q-1)-(4m-\theta))}\leqslant C\mathcal{S}(t)$ for all $q>1+\frac{4m-\theta}{n}$. On the other hand,

$$\int_0^{t/2}\big\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\big\|_p\,\mathrm{d}s\leqslant\int_0^{t/2}\big\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)\big\|_p\big\|\psi\big(u(s)\big)\big\|_1\,\mathrm{d}s$$

$$\leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}}\int_0^{t/2}(s+1)^{-\frac{n}{4m}(q-1)}\,\mathrm{d}s$$

$$\leqslant Ct^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\mathcal{S}(t)$$

since $t^{-\frac{\theta}{4m}}\int_0^{t/2}(s+1)^{-\frac{n}{4m}(q-1)}\,\mathrm{d}s \leqslant C\mathcal{S}(t)$ for all $q > 1+\frac{4m-\theta}{n}$.

Finally, (22) is proved for $u_0 \in L^1(\mathbb{R}^n; 1+|x|)\cap L^\infty(\mathbb{R}^n)$ and hence for $u_0 \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ by density. □

Remark 2.2. Note that by adding a condition on ψ', the techniques used in Section 2.2 allow to extend the result above to the case $|\gamma|+\theta = 2m$.

Theorem 2.3 can be generalized as follows

Proposition 2.2 *(Generalization). Let u_0 be as in Theorem 2.3 and ψ be such that*

$$\left|\psi(t)\right| \leqslant C|t|^{1+\frac{4m-\theta}{n}}\xi(t),$$

where ξ is a continuous and nondecreasing function such that $\lim_{t\to 0}\xi(t) = 0$.

Then (22) *holds with*

$$\mathcal{S}(t) = \begin{cases} \max\left(t^{-\frac{1}{4m}}, t^{-\frac{\theta}{4m}}\int_0^{t/2}\xi\big((s+1)^{-\frac{n}{4m}}\big)(s+1)^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s, \xi\left(\left(\frac{t}{2}+1\right)^{-\frac{n}{4m}}\right)\right) \\ \quad \text{if } |\gamma|+\theta < 2m-1, \\ \max\left(t^{-\frac{\nu}{8m}}, t^{-\frac{\theta}{4m}}\int_0^{t/2}\xi\big((s+1)^{-\frac{n}{4m}}\big)(s+1)^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s, \xi\left(\left(\frac{t}{2}+1\right)^{-\frac{n}{4m}}\right)\right) \\ \quad \text{if } |\gamma|+\theta = 2m-1. \end{cases}$$

Proof. Indeed, if we go over the proof of Theorem 2.3, then thanks to Lemma 2.2 and the properties of ξ we get

$$\int_{t/2}^{t}\left\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\right\|_p\mathrm{d}s \leqslant C\int_{t/2}^{t}\left\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)\right\|_1\left\|\xi\big(u(s)\big)\right\|_\infty\left\|u(s)\right\|_{p(n+4m-\theta)/n}^{(n+4m-\theta)/n}\mathrm{d}s$$

$$\leqslant C\int_{t/2}^{t}(t-s)^{-\frac{|\gamma|+\theta}{4m}}\xi\big((s+1)^{-\frac{n}{4m}}\big)(s+1)^{-\frac{1}{4mp}(p(n+4m-\theta)-n)}\,\mathrm{d}s$$

$$\leqslant C\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}}\xi\left(\left(\frac{t}{2}+1\right)^{-\frac{n}{4m}}\right)$$

and

$$\int_0^{t/2}\left\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)*\psi\big(u(s)\big)\right\|_p\mathrm{d}s \leqslant C\int_0^{t/2}\left\|a\cdot\nabla^\theta\big(D^\gamma K(t-s)\big)\right\|_p\left\|\xi\big(u(s)\big)\right\|_\infty\left\|u(s)\right\|_{(n+4m-\theta)/n}^{(n+4m-\theta)/n}\mathrm{d}s$$

$$\leqslant C\left(\frac{t}{2}\right)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|}{4m}-\frac{\theta}{4m}}\int_0^{t/2}\xi\big((s+1)^{-\frac{n}{4m}}\big)(s+1)^{-(1-\frac{\theta}{4m})}\,\mathrm{d}s.$$

Proposition 2.2 is completely proved. □

2.3.2. Asymptotics related to nonlinear effects

Theorem 2.4. *Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and the function ψ satisfies* (21) *for $q > 1 + \frac{4m}{n}$. Then the solution u of* (1) *verifies for all $p \in (1, \infty]$ and all multi-index $\gamma \in \mathbb{N}^n$ such that $|\gamma| + \theta \leqslant 2m - 1$,*

$$\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|+\theta}{4m}} \left\| D_x^\gamma u(t) - D_x^\gamma \mathrm{e}^{-t\mathcal{T}} u_0 + \mathcal{M}\, a\nabla^\theta\left(D_x^\gamma \mathrm{e}^{-t\mathcal{T}}\right)\right\|_p = 0,$$

where $\mathcal{M} = \iint_{\mathcal{D}} \psi(u(y,s))\,\mathrm{d}y\,\mathrm{d}s$ and $\mathcal{D} = [0,\infty] \times \mathbb{R}^n$.

Proof. Since $D^\gamma u(t) = D^\gamma \mathrm{e}^{-t\mathcal{T}} u_0 - \int_0^t a\nabla^\theta(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}})\psi(u(s))\,\mathrm{d}s$, it suffices to estimate $\|W(t)\|_p$ where

$$W(t) = \mathcal{M} a\nabla^\theta\left(D^\gamma \mathrm{e}^{-t\mathcal{T}}\right) - \int_0^t a\nabla^\theta\left(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\right)\psi\left(u(s)\right)\mathrm{d}s := I_1(0,\infty) - I_2(0,t),$$

with

$$I_1(c,d) = \left(\int_c^d \int_{\mathbb{R}^n} \psi\left(u(y,s)\right)\mathrm{d}y\,\mathrm{d}s\right) a\nabla^\theta\left(D^\gamma e^{-t\mathcal{T}}\right),$$

$$I_2(c,d) = \int_c^d a\nabla^\theta\left(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\right)\psi\left(u(s)\right)\mathrm{d}s.$$

Notice first that according to (G),

$$\left\|D^\lambda \mathrm{e}^{-t\mathcal{T}}\right\|_{1,p} \leqslant C_{p,m} t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}} \quad \text{for } |\lambda| \leqslant 2m-1, \tag{23}$$

and in view of the hypothesis on ψ, Remarks 2.1.2, Lemma 2.2 and the fact that $q > 1 + 4m/n$, we obtain

$$\iint_{\mathcal{D}} \left|\psi\left(u(y,s)\right)\right|\mathrm{d}y\,\mathrm{d}s \leqslant C \iint_{\mathcal{D}} \left|u(y,s)\right|^q \mathrm{d}y\,\mathrm{d}s \leqslant C\int_0^\infty (1+s)^{-\frac{n}{4m}(q-1)}\,\mathrm{d}s < \infty. \tag{24}$$

1. L^p-estimates of $I_1(t/2,\infty)$.

From (23) we easily derive the estimates

$$\left\|I_1(t/2,\infty)\right\|_p \leqslant C t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \int_{t/2}^\infty \int_{\mathbb{R}^n} \left|\psi\left(u(y,s)\right)\right|\mathrm{d}y\,\mathrm{d}s$$

and then $\lim_{t\to+\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|+\theta}{4m}} \|I_1(t/2,\infty)\|_p = 0$ since $\lim_{t\to+\infty} \int_{t/2}^\infty \int_{\mathbb{R}^n} |\psi(u(y,s))|\,\mathrm{d}y\,\mathrm{d}s = 0$ thanks to (24).

2. L^p-estimates of $I_2(t/2,t)$.

The same computations used in the proof of Theorem 2.3 and (23) imply that

$$\left\|I_2(t/2,t)\right\|_p \leqslant C t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} t^{-\frac{1}{4m}(q-(1+4m/n))}.$$

Hence, $\lim_{t\to+\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|+\theta}{4m}} \|I_2(t/2,t)\|_p = 0$ since $q > 1 + 4m/n$.

3. L^p-estimates of $(I_1(0,t/2) - I_2(0,t/2))$.

Writing

$$I_2(0,t/2) = \int_0^{t/2} a\nabla^\theta\left(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\right)\psi\left(u(s)\right)\mathrm{d}s = \int_0^{t/2}\int_{\mathbb{R}^n} a\nabla^\theta\left(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\right)(\cdot - y)\psi\left(u(y,s)\right)\mathrm{d}y\,\mathrm{d}s.$$

Then

$$\left\| I_1(0,t/2) - I_2(0,t/2) \right\|_p = \left\| \int_0^{t/2} \int_{\mathbb{R}^n} a \nabla^\theta \left(\left(D^\gamma \, \mathrm{e}^{-(t-s)\mathcal{T}} \right)(\cdot - y) - \left(D^\gamma \, \mathrm{e}^{-t\mathcal{T}} \right)(\cdot) \right) \psi\left(u(y,s)\right) \mathrm{d}y \, \mathrm{d}s \right\|_p .$$

Let us first estimate the term $\| I_1(0,t/2) - I_2(0,t/2) \|_p$ for $p = 1$. We have

$$\begin{aligned}
\left\| I_1(0,t/2) - I_2(0,t/2) \right\|_1 &\leqslant C \left\| \int_0^{t/2} \int_{\mathbb{R}^n} \nabla^\theta \left(\left(D^\gamma \, \mathrm{e}^{-(t-s)\mathcal{T}} \right) \right)(\cdot - y) \psi\left(u(y,s)\right) \mathrm{d}y \, \mathrm{d}s \right\|_1 \\
&\quad + C \left\| \nabla^\theta \left(D^\gamma \, \mathrm{e}^{-t\mathcal{T}} \right)(\cdot) \right\|_1 \int_0^\infty \int_{\mathbb{R}^n} \left| \psi\left(u(y,s)\right) \right| \mathrm{d}y \, \mathrm{d}s \\
&\leqslant C t^{-\frac{|\gamma|+\theta}{4m}}
\end{aligned}$$

by using (23) and the same computations as in the proof of Theorem 2.3.

Now, suppose that $p > 1$ and write

$$\begin{aligned}
\left\| I_1(0,t/2) - I_2(0,t/2) \right\|_p &= \left\| \int_0^{t/2} \int_{\mathbb{R}^n} a \nabla^\theta \left(\left(D^\gamma \, \mathrm{e}^{-(t-s)\mathcal{T}} \right)(\cdot - y) - \left(D^\gamma \, \mathrm{e}^{-t\mathcal{T}} \right)(\cdot) \right) \psi\left(u(y,s)\right) \mathrm{d}y \, \mathrm{d}s \right\|_p \\
&\leqslant \left\| \iint_{\mathcal{D}_1} \cdots \right\|_p + \left\| \iint_{\mathcal{D}_2} \cdots \right\|_p := \mathcal{X}_1(t) + \mathcal{X}_2(t),
\end{aligned}$$

where, for a fixed $\kappa \in (0, 1/2)$,

$$\begin{aligned}
&\mathcal{D}_1 = [0, \kappa t] \times \left\{ y \in \mathbb{R}^n \mid |y| \leqslant \kappa t^{1/4m} \right\}, \\
&\mathcal{D}_2 = \left([0, t/2] \times \mathbb{R}^n \right) \backslash \mathcal{D}_1.
\end{aligned}$$

We easily obtain estimates on $\mathcal{X}_2(t)$ as follows.

3.1. Estimation of $\mathcal{X}_2(t)$

We have

$$\begin{aligned}
\mathcal{X}_2(t) &\leqslant C \iint_{\mathcal{D}_2} \left(\left\| \nabla^\theta \left(D^\gamma \, \mathrm{e}^{-(t-s)\mathcal{T}} \right)(\cdot - y) \right\|_p + \left\| \nabla^\theta \left(D^\gamma \, \mathrm{e}^{-t\mathcal{T}} \right)(\cdot) \right\|_p \right) \left| \psi\left(u(y,s)\right) \right| \mathrm{d}y \, \mathrm{d}s \\
&\leqslant C \left(\iint_{\mathcal{D}_2} (t-s)^{-\frac{n}{4m}(1-\frac{1}{p}) - \frac{|\gamma|+\theta}{4m}} \left| \psi\left(u(y,s)\right) \right| \mathrm{d}y \, \mathrm{d}s + t^{-\frac{n}{4m}(1-\frac{1}{p}) - \frac{|\gamma|+\theta}{4m}} \iint_{\mathcal{D}_2} \left| \psi\left(u(y,s)\right) \right| \mathrm{d}y \, \mathrm{d}s \right) \\
&\leqslant C t^{-\frac{n}{4m}(1-\frac{1}{p}) - \frac{|\gamma|+\theta}{4m}} \iint_{\mathcal{D}_2} \left| \psi\left(u(y,s)\right) \right| \mathrm{d}y \, \mathrm{d}s.
\end{aligned}$$

Therefore, $\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p}) + \frac{|\gamma|+\theta}{4m}} \mathcal{X}_2(t) = 0$ since $\lim_{t\to\infty} \iint_{\mathcal{D}_2} |\psi(u(y,s))| \, \mathrm{d}y \, \mathrm{d}s = 0$.

3.2. Estimation of $\mathcal{X}_1(t)$

We distinguish two cases. We estimate $\mathcal{X}_1(t)$ for the case $p \in [2, \infty]$ and we derive the estimates for $p \in (1, 2)$ by interpolation.

The argument relies on the following result which will be obtained by using the Hausdorff–Young inequality.

Proposition 3.3. *For a fixed $\kappa \in (0, 1/2)$, the inequality*

$$\sup_{|y|\leqslant \kappa t^{1/4m},\ 0<s\leqslant \kappa t} \big\| \nabla^\theta \big(\big(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\big)(\cdot - y) - \big(D^\gamma \mathrm{e}^{-t\mathcal{T}}\big)(\cdot) \big) \big\|_p \leqslant C\kappa t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}}$$

holds for all $p \in [2, \infty]$ and all $t > 0$.

Proof. Let

$$\mathcal{A}(\kappa, t) := \sup_{|y|\leqslant \kappa t^{1/4m},\ 0<s\leqslant \kappa t} \big\| \nabla^\theta \big(\big(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\big)(\cdot - y) - \big(D^\gamma \mathrm{e}^{-t\mathcal{T}}\big)(\cdot) \big) \big\|_p.$$

We have

$$\begin{aligned}
\mathcal{A}(\kappa, t) &\leqslant \sup_{0<s\leqslant \kappa t} \big\| \nabla^\theta \big(D^\gamma \mathrm{e}^{-(t-s)\mathcal{T}}\big)(\cdot) - \nabla^\theta \big(D^\gamma \mathrm{e}^{-t\mathcal{T}}\big)(\cdot) \big\|_p \\
&\quad + \sup_{|y|\leqslant \kappa t^{1/4m}} \big\| \nabla^\theta \big(D^\gamma \mathrm{e}^{-t\mathcal{T}}\big)(\cdot - y) - \nabla^\theta \big(D^\gamma \mathrm{e}^{-t\mathcal{T}}\big)(\cdot) \big\|_p \\
&:= \mathcal{R}_1(\kappa, t) + \mathcal{R}_2(\kappa, t).
\end{aligned}$$

To estimate $\mathcal{R}_i(\kappa, t)$, $i = 1, 2$, we will use the Hausdorff–Young inequality $\|\hat{f}\|_p \leqslant \|f\|_q$ verified for all p, q such that $1 \leqslant q \leqslant 2 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$.

By applying the Fourier transform we obtain

$$\mathcal{R}_1(\kappa, t) = \sup_{0<s\leqslant \kappa t} \bigg\| (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathrm{i}\zeta)^{|\gamma|+\theta} \big(\mathrm{e}^{-(t-s)\mathcal{F}(\zeta)} - \mathrm{e}^{-t\mathcal{F}(\zeta)}\big) \mathrm{e}^{\mathrm{i}x\zeta}\, \mathrm{d}\zeta \bigg\|_p$$

since $\nabla^\theta (D^\gamma \mathrm{e}^{-t\mathcal{T}})(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathrm{i}\zeta)^{|\gamma|+\theta} \mathrm{e}^{-t\mathcal{F}(\zeta)+\mathrm{i}x\zeta}\, \mathrm{d}\zeta$, and where $\widehat{\mathcal{T}v}(\zeta) = \mathcal{F}(\zeta)\hat{v}(\zeta)$. Note that, in view of the properties of the function A, there exist constants $c_1, c_2 > 0$ such that

$$c_1|\zeta|^{4m} \leqslant \mathcal{F}(\zeta) \leqslant c_2|\zeta|^{4m}. \tag{25}$$

Now, using the Hausdorff–Young inequality implies that $\mathcal{R}_1(\kappa, t) \leqslant C \sup_{0<s\leqslant \kappa t} \|\mathcal{H}_s\|_p$ where $\mathcal{H}_s(\zeta) = \zeta^{|\gamma|+\theta} \times (\mathrm{e}^{-(t-s)\mathcal{F}(\zeta)} - \mathrm{e}^{-t\mathcal{F}(\zeta)})$ and then

$$\big(\mathcal{R}_1(\kappa, t)\big)^q \leqslant C \sup_{0<s\leqslant \kappa t} \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta)} \big|s\mathcal{F}(\zeta)\big|^q \mathrm{e}^{-q(t-s)\mathcal{F}(\zeta)}\, \mathrm{d}\zeta$$

thanks to the inequality $|\mathrm{e}^{-a} - \mathrm{e}^{-b}| \leqslant |a - b|\,\mathrm{e}^{-a}$ verified for all $0 < a \leqslant b$. On the other hand, in view of (25)

$$\big(\mathcal{R}_1(\kappa, t)\big)^q \leqslant C(\kappa t)^q \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta)} |\zeta|^{4mq} \mathrm{e}^{-c_1 q(1-\kappa)t|\zeta|^{4m}}\, \mathrm{d}\zeta \leqslant C\kappa^q t^{-\frac{n+q(|\gamma|+\theta)}{4m}},$$

i.e., $\mathcal{R}_1(\kappa, t) \leqslant C\kappa t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}}$.

In the same manner, we estimate $\mathcal{R}_2(\kappa, t)$ by using the Hausdorff–Young inequality and (25) as follows

$$\begin{aligned}
\big(\mathcal{R}_2(\kappa, t)\big)^q &\leqslant C \sup_{|y|\leqslant \kappa t^{1/4m}} \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta)} \big|\mathrm{e}^{\mathrm{i}y\zeta} - 1\big|^q \mathrm{e}^{-qt\mathcal{F}(\zeta)}\, \mathrm{d}\zeta \\
&\leqslant C \sup_{|y|\leqslant \kappa t^{1/4m}} \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta)} |y\zeta|^q \mathrm{e}^{-c_1 qt|\zeta|^{4m}}\, \mathrm{d}\zeta \quad \big(\text{since } \big|\mathrm{e}^{\mathrm{i}y\zeta} - 1\big| \leqslant |y\zeta|\big) \\
&\leqslant C \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta+1)} \big|\kappa t^{1/4m}\big|^q \mathrm{e}^{-c_1 qt|\zeta|^{4m}}\, \mathrm{d}\zeta \\
&= C\kappa^q t^{q/4m} \int_{\mathbb{R}^n} |\zeta|^{q(|\gamma|+\theta+1)} \mathrm{e}^{-c_1 qt|\zeta|^{4m}}\, \mathrm{d}\zeta \\
&\leqslant C\kappa^q t^{-\frac{n+q(|\gamma|+\theta)}{4m}},
\end{aligned}$$

that is $\mathcal{R}_2(\kappa, t) \leqslant C\kappa t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}}$. Therefore $\mathcal{A}(\kappa, t) \leqslant C\kappa t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}}$ and Proposition 3.3 is proved. □

Now, let us come back to $\mathcal{X}_1(t)$. We get from Proposition 3.3,

$$\mathcal{X}_1(t) \leqslant C\mathcal{A}(\kappa, t) \iint\limits_{\mathcal{D}_1} \left|\psi\big(u(y,s)\big)\right| \mathrm{d}y\, \mathrm{d}s \leqslant C\kappa t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\gamma|+\theta}{4m}} \iint\limits_{\mathcal{D}_1} \left|\psi\big(u(y,s)\big)\right| \mathrm{d}y\, \mathrm{d}s$$

and then $\lim_{t\to\infty} t^{\frac{n}{4m}(1-\frac{1}{p})+\frac{|\gamma|+\theta}{4m}} \mathcal{X}_1(t) = 0$ for all $p \in [2, \infty]$.

Eventually, we derive the estimates for the case $p \in (1, 2)$ by the classical interpolation inequality

$$\left\| I_1(0, t/2) - I_2(0, t/2)\right\|_p \leqslant \left\| I_1(0, t/2) - I_2(0, t/2)\right\|_1^{1/p} \left\| I_1(0, t/2) - I_2(0, t/2)\right\|_\infty^{1-1/p}.$$

Theorem 2.4 is now completely proved. □

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