# The space $B V\left(S^{2}, S^{1}\right)$ : minimal connection and optimal lifting 

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## Abstract

We show that topological singularities of maps in $B V\left(S^{2}, S^{1}\right)$ can be detected by its distributional Jacobian. As an application, we construct an optimal lifting and we compute its total variation.
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## Résumé

On montre que le jacobien d'une fonction $u \in B V\left(S^{2}, S^{1}\right)$ permet de localiser les singularités topologiques de $u$. On applique ce résultat à la construction d'un relèvement optimal et on calcule sa variation totale.
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## 1. Introduction

Let $u \in B V\left(S^{2}, S^{1}\right)$, i.e. $u=\left(u_{1}, u_{2}\right) \in L^{1}\left(S^{2}, \mathbb{R}^{2}\right),|u(x)|=1$ for a.e. $x \in S^{2}$ and the derivative of $u$ (in the sense of the distributions) is a finite $2 \times 2$-matrix Radon measure

$$
\int_{S^{2}}|D u|=\sup \left\{\int_{S^{2}} \sum_{k=1}^{2} u_{k} \operatorname{div} \zeta_{k} \mathrm{~d} \mathcal{H}^{2}: \zeta_{k} \in C^{1}\left(S^{2}, \mathbb{R}^{2}\right), \sum_{k=1}^{2}\left|\zeta_{k}(x)\right|^{2} \leqslant 1, \forall x \in S^{2}\right\}<\infty
$$

where the norm in $\mathbb{R}^{2}$ is the Euclidean norm. Observe that the total variation of $D u$ is independent of the choice of the orthonormal frame $(x, y)$ on $S^{2}$; a frame $(x, y)$ is always taken such that $(x, y, e)$ is direct, where $e$ is the outward normal to the sphere $S^{2}$.

[^0]We begin with the notion of minimal connection between point singularities of $u$. The concept of a minimal connection associated to a function from $\mathbb{R}^{3}$ into $S^{2}$ was originally introduced by Brezis, Coron and Lieb [3]. Following the ideas in [3] and [6], Brezis, Mironescu and Ponce [4] studied the topological singularities of functions $g \in W^{1,1}\left(S^{2}, S^{1}\right)$. They show that the distributional Jacobian of $g$ describes the location and the topological charge of the singular set of $g$. More precisely, let $T(g) \in \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$ be defined as

$$
T(g)=2 \operatorname{det}(\nabla g)=-\left(g \wedge g_{x}\right)_{y}+\left(g \wedge g_{y}\right)_{x}
$$

then there exist two sequences of points $\left(p_{k}\right),\left(n_{k}\right)$ in $S^{2}$ such that

$$
\sum_{k}\left|p_{k}-n_{k}\right|<\infty \quad \text { and } \quad T(g)=2 \pi \sum_{k}\left(\delta_{p_{k}}-\delta_{n_{k}}\right)
$$

Our aim is to extend these notions for functions $u \in B V\left(S^{2}, S^{1}\right)$. In this case, the difficulty of the analysis of the singular set arises from the existence of more than one type of singularity: besides the point singularities carrying a degree, the jump singularities of $u$ should be taken into account.

We start by introducing some notation. Write the finite Radon $2 \times 2$-matrix measure $D u$ as

$$
D u=D^{a} u+D^{c} u+D^{j} u
$$

where $D^{a} u, D^{c} u$ and $D^{j} u$ are the absolutely continuous part, the Cantor part and the jump part of $D u$ (see e.g. [1]). We recall that $D^{j} u$ can be written as

$$
D^{j} u=\left(u^{+}-u^{-}\right) \otimes v_{u} \mathcal{H}^{1}\llcorner S(u)
$$

where $S(u)$ denotes the set of jump points of $u ; S(u)$ is a countably $\mathcal{H}^{1}$-rectifiable set on $S^{2}$ oriented by the Borel map $v_{u}: S(u) \rightarrow S^{1}$. The Borel functions $u^{+}, u^{-}: S(u) \rightarrow S^{1}$ are the traces of $u$ on the jump set $S(u)$ with respect to the orientation $v_{u}$. Throughout the paper we identify $u$ by its precise representative that is defined $\mathcal{H}^{1}$-a.e. in $S^{2} \backslash S(u)$.

We now introduce the distribution $T(u) \in \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$ as

$$
\begin{equation*}
\langle T(u), \zeta\rangle=\int_{S^{2}} \nabla^{\perp} \zeta \cdot\left(u \wedge\left(D^{a} u+D^{c} u\right)\right)+\int_{S(u)} \rho\left(u^{+}, u^{-}\right) v_{u} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}, \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right) \tag{1}
\end{equation*}
$$

Here, $\nabla^{\perp} \zeta=\left(\zeta_{y},-\zeta_{x}\right)$,

$$
\binom{u_{1}}{u_{2}} \wedge\left(\begin{array}{ll}
a_{1} & b_{1} \\
a_{2} & b_{2}
\end{array}\right)=(u \wedge a, u \wedge b)=\left(u_{1} a_{2}-u_{2} a_{1}, u_{1} b_{2}-u_{2} b_{1}\right)
$$

where $a=\binom{a_{1}}{a_{2}}$ and $b=\binom{b_{1}}{b_{2}}$. The function $\rho(\cdot, \cdot): S^{1} \times S^{1} \rightarrow[-\pi, \pi]$ is the signed geodesic distance on $S^{1}$ defined as

$$
\rho\left(\omega_{1}, \omega_{2}\right)=\left\{\begin{array}{ll}
\operatorname{Arg}\left(\frac{\omega_{1}}{\omega_{2}}\right) & \text { if } \frac{\omega_{1}}{\omega_{2}} \neq-1 \\
\operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right) & \text { if } \frac{\omega_{1}}{\omega_{2}}=-1
\end{array} \quad \forall \omega_{1}, \omega_{2} \in S^{1}\right.
$$

where $\operatorname{Arg}(\omega) \in(-\pi, \pi]$ stands for the argument of the unit complex number $\omega \in S^{1} . T(u)$ represents the distributional determinant of the absolutely continuous part and the Cantor part of $D u$ which is adjusted on $S(u)$ by the tangential derivative of $\rho\left(u^{+}, u^{-}\right)$. The second term in the RHS of (1) is motivated by the study of $B V\left(S^{1}, S^{1}\right)$ functions (see [9]): we defined there a similar quantity that represents a pseudo-degree for $B V\left(S^{1}, S^{1}\right)$ functions.

Remark 1. (i) The integrand in (1) is computed pointwise in any orthonormal frame ( $x, y$ ) and the corresponding quantity is frame-invariant.
(ii) The 2-vector measure

$$
\mu=\left(\mu_{1}, \mu_{2}\right)=u \wedge\left(D^{a} u+D^{c} u\right)=\left(u \wedge\left(\left(u_{x}\right)^{a}+\left(u_{x}\right)^{c}\right), u \wedge\left(\left(u_{y}\right)^{a}+\left(u_{y}\right)^{c}\right)\right)
$$

is well-defined since $D^{a} u+D^{c} u$ vanishes on sets which are $\sigma$-finite with respect to $\mathcal{H}^{1}$.
(iii) Notice that the function $\rho$ is antisymmetric, i.e.

$$
\rho\left(\omega_{1}, \omega_{2}\right)=-\rho\left(\omega_{2}, \omega_{1}\right), \quad \forall \omega_{1}, \omega_{2} \in S^{1}
$$

and therefore, $T(u)$ does not depend of the choice of the orientation $v_{u}$ on the jump set $S(u)$. By Lemma 5 (see below), we obtain

$$
|\langle T(u), \zeta\rangle| \leqslant|u|_{B V S^{1}}, \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right) \text { with }|\nabla \zeta| \leqslant 1
$$

where $|u|_{B V S^{1}}=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S(u)} d_{S^{1}}\left(u^{+}, u^{-}\right) \mathrm{d} \mathcal{H}^{1}$ and $d_{S^{1}}$ stands for the geodesic distance on $S^{1}$. Therefore, $T(u)$ is indeed a distribution (of order 1) on $S^{2}$.

For a compact Riemannian manifold $X$ with the induced distance $d$, define

$$
\mathcal{Z}(X)=\left\{\Lambda \in\left[C^{1}(X)\right]^{*}: \exists\left(p_{k}\right),\left(n_{k}\right) \subset X, \sum_{k} d\left(p_{k}, n_{k}\right)<\infty \text { and } \Lambda=2 \pi \sum_{k}\left(\delta_{p_{k}}-\delta_{n_{k}}\right)\right\} .
$$

$\mathcal{Z}(X)$ is the set of distributions that can be written as a countable sum of dipoles.
Remark 2. (i) In general, $\Lambda \in \mathcal{Z}(X)$ is not a measure. In fact, it can be shown that $\Lambda$ is a measure if and only if $\Lambda$ is a finite sum of dipoles (see Smets [11] and also Ponce [10]).
(ii) $\Lambda \in \mathcal{Z}(X)$ has always infinitely many representations as a sum of dipoles and these representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_{p}-\delta_{n}$ may be represented as $\delta_{p}-\delta_{n_{1}}+\sum_{k \geqslant 1}\left(\delta_{n_{k}}-\delta_{n_{k+1}}\right)$ for any sequence $\left(n_{k}\right)$ rapidly converging to $n$.

For each $\Lambda \in \mathcal{Z}(X)$, the length of a minimal connection between the singularities is defined as

$$
\|\Lambda\|=\sup _{\substack{\zeta \in C^{1}(X) \\|\nabla \zeta| \leqslant 1}}\langle\Lambda, \zeta\rangle
$$

For example, when $\Lambda=2 \pi \sum_{k=1}^{m}\left(\delta_{p_{k}}-\delta_{n_{k}}\right)$ is a finite sum of dipoles, Brezis, Coron and Lieb [3] showed that

$$
\|\Lambda\|=2 \pi \min _{\sigma \in S_{m}} \sum_{k=1}^{m} d\left(p_{k}, n_{\sigma(k)}\right)
$$

where $S_{m}$ denotes the group of permutation of $\{1,2, \ldots, m\}$. In general, for an arbitrary $\Lambda \in \mathcal{Z}(X)$, Bourgain, Brezis and Mironescu [2] proved the following characterization of the length of a minimal connection:

$$
\begin{equation*}
\|\Lambda\|=\inf _{\left(p_{k}\right),\left(n_{k}\right)}\left\{2 \pi \sum_{k} d\left(p_{k}, n_{k}\right): \Lambda=2 \pi \sum_{k}\left(\delta_{p_{k}}-\delta_{n_{k}}\right) \text { and } \sum_{k} d\left(p_{k}, n_{k}\right)<\infty\right\} . \tag{2}
\end{equation*}
$$

From (2), one can deduce that $\mathcal{Z}(X)$ is a complete metric space with respect to the distance induced by $\|\cdot\|$ (see e.g. [10]).

Our first theorem states that $T(u)$ is a countable sum of dipoles. It is the extension to the $B V$ case of the result in [4] mentioned in the beginning.

Theorem 1. For every $u \in B V\left(S^{2} ; S^{1}\right)$, we have $T(u) \in \mathcal{Z}\left(S^{2}\right)$, i.e. there exist $\left(p_{k}\right),\left(n_{k}\right)$ in $S^{2}$ such that

$$
\sum_{k}\left|p_{k}-n_{k}\right|<\infty \quad \text { and } \quad T(u)=2 \pi \sum_{k}\left(\delta_{p_{k}}-\delta_{n_{k}}\right)
$$

The proof relies on the fact that the derivative (in the sense of distributions) of the characteristic function of a bounded measurable set in $\mathbb{R}$ can be written as a sum of differences between Dirac masses:

Lemma 1. Let $I \subset \mathbb{R}$ be a compact interval and $f: I \rightarrow 2 \pi \mathbb{Z}$ be an integrable function. Define

$$
\left\langle\frac{\mathrm{d} f}{\mathrm{~d} t}, \zeta\right\rangle:=-\int_{I} f(t) \zeta^{\prime}(t) \mathrm{d} t, \quad \forall \zeta \in C^{1}(I) .
$$

Then

$$
\frac{\mathrm{d} f}{\mathrm{~d} t} \in \mathcal{Z}(I) \quad \text { and } \quad\left\|\frac{\mathrm{d} f}{\mathrm{~d} t}\right\|=\int_{I}|f| \mathrm{d} t
$$

The same property is valid for the distributional tangential derivative of an integrable function taking values in $2 \pi \mathbb{Z}$ and defined on a $C^{1} 1$-graph (see Remark 3). Since every countably $\mathcal{H}^{1}$-rectifiable set $S \subset S^{2}$ can be covered $\mathcal{H}^{1}$-a.e. by a sequence of $C^{1} 1$-graphs, it makes sense to define for every $\Lambda \in \mathcal{Z}\left(S^{2}\right)$ the set

$$
\begin{aligned}
& \mathcal{J}(\Lambda)=\left\{(f, S, v): S \text { is a countably } \mathcal{H}^{1} \text {-rectifiable set in } S^{2}, v \text { is an orientation on } S,\right. \\
& \\
& \left.\quad f \in L^{1}(S, 2 \pi \mathbb{Z}) \text { is such that } \int_{S} f v \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=\langle\Lambda, \zeta\rangle, \forall \zeta \in C^{1}\left(S^{2}\right)\right\} .
\end{aligned}
$$

We have the following reformulation of (2):
Lemma 2. For every $\Lambda \in \mathcal{Z}\left(S^{2}\right)$, we have

$$
\|\Lambda\|=\min _{(f, S, v) \in \mathcal{J}(\Lambda)} \int_{S}|f| \mathrm{d} \mathcal{H}^{1} .
$$

It is known that the infimum in (2) is not achieved in general (see [10]); the advantage of the above formula is that the minimum is always attained. It means that the length of $\Lambda$ represents the minimal mass that an $\mathcal{H}^{1}$-integrable function with values into $2 \pi \mathbb{Z}$ could carry between the dipoles of $\Lambda$.

In the sequel we are concerned with the lifting of $u \in B V\left(S^{2}, S^{1}\right)$. We call $B V$ lifting of $u$ every function $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ such that

$$
u=\mathrm{e}^{i \varphi} \quad \text { a.e. in } S^{2} .
$$

The existence of a $B V$ lifting for functions $u \in B V\left(S^{2}, S^{1}\right)$ was initially shown by Giaquinta, Modica and Souček [8]. Later, Dávila and Ignat [5] proved the existence of a lifting $\varphi \in B V \cap L^{\infty}\left(S^{2}, \mathbb{R}\right)$ such that

$$
\begin{equation*}
\int_{S^{2}}|D \varphi| \leqslant 2 \int_{S^{2}}|D u| \tag{3}
\end{equation*}
$$

moreover, the constant 2 in (3) is the best constant (see Example 1 and Proposition 3 below).
We give the following characterization for a lifting of $u$ :

Lemma 3. Let $u \in B V\left(S^{2}, S^{1}\right)$. For every lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$, there exists $(f, S, v) \in \mathcal{J}(T(u))$ such that

$$
\begin{equation*}
D \varphi=u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-f \nu \mathcal{H}^{1}\llcorner S .\right. \tag{4}
\end{equation*}
$$

Conversely, for every triple $(f, S, v) \in \mathcal{J}(T(u))$ there exists a lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ such that (4) holds.
In this framework, it is natural to investigate the quantity

$$
\begin{equation*}
E(u)=\inf \left\{\int_{S^{2}}|D \varphi|: \varphi \in B V\left(S^{2}, \mathbb{R}\right), \mathrm{e}^{\mathrm{i} \varphi}=u \text { a.e. in } S^{2}\right\} \tag{5}
\end{equation*}
$$

The infimum from above is achieved and it is equal to the relaxed energy

$$
\begin{equation*}
E_{\text {rel }}(u)=\inf \left\{\liminf _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla u_{k}\right| \mathrm{d} \mathcal{H}^{2}: u_{k} \in C^{\infty}\left(S^{2}, S^{1}\right), u_{k} \rightarrow u \text { a.e. in } S^{2}\right\} \tag{6}
\end{equation*}
$$

(see Remark 4). A lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ is called optimal if

$$
E(u)=\int_{S^{2}}|D \varphi| .
$$

An optimal lifting need not be unique (see Proposition 3). Remark also that for $u \in B V\left(S^{2}, S^{1}\right)$, there could be no optimal $B V$ lifting of $u$ that belongs to $L^{\infty}$ (see Example 3).

Our aim is to compute the total variation $E(u)$ of an optimal lifting and to construct an optimal lifting. Theorem 2 establishes the formula for $E(u)$ using the distribution $T(u)$.
Theorem 2. For every $u \in B V\left(S^{2}, S^{1}\right)$, we have

$$
\begin{equation*}
E(u)=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\min _{(f, S, v) \in \mathcal{J}(T(u))} \int_{S \cup S(u)}\left|f v \chi_{S}-\rho\left(u^{+}, u^{-}\right) v_{u} \chi_{S(u)}\right| \mathrm{d} \mathcal{H}^{1} . \tag{7}
\end{equation*}
$$

We refer the reader to [8] for related results in terms of Cartesian currents.
As a consequence of Theorem 2, we recover the result of Brezis, Mironescu and Ponce [4] about the total variation of an optimal $B V$ lifting for functions $g \in W^{1,1}\left(S^{2}, S^{1}\right)$ : the gap

$$
E(g)-\int_{S^{2}}|\nabla g| \mathrm{d} \mathcal{H}^{2}
$$

is equal to the length of a minimal connection connecting the topological singularities of $g$.
Corollary 1. For every $g \in W^{1,1}\left(S^{2}, S^{1}\right)$, we have

$$
E(g)=\int_{S^{2}}|\nabla g| \mathrm{d} \mathcal{H}^{2}+\|T(g)\| .
$$

From (7), we deduce an estimate for $E(u)$ (which is a weaker form of inequality (3)):
Corollary 2. For every $u \in B V\left(S^{2}, S^{1}\right)$, we have

$$
E(u) \leqslant 2|u|_{B V S^{1}}
$$

In the spirit of [4], we have the following interpretation of $\|T(u)\|$ as a distance:

Theorem 3. For every $u \in B V\left(S^{2}, S^{1}\right)$, we have

$$
\begin{equation*}
\|T(u)\|=\min _{\psi \in B V\left(S^{2}, \mathbb{R}\right)} \int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)-D \psi \mid \tag{8}
\end{equation*}
$$

Moreover, there is at least one minimizer $\psi \in B V\left(S^{2}, \mathbb{R}\right)$ of (8) that is a lifting of $u$.
Remark that in general, $\|T(u)\|$ is not the distance of the measure

$$
u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)
$$

to the class of gradient maps. In Example 4, we construct a function $u \in B V\left(S^{2}, S^{1}\right)$ such that

$$
\|T(u)\|<\inf _{\psi \in C^{\infty}\left(S^{2}, \mathbb{R}\right)} \int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)-D \psi \mid
$$

In Section 2, we present the proofs of Lemmas 1, 2 and 3, Theorems 1, 2 and 3 and Corollaries 1 and 2. Some examples and interesting properties of $T(u)$ are given in Section 3. Among other things, we show that $T: B V\left(S^{2}, S^{1}\right) \rightarrow \mathcal{Z}\left(S^{2}\right)$ is discontinuous and we analyze some algebraic properties of $T(u)$. We also discuss the meaning of the point singularities of $T(u)$ and about their location on $S^{2}$.

All the results included here can be easily adapted for functions in $B V\left(\Omega, S^{1}\right)$ where $\Omega$ is a more general simply connected Riemannian manifold of dimension 2.

## 2. Remarks and proofs of the main results

We start by proving Lemma 1 :
Proof of Lemma 1. Firstly, let us suppose that $f=2 \pi \chi_{A}$ where $A \subset I$ is an open set. Write $A=\bigcup_{j \in \mathbb{N}}\left(a_{j}, b_{j}\right)$ as a countable reunion of disjoint intervals. It is clear that

$$
\left\langle\frac{\mathrm{d} \chi_{A}}{\mathrm{~d} t}, \zeta\right\rangle=\sum_{j \in \mathbb{N}}\left(\zeta\left(a_{j}\right)-\zeta\left(b_{j}\right)\right), \quad \forall \zeta \in C^{1}(I)
$$

and $\sum_{j \in \mathbb{N}}\left(b_{j}-a_{j}\right)=\mathcal{H}^{1}(A)$. Thus $2 \pi \frac{\mathrm{~d} \chi_{A}}{\mathrm{~d} t} \in \mathcal{Z}(I)$ and

$$
\left\|\frac{\mathrm{d} f}{\mathrm{~d} t}\right\|=2 \pi \sup _{\substack{\zeta \in C^{1}(I) \\\left|\zeta^{\prime}\right| \leqslant 1}} \int_{I} \chi_{A} \zeta^{\prime} \mathrm{d} t=2 \pi \sup _{\substack{\psi \in C(I) \\|\psi| \leqslant 1}} \int_{I} \chi_{A} \psi \mathrm{~d} t=2 \pi \mathcal{H}^{1}(A)
$$

Moreover, let $A \subset I$ be a Lebesgue measurable set and $f=2 \pi \chi_{A}$. Using the regularity of the Lebesgue measure, there exists a decreasing sequence of open sets $A \subset A_{k+1} \subset A_{k} \subset I, k \in \mathbb{N}$ such that $\lim _{k \rightarrow \infty} \mathcal{H}^{1}\left(A_{k}\right)=\mathcal{H}^{1}(A)$. Observe that $\frac{\mathrm{d} \chi_{A_{k}}}{\mathrm{~d} t} \rightarrow \frac{\mathrm{~d} \chi_{A}}{\mathrm{~d} t}$ in $\left[C^{1}(I)\right]^{*}$. Since $\mathcal{Z}(I)$ is a complete metric space, we conclude that $2 \pi \frac{\mathrm{~d} \chi_{A}}{\mathrm{~d} t} \in \mathcal{Z}(I)$ and $\left\|2 \pi \frac{\mathrm{~d} \chi_{A}}{\mathrm{~d} t}\right\|=2 \pi \mathcal{H}^{1}(A)$. In the general case of an integrable function $f: I \rightarrow 2 \pi \mathbb{Z}$, write

$$
\begin{equation*}
f=2 \pi \sum_{k \in \mathbb{Z}} k \chi_{E_{k}} \quad \text { in } L^{1} \tag{9}
\end{equation*}
$$

where $E_{k}=\{x \in I: f(x)=2 \pi k\}$. Notice that $2 \pi \frac{\mathrm{~d}\left(k \chi_{E_{k}}\right)}{\mathrm{d} t} \in \mathcal{Z}(I)$ and the series $\sum_{k \in \mathbb{Z}} 2 \pi \frac{\mathrm{~d}\left(k \chi_{E_{k}}\right)}{\mathrm{d} t}$ converges ab-
solutely; indeed, we have

$$
\sum_{k \in \mathbb{Z}}\left\|2 \pi \frac{\mathrm{~d}\left(k \chi_{E_{k}}\right)}{\mathrm{d} t}\right\|=2 \pi \sum_{k \in \mathbb{Z}}|k| \mathcal{H}^{1}\left(E_{k}\right)=\int_{I}|f| \mathrm{d} t<\infty
$$

By (9), we conclude that $\frac{\mathrm{d} f}{\mathrm{~d} t} \in \mathcal{Z}(I)$ and

$$
\left\|\frac{\mathrm{d} f}{\mathrm{~d} t}\right\|=\sup _{\substack{\zeta \in C^{1}(I) \\\left|\zeta^{\prime}\right| \leqslant 1}} \int_{I} f \zeta^{\prime} \mathrm{d} t=\sup _{\substack{\psi \in C(I) \\|\psi| \leqslant 1}} \int_{I} f \psi \mathrm{~d} t=\int_{I}|f| \mathrm{d} t
$$

Remark 3. The conclusion of Lemma 1 is also true for $\mathcal{H}^{1}$-integrable functions with values in $2 \pi \mathbb{Z}$ that are defined on $C^{1} 1$-graphs. For simplicity, we restrict to $C^{1} 1$-graphs in $S^{2}$, i.e. for an orthonormal frame $(x, y)$ on $S^{2}$, we consider the set

$$
\Gamma=\{(x, y): \phi(x)=y\}
$$

where $\phi$ is a $C^{1}$ function. Suppose $c:[0,1] \rightarrow \Gamma$ is a parameterization of $\Gamma$ and set $\tau(c(t))=\frac{c^{\prime}(t)}{\left|c^{\prime}(t)\right|}$ the tangent unit vector to the curve $\Gamma$ at $c(t), \forall t \in(0,1)$. Let $f: \Gamma \rightarrow 2 \pi \mathbb{Z}$ be an $\mathcal{H}^{1}$-integrable function on $\Gamma$. Define

$$
\left\langle\frac{\partial f}{\partial \tau}, \zeta\right\rangle:=-\int_{0}^{1} f \circ c(t)(\zeta \circ c)^{\prime}(t) \mathrm{d} t, \quad \forall \zeta \in C^{1}(\Gamma)
$$

By Lemma 1, we have

$$
\frac{\partial f}{\partial \tau} \in \mathcal{Z}(\Gamma) \quad \text { and } \quad\left\|\frac{\partial f}{\partial \tau}\right\|=\int_{0}^{1}|f|(c(t))\left|c^{\prime}(t)\right| \mathrm{d} t
$$

Before proving Lemma 3, we give the following result:
Lemma 4. For every $u \in B V\left(S^{2}, S^{1}\right)$, we have

$$
\begin{aligned}
& u \wedge\left(D^{a} u+D^{c} u\right)=\frac{1}{\mathrm{i}} \bar{u}\left(D^{a} u+D^{c} u\right) \\
\text { and } & \left|u \wedge\left(D^{a} u+D^{c} u\right)\right|=\left|D^{a} u\right|+\left|D^{c} u\right| .
\end{aligned}
$$

Proof. Write $u=\left(u_{1}, u_{2}\right)=u_{1}+\mathrm{i} u_{2}$. We can consider the $2 \times 2$ matrix of real measures $D u$ as a 2 -vector of complex measures, i.e. $D u=D u_{1}+\mathrm{i} D u_{2}$. Since $u_{1}^{2}+u_{2}^{2}=1$, it results $D\left(u_{1}^{2}+u_{2}^{2}\right)=0$. By the chain rule (see e.g. [1]), we obtain

$$
u_{1}\left(D^{a} u_{1}+D^{c} u_{1}\right)+u_{2}\left(D^{a} u_{2}+D^{c} u_{2}\right)=0,
$$

i.e. the real part of the $\mathbb{C}^{2}$-measure $\bar{u}\left(D^{a} u+D^{c} u\right)$ vanishes. Therefore,

$$
u \wedge\left(D^{a} u+D^{c} u\right)=\frac{1}{\mathrm{i}} \bar{u}\left(D^{a} u+D^{c} u\right) .
$$

Hence, using the fact that the absolutely continuous part and the Cantor part of $D u$ are mutually singular, we conclude that

$$
\left|u \wedge\left(D^{a} u+D^{c} u\right)\right|=|u|\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)=\left|D^{a} u\right|+\left|D^{c} u\right| .
$$

Proof of Lemma 3. Let $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ be a lifting of $u$. Write

$$
D \varphi=D^{a} \varphi+D^{c} \varphi+\left(\varphi^{+}-\varphi^{-}\right) \nu_{\varphi} \mathcal{H}^{1}\llcorner S(\varphi) .
$$

By the chain rule and Lemma 4, we obtain

$$
D^{a} \varphi+D^{c} \varphi=\frac{1}{\mathrm{i}} \bar{u}\left(D^{a} u+D^{c} u\right)=u \wedge\left(D^{a} u+D^{c} u\right)
$$

Since $u=\mathrm{e}^{\mathrm{i} \varphi}$ a.e. in $S^{2}$, we have that $S(u) \subset S(\varphi)$ and by changing the orientation $v_{\varphi}$, we may assume

$$
\left\{\begin{array}{l}
v_{\varphi}=v_{u} \\
\mathrm{e}^{i} \mathrm{i}+u^{+} \quad \mathcal{H}^{1} \text {-a.e. on } S(u) . \\
\mathrm{e}^{\mathrm{i} \varphi-}=u^{-}
\end{array}\right.
$$

Therefore,

$$
\begin{array}{ll} 
& \varphi^{+}-\varphi^{-} \equiv \rho\left(u^{+}, u^{-}\right) \quad(\bmod 2 \pi) \mathcal{H}^{1} \text {-a.e. in } S(u) \\
\text { and } & \varphi^{+}-\varphi^{-} \equiv 0 \quad(\bmod 2 \pi) \mathcal{H}^{1} \text {-a.e. in } S(\varphi) \backslash S(u) .
\end{array}
$$

Hence, there exists $f_{\varphi}: S(\varphi) \rightarrow 2 \pi \mathbb{Z}$ a measurable function such that

$$
\begin{equation*}
D \varphi=u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-f_{\varphi} v_{\varphi} \mathcal{H}^{1}\llcorner S(\varphi) .\right. \tag{10}
\end{equation*}
$$

Observe that $f_{\varphi}$ is an $\mathcal{H}^{1}$-integrable function since

$$
\left|\rho\left(u^{+}, u^{-}\right)\right|=d_{S^{1}}\left(u^{+}, u^{-}\right) \leqslant \frac{\pi}{2}\left|u^{+}-u^{-}\right| .
$$

Since $D \varphi$ is a measure, we have

$$
\operatorname{curl} D \varphi=0 \quad \text { in } \mathcal{D}^{\prime},
$$

i.e. for every $\zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)$,

$$
\int_{S^{2}} \nabla^{\perp} \zeta D \varphi=0 .
$$

By (10), it yields

$$
\langle T(u), \zeta\rangle=\int_{S(\varphi)} f_{\varphi} \nabla^{\perp} \zeta \cdot v_{\varphi} \mathrm{d} \mathcal{H}^{1}, \quad \forall \zeta \in C^{1}\left(S^{2}\right)
$$

and therefore, $\left(f_{\varphi}, S(\varphi), v_{\varphi}\right) \in \mathcal{J}(T(u))$.
Conversely, take $(f, S, v) \in \mathcal{J}(T(u))$. Without loss of generality, we may consider $S=\{f \neq 0\}$. Consider the finite Radon $\mathbb{R}^{2}$-valued measure

$$
\mu=u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-f \nu \mathcal{H}^{1}\llcorner S .\right.
$$

We check that curl $\mu=0$ in $\mathcal{D}^{\prime}\left(S^{2}\right)$. Indeed, for every $\zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)$,

$$
-\langle\operatorname{curl} \mu, \zeta\rangle=\int_{S^{2}} \nabla^{\perp} \zeta \mathrm{d} \mu=\langle T(u), \zeta\rangle-\int_{S} f \nabla^{\perp} \zeta \cdot v \mathrm{~d} \mathcal{H}^{1}=0
$$

By the $B V$ version of Poincare's lemma, there exists $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ such that $D \varphi=\mu$ in $\mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}^{2}\right)$. Here, $S \cup S(u)$ is the jump set of $\varphi$. On the set $S \cup S(u)$, we choose an orientation $v_{\varphi}$ such that $v_{\varphi}=v_{u}$ on $S(u)$. We have

$$
\left\{\begin{array}{l}
D^{a} \varphi+D^{c} \varphi=u \wedge\left(D^{a} u+D^{c} u\right)=\frac{1}{\mathrm{i}} \bar{u}\left(D^{a} u+D^{c} u\right), \\
\varphi^{+}-\varphi^{-} \equiv \rho\left(u^{+}, u^{-}\right) \quad(\bmod 2 \pi) \mathcal{H}^{1} \text {-a.e. in } S(u), \\
\varphi^{+}-\varphi^{-} \equiv 0 \quad(\bmod 2 \pi) \mathcal{H}^{1} \text {-a.e. in } S \backslash S(u) .
\end{array}\right.
$$

We now show that

$$
D\left(u \mathrm{e}^{-\mathrm{i} \varphi}\right)=0 .
$$

By the chain rule, we get

$$
\begin{aligned}
D\left(\mathrm{e}^{-\mathrm{i} \varphi}\right) & =-\mathrm{i} \mathrm{e}^{-\mathrm{i} \varphi}\left(D^{a} \varphi+D^{c} \varphi\right)+\left(\mathrm{e}^{-\mathrm{i} \varphi^{+}}-\mathrm{e}^{-\mathrm{i} \varphi^{-}}\right) \otimes v_{u} \mathcal{H}^{1}\llcorner S(u) \\
& =-\mathrm{e}^{-\mathrm{i} \varphi} \bar{u}\left(D^{a} u+D^{c} u\right)+\left(\mathrm{e}^{-\mathrm{i} \varphi^{+}}-\mathrm{e}^{-\mathrm{i} \varphi^{-}}\right) \otimes v_{u} \mathcal{H}^{1}\llcorner S(u) .
\end{aligned}
$$

Remark that the space $B V\left(S^{2}, \mathbb{C}\right) \cap L^{\infty}$ is an algebra. Differentiating the product $u \mathrm{e}^{-\mathrm{i} \varphi}$, we obtain

$$
D\left(u \mathrm{e}^{-\mathrm{i} \varphi}\right)=\mathrm{e}^{-\mathrm{i} \varphi}\left(D^{a} u+D^{c} u\right)-u \mathrm{e}^{-\mathrm{i} \varphi} \bar{u}\left(D^{a} u+D^{c} u\right)+\left(u^{+} \mathrm{e}^{-\mathrm{i} \varphi^{+}}-u^{-} \mathrm{e}^{-\mathrm{i} \varphi^{-}}\right) \otimes v_{u} \mathcal{H}^{1}\llcorner S(u)=0 .
$$

Thus, up to an additive constant, $\varphi$ is a $B V$ lifting of $u$ and (4) is fulfilled.
Proof of Theorem 1. Let $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ be a lifting of $u$. By Lemma 3, there exists $(f, S, v) \in \mathcal{J}(T(u))$ such that (4) holds. Denote by $\tau: S \rightarrow S^{1}$ the tangent vector at $\mathcal{H}^{1}$-a.e. point of $S$ such that ( $v, \tau, e$ ) is direct. By (4),

$$
\langle T(u), \zeta\rangle=\int_{S} f \nabla^{\perp} \zeta \cdot v \mathrm{~d} \mathcal{H}^{1}=\int_{S} f \frac{\partial \zeta}{\partial \tau} \mathrm{~d} \mathcal{H}^{1}=\sum_{k \in \mathbb{N}} \int_{I_{k}} \chi_{S} f \frac{\partial \zeta}{\partial \tau} \mathrm{~d} \mathcal{H}^{1}, \quad \forall \zeta \in C^{1}\left(S^{2}\right)
$$

where $\left\{I_{k}\right\}_{k \in \mathbb{N}}$ is a family of disjoint compact $C^{1} 1$-graphs that covers $\mathcal{H}^{1}$-almost all of the countably rectifiable set $S$, i.e.

$$
\mathcal{H}^{1}\left(S \backslash \bigcup_{k \in \mathbb{N}} I_{k}\right)=0
$$

According to Lemma 1 and Remark 3, we conclude $T(u) \in \mathcal{Z}\left(S^{2}\right)$ and $\|T(u)\| \leqslant \int_{S}|f| \mathrm{d} \mathcal{H}^{1}$.
Before proving Theorem 2, let us make some remarks about $E(u)$ and $E_{\text {rel }}(u)$ for $u \in B V\left(S^{2}, S^{1}\right)$ (see also [4]):
Remark 4. (i) $E(u)<\infty$ and $E_{\text {rel }}(u)<\infty$;
(ii) The infimum in (5) is achieved; indeed, let $\varphi_{k} \in B V\left(S^{2}, \mathbb{R}\right), \mathrm{e}^{\mathrm{i} \varphi_{k}}=u$ a.e. in $S^{2}$, be such that

$$
\lim _{k \rightarrow \infty} \int_{S^{2}}\left|D \varphi_{k}\right|=E(u)<\infty
$$

By Poincaré's inequality, there exists a universal constant $C>0$ such that

$$
\int_{S^{2}}\left|\varphi_{k}-f_{S^{2}} \varphi_{k}\right| \mathrm{d} \mathcal{H}^{2} \leqslant C \int_{S^{2}}\left|D \varphi_{k}\right|, \quad \forall k \in \mathbb{N}
$$

(where $f_{S^{2}}$ stands for the average). Therefore, by subtracting a suitable integer multiple of $2 \pi$, we may assume that $\left(\varphi_{k}\right)_{k \in \mathbb{N}}$ is bounded in $B V\left(S^{2}, \mathbb{R}\right)$. After passing to a subsequence if necessary, we may assume that $\varphi_{k} \rightarrow \varphi$ a.e. and $L^{1}$ for some $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$. It follows that $\varphi$ is a lifting of $u$ on $S^{2}$ and

$$
E(u)=\lim _{k \rightarrow \infty} \int_{S^{2}}\left|D \varphi_{k}\right| \geqslant \int_{S^{2}}|D \varphi| \geqslant E(u)
$$

(iii) The infimum in (6) is also achieved; take $u_{k}^{m} \in C^{\infty}\left(S^{2}, S^{1}\right)$ such that for each $k \in \mathbb{N}$,

$$
u_{k}^{m} \rightarrow u \quad \text { a.e. in } S^{2} \quad \text { and } \quad \int_{S^{2}}\left|\nabla u_{k}^{m}\right| \mathrm{d} \mathcal{H}^{2} \searrow a_{k} \in \mathbb{R} \quad \text { as } m \rightarrow \infty
$$

and $\lim _{k \rightarrow \infty} a_{k}=E_{\text {rel }}(u)$. Subtracting a subsequence, we may assume that for each $k \in \mathbb{N}$,

$$
\int_{S^{2}}\left|u_{k}^{m}-u\right| \mathrm{d} \mathcal{H}^{2}<\frac{1}{k} \quad \text { and } \quad \int_{S^{2}}\left|\nabla u_{k}^{m}\right| \mathrm{d} \mathcal{H}^{2}-a_{k}<\frac{1}{k}, \quad \forall m \geqslant 1 .
$$

Therefore, $u_{k}^{k} \rightarrow u$ in $L^{1}$ and

$$
\lim _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla u_{k}^{k}\right| \mathrm{d} \mathcal{H}^{2}=E_{\text {rel }}(u)
$$

(iv) $E(u)=E_{\text {rel }}(u)$. For " $\leqslant$ ", take $u_{k} \in C^{\infty}\left(S^{2}, S^{1}\right), \forall k \in \mathbb{N}$ such that $u_{k} \rightarrow u$ a.e. in $S^{2}$ and

$$
\sup _{k \in \mathbb{N}} \int_{S^{2}}\left|\nabla u_{k}\right| \mathrm{d} \mathcal{H}^{2}<\infty
$$

Since $S^{2}$ is simply connected, there exists $\varphi_{k} \in C^{\infty}\left(S^{2}, \mathbb{R}\right)$ such that $\mathrm{e}^{\mathrm{i} \varphi_{k}}=u_{k}$. Moreover,

$$
\int_{S^{2}}\left|\nabla \varphi_{k}\right| \mathrm{d} \mathcal{H}^{2}=\int_{S^{2}}\left|\nabla u_{k}\right| \mathrm{d} \mathcal{H}^{2}
$$

Using the same argument as in ii), we may assume that $\varphi_{k} \rightarrow \varphi$ a.e. and $L^{1}$ for some $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$. Therefore, $\mathrm{e}^{\mathrm{i} \varphi}=u$ a.e. in $S^{2}$ and

$$
E(u) \leqslant \int_{S^{2}}|D \varphi| \leqslant \liminf _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla \varphi_{k}\right| \mathrm{d} \mathcal{H}^{2}=\liminf _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla u_{k}\right| \mathrm{d} \mathcal{H}^{2} .
$$

For " $\geqslant$ ", consider a $B V$ lifting $\varphi$ of $u$ and take an approximating sequence $\varphi_{k} \in C^{\infty}\left(S^{2}, \mathbb{R}\right)$ such that $\varphi_{k} \rightarrow \varphi$ a.e. and $|D \varphi|\left(S^{2}\right)=\lim _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla \varphi_{k}\right| \mathrm{d} \mathcal{H}^{2}$. With $u_{k}=\mathrm{e}^{\mathrm{i} \varphi_{k}} \in C^{\infty}\left(S^{2}, S^{1}\right)$, we have $u_{k} \rightarrow u$ a.e. in $S^{2}$ and

$$
E_{\text {rel }}(u) \leqslant \lim _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla u_{k}\right| \mathrm{d} \mathcal{H}^{2}=\lim _{k \rightarrow \infty} \int_{S^{2}}\left|\nabla \varphi_{k}\right| \mathrm{d} \mathcal{H}^{2}=\int_{S^{2}}|D \varphi| .
$$

Proof of Theorem 2. For " $\leqslant$ ", take $(f, S, v) \in \mathcal{J}(T(u))$. By Lemma 3, there exists a lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ such that (4) holds. It follows that

$$
E(u) \leqslant \int_{S^{2}}|D \varphi|=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S \cup S(u)}\left|f v \chi_{S}-\rho\left(u^{+}, u^{-}\right) v_{u} \chi_{S(u)}\right| \mathrm{d} \mathcal{H}^{1} .
$$

Let us prove now " $\geqslant$ ". By Remark 4, there is an optimal $B V$ lifting $\varphi$ of $u$, i.e. $E(u)=\int_{S^{2}}|D \varphi|$. By Lemma 3, there exists $(f, S, v) \in \mathcal{J}(T(u))$ such that (4) holds. It results that

$$
E(u)=\int_{S^{2}}|D \varphi|=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S \cup S(u)}\left|f v \chi_{S}-\rho\left(u^{+}, u^{-}\right) v_{u} \chi_{S(u)}\right| \mathrm{d} \mathcal{H}^{1} .
$$

From here, we also deduce that the minimum inside the RHS of (7) is achieved.

Remark 5 (Construction of an optimal lifting). Take $(f, S, v) \in \mathcal{J}(T(u))$ that achieves the minimum

$$
\begin{equation*}
\min _{(f, S, v) \in \mathcal{J}(T(u))} \int_{S \cup S(u)}\left|f v \chi_{S}-\rho\left(u^{+}, u^{-}\right) v_{u} \chi_{S(u)}\right| \mathrm{d} \mathcal{H}^{1} \tag{11}
\end{equation*}
$$

By Lemma 3, there exists a lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ such that (4) holds. Then

$$
\int_{S^{2}}|D \varphi|=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S \cup S(u)}\left|f v \chi_{S}-\rho\left(u^{+}, u^{-}\right) v_{u} \chi_{S(u)}\right| \mathrm{d} \mathcal{H}^{1}=E(u)
$$

and therefore, $\varphi$ is an optimal lifting of $u$.
Proof of Lemma 2. For " $\leqslant$ ", it is easy to see that if $(f, S, v) \in \mathcal{J}(\Lambda)$ then for every $\zeta \in C^{1}\left(S^{2}\right)$ with $|\nabla \zeta| \leqslant 1$,

$$
\langle\Lambda, \zeta\rangle=\int_{S} f \nu \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1} \leqslant \int_{S}|f| \mathrm{d} \mathcal{H}^{1}
$$

For " $\geqslant$ ", we use characterization (2) of the distribution $\Lambda \in \mathcal{Z}\left(S^{2}\right)$. We denote by $d_{S^{2}}$ the geodesic distance on $S^{2}$. Let $\Lambda=2 \pi \sum_{k}\left(\delta_{p_{k}}-\delta_{n_{k}}\right)$ where $\left(p_{k}\right)_{k \in \mathbb{N}},\left(n_{k}\right)_{k \in \mathbb{N}}$ belong to $S^{2}$ such that $\sum_{k} d_{S^{2}}\left(p_{k}, n_{k}\right)<\infty$. For every $k \in \mathbb{N}$, consider ${ }_{n_{k}}{ }^{p_{k}}$ a geodesic arc on $S^{2}$ oriented from $n_{k}$ to $p_{k}$. Take $v_{k}$ the normal vector to $\widetilde{n}_{k}{ }^{p} p_{k}$ in the frame $(x, y)$. Set $S=\bigcup_{k} n_{k} \widehat{p}_{k}$. Since $\sum_{k} d_{S^{2}}\left(p_{k}, n_{k}\right)<\infty$, there exist an orientation $v: S \rightarrow S^{1}$ on $S$ and an $\mathcal{H}^{1}$-integrable function $f: S \rightarrow 2 \pi \mathbb{Z}$ such that

$$
\begin{equation*}
f v \chi_{S}=\sum_{k} 2 \pi v_{k} \chi_{n_{k} \hat{p}_{k}} \quad \text { in } L^{1}\left(S, \mathbb{R}^{2}\right) \tag{12}
\end{equation*}
$$

Then

$$
\int_{S} f v \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=2 \pi \sum_{k} \int_{n_{k} p_{k}} v_{k} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=2 \pi \sum_{k}\left(\zeta\left(p_{k}\right)-\zeta\left(n_{k}\right)\right)=\langle\Lambda, \zeta\rangle, \quad \forall \zeta \in C^{1}\left(S^{2}\right)
$$

It follows that $(f, S, \nu) \in \mathcal{J}(\Lambda)$ and by (12),

$$
\int_{S}|f| \mathrm{d} \mathcal{H}^{1} \leqslant \sum_{k} 2 \pi d_{S^{2}}\left(n_{k}, p_{k}\right)
$$

Minimizing after all suitable pairs $\left(p_{k}, n_{k}\right)_{k \in \mathbb{N}}$, it follows by (2),

$$
\begin{equation*}
\|\Lambda\|=\inf _{(f, S, v) \in \mathcal{J}(\Lambda)} \int_{S}|f| \mathrm{d} \mathcal{H}^{1} \tag{13}
\end{equation*}
$$

We now show that the infimum in (13) is indeed achieved. By a dipole construction (see [2], Lemma 16), there exists $u \in W^{1,1}\left(S^{2}, S^{1}\right)$ such that $\Lambda=T(u)$. We choose $\left(f_{k}, S_{k}, v_{k}\right) \in \mathcal{J}(T(u))$ such that

$$
\|T(u)\|=\lim _{k} \int_{S_{k}}\left|f_{k}\right| \mathrm{d} \mathcal{H}^{1}
$$

By Lemma 3, we construct a lifting $\varphi_{k} \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ such that

$$
D \varphi_{k}=u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-f_{k} \nu_{k} \mathcal{H}^{1}\left\llcorner S_{k} .\right.\right.
$$

Remark that

$$
\int_{S^{2}}\left|D \varphi_{k}\right| \leqslant \int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S(u)}\left|\rho\left(u^{+}, u^{-}\right)\right| \mathrm{d} \mathcal{H}^{1}+\int_{S_{k}}\left|f_{k}\right| \mathrm{d} \mathcal{H}^{1} .
$$

Subtracting a suitable number in $2 \pi \mathbb{Z}$, we may assume that $\left(\varphi_{k}\right)$ is a bounded sequence in $B V\left(S^{2}, \mathbb{R}\right)$. Up to a subsequence, we find $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ such that

$$
\varphi_{k} \rightarrow \varphi \quad \text { a.e. in } S^{2} \quad \text { and } \quad D \varphi_{k} \stackrel{*}{\rightharpoonup} D \varphi \text { in the measure sense. }
$$

Therefore, $\varphi$ is a $B V$ lifting of $u$ and by Lemma 3, there exists $(f, S, v) \in \mathcal{J}(T(u))$ such that

$$
D \varphi=u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) \nu_{u} \mathcal{H}^{1}\left\llcorner S(u)-f \nu \mathcal{H}^{1}\llcorner S .\right.
$$

We conclude

$$
\begin{aligned}
\int_{S}|f| \mathrm{d} \mathcal{H}^{1} & =\int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)-D \varphi \mid \\
& \leqslant \liminf _{k} \int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-D \varphi_{k} \mid\right. \\
& =\lim _{k} \int_{S_{k}}\left|f_{k}\right| \mathrm{d} \mathcal{H}^{1}=\|T(u)\| .
\end{aligned}
$$

Proof of Theorem 3. Let $\psi \in B V\left(S^{2}, \mathbb{R}\right)$ and $\zeta \in C^{1}\left(S^{2}\right)$ be such that $|\nabla \zeta| \leqslant 1$. Then

$$
\int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-D \psi \mid \geqslant\langle T(u), \zeta\rangle-\int_{S^{2}} D \psi \cdot \nabla^{\perp} \zeta=\langle T(u), \zeta\rangle .\right.
$$

By taking the supremum over $\zeta$, we obtain

$$
\int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)-D \psi \mid \geqslant\|T(u)\| .
$$

We now show that there is a lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ such that the minimum in (8) is achieved. By Lemma 2 , choose $(f, S, v) \in \mathcal{J}(T(u))$ such that

$$
\|T(u)\|=\int_{S}|f| \mathrm{d} \mathcal{H}^{1}
$$

Using Lemma 3, we construct a lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ such that (4) holds. Thus,

$$
\|T(u)\|=\int_{S}|f| \mathrm{d} \mathcal{H}^{1}=\int_{S^{2}} \mid u \wedge\left(D^{a} u+D^{c} u\right)+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\llcorner S(u)-D \varphi \mid .
$$

Proof of Corollary 1. The result is a straightforward consequence of Theorem 2 and Lemma 2.
In order to prove Corollary 2, we need the following estimation of $\|T(u)\|$ in terms of the seminorm $|u|_{B V S^{1}}$ :
Lemma 5. We have $\|T(u)\| \leqslant|u|_{B V S^{1}}, \forall u \in B V\left(S^{2}, S^{1}\right)$.

Proof. By Lemma 4, it results that for every $\zeta \in C^{1}\left(S^{2}\right)$ with $|\nabla \zeta| \leqslant 1$,

$$
\begin{aligned}
|\langle T(u), \zeta\rangle| & \leqslant \int_{S^{2}}\left|u \wedge\left(D^{a} u+D^{c} u\right)\right|+\int_{S(u)}\left|\rho\left(u^{+}, u^{-}\right)\right| \mathrm{d} \mathcal{H}^{1} \\
& =\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S(u)} d_{S^{1}}\left(u^{+}, u^{-}\right) \mathrm{d} \mathcal{H}^{1}
\end{aligned}
$$

therefore

$$
\|T(u)\| \leqslant|u|_{B V S^{1}} .
$$

Proof of Corollary 2. By Theorem 2, Lemmas 2 and 5, we conclude that

$$
\begin{aligned}
E(u) & \leqslant \int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S(u)}\left|\rho\left(u^{+}, u^{-}\right)\right| \mathrm{d} \mathcal{H}^{1}+\min _{(f, S, v) \in \mathcal{J}(T(u))} \int_{S}|f| \mathrm{d} \mathcal{H}^{1} \\
& =|u|_{B V S^{1}}+\|T(u)\| \leqslant 2|u|_{B V S^{1}} .
\end{aligned}
$$

Let $|u|_{B V}=\int_{S^{2}}|D u|=\int_{S^{2}}\left(\left|D^{a} u\right|+\left|D^{c} u\right|\right)+\int_{S(u)}\left|u^{+}-u^{-}\right| \mathrm{d} \mathcal{H}^{1}$; we deduce that

$$
|u|_{B V} \leqslant|u|_{B V S^{1}} \leqslant \frac{\pi}{2}|u|_{B V}, \quad \forall u \in B V\left(S^{2}, S^{1}\right) .
$$

Therefore, Corollary 2 is a weaker estimate of $E(u)$ than inequality (3) obtained in [5].

## 3. Some other properties of the distribution $T$

We start by observing that $T: B V\left(S^{2}, S^{1}\right) \rightarrow \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$ is not continuous, i.e. there exists a sequence of functions $u_{k} \in B V\left(S^{2}, S^{1}\right)$ such that $u_{k} \rightarrow u$ strongly in $B V\left(S^{2}, S^{1}\right)$ and $T\left(u_{k}\right) \nrightarrow T(u)$ in $\mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$. The reason for that is the discontinuity of the function $\rho$ that enters in the definition of $T$.

Proposition 1. The map $T: B V\left(S^{2}, S^{1}\right) \rightarrow \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$ is discontinuous.
Proof. Write

$$
S^{2}=\{(\cos \theta \sin \alpha, \sin \theta \sin \alpha, \cos \alpha): \alpha \in[0, \pi], \theta \in(0,2 \pi]\} .
$$

In the spherical coordinates $(\alpha, \theta) \in[0, \pi] \times[0,2 \pi]$, consider the $B V$ functions $\varphi$ and $u$ defined as

$$
\varphi(\alpha, \theta)=\left\{\begin{array}{ll}
-2 \theta & \text { if } \theta \in\left(0, \frac{\pi}{2}\right), \alpha \in\left(0, \frac{\pi}{2}\right),  \tag{14}\\
-\pi & \text { if } \theta \in\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right), \alpha \in\left(0, \frac{\pi}{2}\right), \\
2(\theta-2 \pi) & \text { if } \theta \in\left(\frac{3 \pi}{2}, 2 \pi\right), \alpha \in\left(0, \frac{\pi}{2}\right), \\
0 & \text { if } \theta \in(0,2 \pi), \alpha \in\left(\frac{\pi}{2}, \pi\right)
\end{array} \text { and } \quad u=\mathrm{e}^{\mathrm{i} \varphi}\right.
$$

We have that the jump set of $u$ and $\varphi$ is concentrated on the equator $\left\{\alpha=\frac{\pi}{2}\right\}$ of the sphere $S^{2}$, i.e.

$$
S(\varphi)=S(u)=\left\{\alpha=\frac{\pi}{2}\right\} .
$$

On the equator we choose the orientation given by the normal vector $\vec{\alpha}$ oriented from the north to the south; so $(\vec{\alpha}, \vec{\theta}, \vec{e})$ is direct. We show that

$$
\begin{equation*}
T(u)=2 \pi\left(\delta_{p}-\delta_{n}\right), \tag{15}
\end{equation*}
$$

where $n=\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right)$ and $p=\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ in the frame $(\alpha, \theta)$. Indeed, we remark that

$$
\varphi^{+}-\varphi^{-}=\rho\left(u^{+}, u^{-}\right)+2 \pi \chi_{\widehat{n p}} \quad \text { in } S(u)
$$

by Lemma 3, we obtain

$$
D \varphi=u \wedge \nabla u \mathcal{H}^{2}+\rho\left(u^{+}, u^{-}\right) \vec{\alpha} \mathcal{H}^{1}\left\llcorner S(u)+2 \pi \vec{\alpha} \mathcal{H}^{1}\llcorner\widehat{n p}\right.
$$

and it yields

$$
\langle T(u), \zeta\rangle=-2 \pi \int_{\widehat{n p}} \vec{\alpha} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=-2 \pi \int_{p}^{n} \frac{\partial \zeta}{\partial \theta} \mathrm{~d} \mathcal{H}^{1}=2 \pi(\zeta(p)-\zeta(n)), \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)
$$

Construct the approximation sequence $\varphi_{\varepsilon} \in B V\left(S^{2}, \mathbb{R}\right), \varepsilon \in(0,1)$ defined (in the spherical coordinates) as

$$
\varphi_{\varepsilon}(\alpha, \theta)= \begin{cases}-2 \theta & \text { if } \theta \in\left(0, \frac{\pi-\varepsilon}{2}\right), \alpha \in\left(0, \frac{\pi}{2}\right) \\ -\pi+\varepsilon & \text { if } \theta \in\left(\frac{\pi-\varepsilon}{2}, \frac{3 \pi+\varepsilon}{2}\right), \alpha \in\left(0, \frac{\pi}{2}\right), \\ 2(\theta-2 \pi) & \text { if } \theta \in\left(\frac{3 \pi+\varepsilon}{2}, 2 \pi\right), \alpha \in\left(0, \frac{\pi}{2}\right), \\ 0 & \text { if } \theta \in(0,2 \pi), \alpha \in\left(\frac{\pi}{2}, \pi\right)\end{cases}
$$

and set $u_{\varepsilon}=\mathrm{e}^{\mathrm{i} \varphi_{\varepsilon}}$. An easy computation shows that $\varphi_{\varepsilon} \rightarrow \varphi$ strongly in $B V$; therefore, $u_{\varepsilon} \rightarrow u$ strongly in $B V$ as $\varepsilon \rightarrow 0$. As before, we have

$$
S\left(\varphi_{\varepsilon}\right)=S\left(u_{\varepsilon}\right)=\left\{\alpha=\frac{\pi}{2}\right\} \quad \text { and } \quad \varphi_{\varepsilon}^{+}-\varphi_{\varepsilon}^{-}=\rho\left(u_{\varepsilon}^{+}, u_{\varepsilon}^{-}\right) \quad \text { in }\left\{\alpha=\frac{\pi}{2}\right\} .
$$

It follows that $T\left(u_{\varepsilon}\right)=0$ and we conclude

$$
T\left(u_{\varepsilon}\right) \nrightarrow T(u) \quad \text { in } \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)
$$

As Brezis, Mironescu and Ponce proved in [4], if we restrict ourselves to $W^{1,1}\left(S^{2}, S^{1}\right)$, then the map $\left.T\right|_{W^{1,1}\left(S^{2}, S^{1}\right)}: W^{1,1}\left(S^{2}, S^{1}\right) \rightarrow \mathcal{Z}\left(S^{2}\right)$ is continuous, i.e. if $g, g_{k} \in W^{1,1}\left(S^{2}, S^{1}\right)$ such that $g_{k} \rightarrow g$ in $W^{1,1}$ then $\left\|T\left(g_{k}\right)-T(g)\right\| \rightarrow 0$ as $k \rightarrow \infty$. It is natural to ask if one could change the antisymmetric function $\rho$ in order that the corresponding map $T$ become continuous. The answer is negative:

Proposition 2. There is no antisymmetric function $\gamma: S^{1} \times S^{1} \rightarrow \mathbb{R}$ such that the map $T_{\gamma}: B V\left(S^{2}, S^{1}\right) \rightarrow \mathcal{Z}\left(S^{2}\right)$ given for every $u \in B V\left(S^{2}, S^{1}\right)$ as

$$
\left\langle T_{\gamma}(u), \zeta\right\rangle=\int_{S^{2}} \nabla^{\perp} \zeta \cdot\left(u \wedge\left(D^{a} u+D^{c} u\right)\right)+\int_{S(u)} \gamma\left(u^{+}, u^{-}\right) v_{u} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}, \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)
$$

is well-defined and continuous.
Proof. By contradiction, suppose that there exists such a function $\gamma$. First we show that

$$
\begin{equation*}
\gamma\left(\omega_{1}, \omega_{2}\right) \equiv \operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right) \quad(\bmod 2 \pi), \quad \forall \omega_{1}, \omega_{2} \in S^{1} . \tag{16}
\end{equation*}
$$

Indeed, fix $\omega_{1}, \omega_{2} \in S^{1}$. Take $f:[0,2 \pi] \rightarrow \mathbb{R}$ the linear function satisfying $f(0)=\operatorname{Arg}\left(\omega_{1}\right)$ and $f(2 \pi)=\operatorname{Arg}\left(\omega_{2}\right)$; define $u \in B V\left(S^{2}, S^{1}\right)$ as

$$
u(\alpha, \theta)=\mathrm{e}^{\mathrm{i} f(\theta)}, \quad \forall \alpha \in(0, \pi), \theta \in(0,2 \pi) .
$$

Consider the lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ given by

$$
\varphi(\alpha, \theta)=f(\theta), \quad \forall \alpha \in(0, \pi), \theta \in(0,2 \pi)
$$

If $\omega_{1} \neq \omega_{2}$, the jump set of $u$ and $\varphi$ is concentrated on the meridian $\{\theta=0\}$ orientated counterclockwise by the unit vector $\vec{\theta}$. We have that

$$
D \varphi=u \wedge \nabla u \mathcal{H}^{2}+\left(\operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right)\right) \vec{\theta} \mathcal{H}^{1}\llcorner\{\theta=0\} .
$$

Since curl $D \varphi=0$ in $\mathcal{D}^{\prime}$, it yields

$$
\begin{aligned}
\int_{S^{2}} u \wedge \nabla u \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{2} & =-\int_{\{\theta=0\}}\left(\operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right)\right) \vec{\theta} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1} \\
& =\left(\operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right)\right) \int_{p}^{n} \frac{\partial \zeta}{\partial \alpha} \mathrm{~d} \mathcal{H}^{1} \\
& =\left(\operatorname{Arg}\left(\omega_{2}\right)-\operatorname{Arg}\left(\omega_{1}\right)\right)(\zeta(p)-\zeta(n)), \quad \forall \zeta \in C^{1}\left(S^{2}\right)
\end{aligned}
$$

where $p=(0,0)$ and $n=(\pi, 0)$ (in the spherical coordinates) are the north and the south pole of $S^{2}$. We obtain that

$$
\begin{aligned}
\left\langle T_{\gamma}(u), \zeta\right\rangle & =\int_{S^{2}} \nabla^{\perp} \zeta \cdot(u \wedge \nabla u) \mathrm{d} \mathcal{H}^{2}+\gamma\left(\omega_{1}, \omega_{2}\right) \int_{\{\theta=0\}} \vec{\theta} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1} \\
& =\left(\operatorname{Arg}\left(\omega_{2}\right)-\operatorname{Arg}\left(\omega_{1}\right)+\gamma\left(\omega_{1}, \omega_{2}\right)\right)(\zeta(p)-\zeta(n)), \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)
\end{aligned}
$$

From the definition we know that $T_{\gamma}(u) \in \mathcal{Z}\left(S^{2}\right)$ and therefore, (16) holds. If $\omega_{1}=\omega_{2}$, by the antisymmetry of $\gamma$, we have $\gamma\left(\omega_{1}, \omega_{2}\right)=0$ and so, (16) is obvious.

Second we prove that the continuity of $T_{\gamma}$ implies that $\gamma$ is continuous on $S^{1} \times S^{1}$. Indeed, let $\left(\omega_{1}^{\varepsilon}\right)$ and $\left(\omega_{2}^{\varepsilon}\right)$ be two sequences in $S^{1}$ such that $\omega_{1}^{\varepsilon} \rightarrow \omega_{1}$ and $\omega_{2}^{\varepsilon} \rightarrow \omega_{2}$. We want that

$$
\begin{equation*}
\gamma\left(\omega_{1}^{\varepsilon}, \omega_{2}^{\varepsilon}\right) \rightarrow \gamma\left(\omega_{1}, \omega_{2}\right) . \tag{17}
\end{equation*}
$$

Take $\beta \in[0,2 \pi)$ such that $\mathrm{e}^{\mathrm{i} \beta}$ is different from $\omega_{1}$ and $\omega_{2}$. For each $\omega \in S^{1}$ denote by $\arg _{\beta}(\omega) \in(\beta-2 \pi, \beta]$ the argument of $\omega$, i.e.

$$
\begin{equation*}
\mathrm{e}^{\mathrm{i} \arg _{\beta}(\omega)}=\omega \tag{18}
\end{equation*}
$$

As above, define $f_{\varepsilon}:[0,2 \pi] \rightarrow \mathbb{R}$ as the linear function satisfying $f_{\varepsilon}(0)=\arg _{\beta}\left(\omega_{1}^{\varepsilon}\right)$ and $f_{\varepsilon}(2 \pi)=\arg _{\beta}\left(\omega_{2}^{\varepsilon}\right)$ and consider $u_{\varepsilon} \in B V\left(S^{2}, S^{1}\right)$ such that

$$
u_{\varepsilon}(\alpha, \theta)=\mathrm{e}^{\mathrm{i} f_{\varepsilon}(\theta)}, \quad \forall \alpha \in(0, \pi), \theta \in(0,2 \pi) .
$$

It is easy to check that $u_{\varepsilon} \rightarrow u$ strongly in $B V$, where $u(\alpha, \theta)=\mathrm{e}^{\mathrm{i} f(\theta)}$ and $f$ is the linear function satisfying $f(0)=\arg _{\beta}\left(\omega_{1}\right)$ and $f(2 \pi)=\arg _{\beta}\left(\omega_{2}\right)$. As before, we obtain

$$
\begin{array}{ll} 
& T_{\gamma}\left(u_{\varepsilon}\right)=\left(\arg _{\beta}\left(\omega_{2}^{\varepsilon}\right)-\arg _{\beta}\left(\omega_{1}^{\varepsilon}\right)+\gamma\left(\omega_{1}^{\varepsilon}, \omega_{2}^{\varepsilon}\right)\right)\left(\delta_{p}-\delta_{n}\right) \\
\text { and } & T_{\gamma}(u)=\left(\arg _{\beta}\left(\omega_{2}\right)-\arg _{\beta}\left(\omega_{1}\right)+\gamma\left(\omega_{1}, \omega_{2}\right)\right)\left(\delta_{p}-\delta_{n}\right) .
\end{array}
$$

Since $T_{\gamma}$ and $\arg _{\beta}$ are continuous on $B V\left(S^{2}, S^{1}\right)$, respectively on $S^{1} \backslash\left\{\mathrm{e}^{\mathrm{i} \beta}\right\}$, we deduce that (17) holds.
Observe now that the function

$$
\left(\omega_{1}, \omega_{2}\right) \mapsto \gamma\left(\omega_{1}, \omega_{2}\right)-\operatorname{Arg}\left(\omega_{1}\right)+\operatorname{Arg}\left(\omega_{2}\right)
$$

is continuous on the connected set $S^{1} \backslash\{-1\} \times S^{1} \backslash\{-1\}$ and takes values in $2 \pi \mathbb{Z}$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$
\gamma\left(\omega_{1}, \omega_{2}\right)=\operatorname{Arg}\left(\omega_{1}\right)-\operatorname{Arg}\left(\omega_{2}\right)-2 \pi k \quad \text { in } S^{1} \backslash\{-1\} \times S^{1} \backslash\{-1\}
$$

In fact, $k=0$ if one takes $\omega_{1}=\omega_{2}$. But $\operatorname{Arg}(\cdot)$ is not a continuous map on $S^{1}$ which is a contradiction with the continuity of $\gamma$ on $S^{1} \times S^{1}$.

The algebraic properties of $T$ restricted to $W^{1,1}\left(S^{2}, S^{1}\right)$ (see [4], Lemma 1) do not hold in general for $B V\left(S^{2}, S^{1}\right)$ functions.

Remark 6. (a) There exists $u \in B V\left(S^{2}, S^{1}\right)$ such that $T(\bar{u}) \neq-T(u)$. Indeed, take the function $u$ defined in (14). A similar computation gives us that $T(\bar{u})=0 \neq-T(u)$.
(b) The relation $T(g h)=T(g)+T(h), \forall g, h \in W^{1,1}\left(S^{2}, S^{1}\right)$ need not hold for $B V\left(S^{2}, S^{1}\right)$ functions. As before, consider the function $u$ in (14). Then $T(-u)=0$. Since $T(-1)=0$, we conclude $T(-u) \neq T(u)+T(-1)$.

In the following we discuss the nature of the singularities of the distribution $T(u)$. As it was mentioned in the beginning, we deal with two types of singularity:
(i) topological singularities carrying a degree which are created by the absolutely continuous part and the Cantor part of the distributional determinant of $u$;
(ii) point singularities coming from the jump part of the derivative $D u$.

We give some examples in order to point out these two different kind of singularity. In Example 1, $T(u)$ is a dipole made up by two vortices of degree +1 and -1 ; these two vortices are generated by the absolutely continuous part of $\operatorname{det}(\nabla u)$ in (a), respectively by the Cantor part of the distributional Jacobian of $u$ in (b).

Example 1. (a) Let us analyze the function $g \in W^{1,1}\left(S^{2}, S^{1}\right)$,

$$
g(\alpha, \theta)=\mathrm{e}^{\mathrm{i} \theta}, \quad \forall \alpha \in(0, \pi), \theta \in[0,2 \pi) .
$$

Denote $p$ and $n$ the north and respectively the south pole of the unit sphere. We consider the lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ given by $\varphi(\alpha, \theta)=\theta$ for every $\alpha \in(0, \pi), \theta \in(0,2 \pi)$. Then the jump set of $\varphi$ is concentrated on the meridian $\{\theta=0\}$ oriented counterclockwise by the unit vector $\vec{\theta}$. We have

$$
D \varphi=g \wedge \nabla g \mathcal{H}^{2}-2 \pi \vec{\theta} \mathcal{H}^{1}\llcorner\overparen{\sim p} .
$$

Therefore, $T(g)=2 \pi\left(\delta_{p}-\delta_{n}\right)$. The two poles are the vortices of the function $g$.
(b) The same situation may occur for some purely Cantor functions. Let us consider the standard Cantor function $f:[0,1] \rightarrow[0,1] ; f$ is a continuous, nondecreasing function with $f(0)=0, f(1)=1$ and $f^{\prime}(x)=0$ for a.e. $x \in(0,1)$. Take $v \in B V\left(S^{2}, S^{1}\right)$ defined as

$$
v(\alpha, \theta)=\mathrm{e}^{2 \pi \mathrm{i} f(\theta / 2 \pi)}, \quad \forall \alpha \in(0, \pi), \theta \in[0,2 \pi)
$$

The lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ given by $\varphi(\alpha, \theta)=2 \pi f(\theta / 2 \pi)$ for every $\alpha \in(0, \pi), \theta \in(0,2 \pi)$ has the jump set concentrated on the meridian $\{\theta=0\}$ and

$$
D \varphi=v \wedge D^{c} v-2 \pi \vec{\theta} \mathcal{H}^{1}\llcorner\widehat{n p} .
$$

As before, we obtain that $T(v)=2 \pi\left(\delta_{p}-\delta_{n}\right)$ where $p$ and $n$ are the poles of $S^{2}$.
Remark also that for the two functions constructed in Example 1, the constant 2 in inequality (3) is optimal and we have a specific structure for an optimal lifting:

Proposition 3. Let $u \in B V\left(S^{2}, S^{1}\right)$ be one of the two functions defined in Example 1. Then for every lifting $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ of $u$ we have

$$
\int_{S^{2}}|D \varphi| \geqslant 2 \int_{S^{2}}|D u| .
$$

Moreover, the set of all optimal liftings of $u$ is given by

$$
\left\{\arg _{\beta}(u)+2 \pi k: \beta \in[0,2 \pi), k \in \mathbb{Z}\right\}
$$

where $\arg _{\beta}(\omega) \in(\beta-2 \pi, \beta]$ stands for the argument of $\omega \in S^{1}$ (as in (18)).
Proof. First we notice that

$$
\int_{S^{2}}|D u|=2 \pi^{2} \quad \text { and } \quad\|T(u)\|=2 \pi d_{S^{2}}(n, p)=2 \pi^{2}
$$

where $n$ and $p$ are the two poles of $S^{2}$.
Let $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ be a lifting of $u$. By Theorem 2 and Lemma 2, we obtain

$$
\int_{S^{2}}|D \varphi| \geqslant E(u)=\int_{S^{2}}|D u|+\|T(u)\|=4 \pi^{2}=2 \int_{S^{2}}|D u| .
$$

Take now $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ an optimal lifting of $u$. By Lemma 3, there exists $(f, S, v) \in \mathcal{J}(T(u))$ that achieves the minimum in (11) and satisfies

$$
D \varphi=u \wedge D u-f \nu \mathcal{H}^{1}\llcorner S
$$

That means

$$
\begin{equation*}
D^{j} \varphi=-f \nu \mathcal{H}^{1}\left\llcorner S \quad \text { and } \quad \int_{S}|f|=2 \pi d_{S^{2}}(n, p)\right. \tag{19}
\end{equation*}
$$

We may assume here that $S=\{f \neq 0\}$. For every $\alpha \in(0, \pi)$ we denote $L_{\alpha}$ the latitude on $S^{2}$ corresponding to $\alpha$ and $\varphi_{\alpha}: L_{\alpha} \rightarrow \mathbb{R}$ the restriction of $\varphi$ to $L_{\alpha}$. Using the Characterization Theorem of $B V$ functions by sections and Theorem 3.108 in [1], it results that for a.e. $\alpha \in(0, \pi), \varphi_{\alpha} \in B V\left(L_{\alpha}, \mathbb{R}\right)$ and the discontinuity set of $\varphi_{\alpha}$ is $S \cap L_{\alpha}$. Remark that $\operatorname{deg}\left(u ; L_{\alpha}\right)=1$ for every $\alpha \in(0, \pi)$. Thus, for a.e. $\alpha \in(0, \pi), \varphi_{\alpha}$ will have at least one jump on $L_{\alpha}$ and the length of a jump is not less than $2 \pi$. It yields $\mathcal{H}^{1}(S) \geqslant \pi$ and $|f| \geqslant 2 \pi \mathcal{H}^{1}-$ a.e. in $S$. By (19), we deduce that

$$
|f|=2 \pi \quad \mathcal{H}^{1} \text {-a.e. in } S \quad \text { and } \quad \mathcal{H}^{1}(S)=\pi
$$

We know that

$$
\int_{S} \frac{f}{2 \pi} \nu \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=\zeta(p)-\zeta(n), \quad \forall \zeta \in C^{1}\left(S^{2}\right)
$$

By [7] (Section 4.2.25), it results that $S$ covers $\mathcal{H}^{1}$-almost all of a Lipschitz univalent path $c$ between the two poles. Since $\mathcal{H}^{1}(S)=d_{S^{2}}(n, p)$ we deduce that $S$ is a geodesic arc on $S^{2}$ between $n$ and $p$ and $\frac{f}{2 \pi} v$ is the normal unit vector to the curve $c$. Take $\beta \in[0,2 \pi)$ such that $S=\{\theta=\beta\}$ in the spherical coordinates. We have that $\varphi-\arg _{\beta}(u): S^{2} \backslash S \rightarrow 2 \pi \mathbb{Z}$ is continuous on the connected set $S^{2} \backslash S$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$
\varphi=\arg _{\beta}(u)+2 \pi k
$$

and the conclusion follows.

The appearance of non-topological singularities in the writing of $T(u)$ for $u \in B V\left(S^{2}, S^{1}\right)$ was already seen in the example (14); there the distribution $T(u)$ is a dipole even if the function $u$ does not have any vortex. One should notice that the dipole (15) is created on the jump set of $u$ by the discontinuity of the chosen argument Arg. In Remark 7, we will see that a dipole could disappear if we change the choice of the argument.

Remark 7. Let $\beta \in[0,2 \pi)$. Define the antisymmetric function $\gamma_{\beta}(\cdot, \cdot): S^{1} \times S^{1} \rightarrow[-\pi, \pi]$ as

$$
\gamma_{\beta}\left(\omega_{1}, \omega_{2}\right)=\left\{\begin{array}{ll}
\operatorname{Arg}\left(\frac{\omega_{1}}{\omega_{2}}\right) & \text { if } \frac{\omega_{1}}{\omega_{2}} \neq-1, \\
\arg _{\beta}\left(\omega_{1}\right)-\arg _{\beta}\left(\omega_{2}\right) & \text { if } \frac{\omega_{1}}{\omega_{2}}=-1
\end{array} \quad \forall \omega_{1}, \omega_{2} \in S^{1}\right.
$$

Consider now the distribution $T_{\gamma \beta}(u) \in \mathcal{D}^{\prime}\left(S^{2}, \mathbb{R}\right)$ given as in Proposition 2:

$$
\left\langle T_{\gamma \beta}(u), \zeta\right\rangle=\int_{S^{2}} \nabla^{\perp} \zeta \cdot\left(u \wedge\left(D^{a} u+D^{c} u\right)\right)+\int_{S(u)} \gamma_{\beta}\left(u^{+}, u^{-}\right) v_{u} \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}, \quad \forall \zeta \in C^{1}\left(S^{2}, \mathbb{R}\right)
$$

Observe that $T_{\gamma_{\beta}}$ inherits the properties of $T$ given in Theorems 1, 2 and 3. However, the structure of the singularities of $T_{\gamma \beta}(u)$ may be different from $T(u)$. Indeed, consider $u \in B V\left(S^{2}, S^{1}\right)$ the function constructed in (14). We saw that $T(u)=2 \pi\left(\delta_{p}-\delta_{n}\right)$ where $n=\left(\frac{\pi}{2}, \frac{3 \pi}{2}\right)$ and $p=\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ (in the spherical coordinates). The same computation gives us $T_{\gamma_{\pi / 2}}(u)=0$. The difference between $T(u)$ and $T_{\gamma_{\pi / 2}}(u)$ arises from the choice of the argument.

An interesting phenomenon is observed in Example 2 where the two types of singularity are mixed: some topological vortices may be located on the jump set of $u$.

Example 2. (a) An example that points out the mixture of the two type of singularity is given by functions with pseudo-vortices: define $u \in B V\left(S^{2}, S^{1}\right)$ as

$$
u(\alpha, \theta)=\mathrm{e}^{3 \mathrm{i} \theta / 2}, \quad \forall \alpha \in(0, \pi), \theta \in(0,2 \pi)
$$

The jump set of $u$ is the meridian $\{\theta=0\}$. We have

$$
T(u)=2 \pi\left(\delta_{p}-\delta_{n}\right) \quad \text { and } \quad T_{\gamma_{\pi / 2}}(u)=4 \pi\left(\delta_{p}-\delta_{n}\right)
$$

The two poles $p$ and $n$ arise on the jump set of $u$ and behave like some pseudo-vortices, i.e. after a complete turn, the function $u$ rotates $3 / 2$ times around the poles (with different signs: ' + ' around $p$ and ' - ' around $n$ ). According to the choice of the argument in the definition of $\gamma_{\beta}$, the distribution $T_{\gamma_{\beta}}(u)$ will count once or twice the dipole.
(b) A piecewise constant function $u \in B V\left(S^{2}, S^{1}\right)$ may create a dipole for $T(u)$. Indeed, let us define $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ as

$$
\varphi(\alpha, \theta)= \begin{cases}0 & \text { if } \theta \in(0,2 \pi / 3), \alpha \in(0, \pi) \\ 2 \pi / 3 & \text { if } \theta \in(2 \pi / 3,4 \pi / 3), \alpha \in(0, \pi) \\ 4 \pi / 3 & \text { if } \theta \in(4 \pi / 3,2 \pi), \alpha \in(0, \pi)\end{cases}
$$

and set $u=\mathrm{e}^{\mathrm{i} \varphi}$. The jump set of $u$ and $\varphi$ is the union of three meridians

$$
S(u)=S(\varphi)=\{\theta=0\} \cup\{\theta=2 \pi / 3\} \cup\{\theta=4 \pi / 3\}
$$

We have

$$
\varphi^{+}-\varphi^{-}=\rho\left(u^{+}, u^{-}\right)-2 \pi \chi_{\{\theta=0\}} \quad \text { in } S(\varphi)
$$

We obtain $T(u)=2 \pi\left(\delta_{p}-\delta_{n}\right)$ where $p$ and $n$ are the two poles of the unit sphere. For every $\beta \in[0,2 \pi), T_{\gamma_{\beta}}$ has the same behavior, i.e. $T_{\gamma_{\beta}}(u)=2 \pi\left(\delta_{p}-\delta_{n}\right)$.
(c) Let $u \in B V\left(S^{2}, S^{1}\right)$ be the function defined above in (b) and take $g$ the function constructed in Example 1(a). Set $w=g u \in B V\left(S^{2}, S^{1}\right)$. We have $S(w)=\{\theta=0\} \cup\{\theta=2 \pi / 3\} \cup\{\theta=4 \pi / 3\}$. We show that $T(w)=4 \pi\left(\delta_{p}-\delta_{n}\right)$. Indeed, construct the lifting $\psi \in B V\left(S^{2}, \mathbb{R}\right)$ of $w$ as

$$
\psi(\alpha, \theta)= \begin{cases}\theta & \text { if } \theta \in(0,2 \pi / 3), \alpha \in(0, \pi) \\ \theta+2 \pi / 3 & \text { if } \theta \in(2 \pi / 3,4 \pi / 3), \alpha \in(0, \pi) \\ \theta-2 \pi / 3 & \text { if } \theta \in(4 \pi / 3,2 \pi), \alpha \in(0, \pi)\end{cases}
$$

Observe that

$$
\psi^{+}-\psi^{-}=\rho\left(w^{+}, w^{-}\right)-2 \pi \chi_{\{\theta=0\}}-2 \pi \chi_{\{\theta=4 \pi / 3\}} \quad \text { in } S(w)
$$

and conclude that $T(w)=4 \pi\left(\delta_{p}-\delta_{n}\right)$. So, the north pole $p$ and the south pole $n$ which are the vortices of $g$ remain singularities for the function $w$; they appear now on the jump part of $w$. The same behavior happens to $T_{\gamma_{\beta}}$ for every $\beta \in[0,2 \pi)$, i.e. $T_{\gamma_{\beta}}(w)=4 \pi\left(\delta_{p}-\delta_{n}\right)$.

As we mentioned before, for every $u \in B V\left(S^{2}, S^{1}\right)$ there exists a bounded lifting $\varphi \in B V \cap L^{\infty}\left(S^{2}, \mathbb{R}\right)$ (see [5]). The striking fact is that we can construct functions $u \in B V\left(S^{2}, S^{1}\right)$ such that no optimal lifting belongs to $L^{\infty}$. We give such an example in the following:

Example 3. On the interval $(0,2 \pi)$ we consider

$$
p_{1}=1, \quad n_{k}=p_{k}+\frac{1}{4^{k}} \quad \text { and } \quad p_{k+1}=n_{k}+\frac{1}{2^{k}}, \quad \forall k \geqslant 1
$$

Suppose that this configuration of points lies on the equator $\left\{\frac{\pi}{2}\right\} \times[0,2 \pi]$ (in the spherical coordinates) of $S^{2}$ and we consider that each dipole $\left(p_{k}, n_{k}\right)$ appears $k$ times. Since $\sum_{k \geqslant 1} k d_{S^{2}}\left(p_{k}, n_{k}\right)<\infty$, set

$$
\Lambda=2 \pi \sum_{k \geqslant 1} k\left(\delta_{p_{k}}-\delta_{n_{k}}\right) \in \mathcal{Z}\left(S^{2}\right) .
$$

By [2] (Lemma 16),

$$
T\left(W^{1,1}\left(S^{2}, S^{1}\right)\right)=\mathcal{Z}\left(S^{2}\right)
$$

Thus, take $g \in W^{1,1}\left(S^{2}, S^{1}\right)$ such that $T(g)=\Lambda$. Using (2), it follows that

$$
\|T(g)\|=2 \pi \sum_{k \geqslant 1} k d_{S^{2}}\left(p_{k}, n_{k}\right) .
$$

Let $\varphi \in B V\left(S^{2}, \mathbb{R}\right)$ be an optimal lifting of $g$. Then there is a triple $(f, S, v) \in \mathcal{J}(T(g))$ such that

$$
\begin{equation*}
D \varphi=g \wedge \nabla g \mathcal{H}^{2}-f \nu \mathcal{H}^{1}\left\llcorner S \quad \text { and } \quad \int_{S}|f| \mathrm{d} \mathcal{H}^{1}=\|T(g)\|\right. \text {. } \tag{20}
\end{equation*}
$$

We may assume that $S=\{f \neq 0\}$.
We know that $\int_{S} f v \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=2 \pi \sum_{k \geqslant 1} k\left(\zeta\left(p_{k}\right)-\zeta\left(n_{k}\right)\right), \forall \zeta \in C^{1}\left(S^{2}\right)$. For each $k \geqslant 1$, we denote in the spherical coordinates $V_{k}=(0, \pi) \times\left(p_{k}-\frac{1}{8^{k}}, n_{k}+\frac{1}{8^{k}}\right)$. Then

$$
\int_{S} f v \cdot \nabla^{\perp} \zeta \mathrm{d} \mathcal{H}^{1}=2 \pi k\left(\zeta\left(p_{k}\right)-\zeta\left(n_{k}\right)\right), \quad \forall \zeta \in C^{1}\left(S^{2}\right) \text { with } \operatorname{supp} \zeta \subset V_{k}
$$

By (20), it follows that

$$
\int_{S \cap V_{k}}|f| \mathrm{d} \mathcal{H}^{1}=2 \pi k d_{S^{2}}\left(p_{k}, n_{k}\right)
$$

Using the same argument as in the proof of Proposition 3, we deduce that for each $k \in \mathbb{N}$,

$$
S(\varphi) \cap V_{k}=S \cap V_{k}=\overparen{n}_{k} \mathscr{p}_{k} \quad \text { and } \quad\left|\varphi^{+}-\varphi^{-}\right|=|f|=2 k \pi \quad \mathcal{H}^{1} \text {-a.e. on } \overparen{n}_{k} \overparen{p}_{k}
$$

where $\overparen{n}_{k} \widetilde{p}_{k}$ is the geodesic arc connecting $n_{k}$ and $p_{k}$. It yields that $\varphi \notin L^{\infty}$. So, every optimal $B V$ lifting of $g$ does not belong to $L^{\infty}$.

In the next example, we show that Theorem 3 fails if we minimize the energy in (8) just over the class of gradient maps:

Example 4. Let $u \in B V\left(S^{2}, S^{1}\right)$ be defined as

$$
u(\alpha, \theta)=\mathrm{e}^{\mathrm{i} \theta / 3}, \quad \forall \alpha \in(0, \pi), \theta \in(0,2 \pi)
$$

The jump set of $u$ is the meridian $\{\theta=0\}$ oriented counterclockwise and $\rho\left(u^{+}, u^{-}\right)=-2 \pi / 3$ on $S(u)$. We have that $T(u)=0$. On the other hand, for every $\psi \in C^{\infty}\left(S^{2}, \mathbb{R}\right)$, we have

$$
\begin{aligned}
\int_{S^{2}} \mid u \wedge \nabla u \mathcal{H}^{2}+\rho\left(u^{+}, u^{-}\right) v_{u} \mathcal{H}^{1}\left\llcorner S(u)-\nabla \psi \mathcal{H}^{2} \mid\right. & =\int_{S^{2}}|u \wedge \nabla u-\nabla \psi| \mathrm{d} \mathcal{H}^{2}+\int_{S(u)}\left|\rho\left(u^{+}, u^{-}\right)\right| \mathrm{d} \mathcal{H}^{1} \\
& \geqslant \int_{S(u)} \frac{2 \pi}{3} \mathrm{~d} \mathcal{H}^{1}=\frac{2 \pi^{2}}{3}>\|T(u)\|
\end{aligned}
$$

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