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Non-planar fronts in Boussinesq reactive flows

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Abstract

We consider the reactive Boussinesq equations in a slanted cylinder, with zero stress boundary conditions and arbitrary Rayleigh number. We show that the equations have non-planar traveling front solutions that propagate at a constant speed. We also establish uniform upper bounds on the burning rate and the flow velocity for general front-like initial data for the Cauchy problem.

1. Introduction

The existence of traveling fronts for reaction–diffusion equations and their stability has been extensively studied since the pioneering work of Kolmogorov et al. [26] and Fisher [17]. A large number of results have been obtained during the last decade on the generalization of the notion of a traveling front to reaction–diffusion–advection equations in a prescribed flow. These include non-planar traveling fronts in shear flows [9,10,12], and pulsating traveling fronts in periodic flows [6,39,40], as well as results for monotonic systems in a unidirectional flow [35–37]. One of the main qualitative effects of a flow is the speed-up of front propagation due to front stretching. Various bounds have been obtained for the speed of propagation of fronts in prescribed flows [1–3,7,13,22,25,23,24,30], including variational principles for the front speed [7,8,18,19,21,22]. The homogenization limit in a periodic flow has also been studied [27]. Extensive recent overviews can be found in [5,31,41].

However, those results have been obtained under the assumption that the flow is imposed from outside, and that it is not affected by the evolution of the solution of the reaction–diffusion–advection equation, that is, by the temperature or concentration of the reactant. This is known as the constant density approximation in the combustion literature [42]. A first step in the coupling of the temperature and fluid flow evolution is via the Boussinesq approximation: the density mismatch is so small that the density difference is accounted by a buoyancy force in the equation for an incompressible flow. Recently a number of works considered systems of a reaction–diffusion–advection equation coupled to a flow equation of the Boussinesq type. Global existence and regularity of solutions in two dimensions was studied in [28]. It has been shown that non-planar convective traveling fronts may not exist in a vertical cylinder if the Rayleigh number is too small while for large Rayleigh numbers the planar fronts become unstable [14,32,33]. Moreover, there exists a bifurcation at a critical value $\rho_c > 0$ – non-trivial convective fronts may exist for the Rayleigh numbers close to ρ_c

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[32,33]. Numerical computations [34] show that non-planar convective fronts exist and are stable for a large range of Rayleigh numbers $\rho > \rho_c$. The fingering instability in this regime was investigated in [15].

One of the difficulties in the analysis of the Boussinesq problem at large Rayleigh numbers in a vertical cylinder is the presence of unstable planar fronts that make uniform lower bounds on the front speed quite difficult. However, it has been observed in [4] that such planar fronts cannot exist in a horizontal cylinder. One of the main results of [4] is that non-planar fronts in a horizontal cylinder exist for small Rayleigh numbers. A purpose of the present paper is to extend this result to all positive Rayleigh numbers; we use an approach that is different from [4] and is based on the a priori bounds developed in [14].

The reactive Boussinesq equations for the temperature T and flow $\mathbf{u}$ have the dimensional form

$$\begin{aligned} &T_t + \mathbf{u}\cdot\nabla T = \kappa\Delta T + \frac{v_0^2}{\kappa} f(T), \\ &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = gT\mathbf{e}_z, \\ &\nabla\cdot\mathbf{u} = 0. \end{aligned} \tag{1}$$

Here $\mathbf{e}_z$ is the unit vector in the vertical direction, g is the strength of gravity, the speed v_0 is proportional to the traveling front speed in the absence of gravity, κ is the thermal diffusivity and ν is the fluid viscosity. The temperature is normalized so that $0 \leqslant T \leqslant 1$. The nonlinearity $f(T)$ is assumed to be a Lipschitz function of the ignition type

$$f(T) = 0 \quad \text{for } 0 \leqslant T \leqslant \theta_0 \text{ with } \theta_0 > 0, \qquad f(T) > 0 \quad \text{for } T \in (\theta_0, 1) \quad \text{and} \quad f(1) = 0. \tag{2}$$

We consider Eqs. (1) in a slanted two-dimensional cylinder $x \in \mathbb{R}$, $\alpha x \leqslant z \leqslant \alpha x + H$ with a finite slope $\alpha < \infty$. It is convenient to rotate the cylinder in order to make it horizontal to simplify the notation. Then (1) becomes

$$\begin{aligned} &T_t + \mathbf{u}\cdot\nabla T = \kappa\Delta T + \frac{v_0^2}{\kappa} f(T), \\ &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = gT\hat{\mathbf{e}}, \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{3}$$

where $\mathbf{u}$ is the flow velocity measured relative to the new coordinate system. The gravity on the right points in a direction $\hat{\mathbf{e}}$ that is non-parallel to the x-axis, as the original cylinder was assumed to be non-vertical ($\alpha < \infty$). The new rotated problem is posed in a cylinder $D = \mathbb{R}_x \times [0, L]_z$, $L = H/\sqrt{1+\alpha^2}$. The boundary conditions for the temperature T are set to be front-like:

$$T \to 1 \quad \text{as } x \to -\infty, \quad T \to 0 \quad \text{as } x \to +\infty, \quad \frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0, L. \tag{4}$$

The flow $\mathbf{u} = (v, w)$ satisfies the no stress boundary conditions:

$$\mathbf{u}, \omega \to 0 \quad \text{as } x \to \pm\infty \quad \text{and} \quad w, \omega = 0 \quad \text{at } z = 0, L. \tag{5}$$

Here $\omega = w_x - v_z$ is the flow vorticity so that

$$\Delta v = -\omega_z, \qquad \Delta w = \omega_x.$$

In order to pass to the non-dimensional variables we introduce the laminar front width $\delta = \kappa/v_0$ and reaction time $t_c = \kappa/v_0^2$ and rescale the space and time variables: $\mathbf{x}_{\text{new}} = \mathbf{x}_{\text{old}}/\delta$ and $t_{\text{new}} = t_{\text{old}}/t_c$. We also rescale the flow $\mathbf{u}_{\text{new}} = \mathbf{u}_{\text{old}}/v_0$. Then the Boussinesq equations become

$$\begin{aligned} &T_t + \mathbf{u}\cdot\nabla T = \Delta T + f(T), \\ &\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Delta\mathbf{u} + \nabla p = \rho T\hat{\mathbf{e}}, \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{6}$$

where $\sigma = \nu/\kappa$ is the Prandtl number and $\rho = g\delta^3/\kappa^2$ is the Rayleigh number. The problem is now posed in the strip $D = \mathbb{R}_x \times [0, \lambda]_z$, $\lambda = L/\delta$, with the boundary conditions that come from (4) and (5).

The traveling front solutions of (6) are solutions of the form $T(x - ct, z)$, $\mathbf{u}(x - ct, z)$ with the speed c to be determined. They satisfy

$$
\begin{aligned}
&-cT_x + \mathbf{u}\cdot\nabla T = \Delta T + f(T),\\
&-c\mathbf{u}_x + \mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Delta\mathbf{u} + \nabla p = \rho T\hat{\mathbf{e}},\\
&\nabla\cdot\mathbf{u} = 0,
\end{aligned}
\tag{7}
$$

with the boundary conditions

$$T \to \theta_- \quad \text{as } x \to -\infty, \quad T \to 0 \quad \text{as } x \to +\infty, \quad \frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0, \lambda \tag{8}$$

and

$$w, \omega = 0 \quad \text{at } z = 0, \lambda. \tag{9}$$

Here θ_- is a constant that is not a priori prescribed. We recall that, as has been observed in [4], if the direction of gravity $\hat{\mathbf{e}}$ is not parallel to the x-axis, any traveling front solution of (7) must be non-planar, that is, it must depend on both variables x and z. This is the main difference between the cases of a vertical and slanted cylinder: planar fronts exist in the former case but not in the latter. Our main result is the following theorem.

Theorem 1. *Let the nonlinearity $f(T)$ be of the ignition type* (2). *Then a traveling front solution* $(c, T, \mathbf{u})$ *of* (7) *exists such that it is non-planar*: *$T_z \not\equiv 0$, the flow $\mathbf{u} \not\equiv 0$ and the reaction rate $f(T) \not\equiv 0$. Moreover, the solution satisfies the following properties*: *$c > 0$, $T \in C^{2,\alpha}(D)$, $\nabla T \in L^2(D)$, $\mathbf{u} \in H^1(D) \cap C^{2,\alpha}(D)$. If we assume in addition that*

$$f(T) \leqslant (T - \theta_0)_+^2/\lambda^2, \tag{10}$$

then the left limit is $\theta_- = 1$.

The assumption (10) is of technical nature. It does not involve the Rayleigh number ρ, it is rather a restriction on the channel width λ. We do not address the question of the uniqueness of the traveling front speed or profile in this paper – this problem requires an additional study. Our results can be generalized to the no-slip boundary conditions $\mathbf{u} = 0$ on ∂D at the expense of a more technical proof – we leave this problem for a future publication.

The general idea of the proof follows the method developed in [11] and is as follows. We first consider the problem (7) on a finite domain $D_a = [-a, a]_x \times [0, \lambda]_z$. Solutions $(T_a^c, \mathbf{u}_a^c)$ of the restricted problem exist for all $c \in \mathbb{R}$. We normalize them by the requirement that

$$\max_{x\geqslant 0,\ z\in[0,\lambda]} T_a^c(x, z) = \theta_0. \tag{11}$$

This imposes a restriction on the speed c. In order to show that there exists a speed c_a so that (11) holds we first obtain some a priori bounds on c, T and $\mathbf{u}$ under the condition (11). Then we use the Leray–Schauder topological degree theory and the above a priori bounds to show that c_a exists. The a priori bounds allow us to pass to the limit $a \to \infty$. Finally we show that the right limit of T as $x \to +\infty$ is equal to zero, and that the left limit is equal to one under the additional assumption on $f(T)$ in Theorem 1. This general strategy is similar to that in the proof of existence of traveling fronts in a prescribed decoupled flow, as in, for example, [9,12]. The main difficulty and novelty are in the a priori bounds for the solution of the coupled problem in a bounded domain.

Our second result shows that the solution of the Cauchy problem for (6) propagate with a finite speed and that this speed is close to the speed of the laminar front c_0 when the Rayleigh number is small. Recall that there exists a unique speed c_0 so that a traveling front solution of

$$-c_0\Phi_x = \Phi_{xx} + f(\Phi), \quad \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0$$

exists.

In order to make this precise we define the bulk burning rate $\overline{V}(t)$, the Nusselt number $\overline{N}(t)$ and the average horizontal flow $\overline{U}(t)$ by

$$\overline{V}(t) = \frac{1}{t}\int_0^t V(s)\,\mathrm{d}s, \qquad V(t) = \int f(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \tag{12}$$

$$\overline{N}(t) = \frac{1}{t}\int_0^t N(s)\,\mathrm{d}s, \qquad N(t) = \int |\nabla T|^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \tag{13}$$

$$\overline{U}(t) = \frac{1}{t}\int_0^t \big\|v(s)\big\|_\infty \,\mathrm{d}s. \tag{14}$$

The following theorem provides uniform bounds on these bulk quantities. It also shows that the coupled problem (6) is in a sense a "regular perturbation" of the single reaction–diffusion equation with $\rho = 0$.

Theorem 2. *Assume that there exists R so that $T_0(x,z) = 0$ for $x > R$ and $T_0(x,z) = 1$ for $x < -R$ and that the initial vorticity $\omega_0 \in L^2(D)$. There exists a constant $C > 0$ so that under the above assumptions on the initial data T_0, $\mathbf{u}_0$ we have the following bounds*

$$\begin{aligned} &c_0 - C[\rho + \rho^2] + \mathrm{o}(1) \leqslant \overline{V}(t) \leqslant c_0 + C[\rho + \rho^2] + \mathrm{o}(1), \\ &\overline{N}(t) \leqslant \left[C\rho + \sqrt{\frac{c_0}{2} + C^2\rho^2}\right]^2 + \mathrm{o}(1), \\ &\overline{U}(t) \leqslant C\rho[1+\rho] + \mathrm{o}(1) \end{aligned} \tag{15}$$

as $t \to +\infty$.

This theorem may be interpreted as a stability result for a perturbation of a homogeneous reaction–diffusion equation by the buoyancy coupling. The proof is based on the construction of super- and sub- solutions, and a bound on the decay of the solutions of advection–diffusion equations that is uniform in the advection flow.

The third result of this paper deals with the Boussinesq system in a narrow domain. It has been shown in [14] that if a vertical strip is sufficiently narrow and gravity is sufficiently weak then solutions of the Cauchy data become planar as $t \to +\infty$. The following theorem generalizes this result to inclined cylinders.

Theorem 3. *Let $\hat{\mathbf{e}} = (e_1, e_2)$ be the unit vector in the direction of gravity and let $\rho_j = \rho e_j$, $j = 1, 2$, and let the initial data $(T_0, \mathbf{u}_0)$ be as in Theorem 2. There exist two constants λ_0 and ρ_0 so that if the domain is sufficiently narrow: $\lambda \leqslant \lambda_0$ and gravity is sufficiently small: $\rho \leqslant \rho_0$, then the burning rate is bounded by*

$$\overline{V}(t) \leqslant c_0 + C\rho_2 + \mathrm{o}(1) \quad \text{as } t \to +\infty. \tag{16}$$

Moreover, the front is nearly planar in the sense that

$$\overline{N}_z(t) = \frac{1}{t}\int_0^t \big\|T_z(s)\big\|_2^2 \,\mathrm{d}s \leqslant C\rho_2^2 + \mathrm{o}(1) \quad \text{as } t \to +\infty. \tag{17}$$

The main observation of this theorem is that only the gravity strength in the direction perpendicular to the strip enters in the upper bounds (16) and (17).

The paper is organized as follows: Theorem 1 is proved in Sections 2 and 3. Theorems 2 and 3 are proved in Section 4.

2. The finite domain problem

We consider in this section the approximating problem

$$\begin{aligned} &-cT_x + \mathbf{u}\cdot\nabla T = \Delta T + f(T), \\ &-c\mathbf{u}_x + \mathbf{u}\cdot\nabla\mathbf{u} - \sigma\Delta\mathbf{u} + \nabla p = \rho T\hat{\mathbf{e}}, \\ &\nabla\cdot\mathbf{u} = 0, \end{aligned} \tag{18}$$

in a finite domain $D_a = [-a, a]_x \times [0, \lambda]_z$, $a > 0$, with the boundary conditions

$$T(-a, z) = 1, \quad T(a, z) = 0, \quad \frac{\partial T}{\partial z} = 0 \quad \text{at } z = 0, \lambda \tag{19}$$

and

$$w = 0, \quad \omega = 0 \quad \text{at } z = 0, \lambda \quad \text{and} \quad v(\pm a, z) = \omega(\pm a, z) = 0 \quad \text{at } x = \pm a,\ z \in [0, \lambda]. \tag{20}$$

One can show with the techniques of the present section that a solution T_a, $\mathbf{u}_a$ of (18) in D_a with the boundary conditions (19) and (20) exists for all $c \in \mathbb{R}$. However, given an arbitrary c there is no way to control the limit of T_a and $\mathbf{u}_a$ as $a \to \infty$. Hence, following the standard procedure, we impose an additional constraint (11). This ensures that the non-trivial part of the solution does not escape to infinity when we pass to the limit $a \to \infty$.

Proposition 1. *There exists a speed $c_a \in \mathbb{R}$ so that there exists a solution $(T_a, \mathbf{u}_a)$ of* (18) *in D_a with the boundary conditions* (19) *and* (20) *such that*

$$\max_{x \geqslant 0, z \in [0, \lambda]} T_a(x, z) = \theta_0. \tag{21}$$

We denote the corresponding solution as $(c_a, T_a, \mathbf{u}_a)$. There exists $a_0 > 0$ and a constant $C > 0$ that is independent of a, so that for all $a > a_0$:

$$|c_a| \leqslant C, \tag{22}$$

and

$$\int_{D_a} |\nabla T_a|^2 \, dx \, dz + \int_{D_a} |\nabla \mathbf{u}_a|^2 \, dx \, dz + \|\mathbf{u}_a\|_\infty \leqslant C. \tag{23}$$

Moreover, the uniform Hölder estimates hold: there exists $a_0 > 0$ and a constant $C > 0$ independent of a so that for all $a > a_0$ we have

$$\|\omega_a\|_{C^{1,\alpha}(D_a)} + \|\mathbf{u}_a\|_{C^{1,\alpha}(D_a)} + \|T_a\|_{C^{1,\alpha}(D_a)} \leqslant C \tag{24}$$

provided that $0 < \alpha < 1$.

Proof. The proof consists of two parts. First, we introduce a family of problems depending on a parameter $\tau \in [0, 1]$ so that at $\tau = 0$ we have a simple linear problem without advection or coupling and at $\tau = 1$ we have the full problem (18) with the correct boundary conditions. The normalization condition (21) is imposed for all $\tau \in [0, 1]$. We obtain the a priori bounds as in (22), (23) and (24) for such solutions that are uniform in $\tau \in [0, 1]$. In the second step we use the a priori bounds, the Leray–Schauder topological degree argument and the information on the linear problem at $\tau = 0$ to show that solutions of the nonlinear coupled problem at $\tau = 1$ exist. We drop the subscript a throughout the proof to make the notation less cumbersome.

Step 1. A priori bounds for solutions. Let us first define a one-parameter (homotopy) family of finite domain Boussinesq problems in the vorticity formulation

$$\begin{aligned} &-c^\tau T_x^\tau + \tau \mathbf{u}^\tau \cdot \nabla T^\tau = \Delta T^\tau + \tau f(T^\tau), \\ &-c^\tau \omega_x^\tau + \mathbf{u}^\tau \cdot \nabla \omega^\tau - \sigma \Delta \omega^\tau = \tau \rho \hat{\mathbf{e}} \cdot \nabla^\perp T := \rho \tau [e_2 T_x^\tau - e_1 T_z^\tau], \\ &\omega^\tau = w_x^\tau - v_z^\tau, \qquad \nabla \cdot \mathbf{u}^\tau = 0. \end{aligned} \tag{25}$$

As mentioned above, τ is the homotopy parameter: $\tau \in [0, 1]$, with $\tau = 0$ corresponding to the linear problem, and $\tau = 1$ to the full problem (18)–(20). The problem (25) is posed in D_a with the same boundary conditions

$$\frac{\partial T^\tau}{\partial z} = 0, \quad w^\tau = \omega^\tau = 0 \quad \text{for } z = 0, \lambda \tag{26}$$

and

$$T^\tau(-a, z) = 1, \quad T^\tau(a, z) = 0, \quad v^\tau(\pm a, z) = \omega^\tau = (\pm a, z) = 0 \quad \text{for } x = \pm a, \tag{27}$$

as (18). We also require that

$$\max_{x\geqslant 0,z} T^\tau(x,z)=\theta_0 \tag{28}$$

and obtain a priori bounds on c^τ, T^τ and ω^τ. We drop the superscript τ below wherever it causes no confusion. The general plan is as follows. First, we bound the speed c above and below by a linear function of $\|v\|_\infty$ in Lemma 1. Next we bound $\|\mathbf{u}\|_\infty$ from above by a linear function of $\|\nabla T\|_2$ in Lemma 3. The inequality in the other direction, a bound on $\|\nabla T\|_2^2$ in terms of a linear function of $\|\mathbf{u}\|_\infty$ is established in Lemmas 4 and 5. Since the latter bound is quadratic in $\|\nabla T\|_2$, the last estimates allow to obtain a uniform bound on this quantity, from which all other a priori bounds follow in a fairly straightforward manner: see Corollary 1 and Lemma 6.

We begin with a lemma that bounds the speed c in terms of the horizontal flow velocity $\|v\|_{L^\infty(D_a)}$.

Lemma 1. *Let* $(c,T,\mathbf{u})$ *satisfy* (25)–(27) *with the normalization* (28) *and let* $\mathbf{u}=(v,w)$. *Then there exists* $a_0>0$ *so that for all* $a\geqslant a_0$ *we have*

$$-1-\tau\|v\|_\infty\leqslant c\leqslant 1+M\tau+\tau\|v\|_\infty. \tag{29}$$

Proof. First, we observe that the function $\psi_A(x)=A\,\mathrm{e}^{-\alpha(x+a)}$ is a super-solution for the reaction–diffusion–advection equation (with the flow $\mathbf{u}$ fixed) if $A>1$ and

$$c\geqslant\alpha+\frac{M\tau}{\alpha}+\tau\|v\|_\infty, \tag{30}$$

that is,

$$-c\frac{\partial\psi_A}{\partial x}+\tau\mathbf{u}\cdot\nabla\psi_A\geqslant\Delta\psi_A+\tau f(\psi_A), \tag{31}$$

provided that (30) holds with

$$M=\sup_{0\leqslant T\leqslant 1}\frac{f(T)}{T}.$$

Furthermore, we have

$$T(-a,z)=1<A=\psi_A(-a),\qquad T(a,z)=0<\psi_A(a) \tag{32}$$

at the two ends of the domain D_a. We now show that this together with (31) implies that

$$T(x,z)\leqslant\psi_A(x) \tag{33}$$

for all $(x,z)\in D_a$ and $A>1$. Indeed, consider the family of functions $\psi_A(x)$. Then all ψ_A are super-solutions in the sense that the inequality (31) holds. Moreover, as the maximum principle implies that $0\leqslant T\leqslant 1$, for $A>5\,\mathrm{e}^{2\alpha a}$ sufficiently large we have $\psi_A(x)>5>T(x,z)$ for all $(x,z)\in D_a$. We define

$$A_0=\inf\bigl\{A\in\mathbb{R}:\ \psi_A(x)\geqslant T(x,z)\text{ for all }(x,z)\in D_a\bigr\}.$$

The previous argument implies that A_0 is finite, $A_0\leqslant 5\,\mathrm{e}^{2\alpha a}$ and, moreover, clearly $A_0>0$. Observe that since the domain D_a is compact, we should have $\psi_{A_0}(x)\geqslant T(x,z)$ – otherwise this inequality would be violated for A slightly larger than A_0 at some point in D_a. Moreover, the equation $\psi_{A_0}(x)=T(x,z)$ should have a solution. We claim that $A_0=1$. Indeed, otherwise the point (x_0,z_0) that solves $\psi_{A_0}(x_0)=T(x_0,z_0)$ cannot be at the boundary of D_a because of the boundary conditions on the function T. Hence this point has to lie in the interior of D_a. The continuity of $\psi_A(x)$ with respect to A implies that the graphs of $\psi_{A_0}(x)$ and $T(x,z)$ are tangent at (x_0,z_0). Then the strong maximum principle implies that $\psi_{A_0}(x)\equiv T(x,z)$ which is a contradiction, as they differ on the boundary. Hence, we conclude that $A_0=1$ and (33) holds for all $A>1$ and thus for $A=1$, so that

$$T(x,z)\leqslant\mathrm{e}^{-\alpha(x+a)}. \tag{34}$$

However, the existence of such a super-solution contradicts the normalization condition (28) if $\alpha\geqslant\ln(\theta_0^{-1})/a$ because (28) implies that there exists z_0 so that $T(0,z_0)=\theta_0$. Therefore, the existence of a solution T that satisfies (28) implies

$$c \leqslant \inf_{\alpha \geqslant \ln(\theta_0^{-1})/a} \left(\alpha + \frac{M\tau}{\alpha}\right) + \tau\|v\|_\infty \leqslant 1 + M\tau + \tau\|v\|_\infty \tag{35}$$

provided that $a \geqslant \ln(1/\theta_0)$. This proves the upper bound in (29). In order to prove the lower bound we observe that the function $\phi = 1 - e^{\alpha(x-a)}$ is a sub-solution for T with the flow $\mathbf{u}$ fixed if

$$c \leqslant -\alpha - \tau\|v\|_\infty. \tag{36}$$

That is, if (36) holds, then $T(x,z) \geqslant 1 - e^{\alpha(x-a)}$. This is shown in a way similar to the proof of (34) under the assumption (30) above. However, $\phi(0) = 1 - e^{-\alpha a} > \theta_0$ for

$$a > \frac{\ln((1-\theta_0)^{-1})}{\alpha}. \tag{37}$$

This implies that $\max_{x \geqslant 0} T(x,z) \geqslant \phi(0) > \theta_0$ provided that both (36) and (37) hold. Hence, in order for (28) to be possible we need

$$c \geqslant \sup_{\alpha > (\ln((1-\theta_0)^{-1}))/a} \left[-\alpha - \tau\|v\|_\infty\right] \geqslant -1 - \tau\|v\|_\infty \tag{38}$$

provided that $a \geqslant \ln((1-\theta_0)^{-1})$. This is the lower bound in (29) and the proof of Lemma 1 is complete. □

Next, we establish a bound on $\|\mathbf{u}\|_{L^\infty(D_a)}$ and $\|\omega\|_{L^\infty(D_a)}$ in terms of $\|\nabla T\|_{L^2(D_a)}$. These bounds are all obtained from the following type of estimates.

Lemma 2. *Let $S_a = [-a,a]_x \times \Omega_y$ be a finite cylinder with a smooth bounded cross-section $\Omega \in \mathbb{R}^d$, $d = 1,2$. Let ϕ be a function that satisfies one of the following three conditions*: (i) $\phi(x,y) = 0$ *on the whole boundary* ∂S_a, (ii) $\phi(x,y) = 0$ *for* $y \in \partial\Omega$ *and* $\frac{\partial\phi(x,y)}{\partial x} = 0$ *for* $x = -a, a$, *or* (iii) $\frac{\partial\phi(x,y)}{\partial n} = 0$ *for* $y \in \partial\Omega$, *and* $\phi(x,y) = 0$ *for* $x = -a, a$. *Then there exists a constant C that depends only on the domain Ω, but not on the cylinder length a, so that we have*

$$\|\phi\|_{L^\infty(S_a)} \leqslant C\left[\|\Delta\phi\|_{L^2(S_a)} + \|\phi\|_{L^2(S_a)}\right]. \tag{39}$$

Proof. Let Q be any cylinder of the form $[x_0, x_0+1] \times \Omega \subset S_a$ with $-a \leqslant x_0 \leqslant a-1$. The standard interior elliptic estimates up to the boundary [20] can be applied to Q in all the three cases (i)–(iii). The corners at $x = \pm a$ are not an obstacle. Indeed, both in the case of the Dirichlet and Neumann boundary conditions prescribed on the lines $x = \pm a$, one can extend the solution to a larger cylinder $[-a-1, a+1] \times \Omega$ by reflecting the solution across the line $x = \pm a$, either in the even or odd way, respectively. Hence the usual elliptic estimates up to the boundary can be applied to all such cylinders Q to obtain

$$\|\phi\|_{H^2(Q)} \leqslant C\left[\|\Delta\phi\|_{L^2(S_a)} + \|\phi\|_{L^2(S_a)}\right] \tag{40}$$

in all three cases (i)–(iii). Then the Sobolev embedding theorem in dimensions $d = 2, 3$ implies that

$$\|\phi\|_{L^\infty(Q)} \leqslant C\|\phi\|_{H^2(Q)} \leqslant C\left[\|\Delta\phi\|_{L^2(Q)} + \|\phi\|_{L^2(Q)}\right] \leqslant C\left[\|\Delta\phi\|_{L^2(S_a)} + \|\phi\|_{L^2(S_a)}\right]$$

with the constant C that only depends on the domain Ω. □

This lemma can be easily extended to higher dimensions using the appropriate Sobolev embeddings. It implies immediately the following bounds on $\|\mathbf{u}\|_\infty$ and $\|\omega\|_\infty$ in terms of $\|\nabla T\|_{L^2(D_a)}$.

Lemma 3. *Let* $(c, T, \mathbf{u})$ *satisfy* (25)–(27) *with the normalization* (28). *There exists $a_0 > 0$ and a constant $C > 0$ so that, for all $a > a_0$, the following estimates hold*:

$$\|\mathbf{u}\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)} \tag{41}$$

and

$$\|\omega\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\left[1 + \|\nabla T\|_{L^2(D_a)}\right]. \tag{42}$$

Moreover, $\nabla\mathbf{u}$ *satisfies the same bound*:

$$\|\nabla\mathbf{u}\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\left[1 + \|\nabla T\|_{L^2(D_a)}\right]. \tag{43}$$

Proof. We use the vorticity equation

$$-c\omega_x + \mathbf{u}\cdot\nabla\omega - \sigma\Delta\omega = \rho\tau(\hat{\mathbf{e}}\cdot\nabla^{\perp}T), \qquad \omega = 0 \quad \text{on } \partial D_a. \tag{44}$$

Case (i) of Lemma 2 implies that

$$\|\omega\|_{L^\infty(D_a)} \leqslant C\big[\|\nabla T\|_{L^2(D_a)} + \big(|c| + \|\mathbf{u}\|_\infty\big)\|\nabla\omega\|_{L^2(D_a)} + \|\omega\|_{L^2(D_a)}\big]. \tag{45}$$

Here, the constant C depends only on ρ and λ. Note that multiplying the vorticity equation by ω and integrating by parts, using the boundary conditions, we obtain

$$\int_{D_a} |\nabla\omega|^2\,\mathrm{d}x\,\mathrm{d}z = \tau\rho\int(\hat{\mathbf{e}}\cdot\nabla^{\perp}T)\omega\,\mathrm{d}x\,\mathrm{d}z \leqslant \tau\rho\|\nabla T\|_2\|\omega\|_2.$$

The Dirichlet boundary conditions for ω imply that the Poincaré inequality applies to ω so that $\|\omega\|_{L^2(D_a)} \leqslant (\lambda/\pi)\|\nabla\omega\|_{L^2(D_a)}$. Hence we obtain

$$\|\nabla\omega\|_2 \leqslant \frac{\lambda}{\pi}\tau\rho\|\nabla T\|_2 \tag{46}$$

and thus

$$\|\omega\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}, \tag{47}$$

with the constant C independent of the cylinder length a. This, together with (45) and the bound (29) on the speed c, implies (42), provided that we show (41).

We now prove (41). The horizontal flow component v satisfies the Poisson equation

$$\Delta v = -\omega_z, \quad v(\pm a, z) = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \text{at } z = 0, \lambda. \tag{48}$$

The boundary conditions at $z = 0, \lambda$ are obtained from $v_z = w_x - \omega = 0$ as follows from (26). The third case (iii) of Lemma 2 implies that

$$\|v\|_{L^\infty(D_a)} \leqslant C\big[\|\nabla\omega\|_{L^2(D_a)} + \|v\|_{L^2(D_a)}\big]. \tag{49}$$

The first term in the right-hand side is bounded by (46). In order to bound the second one, we multiply (48) by v and integrate to obtain, using the boundary conditions and (46)

$$\int_{D_a} |\nabla v|^2\,\mathrm{d}x\,\mathrm{d}z = \int_{D_a} \omega_z(x,z)v(x,z)\,\mathrm{d}x\,\mathrm{d}z \leqslant \|\omega_z\|_2\|v\|_2 \leqslant C\|\nabla T\|_2\|v\|_2. \tag{50}$$

Now, observe that (48), the Neumann boundary conditions for v and the Dirichlet boundary condition for ω at $z = 0, \lambda$ imply that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int v(x,z)\,\mathrm{d}z = 0.$$

It follows then from the Dirichlet boundary conditions for v at $x = \pm a$ that

$$\int_0^\lambda v(x,z)\,\mathrm{d}z = 0 \tag{51}$$

for all x. One may alternatively deduce (51) from incompressibility of the flow $\mathbf{u}$ and the boundary conditions. Therefore, it follows from the Poincaré inequality that $\|v\|_{L^2(D_a)} \leqslant (\lambda/2\pi)\|\nabla v\|_{L^2(D_a)}$. Thus, (50) implies that both $\|\nabla v\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}$ and $\|v\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}$ with a constant independent of a. Hence, (49) implies (41) for the horizontal flow component.

The vertical flow component satisfies

$$\Delta w = \omega_x, \quad w(x,0) = w(x,\lambda) = 0, \quad \frac{\partial w}{\partial x}(\pm a, z) = 0. \tag{52}$$

The Neumann boundary condition at $x = \pm a$ is deduced from the relation $w_x = \omega + v_z$ and the Dirichlet boundary conditions for v and ω at $x = \pm a$. The case (ii) in Lemma 2 implies that

$$\|w\|_{L^\infty(D_a)} \leqslant C\big[\|\nabla\omega\|_{L^2(D_a)} + \|w\|_{L^2(D_a)}\big]. \tag{53}$$

As before, we use (46) to bound the first term in the right side. In order to bound the second we multiply (52) by w and integrate, using the boundary conditions and (46) again, to obtain that

$$\int\limits_{D_a} |\nabla w|^2 \,\mathrm{d}x\,\mathrm{d}z = -\int\limits_{D_a} \omega_x(x,z)w(x,z)\,\mathrm{d}x\,\mathrm{d}z \leqslant \|\omega_x\|_2\|w\|_2 \leqslant C\|\nabla T\|_2\|w\|_2. \tag{54}$$

The Dirichlet boundary conditions for w at $z = 0, \lambda$ imply that $\|w\|_{L^2(D_a)} \leqslant \lambda/\pi\|\nabla w\|_{L^2(D_a)}$. Thus, (54) implies that

$$\|\nabla w\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}, \tag{55}$$

and hence $\|w\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}$ with a constant independent of a. Therefore, now (53) implies (41) for the vertical flow component. Thus, the proof of (41) is complete. We recall that then (42) follows as well, as explained in the paragraph below (47).

In order to complete the proof of Lemma 3 it remains to bound the derivatives of $\mathbf{u}$. First, we observe that the function $\psi = v_z$ satisfies the boundary value problem

$$-\Delta\psi = \omega_{zz}, \quad \psi = 0 \quad \text{on } \partial D_a. \tag{56}$$

Hence, case (i) of Lemma 2 applies to the function ψ. Moreover, the elliptic estimates for ω, as in (40), imply that $\|\omega_{zz}\|_{L^2(D_a)} \leqslant \|\Delta\omega\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}(1 + \|\nabla T\|_{L^2(D_a)})$. Hence, the same proof as in the derivation of the bound (42) applies to ψ and we obtain that

$$\|v_z\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\big(1 + \|\nabla T\|_{L^2(D_a)}\big).$$

This, together with (42) implies that

$$\|w_x\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\big(1 + \|\nabla T\|_{L^2(D_a)}\big).$$

The other pair of derivatives, v_x and w_z, do not satisfy a homogeneous boundary condition on the lines $x = \pm a$. Therefore, one cannot apply the standard elliptic estimates up to the boundary to the function $\eta = w_z = -v_x$ (the second equality follows from the incompressibility of the flow). In order to circumvent this difficulty, we extend the function w to a larger cylinder $D_{a+1} = [-a-1, a+1] \times [0, \lambda]$ by setting $w(-a-x, z) = w(-a+x, z)$ and $w(a+x, z) = w(a-x, z)$ for $0 \leqslant x \leqslant 1$. The resulting function is of a class $C^2(D_{a+1})$ since $w(x,z)$ satisfies the Neumann boundary condition at $x = \pm a$. This also extends the function $\eta = w_z$ to the larger cylinder. Moreover, η satisfies the Neumann boundary condition along the horizontal lines $z = 0, \lambda$:

$$\eta_z = w_{zz} = -v_{zx} = 0 \quad \text{on } z = 0, \lambda,$$

and

$$\Delta\eta = \omega_{xz},$$

with the function ω extended to the larger cylinder by the same reflection. Hence, the interior elliptic estimates up to the boundary for solutions of the Neumann problem imply that

$$\|\eta\|_{H^2(Q)} \leqslant \|\Delta\eta\|_{L^2(D_a)} + \|\eta\|_{L^2(D_a)}$$

for any rectangle $Q = [x_0, x_0 + 1] \times [0, \lambda]$ that is strictly contained inside the larger cylinder D_{a+1}. Therefore, the Sobolev embedding theorem together with the above estimates imply that

$$\|\eta\|_{L^\infty(D_a)} \leqslant C\big[\|\Delta\eta\|_{L^2(D_a)} + \|\eta\|_{L^2(D_a)}\big] = C[\|\omega_{xz}\|_{L^2(D_a)} + \|\eta\|_{L^2(D_a)}]. \tag{57}$$

However, as the function ω satisfies the Dirichlet boundary conditions in D_a, we can apply the estimate (40) to the function ω up to boundary, to obtain

$$\|\omega\|_{H^2(D_a)} \leqslant C\big[\|\Delta\omega\|_{L^2(D_a)} + \|\omega\|_{L^2(D_a)}\big].$$

We now use the vorticity equation (44) to bound $\|\Delta\omega\|_{L^2(D_a)}$ and the estimate (47) to estimate $\|\omega\|_{L^2(D_a)}$, and conclude that

$$\|\omega_{xz}\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\big(1+\|\nabla T\|_{L^2(D_a)}\big). \tag{58}$$

Furthermore, the estimate (55) for $\|\nabla w\|_{L^2(D_a)}$ implies that

$$\|\eta\|_{L^2(D_a)} = \|w_z\|_{L^2(D_a)} \leqslant \|\nabla w\|_{L^2(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}. \tag{59}$$

We infer from the bounds (57), (58) and (59) that

$$\|\eta\|_{L^\infty(D_a)} \leqslant C\|\nabla T\|_{L^2(D_a)}\big(1+\|\nabla T\|_{L^2(D_a)}\big).$$

This proves the uniform bound on w_z and hence the proof of Lemma 3 is complete. □

Let us now proceed to estimate $\|\nabla T\|_{L^2(D_a)}$ in terms of $\|v\|_{L^\infty(D_a)}$, a bound in the direction opposite to that in Lemma 3. More important, we will bound the square $\|\nabla T\|_2^2$ in terms of a linear function of $\|v\|_\infty$. As we are unable to obtain such bound by the standard elliptic estimates, we have to proceed with an explicit calculation. As a preliminary step we show the following.

Lemma 4. *Let* $(c, T, \mathbf{u})$ *satisfy* (25)–(27) *with the normalization* (28). *Then, there exists a constant* $C > 0$ *and a constant* $a_0 > 0$ *so that we have for all* $a > a_0$ *and* $0 \leqslant \tau \leqslant 1$

$$\int_{D_a} |\nabla T|^2\,dx\,dz + \int_0^\lambda T_x(a,z)\,dz \leqslant C\big[1+\|v\|_\infty\big]. \tag{60}$$

Proof. Recall that the function T satisfies

$$-cT_x + \tau\mathbf{u}\cdot\nabla T = \Delta T + \tau f(T) \tag{61}$$

with the boundary conditions

$$T(-a,z)=1,\quad T(a,z)=0,\quad \frac{\partial T}{\partial z}=0 \quad \text{at } z=0,\lambda. \tag{62}$$

We multiply (61) by $(1-T)$ and use the boundary conditions and incompressibility of the flow $\mathbf{u}$ to obtain

$$\frac{c\lambda}{2} = \int_0^\lambda T_x(a,z)\,dz + \int_{D_a} |\nabla T|^2\,dx\,dz + \tau\int (1-T)f(T)\,dx\,dz. \tag{63}$$

Hence, Lemma 1 and the fact that $(1-T)f(T)\geqslant 0$ imply that

$$\int_{D_a} |\nabla T|^2\,dx\,dz + \int_0^\lambda T_x(a,z)\,dz \leqslant \frac{c\lambda}{2} \leqslant C\big[1+\|v\|_\infty\big] \tag{64}$$

and Lemma 4 is proved. □

In order to close the bounds (29), (41) and (60) we need to bound the integral of T_x in (60). This is done in the next Lemma.

Lemma 5. *Let* $(c, T, \mathbf{u})$ *satisfy* (25)–(27) *with the normalization* (28). *There exists a constant* $C > 0$ *and a constant* a_0 *so that we have for all* $a \geqslant a_0$ *and* $0 \leqslant \tau \leqslant 1$

$$0 \leqslant -\int_0^\lambda T_x(a,z)\,dz \leqslant C\big[1+\|\nabla T\|_2\big]. \tag{65}$$

Proof. In order to find a bound for $\int_0^\lambda T_x(x=\pm a,z)\,\mathrm{d}z$ we introduce

$$I(x)=\frac{1}{\lambda}\int_0^\lambda T(x,z)\,\mathrm{d}z$$

and integrate Eq. (25) for T in z. Using the boundary conditions we obtain

$$-I_{xx}=G(x),\quad I(-a)=1,\quad I(a)=0,\quad G(x)=\frac{\tau}{\lambda}\int f\bigl(T(x,z)\bigr)\,\mathrm{d}z-\int(\tau\mathbf{u}\cdot\nabla T-cT_x)\frac{\mathrm{d}z}{\lambda}. \tag{66}$$

This equation can be solved explicitly:

$$I(x)=-\int_{-a}^{x}(x-s)G(s)\,\mathrm{d}s+Ax+B$$

with constants

$$A=-\frac{1}{2a}+\frac{1}{2a}\int_{-a}^{a}(a-s)G(s)\,\mathrm{d}s,\qquad B=\frac{1}{2}+\frac{1}{2}\int_{-a}^{a}(a-s)G(s)\,\mathrm{d}s$$

that are determined from the boundary conditions. Thus, we have

$$I_x(-a)=A,\qquad I_x(a)=A-\int_{-a}^{a}G(s)\,\mathrm{d}s.$$

Using the expression for the function $G(x)$ in (66), we now infer that

$$\begin{aligned}0\leqslant -I_x(a)&=\frac{1}{2a}+\frac{1}{2a}\int_{-a}^{a}(a+s)G(s)\,\mathrm{d}s=\frac{1}{2a}+\frac{\tau}{2a}\int_{-a}^{a}\int_0^\lambda(a+x)f\bigl(T(x,z)\bigr)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}\\&\quad-\frac{\tau}{2a}\int_{-a}^{a}\int_0^\lambda(a+x)\mathbf{u}\cdot\nabla T(x,z)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}+\frac{c}{2a}\int_{-a}^{a}\int_0^\lambda(a+x)T_x\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}.\end{aligned}$$

Integrating by parts, using the boundary conditions and incompressibility of $\mathbf{u}$, we obtain

$$\begin{aligned}0\leqslant -I_x(a)&=\frac{1}{2a}+\frac{\tau}{2}\int_{-a}^{a}\int_0^\lambda f\bigl(T(x,z)\bigr)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}+\frac{\tau}{2}\int_{-a}^{a}\int_0^\lambda \frac{x}{a}f\bigl(T(x,z)\bigr)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}\\&\quad+\frac{\tau}{2a}\int_{-a}^{a}\int_0^\lambda v(x,z)T(x,z)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}-\frac{c}{2a}\int_{-a}^{a}\int_0^\lambda T\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}.\end{aligned}$$

However, the normalization condition (28) implies that $f(T(x,z))=0$ for $x\geqslant 0$ since there is no reaction to the right of $x=0$. Therefore, we can drop the third term above. This is one of the crucial points in the proof of the current lemma. Hence, we conclude that

$$\begin{aligned}0\leqslant -I_x(a)&\leqslant\frac{1}{2a}+\frac{\tau}{2}\int_{-a}^{a}\int_0^\lambda f\bigl(T(x,z)\bigr)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}+\frac{\tau}{2a}\int_{-a}^{a}\int_0^\lambda v(x,z)T(x,z)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}\\&\quad-\frac{c}{2a}\int_{-a}^{a}\int_0^\lambda T\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}\leqslant\frac{1}{2a}+\frac{\tau}{2}\int_{-a}^{a}\int_0^\lambda f\bigl(T(x,z)\bigr)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}+\tau\|v\|_\infty+|c|.\end{aligned} \tag{67}$$

We used the fact that $0 \leqslant T \leqslant 1$ to bound the last term above. Next, we look at $I_x(-a)$:

$$0 \leqslant -I_x(-a) = \frac{1}{2a} - \frac{1}{2a}\int_{-a}^{a}(a-s)G(s)\,\mathrm{d}s = \frac{1}{2a} - \frac{\tau}{2a}\int_{-a}^{a}\int_{0}^{\lambda}(a-x)f\big(T(x,z)\big)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} + \frac{\tau}{2a}\int_{-a}^{a}\int_{0}^{1}(a-x)\mathbf{u}\cdot\nabla T(x,z)\,\mathrm{d}z\,\mathrm{d}x - \frac{c}{2a}\int_{-a}^{a}\int_{0}^{\lambda}(a-x)T_x\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda}.$$

We can drop the second term above, as $(a-x)f(T) \geqslant 0$, so that, after integration by parts, we get

$$0 \leqslant -I_x(-a) \leqslant \frac{1}{2a} + \frac{\tau}{2a}\int_{-a}^{a}\int_{0}^{\lambda}v(x,z)T(x,z)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} - \frac{c}{2a}\int_{-a}^{a}\int_{0}^{\lambda}T\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} + |c| \leqslant \frac{1}{2a} + \tau\|v\|_\infty + |c|. \tag{68}$$

Let us now put together (67), (68) and (63). We observe that, with $F = \tau\iint f(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}$, we have the following three inequalities:

$$c = I_x(a) - I_x(-a) + F,$$
$$0 \leqslant -I_x(a) \leqslant \frac{1}{2a} + \frac{F}{2} + \tau\|v\|_\infty + |c|,$$
$$0 \leqslant -I_x(-a) \leqslant \frac{1}{2a} + \tau\|v\|_\infty + |c|.$$

This implies that

$$F \leqslant \frac{1}{a} + 2\tau\|v\|_\infty + 4|c| \tag{69}$$

and thus

$$-I_x(a) \leqslant 3|c| + \frac{1}{a} + 2\tau\|v\|_\infty \quad \text{for } a \geqslant a_0.$$

Lemma 1 implies then that

$$-I_x(a) \leqslant C\left[1 + \tau\|v\|_\infty + \frac{1}{a}\right].$$

Finally, Lemma 3 implies that

$$-I_x(a) \leqslant C\left[1 + \|\nabla T\|_2 + \frac{1}{a}\right] \quad \text{for } a \geqslant a_0.$$

Thus, Lemma 5 is proved. □

The previous lemmas imply uniform bounds that we summarize as follows.

Corollary 1. *Let* $(c, T, \mathbf{u})$ *satisfy* (25)–(27) *with the normalization* (28). *There exists a constant* $C > 0$ *and* $a_0 > 0$ *so that we have for all* $a \geqslant a_0$ *and* $0 \leqslant \tau \leqslant 1$

$$\|\mathbf{u}\|_{L^\infty(D_a)} + \|\nabla\mathbf{u}\|_{L^\infty(D_a)} + \|\omega\|_{L^2(D_a)} + |c| + \|\nabla T\|_{L^2(D_a)} + \tau\int_{D_a} f(T)\,\mathrm{d}x\,\mathrm{d}z \leqslant C. \tag{70}$$

In particular, as a consequence we also have

$$\|\nabla\mathbf{u}\|_{L^2(D_a)} + \|\omega\|_{L^\infty(D_a)} \leqslant C. \tag{71}$$

Proof. Lemmas 5 and 4 imply that

$$\int |\nabla T|^2 \,\mathrm{d}x\,\mathrm{d}z \leqslant C\big[1+\tau\|v\|_\infty + \|\nabla T\|_{L^2(D_a)}\big].$$

Then Lemma 3 implies that

$$\|\nabla T\|^2_{L^2(D_a)} \leqslant C\big(1+\|\nabla T\|_{L^2(D_a)}\big)$$

and thus the estimate on $\|\nabla T\|_{L^2(D_a)}$ in (70) holds. Then Lemma 3 implies the bounds on $\|\mathbf{u}\|_{L^\infty(D_a)}$, $\|\nabla\mathbf{u}\|_{L^\infty(D_a)}$ and $\|\omega\|_{L^2(D_a)}$. The bound on $|c|$ in (70) now follows from Lemma 1. Finally, the estimate on the total reaction rate follows from the above bounds and (69). One can eliminate the factor τ in front of the total reaction rate in (70): actually, one can show that it remains bounded as $\tau \to 0$. However, unlike the other estimates in (70), we will use the bound on the total reaction rate only at $\tau = 1$. □

It remains to prove the uniform Hölder $C^{1,\alpha}$-estimates for $T(x,z)$, ω and $\mathbf{u}$ in order to finish the proof of Proposition 1.

Lemma 6. *There exist two constants $C>0$ and $a_0>0$ so that the following bound holds for all $a \geqslant a_0$:*

$$\|\omega\|_{C^{1,\alpha}(D_a)} + \|\mathbf{u}\|_{C^{1,\alpha}(D_a)} + \|T\|_{C^{1,\alpha}(D_a)} \leqslant C \tag{72}$$

provided that $0 \leqslant \alpha < 1$.

Proof. The bound for T follows from the standard elliptic local regularity estimates up to the boundary [20], the C^1-bound on the flow $\mathbf{u}$ and the uniform bound on the speed c in Corollary 1. The Hölder estimate for ω follows then from the vorticity equation (44) with the Dirichlet boundary conditions, the above mentioned $C^{1,\alpha}$-bound on T, the same uniform estimates in Corollary 1 and the same results of [20]. Finally, the Hölder bounds on $\mathbf{u}$ follow from the Poisson equations (48) and (52) on the horizontal and vertical flow components, respectively, and the Hölder estimates for ω. □

This completes the proof of the a priori bounds in Proposition 1. We now turn to the proof of the existence part of this proposition.

Step 2. The degree argument. The a priori bounds proved in the first step of the proof allow us to use the Leray–Schauder topological degree argument to establish existence of solutions to the problem (25)–(27) with the normalization (28) in the bounded domain D_a. This method of construction of traveling wave solutions goes back to [11]. We introduce a map

$$\mathcal{K}_\tau : (c,\omega,T) \to (\theta^\tau, \Omega^\tau, Z^\tau)$$

as the solution operator of the linear system

$$\begin{aligned} &-cZ^\tau_x + \tau\mathbf{u}\cdot\nabla Z^\tau = \Delta Z^\tau + \tau f(T), \\ &-c\Omega^\tau_x + \mathbf{u}\cdot\nabla\Omega^\tau - \sigma\Delta\Omega^\tau = \tau\rho[e_2T_x - e_1T_z] \end{aligned} \tag{73}$$

in D_a with the no stress boundary conditions

$$\frac{\partial Z^\tau}{\partial z} = 0, \quad \widetilde{w}^\tau = \Omega^\tau = 0 \quad \text{at } z = 0, \lambda \tag{74}$$

and

$$Z^\tau(-a,y) = 1, \quad Z^\tau(a,y) = 0, \quad \tilde{u}^\tau = \Omega^\tau = 0 \quad \text{at } x = \pm a. \tag{75}$$

Here the unknown flow $\tilde{\mathbf{u}}^\tau = (\tilde{u}^\tau, \widetilde{w}^\tau)$ and the given flow $\mathbf{u}$ are the incompressible flows corresponding to the vorticities Ω^τ and ω, respectively, and satisfying the no-stress boundary conditions. The number θ^τ is defined by

$$\theta^\tau = \theta_0 - \max_{x\geqslant 0} T(x,z) + c.$$

The operator $\mathcal{K}_\tau$ is a mapping of the Banach space $X = \mathbb{R} \times C^{1,\alpha}(D_a) \times C^{1,\alpha}(D_a)$, equipped with the norm $\|(c,\omega,T)\|_X = \max(|c|, \|\omega\|_{C^{1,\alpha}(D_a)}, \|T\|_{C^{1,\alpha}(D_a)})$, into itself. A solution $\mathbf{q}^\tau = (c^\tau, \omega^\tau, T^\tau)$ of (25)–(27) is a fixed point of $\mathcal{K}_\tau$ and satisfies $\mathcal{K}_\tau \mathbf{q}^\tau = \mathbf{q}^\tau$, and vice versa: a fixed point of $\mathcal{K}_\tau$ provides a solution to (25)–(27). Hence, in order to show that (25)–(27) has a traveling front solution, it suffices to show that the kernel of the operator $\mathcal{F}_\tau = \mathrm{Id} - \mathcal{K}_\tau$ is not trivial. The standard elliptic regularity results in [20] imply that the operator $\mathcal{K}_\tau$ is compact and depends continuously on the parameter $\tau \in [0,1]$. Thus the Leray–Schauder topological degree theory can be applied. Let us introduce a ball $B_M = \{\|(c,\omega,T)\|_X \leqslant M\}$. Then Lemmas 6 and 1 show that the operator $\mathcal{F}_\tau$ does not vanish on the boundary ∂B_M with M sufficiently large for any $\tau \in [0,1]$. It remains only to show that the degree $\deg(\mathcal{F}_1, B_M, 0)$ in $\overline{B}_M$ is not zero. However, the homotopy invariance property of the degree implies that $\deg(\mathcal{F}_\tau, B_M, 0) = \deg(\mathcal{F}_0, B_M, 0)$ for all $\tau \in [0,1]$. Moreover, the degree at $\tau = 0$ can be computed explicitly as the operator $\mathcal{F}_0$ is given by

$$\mathcal{F}_0(c,\omega,T) = \Big(\max_{x \geqslant 0} T(x,y) - \theta_0, \omega, T - T_0^c\Big).$$

Here the function $T_0(x)$ solves

$$\frac{\mathrm{d}^2 T_0^c}{\mathrm{d}x^2} + c\frac{\mathrm{d}T_0^c}{\mathrm{d}x} = 0, \quad T_0^c(-a) = 1, \quad T_0^c(a) = 0$$

and is given by

$$T_0^c(x) = \frac{\mathrm{e}^{-cx} - \mathrm{e}^{-ca}}{\mathrm{e}^{ca} - \mathrm{e}^{-ca}}.$$

The mapping $\mathcal{F}_0$ is homotopic to

$$\Phi(c,\omega,T) = \Big(\max_{x \geqslant 0} T_0^c(x,y) - \theta_0, \omega, T - T_0^c\Big)$$

that, in turn, is homotopic to

$$\tilde{\Phi}(c,\omega,T) = \big(T_0^c(0) - \theta_0, \omega, T - T_0^{c_*^0}\big),$$

where c_*^0 is the unique number so that $T_0^{c_*}(0) = \theta_0$. The degree of the mapping $\tilde{\Phi}$ is the product of the degrees of each component. The last two have degree equal to one, and the first to -1, as the function $T_0^c(0)$ is decreasing in c. Thus $\deg \mathcal{F}_0 = -1$ and hence $\deg \mathcal{F}_1 = -1$ so that the kernel of $\mathrm{Id} - \mathcal{K}_1$ is not empty. This finishes the proof of Proposition 1. □

Remark 1. Observe that the $C^{1,\alpha}$-regularity of T, $\mathbf{u}$ and ω can be bootstrapped to $C^{2,\alpha}$-regularity: we have

$$\|\omega\|_{C^{2,\alpha}(D_a)} + \|\mathbf{u}\|_{C^{2,\alpha}(D_a)} + \|T\|_{C^{2,\alpha}(D_a)} \leqslant C \tag{76}$$

provided that $0 \leqslant \alpha < 1$.

3. Identification of the limit

In order to finish the proof of Theorem 1 we consider the solutions $(c^a, T^a, \mathbf{u}^a)$ constructed in Proposition 1 and pass to the limit $a \to +\infty$. The a priori estimates in the same proposition imply that we can choose a subsequence $a_n \to \infty$ so that $T_n(x,z) = T_{a_n}(x,z)$ converges uniformly on compact sets to a function $T(x,z)$, while the flow $\mathbf{u}_n(x,z) = \mathbf{u}_{a_n}(x,z)$ converges to a flow $\mathbf{u}(x,z) = (v,w)$ and the front speeds also converge: $c_n = c_{a_n} \to c$. The vorticity functions $\omega_n(x,z) = \omega_{a_n}(x,z)$ converge to the limit $\omega = w_x - v_z$. The limits satisfy the uniform bounds

$$|c| + \|\mathbf{u}\|_\infty + \|\omega\|_\infty + \int |\nabla T|^2 \,\mathrm{d}x\,\mathrm{d}z + \int f(T)\,\mathrm{d}x\,\mathrm{d}z + \int |\nabla \mathbf{u}|^2 \,\mathrm{d}x\,\mathrm{d}z \leqslant C \tag{77}$$

that follow from Corollary 1 and the Hölder estimates (24) and (76). The regularity estimates on $(T^a, \mathbf{u}^a)$ imply that the limit functions T and $\mathbf{u}$ satisfy the Boussinesq system

$$\begin{aligned}
&-cT_x + \mathbf{u}\cdot\nabla T = \Delta T + f(T),\\
&-c\omega_x + \mathbf{u}\cdot\nabla\omega - \sigma\Delta\omega = \rho(\hat{\mathbf{e}}\cdot\nabla^\perp T),\\
&\omega = w_x - v_z.
\end{aligned} \tag{78}$$

Moreover, the boundary conditions on the lateral boundaries hold for T and $\mathbf{u}$:

$$\frac{\partial T}{\partial z} = 0, \quad w = \omega = 0 \quad \text{on } z = 0, \lambda. \tag{79}$$

The normalization condition

$$\max_{x \geqslant 0} T(x, z) = \theta_0 \tag{80}$$

is also satisfied.

Therefore, to finish the proof of Theorem 1, it remains only to show that (i) T converges to a constant θ_- as $x \to -\infty$ and $T \to 0$ as $x \to +\infty$, (ii) $\mathbf{u} \to 0$ as $x \to +\infty$, and (iii) $\theta_- = 1$ if the reaction rate satisfies $f(T) \leqslant (T - \theta_0)_+^2/\lambda^2$. First, we note that the uniform L^2-bound on ∇T in (77) implies that T converges to two constants θ_- and θ_+ as $x \to \pm\infty$, possibly passing to a subsequence $x_n \to \pm\infty$. The elliptic regularity results imply that actually T converges to these constants as $x \to \pm\infty$. Moreover, the bound for the total reaction rate $\int f(T)\,\mathrm{d}x\,\mathrm{d}z$ in (77) implies that $f(\theta_-) = f(\theta_+) = 0$. Furthermore, integrating (78) we obtain

$$c(\theta_- - \theta_+) = \int f(T) \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}. \tag{81}$$

In order to identify the limits $\theta_\pm$ we will make use of the following lemmas that provide some additional information on solutions on a finite domain before the passage to the limit. The first result describes the behavior near the right end $x = a_n$.

Lemma 7. *There exists a sequence $a_n \to \infty$ so that*

$$\left| \frac{\partial T_n(a_n, z)}{\partial x} \right| \to 0 \tag{82}$$

as $n \to \infty$, uniformly in z. Moreover, we have $\lim_{n\to\infty} T_n(a_n - x_0, z) = 0$ for all $x_0 \in \mathbb{R}$.

Proof. We introduce a shifted solution $\Phi_n(x, z) = T_n(x + a_n, z)$, $\mathbf{v}_n = \mathbf{u}_n(x + a_n, z)$ defined in the domain $-2a_n \leqslant x \leqslant 0$. The functions Φ_n and $\mathbf{v}_n$ satisfy the same a priori bounds (77) as T_n and $\mathbf{u}_n$ and hence they converge as $n \to \infty$ to some limits Φ and $\mathbf{v}$ that satisfy

$$-c\Phi_x + \mathbf{v} \cdot \nabla\Phi = \Delta\Phi, \quad \text{in } x \leqslant 0, \qquad \Phi(0, z) = 0. \tag{83}$$

as $f(\Phi_n) = 0$ for $x > -a_n$ and thus in the limit $f(\Phi) = 0$. The function Φ satisfies the Neumann boundary conditions at $z = 0, \lambda$. The uniform upper bound on $\|\nabla\Phi_n\|_2$ together with the elliptic regularity results imply that Φ has to converge to a constant Φ_- as $x \to -\infty$ along a subsequence. We note that, as $0 \leqslant \Phi(x, z) \leqslant \theta_0$, the constant Φ_- satisfies the same bounds:

$$0 \leqslant \Phi_- \leqslant \theta_0.$$

Integrating (83) we obtain

$$c\lambda\Phi_- = \int \Phi_x(0, z)\,\mathrm{d}z \leqslant 0. \tag{84}$$

Hence, either $\Phi_- = 0$ or $c \leqslant 0$. In the former case $\Phi \equiv 0$ and hence $\Phi_x(0, z) = 0$ for all z. That implies that both $T_x^n(a_n, z) \to 0$ as $n \to \infty$ and $T_x^n(a_n - x_0, z) = \Phi_x^n(-x_0, z) \to 0$ as $n \to \infty$, as claimed in Lemma 7. It remains to rule out the second case, $c \leqslant 0$. This is done in the next lemma that provides a crucial lower bound on the speed c_n. In particular it shows that $c > 0$ – this will conclude the proof of Lemma 7. □

Lemma 8. *The front speed is positive*: $c > 0$.

Proof. Integrating the temperature equation in (78) for T_n, we obtain

$$c_n\lambda = \int_0^\lambda \frac{\partial T_n}{\partial x}(a_n, z)\,\mathrm{d}z - \int_0^\lambda \frac{\partial T_n}{\partial x}(-a_n, z)\,\mathrm{d}z + \int f(T_n)\,\mathrm{d}x\,\mathrm{d}z \geqslant \int_0^\lambda \frac{\partial T_n}{\partial x}(a_n, z)\,\mathrm{d}z + \int f(T_n)\,\mathrm{d}x\,\mathrm{d}z. \tag{85}$$

Observe also that we have a uniform bound

$$\left(\int f(T_n)\,\mathrm{d}x\,\mathrm{d}z\right)\left(\int |\nabla T_n|^2\,\mathrm{d}x\,\mathrm{d}z\right) \geqslant C \tag{86}$$

that follows from the fact that $T_n(0,z) \leqslant \theta_0$ and $T_n(-a_n,z) = 1$. The proof of (86) is as in [13]: there exists $z_0 \in (0,\lambda)$ such that both

$$\int_{-a_n}^{0} \left|\nabla T_n(x,z_0)\right|^2 \mathrm{d}x \leqslant 3\int_{D_n} |\nabla T_n|^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \quad D_n = [-a_n,a_n]_x \times [0,\lambda]_z,$$

and

$$\int_{-a_n}^{0} f\big(T_n(x,z_0)\big)\,\mathrm{d}x \leqslant 3\int_{D_n} f\big(T_n(x,z)\big)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}.$$

Let x_1 be the left-most point so that $T_n(x_1,z_0) = 1-(1-\theta_0)/4$:

$$x_1 = \inf\left\{x \in (-a_n,0)\colon\ T_n(x,z_0) = 1 - \frac{1-\theta_0}{4}\right\}$$

and $x_2 > x_1$ be the left-most point so that $T_n(x_2,z_0) = \theta_0 + (1-\theta_0)/4$:

$$x_2 = \inf\left\{x \in (-a_n,0)\colon\ T_n(x,z_0) = \theta_0 + \frac{1-\theta_0}{4}\right\}.$$

Existence of x_1 and x_2 is guaranteed by the fact that $T_n(-a_n,z) = 1$ and $T_n(0,z) \leqslant \theta_0$ for all $z \in [0,\lambda]$. Then the reaction rate $f(T_n(x,z_0)) > C$ for $x_1 \leqslant x \leqslant x_2$ so that

$$C|x_1-x_2| \leqslant \int_{x_1}^{x_2} f\big(T_n(x,z_0)\big)\,\mathrm{d}x \leqslant 3\int_{D_n} f\big(T_n(x,z)\big)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}$$

and

$$\frac{(1-\theta_0)^2}{4|x_1-x_2|} \leqslant \int_{x_1}^{x_2} \left|\nabla T_n(x,z_0)\right|^2 \mathrm{d}x \leqslant 3\int_{D_n} |\nabla T_n|^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}.$$

Multiplying these two inequalities, we arrive at (86).

The estimate (86) and the uniform upper bound on $\|\nabla T_n\|_2$ in Corollary 1 imply that

$$\int_{D_n} f(T_n)\,\mathrm{d}x\,\mathrm{d}z \geqslant C. \tag{87}$$

Then, passing to the limit in (85), and using (84) we obtain

$$c\lambda(1-\Phi_-) \geqslant C > 0,$$

as

$$\int \frac{\partial T_n}{\partial x}(a_n,z)\,\mathrm{d}z \to \int \Phi_x(0,z)\,\mathrm{d}z,$$

with the function Φ as in the proof of Lemma 7. Now, we recall that $\Phi_- \leqslant \theta_0 < 1$ and thus the front speed $c > 0$. This finishes the proof of Lemma 8 and hence also that of Lemma 7. □

Lemma 8 and (81) imply that $\theta_- \geqslant \theta_+$. However, if $\theta_- = \theta_+$ we have $f(T) = 0$ everywhere and hence (78) is a linear equation. The maximum principle implies that T is constant in this case. The last condition in (77) implies that this constant has to be equal to θ_0. Hence, either $\theta_- > \theta_+$ or $T \equiv \theta_0$.

Let us now rule out the special case $\theta_- = \theta_+ = \theta_0$.

Lemma 9. *The left and right limits θ_- and θ_+ satisfy $\theta_- > \theta_+$.*

Proof. We have already shown that $\theta_- \geqslant \theta_+$ and, moreover, if $\theta_- = \theta_+$ then

$$\theta_- = \theta_+ = \theta_0. \tag{88}$$

Hence, it suffices to show that the latter is impossible. Let us assume that (88) holds. As we have explained above, then

$$T_n \to \theta_0, \quad \text{and } \partial T_n/\partial x \to 0 \quad \text{uniformly on compact sets.} \tag{89}$$

Then, integrating the equation

$$-c_n \frac{\partial T_n}{\partial x} + \mathbf{u}_n \cdot \nabla T_n = \Delta T_n + f(T_n)$$

between $x = 0$ and $x = a_n$ we obtain, as $f(T_n) = 0$ in this region,

$$c_n \int_0^\lambda T_n(0,z)\,\mathrm{d}z - \int_0^\lambda v_n(0,z) T_n(0,z)\,\mathrm{d}z = \int_0^\lambda \frac{\partial T_n}{\partial x}(a_n,z)\,\mathrm{d}z - \int_0^\lambda \frac{\partial T_n}{\partial x}(0,z)\,\mathrm{d}z. \tag{90}$$

We now pass to the limit $n \to \infty$ in (90). The first term on the left converges to $c\theta_0\lambda$, as we have assumed that T converges uniformly to θ_0 on compact intervals. The second term on the left converges to

$$\int_0^\lambda v(0,z)\theta_0\,\mathrm{d}z = 0,$$

as incompressibility of the flow $\mathbf{u}_n$ and the boundary conditions at $x = \pm a$ imply that

$$\int v_n(0,z)\,\mathrm{d}z = 0.$$

The limit (82) in Lemma 7 implies that the first term on the right side of (90) converges to zero. Finally, the last term on the right side of (90) converges to zero because of (89). Therefore, we obtain

$$c\lambda\theta_0 = 0.$$

However, this implies that $c = 0$ which contradicts Lemma 8. Hence, the case $\theta_- = \theta_+ = \theta_0$ is ruled out and thus $\theta_- > \theta_+$. $\square$

We continue the analysis of the behavior of the solution at the right end of the domain.

Lemma 10. *The gradient ∇T_n converges to zero "as $x \to +\infty$" uniformly in n, that is, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ and R so that $|\nabla T_n(x,z)| < \varepsilon$ for all $n \geqslant N$ and all $R \leqslant x \leqslant a_n$.*

Proof. Let us assume that this is not the case. Then there exists $\varepsilon_0 > 0$ and a sequence $b_n \to +\infty$ so that $|\nabla T_n(b_n, z_n)| \geqslant \varepsilon_0$ for some $z_n \in [0,\lambda]$. Note that Lemma 7 implies that

$$|a_n - b_n| \to \infty. \tag{91}$$

Let us define the shifted solution $\Psi_n(x,z) = T_n(x - b_n, z)$, $\mathbf{v}_n(x,z) = \mathbf{u}_n(x - b_n, z)$ on the domain $x \in [-a_n - b_n, a_n - b_n]$. Then Ψ_n and $\mathbf{v}_n$ satisfy the same uniform bounds as T_n and $\mathbf{u}_n$ and thus they converge to a pair of functions Ψ, $\mathbf{v}$ uniformly on compact intervals, together with their derivatives. The functions Ψ and $\mathbf{v}$ are defined on the whole real line because of (91). Moreover, the function Ψ has left and right limits $\Psi_\pm$ as $x \to \pm\infty$. Hence, the same argument as in the proof of Lemma 7 shows that Ψ must be equal a constant, as it has left and right limits and satisfies

$$-c\Psi_x + \mathbf{v} \cdot \nabla\Psi = \Delta\Psi.$$

However, this contradicts the fact that $\max_z |\nabla\Psi(0,z)| \geqslant \varepsilon_0$. $\square$

The decay of the gradient of T_n implies that the flow ahead of the front goes to zero for large x, uniformly in n.

Lemma 11. *The flow $\mathbf{u}_n(x,z)$ converges to zero on the right uniformly in n, that is, for any $\varepsilon > 0$ there exists $R > 0$ and $N \in \mathbb{N}$ so that $|\mathbf{u}_n(x,z)| < \varepsilon$ for all $R < x \leqslant a_n$ and $n \geqslant N$.*

Proof. We choose N and R so that $|\nabla T_n| < \varepsilon$ for all $n \geqslant N$ and $x \in [R, a_n]$. Then, we decompose $T_n = T_n^{\mathrm{in}} + T_n^{\mathrm{out}}$ with $\operatorname{supp} T_n^{\mathrm{in}} \subset \{x \leqslant R+1\}$ and $\operatorname{supp} T_n^{\mathrm{out}} \subset \{x \geqslant R\}$. We also require that both T_n^{in} and T_n^{out} satisfy the same uniform gradient bounds as T_n. Moreover, we have $|\nabla T_n^{\mathrm{out}}| < \varepsilon$. We also split $\omega_n = \omega_n^{\mathrm{in}} + \omega_n^{\mathrm{out}}$ and $\mathbf{u}_n = \mathbf{u}_n^{\mathrm{in}} + \mathbf{u}_n^{\mathrm{out}}$ accordingly:

$$-c_n\omega_n^{\mathrm{in}} + \mathbf{u}_n\cdot\nabla\omega_n^{\mathrm{in}} - \sigma\Delta\omega_n^{\mathrm{in}} = \rho\left(e_2\frac{\partial T_n^{\mathrm{in}}}{\partial x} - e_1\frac{\partial T_n^{\mathrm{in}}}{\partial z}\right), \quad \omega_n^{\mathrm{in}} = 0 \quad \text{on } \partial D_{a_n},$$

and similarly for ω_n^{out}.

We now bound $|\mathbf{u}_n^{\mathrm{in}}|$ and $|\mathbf{u}_n^{\mathrm{out}}|$ separately for sufficiently large x. First, we look at $\mathbf{u}_n^{\mathrm{in}} = (v_n^{\mathrm{in}}, w_n^{\mathrm{in}})$. The function ω_n^{in} satisfies a homogeneous equation

$$-c_n\frac{\partial\omega_n^{\mathrm{in}}}{\partial x} + \mathbf{u}_n\cdot\nabla\omega_n^{\mathrm{in}} - \sigma\Delta\omega_n^{\mathrm{in}} = 0 \tag{92}$$

in the rectangle $D_{R+2,a_n} = \{R+2 \leqslant x \leqslant a_n,\ 0 \leqslant z \leqslant \lambda\}$, as T_n^{in} vanishes in D_{R+2,a_n}. The function ω_n^{in} satisfies a uniform $C^{2,\alpha}$-bound – this is shown in the same way as the $C^{2,\alpha}$-bound for the full vorticity function ω in (76). This in turn implies that the function $\psi(z) = \omega_n^{\mathrm{in}}(R+2,z)$ is uniformly bounded in $C^2[0,\lambda]$. Let $g(x)$ be a smooth monotonic and positive cut-off function so that

$$g(x) = 1 \quad \text{for } R+2 \leqslant x \leqslant R+3 \quad \text{and} \quad g(x) = 0 \quad \text{for } x \geqslant R+4. \tag{93}$$

Then the function ω_n^{in} can be decomposed as

$$\omega_n^{\mathrm{in}}(x,z) = \psi(z)g(x) + \zeta_n.$$

The function ζ_n satisfies

$$-c_n\frac{\partial\zeta_n}{\partial x} + \mathbf{u}_n\cdot\nabla\zeta_n - \sigma\Delta\zeta_n = f_n \quad \text{in } D_{R+2,a_n}, \quad \zeta_n = 0 \quad \text{on } \partial D_{R+2,a_n}. \tag{94}$$

The right side f_n is given by

$$f_n := \sigma\psi''(z)g(x) + \sigma\psi(z)g''(x) - c_n\psi(z)g'(x) - v_n\psi(z)g'(x) - w_n\psi'(z)g(x).$$

It is supported in $R+2 \leqslant x \leqslant R+4$ and is uniformly bounded since $\|\omega_n^{\mathrm{in}}\|_{C^{2,\alpha}(D_a)} \leqslant C$. Let us choose $\alpha > 0$ sufficiently small, then the function $\xi_n(x,z) = \zeta_n(x,z)\,\mathrm{e}^{\alpha x}$ satisfies

$$\begin{aligned}&-c_n\frac{\partial\xi_n}{\partial x} + \alpha c\xi_n + \mathbf{u}_n\cdot\nabla\xi_n - \alpha v_n\xi_n - \sigma\Delta\xi_n + 2\sigma\alpha\frac{\partial\xi_n}{\partial x} - \sigma\alpha^2\xi_n = g_n \quad \text{in } D_{R+2,a_n},\\ &\xi_n = 0 \quad \text{on } \partial D_{R+2,a_n}\end{aligned} \tag{95}$$

with $g_n = f_n(x)\,\mathrm{e}^{\alpha x}$. Multiplying (95) by ξ_n and integrating by parts, using the boundary conditions, we obtain

$$\sigma\int_{D_{R+2,a_n}} |\nabla\xi_n|^2\,\mathrm{d}x\,\mathrm{d}z + \left(c\alpha - \alpha\|v\|_\infty - \sigma\alpha^2\right)\int_{D_{R+2,a_n}} |\xi_n|^2\,\mathrm{d}x\,\mathrm{d}z \leqslant \|g_n\|_2\|\xi_n\|_2. \tag{96}$$

However, as the function ξ_n vanishes at $z = 0, \lambda$, the Poincaré inequality implies that

$$\int_{D_{R+2,a_n}} |\nabla\xi_n|^2\,\mathrm{d}x\,\mathrm{d}z \geqslant \frac{\pi^2}{\lambda^2}\int_{D_{R+2,a_n}} |\xi_n|^2\,\mathrm{d}x\,\mathrm{d}z.$$

Hence, the following upper bound holds

$$\int_{D_{R+2,a_n}} |\xi_n|^2\,\mathrm{d}x\,\mathrm{d}z \leqslant \|g_n\|_2^2 \leqslant C,$$

provided that α is sufficiently small, since $\|v\|_\infty \leqslant C$. Using (96) once again we conclude that

$$\int\limits_{D_{R+2,a_n}} |\nabla \xi_n|^2 \,\mathrm{d}x\,\mathrm{d}z \leqslant C.$$

Therefore, the function ζ_n satisfies

$$\int\limits_{D_{R+2,a_n}} \left[|\nabla \zeta_n|^2 + |\zeta_n|^2\right] \mathrm{e}^{2\alpha x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C.$$

This, in turn implies the same bound for the function ω_n^{in}:

$$\int\limits_{R+2}^{a_n} \int\limits_0^{\lambda} \left|\omega_n^{\mathrm{in}}\right|^2 \mathrm{e}^{2\alpha x} \,\mathrm{d}x\,\mathrm{d}z + \int\limits_{R+2}^{a_n} \int\limits_0^{\lambda} \left|\nabla\omega_n^{\mathrm{in}}\right|^2 \mathrm{e}^{2\alpha x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C. \tag{97}$$

It follows that the L^2-norm of ω_n^{in} decays uniformly in n:

$$\int\limits_{r_0}^{a_n}\int\limits_0^{\lambda} \left|\omega_n^{\mathrm{in}}\right|^2 \mathrm{d}x\,\mathrm{d}z \leqslant \mathrm{e}^{-\alpha r_0} \int\limits_{r_0}^{a_n}\int\limits_0^{\lambda} \left|\omega_n^{\mathrm{in}}\right|^2 \mathrm{e}^{2\alpha x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C\mathrm{e}^{-\alpha r_0} \tag{98}$$

for $r_0 > R+5$, and the same bound holds for $\nabla\omega_n^{\mathrm{in}}$:

$$\int\limits_{r_0}^{a_n}\int\limits_0^{\lambda} \left|\nabla\omega_n^{\mathrm{in}}\right|^2 \mathrm{d}x\,\mathrm{d}z \leqslant \mathrm{e}^{-\alpha r_0} \int\limits_{r_0}^{a_n}\int\limits_0^{\lambda} \left|\nabla\omega_n^{\mathrm{in}}\right|^2 \mathrm{e}^{2\alpha x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C\mathrm{e}^{-\alpha r_0}. \tag{99}$$

As the function ω_n^{in} satisfies the homogeneous equation (92) for $x \geqslant R+1$ with a bounded flow $\mathbf{u}$, the standard local elliptic estimates now imply that

$$\left|\omega_n^{\mathrm{in}}(x,z)\right| \leqslant C\,\mathrm{e}^{-\alpha x} \quad \text{for } x \geqslant R+5. \tag{100}$$

The $W^{2,p}$ elliptic estimates then imply the uniform decay of the gradient of ω_n^{in}:

$$\left|\nabla\omega_n^{\mathrm{in}}(x,z)\right| \leqslant C\,\mathrm{e}^{-\alpha x} \quad \text{for } x \geqslant R+5. \tag{101}$$

Now we can bound the flow $\mathbf{u}_n^{\mathrm{in}} = (v_n^{\mathrm{in}}, w_n^{\mathrm{in}})$ itself. First, we look at the horizontal component v_n^{in}. It satisfies the following Poisson equation in D_{R+2,a_n}:

$$-\Delta v_n^{\mathrm{in}} = \frac{\partial \omega_n^{\mathrm{in}}}{\partial z} \quad \text{in } D_{R+2,a_n}, \quad \frac{\partial v_n^{\mathrm{in}}}{\partial z} = 0 \quad \text{on } z = 0, \lambda, \quad v_n^{\mathrm{in}} = 0 \quad \text{on } x = a_n.$$

Moreover, the $C^{2,\alpha}$-regularity of $\mathbf{u}_n$ implies that the boundary value $\phi(z) = v_n^{\mathrm{in}}(R+2,z)$ is bounded in $C^2[0,\lambda]$. Therefore, as we did with ω_n^{in}, we represent $v_n^{\mathrm{in}}(x,z) = \phi(z)g(x) + \bar{v}_n^{\mathrm{in}}(x,z)$ with the cut-off function $g(x)$ as in (93). The function $\bar{v}_n^{\mathrm{in}}$ satisfies

$$-\Delta \bar{v}_n^{\mathrm{in}} = \bar{f}_n := -\phi_{zz}(z)g(x) - \phi(z)g''(x) + \frac{\partial \omega_n^{\mathrm{in}}}{\partial z} \quad \text{in } D_{R+2,a_n},$$

with an exponentially decaying function $\bar{f}_n$, as follows from (101). The boundary conditions are

$$\frac{\partial \bar{v}_n^{\mathrm{in}}}{\partial z} = 0 \quad \text{on } z = 0, \lambda, \quad \bar{v}_n^{\mathrm{in}} = 0 \quad \text{on } x = R+2, a_n.$$

The same argument as we used to obtain (97) implies that

$$\int\limits_{R+2}^{a_n}\int\limits_0^{\lambda} |v_n^{\mathrm{in}}|^2 \mathrm{e}^{2\beta x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C \int \left|\bar{f}_n(x,z)\right|^2 \mathrm{e}^{2\beta x} \,\mathrm{d}x\,\mathrm{d}z \leqslant C \tag{102}$$

with a sufficiently small $0 < \beta < \alpha$. Therefore, in the same vein as we have obtained (100) and (101), we conclude that

$$\left|v_n^{\text{in}}(x,z)\right| \leqslant C\,\mathrm{e}^{-\alpha x} \quad \text{for } x \geqslant R+5 \tag{103}$$

and

$$\left|\nabla v_n^{\text{in}}(x,z)\right| \leqslant C\,\mathrm{e}^{-\alpha x} \quad \text{for } x \geqslant R+5. \tag{104}$$

The uniform bound on w_n^{in} now follows, as it satisfies the Dirichlet boundary condition $w_n^{\text{in}}(x,0) = w_n^{\text{in}}(x,\lambda) = 0$ and the derivative $\partial w_n^{\text{in}}/\partial z = -\partial v_n^{\text{in}}/\partial x$ is exponentially decaying (104). We infer that

$$\left|w_n^{\text{in}}(x,z)\right| \leqslant C\,\mathrm{e}^{-\alpha x} \quad \text{for } x \geqslant R+5. \tag{105}$$

Now we bound $\mathbf{u}_n^{\text{out}}$. The corresponding vorticity satisfies

$$-c_n\frac{\partial \omega_n^{\text{out}}}{\partial x} + \mathbf{u}_n\cdot\nabla\omega_n^{\text{out}} - \sigma\Delta\omega_n^{\text{out}} = \rho\big(\hat{\mathbf{e}}\cdot\nabla^{\perp}T_n^{\text{out}}\big) \quad \text{in } D_{a_n}, \quad \omega = 0 \quad \text{on } D_{a_n}.$$

However,

$$\left|\nabla T_n^{\text{out}}\right| \leqslant \varepsilon \tag{106}$$

by construction, hence the maximum principle implies that

$$\left|\omega_n^{\text{out}}(x,z)\right| \leqslant \varepsilon\rho q(z) \leqslant C\varepsilon. \tag{107}$$

Here the non-negative function $q(z)$ satisfies the boundary value problem

$$-\sigma q''(z) + w_n(z)q'(z) = 1, \quad q(0) = q(\lambda) = 0.$$

We infer from the standard local elliptic estimates up to the boundary, (107) and (106) that

$$\left|\nabla\omega_n^{\text{out}}(x,z)\right| \leqslant C\varepsilon \quad \text{in } D_{a_n} \tag{108}$$

as well. The vertical flow component satisfies

$$\Delta w_n^{\text{out}} = \frac{\partial\omega_n^{\text{out}}}{\partial x} \quad \text{in } D_{a_n}, \quad w_n^{\text{out}}(x,0) = w_n^{\text{out}}(x,\lambda) = 0, \quad \frac{\partial w_n^{\text{out}}(\pm a_n, z)}{\partial x} = 0.$$

Therefore, the maximum principle implies once again that

$$\left|w_n^{\text{out}}(x,z)\right| \leqslant \frac{C\rho\varepsilon}{2} z(\lambda - z) \leqslant C\varepsilon \quad \text{in } D_{a_n}.$$

Hence, the same local elliptic regularity results allow us to conclude that

$$\left|\nabla w_n^{\text{out}}(x,z)\right| \leqslant C\varepsilon \quad \text{in } D_{a_n}.$$

In order to bound the horizontal flow component v_n^{out} and conclude the proof of Lemma 11 we observe that $\partial v_n^{\text{out}}/\partial z = \partial w_n^{\text{out}}/\partial z - \omega_n^{\text{out}}$ so that $|\partial v_n^{\text{out}}/\partial z| \leqslant C\varepsilon$ in D_{a_n}. However, v_n^{out} also satisfies the mean-zero condition

$$\int_0^\lambda v_n^{\text{out}}(x,z) = 0 \quad \text{for all } -a_n \leqslant x \leqslant a_n.$$

Hence, we have $|v_n^{\text{out}}(x,z)| \leqslant C\varepsilon$ in D_{a_n}, and the proof of Lemma 11 is now complete. □

The next lemma implies that the right limit satisfies $\theta_+ = 0$.

Lemma 12. *The right limit is* $\theta_+ = 0$.

Proof. Let us choose R independent of n so that $c_{a_n} > \sup_{x>R} |v_n(x,z)|$ for all n. Lemma 8 implies that the speeds c_n are uniformly bounded below by a positive constant, thus it follows from Lemma 11 that we can find such $R > 0$. Then the function $\phi(x) = A\,\mathrm{e}^{-\alpha x}$, with a sufficiently small $\alpha > 0$, satisfies

$$-c_n\phi_x + \mathbf{u}_n \cdot \nabla\phi \geqslant \Delta\phi.$$

An argument as in the proof of Lemma 1 shows that if A is chosen so that $A\,\mathrm{e}^{-\alpha R} > 1$ then $T_n(x,z) \leqslant A\,\mathrm{e}^{-\alpha x}$ on the domain $x \in [R, a_n]$. Therefore, the limit $T(x,z)$ obeys the same bound, which in turn implies that $\theta_+ = 0$. □

Finally, we show that under the additional assumption (10) the left limit $\theta_- = 1$. This is the only place in the proof where assumption (10) is used.

Lemma 13. *Let us assume that $f(T) \leqslant (T-\theta_0)_+^2/\lambda^2$. Then the left limit is $\theta_- = 1$.*

Proof. We note that we have for each $x \in \mathbb{R}$

$$\int \left|\nabla T(x,z)\right|^2 \mathrm{d}z \geqslant \frac{(M(x)-m(x))^2}{\lambda}$$

with $M(x) = \max_z T(x,z)$ and $m(x) = \min_z T(x,z)$. It follows from the maximum principle that the function $m(x)$ is non-increasing. Indeed, the maximum principle implies that $m(x)$ cannot achieve an interior minimum. Moreover, for each finite a, the function $m_a(x)$ attains its maximum (equal to 1) at the point $x = -a$. Therefore, $m_a(x)$ is decreasing immediately to the right of $x = -a$. As $m_a(x)$ cannot achieve an internal minimum, it follows that $m_a(x)$ is a decreasing function of x for all $a > 0$. Thus, the function $m(x)$, which is the limit of $m_a(x)$ as $a \to +\infty$, is also a decreasing function. Let us assume that $\theta_- \leqslant \theta_0$, then monotonicity of $m(x)$ implies that $m(x) < \theta_0$ for all $x \in \mathbb{R}$. Then we have

$$\int \left|\nabla T(x,z)\right|^2 \mathrm{d}x\,\mathrm{d}z \geqslant \int \bigl(M(x)-m(x)\bigr)^2 \frac{\mathrm{d}x}{\lambda} \geqslant \int \bigl(T(x,z)-\theta_0\bigr)_+^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda^2}.$$

We also observe that

$$c\theta_- = \int f(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \qquad \frac{c\theta_-^2}{2} + \int |\nabla T|^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda} = \int T f(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}$$

so that

$$\int |\nabla T|^2\,\mathrm{d}x\,\mathrm{d}z = \int \left(T - \frac{\theta_-}{2}\right) f(T)\,\mathrm{d}x\,\mathrm{d}z.$$

Hence we obtain using (10)

$$\int \left(T - \frac{\theta_-}{2}\right)(T-\theta_0)_+^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda^2} \geqslant \int \left(T - \frac{\theta_-}{2}\right) f(T)\,\mathrm{d}x\,\mathrm{d}z \geqslant \int \bigl(T(x,z)-\theta_0\bigr)_+^2 \frac{\mathrm{d}x\,\mathrm{d}z}{\lambda^2}.$$

However, the left side is smaller than the right side unless $T \equiv \theta_0$, the case that we have already ruled out. □

This finishes the proof of Theorem 1.

4. Bounds for the initial value problem

We consider in this section the solutions of the Cauchy problem with general front-like initial data and obtain the uniform bounds on the bulk burning rate and other average quantities stated in Theorems 2 and Theorem 3. We prove the first result, and the proof of the second result is presented in Section 4.2.

4.1. Bounds in an arbitrary strip

We prove in this section Theorem 2. Let $T(t,x,z)$, $u(t,x,z)$ be the solution of the Cauchy problem

$$T_t + \mathbf{u} \cdot \nabla T = \Delta T + f(T), \tag{109}$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \sigma \Delta \mathbf{u} + \nabla p = \rho T \hat{\mathbf{e}},$$

$$\nabla \cdot \mathbf{u} = 0, \tag{110}$$

with initial data $T_0(x,z)$, $\mathbf{u}_0(\mathbf{x})$. We assume that there exists $R > 0$ so that $T_0(x,z) = 0$ for $x > R$ and $T_0(x,z) = 1$ for $x < -R$, and that the initial vorticity is bounded in L^2:

$$\int \left|\omega_0(x,z)\right|^2 \mathrm{d}x\,\mathrm{d}z < +\infty.$$

The assumptions on the initial temperature T_0 can be relaxed – it simply has to approach one and zero at the two ends of the domain sufficiently fast.

We recall that the bulk burning rate $\overline{V}(t)$, the Nusselt number $\overline{N}(t)$ and the average horizontal flow $\overline{U}(t)$ are defined by

$$\overline{V}(t) = \frac{1}{t}\int_0^t V(s)\,\mathrm{d}s, \qquad V(t) = \int f(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \tag{111}$$

$$\overline{N}(t) = \frac{1}{t}\int_0^t N(s)\,\mathrm{d}s, \qquad N(t) = \int |\nabla T|^2\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}, \tag{112}$$

$$\overline{U}(t) = \frac{1}{t}\int_0^t \left\|v(s)\right\|_\infty \mathrm{d}s. \tag{113}$$

The laminar front speed c_0 is defined as the unique c so that equation

$$-c\Phi' = \Phi'' + f(\Phi), \quad \Phi(-\infty) = 1, \quad \Phi(+\infty) = 0$$

has a solution $0 < \Phi < 1$. We recall the statement of Theorem 2.

Theorem 4. *There exists a constant $C > 0$ so that under the above assumptions on the initial data T_0, $\mathbf{u}_0$, the following bounds hold*

$$c_0 - C[\rho + \rho^2] + \mathrm{o}(1) \leqslant \overline{V}(t) \leqslant c_0 + C[\rho + \rho^2] + \mathrm{o}(1), \tag{114}$$

$$\overline{N}(t) \leqslant \left[C\rho + \sqrt{\frac{c_0}{2} + C^2\rho^2}\right]^2 + \mathrm{o}(1),$$

$$\overline{U}(t) \leqslant C\rho[1+\rho] + \mathrm{o}(1)$$

as $t \to +\infty$.

Proof. First, we prove the following bounds on $\overline{N}(t)$ and $\overline{V}(t)$ in terms of $\overline{U}(t)$.

Lemma 14. *There exists a constant C_0 that depends only on the initial data T_0 so that*

$$\overline{N}(t) \leqslant \frac{1}{2}\overline{V}(t) + \overline{U}(t) + C_0\left[\frac{1}{t} + \frac{1}{\sqrt{t}}\right], \tag{115}$$

and

$$\overline{V}(t) \leqslant c_0 + \overline{U}(t) + C_0\left[\frac{1}{t} + \frac{1}{\sqrt{t}}\right]. \tag{116}$$

Proof. Define $g(T) = T(1-T)$ and its integral

$$R(t) = \int g(T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}.$$

The idea of using a concave function $g(T)$ in a related context is due to B. Winn [38]. We observe that

$$\frac{dR}{dt} = \int g'(T)\Delta T \frac{dx\,dz}{\lambda} + \int g'(T) f(T) \frac{dx\,dz}{\lambda} \geqslant -\int g''(T)|\nabla T|^2 \frac{dx\,dz}{\lambda} - V(t) \tag{117}$$

with the burning rate $V(t)$ defined in (111). Thus

$$\frac{dR}{dt} + V(t) \geqslant 2\int |\nabla T|^2 \frac{dx\,dz}{\lambda} = 2N(t),$$

which after averaging in time becomes

$$\frac{R(t)}{t} + \overline{V}(t) \geqslant 2\overline{N}(t). \tag{118}$$

In order to obtain an upper bound for the potentially small term $R(t)/t$ in (118) we construct sub- and super-solutions for $T(t,x,z)$. This construction follows [40]. We look for a sub-solution for T of the form

$$\psi_l(t,x,z) = \Phi_0\big(x - c_0 t + x_1 + \xi_1(t)\big) - q_1(t,x,z).$$

Here Φ_0 is the traveling wave in the absence of convection, at $\rho = 0$, normalized so that $\Phi_0(0) = \theta_0$. It is the unique solution of

$$-c_0\Phi_0' = \Phi_0'' + f(\Phi_0), \quad \Phi_0(0) = \theta_0, \quad \Phi_0(-\infty) = 1, \quad \Phi_0(+\infty) = 0.$$

The functions $\xi_1(t)$ and $q_1(t,x,z)$ are to be chosen. In order for ψ_l to be a sub-solution we need

$$G[\psi_l] = \frac{\partial \psi_l}{\partial t} + \mathbf{u}\cdot\nabla\psi_l - \Delta\psi_l - f(\psi_l) \leqslant 0.$$

We have

$$G[\psi_l] = \dot{\xi}_1\Phi_0' + u\Phi_0' - \frac{\partial q_1}{\partial t} - \mathbf{u}\cdot\nabla q_1 + \Delta q_1 + f(\Phi_0) - f(\Phi_0 - q_1).$$

With an appropriate choice of x_1, that is, by shifting Φ_0 sufficiently to the left we can ensure that $T_0(x,z) \geqslant \Phi_0(x) - q_{10}(x)$ with $0 \leqslant q_{10}(x) \leqslant (1-\theta_0)/2$ and $q_{10}(x) \in L^1(\mathbb{R})$. Then we choose $q_1(t,x,z)$ to be the solution of

$$\frac{\partial q_1}{\partial t} + \mathbf{u}\cdot\nabla q_1 = \Delta q_1, \quad q_1(0,x,z) = q_{10}(x), \quad \frac{\partial q_1}{\partial z} = 0 \quad \text{at } z = 0, \lambda. \tag{119}$$

The following lemma from [16] provides a uniform L^1–L^∞ decay estimate for q_1 that is independent of the advection term.

Lemma 15. *There exists a constant $C > 0$ that is independent of the* (*incompressible*) *flow* $\mathbf{u}$ *so that*

$$\|q_1(t)\|_\infty \leqslant \frac{C}{\lambda\sqrt{t}}\|q_{10}\|_{L^1(D)} \tag{120}$$

for $t \geqslant 1$.

As mentioned above, the main point of the above result is the independence of the constant in (120) from the flow $\mathbf{u}$. We also note that this L^1–L^∞ estimate behaves in a one-dimensional way for large times, as one would expect for a strip. The factor of λ in the denominator is compensated by the fact that the L^1-norm is taken over the strip and not only in x. We postpone the proof of Lemma 15 till the end of this section.

We can find $\delta > 0$ so that if $\Phi_0 \in (1-\delta, 1)$ and $q_1 \in (0, (1-\theta_0)/2)$ then $f(\Phi_0) \leqslant f(\Phi_0 - \delta)$. Hence we have in this range of Φ_0:

$$G[\psi_l] \leqslant \dot{\xi}_1\Phi_0' + v\Phi_0'. \tag{121}$$

Furthermore, if $\Phi_0 \in (0,\delta)$ then $f(\Phi_0) = f(\Phi_0 - \delta) = 0$ and hence in this range of Φ_0 we have (121) with the equality sign. Finally, if $\Phi_0 \in (\delta, 1-\delta)$ then $|f(\Phi_0) - f(\Phi_0 - q)| \leqslant K|q|$ and $\Phi_0' \leqslant -\beta$. Hence $G[\psi_l] \leqslant 0$ everywhere provided that

$$\dot{\xi}_1(t) \geqslant \|v(t)\|_\infty + \frac{K\|q(t)\|_\infty}{\beta}. \tag{122}$$

Thus choose

$$\xi_1(t) = \overline{U}(t)t + C\sqrt{t}. \tag{123}$$

Therefore we obtain a lower bound for T:

$$T(t,x,z) \geqslant \Phi_0(x - c_0 t + \overline{U}(t)t + C\sqrt{t}) - q_1(t,x,z). \tag{124}$$

In order to obtain an upper bound we set $\psi_u = \Phi_0(x - c_0 t - x_2 - \xi_2(t)) + q_2(t,x,z)$ and look for $\xi_2(t)$ and $q_2(t,x,z)$ so that $G[\psi_u] \geqslant 0$. The constant x_2 is chosen so that

$$T_0(x,z) \leqslant \Phi_0(x - x_2) + q_2(0,x,z)$$

with $q_2(0,x,z) \in L^1(D)$ and $0 \leqslant q_2(0,x,z) \leqslant \theta_0/2$, as with $q_1(0,x,z)$. The function $q_2(t,x,z)$ is then chosen to satisfy the same advection–diffusion equation (119) similarly to q_1. Hence it obeys the same time decay bounds as q_1. With the above choice of q_2 we have

$$G(\psi_u) = -\dot{\xi}_2\Phi_0' + v\Phi_0' + f(\Phi_0) - f(\Phi_0 + q_2).$$

Once again, we consider three regions of values for Φ_0. First, if $1 - \delta \leqslant \Phi_0 \leqslant 1$ with a sufficiently small $\delta > 0$ then $f(\Phi_0) - f(\Phi_0 + q_2) \geqslant 0$, as $q_2 \geqslant 0$. Hence $G[\psi_u] \geqslant 0$ in this region provided that $\dot{\xi}_2 \geqslant 0$. Second, as $q_2 \leqslant \theta_0/2$ we have $f(\Phi_0) = f(\Phi_0 + q_2) = 0$ if $0 \leqslant \Phi_0 \leqslant \delta$ with a sufficiently small $\delta > 0$. Hence $G[\psi_u] \geqslant 0$ in that region under the same condition $\dot{\xi}_2 \geqslant 0$. Finally, if $\Phi_0 \in (\delta, 1-\delta)$ then $\Phi_0' \leqslant -\beta$ with $\beta > 0$ and $|f(\Phi_0) - f(\Phi_0 + q_2)| \leqslant K\|q_2\|_\infty$. That means that $G[\psi_u] \geqslant 0$ if we choose ξ_2 so that

$$\dot{\xi}_2 \geqslant \big\|v(t)\big\|_\infty + \frac{K\|q_2\|_\infty}{\beta}.$$

Therefore we choose

$$\xi_2(t) = \overline{U}(t)t + C\sqrt{t},$$

as with $\xi_1(t)$. Then, we obtain upper and lower bounds

$$\Phi_0\big(x - c_0 t + \xi_1(t) + x_1\big) - q_1(t,x,z) \leqslant T(t,x,z) \leqslant \Phi_0\big(x - c_0 t - \xi_2(t) - x_2\big) + q_2(t,x,z) \tag{125}$$

that imply in particular that

$$\Phi_0\big(x - c_0 t + \overline{U}(t)t + C_0[1 + \sqrt{t}]\big) - \frac{C_0}{\sqrt{t}} \leqslant T(t,x,z) \leqslant \Phi_0\big(x - c_0 t - \overline{U}(t)t - C_0[1 + \sqrt{t}]\big) + \frac{C_0}{\sqrt{t}} \tag{126}$$

with a constant C_0 determined by the initial conditions. Hence, using (125)–(126) and the L^1-bounds

$$\big\|q_j(t)\big\|_{L^1(D)} \leqslant C_0, \quad j = 1, 2,$$

we obtain

$$\begin{aligned} R(t) &= \int T(1-T)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda} = \int\limits_{-\infty}^{c_0 t - \xi_2(t) - x_2} \int T(1-T)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} + \int\limits_{c_0 t - \xi_2(t) - x_2}^{c_0 t + \xi_1(t) + x_1} \int T(1-T)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} \\ &\quad + \int\limits_{c_0 t + \xi_1(t) + x_1}^{\infty} \int T(1-T)\frac{\mathrm{d}z\,\mathrm{d}x}{\lambda} \leqslant C_0 + \int\limits_{-\infty}^{0} (1 - \Phi_0)\,\mathrm{d}x + \big(\xi_1(t) + \xi_2(t)\big) + \int\limits_{0}^{\infty} \Phi_0(x)\,\mathrm{d}x \\ &\leqslant C_0(1 + \sqrt{t}) + 2t\overline{U}(t). \end{aligned}$$

This together with (118) implies that

$$\overline{V}(t) + 2\overline{U}(t) + C_0\left[\frac{1}{t} + \frac{1}{\sqrt{t}}\right] \geqslant 2\overline{N}(t) \tag{127}$$

so that (115) holds.

Moreover, we have

$$\begin{aligned}\overline{V}(t) &= \frac{1}{t}\int_0^t\left(\int f\big(T(s,x,z)\big)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}\right)\mathrm{d}s = \frac{1}{t}\int_0^t\left(\int T_t(s,x,z)\right)\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda}\,\mathrm{d}s \\ &= \frac{1}{t}\int\big[T(t,x,z)-T_0(x,z)\big]\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda} \\ &\leqslant \frac{1}{t}\int\big[\Phi_0\big(x-c_0t-x_2-\xi_2(t)\big)+q_2(t,x,z)-\Phi_0(x+x_1)+q_1(0,x,z)\big]\frac{\mathrm{d}x\,\mathrm{d}z}{\lambda} \\ &\leqslant \frac{C_0}{t}+\frac{1}{t}\int_{-\infty}^{c_0t+\xi_2(t)+x_2}\big(1-\Phi_0(x)\big)\,\mathrm{d}x+\frac{1}{t}\int_0^\infty\Phi_0(x)\,\mathrm{d}x \leqslant c_0+\overline{U}(t)+C_0\left[\frac{1}{\sqrt{t}}+\frac{1}{t}\right]\end{aligned} \tag{128}$$

as follows from (125), (126). This proves (116) and finishes the proof of Lemma 14. □

On the other hand we have the following upper bound for $\overline{U}(t)$ in terms of $\overline{N}(t)$.

Lemma 16. *There exists a constant $C>0$ so that for all $t>0$ the following inequality holds*

$$\overline{U}(t) \leqslant C\left[\rho\sqrt{\overline{N}(t)}+\frac{1}{\sqrt{t}}\|\omega_0\|_{L^2}\right]. \tag{129}$$

Proof. We multiply the vorticity equation

$$\frac{\partial\omega}{\partial t}+\mathbf{u}\cdot\nabla\omega-\sigma\omega=\rho(\hat{\mathbf{e}}\cdot\nabla^\perp T)$$

by ω and integrate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big|\omega(t,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z+\sigma\int\big|\nabla\omega(t,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z=\rho\int\omega(t,x,z)(\hat{\mathbf{e}}^\perp\cdot\nabla T)\,\mathrm{d}x\,\mathrm{d}z, \tag{130}$$

with $\hat{\mathbf{e}}^\perp=(e_2,-e_1)$.

The Poincaré inequality for $\omega(t,x,z)$ implies then that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big|\omega(t,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z+\frac{\sigma}{2}\int\big|\nabla\omega(t,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z\leqslant C\rho^2\int\big|\nabla T(t,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z.$$

Integrating this equation in time we conclude that

$$\frac{1}{t}\int_0^t\int\big|\nabla\omega(s,x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}s\leqslant C\left[\rho^2\overline{N}(t)+\frac{1}{t}\|\omega_0\|_{L^2}^2\right]. \tag{131}$$

However, as in the proof of Lemma 3, we have $\|v(t)\|_{L^\infty(D)}\leqslant C\|\nabla\omega(t)\|_{L^2(D)}$. This, together with (131) implies (129). □

Putting the bounds (129) and (115), (116) together and using the Cauchy–Schwartz inequality we arrive at

$$2\overline{N}(t)\leqslant c_0+C_0\left[\frac{1}{t}+\frac{1}{\sqrt{t}}\right]+\frac{C\rho}{t}\int_0^t\sqrt{N(s)}\,\mathrm{d}s\leqslant c_0+C_0\left[\frac{1}{t}+\frac{1}{\sqrt{t}}\right]+C\rho\sqrt{\overline{N}(t)}.$$

Hence we obtain an upper bound

$$\overline{N}(t)\leqslant\left[C\rho+\sqrt{\frac{c_0}{2}+C^2\rho^2}\right]^2+\mathrm{o}(1). \tag{132}$$

This, together with (129) implies that

$$\overline{U}(t)\leqslant C\rho\big[1+\rho\big]+\mathrm{o}(1).$$

It follows then from (128) that

$$\overline{V}(t) \leqslant c_0 + C\rho\big[1+\rho\big] + \mathrm{o}(1).$$

The lower bound on $\overline{V}(t)$ in (114) is proved similarly. This finishes the proof of Theorem 4. It remains only to prove Lemma 15. □

Proof of Lemma 15. We will show that there exists a universal constant $C > 0$ so that the solution of

$$\begin{aligned}&\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi = \sigma\Delta\psi,\\ &\psi(0,x,z) = \psi_0(x,z) \geqslant 0,\end{aligned} \tag{133}$$

with the Neumann boundary conditions at $z = 0$ and $z = \lambda$, and $\mathbf{u}$ sufficiently regular, satisfies

$$\big\|\psi(t)\big\|_{L^\infty(D)} \leqslant Cn^2(t)\|\psi_0\|_{L^1(D)}. \tag{134}$$

Here $n(t)$ is the unique solution of

$$\frac{n^4(t)}{1+n^3(t)\lambda^3} = \frac{C}{\sigma\lambda^2 t}. \tag{135}$$

We multiply (133) by ψ and integrate over the domain D to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_2^2 = -\sigma\|\nabla\psi\|_2^2. \tag{136}$$

We now prove the following version of the Nash inequality [29] for a strip of width λ in two dimensions:

$$\|\nabla\psi\|_2^2 \geqslant C\frac{\lambda^2\|\psi\|_2^6}{\|\psi\|_1^4 + \lambda^3\|\psi\|_1\|\psi\|_2^3}. \tag{137}$$

The proof of (137) is similar to that of the usual Nash inequality. We represent ψ in terms of its Fourier series-integral:

$$\psi(x,z) = \sum_{n\geqslant 0}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}kx}\cos\left(\frac{\pi nz}{\lambda}\right)\hat{\psi}_n(k)\frac{\mathrm{d}k}{2\pi},$$

where

$$\hat{\psi}_n(k) = \frac{2}{\lambda}\int_D \mathrm{e}^{-\mathrm{i}kx}\cos\left(\frac{\pi nz}{\lambda}\right)\psi(x,z)\,\mathrm{d}x\,\mathrm{d}z.$$

Therefore we have

$$\big|\hat{\psi}_n(k)\big| \leqslant \frac{2}{\lambda}\|\psi\|_{L^1}. \tag{138}$$

The Plancherel formula becomes

$$\begin{aligned}\int_D \big|\psi(x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z &= \sum_{n,m\geqslant 0}\int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}kx-\mathrm{i}px}\cos\left(\frac{\pi nz}{\lambda}\right)\cos\left(\frac{\pi mz}{\lambda}\right)\hat{\psi}_n(k)\hat{\psi}_m^*(p)\frac{\mathrm{d}k\,\mathrm{d}p\,\mathrm{d}x\,\mathrm{d}z}{(2\pi)^2}\\ &= \frac{\lambda}{2}\sum_{n\geqslant 0}\int \big|\hat{\psi}_n(k)\big|^2\frac{\mathrm{d}k}{2\pi}\end{aligned}$$

and similarly

$$\int_D \big|\nabla\psi(x,z)\big|^2\,\mathrm{d}x\,\mathrm{d}z = \frac{\lambda}{2}\sum_{n\geqslant 0}\int_{\mathbb{R}}\left(k^2 + \frac{\pi^2 n^2}{\lambda^2}\right)\big|\hat{\psi}_n(k)\big|^2\frac{\mathrm{d}k}{2\pi}.$$

Let $\rho > 0$ be a positive number to be chosen later. Then using the above Plancherel formula we write

$$\|\psi\|_2^2 = I + II,$$

with the first term that is bounded using (138)

$$I = \frac{\lambda}{2} \sum_{0 \leqslant n \leqslant \rho\lambda} \int_{|k| \leqslant \rho} \left|\hat{\psi}_n(k)\right|^2 \frac{\mathrm{d}k}{2\pi} \leqslant \frac{C\lambda\rho([\lambda\rho]+1)}{\lambda^2} \|\psi\|_1^2 \leqslant \frac{C\rho(\lambda\rho+1)}{\lambda} \|\psi\|_1^2.$$

The rest is bounded by

$$II \leqslant \frac{C\lambda}{\rho^2} \sum_{n \geqslant 0} \int_{k \in \mathbb{R}} \left(k^2 + \frac{4\pi^2 n^2}{\lambda^2}\right) \left|\hat{\psi}_n(k)\right|^2 \mathrm{d}k \leqslant \frac{C}{\rho^2} \|\nabla\psi\|_2^2.$$

Therefore we have for all $\rho > 0$:

$$\|\psi\|_2^2 \leqslant \frac{C\rho(\lambda\rho+1)}{\lambda} \|\psi\|_1^2 + \frac{C}{\rho^2} \|\nabla\psi\|_2^2.$$

We choose ρ so that

$$\rho^3 = \frac{\lambda \|\nabla\psi\|_2^2}{\|\psi\|_1^2}$$

and obtain

$$\begin{aligned} \|\psi\|_2^2 &\leqslant \frac{C\|\nabla\psi\|_2^{2/3}}{\lambda^{2/3}\|\psi\|_1^{2/3}} \left(\frac{\lambda^{4/3}\|\nabla\psi\|_2^{2/3}}{\|\psi\|_1^{2/3}} + 1\right) \|\psi\|_1^2 + \frac{C\|\nabla\psi\|_2^2 \|\psi\|_1^{4/3}}{\lambda^{2/3}\|\nabla\psi\|_2^{4/3}} \\ &= \frac{C}{\lambda^{2/3}} \|\psi\|_1^{4/3} \|\nabla\psi\|_2^{2/3} + C\lambda^{2/3} \|\nabla\psi\|_2^{4/3} \|\psi\|_1^{2/3}. \end{aligned}$$

This is a quadratic inequality $ax^2 + bx - c \geqslant 0$ with $x = \|\nabla\psi\|_2^{2/3}$, $a = C\lambda^{2/3}\|\psi\|_1^{2/3}$, $b = (C/\lambda^{2/3})\|\psi\|_1^{4/3}$, and $c = \|\psi\|_2^2$ and hence

$$x \geqslant \frac{-b + \sqrt{b^2 + 4ac}}{2a} = \frac{2c}{b + \sqrt{b^2 + 4ac}} \geqslant \frac{c}{\sqrt{b^2 + 4ac}}.$$

This implies that

$$\|\nabla\psi\|_2^{2/3} \geqslant C\|\psi\|_2^2 \left(\frac{\|\psi\|_1^{8/3}}{\lambda^{4/3}} + \lambda^{2/3} \|\psi\|_1^{2/3} \|\psi\|_2^2\right)^{-1/2}$$

and therefore

$$\begin{aligned} \|\nabla\psi\|_2^2 &\geqslant C\|\psi\|_2^6 \left(\frac{4\|\psi\|_1^{8/3}}{\lambda^{4/3}} + 4\lambda^{2/3} \|\psi\|_1^{2/3} \|\psi\|_2^2\right)^{-3/2} \geqslant C\|\psi\|_2^6 \left(\frac{\|\psi\|_1^4}{\lambda^2} + \lambda \|\psi\|_1 \|\psi\|_2^3\right)^{-1} \\ &\geqslant \frac{C\lambda^2 \|\psi\|_2^6}{\|\psi\|_1^4 + \lambda^3 \|\psi\|_1 \|\psi\|_2^3}. \end{aligned}$$

Hence (137) indeed holds.

We insert (137) into (136) and use the conservation of the L^1-norm of ψ (recall that the initial data is non-negative) obtain

$$\frac{\mathrm{d}\|\psi\|_2}{\mathrm{d}t} \leqslant -\frac{C\sigma\lambda^2 \|\psi\|_2^5}{\|\psi_0\|_1^4 + \lambda^3 \|\psi_0\|_1 \|\psi\|_2^3}. \tag{139}$$

Integrating (139) in time we have

$$C\sigma\lambda^2 t \leqslant \frac{\|\psi_0\|_1^4}{\|\psi\|_2^4} + \frac{\lambda^3 \|\psi_0\|_1}{\|\psi\|_2} \leqslant \frac{1}{z(t)} \left[\lambda^3 + \frac{1}{z^3(t)}\right],$$

where $z(t) = \|\psi(t)\|_2 / \|\psi_0\|_1$, and thus

$$\frac{z^4(t)}{1 + \lambda^3 z^3(t)} \leqslant \frac{1}{C\sigma\lambda^2 t}. \tag{140}$$

The function on the left side of (140) is monotonically increasing and hence we have

$$\|\psi(t)\|_2 \leqslant n(t)\|\psi_0\|_1, \tag{141}$$

where $n(t)$ is the solution of (135).

Let us denote by $\mathcal{P}_t$ the solution operator for (133): $\psi(t) = \mathcal{P}_t\psi_0$. Then (141) implies that $\|\mathcal{P}_t\|_{L^1\to L^2} \leqslant n(t)$. The adjoint operator $\mathcal{P}_t^*$ is the solution operator for

$$\begin{aligned} &\frac{\partial\tilde\psi}{\partial t} - \mathbf{u}\cdot\nabla\tilde\psi = \sigma\Delta\tilde\psi, \\ &\tilde\psi(0,x) = \tilde\psi_0(x), \quad x \in D \end{aligned} \tag{142}$$

with the Neumann boundary conditions at $z = 0, \lambda$. Note that the preceding estimates rely only on the anti-symmetry of the convection operator $\mathbf{u}\cdot\nabla$. Therefore we have the bound $\|\mathcal{P}_t^*\|_{L^1\to L^2} \leqslant n(t)$ and hence $\|\mathcal{P}_t\|_{L^2\to L^\infty} \leqslant n(t)$ so that

$$\|\psi(t)\|_{L^\infty} \leqslant n(t/2)\|\psi(t/2)\|_{L^2} \leqslant n^2(t/2)\|\psi_0\|_{L^1}$$

and thus (134) indeed holds.

The estimate (120) follows from the observation that for large $t \gg 1$, when $n(t)$ is small, we have the bound

$$n^2(t) \leqslant \frac{C}{(\sigma t)^{1/2}\lambda}.$$

This finishes the proof of Lemma 15. □

4.2. *Bounds on the burning rate in a narrow domain*

We recall that no non-planar traveling fronts do exist in the reactive Boussinesq problem in a narrow vertical strip when gravity is sufficiently small [32,33,14]. Moreover, solutions with general front-like initial data become asymptotically planar in the long time limit [14]. We extend now this result to the inclined cylinders. More precisely, we have the following result (this is a re-statement of Theorem 3).

Theorem 5. *Let $\hat{\mathbf{e}} = (e_1, e_2)$ be the unit vector in the direction of gravity and let $\rho_j = \rho e_j$, $j = 1, 2$. There exist two constants λ_0 and ρ_0 so that if the domain is sufficiently narrow: $\lambda \leqslant \lambda_0$ and gravity is sufficiently small: $\rho \leqslant \rho_0$ then the burning rate is bounded by*

$$\overline{V}(t) \leqslant c_0 + C\rho_2 + \mathrm{o}(1) \quad \text{as } t \to +\infty. \tag{143}$$

Moreover, the front is nearly planar in the sense that

$$\overline{N}_z(t) = \frac{1}{t}\int_0^t \|T_z(s)\|_2^2\,\mathrm{d}s \leqslant C\rho_2^2 + \mathrm{o}(1) \quad \text{as } t \to +\infty. \tag{144}$$

The key point in Theorem 5 is that the bounds in (143) and (144) are independent of the gravity strength ρ_1 in the direction parallel to the cylinder.

Proof. Multiplying the vorticity equation by ω and integrating by parts we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega|^2\,\mathrm{d}x\,\mathrm{d}z + \sigma\int|\nabla\omega|^2\,\mathrm{d}x\,\mathrm{d}z = \rho_2\int T_x\omega\,\mathrm{d}x\,\mathrm{d}z - \rho_1\int T_z\omega\,\mathrm{d}x\,\mathrm{d}z. \tag{145}$$

The Poincaré inequality applies to $\omega(x,z)$ with the Poincaré constant proportional to $1/\lambda$. Hence, if $\lambda < \lambda_0$ and $\rho_j < \rho_0$, (145) implies that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\omega|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{\sigma}{2}\int|\nabla\omega|^2\,\mathrm{d}x\,\mathrm{d}z \leqslant C\rho_2^2\int|T_x|^2\,\mathrm{d}x\,\mathrm{d}z + C\rho_1^2\int|T_z|^2\,\mathrm{d}x\,\mathrm{d}z. \tag{146}$$

We now differentiate the equation for $T(t,x,z)$ in z to get

$$\frac{\partial T_z}{\partial t} + \mathbf{u}\cdot\nabla T_z + \mathbf{u}_z\cdot\nabla T = \Delta T_z + f'(T)T_z.$$

Multiplying this equation by T_z we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |T_z|^2\,\mathrm{d}x\,\mathrm{d}z + \int |\nabla T_z|^2\,\mathrm{d}x\,\mathrm{d}z + \int T_z\mathbf{u}_z\cdot\nabla T\,\mathrm{d}x\,\mathrm{d}z = \int f'(T)T_z^2 \leqslant M\int T_z^2\,\mathrm{d}x\,\mathrm{d}z. \tag{147}$$

The last integral on the left side is bounded by

$$\left|\int T_z\mathbf{u}_z\cdot\nabla T\,\mathrm{d}x\,\mathrm{d}z\right| = \left|\int T\mathbf{u}_z\cdot\nabla T_z\,\mathrm{d}x\,\mathrm{d}z\right| \leqslant 2\int |\mathbf{u}_z|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{2}|\nabla T_z|^2\,\mathrm{d}x\,\mathrm{d}z.$$

This, together with incompressibility of $\mathbf{u}$, the Poincaré inequality for T_z and (147) imply that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |T_z|^2\,\mathrm{d}x\,\mathrm{d}z + \frac{1}{4}\int |\nabla T_z|^2\,\mathrm{d}x\,\mathrm{d}z \leqslant 4\int |\omega|^2\,\mathrm{d}x\,\mathrm{d}z, \tag{148}$$

provided that $\lambda < \lambda_0$. Combining (146) and (148) and using the Poincaré inequality for ω and T_z once again, we obtain the following inequalities for $\Omega(t) = \|\omega(t)\|_2^2$ and $N_z(t) = \|T_z\|_2^2$:

$$\frac{1}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}t} + \frac{C}{\lambda^2}\Omega \leqslant C\rho_2^2 N_x(t) + C\rho_1^2 N_z(t)$$

and

$$\frac{1}{2}\frac{\mathrm{d}N_z}{\mathrm{d}t} + \frac{C}{\lambda^2}N_z \leqslant 4\Omega.$$

Hence, the function $Q = N_z + \Omega$ satisfies

$$\frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}t} + \left[\frac{C}{\lambda^2} - 4 - C\rho_1^2\right]Q \leqslant C\rho_2^2 N_x(t).$$

Therefore, we have

$$Q(t) \leqslant Q_0\,\mathrm{e}^{-\gamma t} + C\rho_2^2\int_0^t \mathrm{e}^{-\gamma(t-s)}N_x(s)\,\mathrm{d}s$$

with $\gamma > 0$ provided that $C/\lambda^2 > 5$ and $C\rho_1^2 < 1$. We conclude that

$$\begin{aligned}\bar{Q}(t) := \frac{1}{t}\int_0^t Q(\tau)\,\mathrm{d}\tau &\leqslant \frac{Q_0}{t}\int_0^t \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau + \frac{C\rho_2}{t}\int_0^t\int_0^\tau \mathrm{e}^{-\gamma(\tau-s)}N_x(s)\,\mathrm{d}s\,\mathrm{d}\tau \\ &\leqslant \frac{C_0}{t} + \frac{C\rho_2^2}{t}\int_0^t N_x(s)\,\mathrm{e}^{\gamma s}\int_s^t \mathrm{e}^{-\gamma\tau}\,\mathrm{d}\tau\,\mathrm{d}s \leqslant \frac{C_0}{t} + \frac{C\rho_2^2}{\gamma}\overline{N}_x(t) \leqslant C\rho_2^2 + \frac{C_0}{t}.\end{aligned}$$

The last inequality above follows from the bound on $\overline{N}(t)$ in Theorem 4. Now, the bound (144) in Theorem 5 follows. Then, (146) together with (144) and the same uniform bound on $\overline{N}(t)$ in Theorem 4 yield

$$\frac{1}{t}\int_0^t \|\nabla\omega(s)\|_2^2\,\mathrm{d}s \leqslant C\rho_2^2 + \frac{C_0}{t}.$$

We recall that $\|v(t)\|_{L^\infty(D)} \leqslant C\|\nabla\omega\|_{L^2(D)}$ – this, together with the above, show that

$$\overline{U}(t) \leqslant C\rho_2 + \frac{C_0}{\sqrt{t}}. \tag{149}$$

Finally, using (128) and (149) we obtain (143). □

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