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On the existence of blowing-up solutions for a mean field equation [☆]

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Abstract

In this paper we construct single and multiple blowing-up solutions to the mean field equation:

$$\begin{cases} -\Delta u = \lambda \frac{V(x)e^u}{\int_{\Omega} V(x)e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , V is a smooth function positive somewhere in Ω and λ is a positive parameter.

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Résumé

Dans ce papier nous construisons des solutions qui explosent pour l'équation de champ moyen :

$$\begin{cases} -\Delta u = \lambda \frac{V(x)e^u}{\int_{\Omega} V(x)e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

où Ω est un domaine borné dans \mathbb{R}^2 , V est une fonction positive dans Ω et λ est un paramètre positif.

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1. Introduction

We consider the problem:

$$\begin{cases} -\Delta u = \lambda \frac{V(x)e^u}{\int_{\Omega} V(x)e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in \mathbb{R}^2 , V is a smooth function positive somewhere in Ω and $\lambda \in \mathbb{R}$ is a positive parameter.

This equation occurs in various context such as: conformal geometry (cf. [2]), statistical mechanics (cf. [9,10]) and several other area of applied mathematics (cf. e.g. [6,13,23,34]). In statistical mechanics it is referred as the “mean field equation”. In all those contexts, there is a definite interest to construct solutions which “blow-up” and “concentrate” at a set of given points, whose location carries relevant information about the geometrical/physical properties of the problem under exam.

We are interested in finding solutions u_{λ} to (1.1) which blow-up at k different points q_1, \dots, q_k in Ω' along a sequence $\lambda \rightarrow 8\pi k$, in the following sense:

$$\lambda \frac{V(x)e^{u_{\lambda}}}{\int_{\Omega} V(x)e^{u_{\lambda}}} \rightharpoonup 8\pi \sum_{i=1}^k \delta_{q_i} \quad \text{in the sense of measures in } \overline{\Omega}, \tag{1.2}$$

where δ_p denotes the Dirac measure at p . Here $\Omega' = \{q \in \Omega : V(q) > 0\}$.

In order to state some new and old results it is useful to introduce some notation. Let G_{Ω} denote the Green’s function of $-\Delta$ with Dirichlet boundary condition on Ω , namely for any $y \in \Omega$ it holds

$$\begin{cases} -\Delta_x G_{\Omega}(x, y) = \delta_y(x) & \text{if } x \in \Omega, \\ G_{\Omega}(x, y) = 0 & \text{if } x \in \partial\Omega, \end{cases}$$

and let $H_{\Omega}(x, y) = G_{\Omega}(x, y) + \frac{1}{2\pi} \log|x - y|$ be its regular part. We will refer to G_{Ω} , H_{Ω} simply as G and H respectively when the dependence in Ω is not relevant.

First of all, let us point out that if $\inf_{\Omega} V > 0$, it is known that if the sequence u_{λ} is a family of solutions to (1.1) which is not uniformly bounded from above for λ bounded, then u_{λ} blows-up at k different points q_1, \dots, q_k in Ω along a sequence $\lambda \rightarrow 8\pi k$ and (q_1, \dots, q_k) is a critical point for the function:

$$\mathcal{F}(\xi_1, \dots, \xi_k) = \sum_{i=1}^k H_{\Omega}(\xi_i, \xi_i) + \sum_{\substack{i,j=1 \\ i \neq j}}^k G_{\Omega}(\xi_i, \xi_j) + \frac{1}{4\pi} \sum_{i=1}^k \log V(\xi_i) \tag{1.3}$$

(cf. [8,22,28]; see also [35,38]). Secondly we note that, if $\lambda \in (0, 8\pi)$ problem (1.1) admits a solution $u_{\min,\lambda}$ corresponding to a global minimum point for the functional

$$J_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda \log \left(\int_{\Omega} V(x)e^u \right), \quad u \in H_0^1(\Omega),$$

which is coercive by the Moser–Trudinger inequality (cf. [32,40]).

Let us consider the case $V(x) \equiv 1$. In [9,10] the authors study the asymptotic behavior of $u_{\min,\lambda}$ as $\lambda \rightarrow 8\pi^-$, and show that either $u_{\min,\lambda}$ converges to a minimum of $J_{8\pi}$ or admits a single blow-up point $q \in \Omega$, corresponding to a maximum point of the Robin’s function $H_{\Omega}(\cdot, \cdot)$. They observe that both possibilities can occur. For instance, if Ω is a disk or a simply connected domain sufficiently close to a disk, then concentration does occur as $\lambda \rightarrow 8\pi^-$. On the contrary, there are simply connected regions (for instance, rectangles with large ratio between the sides) for which concentration cannot occur, as $\lambda \rightarrow 8\pi^-$. We also mention that Suzuki in [39] proves that the solution of (1.1) is unique when $\lambda \in (0, 8\pi)$ and Ω is a simply connected domain. If $\lambda \geq 8\pi$ the situation becomes more

complex. As well known (see [4]), if Ω is the unit disk and $\lambda \geq 8\pi$, then there are no solutions to (1.1). In [41] the author constructs a sequence of solutions on simply connected domains “blowing-up” at a critical point of the Robin’s function (see also [29,30] and [31] for similar results in a more general setting). Non-simply connected domains are considered in [19].

The general case concerning the existence of solutions with multiple blow-up points so far has been treated only by Baraket–Pacard in [5]. They prove that any nondegenerate critical point (q_1, \dots, q_k) of the function \mathcal{F} defined in (1.3) generates a family of solutions u_λ which blow-up at q_1, \dots, q_k as $\lambda \rightarrow 8\pi k$. In [21] the author extends the previous result by allowing a general weight function $V(x)$. We are aware of only another successful approach to handle multiple peak blow-up solutions which has been introduced by Chen and Lin in [16] for the annulus. A general degree formula for the corresponding Fredholm map can be found in [15].

Perturbative problems with exponential nonlinearities in dimension two seem to be much more difficult to handle (beside [5], see also [11,12] and [36]) in contrast to similar problems in higher dimensions. Baraket–Pacard’s method is too demanding in terms of assumptions (the non-degeneracy of the critical point) and functional framework, and pays in return with a very accurate control on the asymptotics of the solutions. Instead, in the spirit of some perturbation methods available in higher dimension (see e.g. [3] and [37]), we propose an alternative approach to the existence of blowing-up solutions by introducing for our problem a perturbation setting in the space $H_0^1(\Omega)$. This more flexible approach allows to replace the non-degeneracy assumption of Baraket–Pacard, by showing that “stable” (in a suitable sense) critical points of the function \mathcal{F} in (1.3) generate solutions for (1.1) which blow-up at those points. It is important to point out that this “weaker” stability of critical points enables us to construct some domains where a large number of blowing-up solutions exists.

Let us state now the main results of the paper. Recall $\Omega' = \{q \in \Omega : V(q) > 0\}$ and let $\Delta = \{(q_1, \dots, q_k) \in \Omega^k : q_i = q_j \text{ for some } i \neq j\}$.

Theorem 1.1. *Assume that K is a stable critical set for \mathcal{F} in $(\Omega')^k \setminus \Delta$ (see (1.3) and Definition 5.1). Then there exists a family of solutions of (1.1) which blow-up at points q_1, \dots, q_k with the property $(q_1, \dots, q_k) \in K$, along a sequence $\lambda \rightarrow 8\pi k$, in the sense that (1.2) holds.*

In Section 5 we take $V(x) \equiv 1$ and we exhibit some simply connected domains where many solutions to (1.1) blowing-up at one or more points exist: see Theorems 5.4, 5.5 and Corollary 5.6. This result is in striking contrast with Suzuki’s uniqueness result [39] for the case $\lambda < 8\pi$.

As a consequence of Theorem 1.1, changing sign weight functions $V(x)$ generate many solutions as we show in the following result:

Theorem 1.2. *Let $\Omega'_1, \dots, \Omega'_v$ be the connected components of Ω' in Ω . Then there exist at least v families of solutions of (1.1) which blow-up at a maximum point ξ_i of the function \mathcal{F} in Ω'_i for $i = 1, \dots, v$, along a sequence $\lambda \rightarrow 8\pi$, in the sense that (1.2) holds.*

As far as it concerns the existence of solutions blowing-up at a point when $V(x) \equiv 1$, we would like also to quote the following existence result:

Theorem 1.3. *Let $V(x) \equiv 1$. Assume that c is a stable critical value for the Robin’s function $H_\Omega(\cdot, \cdot)$ (see Definition 6.1). Then there exists a family of solutions of (1.1) which blow-up at a point q with the properties $H_\Omega(q, q) = c$ and $\nabla H_\Omega(q, q) = 0$, along a sequence $\lambda \rightarrow 8\pi$, in the sense that (1.2) holds.*

It allows to find solutions blowing-up at critical points of the Robin’s function of “saddle-type”, which a priori are not stable according to Definition 5.1. In Section 6 we exhibit some domains where this kind of solutions appears: see Theorems 6.2 and 6.3.

Let us sketch the main ideas involved in the proof. A crucial role in the construction of our solution is played by the problem:

$$\begin{cases} -\Delta U = \rho^2 e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^U < +\infty, \end{cases} \tag{1.4}$$

where $\rho \neq 0$ is a fixed parameter. In [14] (see also the Liouville representation formula in [27]) it is shown that all the solutions to (1.4) take the form:

$$U_{\tau,\xi}(x) = \log \frac{8\tau^2}{(\tau^2\rho^2 + |x - \xi|^2)^2}, \quad x \in \mathbb{R}^2, \tag{1.5}$$

for $\tau > 0$ and $\xi \in \mathbb{R}^2$. Following Bahri’s idea (see [3] and also [37]) we look for solutions to (1.1) in the form

$$u(x) = \sum_{i=1}^k PU_{\tau_{\lambda,i},\xi_{\lambda,i}}(x) + \phi^\lambda(x)$$

for suitable positive parameters $\tau_\lambda = (\tau_{\lambda,1}, \dots, \tau_{\lambda,k})$ and points $\xi_\lambda = (\xi_{\lambda,1}, \dots, \xi_{\lambda,k})$. Here, the remainder term ϕ^λ goes to zero in $H_0^1(\Omega)$ as λ goes to $8\pi k$ and $PU_{\tau,\xi}$ denotes the projection of $U_{\tau,\xi}$ into $H_0^1(\Omega)$, in other words, $PU_{\tau,\xi}$ is uniquely defined as satisfying:

$$\begin{cases} -\Delta PU_{\tau,\xi} = -\Delta U_{\tau,\xi} = \rho^2 e^{U_{\tau,\xi}} & \text{in } \Omega, \\ PU_{\tau,\xi} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

In order to determine τ_λ and ξ_λ in Section 4 we reduce the problem to a finite dimensional one. Although this problem has many similarities to equations with critical growth in smooth domains of \mathbb{R}^N , $N \geq 3$, our proof displays important differences. First of all, we point out that the parameter τ_λ will be *a priori* prescribed in terms of the point ξ_λ (see formula (2.6)). An important consequence is that the function ϕ^λ will be found in a space of codimension $2k$. We recall that the standard procedure in elliptic problems involving the critical Sobolev exponent takes place in spaces of codimension $(N + 1)k$. This different choice of the space will require delicate computations in order to establish some invertibility property of the corresponding linearized operator, which plays a crucial role in the finite dimensional reduction. This analysis will be carried out in Section 3. In Section 4 we study the reduced problem and give the expansion of the functional associated to the problem (1.1). We have collected some technical computations in Appendix A, B, C, D.

We learnt that related results to this paper have been proved at the same time, independently and using different arguments by Del Pino, Kowalczyk and Musso (see [17]).

2. Setting of the problem

In order to prove the existence of blow-up solutions to (1.1), we will consider the problem:

$$\begin{cases} -\Delta u = \rho^2 V(x) e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and seek solutions such that

$$\rho^2 V(x) e^u \rightharpoonup 8\pi \sum_{i=1}^k \delta_{q_i} \quad \text{in the sense of measures in } \overline{\Omega}, \text{ as } \rho \rightarrow 0,$$

for some point $q = (q_1, \dots, q_k) \in (\Omega')^k \setminus \Delta$. Via the transformation $\rho^2 = \lambda / (\int_\Omega V(x) e^u)$ it is possible to show that for concentrating solutions problem (2.1) is equivalent to (1.1).

It will be useful to rewrite problem (2.1) in a more convenient setting. For this purpose, let us introduce the following definition.

Definition 2.1. For $p > 1$, let $i_p^* : L^p \rightarrow H_0^1(\Omega)$ be the adjoint operator relative to the immersion $i : H_0^1(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega)$ and let $i^* : \bigcup_{p>1} L^p \rightarrow H_0^1(\Omega)$ be defined by the property $i^*|_{L^p} = i_p^*$ for any $p > 1$.

By definition, $v = i^*(u)$ holds if and only if

$$(v, \varphi) = \int_{\Omega} u \varphi \, dx, \quad \forall \varphi \in H_0^1(\Omega),$$

and the following estimate holds:

Lemma 2.2. *For any $p > 1$ there exists $c_p > 0$ such that*

$$\|i^*(u)\| \leq c_p \|u\|_{L^p}^p, \quad \forall u \in H_0^1(\Omega).$$

$H_0^1(\Omega)$ is an Hilbert space equipped with the usual inner product

$$(u, v) = \int_{\Omega} \nabla u \nabla v \, dx$$

and induced norm

$$\|u\| = \left(\int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}.$$

Next, we recall the Moser–Trudinger inequality (cf. [32,40]):

Lemma 2.3. *There exists a constant $c > 0$ such that for any smooth bounded domain $\Omega \subset \mathbb{R}^2$ we have:*

$$\int_{\Omega} e^{4\pi u^2 / \|u\|_{H_0^1}^2} \leq c |\Omega|, \quad \forall u \in H_0^1(\Omega).$$

In particular, by Lemma 2.3 we deduce the following useful estimate:

Remark 2.4. There exists a constant $c > 0$ such that for any $\eta > 0$,

$$\int_{\Omega} e^{\eta u} \leq c |\Omega| e^{\frac{\eta^2}{16\pi} \|u\|_{H_0^1}^2}, \quad \forall u \in H_0^1(\Omega).$$

In particular, the map:

$$\begin{aligned} H_0^1(\Omega) &\rightarrow L^p(\Omega), \\ u &\rightarrow e^u \end{aligned}$$

is continuous for every $p > 1$ and we can rewrite problem (2.1) in the following equivalent form:

$$\begin{cases} u = i^*(\rho^2 V(x) e^u) & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{2.2}$$

Let $PU_{\tau,\xi}$ be the projection onto $H_0^1(\Omega)$ of $U_{\tau,\xi}$ (see (1.5), (1.6)). Thus, by Definition 2.1 we have

$$PU_{\tau,\xi} := i^*(\rho^2 e^{U_{\tau,\xi}}).$$

We are looking for solutions to (2.2) of the form

$$u(x) = \sum_{i=1}^k P U_{\tau_{\rho,i}, \xi_{\rho,i}}(x) + \phi^\rho(x)$$

for suitable positive parameters $\tau_\rho = (\tau_{\rho,1}, \dots, \tau_{\rho,k})$ and points $\xi_\rho = (\xi_{\rho,1}, \dots, \xi_{\rho,k})$. The remainder term ϕ^ρ belongs to a suitable space which will be defined as follows. Let

$$\psi_{\tau,\xi}^0(x) = \frac{\partial U_{\tau,\xi}}{\partial \tau}(x) = \frac{2|x-\xi|^2 - \tau^2 \rho^2}{\tau|x-\xi|^2 + \tau^2 \rho^2}, \quad x \in \Omega, \tag{2.3}$$

and for $j = 1, 2$ let

$$\psi_{\tau,\xi}^j(x) = \frac{\partial U_{\tau,\xi}}{\partial \xi_j}(x) = 4 \frac{(x-\xi)_j}{\tau^2 \rho^2 + |x-\xi|^2}, \quad x \in \Omega. \tag{2.4}$$

For $j = 0, 1, 2$, the function $\psi_{\tau,\xi}^j$ is a solution for the equation $-\Delta \psi = \rho^2 e^{U_{\tau,\xi}} \psi$ in \mathbb{R}^2 . Let $P\psi_{\tau,\xi}^j$ be the projection into $H_0^1(\Omega)$ of $\psi_{\tau,\xi}^j$, i.e.

$$\begin{cases} -\Delta P\psi_{\tau,\xi}^j = \rho^2 e^{U_{\tau,\xi}} \psi_{\tau,\xi}^j & \text{in } \Omega, \\ P\psi_{\tau,\xi}^j = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

For any $\xi = (\xi_1, \dots, \xi_k) \in (\Omega')^k \setminus \Delta$, set

$$\tau_i(\xi) = \sqrt{\frac{V(\xi_i)}{8}} e^{4\pi(H_\Omega(\xi_i, \xi_i) + \sum_{j \neq i} G_\Omega(\xi_i, \xi_j))}. \tag{2.6}$$

Set $U_i = U_{\tau_i(\xi), \xi_i}$ and $\psi_i^j := \psi_{\tau_i(\xi), \xi_i}^j$ for $j = 0, 1, 2, i = 1, \dots, k$ and $\xi \in (\Omega')^k \setminus \Delta$. We consider the subspace of $H_0^1(\Omega)$:

$$K_\xi = \text{span}\{P\psi_i^j, j = 1, 2, i = 1, \dots, k\},$$

and its complement

$$K_\xi^\perp = \{\phi \in H_0^1(\Omega) \mid (\phi, P\psi_i^j) = 0, j = 1, 2, i = 1, \dots, k\},$$

and consider the corresponding orthogonal projections:

$$\Pi_\xi : H_0^1(\Omega) \rightarrow K_\xi \quad \text{and} \quad \Pi_\xi^\perp : H_0^1(\Omega) \rightarrow K_\xi^\perp.$$

3. The linear problem: a key lemma

Let us introduce the linear operator $L_\xi^\rho : K_\xi^\perp \rightarrow K_\xi^\perp$ defined as follows:

$$L_\xi^\rho(\phi) = \Pi_\xi^\perp \{ \phi - i^* [\rho^2 V(x) e^{\sum_{i=1}^k P U_i} \phi] \}.$$

In order to solve Eq. (4.1), a crucial ingredient is given by the following result, which is similar in spirit to an invertibility property established in [11]:

Proposition 3.1. *Let $\xi \in (\Omega')^k \setminus \Delta$. There exists $\rho_0 > 0$ and a constant $c > 0$ such that for any $\rho \in (0, \rho_0)$ we have*

$$\|L_\xi^\rho(\phi)\| \geq \frac{c}{|\log \rho|} \|\phi\|, \quad \forall \phi \in K_\xi^\perp.$$

In particular, the operator L_ξ^ρ is invertible and $\|(L_\xi^\rho)^{-1}\| \leq |\log \rho|/c$.

Moreover, the estimates are uniform in compact sets of $(\Omega')^k \setminus \Delta$.

Proof. We need only to establish the validity of the following estimate:

$$\|L_\xi^\rho(\phi)\| \geq \frac{c}{|\log \rho|} \|\phi\| \quad \forall \phi \in K_\xi^\perp, \tag{3.1}$$

for some uniform constant $c > 0$. Indeed, by (3.1) we deduce that L_ξ^ρ is an injective operator with closed image. Since L_ξ^ρ is selfadjoint, we also conclude that it is onto and its inverse $(L_\xi^\rho)^{-1}$ satisfies $\|(L_\xi^\rho)^{-1}\| \leq |\log \rho|/c$.

To prove (3.1), we argue by contradiction. Suppose there exist sequences

$$\rho \rightarrow 0, \quad \xi \rightarrow \xi_0 \in (\Omega')^k \setminus \Delta, \quad \phi \in K_\xi^\perp: \|\phi\| = 1 \text{ and } \|L_\xi^\rho(\phi)\| = o\left(\frac{1}{|\log \rho|}\right). \tag{3.2}$$

Write,

$$\phi - i^*[\rho^2 V(x) e^{\sum_{i=1}^k P U_i} \phi] = \psi + w, \tag{3.3}$$

where $\psi \in K_\xi^\perp$, $w \in K_\xi$ and $\|\psi\| = o(1/|\log \rho|) \rightarrow 0$. Equivalently (in a weak sense), there holds

$$\begin{cases} -\Delta \phi = \rho^2 V(x) e^{\sum_{i=1}^k P U_i} \phi - \Delta(\psi + w) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Step 1. For any $p \in (1, 2)$,

$$\|w\| = O(\rho^{(2-p)/p}) \rightarrow 0. \tag{3.4}$$

Let $w = \sum_{h=1}^k \sum_{l=1}^2 c_{hl} P \psi_h^l$. We multiply (3.3) by $P \psi_s^j$, $j = 1, 2$ and $s = 1, \dots, k$, and get

$$\sum_{h=1}^k \sum_{l=1}^2 c_{hl} (P \psi_h^l, P \psi_s^j)_{H_0^1} = -\rho^2 \int_\Omega V(x) e^{\sum_{i=1}^k P U_i} \phi P \psi_s^j. \tag{3.5}$$

By Lemma A.4 the L.H.S. of (3.5) is estimated as follows:

$$\text{L.H.S.} = \frac{D}{\tau^2 \rho^2} c_{sj} + O\left(\sum_{h=1}^k \sum_{l=1}^2 |c_{hl}|\right). \tag{3.6}$$

Moreover, the R.H.S. of (3.5) takes the form:

$$\begin{aligned} & -\rho^2 \int_\Omega V(x) e^{\sum_{i=1}^k P U_i} \phi P \psi_s^j \\ &= \int_\Omega \left[\rho^2 \sum_{i=1}^k e^{U_i} - \rho^2 V(x) e^{\sum_{i=1}^k P U_i} \right] \phi P \psi_s^j - \rho^2 \int_\Omega e^{U_s} \phi [P \psi_s^j - \psi_s^j] - \rho^2 \sum_{i \neq s} \int_\Omega e^{U_i} \phi P \psi_s^j, \end{aligned} \tag{3.7}$$

where we have used (2.5) and $\phi \in K_\xi^\perp$. Fix $p \in (1, 2)$ and use Lemmata A.4 and B.1, together with Hölder's inequality (here $1/q + 1/p < 1$), to get

$$\begin{aligned} & \left| \int_\Omega \left[\rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right] \phi P \psi_s^j \right| \\ & \leq \left\| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right\|_{L^p} \|\phi\|_{L^q} \|P \psi_s^j\| = O(\rho^{2(1-p)/p}). \end{aligned} \tag{3.8}$$

Using Lemma A.2, Proposition A.3 and Hölder’s inequality (with $1/q + 1/p = 1$), we get,

$$\left| \rho^2 \int_{\Omega} e^{U_s} \phi [P \psi_s^j - \psi_s^j] \right| = O \left(\|\rho^2 e^{U_s}\|_{L^p} \|\phi\|_{L^q} \right) = O(\rho^{2(1-p)/p}). \tag{3.9}$$

Using Lemma C.6 and Hölder’s inequality (with $1/q + 1/p = 1$), we obtain,

$$\left| \rho^2 \sum_{i \neq s} \int_{\Omega} e^{U_i} \phi P \psi_s^j \right| = O \left(\sum_{i \neq s} \|\rho^2 e^{U_i} P \psi_s^j\|_{L^p} \|\phi\|_{L^q} \right) = O(\rho^{2/p}). \tag{3.10}$$

Then, by (3.7)–(3.10) it follows that the R.H.S. of (3.5) satisfies:

$$\text{R.H.S.} = O(\rho^{2(1-p)/p}). \tag{3.11}$$

Inserting the estimates (3.6) and (3.11) into (3.5), we deduce that

$$\sum_{h=1}^k \sum_{l=1}^2 |c_{hl}| = O(\rho^{2/p}). \tag{3.12}$$

Finally, by Lemma A.4 and by (3.12) we deduce that

$$\|w\| = O(\rho^{(2-p)/p}) \rightarrow 0,$$

and claim (3.4) is proved.

Step 2. For any $i = 1, \dots, k$

$$(\phi, P \psi_i^0)_{H_0^1} \rightarrow 0. \tag{3.13}$$

Let $i = 1, \dots, k$. Note that, for $\tau_i > 0$ and $\xi_i \in \Omega$, the functions

$$w_i(x) = \frac{4}{3\tau_i} \log(\tau_i^2 \rho^2 + |x - \xi_i|^2) \frac{\tau_i^2 \rho^2 - |x - \xi_i|^2}{\tau_i^2 \rho^2 + |x - \xi_i|^2} + \frac{8}{3} \frac{\tau_i \rho^2}{\tau_i^2 \rho^2 + |x - \xi_i|^2}$$

and

$$t_i(x) = -2 \frac{\tau_i \rho^2}{\tau_i^2 \rho^2 + |x - \xi_i|^2}$$

satisfy: $-\Delta w_i - \rho^2 e^{U_{\tau_i, \xi_i}} w_i = \rho^2 e^{U_{\tau_i, \xi_i}} \psi_{\tau_i, \xi_i}^0$ and $-\Delta t_i - \rho^2 e^{U_{\tau_i, \xi_i}} t_i = \rho^2 e^{U_{\tau_i, \xi_i}}$ in \mathbb{R}^2 respectively. A straightforward calculation shows that,

$$\int_{\Omega} |\nabla w_i|^2 = M_i^2 (1 + o(1)) (\log \rho)^2, \quad \int_{\Omega} |\nabla t_i|^2 = O(1) \quad \text{as } \rho \rightarrow 0 \tag{3.14}$$

with $M_i = \frac{32}{3\tau_i} (\int_{\mathbb{R}^2} \frac{|y|^2}{(1+|y|^2)^4})^{1/2}$, and (3.14) holds uniformly in $\text{dist}(\xi_i, \partial\Omega) \geq \varepsilon$, $\varepsilon > 0$, and τ_i bounded away from zero.

Let now $\tau_i = \tau_i(\xi)$ and ξ as specified in (3.2). The projection $Pu_i \in H_0^1(\Omega)$ of the function

$$u_i = w_i + \frac{16\pi}{3\tau_i} H(\xi_i, \xi_i) t_i$$

satisfies

$$-\Delta Pu_i - \rho^2 V(x) e^{\sum_{h=1}^k Pu_h} Pu_i = \rho^2 e^{U_i} \psi_i^0 - \rho^2 \sum_{h \neq i} e^{U_h} \frac{16\pi}{3\tau_i} G(\xi_h, \xi_i) + F_i,$$

where

$$\begin{aligned}
 F_i(x) = & -\frac{16\pi}{3\tau_i} (H(x, \xi_i) - H(\xi_i, \xi_i)) \rho^2 e^{U_i} + \left(u_i - Pu_i + \frac{16\pi}{3\tau_i} H(x, \xi_i) \right) \rho^2 e^{U_i} \\
 & + \left(\rho^2 \sum_{h=1}^k e^{U_h} - \rho^2 V(x) e^{\sum_{h=1}^k PU_h} \right) Pu_i + \rho^2 \sum_{h \neq i} e^{U_h} \left(\frac{16\pi}{3\tau_i} G(\xi_h, \xi_i) - Pu_i \right).
 \end{aligned} \tag{3.15}$$

Since $u_i - Pu_i + \frac{16\pi}{3\tau_i} H(x, \xi_i)$ is an harmonic function with boundary values satisfying

$$\left\| u_i - Pu_i + \frac{16\pi}{3\tau_i} H(x, \xi_i) \right\|_{2,\alpha,\partial\Omega} \leq C\rho^2$$

for any $\alpha \in (0, 1)$, by elliptic regularity theory it follows that

$$\left\| u_i - Pu_i + \frac{16\pi}{3\tau_i} H(x, \xi_i) \right\|_{2,\alpha,\Omega} \leq C\rho^2$$

for any $\alpha \in (0, 1)$. Hence, we get that

$$Pu_i(x) = \frac{16\pi}{3\tau_i} G(x, \xi_i) + O(\rho^2) \text{ in } C^0_{\text{loc}}(\bar{\Omega} \setminus \{\xi_i\}), \tag{3.16}$$

$$\|Pu_i\| = M_i(1 + o(1)) |\log \rho|, \tag{3.17}$$

and by means of Lemmata A.2, B.1, from (3.15) we deduce that $f_i = i^*(F_i)$ satisfies:

$$\|f_i\| = o(1), \quad \text{as } \rho \rightarrow 0.$$

Therefore,

$$Pu_i - i^*(\rho^2 V(x) e^{\sum_{h=1}^k PU_h} Pu_i) = P\psi_i^0 - \frac{16\pi}{3\tau_i} \sum_{h \neq i} G(\xi_h, \xi_i) PU_h + f_i \tag{3.18}$$

with $\|f_i\| \rightarrow 0$. Multiply (3.3) by Pu_i and use (3.18) to obtain:

$$\begin{aligned}
 (\phi, P\psi_i^0)_{H_0^1} &= \frac{16\pi}{3\tau_i} \sum_{h \neq i} G(\xi_h, \xi_i) (\phi, PU_h)_{H_0^1} + o\left(\frac{\|Pu_i\|}{|\log \rho|}\right) + o(1) \\
 &= \frac{16\pi}{3\tau_i} \sum_{h \neq i} G(\xi_h, \xi_i) (\phi, PU_h)_{H_0^1} + o(1),
 \end{aligned}$$

where we have taken in account (3.2), (3.17) and the estimate in Step 1. In conclusion, we need to show that $(\phi, PU_h)_{H_0^1} = o(1)$ for any $h = 1, \dots, k$. Multiply (3.3) by $P\psi_i^0$ and use Lemmata A.4, B.1 to obtain:

$$\rho^2 \int_{\Omega} e^{U_i} \phi (\psi_i^0 - P\psi_i^0) = \rho^2 \sum_{h \neq i} \int_{\Omega} e^{U_h} \phi P\psi_i^0 + o(1).$$

By (A.3), (A.4) and Lemmata A.2, A.4, finally we deduce that

$$(\phi, PU_i)_{H_0^1} = \rho^2 \int_{\Omega} e^{U_i} \phi = o(1),$$

and (3.13) is completely established.

Define the spaces

$$L := \left\{ \Phi : \left\| \frac{\Phi}{1 + |y|^2} \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\},$$

$$H := \left\{ \Phi : \|\nabla\Phi\|_{L^2(\mathbb{R}^2)} + \left\| \frac{\Phi}{1 + |y|^2} \right\|_{L^2(\mathbb{R}^2)} < +\infty \right\}$$

endowed respectively with the norm $\|\Phi\|_L = \|\Phi/(1 + |y|^2)\|_{L^2(\mathbb{R}^2)}$ and $\|\Phi\|_H = (\|\nabla\Phi\|_{L^2(\mathbb{R}^2)}^2 + \|\Phi\|_L^2)^{1/2}$. If S^2 denotes the unit sphere in \mathbb{R}^3 with the standard metric and π_N is the stereographic projection through the north pole, let us point out that the map $\Phi \rightarrow \Phi \circ \pi_N$ is an isometry from L to $L^2(S^2)$ and from H to $H^1(S^2)$. Hence, by the compactness of the embedding $H^1(S^2) \hookrightarrow L^2(S^2)$, we get the compactness of the embedding $H \hookrightarrow L$.

Assume that $\xi \in \mathcal{O}_\varepsilon$ for some $\varepsilon > 0$, where

$$\mathcal{O}_\varepsilon = \{(q_1, \dots, q_k) \in \Omega^k : \text{dist}(q_i, \partial\Omega) \geq 2\varepsilon, |q_i - q_j| \geq 2\varepsilon \text{ for } i \neq j\}.$$

Let $\chi : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function such that $\chi(x) = 1$ if $|x| \leq \varepsilon/2$, $\chi(x) = 0$ if $|x| \geq \varepsilon$. For any $i = 1, \dots, k$ set

$$\tilde{\phi}^i(y) = \phi^i(y)\chi^i(y), \quad y \in \Omega^i := \frac{\Omega - \xi_i}{\tau_i \rho},$$

where $\phi^i(y) = \phi(\tau_i \rho y + \xi_i)$ and $\chi^i(y) = \chi(\tau_i \rho y)$. We will always consider $\tilde{\phi}^i$ extended to be zero outside Ω^i .

Step 3. For any $i = 1, \dots, k$

$$\tilde{\phi}^i \rightharpoonup 0 \quad \text{weakly in } H. \tag{3.19}$$

First of all, we remark that the function $\tilde{\phi}^i$ satisfies (in a weak sense),

$$\begin{cases} -\Delta\tilde{\phi}^i = a^i(y)\tilde{\phi}^i - \Delta\tilde{z}^i + \tilde{v}^i & \text{in } \Omega^i, \\ \tilde{\phi}^i = 0 & \text{on } \partial\Omega^i, \end{cases} \tag{3.20}$$

where

$$\begin{aligned} a^i(y) &= \tau_i^2 \rho^4 V(\tau_i \rho y + \xi_i) e^{\sum_{h=1}^k P U_h(\tau_i \rho y + \xi_i)}, \\ \tilde{z}^i(y) &= z^i(y)\chi^i(y), \quad z^i(y) = (\psi + w)(\tau_i \rho y + \xi_i), \\ \tilde{v}^i(y) &= -\Delta\chi^i(y)(\phi^i - z^i) - 2\nabla\chi^i(y)\nabla(\phi^i - z^i). \end{aligned}$$

Observe that by Lemma B.1 we have,

$$\begin{aligned} \rho^2 \sum_{h=1}^k \int_{\Omega} e^{U_h} \phi^2 &= \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} \phi^2 + O\left(\int_{\Omega} \left| \rho^2 \sum_{h=1}^k e^{U_h} - \rho^2 V(x) e^{\sum_{h=1}^k P U_h} \right| \phi^2\right) \\ &= \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} \phi^2 + o(1). \end{aligned} \tag{3.21}$$

Multiplying (3.3) by ϕ , we obtain that

$$\rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} \phi^2 = \int_{\Omega} |\nabla\phi|^2 - \int_{\Omega} \nabla\psi \nabla\phi = O(1). \tag{3.22}$$

By (3.21), (3.22) we get that $\tilde{\phi}^i$ is bounded in H , since

$$\int_{\Omega^i} |\nabla\tilde{\phi}^i|^2 = O\left(\int_{\Omega} |\nabla\phi|^2\right) = O(1)$$

and

$$\int_{\Omega^i} \frac{(\tilde{\phi}^i(y))^2}{(1+|y|^2)^2} dy = \frac{\rho^2}{8} \int_{\Omega} e^{U_i} \phi^2 \chi^2(x - \xi_i) = O(1).$$

Hence, we can assume (up to a subsequence) that $\tilde{\phi}^i \rightharpoonup \phi_0^i$ in H and strongly in L . On the other hand, by Proposition A.1 it follows that

$$a^i(y) = \frac{V(\tau_i \rho y + \xi_i)}{\tau_i^2 (1+|y|^2)^2} e^{8\pi(H(\tau_i \rho y + \xi_i, \xi_i) + \sum_{h \neq i} G(\tau_i \rho y + \xi_i, \xi_h)) + O(\rho^2)} \rightarrow \frac{8}{(1+|y|^2)^2} \tag{3.23}$$

uniformly on compact sets of \mathbb{R}^2 . Furthermore,

$$\|\nabla \tilde{z}^i\|_{L^2(\mathbb{R}^2)}^2 = O\left(\int_{\Omega} |\nabla \psi|^2 + |\nabla w|^2\right) \rightarrow 0 \tag{3.24}$$

and, for any $\Psi \in C_0^\infty(\mathbb{R}^2)$ we have,

$$\int_{\mathbb{R}^2} \tilde{v}^i \Psi = \int_{\Omega} [-\Delta \chi(x - \xi_i)(\phi - \psi - w)(x) - 2\nabla \chi(x - \xi_i) \nabla(\phi - \psi - w)(x)] \Psi\left(\frac{x - \xi_i}{\tau_i \rho}\right) dx = 0 \tag{3.25}$$

if $\varepsilon/(2\tau_i \rho) \geq \text{dist}(\text{Supp } \Psi, 0)$. Finally, for $j = 1, 2$ we see that,

$$\int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(|y|^2 + 1)^3} \phi_0^i dy = \int_{\mathbb{R}^2} \frac{y_j}{(|y|^2 + 1)^3} \phi_0^i dy = 0. \tag{3.26}$$

Indeed,

$$\frac{16}{\tau_i} \int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(1+|y|^2)^3} \tilde{\phi}^i dy = \rho^2 \int_{\Omega} e^{U_i} \psi_i^0 \phi \chi(x - \xi_i) = (\phi, P\psi_i^0)_{H_0^1} + O(\rho^2) \rightarrow 0$$

(by Step 2), and the orthogonality conditions give,

$$32 \int_{\mathbb{R}^2} \frac{y_j}{(1+|y|^2)^3} \tilde{\phi}^i dy = \tau_i \rho^3 \int_{\Omega} e^{U_i} \psi_i^j \phi \chi(x - \xi_i) = O(\rho^3) \rightarrow 0, \quad j = 1, 2,$$

and so (3.26) follows by observing that $\tilde{\phi}^i/(1+|y|^2) \rightarrow \phi_0^i/(1+|y|^2)$ in $L^2(\mathbb{R}^2)$. In conclusion, by (3.20)–(3.23), (3.25) we obtain that $\phi_0^i \in H$ is a (distributional) solution for

$$-\Delta \phi_0^i = \frac{8}{(1+|y|^2)^2} \phi_0^i \quad \text{in } \mathbb{R}^2 \tag{3.27}$$

satisfying:

$$\int_{\mathbb{R}^2} \frac{|y|^2 - 1}{(|y|^2 + 1)^3} \phi_0^i dy = \int_{\mathbb{R}^2} \frac{y_j}{(|y|^2 + 1)^3} \phi_0^i dy = 0 \quad \text{for } j = 1, 2.$$

The isometry between H and $H^1(S^2)$ and elliptic regularity theory (see [24]) imply that ϕ_0^i is actually a regular solution of (3.27). By Lemma D.1 we get that necessarily $\phi_0^i = 0$ and (3.19) is established.

Step 4. A contradiction arises!

By the compactness of the embedding $L \hookrightarrow H$, we have that $\tilde{\phi}^i \rightarrow 0$ in L for any $i = 1, \dots, k$. By (3.21) and $\|\psi\| = o(1)$ we have,

$$\begin{aligned} \rho^2 \int_{\Omega} V(x) e^{\sum_{i=1}^k PU_i} \phi^2 + \int_{\Omega} \nabla \psi \nabla \phi &= \rho^2 \sum_{i=1}^k \int_{\Omega} e^{U_i} \phi^2 \chi^2(x - \xi_i) + o(1) \\ &= \sum_{i=1}^k \int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} (\tilde{\phi}^i)^2 dy + o(1) \rightarrow 0, \end{aligned}$$

in contradiction with the relation:

$$1 = \int_{\Omega} |\nabla \phi|^2 = \rho^2 \int_{\Omega} V(x) e^{\sum_{i=1}^k PU_i} \phi^2 + \int_{\Omega} \nabla \psi \nabla \phi.$$

This completes the proof of Proposition 3.1. \square

Remark 3.2. Let us remark that the result in Proposition 3.1 is “sharp” in the sense that in [22] it is shown that for $k = 1$ we have also $\|(L_{\xi}^{\rho})^{-1}\| \geq c_1 |\log \rho|$ for some positive constant $c_1 < c$.

4. The reduced problem and the functional expansion

Our first goal will be to prove that, for any $\rho > 0$ small enough and for any point $\xi \in (\Omega')^k \setminus \Delta$, there exists $\phi_{\xi}^{\rho} \in K_{\xi}^{\perp}$ such that

$$\Pi_{\xi}^{\perp} \left[\sum_{i=1}^k PU_i + \phi_{\xi}^{\rho} - i^* (\rho^2 V(x) e^{\sum_{i=1}^k PU_i + \phi_{\xi}^{\rho}}) \right] = 0. \tag{4.1}$$

Proposition 4.1. *Let $\xi = (\xi_1, \dots, \xi_k)$ in a compact set of $(\Omega')^k \setminus \Delta$ and $\tau_i = \tau_i(\xi)$ be given in (2.6). For any $p \in (1, \frac{4}{3})$ there exists $\rho_0 > 0$ and $R > 0$ (uniformly in ξ) such that for any $\rho \in (0, \rho_0)$ there exists a unique $\phi_{\xi}^{\rho} \in K_{\xi}^{\perp}$ such that*

$$\Pi_{\xi}^{\perp} \left[\sum_{i=1}^k PU_i + \phi_{\xi}^{\rho} - i^* (\rho^2 V(x) e^{\sum_{i=1}^k PU_i + \phi_{\xi}^{\rho}}) \right] = 0 \tag{4.2}$$

and

$$\|\phi_{\xi}^{\rho}\| \leq R \rho^{(2-p)/p} |\log \rho|. \tag{4.3}$$

Proof. According to Proposition 3.1, for ρ small $(L_{\xi}^{\rho})^{-1}$ is a linear operator from K_{ξ}^{\perp} into itself such that $\|(L_{\xi}^{\rho})^{-1}\| \leq c_2 |\log \rho|$ uniformly in ξ . Let us point out that ϕ is a solution of (4.2) if and only if it is a fixed point of the operator $T_{\xi}^{\rho} : K_{\xi}^{\perp} \rightarrow K_{\xi}^{\perp}$ defined by

$$\begin{aligned} T_{\xi}^{\rho}(\phi) &= [(L_{\xi}^{\rho})^{-1} \circ \Pi_{\xi}^{\perp} \circ i^*] M_{\xi}^{\rho}(\phi), \\ M_{\xi}^{\rho}(\phi) &= \rho^2 V(x) e^{\sum_{i=1}^k PU_i} [e^{\phi} - 1 - \phi] + \left[\rho^2 V(x) e^{\sum_{i=1}^k PU_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right]. \end{aligned}$$

We prove that, for ρ small enough and R large enough (but independent of ρ), T_{ξ}^{ρ} defines a contraction map from $\{\|\phi\| \leq R \rho^{(2-p)/p} |\log \rho|\}$ into itself.

Let us fix some $r > 1$ such that $r(3p - 2) < 2$. By Lemma 2.2 and Proposition 3.1 we have that $\|T_\xi^\rho(\phi)\| \leq C|\log \rho| \|M_\xi^\rho(\phi)\|_L^p$. Consequently, by Lemmata B.1, B.2 we deduce,

$$\begin{aligned} \|T_\xi^\rho(\phi)\| &\leq C|\log \rho| \left[\left\| \rho^2 V(x) e^{\sum_{i=1}^k PU_i} (e^\phi - 1 - \phi) \right\|_L^p + \left\| \rho^2 V(x) e^{\sum_{i=1}^k PU_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right\|_L^p \right] \\ &\leq C' \|\phi\|^2 e^{c_2 \|\phi\|^2} \rho^{2(1-pr)/(pr)} |\log \rho| + C' \rho^{(2-p)/p} |\log \rho|. \end{aligned}$$

Hence, for R suitably large we get that

$$\|\phi\| \leq R\rho^{(2-p)/p} |\log \rho| \Rightarrow \|T_\xi^\rho(\phi)\| \leq R\rho^{(2-p)/p} |\log \rho| \tag{4.4}$$

for $\rho > 0$ sufficiently small. Similarly, by Lemma B.2 we deduce

$$\begin{aligned} \|T_\xi^\rho(\phi_1) - T_\xi^\rho(\phi_2)\| &\leq C|\log \rho| \left\| \rho^2 V(x) e^{\sum_{i=1}^k PU_i} [e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)] \right\|_L^p \\ &\leq C' \rho^{2+(2-3pr)/(pr)} (\log \rho)^2 \|\phi_1 - \phi_2\| \leq L \|\phi_1 - \phi_2\| \end{aligned} \tag{4.5}$$

for some $L < 1$, provided $\rho > 0$ is small enough. By (4.4) and (4.5) we get that T_ξ^ρ is a contraction map from $\{\|\phi\| \leq R\rho^{(2-p)/p} |\log \rho|\}$ into itself for every $\rho \in (0, \rho_0)$, provided $\rho_0 > 0$ is sufficiently small. \square

Let us introduce the functional $E_\rho : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$E_\rho(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \rho^2 \int_\Omega V(x) e^u. \tag{4.6}$$

We have the following expansion:

Proposition 4.2. *Let $\tilde{E}_\rho(\xi) := E_\rho(\sum_{i=1}^k PU_{\tau_i(\xi), \xi_i} + \phi_\xi^\rho)$. It holds*

$$\tilde{E}_\rho(\xi) = (-16\pi \log \rho + 24\pi \log 2 - 8\pi(k + 1)) - 32\pi^2 \mathcal{F}(\xi) + o(1) \tag{4.7}$$

as $\rho \rightarrow 0$, C^1 -uniformly in compact sets of $(\Omega')^k \setminus \Delta$ (see (1.3)).

Proof. Set $PU_i := PU_{\tau_i(\xi), \xi_i}$ and $\phi := \phi_\xi^\rho$. It holds

$$\begin{aligned} \tilde{E}_\rho(\xi) &= \frac{1}{2} \int_\Omega \left| \nabla \left(\sum PU_i + \phi \right) \right|^2 - \rho^2 \int_\Omega V(x) e^{\sum_{i=1}^k PU_i + \phi} \\ &= \frac{1}{2} \left(\sum_i \int_\Omega |\nabla PU_i|^2 + \sum_{i \neq j} \int_\Omega \nabla PU_i \nabla PU_j \right) - \rho^2 \int_\Omega V(x) e^{\sum_{i=1}^k PU_i} \\ &\quad + \frac{1}{2} \left(\int_\Omega |\nabla \phi|^2 + \sum_i \int_\Omega \nabla \phi \nabla PU_i \right) - \rho^2 \int_\Omega V(x) (e^{\sum_{i=1}^k PU_i + \phi} - e^{\sum_{i=1}^k PU_i}). \end{aligned} \tag{4.8}$$

By the inequality $|e^s - 1| \leq e^{|s|} |s|$ for any $s \in \mathbb{R}$ and arguing as in Lemma B.2 we obtain

$$\left| \rho^2 \int_\Omega V(x) (e^{\sum_{i=1}^k PU_i + \phi} - e^{\sum_{i=1}^k PU_i}) \right| = O \left(\rho^2 \left(\int_\Omega e^{r \sum_{i=1}^k PU_i} \right)^{1/r} \left(\int_\Omega e^{s|\phi|} \right)^{1/s} \|\phi\|_{L^r} \right) = o(1) \tag{4.9}$$

for $r > 1$ sufficiently close to 1 and $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$ (using (4.3)). By Lemmata C.1, C.2 and (4.8), (4.9) we deduce that expansion (4.7) holds C^0 -uniformly in compact sets of $(\Omega')^k \setminus \Delta$.

To conclude, let us make some remarks:
 – as a consequence of (4.2) we have that,

$$\sum_{i=1}^k P U_i + \phi - i^*(\rho^2 V(x) e^{\sum_{i=1}^k P U_i + \phi}) = \sum_{i=1}^k \sum_{j=1}^2 c_{ij}^\rho P \psi_i^j, \tag{4.10}$$

for some coefficients c_{ij}^ρ . Inserting (4.10) into (C.15), by Lemma A.4 we deduce that

$$\sum_{i=1}^k \sum_{j=1}^2 |c_{ij}^\rho| = O(\rho^2); \tag{4.11}$$

– the map $\xi \rightarrow \phi = \phi_\xi^\rho$ is a C^1 -function as it follows by the IFT applied to the equation

$$F(\xi, u) := \Pi_\xi^\perp \left[\sum_{i=1}^k P U_i + \Pi_\xi^\perp u - i^*(\rho^2 V(x) e^{\sum_{i=1}^k P U_i + \Pi_\xi^\perp u}) \right] + \Pi_\xi u = 0.$$

Indeed, $F(\xi, \phi) = 0$ and the linearized operator:

$$\frac{\partial F}{\partial u}(\xi, \phi) = \Pi_\xi^\perp [\text{Id} - i^*(\rho^2 V(x) e^{\sum_{i=1}^k P U_i + \phi} \Pi_\xi^\perp)] + \Pi_\xi \tag{4.12}$$

is invertible for ρ small. In fact, by (4.3) and arguing as in Lemma B.2, it is easy to derive the estimate,

$$\| \Pi_\xi^\perp i^*(\rho^2 V(x) e^{\sum_{i=1}^k P U_i} (e^\phi - 1) \Pi_\xi^\perp) \| \| (L_\xi^\rho)^{-1} \| < 1,$$

for ρ small. Hence, in view of Proposition 3.1 the invertibility of the operator in (4.12) easily follows.

To complete the C^1 -expansion, for any $j = 1, \dots, k$ and $h = 1, 2$ we write

$$\begin{aligned} & \partial_{(\xi_j)_h} E_\rho \left(\sum_{i=1}^k P U_i + \phi \right) \\ &= \left(\sum_{i=1}^k P U_i + \phi - i^*(\rho^2 V(x) e^{\sum_{i=1}^k P U_i + \phi}), P \psi_j^h + \sum_{i=1}^k \partial_{(\xi_j)_h} \tau_i(\xi) P \psi_i^0 + \partial_{(\xi_j)_h} \phi \right)_{H_0^1(\Omega)} \\ &= -32\pi^2 \frac{\partial \mathcal{F}}{\partial (\xi_j)_h}(\xi_1, \dots, \xi_k) + \sum_{s=1}^k \sum_{m=1}^2 c_{sm}^\rho \left(P \psi_s^m, \sum_{i=1}^k \partial_{(\xi_j)_h} \tau_i(\xi) P \psi_i^0 + \partial_{(\xi_j)_h} \phi \right)_{H_0^1(\Omega)} + o(1) \end{aligned} \tag{4.13}$$

in view of (C.15) and (4.10). Lemma A.4 implies that

$$(P \psi_s^m, P \psi_i^0)_{H_0^1(\Omega)} = O\left(\frac{1}{\rho}\right),$$

and the orthogonality condition $(P \psi_s^m, \phi)_{H_0^1(\Omega)} = 0$ implies that,

$$(P \psi_s^m, \partial_{(\xi_j)_h} \phi)_{H_0^1(\Omega)} = O(\|\phi\| \|\partial_{(\xi_j)_h} P \psi_s^m\|) = O\left(\frac{\|\phi\|}{\rho^2}\right)$$

since, by a straightforward computation, we see that,

$$\|\partial_{(\xi_j)_h} P \psi_s^m\| \leq \|\partial_{(\xi_j)_h} \psi_s^m\| = O\left(\frac{1}{\rho^2}\right).$$

Hence, by (4.3) we get

$$\begin{aligned} & \left(\sum_{i=1}^k PU_i + \phi - i^*(\rho^2 V(x) e^{\sum_{i=1}^k PU_i + \phi}), \sum_{i=1}^k \partial_{(\xi_j)_h} \tau_i(\xi) P\psi_i^0 + \partial_{(\xi_j)_h} \phi \right)_{H_0^1(\Omega)} \\ & = o\left(\frac{1}{\rho^2} \sum_{s=1}^k \sum_{m=1}^2 |c_{sm}^\rho|\right). \end{aligned} \tag{4.14}$$

Finally, recalling (4.11), by (4.13), (4.14) we deduce

$$\partial_{(\xi_j)_h} E_\rho \left(\sum_{i=1}^k PU_i + \phi \right) = -32\pi^2 \frac{\partial \mathcal{F}}{\partial (\xi_j)_h}(\xi_1, \dots, \xi_k) + o(1) \tag{4.15}$$

as $\rho \rightarrow 0$, and the proof is complete. \square

The finite dimensional reduction is meaningful in view of the property:

Proposition 4.3. *The function $\sum_{i=1}^k PU_{\tau_i(\xi), \xi_i} + \phi_\xi^\rho$ is a solution of (2.1) if and only if ξ is a critical point of the reduced map $\xi \rightarrow E_\rho(\sum_{i=1}^k PU_{\tau_i(\xi), \xi_i} + \phi_\xi^\rho)$.*

Proof. Assume that ξ is a critical point of the reduced map. Then, ξ satisfies the relations

$$\left(\sum_{i=1}^k PU_i + \phi - i^*(\rho^2 V(x) e^{\sum_{i=1}^k PU_i + \phi}), P\psi_j^h + \sum_{i=1}^k \partial_{(\xi_j)_h} \tau_i(\xi) P\psi_i^0 + \partial_{(\xi_j)_h} \phi \right)_{H_0^1(\Omega)} = 0$$

for any $j = 1, \dots, k$ and $h = 1, 2$. By (4.10) and (4.14) we get that,

$$\sum_{s=1}^k \sum_{m=1}^2 c_{sm}^\rho (P\psi_s^m, P\psi_j^h)_{H_0^1(\Omega)} + o\left(\frac{1}{\rho^2} \sum_{s=1}^k \sum_{m=1}^2 |c_{sm}^\rho|\right) = 0.$$

By Lemma A.4, we conclude that $c_{sm}^\rho = 0$ for any $s = 1, \dots, k$ and $m = 1, 2$, and then, by (4.10) we get

$$\sum_{i=1}^k PU_i + \phi - i^*(\rho^2 V(x) e^{\sum_{i=1}^k PU_i + \phi}) = 0.$$

The proposition now follows since the converse property is always true. \square

5. Proofs of Theorem 1.1, Theorem 1.2 and some examples

Let us recall the following notion of stability of critical points introduced in [26]:

Definition 5.1. Let $F : D \rightarrow \mathbb{R}$ be a C^1 -function and let K be a bounded set of critical points of F . We say that K is a C^1 -stable critical set of F if for any $\mu > 0$ there exists $\delta > 0$ such that, if $G : D \rightarrow \mathbb{R}$ is a C^1 -function with

$$\max_{\text{dist}(x, K) \leq \mu} (|G(x) - F(x)| + |\nabla G(x) - \nabla F(x)|) \leq \delta,$$

then G has at least one critical point x with $\text{dist}(x, K) \leq \mu$.

Remark 5.2. It is easy to see that a bounded set of critical point of F is stable if one of the following condition is satisfied:

- (i) K is a strict local maximum set of F , i.e. $F(x) = F(y)$ for any $x, y \in K$ and for some open neighborhood U of K it holds $F(x) > F(y)$ for any $x \in K$ and $y \in U \setminus K$;
- (ii) K is a strict local minimum set of F ;
- (iii) the Brouwer degree $\text{deg}(\nabla F, U_\mu, 0) \neq 0$ for any $\mu > 0$ small, where $U_\mu = \{x \in \mathbb{R}^N : \text{dist}(x, K) \leq \mu\}$.

Let us give the proof of Theorem 1.1:

Proof of Theorem 1.1. By Propositions 4.2 and 4.3, we can find a point $\xi_\rho = (\xi_{1,\rho}, \dots, \xi_{k,\rho})$ such that $u_\rho = \sum_{i=1}^k P U_{\tau_i(\xi_\rho), \xi_{i,\rho}} + \phi_{\xi_\rho}^\rho$ is a solution of (2.1). Moreover, u_ρ satisfies the “concentration” property:

$$\rho^2 \int_{\Omega} V(x) e^{u_\rho} \Psi(x) \rightarrow 8\pi \sum_{i=1}^k \Psi(q_i) \quad \text{as } \rho \rightarrow 0, \quad \forall \Psi \in C(\overline{\Omega}). \tag{5.1}$$

In fact, by the inequality $|e^s - 1| \leq e^{|s|} |s|$ for any $s \in \mathbb{R}$, we can proceed as in Proposition 4.2 to obtain

$$\rho^2 \int_{\Omega} V(x) e^{u_\rho} \Psi(x) = \rho^2 \int_{\Omega} V(x) e^{\sum_{i=1}^k P U_i} \Psi(x) + o(1) = 8\pi \sum_{i=1}^k \Psi(q_i) + o(1).$$

Along a sequence $\rho \rightarrow 0$, we can assume that $\text{dist}(\xi_\rho, K) \rightarrow 0$ and the proof is completed. \square

Let us stress that Theorem 1.1 allows to handle the cases where the maximum of the Robin’s function is not isolated (for example, in an annulus).

We are now in position to give the proof of Theorem 1.2:

Proof of Theorem 1.2. Let $k = 1$. It is easy to see that \mathcal{F} has a strict local maximum set in each connected component Ω'_i for any $i = 1, \dots, \nu$. In fact, since $H(\xi, \xi) \rightarrow -\infty$ as $\xi \rightarrow \partial\Omega$ and $\log V(\xi) \rightarrow -\infty$ as $V(\xi) \rightarrow 0$, it follows that $\mathcal{F}(\xi) \rightarrow -\infty$ as $\xi \rightarrow \partial\Omega'_i$. Therefore, the claim follows by Theorem 1.1. \square

In next examples, we follow some ideas introduced in [33] in order to get some contractible domains where a large number of solutions to problem (1.1) exists, provided $V(x) \equiv 1$.

Let $\Omega_0 = \Omega_1 \cup \dots \cup \Omega_h$, where $\Omega_1, \dots, \Omega_h$ are h smooth bounded domains in \mathbb{R}^2 such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset$ if $i \neq j$. Assume that

$$\Omega_i \subset \{(x_1, x_2) \in \mathbb{R}^2 : a_i \leq x_1 \leq b_i\}, \quad \Omega_i \cap \{x_2 = 0\} \neq \emptyset,$$

for some $b_i < a_{i+1}$ and $i = 1, \dots, h$. Let

$$C_\epsilon = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_2| \leq \epsilon, x_1 \in (a_1, b_h)\}.$$

Let Ω_ϵ be a smooth simply connected domain such that

$$\Omega_0 \subset \Omega_\epsilon \subset \Omega_0 \cup C_\epsilon.$$

We say that

$$\Omega_\epsilon \text{ is a } h\text{-dumbell.} \tag{5.2}$$

Lemma 5.3. *We have that*

$$\lim_{\epsilon \rightarrow 0} H_{\Omega_\epsilon}(x, x) = H_{\Omega_0}(x, x) \tag{5.3}$$

C^1 -uniformly on compact sets of Ω_0 and

$$\lim_{\epsilon \rightarrow 0} G_{\Omega_\epsilon}(x, y) = G_{\Omega_0}(x, y) \tag{5.4}$$

C^1 -uniformly on compact sets of $\Omega_0 \times \Omega_0 \setminus \{(x, y): x = y\}$.

Proof. We argue as in Lemma 3.2 of [33]. \square

Theorem 5.4. For any $h \geq 2$ there exists an h -dumbbell Ω for which problem (1.1) has at least h different families of solutions which blow-up at a point q_i in Ω , $i = 1, \dots, h$, along sequences $\lambda \rightarrow 8\pi$.

Proof. In virtue of Theorem 1.1, it is enough to find an h -dumbbell Ω , constructed as in (5.2), so that the Robin’s function $H_\Omega(\cdot, \cdot)$ has at least h disjoint stable critical sets. Since the Robin’s function $H_{\Omega_i}(\cdot, \cdot)$ has a strict local maximum set in Ω_i for any $i = 1, \dots, h$, by Lemma 5.3 we deduce that if ϵ is small enough, the Robin’s function $H_{\Omega_\epsilon}(\cdot, \cdot)$ has h different strict local maximum sets, each contained in the corresponding Ω_i . Since they are stable according to Definition 5.1, the claim follows. \square

Theorem 5.5. For any $h \geq 2$ there exists an h -dumbbell Ω for which problem (1.1) has at least one family of solutions which blow-up at h points q_1, \dots, q_h in Ω , along a sequence $\lambda \rightarrow 8\pi h$.

Proof. In virtue of Theorem 1.1, it is enough to find an h -dumbbell Ω , constructed as in (5.2), so that the function \mathcal{F}_Ω , defined as

$$\mathcal{F}_\Omega(\xi_1, \dots, \xi_h) = \sum_{i=1}^h H_\Omega(\xi_i, \xi_i) + \sum_{j \neq i} G_\Omega(\xi_j, \xi_i),$$

has a stable critical set according to Definition 5.1. It is easy to check that the function \mathcal{F}_{Ω_0} in the connected component $\Omega_1 \times \dots \times \Omega_h$ of the set $(\Omega_0)^h$ reduces to take the form:

$$\mathcal{F}_{\Omega_0}(\xi_1, \dots, \xi_h) = \sum_{i=1}^h H_{\Omega_i}(\xi_i, \xi_i),$$

because $H_{\Omega_0}(x) = H_{\Omega_i}(x)$ if $x \in \Omega_i$, $G_{\Omega_0}(x, y) = G_{\Omega_i}(x, y)$ if $(x, y) \in \Omega_i \times \Omega_i$ and $G_{\Omega_0}(x, y) = 0$ if $(x, y) \in \Omega_i \times \Omega_j$, $i \neq j$. Therefore, \mathcal{F}_{Ω_0} has a strict local maximum set. By Lemma 5.3 we deduce that if ϵ is small enough, the function $\mathcal{F}_{\Omega_\epsilon}$ also has a strict local maximum set, which is stable according to Definition 5.1, and the claim follows. \square

As in the previous results, we easily get:

Corollary 5.6. For any $h \geq 2$ there exists an h -dumbbell Ω for which problem (1.1) has at least $\binom{h}{i}$ families of solutions which blow-up at i points in Ω along sequences $\lambda \rightarrow 8\pi i$, for any $i = 1, \dots, h$.

6. Proof of Theorem 1.3 and some examples

In this section, we will restrict to the case $V(x) \equiv 1$. Let S , ∂Q and Q be compact subsets of a domain D . We will say that S links Q via ∂Q by homotopy in D if $\partial Q \subset Q$, $S \cap \partial Q = \emptyset$ and $\gamma_1(Q) \cap S \neq \emptyset$ for any $\gamma \in \Gamma$, where

$$\Gamma := \{\gamma \in C^0([0, 1] \times Q, D): \gamma_0 = \text{Id}, \gamma_t|_{\partial Q} = \text{Id} \ \forall t \in [0, 1]\}.$$

Now, let us recall the following notion of stability of critical values introduced in [25]. Let $F : D \rightarrow \mathbb{R}$ be a C^1 -function. We say that:

Definition 6.1. c is a stable critical value of F , if there exist compact subsets $S, \partial Q$ and Q of D such that S links Q via ∂Q by homotopy in D ,

$$\max_{\partial Q} F < \min_S F \tag{6.1}$$

and the set $\{x \in D: c - \varepsilon \leq F(x) \leq c + \varepsilon\}$ is complete for some $\varepsilon > 0$, where

$$c := \inf_{\gamma \in \Gamma} \max_{u \in Q} F(\gamma_1(u)). \tag{6.2}$$

Proof of Theorem 1.3. By Proposition 4.3 it is enough to prove that the function \tilde{E}_ρ given in Proposition 4.2 has a critical point. Using the stability of c , the expansion (4.7) and arguing exactly as in Theorem 3.1 in [25], we can prove that if ρ is small enough there exists a critical point ξ_ρ of the function \tilde{E}_ρ such that $\xi_\rho \rightarrow \xi_0$ as $\rho \rightarrow 0$, with $H(\xi_0, \xi_0) = c$ and $\nabla H(\xi_0, \xi_0) = 0$. The claim follows as in the proof of Theorem 1.1. \square

Let us prove that in a domain with “a hole” problem (1.1) has at least two families of solutions blowing-up at one point.

Theorem 6.2. *Let $\Omega_1 \subset\subset \Omega_2$ be smooth bounded domains and let $\Omega = \Omega_2 \setminus \Omega_1$. Assume that there exists a curve σ joining $\partial\Omega_1$ and $\partial\Omega_2$ such that*

$$\max_{\xi \in \sigma} H_\Omega(\xi, \xi) < \max_{\xi \in \Omega} H_\Omega(\xi, \xi). \tag{6.3}$$

Then (1.1) has at least two families of solutions which blow-up at one point in Ω as $\lambda \rightarrow 8\pi$.

Proof. By Theorem 1.3 it is enough to prove that the Robin’s function $H_\Omega(\cdot, \cdot)$ has at least two different stable critical values. Let S be a curve contained in Ω around the hole Ω_1 . Since $H(\xi, \xi) \rightarrow -\infty$ as $\xi \rightarrow \partial\Omega$, it is clear that there exist two points ξ_1 and ξ_2 in σ very close to $\partial\Omega_1$ and $\partial\Omega_2$, respectively, such that if Q is a curve contained in σ with endpoints ξ_1 and ξ_2 and $\partial Q := \{\xi_1, \xi_2\}$ then

$$\max_{\xi \in \partial Q} H(\xi, \xi) < \min_{\xi \in S} H(\xi, \xi) \tag{6.4}$$

and moreover the curve S links the segment Q via ∂Q by homotopy in Ω . Moreover, since the Palais–Smale condition in Ω is satisfied, the Robin’s function $H_\Omega(\cdot, \cdot)$ has a critical value c such that

$$\min_{\xi \in S} H(\xi, \xi) \leq c \leq \max_{\xi \in Q} H(\xi, \xi) < \max_{\xi \in \Omega} H(\xi, \xi) \tag{6.5}$$

in view of (6.3). Therefore, c is stable according to Definition 6.1. Finally, the claim follows, since the maximum value of the Robin’s function is stable according to Definition 6.1. \square

Let us show that in a “2-dumbbell” domain problem (1.1) has at least three families of solutions blowing-up at one point.

Theorem 6.3. *Let Ω_ε be a 2-dumbbell as in (5.2). If ε is small enough then problem (1.1) has at least three families of solutions which blow-up at one point in Ω_ε as $\lambda \rightarrow 8\pi$.*

Proof. By Theorem 1.1 and Theorem 1.3 it is enough to prove that the function $h_\varepsilon(\xi) := -H_{\Omega_\varepsilon}(\xi, \xi)$ has at least two different stable critical points and one stable critical value. First of all, let us point out that there exist opens

sets $B_1 \subset \Omega_1$, $B_2 \subset \Omega_2$, points $\xi_1 \in B_1$, $\xi_2 \in B_2$ and a set $S = \{x \in \Omega_1 \mid \text{dist}(x, \partial\Omega_1) = \mu\}$, with $\mu > 0$ small enough such that

$$h_0(\xi_i) = \min_{\Omega_i} h_0 < \min_{\partial B_i} h_0, \quad i = 1, 2, \quad \max \left\{ \min_{\partial B_1} h_0, \min_{\partial B_2} h_0 \right\} < \min_S h_0.$$

By Lemma 5.3 it follows that for ε small enough it holds

$$h_\varepsilon(\xi_i) < \min_{\partial B_i} h_\varepsilon, \quad i = 1, 2, \quad \max \left\{ \min_{\partial B_1} h_\varepsilon, \min_{\partial B_2} h_\varepsilon \right\} < \min_S h_\varepsilon. \tag{6.6}$$

Let Q be a curve joining ξ_1 and ξ_2 in Ω_ε and $\partial Q = \{\xi_1, \xi_2\}$. It is clear that the curve S links the curve Q via ∂Q by homotopy in Ω_ε and property (6.1) holds. Moreover, since the Palais–Smale condition in Ω_ε is satisfied, by (6.6) it follows that the function h_ε has in B_1 a minimum point $\xi_{1\varepsilon}$ and in B_2 a minimum point $\xi_{2\varepsilon}$ (which provide two different strict local minimum sets of h_ε) and a stable critical value c_ε such that $c_\varepsilon > \max\{h_\varepsilon(\xi_{1\varepsilon}), h_\varepsilon(\xi_{2\varepsilon})\}$. The claim is completely established. \square

Appendix A

Define $\Omega_\varepsilon = \{\xi \in \Omega : \text{dist}(\xi, \partial\Omega) \geq 2\varepsilon\}$ for $\varepsilon > 0$. Let $PU_{\tau,\xi}$ be the projection of $U_{\tau,\xi}$ into $H_0^1(\Omega)$ (see (1.5) and (1.6)).

We will estimate $PU_{\tau,\xi}$ in terms of the Robin’s function and of the Green’s function. We assume all along this section that ε is a fixed positive number and τ varies in $(0, +\infty)$. Arguing as in [37] we can prove the following crucial estimates:

Proposition A.1. *We have*

$$PU_{\tau,\xi}(x) = U_{\tau,\xi}(x) + 8\pi H(x, \xi) - \log(8\tau^2) + O(\rho^2) \tag{A.1}$$

in $C^0(\Omega)$ and in $C_{\text{loc}}^2(\Omega)$ as $\rho \rightarrow 0$,

$$PU_{\tau,\xi}(x) = 8\pi G(x, \xi) + O(\rho^2) \tag{A.2}$$

in $C_{\text{loc}}^0(\overline{\Omega} \setminus \{\xi\})$ and in $C_{\text{loc}}^2(\Omega \setminus \{\xi\})$ as $\rho \rightarrow 0$, and the convergences are uniform in $\xi \in \Omega_\varepsilon$ and τ bounded away from zero.

Proof. Let $\eta_{\tau,\xi}(x) = PU_{\tau,\xi}(x) - U_{\tau,\xi}(x) + \log(8\tau^2)$. We have that $\eta_{\tau,\xi}$ satisfies

$$\begin{cases} \Delta \eta_{\tau,\xi} = 0 & \text{in } \Omega, \\ \eta_{\tau,\xi}(x) = 2 \log(\tau^2 \rho^2 + |x - \xi|^2) & \text{on } \partial\Omega. \end{cases}$$

Recalling that $\eta_\xi(x) = 8\pi H(x, \xi)$ verifies

$$\begin{cases} \Delta \eta_\xi = 0 & \text{in } \Omega, \\ \eta_\xi(x) = 2 \log |x - \xi|^2 & \text{on } \partial\Omega, \end{cases}$$

we get

$$\begin{cases} \Delta(\eta_{\tau,\xi} - \eta_\xi) = 0 & \text{in } \Omega, \\ \eta_{\tau,\xi}(x) - \eta_\xi(x) = -2 \log \frac{|x - \xi|^2}{\tau^2 \rho^2 + |x - \xi|^2} & \text{on } \partial\Omega. \end{cases}$$

Since $|x - \xi| \geq 2\varepsilon$ for $x \in \partial\Omega$, by the maximum principle we get

$$\max_{\overline{\Omega}} |\eta_{\tau,\xi} - \eta_\xi| = \max_{\partial\Omega} |\eta_{\tau,\xi} - \eta_\xi| = O(\rho^2)$$

uniformly for $\xi \in \Omega_\varepsilon$ and τ bounded away from zero. By the representation formula for harmonic function we derive the convergence in $C^2_{\text{loc}}(\Omega)$. By (A.1) we deduce (A.2). \square

We point out the following useful estimate:

Lemma A.2. *If $p \geq 1$, then $\|\rho^2 e^{U_{\tau,\xi}}\|_{L^p} = O(\rho^{2(1-q)/p})$ uniformly in $\xi \in \Omega_\varepsilon$ and τ bounded away from zero.*

Let $P\psi_{\tau,\xi}^j$ be the projection into $H^1_0(\Omega)$ of $\psi_{\tau,\xi}^j$ for $j = 0, 1, 2$, (see (2.3)–(2.5)). Arguing as in the proof of Proposition A.1, we can prove the following crucial estimates:

Proposition A.3. *If $j = 0$ we have*

$$P\psi_{\tau,\xi}^0(x) = \psi_{\tau,\xi}^0(x) - \frac{2}{\tau} + O(\rho^2) = -4 \frac{\tau\rho^2}{\tau^2\rho^2 + |x - \xi|^2} + O(\rho^2) \tag{A.3}$$

in $C^0(\Omega)$ and in $C^2_{\text{loc}}(\Omega)$ as $\rho \rightarrow 0$,

$$P\psi_{\tau,\xi}^0(x) = O(\rho^2) \tag{A.4}$$

in $C^0(\overline{\Omega} \setminus \{\xi\})$ and in $C^2_{\text{loc}}(\Omega \setminus \{\xi\})$ as $\rho \rightarrow 0$, and the convergences are uniform in $\xi \in \Omega_\varepsilon$ and τ bounded away from zero. If $j = 1, 2$ we have

$$P\psi_{\tau,\xi}^j(x) = \psi_{\tau,\xi}^j(x) + 8\pi \frac{\partial H}{\partial \xi_j}(x, \xi) + O(\rho^2) \tag{A.5}$$

in $C^0(\Omega)$ and in $C^2_{\text{loc}}(\Omega)$ as $\rho \rightarrow 0$,

$$P\psi_{\tau,\xi}^j(x) = 8\pi \frac{\partial G}{\partial \xi_j}(x, \xi) + O(\rho^2) \tag{A.6}$$

in $C^0(\overline{\Omega} \setminus \{\xi\})$ and in $C^2_{\text{loc}}(\Omega \setminus \{\xi\})$ as $\rho \rightarrow 0$, and the convergences are uniform in $\xi \in \Omega_\varepsilon$ and τ bounded away from zero.

Finally, we have the following ‘‘orthogonality’’ relations:

Lemma A.4. *Let $j, l = 1, 2$. We have that*

$$\begin{aligned} \|P\psi_{\tau,\xi}^0\|^2 &= \frac{D_0}{\tau^2} [1 + O(\rho^2)], \\ (P\psi_{\tau,\xi}^j, P\psi_{\tau,\xi}^l)_{H^1_0(\Omega)} &= \frac{D}{\tau^2\rho^2} [\delta_{jl} + O(\rho^2)], \\ (P\psi_{\tau_1,\xi_1}^j, P\psi_{\tau_2,\xi_2}^l)_{H^1_0(\Omega)} &= O(1) \end{aligned}$$

as $\rho \rightarrow 0$, uniformly in $\xi, \xi_1, \xi_2 \in \Omega_\varepsilon$ with $|\xi_1 - \xi_2| \geq 2\varepsilon$ and τ, τ_1, τ_2 bounded away from zero. Here, $D_0 = 64 \int_{\mathbb{R}^2} (1 - |y|^2)/(1 + |y|^2)^4$, $D = 64 \int_{\mathbb{R}^2} |y|^2/(1 + |y|^2)^4$ are positive constants and δ_{jl} denote the Kronecker’s symbols.

Proof. Note that for $i, k = 0, 1, 2$

$$\begin{aligned}
 (P\psi_{\tau,\xi}^i, P\psi_{\tau,\xi}^k)_{H_0^1(\Omega)} &= \rho^2 \int_{\Omega} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^i P\psi_{\tau,\xi}^k \\
 &= \rho^2 \int_{B(\xi,\varepsilon)} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^i P\psi_{\tau,\xi}^k + \rho^2 \int_{\Omega \setminus B(\xi,\varepsilon)} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^i P\psi_{\tau,\xi}^k.
 \end{aligned}
 \tag{A.7}$$

We deduce by (A.3) that

$$\begin{aligned}
 \rho^2 \int_{B(\xi,\varepsilon)} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^0 P\psi_{\tau,\xi}^0 &= 16\tau\rho^2 \int_{B(\xi,\varepsilon)} \frac{|x-\xi|^2 - \tau^2\rho^2}{(\tau^2\rho^2 + |x-\xi|^2)^3} \left(-\frac{4\tau\rho^2}{\tau^2\rho^2 + |x-\xi|^2} + O(\rho^2) \right) \\
 &= \frac{64}{\tau^2} \int_{B(0,\varepsilon/\tau\rho)} \frac{1-|y|^2}{(1+|y|^2)^4} + O(\rho^2) = \frac{D_0}{\tau^2} + O(\rho^2),
 \end{aligned}
 \tag{A.8}$$

and by (A.5) that

$$\begin{aligned}
 \rho^2 \int_{B(\xi,\varepsilon)} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^j P\psi_{\tau,\xi}^l &= 32\tau^2\rho^2 \int_{B(\xi,\varepsilon)} \frac{(x-\xi)_j}{(\tau^2\rho^2 + |x-\xi|^2)^3} \left(4\frac{(x-\xi)_l}{\tau^2\rho^2 + |x-\xi|^2} + 8\pi \frac{\partial H}{\partial \xi_l}(\xi, \xi) + O(|x-\xi| + \rho^2) \right) \\
 &= \frac{128}{\tau^2\rho^2} \int_{B(0,\varepsilon/\tau\rho)} \frac{y_j y_l}{(1+|y|^2)^4} + O(1) = \frac{D}{\tau^2\rho^2} \delta_{jl} + O(1).
 \end{aligned}
 \tag{A.9}$$

On the other hand, by (A.4) and (A.6) we get that for $i = 1, \dots, k$

$$\rho^2 \int_{\Omega \setminus B(\xi,\varepsilon)} e^{U_{\tau,\xi}} \psi_{\tau,\xi}^i P\psi_{\tau,\xi}^i = O(\rho^2).
 \tag{A.10}$$

The first two estimates follow inserting (A.8)–(A.10) into (A.7). As far as the third estimate, we can write

$$(P\psi_{\tau_1,\xi_1}^j, P\psi_{\tau_2,\xi_2}^l)_{H_0^1(\Omega)} = \rho^2 \int_{B(\xi_1,\varepsilon)} e^{U_{\tau_1,\xi_1}} \psi_{\tau_1,\xi_1}^j P\psi_{\tau_2,\xi_2}^l + \rho^2 \int_{\Omega \setminus B(\xi_1,\varepsilon)} e^{U_{\tau_1,\xi_1}} \psi_{\tau_1,\xi_1}^j P\psi_{\tau_2,\xi_2}^l,
 \tag{A.11}$$

and by (A.6) we can estimate

$$\begin{aligned}
 \rho^2 \int_{B(\xi_1,\varepsilon)} e^{U_{\tau_1,\xi_1}} \psi_{\tau_1,\xi_1}^j P\psi_{\tau_2,\xi_2}^l &= 8\pi\rho^2 \int_{B(\xi_1,\varepsilon)} e^{U_{\tau_1,\xi_1}} \psi_{\tau_1,\xi_1}^j \left(\frac{\partial G}{\partial(\xi_2)_l}(x, \xi_2) - \frac{\partial G}{\partial(\xi_2)_l}(\xi_1, \xi_2) \right) + O(\rho) = O(1),
 \end{aligned}
 \tag{A.12}$$

and

$$\rho^2 \int_{\Omega \setminus B(\xi_1,\varepsilon)} e^{U_{\tau_1,\xi_1}} \psi_{\tau_1,\xi_1}^j P\psi_{\tau_2,\xi_2}^l = O(\rho^2 \|P\psi_{\tau_2,\xi_2}^l\|) = O(\rho).
 \tag{A.13}$$

Inserting (A.12), (A.13) into (A.11) we obtain the last estimate. \square

Appendix B

In this appendix we prove some technical lemmata which allow to prove Proposition 4.1. By Proposition A.1 we deduce the following useful estimate which will give in particular an estimate of the remainder term ϕ_ξ^ρ in the finite dimensional reduction (see (4.3)).

Lemma B.1. *Let $\xi = (\xi_1, \dots, \xi_k) \in (\Omega')^k \setminus \Delta$ and $\tau_i(\xi)$ be defined as in (2.6). For any $p \geq 1$ there exists a positive constant $c = c(p)$, uniform in compact sets of $(\Omega')^k \setminus \Delta$, such that for any $\rho > 0$ we have:*

$$\|\rho^2 V(x) e^{\sum_{i=1}^k P U_{\tau_i(\xi), \xi_i}} - \rho^2 \sum_{i=1}^k e^{U_{\tau_i(\xi), \xi_i}}\|_L^p \leq c \rho^{(2-p)/p}.$$

Proof. Set $U_i = U_{\tau_i(\xi), \xi_i}$, $i = 1, \dots, k$, and assume that $\xi \in \mathcal{O}_\varepsilon$ for some $\varepsilon > 0$. We write,

$$\begin{aligned} & \int_\Omega \left| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right|^p \\ &= \sum_{j=1}^k \int_{B(\xi_j, \varepsilon)} \left| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right|^p + \int_{\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \varepsilon)} \left| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right|^p. \end{aligned} \tag{B.1}$$

By (A.2) we deduce that,

$$\int_{\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \varepsilon)} \left| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right|^p = O(\rho^{2p}). \tag{B.2}$$

Moreover, for any $j = 1, \dots, k$ we have that

$$\int_{B(\xi_j, \varepsilon)} \left| \rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 \sum_{i=1}^k e^{U_i} \right|^p = O\left(\int_{B(\xi_j, \varepsilon)} |\rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 e^{U_j}|^p + \rho^{2p} \right). \tag{B.3}$$

By (A.1), (A.2) for $x \in B(\xi_j, \varepsilon)$, $j = 1, \dots, k$, we get

$$\sum_{i=1}^k P U_i - U_j = 8\pi \left(H(x, \xi_j) - \log(8\tau_j^2) + \sum_{i \neq j} G(x, \xi_i) \right) + O(\rho^2) = -\log V(\xi_j) + O(|x - \xi_j| + \rho^2),$$

and then, by the boundedness of $(e^x - 1)/x$ on bounded sets, finally we obtain

$$\begin{aligned} \int_{B(\xi_j, \varepsilon)} |\rho^2 V(x) e^{\sum_{i=1}^k P U_i} - \rho^2 e^{U_j}|^p &= O\left(\rho^{2p} \int_{B(\xi_j, \varepsilon)} e^{P U_j} (|x - \xi_j| + \rho^2)^p \right) \quad (\text{setting } x = \tau_j \rho y + \xi_j) \\ &= O\left(\rho^{2-p} \int_{B(0, \varepsilon/\tau_j \rho)} \frac{(|y| + \rho)^p}{(1 + |y|^2)^{2p}} \right) = O(\rho^{2-p}). \end{aligned} \tag{B.4}$$

The claim follows by (B.1)–(B.4). \square

Lemma B.2. *Let $\varepsilon > 0$. For any $p \geq 1$ and $r > 1$ there exists positive constants c_1, c_2 such that for any $\rho > 0$ it holds*

$$\|\rho^2 V(x) e^{\sum_{i=1}^k P U_{\tau_i, \xi_i}} (e^{\phi_1} - 1 - \phi_1)\|_{L^p} \leq c_1 e^{c_2 \|\phi_1\|^2} \rho^{(2-2pr)/(pr)} \|\phi_1\|^2, \tag{B.5}$$

$$\begin{aligned} & \|\rho^2 V(x) e^{\sum_{i=1}^k P U_{\tau_i, \xi_i}} (e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2))\|_{L^p} \\ & \leq c_1 e^{c_2(\|\phi_1\|^2 + \|\phi_2\|^2)} \rho^{(2-2pr)/(pr)} \|\phi_1 - \phi_2\| (\|\phi_1\| + \|\phi_2\|) \end{aligned} \tag{B.6}$$

for any $\phi_1, \phi_2 \in H_0^1(\Omega)$, uniformly for $\xi \in \mathcal{O}_\varepsilon$ and τ_i bounded away from zero.

Proof. Let us remark that (B.5) follows by choosing $\phi_2 = 0$ in (B.6). Let us prove (B.6). By the mean value theorem we see that, for any $\phi_1, \phi_2 \in \mathbb{R}$

$$|e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)| \leq e^{|\phi_1| + |\phi_2|} |\phi_1 - \phi_2| (|\phi_1| + |\phi_2|).$$

Hence, by Hölder’s inequality with $\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1$ and Remark 2.4, we find that

$$\begin{aligned} & \left(\int_{\Omega} e^{p \sum_{i=1}^k P U_{\tau_i, \xi_i}} |e^{\phi_1} - e^{\phi_2} - (\phi_1 - \phi_2)|^p \right)^{1/p} \\ & \leq C \sum_{j=1}^2 \left(\int_{\Omega} e^{p \sum_{i=1}^k P U_{\tau_i, \xi_i}} e^{p|\phi_1| + p|\phi_2|} |\phi_1 - \phi_2|^p |\phi_j|^p \right)^{1/p} \\ & \leq C \sum_{j=1}^2 \left(\int_{\Omega} e^{pr \sum_{i=1}^k P U_{\tau_i, \xi_i}} \right)^{1/(pr)} \left(\int_{\Omega} e^{ps|\phi_1| + ps|\phi_2|} \right)^{1/(ps)} \left(\int_{\Omega} |\phi_1 - \phi_2|^{pt} |\phi_j|^{pt} \right)^{1/(pt)} \\ & \leq C' \sum_{j=1}^2 \left(\int_{\Omega} e^{pr \sum_{i=1}^k P U_{\tau_i, \xi_i}} \right)^{1/(pr)} e^{(ps)/(8\pi)(\|\phi_1\|^2 + \|\phi_2\|^2)} \|\phi_1 - \phi_2\| \|\phi_j\|. \end{aligned} \tag{B.7}$$

On the other hand, by Proposition A.1 we deduce,

$$\left(\int_{\Omega} e^{pr \sum_{i=1}^k P U_{\tau_i, \xi_i}} \right)^{1/(pr)} \leq C' \left(1 + \sum_{i=1}^k \int_{B(\xi_i, \varepsilon)} \frac{1}{(\tau_i^2 \rho^2 + |x - \xi_i|^2)^{2pr}} \right)^{1/(pr)} \leq C \rho^{(2-4pr)/(pr)}. \tag{B.8}$$

Thus, (B.6) follows by means of (B.7) and (B.8). \square

Appendix C

This last section contains the proof of some technical lemmata. All the results we will state here are uniform in compact sets of $(\Omega')^k \setminus \Delta$. We will assume throughout the section the additional property $\xi \in \mathcal{O}_\varepsilon$ for some $\varepsilon > 0$. For $j = 1, 2, i = 1, \dots, k$ set $U_i := U_{\tau_i(\xi), \xi_i}$ and $\psi_i^j := \psi_{\tau_i(\xi), \xi_i}^j$, where $\tau_i = \tau_i(\xi)$ is defined in (2.6).

Lemma C.1. *We have:*

$$\int_{\Omega} |\nabla P U_i|^2 = (-32\pi \log \rho + 48\pi \log 2 - 16\pi) - 64\pi^2 \mathcal{F}(\xi) - 64\pi^2 \sum_{j \neq i} G(\xi_i, \xi_j) + O(\rho) \tag{C.1}$$

and

$$\int_{\Omega} \nabla P U_i \nabla P U_j = 64\pi^2 G(\xi_i, \xi_j) + O(\rho) \tag{C.2}$$

as $\rho \rightarrow 0$.

Proof. By Proposition A.1 we get that

$$\begin{aligned} \int_{\Omega} |\nabla P U_i|^2 &= \rho^2 \int_{\Omega} e^{U_i} P U_i \\ &= \int_{B(\xi_i, \varepsilon)} \frac{8\tau_i^2(\xi)\rho^2}{(\tau_i^2(\xi)\rho^2 + |x - \xi_i|^2)^2} (-2\log(\tau_i^2(\xi)\rho^2 + |x - \xi_i|^2) \\ &\quad + 8\pi H(\xi_i, \xi_i) + O(|x - \xi_i| + \rho^2)) + O(\rho^2) \quad (\text{setting } x = \tau_i(\xi)\rho y + \xi_i) \\ &= \int_{B(0, \varepsilon/\tau_i(\xi)\rho)} \frac{8}{(1 + |y|^2)^2} (-4\log \rho + 6\log 2 - 8\pi \mathcal{F}(\xi) \\ &\quad - 8\pi \sum_{j \neq i} G(\xi_i, \xi_j) - 2\log(1 + |y|^2)) + O(\rho) \\ &= (-32\pi \log \rho + 48\pi \log 2 - 16\pi) - 64\pi^2 \mathcal{F}(\xi) - 64\pi^2 \sum_{j \neq i} G(\xi_i, \xi_j) + O(\rho), \end{aligned} \tag{C.3}$$

where we have used the fact that we have the relation:

$$\int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} \log(1 + |y|^2) = \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^2} = \pi$$

(by an integration by parts). By (C.3), in view of Proposition A.1 for $i \neq j$ we obtain

$$\begin{aligned} \int_{\Omega} \nabla P U_i \nabla P U_j &= \int_{B(\xi_i, \varepsilon)} \frac{8\tau_i^2(\xi)\rho^2}{(\tau_i^2(\xi)\rho^2 + |x - \xi_i|^2)^2} (8\pi G(\xi_i, \xi_j) + O(|x - \xi_i| + \rho^2)) + O(\rho^2 \|P U_j\|) \\ &= 64\pi^2 G(\xi_i, \xi_j) + O(\rho). \quad \square \end{aligned}$$

Lemma C.2. *We have:*

$$\rho^2 \int_{\Omega} V(x) e^{\sum_i P U_i} = 8\pi k + o(1) \tag{C.4}$$

as $\rho \rightarrow 0$.

Proof. By Proposition A.1 we get

$$\begin{aligned} \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} &= \sum_{i=1}^k \rho^2 \int_{B(\xi_i, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} + O(\rho^2) \\ &= \sum_{i=1}^k \int_{B(\xi_i, \varepsilon)} \frac{\rho^2 V(\xi_i)}{(\tau_i^2 \rho^2 + |x - \xi_i|^2)^2} e^{8\pi(H(\xi_i, \xi_i) + \sum_{h \neq i} G(\xi_i, \xi_h))} \end{aligned}$$

$$\begin{aligned} & \times (1 + O(|x - \xi_i| + \rho^2)) + O(\rho^2) \quad (\text{setting } x = \tau_i \rho y + \xi_i) \\ & = \sum_{i=1}^k \int_{B(0, \varepsilon / \tau_i \rho)} \frac{8}{(1 + |y|^2)^2} dy + o(1) = 8\pi k + o(1). \quad \square \end{aligned}$$

Lemma C.3. *Let $j = 1, 2$ and $i, h = 1, \dots, k$. We have:*

$$\rho^2 \int_{\Omega} e^{U_i} P \psi_h^j = 64\pi^2 \frac{\partial H}{\partial(\xi_i)_j}(\xi_i, \xi_i) \delta_{ih} + 64\pi^2 \frac{\partial G}{\partial(\xi_h)_j}(\xi_i, \xi_h) (1 - \delta_{ih}) + O(\rho) \tag{C.5}$$

as $\rho \rightarrow 0$ (here δ_{ih} is the Kronecker’s symbol).

Proof. Since

$$\rho^2 \int_{\Omega} e^{U_i} P \psi_h^j = \rho^2 \int_{\Omega \setminus B(\xi_i, \varepsilon)} e^{U_i} P \psi_h^j + \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} P \psi_h^j \tag{C.6}$$

and, by Lemma A.4,

$$\rho^2 \int_{\Omega \setminus B(\xi_i, \varepsilon)} e^{U_i} P \psi_h^j = O(\rho), \tag{C.7}$$

we are left to estimate the second term in (C.6). By (A.5) we find,

$$\begin{aligned} \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} P \psi_i^j &= \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} \left(4 \frac{(x - \xi_i)_j}{\tau_i^2 \rho^2 + |x - \xi_i|^2} + 8\pi \frac{\partial H}{\partial(\xi_i)_j}(x, \xi_i) + O(\rho^2) \right) \\ &= 8\pi \frac{\partial H}{\partial(\xi_i)_j}(\xi_i, \xi_i) \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} + O(\rho) = 64\pi^2 \frac{\partial H}{\partial(\xi_i)_j}(\xi_i, \xi_i) + O(\rho) \end{aligned} \tag{C.8}$$

since $\int_{B(\xi_i, \varepsilon)} e^{U_i} (x - \xi_i)_j / (\tau_i^2 \rho^2 + |x - \xi_i|^2) = 0$ by symmetry, and (A.6) provides for $i \neq h$

$$\begin{aligned} \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} P \psi_h^j &= \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} \left(8\pi \frac{\partial G}{\partial(\xi_h)_j}(x, \xi_h) + O(\rho^2) \right) = 8\pi \frac{\partial G}{\partial(\xi_h)_j}(\xi_i, \xi_h) \rho^2 \int_{B(\xi_i, \varepsilon)} e^{U_i} + O(\rho) \\ &= 64\pi^2 \frac{\partial G}{\partial(\xi_h)_j}(\xi_i, \xi_h) + O(\rho). \end{aligned} \tag{C.9}$$

Thus, (C.5) follows by (C.6)–(C.9). \square

Lemma C.4. *Let $j = 1, 2$ and $i = 1, \dots, k$. We have:*

$$\begin{aligned} & \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \\ &= 64\pi^2 \frac{\partial}{\partial(\xi_i)_j} \left(\sum_{l=1}^k H(\xi_l, \xi_l) + \sum_{l \neq h} G(\xi_l, \xi_h) + \frac{1}{8\pi} \sum_{l=1}^k \log V(\xi_l) \right) + O(\rho) \end{aligned} \tag{C.10}$$

as $\rho \rightarrow 0$.

Proof. Write,

$$\begin{aligned} \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j &= \rho^2 \int_{B(\xi_i, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j + \rho^2 \sum_{l \neq i} \int_{B(\xi_l, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \\ &\quad + \rho^2 \int_{\Omega \setminus \bigcup_{l=1}^k B(\xi_l, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j, \end{aligned} \tag{C.11}$$

and by (A.2), Lemma A.4 deduce that,

$$\rho^2 \left| \int_{\Omega \setminus \bigcup_{l=1}^k B(\xi_l, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \right| \leq C \rho^2 \int_{\Omega} |P \psi_i^j| = O(\rho). \tag{C.12}$$

By Propositions A.1, A.3 we can also estimate the first term in (C.11) as follows:

$$\begin{aligned} &\rho^2 \int_{B(\xi_i, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \\ &= \rho^2 \int_{B(\xi_i, \varepsilon)} \frac{V(x) e^{8\pi(H(x, \xi_i) + \sum_{h \neq i} G(x, \xi_h))}}{(\tau_i^2 \rho^2 + |x - \xi_i|^2)^2} \left(4 \frac{(x - \xi_i)_j}{\tau_i^2 \rho^2 + |x - \xi_i|^2} + 8\pi \frac{\partial H}{\partial (\xi_i)_j}(x, \xi_i) + O(\rho) \right) \\ &= \int_{B(0, \varepsilon / (\tau_i \rho))} \frac{V(\tau_i \rho y + \xi_i) e^{8\pi(H(\tau_i \rho y + \xi_i, \xi_i) + \sum_{h \neq i} G(\tau_i \rho y + \xi_i, \xi_h))}}{\tau_i^2 (1 + |y|^2)^2} \left(\frac{4}{\tau_i \rho} \frac{y_j}{1 + |y|^2} + 8\pi \frac{\partial H}{\partial (\xi_i)_j}(\xi_i, \xi_i) \right) \\ &\quad + O(\rho) \\ &= \frac{32}{\tau_i \rho} \int_{B(0, \varepsilon / (\tau_i \rho))} \left[1 + \tau_i \rho \sum_{s=1}^2 y_s \frac{\partial}{\partial (\xi_i)_s} \left(4\pi H(\xi_i, \xi_i) + 8\pi \sum_{h \neq i} G(\xi_i, \xi_h) + \log V(\xi_i) \right) \right] \frac{y_j}{(1 + |y|^2)^3} \\ &\quad + 32\pi \frac{\partial}{\partial (\xi_i)_j} (H(\xi_i, \xi_i)) \int_{B(0, \varepsilon / (\tau_i \rho))} \frac{1}{(1 + |y|^2)^2} + O(\rho) \\ &= 64\pi^2 \frac{\partial}{\partial (\xi_i)_j} \left(H(\xi_i, \xi_i) + \sum_{h \neq i} G(\xi_i, \xi_h) + \frac{1}{8\pi} \log V(\xi_i) \right) + O(\rho). \end{aligned} \tag{C.13}$$

If $l \neq i$, by Propositions A.1, A.3 we get

$$\begin{aligned} \rho^2 \int_{B(\xi_l, \varepsilon)} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j &= \rho^2 \int_{B(\xi_l, \varepsilon)} \frac{V(x) e^{8\pi(H(x, \xi_l) + \sum_{h \neq l} G(x, \xi_h))}}{(\tau_l^2 \rho^2 + |x - \xi_l|^2)^2} \left(8\pi \frac{\partial G}{\partial (\xi_i)_j}(x, \xi_i) + O(\rho^2) \right) \\ &= 8\pi \int_{B(0, \varepsilon / (\tau_l \rho))} \frac{V(\tau_l \rho y + \xi_l) e^{8\pi(H(\tau_l \rho y + \xi_l, \xi_l) + \sum_{h \neq l} G(\tau_l \rho y + \xi_l, \xi_h))}}{\tau_l^2 (1 + |y|^2)^2} \frac{\partial G}{\partial (\xi_i)_j}(\tau_l \rho y + \xi_l, \xi_i) + O(\rho^2) \\ &= 64\pi^2 \frac{\partial G}{\partial (\xi_i)_j}(\xi_l, \xi_i) + O(\rho). \end{aligned} \tag{C.14}$$

In view of (C.11), by (C.12)–(C.14) we derive (C.10). \square

Lemma C.5. *It holds*

$$\left(\sum_{h=1}^k P U_h + \phi - i^*(\rho^2 V(x) e^{\sum_{h=1}^k P U_h + \phi}), P \psi_i^j \right)_{H_0^1(\Omega)} = -32\pi^2 \frac{\partial \mathcal{F}}{\partial (\xi_i)_j}(\xi_1, \dots, \xi_k) + o(1), \tag{C.15}$$

as $\rho \rightarrow 0$, C^1 -uniformly for ξ in a compact set of $(\Omega')^k \setminus \Delta$.

Proof. Let $p \in (1, \frac{6}{5})$ be fixed. We have,

$$\begin{aligned} & \int_{\Omega} \nabla \left(\sum_{h=1}^k P U_h + \phi \right) \nabla P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h + \phi} P \psi_i^j \\ &= \sum_{h=1}^k \int_{\Omega} \nabla P U_h \nabla P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \\ & \quad - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} \phi P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} (e^{\phi} - 1 - \phi) P \psi_i^j \\ &= \rho^2 \sum_{h=1}^k \int_{\Omega} e^{U_h} P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j \\ & \quad + \int_{\Omega} \left(\rho^2 \sum_{h=1}^k e^{U_h} - \rho^2 V(x) e^{\sum_{h=1}^k P U_h} \right) \phi P \psi_i^j + \rho^2 \int_{\Omega} e^{U_i} \phi (\psi_i^j - P \psi_i^j) \\ & \quad - \rho^2 \sum_{h \neq i} \int_{\Omega} e^{U_h} \phi P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} (e^{\phi} - 1 - \phi) P \psi_i^j. \end{aligned} \tag{C.16}$$

In view of Lemmata C.3, C.4 we deduce that

$$\rho^2 \sum_{h=1}^k \int_{\Omega} e^{U_h} P \psi_i^j - \rho^2 \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} P \psi_i^j = -32\pi^2 \frac{\partial \mathcal{F}}{\partial (\xi_i)_j}(\xi_1, \dots, \xi_k) + O(\rho). \tag{C.17}$$

Moreover, with the aid of Hölder’s inequality (here $1/q + 1/p < 1$), we can use Lemmata A.4, B.1 to find,

$$\begin{aligned} \left| \int_{\Omega} \left(\rho^2 \sum_{h=1}^k e^{U_h} - \rho^2 V(x) e^{\sum_{h=1}^k P U_h} \right) \phi P \psi_i^j \right| &\leq c \|\phi\|_{L^q} \left\| \rho^2 \sum_{h=1}^k e^{U_h} - \rho^2 V(x) e^{\sum_{h=1}^k P U_h} \right\|_{L^p} \|P \psi_i^j\|_{L^p} \\ &\leq c \|\phi\| \rho^{2(1-p)/p}; \end{aligned} \tag{C.18}$$

while, by Lemma C.7 we get,

$$\left| \int_{\Omega} \rho^2 e^{U_i} \phi (\psi_i^j - P \psi_i^j) \right| \leq c \|\phi\|_{L^q} \|\rho^2 e^{U_i} (\psi_i^j - P \psi_i^j)\|_{L^p} \leq c \|\phi\| \rho^{2(1-p)/p}, \tag{C.19}$$

and Lemma C.6 provides

$$\left| \rho^2 \sum_{h \neq i} \int_{\Omega} e^{U_h} \phi P \psi_i^j \right| \leq c \|\phi\|_{L^q} \sum_{h \neq i} \|\rho^2 e^{U_h} P \psi_i^j\|_{L^p} \leq c \|\phi\| \rho^{2(1-p)/p}. \tag{C.20}$$

Finally, Lemmata A.4, B.2 give,

$$\begin{aligned} & \rho^2 \left| \int_{\Omega} V(x) e^{\sum_{h=1}^k P U_h} (e^{\phi} - 1 - \phi) P \psi_i^j \right| \\ & \leq c \left\| \rho^2 V(x) e^{\sum_{h=1}^k P U_h} (e^{\phi} - 1 - \phi) \right\|_{L^p} \|P \psi_i^j\|_{L^q} \leq c \|\phi\|^2 \rho^{(2-3pr)/(pr)}, \end{aligned} \tag{C.21}$$

for some $r > 1$. We fix $r > 1$ sufficiently close to 1 so that $2 + 4r - 5pr > 0$. Inserting the estimate (4.3) in (C.18)–(C.21) and taking in account (C.17), by (C.16) the claim in (C.15) follows. \square

Lemma C.6. *Let $j = 1, 2$ and $i, h = 1, \dots, k$. If $p \geq 1$ and $i \neq h$, then*

$$\|\rho^2 e^{U_i} P \psi_h^j\|_{L^p} = O(\rho^{2(1-p)/p})$$

as $\rho \rightarrow 0$.

Proof. We have

$$\rho^{2p} \int_{\Omega} |e^{U_i} P \psi_h^j|^p = \rho^{2p} \int_{B(\xi_i, \varepsilon)} |e^{U_i} P \psi_h^j|^p + \rho^{2p} \int_{\Omega \setminus B(\xi_i, \varepsilon)} |e^{U_i} P \psi_h^j|^p. \tag{C.22}$$

By Lemma A.4 we get

$$\rho^{2p} \int_{\Omega \setminus B(\xi_i, \varepsilon)} |e^{U_i} P \psi_h^j|^p = O(\rho^p). \tag{C.23}$$

By (A.6) and Lemma A.2 we derive

$$\rho^{2p} \int_{B(\xi_i, \varepsilon)} |e^{U_i} P \psi_h^j|^p = O(\|\rho^2 e^{U_i}\|_{L^p}^p) = O(\rho^{2-2p}). \tag{C.24}$$

By (C.22)–(C.24) we obtain the desired estimate. \square

Lemma C.7. *Let $j = 1, 2$ and $i = 1, \dots, k$. If $p \geq 1$, then*

$$\|\rho^2 e^{U_i} (P \psi_i^j - \psi_i^j)\|_{L^p} = O(\rho^{2(1-p)/p})$$

as $\rho \rightarrow 0$.

Proof. By (A.5) we obtain that $P \psi_i^j - \psi_i^j = O(1)$, and by Lemma A.2 we get,

$$\|\rho^2 e^{U_i} (P \psi_i^j - \psi_i^j)\|_{L^p} = O(\|\rho^2 e^{U_i}\|_{L^p}) = O(\rho^{2(1-p)/p}). \quad \square$$

Appendix D

We are interested to prove a result concerning the structure of the solutions of the linearized problem “at infinity”. This information will be crucial in the proof of Proposition 3.1. The corresponding result in higher dimensions is well known (cf. [1,7,37]). Here, we use some ideas of [20] (see also [1] and [18]). We recall that this result was first stated in [14] for solution in $L^\infty(\mathbb{R}^2)$.

Lemma D.1. *Let $U \in C^2(\mathbb{R}^2)$ be a solution of the following problem*

$$\begin{cases} -\Delta U = \frac{8}{(1+|y|^2)^2}U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |\nabla U|^2 dy < +\infty. \end{cases} \tag{D.1}$$

Then,

$$U(y) = a_0 \frac{1-|y|^2}{1+|y|^2} + \sum_{i=1}^2 a_i \frac{y_i}{1+|y|^2}$$

for some real numbers a_i , $i = 0, 1, 2$.

Proof. We expand U in Fourier series as follows,

$$U(y) = a_0(r) + \sum_{k=1}^{\infty} (a_k(r) \cos(k\theta) + b_k(r) \sin(k\theta)), \quad y = r(\cos \theta, \sin \theta).$$

Thus, (D.1) reduces to study, for any $k \geq 0$, the ordinary differential equation

$$-\ddot{u}(r) - \frac{1}{r}\dot{u}(r) + \frac{k^2}{r^2}u(r) = \frac{8}{(1+r^2)^2}u(r) \quad \text{in } (0, +\infty) \tag{D.2}$$

with the integral condition

$$\int_0^{\infty} \dot{u}(r)^2 r dr < +\infty. \tag{D.3}$$

Let us consider the case $k = 0$. A direct computation shows that $\zeta_0(r) = (1 - r^2)/(1 + r^2)$ is a solution of (D.2) $_{|k=0}$ which satisfies (D.3). Let us prove that, if w is a second linearly independent solution of (D.2) $_{|k=0}$, then w verifies $\int_0^{\infty} \dot{w}(r)^2 r dr = +\infty$. Writing $w(r) = c(r)\zeta_0(r)$ for $r < 1$, we get that

$$-\ddot{c}\zeta_0 - \dot{c}(2\dot{\zeta}_0 + \frac{1}{r}\zeta_0) = 0$$

and so,

$$\begin{aligned} \dot{c}(r) &= \frac{C}{r\zeta_0^2(r)} = C \frac{(1+r^2)^2}{r(1-r^2)^2} \sim \frac{C}{r} \quad \text{for } r \text{ small,} \\ c(r) &\sim C \log r \quad \text{for } r \text{ small,} \end{aligned}$$

where $C \neq 0$ is a constant. Hence, $\dot{w}(r) \sim \frac{C}{r}$ for r small and $\int_0^{\infty} \dot{w}(r)^2 r dr = +\infty$. Now we consider the case $k = 1$ in (D.2). Here, we have that $\zeta_1(r) = \frac{r}{1+r^2}$ is a solution of (D.2) $_{|k=1}$ which satisfies (D.3). As in case $k = 0$, we obtain that, for a second linearly independent solution $w(r)$ in the form $w(r) = c(r)\zeta_1(r)$ the function $c(r)$ must satisfy

$$\dot{c}(r) = \frac{C}{r\zeta_1(r)^2} \sim \frac{C}{r^3} \quad \text{and} \quad c(r) \sim -\frac{C}{2r^2} \quad \text{for } r \text{ small,}$$

where $C \neq 0$ is a constant. Hence, $\dot{w}(r) \sim C/(2r^2)$ for r small and $\int_0^{\infty} \dot{w}(r)^2 r dr = +\infty$. Now, let us show that (D.2) $_{|k}$ for $k \geq 2$ has no nontrivial solution satisfying (D.3). In fact,

$$\zeta_k^1(r) = \frac{(k+1) + (k-1)r^2}{1+r^2} r^k \quad \text{and} \quad \zeta_k^2(r) = \frac{(k-1) + (k+1)r^2}{1+r^2} r^{-k}$$

for $k \geq 2$ represent a set of fundamental solutions for $(D.2)|_k$ and they do not satisfy the integral condition (D.3) at infinity and at the origin respectively.

Hence,

$$\begin{aligned} U(y) &= a_0(r) + a_1(r) \cos \theta + b_1(r) \sin \theta = a_0 \zeta_0(r) + a_1 \zeta_1(r) \cos \theta + a_2 \zeta_1(r) \sin \theta \\ &= a_0 \frac{1 - |y|^2}{1 + |y|^2} + \sum_{i=1}^2 a_i \frac{y_i}{1 + |y|^2}, \end{aligned}$$

where a_0, a_1, a_2 are real numbers. \square

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